Exercise 3 - Procrustes

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1 Remarks

To avoid confusion, I'm putting down the equations to convert a rigid pose (R, t), which we compute with Procrustes algorithm, to a rigid pose $(\tilde{R}, \tilde{t})$, which we use in the rest of the framework. We are given a source shape X, a target shape Y, and point matches (x_i, y_i) such that $x_i \in X$ and $y_i \in Y$. Our goal is to estimate $(\tilde{R}, \tilde{t})$ that transform source points x_i into target points y_i as $y_i = \tilde{R}x_i + \tilde{t}$.

1.1 First approach

With Procrustes algorithm we compute the rotation between mean-centered points, i.e. we estimate a rotation matrix R that should satisfy the following equation for every point pair i:

$$R(x_i - \overline{x}) = y_i - \overline{y} \tag{1}$$

On the other hand, we estimate the translation as a difference between the means of both sets of points:

$$t = \overline{y} - \overline{x} \tag{2}$$

From equation 1 (and by inserting result from equation 2) we can get:

$$y_i = R(x_i - \overline{x}) + \overline{y} = R(x_i - \overline{x}) + (t + \overline{x}) = Rx_i + (-R\overline{x} + t + \overline{x}) \tag{3}$$

It follows that $\tilde{R} = R$ and $\tilde{t} = -R\overline{x} + t + \overline{x}$.

1.2 Second approach

Second method to achieve the same result (as mentioned in the exercise) is by applying the Procrustes algorithm in two steps. To transform a source point x_i to a target point y_i, we first apply a translation vector t, resulting in a translated point $x' = x_i + t$. In the next step we apply the rotation R. But since Procrustes algorithm computes rotation between mean-centered points, we can only apply the rotation on the mean-centered point $x' - \overline{y}$, and after application of rotation we add the mean $\overline{y}$ back. That results in the final transformed point:

$$y_i = R(x' - \overline{y}) + \overline{y} = R(x_i + t - \overline{y}) + \overline{y} = Rx_i + (Rt - R\overline{y} + \overline{y}) \tag{4}$$

It follows that $\tilde{R} = R$ and $\tilde{t} = Rt - R\overline{y} + \overline{y}$.

1.3 Result equality

It is easy to show that both results are exactly the same. We can transform from translation in the first approach to the translation in the second approach in the following way (by using equation 2):

$$-R\overline{x} + t + \overline{x} = -R(\overline{y} - t) + (\overline{y} - \overline{x}) + \overline{x} = Rt - R\overline{y} + \overline{y} \tag{5}$$