# **Functional Analysis**

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### Bio:

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2006-2011 Lic. en Matematica at UNC (Cordoba, Argentina)
2011-2016 Ph.D. Applied Mathematics at UMD (Maryland, United States)
2016-2017 Postdoc at ICTP (Trieste, Italy)
2017-2019 Postdoc at Imperial College (London, England)
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2019- Proffessor at PUC (Rio de Janeiro, Brazil)

2020-2022 Hooke Fellow at Oxford (Oxford, England)

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Differential Calculus and L<sup>p</sup> spaces.

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- Embedding Theorems.

## Recomended books

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- Linear Functional Analysis by Rynne, Bryan, Youngson, M.A.
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### **Sobolev Spaces**

- Measure Theory and Fine Properties of Functions, by L.C. Evans and R.F. Gariepy.
- Functional Analysis, Sobolev Spaces and Partial Differential Equations, by Haim Brezis
- Sobolev Spaces, by R.A. Adams and J.J.F. Fournier.
- The Analysis of Partial Differential Operators I, by L. Hörmander.

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That is to say:

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- If  $f \in F(U)$  and  $c \in \mathbb{R}$ , then  $(cf) \in F(U)$ , where (cf)(x) = cf(x).

### Definition

A subspace  $X \subset F(U)$  endowed with topology  $\tau$ , is a Topological Vector Space. If the operations addition and multiplication by a scalar are continuous in the topology  $\tau$ .

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More specifically, we consider summing two functions as an application  $sum: F(U) \times F(U) \to F(U)$ . Then, the definition is asking that sum is continuous when we endow  $F(U) \times F(U)$  with the product topology  $\tau \times \tau$ .

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i.e. whenever we have sequence  $\{f_n\}_{n\in\mathbb{N}}$ ,  $\{g_n\}_{n\in\mathbb{N}}\subset X$ , such that

$$f_n \to^{\tau} f$$
 &  $g_n \to^{\tau} g$ ,

then

$$f_n + g_n \rightarrow^{\tau} f + g$$
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The most studied types of vector spaces:

- **Metrizable/Metric:** There exists a metric/distance  $d_{\tau}(\cdot, \cdot): X \times X \to [0, \infty)$ , satisfying the triangle inequality that induces the topology  $\tau$ .
- **Normed:** There exists a norm  $\|\cdot\|_{\tau}: X \to [0,\infty)$  that induces the metric/distance  $d_{\tau}(f,g) = \|f-g\|_{\tau}$  that induces the topology  $\tau$ .
- Inner Product: There exists an inner product  $\langle \cdot, \cdot \rangle_{\tau} : X \times X \to \mathbb{R}$  that induces a norm  $\langle f, f \rangle_{\tau}^{1/2} = \|f\|_{\tau}$  that induces a metric, that induces the topology  $\tau$ .

Examples: Take  $U=\mathbb{R}^n$  or any other measure space with underlying measure  $\mu$ , we have

$$L^{p}(U) = \{ f \in F(U) \cap \mathcal{M}(U) : \int_{U} |f(x)|^{p} d\mu(x) < \infty \}.$$

- For  $p \ge 1$ ,  $L^p$  is a **Normed** vector space.
- For p = 2,  $L^2$  is an **Inner-Product** space.
- For  $0 , <math>L^p$  is a **Metric** vector space with distance

$$d_p(f,g) = \int_U |f-g|^p d\mu.$$



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$$d(f_n, f_m) < \epsilon$$
.

By the definition there exists a unique  $f_{\infty} \in X$ , such that

$$\lim_{n\to\infty}d(f_n,f_\infty)=0.$$



**Completeness** is such an important property that spaces change their name.

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**Warning:** The set of indexes *I* is not necessarily countable!

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### Corollary

If X is a separable Hilbert space, then it is homeomorphic to  $L^2([0,1])$ .

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Any bounded sequence, admits a convergent subsequence. Alternatively, every closed bounded set is compact.

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In  $\mathbb{R}^n$  all norms induce the same topology.

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$$||e_i - e_j||^2 = \langle e_i - e_j, e_i - e_j \rangle = 2(1 - \langle e_i, e_j \rangle) = 2\delta_{i,j}$$

For  $L^2([0,1])$  this is like taking the Fourier basis

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This is known as the Riemann-Lebesgue Lemma.



Given two **normed** vector spaces X, Y we can consider  $\Omega \subset X$  open, an application

$$F:\Omega\subset X\to Y$$

and the set of continuous linear mappings

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If  $Y = \mathbb{R}$  with the usual topology, then we denote  $\mathcal{L}(X, R) = X^*$ .



#### Definition

A mapping  $F:\Omega\to Y$  is said to be (Frechet) differentiable at  $x_0\in\Omega$  if exists  $L\in\mathcal{L}(X,Y)$  such that

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The idea is to lose fear and realize that most of the properties that you know for differentiable vector valued functions  $F: \mathbb{R}^n \to \mathbb{R}^m$  are still valid in this case.

For instance,

### Theorem (Fundamental Theorem of Calculus)

If  $F:(a,b)\subset\mathbb{R}\to Y$  is  $C^1$  and Y is a **Banach** space, then

$$F(t) = F(s) + \int_{s}^{t} dg(u) \ du,$$

where we define the integral via Riemman sums.

Remark: Mean Value theorem doesn't hold, but MV Inequality does.

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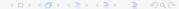
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In this case dF = g.



Given a measure space  $(\Omega, \mathcal{F}, \mu)$  we can define

$$L^p(\Omega;d\mu)=\left\{f:\Omega o\mathbb{R}\; extit{measurable}\;:\;\int_\Omega|f|^p\;d\mu<\infty
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Typical case  $\Omega \subset \mathbb{R}^n$  and  $\mu$  is the Lebesgue measure.



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■ For  $1 <math>L^p$  is reflexive

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■ Hölder inequality or duality pairing for  $X = L^p(\Omega; d\mu)$ 

$$\left| \int_{\Omega} fg \ d\mu \right| = |\langle f, g \rangle_{X, X^*}| \le \|f\|_X \|g\|_{X^*} = \|f\|_{L^p} \|g\|_{L^q},$$

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A useful variation

$$||fg||_{L^r} \leq ||f||_{L^p}||g||_{L^q},$$

if 
$$1/p + 1/q = 1/r$$
.



■ Minkowski's or triangle inequality for the norm

$$||f+g||_{L^p} \le ||f||_{L^p} + ||g||_{L^p}$$

# L<sup>p</sup> spaces

Minkowski's or triangle inequality for the norm

$$||f + g||_{L^p} \le ||f||_{L^p} + ||g||_{L^p}$$

Interpolation

$$||f||_{L^r} = ||f||_{L^p}^{\lambda} ||f||_{L^q}^{1-\lambda}$$

if

$$\frac{\lambda}{p} + \frac{1 - \lambda}{q} = \frac{1}{r}.$$



# $L^p$ spaces

Young's inequality for the convolution

$$||f * g||_{L^r} \le ||f||_{L^p} ||g||_{L^q}$$

if

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}.$$



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$$C_c^{\infty}(\Omega) = \{ \varphi \in C_c^{\infty} : supp \varphi = \{ x \in \Omega : \varphi(x) \neq 0 \} \text{ is compact} \}.$$

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The basic building block is

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1\\ 0 & |x| \ge 1. \end{cases}$$



We can show existence of partitions of unity

#### Lemma

Given an open covering  $\{U_i\}_{i\in I}$  there exists positive functions  $\varphi_i \in C_c^{\infty}(U_i)$  such that

$$\sum_{i\in I}\varphi_i(x)=1\qquad\forall x\in\Omega,$$

such that only a finite number of them are non-zero.

We can show existence of cut-off functions with estimates

#### Lemma

For any V compact subset  $\Omega$ , there exists  $\chi_V \in C_c^{\infty}(\Omega)$ , such that

$$\chi_V(x) = 1$$
 for all  $x \in V$ 

and

$$|D^{\alpha}(\chi_V)(x)| \leq C_{\alpha} d(x, \partial \Omega)^{-|\alpha|}.$$

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A mollification of a function  $f \in L^p$  is given by

$$f^{\varepsilon}(x) = f * \varphi_{\varepsilon}(x) = \int_{\mathbb{D}^n} f(y) \varphi_{\varepsilon}(x-y) dy,$$

where we have extended f by zero outside of  $\Omega$ .

#### **Properties:**

 $lacksquare f^arepsilon\in \mathcal{C}^\infty$  and

$$D^{\alpha}f^{\varepsilon}=\left( D^{\alpha}\varphi_{\varepsilon}\right) \ast f.$$

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 $lacksquare f^arepsilon\in \mathcal{C}^\infty$  and

$$D^{\alpha}f^{\varepsilon}=(D^{\alpha}\varphi_{\varepsilon})*f.$$

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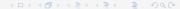
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### Theorem (Monotone Convergence)

If  $f_n \to f$  a.e. is monotone increasing i.e.  $f_n(x) \le f_{n+1}(x)$  for every x and n, then

$$\lim_{n} \int f_{n} = \sup_{n} \int f_{n} = \int \sup_{n} f_{n} = \int f.$$



#### Theorem (Lebesgue Dominated Convergence)

If  $f_n \to f$  a.e. and exists  $g \in L^1$  such that  $|f_n|(x) \le g(x)$  for every n and x then

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#### Theorem (Vitali's Convergence theorem)

Given  $\Omega$  a set of finite measure  $|\Omega| < \infty$ , then  $f_n \to f$  in  $L^p$ , i.e.

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if and only if,

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■ The family  $\{f_n\}$  is equintegrable. i.e. for every  $\varepsilon > 0$  exists a  $\delta$  such that for every measurable set A satisfying  $|A| < \delta$  implies

$$\int_A |f_n|^p \le \varepsilon.$$

### Theorem (Riesz-Kolmogorov)

For  $1 \le p < \infty$ , a family  $\{f_i\}_{i \in I}$  is pre-compact in  $L^p(\mathbb{R}^n)$ , iff,

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■ It is tight: for every  $\varepsilon > 0$  there exists a compact set  $K \subset \mathbb{R}^n$  such that

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■ They have a uniform modulus of continuity. i.e. there exists  $\omega:[0,\infty)\to[0,\infty)$  such that  $\omega$  is increasing and

$$\sup_{x,y,l} d(f_i(x), f_i(y)) \leq \omega(d(x,y)).$$

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- Tightness, you need be careful things are not escaping to infinity.

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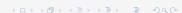
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Bounded sets in  $L^1$  are not compact!



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$$f_n(x) = \frac{1}{n^{1/p}} \chi_{(-n,n)}$$



## Weak convergence

Weak or Weak-\* convergence can be characterized by any dense subset of the space X or  $X^*$ .

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$$|\Omega| < 1$$
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#### Theorem (Riesz-Representation Theorem)

The dual of continuous functions with compact support is locally finite measures:

$$\mathcal{M}_{loc}(\Omega) = (C_c(\Omega))^*.$$

Small caveat with the topology of  $C_c(\Omega)$ ,  $f_n \to f$  if  $||f_n - f||_{\infty} \to 0$  and support of  $\bigcup_n supp f_n$  is compact. More, next class.

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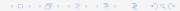
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