Announcements

- Homework 2 is due on 9/17 at 11:59pm ET. Please have your submissions in before then.
- Roshan has office hours on Fridays at 4pm ET.
- Important: No Handwritten Documents are permitted for any submission. If you submit a handwritten or hand-drawn assignment, it will receive a grade of 0.

Additional Examples - Posterior / Conjugate

1. Binomial likelihood and Beta prior

   Likelihood: $X \mid p \sim \text{Bin}(n, p)$; with associated pmf $f(x \mid p) = \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}$

   Prior: $p \sim Be(\alpha, \beta)$; with associated pdf $\pi(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$

   We have

$$
\begin{aligned}
\pi(p \mid x) &\propto f(x \mid p)\pi(p) \\
&= \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x} \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1} \\
&= \begin{pmatrix} n \\ x \end{pmatrix} \frac{1}{B(\alpha,\beta)} p^{\alpha+x-1}(1-p)^{\beta+n-x-1} \\
&= C p^{\alpha+x-1}(1-p)^{\beta+n-x-1}
\end{aligned}
$$

   which is the kernel of a $\text{Beta}(\alpha + x, \beta + n - x)$ distribution.

2. Poisson likelihood and gamma prior

   Likelihood: $x \mid \lambda \sim \text{Poi}(\lambda)$; with pmf $f(x \mid \lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$

   Prior: $\lambda \sim \text{Gamma}(\alpha, \beta)$; with pdf $\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$$
\begin{aligned}
\pi(\lambda \mid x) &\propto f(x \mid \lambda)\pi(\lambda) \\
&= \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)x!} \lambda^x \lambda^{\alpha-1} e^{-\beta\lambda} e^{-\lambda} \\
&= C\lambda^{x+\alpha-1} e^{-(\beta+1)\lambda}
\end{aligned}
$$

   which is the kernel of a $\text{Gamma}(x + \alpha, \beta + 1)$ distribution.

Example: Classical Statistics with MLE

Exponential Distribution with parameter λ:

$$f\left(t_i\right) = \lambda e^{-\lambda t_i}, \lambda > 0$$

The classical statistician would optimize for an unknown parameter given data points $1, \ldots, n$ using Maximum Likelihood Estimation. First we need the likelihood function:

$$L\left(\lambda; t_1, \ldots, t_n\right) = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

Due to the logarithm being strictly increasing, the maximum of $L()$ occurs at the same location as the maximum of $\log(L)$. So we take the logarithm first to make the algebra easier:

$$\begin{aligned}
\ln L\left(\lambda; t_1, \ldots, t_n\right) &= \ln\left(\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}\right) \\
&= n \ln(\lambda) + \ln\left(e^{-\lambda \sum_{i=1}^{n} t_i}\right) \\
&= n \ln(\lambda) - \lambda \sum_{i=1}^{n} t_i
\end{aligned}$$

Next we differentiate with respect to λ and set the result equal to 0:

$$\begin{aligned}
\frac{d}{d\lambda} \ln L\left(\lambda; t_1, \ldots, t_n\right) &= \frac{n}{\lambda} - \sum_{i=1}^{n} t_i \\
&= 0
\end{aligned}$$

Rearranging and solving for λ yields

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}$$

Homework 2 Guidance

Question 1: integration / expectation / LOTUS tasks

Question 2: using the Binomial formula with the specific value of p

Question 3: Conjugate priors in action