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Problem 1.

From OLS: $Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_n X_n + \varepsilon$, $E[\varepsilon] = 0$

we know: $E[Y \mid X_1 = a] = \beta_0 + \beta_1 a + \cdots \beta_n X_n$

From Conditional distribution of Multivariate Normal:

$$\bar{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - \mu_1), \text{ given } X_1 = a$$

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}$$

$$Y \sim N(\bar{\mu}, \bar{\Sigma})$$

$\Rightarrow$ Intuitively, this mean Given $X_1 = a$, Y is normally distributed with mean $= \bar{\mu}$, and variance $= \bar{\Sigma}$

This implies that Given $X_1 = a$, the expected value of Y is $\bar{\mu}$ $\Rightarrow E[Y \mid X] = \bar{\mu}$

Therefore, we know that

$$\boxed{E[Y \mid X_1 = a]} = \beta_0 + \beta_1 a + \cdots + \beta_n X_n = \boxed{\bar{\mu}} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - \mu_1)$$

Thus,

The Value of Conditional distribution of the Multivariate Normal and the value of OLS equation are the same

They are actually the same thing.

The Mathamatical Proof is shown on the next page. $\longrightarrow$

From OLS: $E[Y|X] = \beta_0 + \beta X$ ①

$E[Y] = \beta_0 + \beta E[X]$ ②

$E[Y|X] = E[Y] + \beta[X - E(X)]$ (subtract ② from ①)

times $E[X]$ $\Rightarrow$ $E[XY] = \beta_0 E[X] + \beta E[X^2]$

$\Rightarrow$ $E[Y] E[X] = \beta_0 E[X] + \beta E[X]^2$

subtract these two, and we get

$$\beta = \frac{E[XY] - E[X]E[Y]}{E[X^2] - E[X]^2}$$

$$\beta = \frac{E[(X - E[X]) \cdot (Y - E[Y])]}{E[X^2] - E[X]^2}$$

$$\boxed{\beta = \frac{Cov(X, Y)}{Cov(X, X)}}$$

put it back to the third equation above, we get

$$\Rightarrow E[Y|X] = E[Y] + \frac{Cov(X,Y)}{Cov(X,X)}[X - E[X]]$$

$$\Rightarrow \boxed{\mu_{Y|X} = \mu_Y + \frac{\sigma_{X,Y}}{\sigma_{X,X}}(X - \mu_X)}$$

This exacty the same equation as the conditional expectation of multivariate normal, thus Mathmatically, we can conclude that These two are the same thing.

I also proved this in the coding part, Letting $X = 5$, And the value generated for $E[Y|X=5]$ is equal to the value of $\bar{\mu}$, given $X=5$.