Functional Analysis

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Who is this guy teaching me Functional?

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Bio:

2006-2011	Lic. en Matematica at UNC (Cordoba, Argentina)
2011-2016	Ph.D. Applied Mathematics at UMD (Maryland, United States)
2016-2017	Postdoc at ICTP (Trieste, Italy)
2017-2019	Postdoc at Imperial College (London, England)
2019-	Proffessor at PUC (Rio de Janeiro, Brazil)
2020-2022	Hooke Fellow at Oxford (Oxford, England)

Outline for the course

- Differential Calculus and L^p spaces.

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- Distributions.

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- Sobolev Spaces.
- Embedding Theorems.

Recomended books

Foundations functional analysis and distributions

- Linear Functional Analysis by Rynne, Bryan, Youngson, M.A.
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Sobolev Spaces

- Measure Theory and Fine Properties of Functions, by L.C. Evans and R.F. Gariepy.
- Functional Analysis, Sobolev Spaces and Partial Differential Equations, by Haim Brezis
- Sobolev Spaces, by R.A. Adams and J.J.F. Fournier.
- The Analysis of Partial Differential Operators I, by L. Hörmander.

Not so gentle introduction

Functional Analysis: When Analysis/(point set) Topology meets Linear Algebra.

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- If $f, g \in F(U)$, then $f + g \in F(U)$, where $(f + g)(x) = f(x) + g(x)$.
- If $f \in F(U)$ and $c \in \mathbb{R}$, then $(cf) \in F(U)$, where $(cf)(x) = cf(x)$.

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More specifically, we consider summing two functions as an application $sum : F(U) \times F(U) \to F(U)$. Then, the definition is asking that *sum* is continuous when we endow $F(U) \times F(U)$ with the product topology $\tau \times \tau$.

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i.e. whenever we have sequence $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}} \subset X$, such that

$$f_n \to^{\tau} f \qquad \& \qquad g_n \to^{\tau} g,$$

then

$$f_n + g_n \to^{\tau} f + g.$$

Not so gentle introduction

The most studied types of vector spaces:

- **Metrizable/Metric:** There exists a metric/distance $d_\tau(\cdot,\cdot) : X \times X \to [0,\infty)$, satisfying the triangle inequality that induces the topology τ.
- **Normed:** There exists a norm $\|\cdot\|_\tau : X \to [0,\infty)$ that induces the metric/distance $d_\tau(f,g) = \|f-g\|_\tau$ that induces the topology τ.
- **Inner Product:** There exists an inner product $\langle\cdot,\cdot\rangle_\tau : X \times X \to \mathbb{R}$ that induces a norm $\langle f, f\rangle_\tau^{1/2} = \|f\|_\tau$ that induces a metric, that induces the topology τ.

Not so gentle introduction

Examples: Take $U = \mathbb{R}^n$ or any other measure space with underlying measure μ, we have

$$L^p(U) = \{f \in F(U) \cap \mathcal{M}(U) \ : \ \int_U |f(x)|^p \, d\mu(x) < \infty\}.$$

- For $p \geq 1$, L^p is a **Normed** vector space.
- For $p = 2$, L^2 is an **Inner-Product** space.
- For $0 < p < 1$, L^p is a **Metric** vector space with distance

$$d_p(f, g) = \int_U |f - g|^p \, d\mu.$$

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By the definition there exists a unique $f_\infty \in X$, such that

$$\lim_{n\to\infty} d(f_n, f_\infty) = 0.$$

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Warning: The set of indexes I is not necessarily countable!

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Corollary

If X is a **separable Hilbert** *space, then it is* **homeomorphic** *to $L^2([0,1])$.*

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Corollary

In $\mathbb{R}^n$ all norms induce the same topology.

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$$\|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j\rangle = 2(1 - \langle e_i, e_j\rangle) = 2\delta_{i,j}$$

For $L^2([0,1])$ this is like taking the Fourier basis

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This is known as the Riemann-Lebesgue Lemma.

Differential Calculus

Given two **normed** vector spaces X, Y we can consider $\Omega \subset X$ open, an application

$$F : \Omega \subset X \to Y$$

and the set of continuous linear mappings

$$\mathcal{L}(X, Y) = \{L : X \to Y \ : \ L \text{ is linear and continuous}\}.$$

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If $Y = \mathbb{R}$ with the usual topology, then we denote $\mathcal{L}(X, R) = X^*$.

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The idea is to lose fear and realize that most of the properties that you know for differentiable vector valued functions $F : \mathbb{R}^n \to \mathbb{R}^m$ are still valid in this case.

Differential Calculus

For instance,

Theorem (Fundamental Theorem of Calculus)

If $F : (a, b) \subset \mathbb{R} \to Y$ is C^1 and Y is a **Banach** *space, then*

$$F(t) = F(s) + \int_s^t dg(u)\, du,$$

where we define the integral via Riemman sums.

Remark: Mean Value theorem doesn't hold, but MV Inequality does.

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- F *is continuous.*

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In this case $dF = g$.

L^p spaces

Given a measure space $(\Omega, \mathcal{F}, \mu)$ we can define

$$L^p(\Omega; d\mu) = \left\{ f : \Omega \to \mathbb{R} \textit{ measurable } : \int_\Omega |f|^p \, d\mu < \infty \right\}.$$

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Typical case $\Omega \subset \mathbb{R}^n$ and μ is the Lebesgue measure.

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L^p endowed with the norm

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- For $1 < p < \infty$ L^p is reflexive

$$(L^p)^{**} = L^p.$$

L^p spaces

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L^p spaces

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- Hölder inequality or duality pairing for $X = L^p(\Omega; d\mu)$

$$\left|\int_\Omega fg \, d\mu\right| = |\langle f, g\rangle_{X,X^*}| \leq \|f\|_X \|g\|_{X^*} = \|f\|_{L^p} \|g\|_{L^q},$$

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  A useful variation

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q},$$

  if $1/p + 1/q = 1/r$.

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- Interpolation

$$\|f\|_{L^r} = \|f\|_{L^p}^{\lambda} \|f\|_{L^q}^{1-\lambda}$$

if

$$\frac{\lambda}{p} + \frac{1-\lambda}{q} = \frac{1}{r}.$$

L^p spaces

- Young's inequality for the convolution

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Mollification

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The basic building block is

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Mollification

We can show existence of partitions of unity

Lemma

Given an open covering $\{U_i\}_{i\in I}$ *there exists positive functions* $\varphi_i \in C_c^\infty(U_i)$ *such that*

$$\sum_{i\in I} \varphi_i(x) = 1 \qquad \forall x \in \Omega,$$

such that only a finite number of them are non-zero.

Mollification

We can show existence of cut-off functions with estimates

Lemma

For any V compact subset Ω, there exists $\chi_V \in C_c^\infty(\Omega)$, such that

$$\chi_V(x) = 1 \qquad \textit{for all } x \in V$$

and

$$|D^\alpha(\chi_V)(x)| \le C_\alpha d(x, \partial\Omega)^{-|\alpha|}.$$

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A mollification of a function $f \in L^p$ is given by

$$f^\varepsilon(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)\, dy,$$

where we have extended f by zero outside of Ω.

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Then C_c^∞ dense in L^p.

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Theorem (Fatou's Lemma)

If $f_n \to f$ a.e and $f_n \geq 0$, then

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Theorem (Monotone Convergence)

If $f_n \to f$ a.e. is monotone increasing i.e. $f_n(x) \leq f_{n+1}(x)$ for every x and n, then

$$\lim_n \int f_n = \sup_n \int f_n = \int \sup_n f_n = \int f.$$

Convergence Theorems

Theorem (Lebesgue Dominated Convergence)

If $f_n \to f$ a.e. and exists $g \in L^1$ such that $|f_n|(x) \leq g(x)$ for every n and x then

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Convergence Theorems

Theorem (Vitali's Convergence theorem)

Given Ω *a set of finite measure* $|\Omega| < \infty$, *then* $f_n \to f$ *in* L^p, *i.e.*

$$\|f_n - f\|_{L^p} \to 0,$$

if and only if,

- $f_n \to f$ *in measure, i.e. for every* $\varepsilon > 0$

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- *The family $\{f_n\}$ is equintegrable. i.e. for every $\varepsilon > 0$ exists a δ such that for every measurable set A satisfying $|A| < \delta$ implies*

$$\int_A |f_n|^p \leq \varepsilon.$$

Convergence/Compactness

Theorem (Riesz-Kolmogorov)

For $1 \leq p < \infty$*, a family* $\{f_i\}_{i \in I}$ *is pre-compact in* $L^p(\mathbb{R}^n)$*, iff,*

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- *It is tight: for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that*

$$\sup_I \int_{K^c} |f_i|^p \le \varepsilon.$$

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- *They have a uniform modulus of continuity. i.e. there exists* $\omega : [0,\infty) \to [0,\infty)$ *such that* ω *is increasing and*

$$\sup_{x,y,I} d(f_i(x), f_i(y)) \leq \omega(d(x,y)).$$

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- **Boundedness** in the appropriate norm.
- Uniform **Regularity** measured in the appropriate norm.
- **Tightness**, you need be careful things are not escaping to infinity.

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Bounded sets in L^1 are not compact!

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- **Not tight:**

$$f_n(x) = \frac{1}{n^{1/p}} \chi_{(-n,n)}$$

Weak convergence

Weak or Weak-* convergence can be characterized by any dense subset of the space X or X^*.

Theorem (Dunford-Pettis)

$|\Omega| < 1$ *and* $1 < p < \infty$, *then* $f_n \rightharpoonup f$, *if and only if*

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Given $f \in L^1(\Omega)$, we can consider

$$\mu_f = f d\mathcal{L} \qquad \text{i.e.} \qquad \mu_f(A) = \int_A f \, d\mathcal{L}.$$

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Theorem (Riesz-Representation Theorem)

The dual of continuous functions with compact support is locally finite measures:

$$\mathcal{M}_{loc}(\Omega) = (C_c(\Omega))^*.$$

Small caveat with the topology of $C_c(\Omega)$, $f_n \to f$ if $\|f_n - f\|_\infty \to 0$ and support of $\bigcup_n suppf_n$ is compact. More, next class.

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Finally, consider $\{f_n\} \subset L^1$ such that $\sup_n \|f_n\|_{L^1} < \infty$, then, up to subsequence, there exists $\mu \in \mathcal{M}(\Omega)$ such that

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