Generative Adversarial Networks

Dynamics

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UT Austin
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Creating new people

This person does not exist! Link

Tero Karras, Samuli Laine, and Timo Aila. A style-based generator architecture for generative adversarial networks. In: *Proceedings of the IEEE/CVF conference on computer vision and pattern recognition*. 2019, pp. 4401–4410.

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Starting point 200K samples of HQ headshots: CelebAHQ ▸ Link

Karras, Laine, and Aila 2019.

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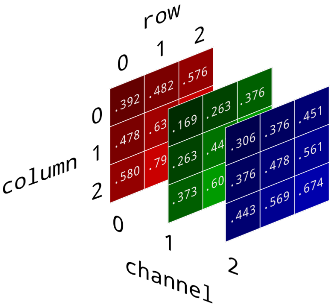
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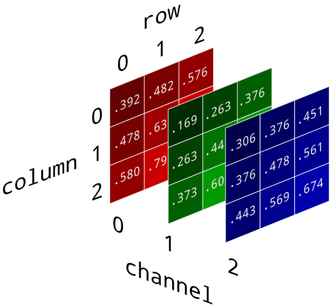

$$\{X_i\}_{i=1}^{200K} \subset \mathbb{R}^{1024 \times 1024 \times 3} = \mathbb{R}^{3145728}$$

Assumptions

1. We have access to infinite data samples that are independent and identically distributed:

$$\{X_i\}_{i=1}^{\infty} \text{ i.i.d. with distribution } P_* \in \mathcal{P}(\mathbb{R}^K)$$

with

$$1 \ll K$$

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2. The effective dimension of the problem is more manageable

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e.g. This person does not exist: $K = 3145728$ and $L = 512$, Parameter size: 310Mb, 40 days of GPU compute time.

Supervised Learning problem

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easy to evaluate, which we call the Generator,

we consider the distribution of the composition

$$g(Z) \sim g\#\mathcal{N} \in \mathcal{P}(\mathbb{R}^K)$$

Supervised Learning problem

Objective:

Find $g : \mathbb{R}^L \to \mathbb{R}^K$ easy to evaluate, such that

$$d(g\#\mathcal{N}, P_*) \quad \text{is small,}$$

for some meaningful metric d on $\mathcal{P}(\mathbb{R}^K)$.

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- If we consider the family $g_\theta(z)$ of parametric function, we can minimize over θ to get a supervised learning problem.
- Catch: We do not have access to the distribution P_*, but only to samples.

Vanilla GAN

Information theory Relative Entropy or Kullback–Leibler divergence

$$\mathcal{H}(g\#\mathcal{N}|P_*) = \begin{cases} \int_{\mathbb{R}^K} \left(\frac{dg\#\mathcal{N}}{dP_*}\right) \log\left(\frac{dg\#\mathcal{N}}{dP_*}\right) dP_* & g\#\mathcal{N} \ll P_* \\ +\infty & g\#\mathcal{N} \not\ll P_* \end{cases}$$

Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In: *Advances in neural information processing systems* 27 (2014).

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We need a way to evaluate it using samples.

Goodfellow, Pouget-Abadie, Mirza, Xu, Warde-Farley, Ozair, Courville, and Bengio 2014.

Duality

Legendre-Fenchel Transform:

$$\mathcal{H}(g\#\mathcal{N}|P_*) = \sup_{f\in C_b(\mathbb{R}^K)} \int_{\mathbb{R}^L} f(g(z))d\mathcal{N}(z) - \log \int_{\mathbb{R}^L} e^{g(x)} dP_*(x),$$

where

$$f : \mathbb{R}^K \to \mathbb{R}$$

is called the Discriminator.

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Advantage: For fixed Discriminator $f \in C_b(\mathbb{R}^K)$, we can sample the integrals:

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Given $m \in \mathbb{N}$ a batch size and $Z_1, ..., Z_m$ i.i.d. with distribution $\mathcal{N}$ and $X_1, ..., X_m$ i.i.d. with distribution P_*

$$\int_{\mathbb{R}^L} f(g(z)) d\mathcal{N}(z) - \log \int_{\mathbb{R}^L} e^{f(x)} dP_*(x)$$

$$\sim$$

$$\frac{1}{m} \sum_{i=1}^{m} f(g(Z_i)) - \log \frac{1}{m} \sum_{i=1}^{m} e^{f(X_i)}$$

For simplicity, we take the batch size $m = 1$ from now on, which is an estimator in expectation.

Degeneracy

If $g\#\mathcal{N} \not\ll P_*$, then $\mathcal{H}(g\#\mathcal{N}|P_*) = \infty$

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We will learn nothing if the distributions are not aligned from the start!

1-Wasserstein distance

Alternative, the 1-Wasserstein distance with Kantorovich's duality

Martin Arjovsky, Soumith Chintala, and Léon Bottou. Wasserstein generative adversarial networks. In: *International conference on machine learning*. PMLR. 2017, pp. 214–223.

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$$d_1(g\#\mathcal{N}, P_*) = \mathbb{E}_{(X,Z)\sim\pi}[\|X - g(Z)\|]$$

Arjovsky, Chintala, and Bottou 2017.

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$$d_1(g\#\mathcal{N}, P_*) = \mathbb{E}_{(X,Z)\sim\pi}[\|X - g(Z)\|]$$

$$= \sup_{f:\|f\|_{lip}\leq 1} \int_{\mathbb{R}^L} f(g(z))d\mathcal{N}(z) - \int_{\mathbb{R}^K} f(x)\, dP_*(x).$$

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The main advantage is that this distance does not degenerate.

Arjovsky, Chintala, and Bottou 2017.

Neural Networks

Introduce, the simplest setting 1 hidden layer Neural Networks:

$$g_\Theta(z) = \frac{1}{N}\sum_{i=1}^{N}\sigma(z;\theta_i) \qquad f_\Omega(x) = \frac{1}{M}\sum_{j=1}^{M}\sigma(x;\omega_i)$$

with $\Theta = (\theta_1, ...\theta_N)$ and $\Omega = (\omega_1, ..., \omega_M)$.

A typical smooth example is the sigmoid

$$\sigma(z;\theta_i) = \begin{pmatrix} \frac{a_i^1}{1+e^{-(b_i^1\cdot z+c_i^1)}} \\ \dots \\ \frac{a_i^K}{1+e^{-(b_i^K\cdot z+c_i^K)}} \end{pmatrix} \in \mathbb{R}^K$$

$$\theta_i = ((a_i^1, b_i^1, c_i^1), \dots, (a_i^K, b_i^K, c_i^K)) \in (\mathbb{R} \times \mathbb{R}^L \times \mathbb{R})^K$$

$$\sigma(x;\omega_j) = \frac{\alpha_j}{1 + e^{-(\beta_j\cdot x+\gamma_j)}} \in \mathbb{R}$$

$$\omega_j = (\alpha_j, \beta_j, \gamma_j) \in \mathbb{R} \times \mathbb{R}^K \times \mathbb{R}$$

Exchangeability

The relative order of the parameters does not affect the output function.

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Without loss of information we can encode

$$(\theta_1, \dots, \theta_N) \to \mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\theta_i} \in \mathcal{P}\left((\mathbb{R} \times \mathbb{R}^L \times \mathbb{R})^K \right)$$

and

$$(\omega_1, \dots, \omega_N) \to \nu_M = \frac{1}{M} \sum_{i=1}^{M} \delta_{\omega_i} \in \mathcal{P}\left(\mathbb{R} \times \mathbb{R}^K \times \mathbb{R} \right).$$

Algorithm

Algorithm 1 WGAN, our proposed algorithm. All experiments in the paper used the default values $\alpha = 0.00005$, $c = 0.01$, $m = 64$, $n_{\text{critic}} = 5$.

Require: : α, the learning rate. c, the clipping parameter. m, the batch size. n_{critic}, the number of iterations of the critic per generator iteration.

Require: : w_0, initial critic parameters. θ_0, initial generator's parameters.

while θ has not converged **do**

  for $t = 0, \ldots, n_{\text{critic}}$ **do**

    Sample $\{x^{(i)}\}_{i=1}^m \sim \mathbb{P}_r$ a batch from the real data.

    Sample $\{z^{(i)}\}_{i=1}^m \sim p(z)$ a batch of prior samples.

    $g_w \leftarrow \nabla_w \left[\frac{1}{m}\sum_{i=1}^m f_w(x^{(i)}) - \frac{1}{m}\sum_{i=1}^m f_w(g_\theta(z^{(i)}))\right]$

    $w \leftarrow w + \alpha \cdot \text{RMSProp}(w, g_w)$

    $w \leftarrow \text{clip}(w, -c, c)$

  end for

  Sample $\{z^{(i)}\}_{i=1}^m \sim p(z)$ a batch of prior samples.

  $g_\theta \leftarrow -\nabla_\theta \frac{1}{m}\sum_{i=1}^m f_w(g_\theta(z^{(i)}))$

  $\theta \leftarrow \theta - \alpha \cdot \text{RMSProp}(\theta, g_\theta)$

end while

Arjovsky, Chintala, and Bottou 2017.

Important parameters

- Learning rate $\alpha = 0.00005$, we consider $\Delta t = \alpha / N$ the fictitious time discretization.

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- $c = 0.01$ the clipping parameter that imposes $\|\omega_i\|_\infty \leq c$ to satisfy a uniform Lipschitz bound.
- RMSProp is a version of SGD that normalizes the gradient sizes componentwise to escape plateaus. For some $\beta \in [0, 1]$:

$$
\begin{aligned}
M_k^i &= (1-\beta) M_{k-1}^i + \beta |\partial_{\theta_i} E(\Theta_k)|^2 \\
\theta_{k+1}^i &= \theta_{k+1}^i - \alpha \frac{\partial_{\theta_i} E(\Theta_k)}{\sqrt{M_k^i}}
\end{aligned}
$$

Supervised learning

Supervised learning:

$$\min_{\Theta} E[\Theta] = \min_{\Theta} \int |g_{\Theta}(x) - g_*(x)|^2 \, dP_*(x) = \min_{\Theta} \int e(\Theta, x) \, dP_*(x)$$

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While Θ has not converged:
  Sample $X_k \sim P_*$

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SGD is a stochastic discretization of

$$\dot{\Theta} = -\nabla_{\Theta} E[\Theta].$$

SGD as a Stochastic discretization

Using, exchangeability

$$g_{\Theta}(x) = \frac{1}{N}\sum_{i=1}^{N} \sigma(x, \theta_i) = \langle g(x, \cdot), \mu_N \rangle$$

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$$g_\Theta(x) = \frac{1}{N}\sum_{i=1}^{N}\sigma(x,\theta_i) = \langle g(x,\cdot),\mu_N\rangle$$

we notice

$$\partial_{\theta_i}E[\theta] = \frac{2}{N}\int (g_\Theta(x) - g_*(x))\partial_2\sigma(x,\theta_i)dP_*(x) = \frac{1}{N}V[\mu_N](\theta_i)$$

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Namely $\dot{\Theta}(t) = -\nabla E[\Theta(t)]$, if and only if,

$$\begin{cases} \partial_t \mu_N(t) + \frac{1}{N}\nabla \cdot (\mu_N(t) V[\mu_N(t)]) = 0 \\ \mu_N(0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\theta_{i,in}} \end{cases}$$

Convergence of the dynamics

Theorem (Law of Large Numbers)

Assume $\{\theta_{i,in}\}$ i.i.d. sampled from μ_{in}. Then, μ_N converges to a deterministic process which concentrates in the unique solution to

$$\begin{cases} \partial_t \mu(t) + \nabla \cdot (\mu(t) V[\mu(t)]) = 0 \\ \mu(0) = \mu_{in} \end{cases} \tag{SGD}$$

Lenaic Chizat and Francis Bach. On the global convergence of gradient descent for over-parameterized models using optimal transport. In: *Advances in neural information processing systems* 31 (2018); Song Mei, Andrea Montanari, and Phan-Minh Nguyen. A mean field view of the landscape of two-layer neural networks. In: *Proceedings of the National Academy of Sciences* 115.33 (2018), E7665–E7671; Justin Sirignano and Konstantinos Spiliopoulos. Mean field analysis of neural networks: A law of large numbers. In: *SIAM Journal on Applied Mathematics* 80.2 (2020), pp. 725–752; Grant Rotskoff and Eric Vanden-Eijnden. Trainability and accuracy of artificial neural networks: An interacting particle system approach. In: *Communications on Pure and Applied Mathematics* 75.9 (2022), pp. 1889–1935.

Gradient flow interpreation

Considering the energy $\mathsf{E} : \mathcal{P} \to \mathbb{R}$, given by

$$\mathsf{E}[\mu] = \frac{1}{2} \int |g_\mu(x) - g_*(x)|^2 \, dP_*(x)$$

we have that (SGD) is the 2-Wasserstein gradient flow of E.

Aggregation Equation

Moreover, expanding the square we obtain the aggregation equation:

$$\mathsf{E}[\mu] = \frac{1}{2}\int W(\theta_1, \theta_2)d\mu(\theta_1)d\mu(\theta_2) + \int V(\theta)d\mu(\theta) + C,$$

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where

$$W(\theta_1, \theta_2) = \int \sigma(x; \theta_1) \sigma(x; \theta_2) dP_*(x)$$

and

$$V(\theta) = -\int g_*(x) \sigma(x; \theta) \; dP_*(x).$$

W-GAN as a discretization

Replacing RMSProp by SGD, we have the algorithm

$$\begin{cases} \theta_i^{k+1} = \theta_i^k + \Delta t \; v_\theta[\mu_N, \nu_M](\theta_i; (X_k, Z_k)) \\ \omega_j^{k+1} = Proy_Q(\omega_j^k + \gamma_c \Delta t \; v_\omega[\mu_N, \nu_M](\omega_j^k; (X_k, Z_k))), \end{cases}$$

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where

$$Q = [-c, c]^{1+L+1}, \qquad \gamma_c = n_c \frac{N}{M}$$

and $\{X_k\}_{k=0}^{\infty}$ and $\{Z_k\}$ i.i.d sampled from P_* and $\mathcal{N}$ respectively.

WGAN as a PDE

The associated PDE is given by

$$\begin{cases} \partial_t \mu - \nabla \cdot (\partial_\mu \Psi[\mu, \nu] \mu) = 0 \\ \partial_t \nu + \gamma_c \nabla \cdot (\mathbf{Proj}_{\Pi_Q} \partial_\nu \Psi[\mu, \nu] \nu) = 0 \end{cases} \tag{WGAN-PDE}$$

where

$$\Psi[\mu, \nu] = \int_{\mathbb{R}^L} f_\nu(g_\mu(z))\, d\mathcal{N}(z) - \int_{\mathbb{R}^K} f_\nu(x)\, dP_*(x)$$

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where

$$\Psi[\mu, \nu] = \int_{\mathbb{R}^L} f_\nu(g_\mu(z))\, d\mathcal{N}(z) - \int_{\mathbb{R}^K} f_\nu(x)\, dP_*(x)$$

Notice that $\mathbf{Proj}_{\Pi_Q} : Q \times \mathbb{R}^K \to \mathbb{R}^K$ is a discontinuous operator on ∂_Q.

Well Posedness and Coagulation at the Boundary

Proposition (jww R. Cabrera & B. Suassuna)

If the activation function is smooth, then (WGAN-PDE) has a unique stable solution:

$$
\begin{aligned}
&d_2(\mu_1(t), \mu_2(t)) + d_2(\nu_1(t), \nu_2(t)) \\
&\leq C(d_4(\mu_{1,in}, \mu_{2,in}) + d_2(\nu_{1,in}, \nu_{2,in}))
\end{aligned}
$$

for any $t \in [0, T]$.

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Observation: If the support of ν hits ∂Q it will flatten, and can never fatten back up.

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Observation: If the support of ν hits ∂Q it will flatten, and can never fatten back up.

In particular, the support it can coagulate to a single point in finite time t_0, and $\nu(t) = \delta_{\omega(t)}$ for any $t > t_0$.

Quantified convergence

Theorem (jww R. Cabrera & B. Suassuna)

Consider $(\mu_N(t), \nu_N(t))$ *the time interpolation of the empirical measures* $\{(\mu_N^k, \nu_N^k)\}_{k=1}^{\infty}$ *given by the WGAN algorithm, then for any fixed time interval* $t \in [0, T]$

$$\mathbb{E} d_2^2((\mu_N(t), \nu_N(t)), (\mu_\infty(t), \nu_\infty(t))) \leq \frac{C}{N}$$

where (μ_∞, ν_∞) *is the unique solution to* (WGAN-PDE) *with initial condition* $\mu_{in} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\theta_i}$, $\nu_{in} = \frac{1}{M}\sum_{j=1}^{M} \delta_{\omega_j}$.

Quantified convergence

Corollary (jww R. Cabrera & B. Suassuna)

If $\{\theta_i\}_{i=1}^N$, $\{\omega_j\}_{j=1}^M$ i.i.d. sampled from $\overline{\mu}_{in}$ and $\overline{\nu}_{in}$ respectively, then

$$\mathbb{E}d_2^2((\mu_N(t),\nu_N(t)),(\overline{\mu}_\infty(t),\overline{\nu}_\infty(t))) \leq \frac{C}{N^{\frac{1}{K(2+L)}}}$$

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Remark: The Wasserstein distance suffers from the curse of dimensionality, when we approximate by samples.

Proof

Compare SGD

$$\theta^{k+1} = \theta^k + \Delta t v(\theta^k, X_k)$$

with (Projected) Forward Euler

$$\tilde{\theta}^{k+1} = \tilde{\theta}^k + \Delta t V(\tilde{\theta}^k)$$

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$$e_{k+1} = \theta^{k+1} - \tilde{\theta}^{k+1} \leq (1 + \Delta t |V|_{lip}) e_k + \Delta t M_k,$$

with

$$M_k = v(\theta^k, X_k) - V(\theta^k)$$

Proof

Compare SGD

$$\theta^{k+1} = \theta^k + \Delta t v(\theta^k, X_k)$$

with (Projected) Forward Euler

$$\tilde{\theta}^{k+1} = \tilde{\theta}^k + \Delta t V(\tilde{\theta}^k)$$

$$e_{k+1} = \theta^{k+1} - \tilde{\theta}^{k+1} \leq (1 + \Delta t|V|_{lip})e_k + \Delta t M_k,$$

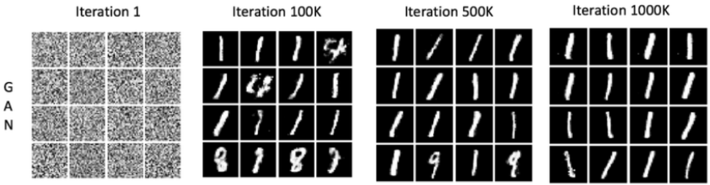
with

$$M_k = v(\theta^k, X_k) - V(\theta^k)$$

Gromwall' inequality, we have

$$\mathbb{E}[|e_k|^2] \leq (\Delta t)^2 \mathbb{E}\left|\sum_{r=0}^{k}(1 + \Delta t|V|_{lip})^{k-r} M_r\right|^2 \leq C\Delta t.$$

Mode Collapse



Mode Collapse

Chat-GPT loves to delve:

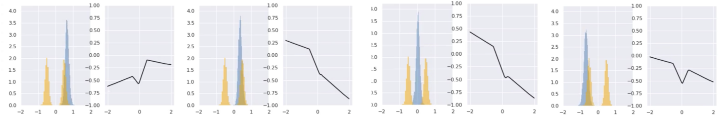
Abstract

Generative Adversarial Networks (GANs) was one of the first Machine Learning algorithms to be able to generate remarkably realistic synthetic images. In this presentation, we **DELVE** into the mechanics of the GAN algorithm and its profound relationship with optimal transport theory. Through a detailed exploration, we illuminate how GAN approximates a system of PDE, particularly evident in shallow network architectures. Furthermore, we investigate the phenomenon of mode collapse, a well-known pathological behavior in GANs, and elucidate its connection to the underlying PDE framework through an illustrative example.

Failure to converge

Example: $K = 1$, $L = 1$

$$P_* = \frac{1}{2}\mathcal{N}(0,-1) + \frac{1}{2}\mathcal{N}(0,1)$$



Video

Toy Example

$K = 1$, $L = 1$, $P_* = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and activation functions

$$g(z;\theta) = \begin{cases} -1 & \text{if } z < \theta \\ 1 & \text{if } z > \theta \end{cases} \qquad f(x,\omega) = (\omega x)_+.$$

Toy Example

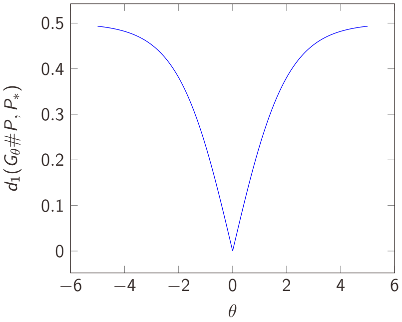
$K = 1$, $L = 1$, $P_* = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and activation functions

$$g(z;\theta) = \begin{cases} -1 & \text{if } z < \theta \\ 1 & \text{if } z > \theta \end{cases} \qquad f(x,\omega) = (\omega x)_+.$$

$$g_\theta \# \mathcal{N} = \Phi(\theta)\delta_{-1} + (1 - \Phi(\theta))\delta_1$$

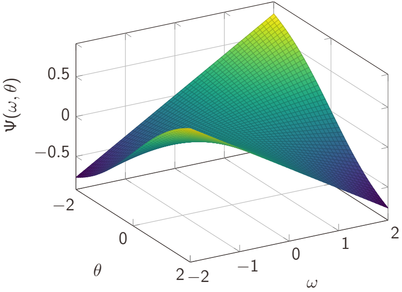
Graphs

$$d_1(g_\theta \# \mathcal{N}, P_*) = \max_{\omega \in [-1,1]} \int f_\omega(g_\theta(z)) d\mathcal{N}(z) - \int f_\omega(x) dP_*(x)$$



Graphs

$$\begin{aligned}\Psi(\omega, \theta) &= \int_{\mathbb{R}} D_\omega(g_\theta(z))\, dP(z) - \int_{\mathbb{R}} D_\omega(x)\, dP_*(x) \\ &= \left(\frac{1}{2} - \Phi(\theta)\right)\omega.\end{aligned}$$



Toy Example:ODE dynamics

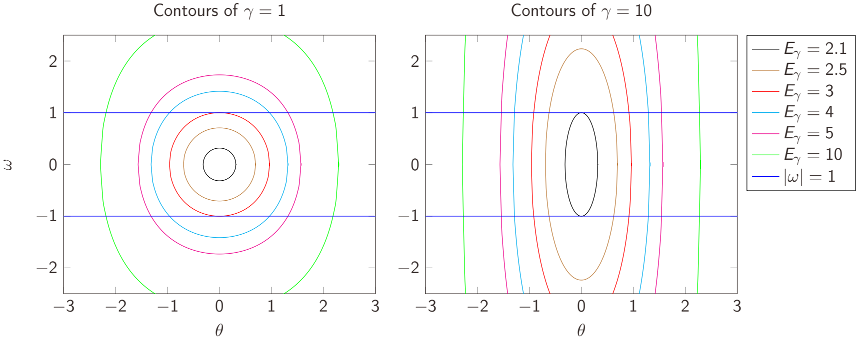
Gradient descent/ascent gives rise to periodic orbits. If we consider

$$E_\gamma[\theta, \omega] = \cosh(\theta) + \frac{1}{\gamma}|\omega|^2,$$

then for all $t > 0$

$$E_\gamma[\theta(t), \omega(t)] = E_\gamma[\theta_{in}, \omega_{in}]$$

Periodic Orbits



Questions?

Thank you!