Machine Learning Homework Sheet 10

Dimensionality Reduction

Problem 1: In this exercise, we use proof by induction to show that the linear projection onto an M-dimensional subspace that maximizes the variance of the projected data is defined by the M eigenvectors of the data covariance matrix S, given by

$$S = \frac{1}{N}\sum_{n=1}^{N}(\boldsymbol{x}_n - \bar{\boldsymbol{x}})(\boldsymbol{x}_n - \bar{\boldsymbol{x}})^T \qquad \bar{\boldsymbol{x}} = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{x}_n$$

corresponding to the M largest eigenvalues. In Section 12.1 in Bishop this result was proven for the case of $M = 1$. Now suppose the result holds for some general value of M and show that it consequently holds for dimensionality $M + 1$. To do this, first set the derivative of the variance of the projected data with respect to a vector $\boldsymbol{u}_{M+1}$ defining the new direction in data space equal to zero. This should be done subject to the constraints that $\boldsymbol{u}_{M+1}$ be orthogonal to the existing vectors $\boldsymbol{u}_1, \dots \boldsymbol{u}_M$, and also that it be normalized to unit length. Use Lagrange multipliers to enforce these constraints. Then make use of the orthonormality properties of the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_M$ to show that the new vector $\boldsymbol{u}_{M+1}$ is an eigenvector of S. Finally, show that the variance is maximized if the eigenvector is chosen to be the one corresponding to eigenvector λ_{M+1} where the eigenvalues have been ordered in decreasing value.

Problem 2: Consider the latent space distribution

$$p(\boldsymbol{z}) = \mathcal{N}(\boldsymbol{z}|\boldsymbol{0}, \boldsymbol{I})$$

and a conditional distribution for the observed variable $\boldsymbol{x} \in \mathbb{R}^d$,

$$p(\boldsymbol{x}|\boldsymbol{z}) = \mathcal{N}(\boldsymbol{x}|\boldsymbol{W}\boldsymbol{z} + \boldsymbol{\mu}, \boldsymbol{\Phi})$$

where $\boldsymbol{\Phi}$ is an arbitrary symmetric, positive-definite noise covariance variable. Now suppose that we make a nonsingular linear transformation of the data variables $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{A}$ is a non-singular $d \times d$ matrix. If $\boldsymbol{\mu}_{ML}$, $\boldsymbol{W}_{ML}$, and $\boldsymbol{\Phi}_{ML}$ represent the maximum likelihood solution corresponding to the original untransformed data, show that $\boldsymbol{A}\boldsymbol{\mu}_{ML}$, $\boldsymbol{A}\boldsymbol{W}_{ML}$, and $\boldsymbol{A}\boldsymbol{\Phi}_{ML}\boldsymbol{A}^T$ will represent the corresponding maximum likelihood solution for the transformed data set. Finally, show that the form of the model is preserved if $\boldsymbol{A}$ is orthogonal and $\boldsymbol{\Phi}$ is proportional to the unit matrix so $\boldsymbol{\Phi} = \sigma^2\boldsymbol{I}$ (i.e. probabilistic PCA). The transformed $\boldsymbol{\Phi}$ matrix remains proportional to the unit matrix, and hence probabilistic PCA is covariant under a rotation of the axes of data space, as is the case for conventional PCA.

Problem 3: Use the SVD shown below. Suppose a new user Leslie assigns rating 3 to Alien and rating 4 to Titanic, giving us a representation of Leslie in the 'original space' of $[0, 3, 0, 0, 4]$. Find the representation of Leslie in concept space. What does that representation predict about how well Leslie would like the other movies appearing in our example data?

Upload a single PDF file with your solution to Moodle by 13.01.2019, 11:59pm CET. We recommend to typeset your solution (using LaTeX or Word), but handwritten solutions are also accepted. If your handwritten solution is illegible, it won't be graded and you waive your right to dispute that.

	Matrix	Alien	Star Wars	Casablanca	Titanic
Joe	1	1	1	0	0
Jim	3	3	3	0	0
John	4	4	4	0	0
Jack	5	5	5	0	0
Jill	0	0	0	4	4
Jenny	0	0	0	5	5
Jane	0	0	0	2	2

Figure 11.6: Ratings of movies by users

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} .14 & 0 \\ .42 & 0 \\ .56 & 0 \\ .70 & 0 \\ 0 & .60 \\ 0 & .75 \\ 0 & .30 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 12.4 & 0 \\ 0 & 9.5 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} .58 & .58 & .58 & 0 & 0 \\ 0 & 0 & 0 & .71 & .71 \end{bmatrix}}_{V^{\mathrm{T}}}$$

Problem 4: Load the notebook `homework_10_dim_reduction.ipynb` from Piazza. Fill in the missing code and run the notebook. Convert the evaluated notebook to pdf and add it to the printout of your homework.

Note: We suggest that you use Anaconda for installing Python and Jupyter, as well as for managing packages. We recommend that you use Python 3.

For more information on Jupyter notebooks consult the Jupyter documentation. Instructions for converting the Jupyter notebooks to PDF are provided within the notebook.