Practical Course: Vision-based Navigation
Summer Semester 2019

Lecture 1. 3D Geometry and Lie Groups

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Contents

- Course contents and preliminary knowledge
- Framework and mathematic form of a SLAM problem
- 3D geometry
- Lie groups

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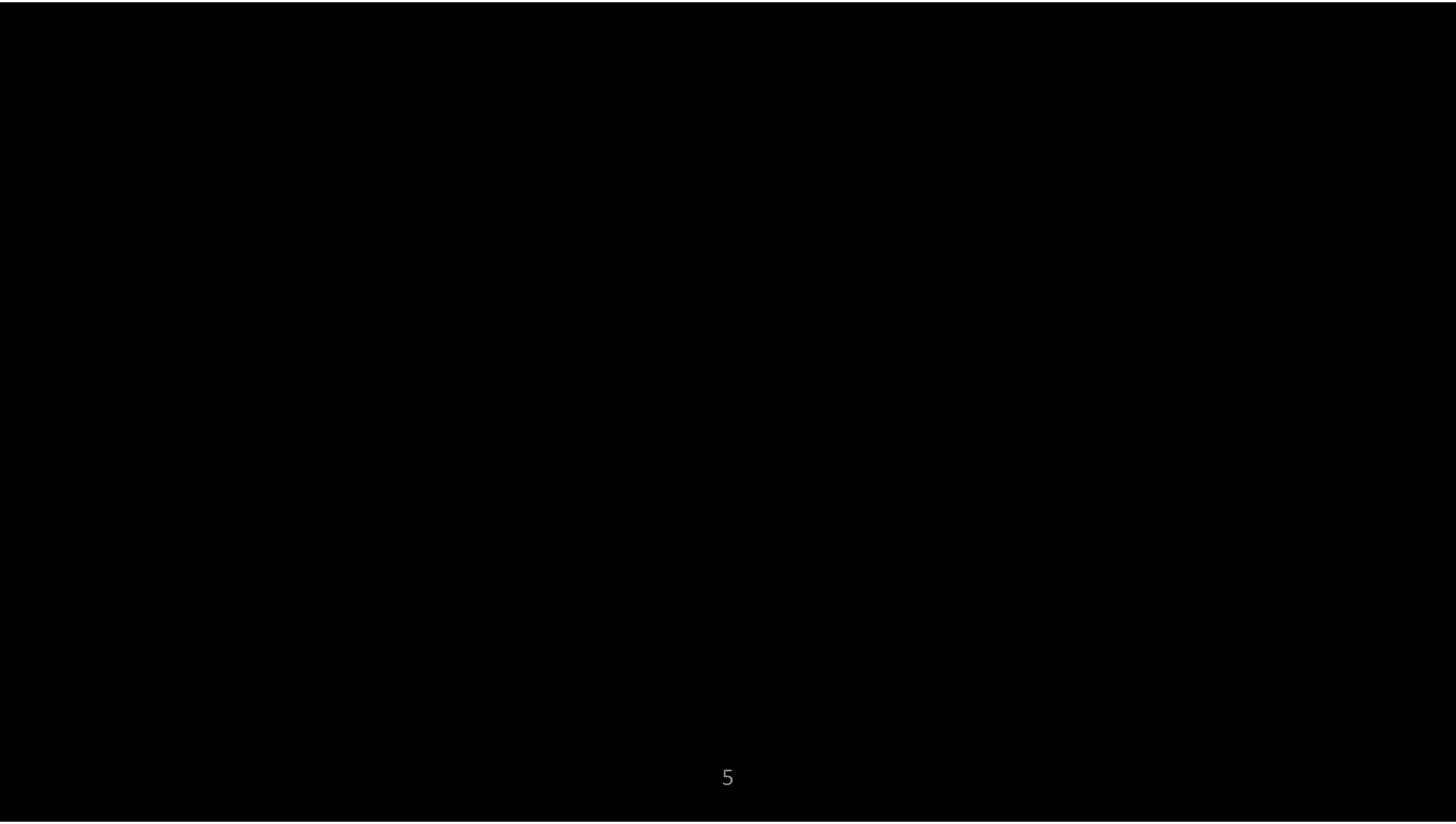
1. Course contents and preliminary knowledge

- General overview of computer vision tasks



1. Course contents and preliminary knowledge

- Computer vision

Real world cameras

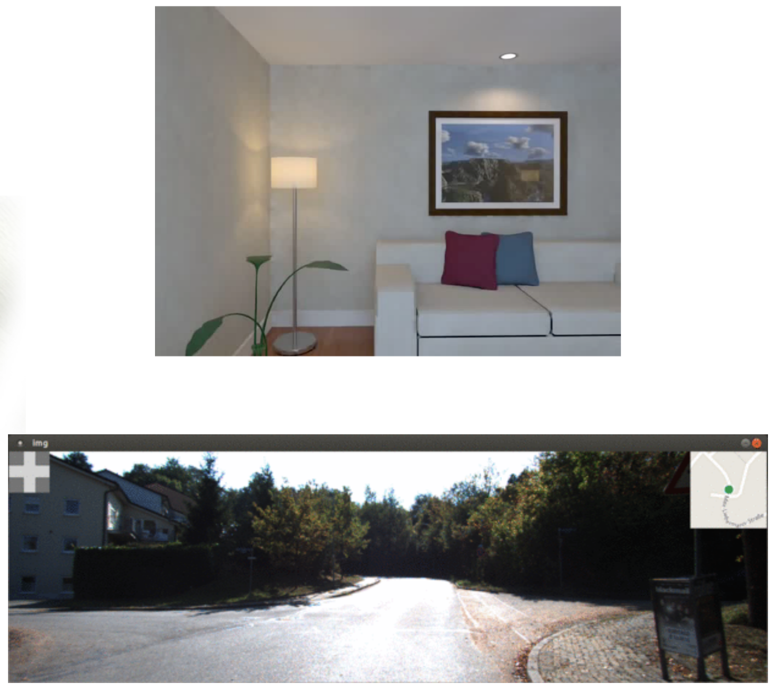


Image and video sequences

Object detection
Object recognition
Object tracking
Segmentation
...
SLAM

CV tasks

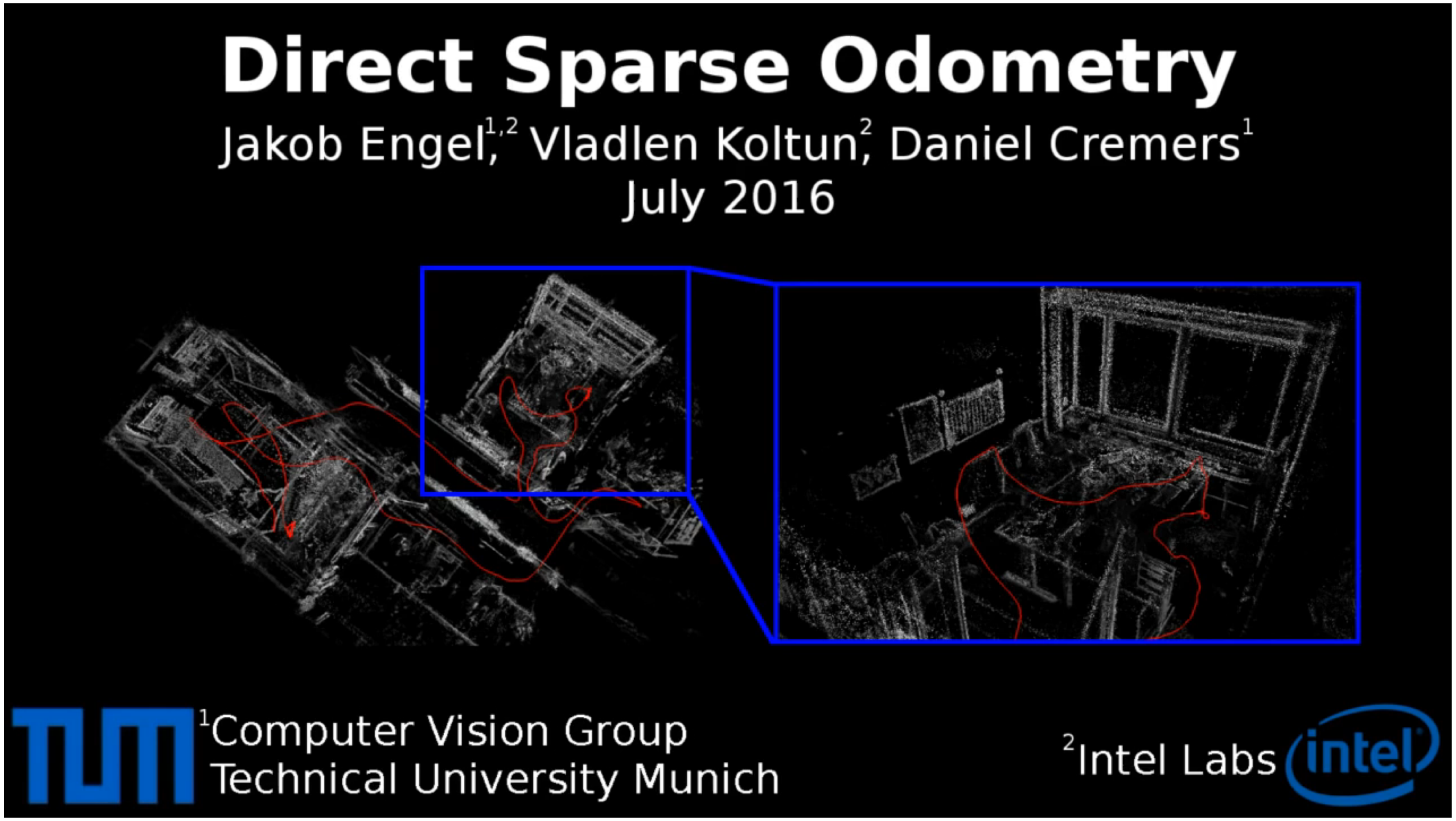
1. Course contents and preliminary knowledge

- What is SLAM? Simultaneous localization and mapping

Indoor/outdoor localization

1. Course contents and preliminary knowledge

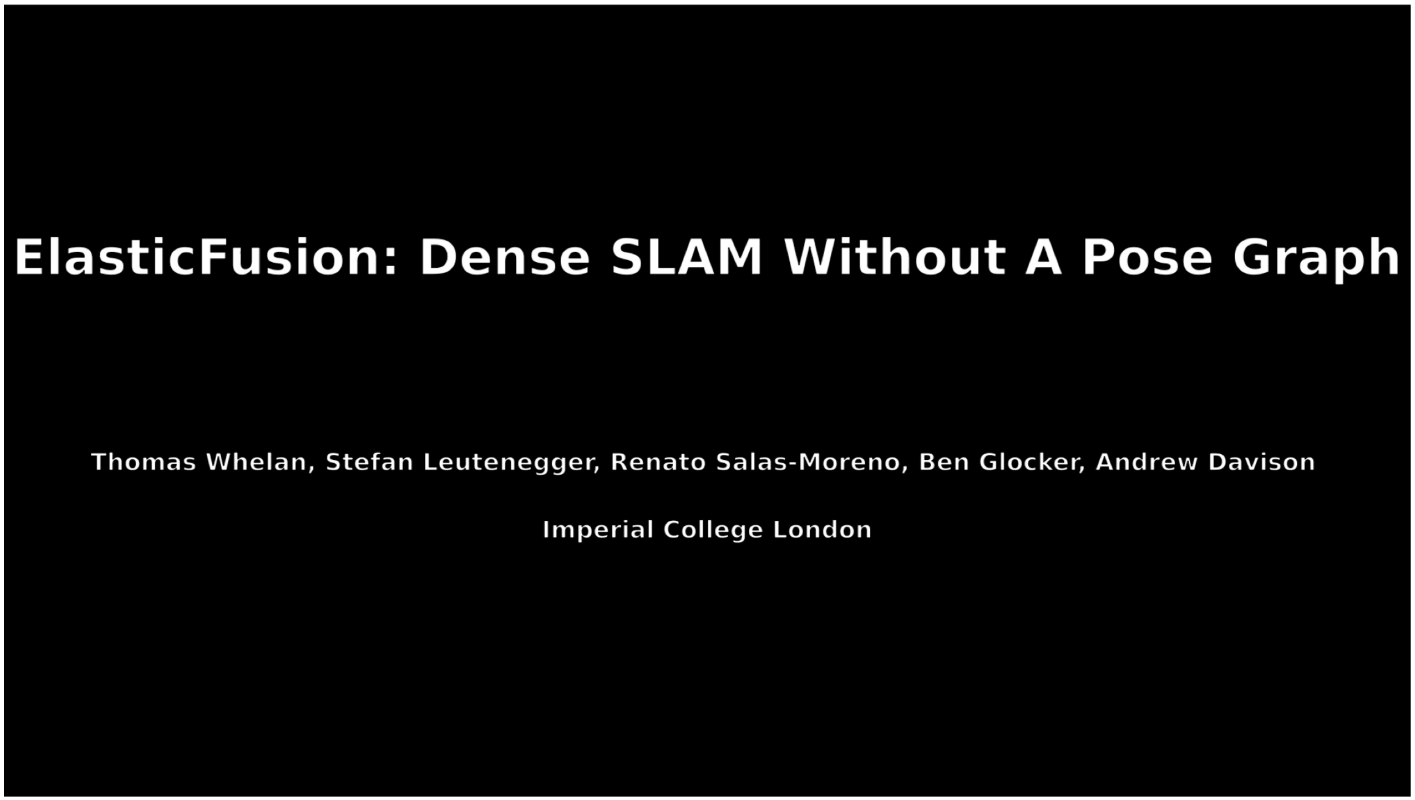
- Computer vision



Dense/semi-dense reconstruction

1. Course contents and preliminary knowledge

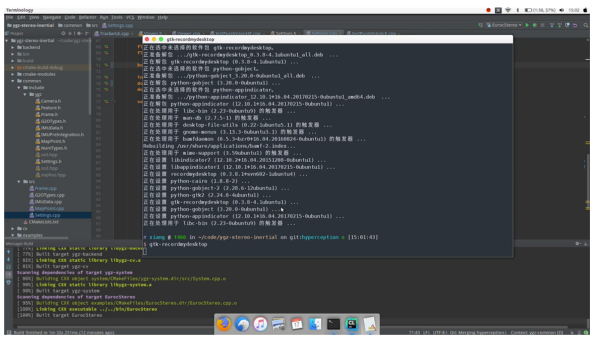
- What is SLAM?

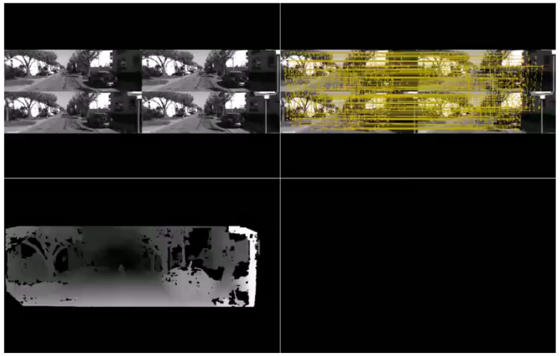


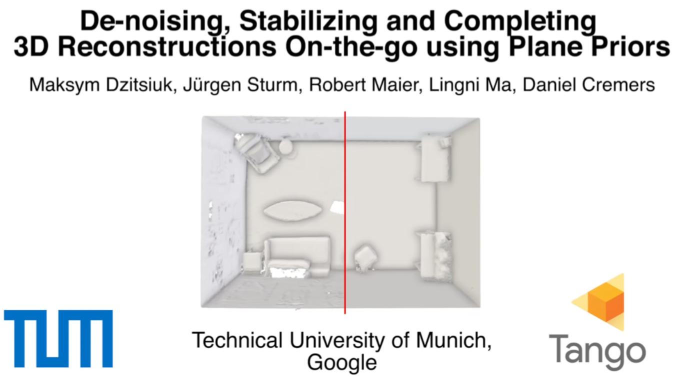
RGB-D dense reconstruction

1. Course contents and preliminary knowledge

- SLAM applications







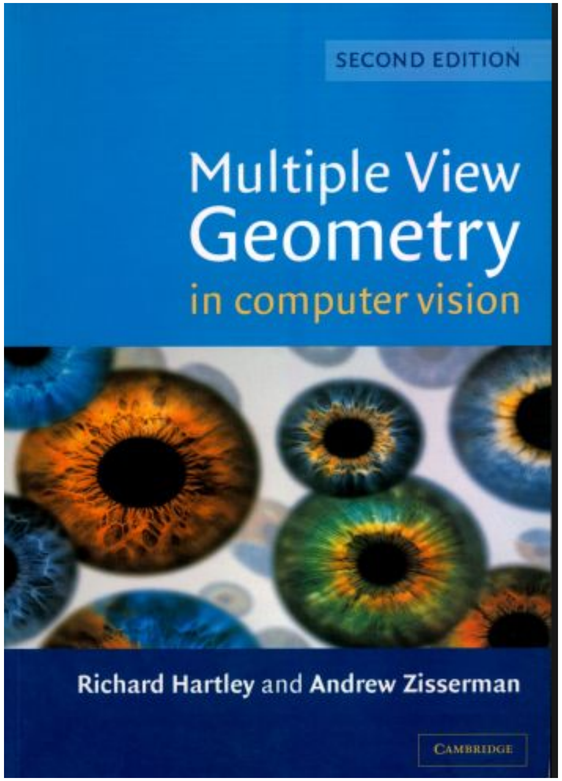
Hand-held devices

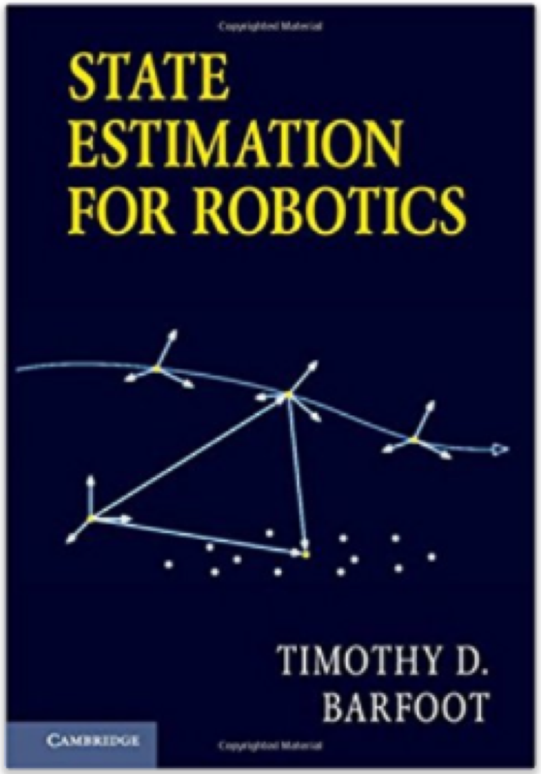
Autonomous driving

Augmented reality/VR

1. Course contents and preliminary knowledge

- Computer vision





Harley and Zisserman, Multiple view geometry in computer vision

Tim Barfoot, State estimation for robotics

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2. Framework of SLAM

- SLAM problem
  - Fundamental problems in intelligent robots
  - Where am I?
    -Localization
  - What is around me?
    -Mapping

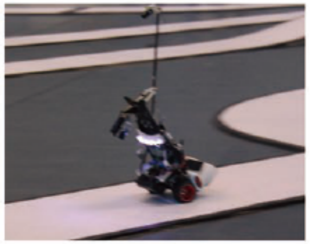
- Chicken and egg problem
  - Localization needs accurate map
  - Mapping needs accurate localization

2. Framework of SLAM

- How to do SLAM?   -Sensors
- Sensor is the way to measure the outside environment
- Interoseptive sensors: accelerometer, gyroscope …
- Exteroceptive sensors: camera, laser rangefinder, GPS …

(a)

(b)



(c)

(d)

(e)

(f)

Some sensors must be placed in a cooperative environment, other can be directly equipped in the robot itself

2. Framework of SLAM

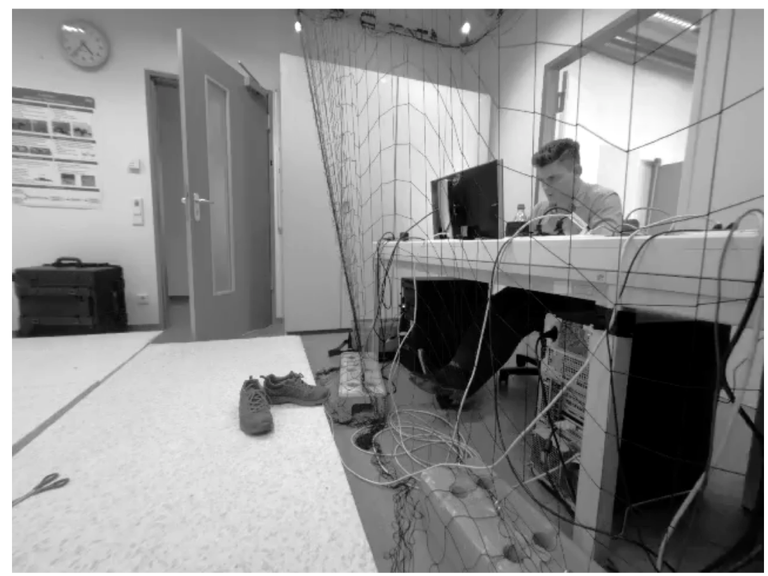
- Visual SLAM
- Cameras
  - Monocular
  - Stereo
  - RGB-D
  - Omnidirectional, Event camera, etc

Monocular camera

RGB-D (depth) camera

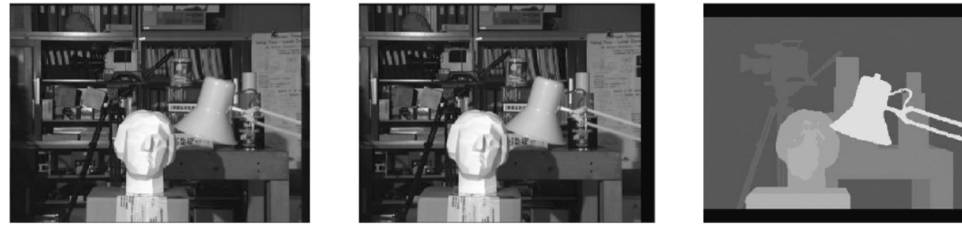
Stereo camera

- Cameras
  - Cheap, rich information
  - Record 2D projected image of the 3D world
  - The 3D-2D projection throws away one dimension: distance



2. Framework of SLAM

- Various kinds of cameras:
- Monocular: image only, need other methods to estimate the depth
- Stereo: disparity to depth
- RGB-D: physical depth measurements



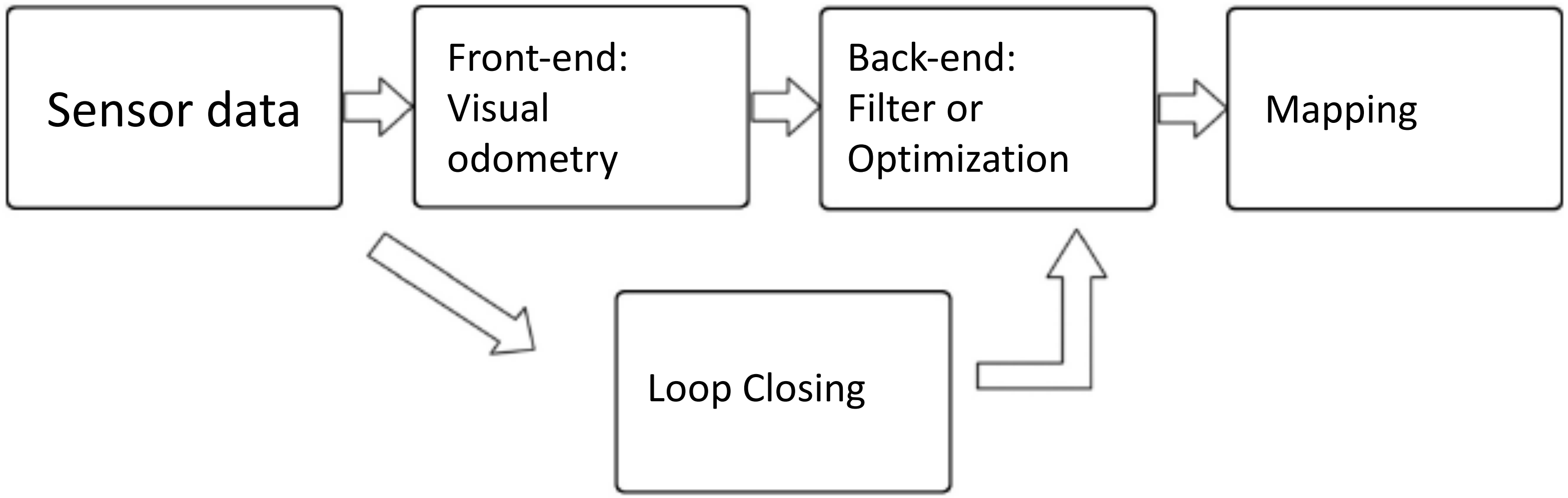
Stereo vision estimates the depth from disparity

Moving stereo: disparity can be estimated in the motion

Ambiguity in mono vision: small + close or large + far away?

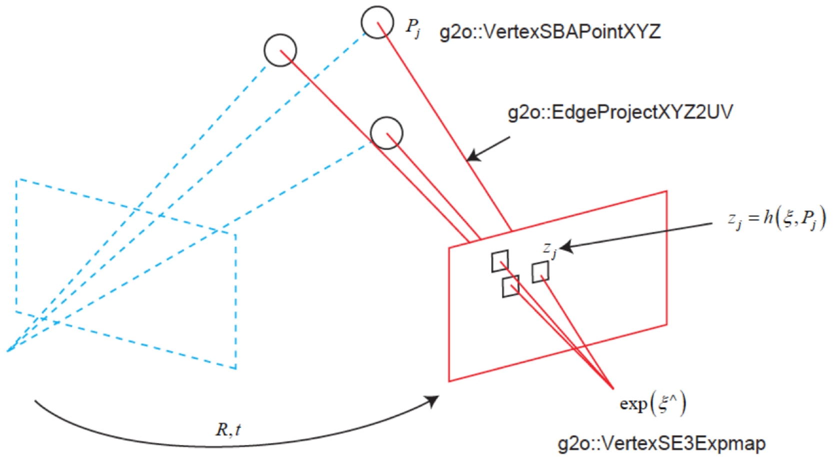
2. Framework of SLAM

- SLAM framework



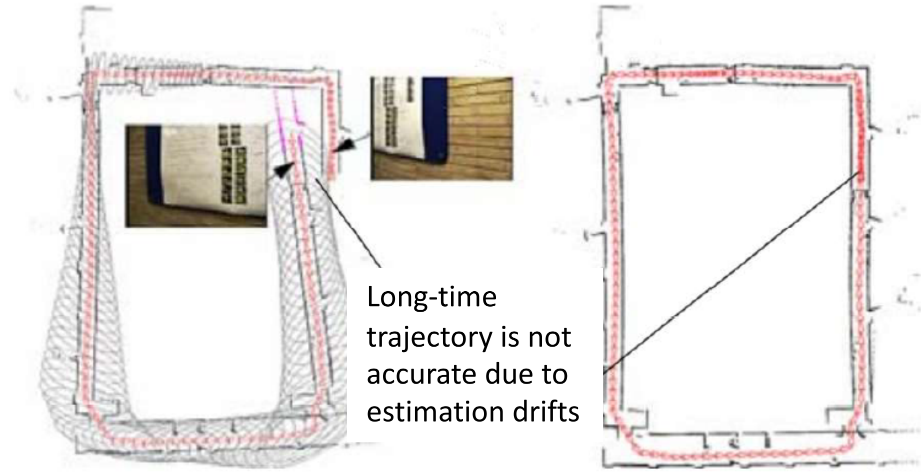
2. Framework of SLAM

- Visual odometry
  - Motion estimation betwee adjacent frames
  - Simplest: two-view geometry
- Method
  - Feature method
  - Direct method
- Backend
  - Long-term trajectory and map estimation
  - MAP: Maximum of a Posteri
  - Filters/Graph Optimization

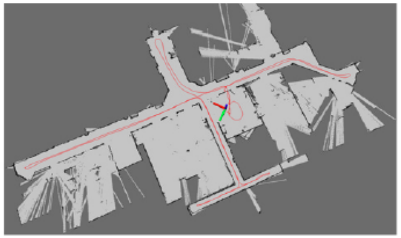


2. Framework of SLAM

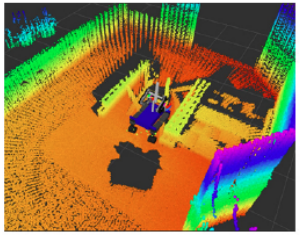
- Loop closing
  - Correct the drift in estimation
  - Loop detection and correction

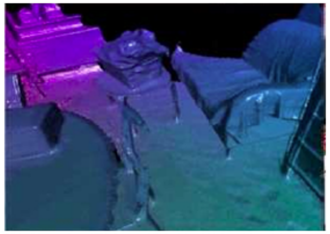


- Mapping
  - Generate globally consisten map for navigation/planning/comm nication/visualization etc
  - Grid/topological/hybrid maps
  - Pointcloud/Mesh/TSDF …


2D grid map


2D topological map


Point cloud maps


TSDF models

2. Framework of SLAM

- Mathematical representation of visual SLAM
- Assume a camera is moving in 3D space
  - But measurements are taken at discrete times:

$$\left\{ \begin{array}{l} \boldsymbol{x}_k = f\left(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k, \boldsymbol{w}_k\right) \\ \boldsymbol{z}_{k,j} = h\left(\boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{v}_{k,j}\right) \end{array} \right. .$$

Motion model

Observation model

$$\begin{cases} x_k = A_k x_{k-1} + B_k u_k + w_k \\ z_{k,j} = C_j y_j + D_k x_k + v_{k,j} \end{cases}$$

Non-linear form

linear form

2. Framework of SLAM

- Questions:

$$\begin{cases} \boldsymbol{x}_k = f\left(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k, \boldsymbol{w}_k\right) \\ \boldsymbol{z}_{k,j} = h\left(\boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{v}_{k,j}\right) \end{cases}.$$

Motion model

Observation model

- How to represent state variables?
  - 3D geometry, Lie group and Lie algebra
- Exact form of motion/observation model?
  - Camera intrinsic and extrinsics
- How to estimate the state given measurement data?
  - State estimation problem
  - Filters and optimization

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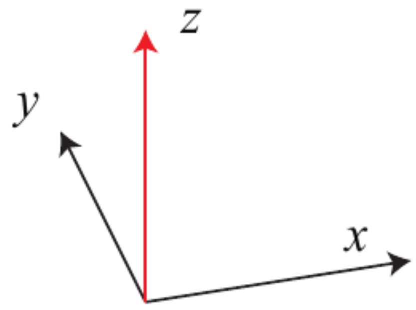
3. 3D geometry

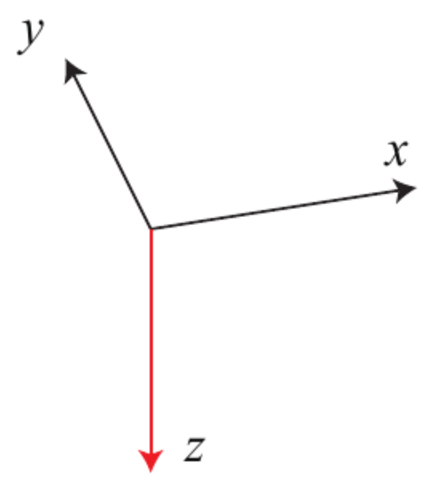
- Point and Coordinate system
- 2D: (x,y) and angle
- 3D?

3. 3D geometry

- 3D coordinate system
- Vectors and their coordinates



Right handed



Left handed

3. 3D geometry

- Vector operations
  - Addition/subtraction
  - Dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b} = \sum_{i=1}^{3} a_i b_i = |\boldsymbol{a}| \, |\boldsymbol{b}| \cos \langle \boldsymbol{a}, \boldsymbol{b} \rangle \, .$$

  - Cross product

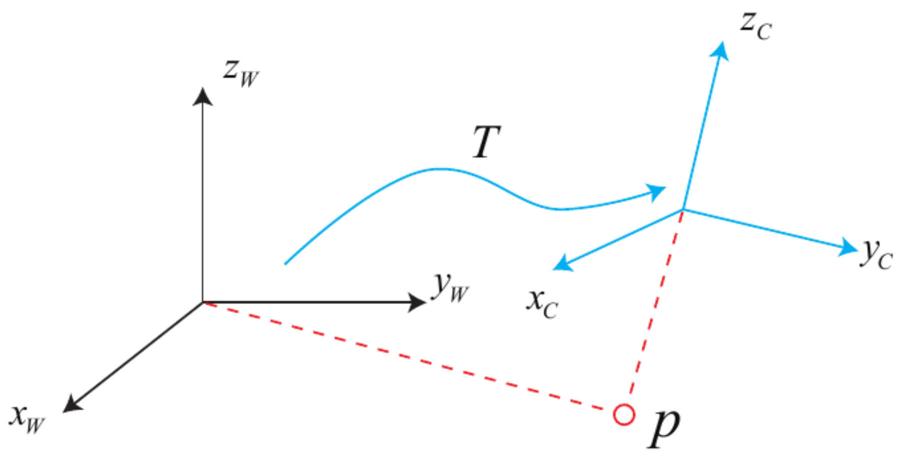
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \boldsymbol{b} \triangleq \boldsymbol{a}^{\wedge} \boldsymbol{b}.$$

Skew-symmetric operator

3. 3D geometry

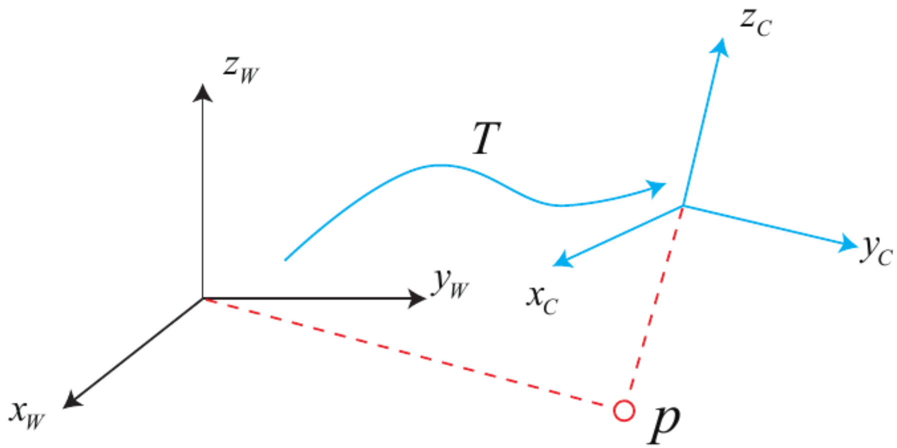
- Questions
  - Compute the coordinates in different systems?

- In SLAM:
  - Fixed world frame
  - Moving camera frame
  - Other sensor frames



3. 3D geometry

- 3D rigid body motion can be described with rotation and translation



- Translation is just a vector addition
- How to represent rotations?

3. 3D geometry

- Rotation
  - Consider coordinate system $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is rotated and become $(\boldsymbol{e}_1^{'}, \boldsymbol{e}_2^{'}, \boldsymbol{e}_3^{'})$
  - Vector $\boldsymbol{a}$ is fixed, then how are its coordinates changed?

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \left[\boldsymbol{e}_1^{'}, \boldsymbol{e}_2^{'}, \boldsymbol{e}_3^{'}\right] \begin{bmatrix} a_1^{'} \\ a_2^{'} \\ a_3^{'} \end{bmatrix}.$$

  - Left multiplied by $[e_1^T, e_2^T, e_3^T]^T$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1^T \boldsymbol{e}_1^{'} & \boldsymbol{e}_1^T \boldsymbol{e}_2^{'} & \boldsymbol{e}_1^T \boldsymbol{e}_3^{'} \\ \boldsymbol{e}_2^T \boldsymbol{e}_1^{'} & \boldsymbol{e}_2^T \boldsymbol{e}_2^{'} & \boldsymbol{e}_2^T \boldsymbol{e}_3^{'} \\ \boldsymbol{e}_3^T \boldsymbol{e}_1^{'} & \boldsymbol{e}_3^T \boldsymbol{e}_2^{'} & \boldsymbol{e}_3^T \boldsymbol{e}_3^{'} \end{bmatrix} \begin{bmatrix} a_1^{'} \\ a_2^{'} \\ a_3^{'} \end{bmatrix} \triangleq \boldsymbol{R}\boldsymbol{a}^{'}.$$

Rotation matrix

3. 3D geometry

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} e_1^T e_1^{'} & e_1^T e_2^{'} & e_1^T e_3^{'} \\ e_2^T e_1^{'} & e_2^T e_2^{'} & e_2^T e_3^{'} \\ e_3^T e_1^{'} & e_3^T e_2^{'} & e_3^T e_3^{'} \end{bmatrix} \begin{bmatrix} a_1^{'} \\ a_2^{'} \\ a_3^{'} \end{bmatrix} \triangleq \boldsymbol{R}\boldsymbol{a}^{'}.$$

- R is rotation matrix, which satisfies:
  - R is orthogonal
  - Det(R) = +1 (if Det(R)=-1 then it's improper rotation)

- Special orthogonal group:

$$SO(n) = \{\boldsymbol{R} \in \mathbb{R}^{n \times n} | \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1\}.$$

- Rotation from frame 2 to 1 can be written as:

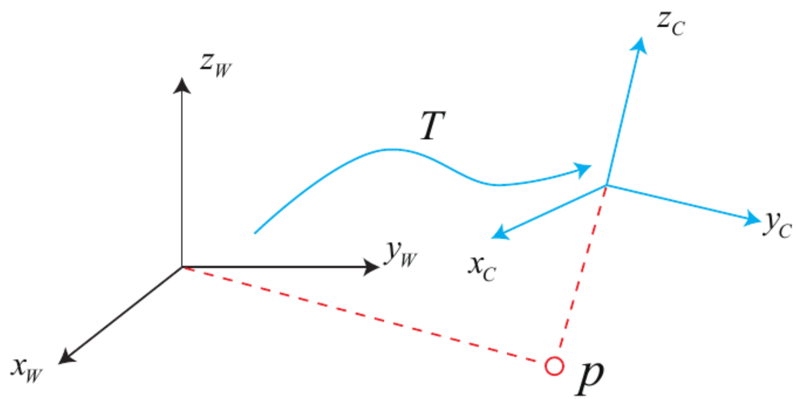
$a_1 = R_{12} a_2$ and vise vesa: $a_2 = R_{21} a_1$

$$\boxed{R_{21} = R_{12}^{-1} = R_{12}^T}$$

3. 3D geometry

- Rotation plus translation:

$$\boldsymbol{a}' = \boldsymbol{R}\boldsymbol{a} + \boldsymbol{t}.$$



- Compounding rotation and translation:
  - $\boldsymbol{b} = \boldsymbol{R}_1\boldsymbol{a} + \boldsymbol{t}_1, \quad \boldsymbol{c} = \boldsymbol{R}_2\boldsymbol{b} + \boldsymbol{t}_2. \quad \Longrightarrow \quad \boldsymbol{c} = \boldsymbol{R}_2\left(\boldsymbol{R}_1\boldsymbol{a} + \boldsymbol{t}_1\right) + \boldsymbol{t}_2.$
- Homogeneous form:

$$\begin{bmatrix} \boldsymbol{a}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ 1 \end{bmatrix} \triangleq \boldsymbol{T} \begin{bmatrix} \boldsymbol{a} \\ 1 \end{bmatrix}.$$

$$\tilde{\boldsymbol{b}} = \boldsymbol{T}_1\tilde{\boldsymbol{a}}, \ \tilde{\boldsymbol{c}} = \boldsymbol{T}_2\tilde{\boldsymbol{b}} \quad \Rightarrow \tilde{\boldsymbol{c}} = \boldsymbol{T}_2\boldsymbol{T}_1\tilde{\boldsymbol{a}}.$$

Inverse: $\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T t \\ \boldsymbol{0}^T & 1 \end{bmatrix}.$

3. 3D geometry

- Homogenous coordinates:

$$\tilde{a} = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$\tilde{a} = \begin{bmatrix} a \\ 1 \end{bmatrix} = k \begin{bmatrix} a \\ 1 \end{bmatrix}$$

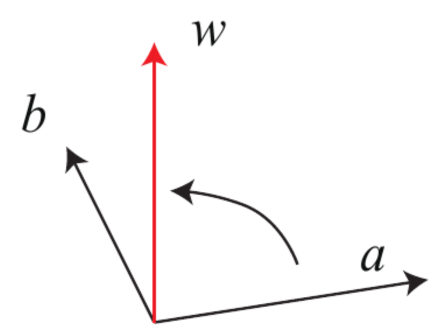
Still keeps equal when multiplying any non-zero factors

- Transform matrix forms Special Euclidean Group

$$SE(3) = \left\{ \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} | \boldsymbol{R} \in SO(3), \boldsymbol{t} \in \mathbb{R}^3 \right\}.$$

3. 3D geometry

- Alternative rotation representations
  - Rotation vectors
  - Euler angles
  - Quaternions



Rotation vectors
Only three parameters

- Rotation vectors
  - Angle + axis: θn
  - Rotation angle θ
  - Rotation axis n
- Rotation vector to rotation matrix: Rodrigues' formula

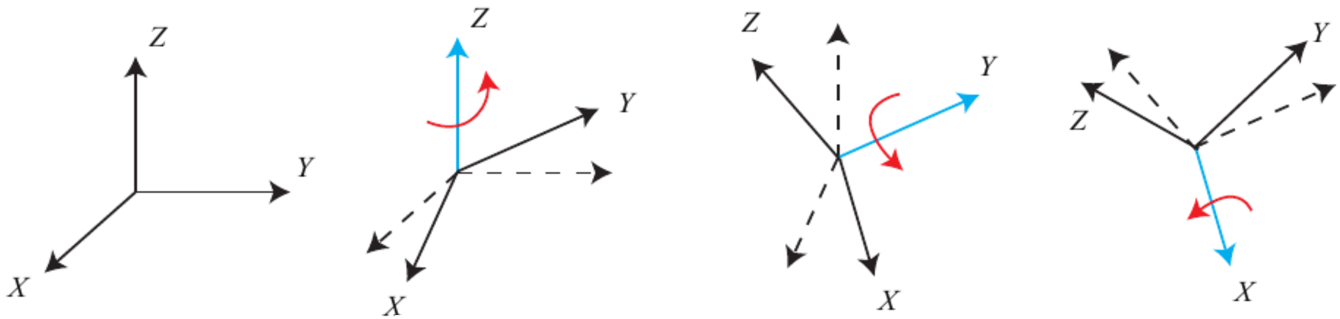
$$\boldsymbol{R} = \cos\theta \boldsymbol{I} + (1 - \cos\theta)\, \boldsymbol{n}\boldsymbol{n}^T + \sin\theta \boldsymbol{n}^\wedge.$$

- Inverse:

$$\theta = \arccos(\frac{\mathrm{tr}(\boldsymbol{R}) - 1}{2}). \qquad \boldsymbol{R}\boldsymbol{n} = \boldsymbol{n}.$$

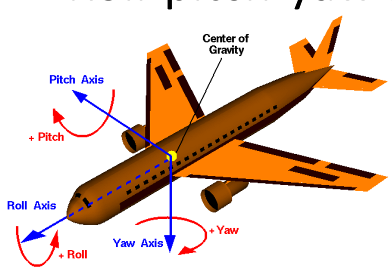
3. 3D geometry

- Euler angles
  - Any rotation can be decomposed into three principal rotations



Original First Second Third

  - However the order of axis can be defined very differently:
  - Roll-pitch-yaw (in navigation) Spin-nutation-precession in mechanics



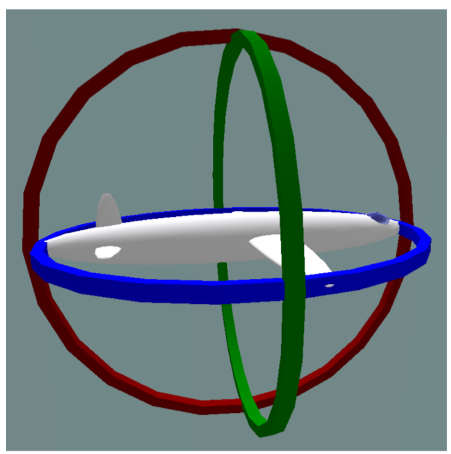
XYZ order

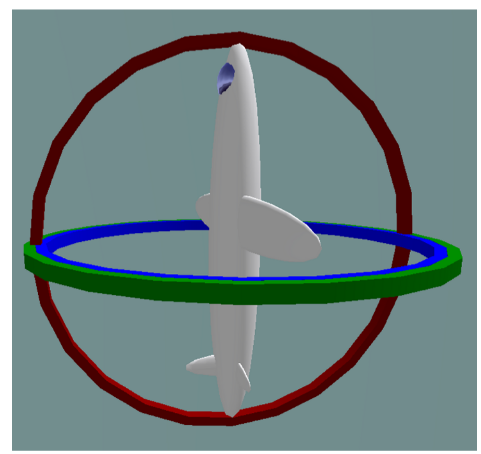
3-1-3 order

3. 3D geometry

- Gimbal lock
  - Singularity always exist if we want to use 3 parameters to describe rotation
  - Degree-of-Freedom is reduced in singular case
  - In yaw-pitch-roll order, when pitch=90 degrees

normal





singular

3. 3D geometry

- Quaternions
  - In 2D case, we can use (unit) complex numbers to denote rotations

$$z = x + iy = \rho e^{i\theta}$$

Multiply i to rotate 90 degrees

  - How about 3D case?
- (Unit) Quaternions
  - Extended from complex numbers
  - Three imaginary and one real part:
  - The imaginary parts satisfy:

$$\boxed{\boldsymbol{q} = q_0 + q_1 i + q_2 j + q_3 k,}$$

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k, ji = -k \\ jk = i, kj = -i \\ ki = j, ik = -j \end{cases}.$$

i,j,k look like complex numbers when multiplying with themselves
And look like cross product when multiply with others

3. 3D geometry

- Quaternions

$$\boldsymbol{q} = q_0 + q_1 i + q_2 j + q_3 k, \qquad \boldsymbol{q} = [s, \boldsymbol{v}]\,, \quad s = q_0 \in \mathbb{R}, \boldsymbol{v} = [q_1, q_2, q_3]^T \in \mathbb{R}^3,$$

- Operations

$$\boldsymbol{q}_a \pm \boldsymbol{q}_b = [s_a \pm s_b, \boldsymbol{v}_a \pm \boldsymbol{v}_b]\,.$$

$$\begin{aligned} \boldsymbol{q}_a \boldsymbol{q}_b &= s_a s_b - x_a x_b - y_a y_b - z_a z_b \\ &+ (s_a x_b + x_a s_b + y_a z_b - z_a y_b)\, i \\ &+ (s_a y_b - x_a z_b + y_a s_b + z_a x_b)\, j \\ &+ (s_a z_b + x_a y_b - y_b x_a + z_a s_b)\, k. \end{aligned}$$

$$\boldsymbol{q}_a \boldsymbol{q}_b = \left[s_a s_b - \boldsymbol{v}_a^T \boldsymbol{v}_b, s_a \boldsymbol{v}_b + s_b \boldsymbol{v}_a + \boldsymbol{v}_a \times \boldsymbol{v}_b\right]\,.$$

$$\boldsymbol{q}_a^* = s_a - x_a i - y_a j - z_a k = [s_a, -\boldsymbol{v}_a].$$

$$\|\boldsymbol{q}_a\| = \sqrt{s_a^2 + x_a^2 + y_a^2 + z_a^2}.$$

$$\boldsymbol{q}^{-1} = \boldsymbol{q}^* / \|\boldsymbol{q}\|^2.$$

$$k\boldsymbol{q} = [ks, k\boldsymbol{v}]\,.$$

$$\boldsymbol{q}_a \cdot \boldsymbol{q}_b = s_a s_b + x_a x_b i + y_a y_b j + z_a z_b k.$$

3. 3D geometry

- From quaternions to angle-axis:

$$\boldsymbol{q} = \left[\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right]^T.$$

- Inverse:

$$\begin{cases} \theta = 2\arccos q_0 \\ [n_x, n_y, n_z]^T = [q_1, q_2, q_3]^T / \sin\frac{\theta}{2} \end{cases}.$$

- Rotate a vector by quaternions:
  - Vector p is rotated by q and become p' , how to write their relationships?
  - Write p as quaternion (pure imaginary): $\boldsymbol{p} = [0, x, y, z] = [0, \boldsymbol{v}].$
  - Then: $\boldsymbol{p}' = \boldsymbol{q}\boldsymbol{p}\boldsymbol{q}^{-1}.$ Also pure imaginary

Contents

- Course contents and preliminary knowledge
- Framework and mathematic form of a SLAM problem
- 3D geometry
- Lie groups

4. Lie Group and Lie Algebra

- Recall the mathematic model of SLAM

$$\begin{cases} \boldsymbol{x}_k = f\left(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k, \boldsymbol{w}_k\right) \\ \boldsymbol{z}_{k,j} = h\left(\boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{v}_{k,j}\right) \end{cases}.$$

Motion model

Observation model

- We use SO(3) and SE(3) to represent the pose of camera
- Let's consider optimizing some function of rotation/transform

$$f(R) \qquad \frac{df}{df(R)} \qquad \frac{f(R+\Delta R)-f(R)}{\Delta R}$$

  - Rotation and transform matrix don't have a plus operator!

4. Lie Group and Lie Algebra

- Group
  - 3D rotation matrix forms the Special Orthogonal Group

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times3} | \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, det(\boldsymbol{R}) = 1\}.$$

  - 3D transform matrix forms the Special Euclidean Group

$$SE(3) = \left\{ \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} | \boldsymbol{R} \in SO(3), \boldsymbol{t} \in \mathbb{R}^3 \right\}.$$

- What is Group?

4. Lie Group and Lie Algebra

- Group
  - Group is a set with an operator $(A, \cdot)$ that satisfies the following:

1. Closure $\forall a_1, a_2 \in A, \quad a_1 \cdot a_2 \in A.$

2. Associativity $\forall a_1, a_2, a_3 \in A, \quad (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3).$

3. Identity $\exists a_0 \in A, \quad s.t. \quad \forall a \in A, \quad a_0 \cdot a = a \cdot a_0 = a.$

4. Invertibility $\forall a \in A, \quad \exists a^{-1} \in A, \quad s.t. \quad a \cdot a^{-1} = a_0.$

- Obviously,
  - $(SO(3), \cdot)$, $(SE(3), \cdot)$ are groups

4. Lie Group and Lie Algebra

- Lie Group
  - Group that is smooth
  - Group that is also a manifold
  - "Locally looks like R^n"
  - Further explanation needs knowledge from topology and differential geometry
  - SO(3) and SE(3) are also Lie groups

- Lie Algebra
  - Tangent space of the Lie group at identity
  - SO(3)->so(3), SE(3)->se(3)

4. Lie Group and Lie Algebra

- Introducing of the Lie Algebra
  - Assume a time-varying rotation matrix $R(t)$
  - It satisfies: $\boldsymbol{R}(t)\boldsymbol{R}(t)^T = \boldsymbol{I}.$
  - Take derivative of time t at both sides:

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T + \boldsymbol{R}(t)\dot{\boldsymbol{R}}(t)^T = 0.$$

  - Rearrange: $\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T = -\left(\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\right)^T.$

Skew-symmetric

4. Lie Group and Lie Algebra

$$\boxed{\boldsymbol{a}^{\wedge} = \boldsymbol{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \boldsymbol{A}^{\vee} = \boldsymbol{a}.}$$

- Denote the skew-symmetric matrix as $\phi(t)^{\wedge}$

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T = -\left(\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\right)^T. \qquad \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T = \boldsymbol{\phi}(t)^{\wedge}.$$

- Put R(t) to the right side: $\dot{\boldsymbol{R}}(t) = \boldsymbol{\phi}(t)^{\wedge}\boldsymbol{R}(t)$
- It looks like when we take the derivative, we will get a $\phi(t)^{\wedge}$ at the left side

- Assume we are close to identity: $t_0 = 0, R(0) = I$
- And $\phi(t)^{\wedge}$ does not change: $\dot{\boldsymbol{R}}(t) = \boldsymbol{\phi}(t_0)^{\wedge}\boldsymbol{R}(t) = \boldsymbol{\phi}_0^{\wedge}\boldsymbol{R}(t).$

- With $R(0) = I$, we solve this ODE: $\boxed{\boldsymbol{R}(t) = \exp\left(\boldsymbol{\phi}_0^{\wedge}t\right).}$

4. Lie Group and Lie Algebra

$$\boldsymbol{R}(t) = \exp\left(\phi_0^{\wedge} t\right).$$

- So, if t is close to 0, then we can always find an R given ϕ
- ϕ is called a Lie algebra
- From a Lie algebra, if we take a Exponential Map, then it becomes a Lie group
- Questions:
  - Lie algebra's definition and constraints?
  - How to compute the exponential map?

4. Lie Group and Lie Algebra

- Lie algebra:
  - We have a Lie algebra for each Lie group, which is a vector space (the tangent space) at the identity
  - Lie algebra has a vector space V over field F together with a binary operator (Lie bracket) $[,]$, that satisfies:
  - Closure: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathbb{V}, [\boldsymbol{X}, \boldsymbol{Y}] \in \mathbb{V}.$
  - Bilinearity: for any $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}, a, b \in \mathbb{F},$

$$[a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}] = a[\boldsymbol{X}, \boldsymbol{Z}] + b[\boldsymbol{Y}, \boldsymbol{Z}], \quad [\boldsymbol{Z}, a\boldsymbol{X} + b\boldsymbol{Y}] = a[\boldsymbol{Z}, \boldsymbol{X}] + b[\boldsymbol{Z}, \boldsymbol{Y}].$$

  - Alternativity: $\forall \boldsymbol{X} \in \mathbb{V}, [\boldsymbol{X}, \boldsymbol{X}] = \mathbf{0}.$
  - Jacobi identity:

$$\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}, [\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] = \mathbf{0}.$$

4. Lie Group and Lie Algebra

- Example: $(R^3, R, \times)$ is a Lie algebra
- Lie algebra so(3): $\mathfrak{so}(3) = \{\phi \in \mathbb{R}^3, \boldsymbol{\Phi} = \phi^\wedge \in \mathbb{R}^{3\times 3}\}.$
  - where

$$\boldsymbol{\Phi} = \phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

  - And the Lie bracket is: $[\phi_1, \phi_2] = (\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1)^\vee.$

4. Lie Group and Lie Algebra

- Similarly, for SE(3) we also have se(3):

$$\mathfrak{se}(3) = \left\{ \xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6, \rho \in \mathbb{R}^3, \phi \in \mathfrak{so}(3), \xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \right\}.$$

- Where

$$\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

and Lie bracket is:

$$[\xi_1, \xi_2] = \left(\xi_1^\wedge \xi_2^\wedge - \xi_2^\wedge \xi_1^\wedge\right)^\vee.$$

NOTE in se(3) this operator is not a skew-symmetric matrix, but we still keeps its form

- Note:
  - The definition of se(3) may be different in literature
  - Vector or matrix are both ok to define a lie algebra

4. Lie Group and Lie Algebra

- Exponential map
  - Operator from Lie algebra to Lie group: $\boxed{R = \exp(\phi^{\wedge})}$
  - Here $\phi^{\wedge}$ is a 3x3 matrix so this exponential map is a matrix operator
  - Take Taylor expansion:

$$\exp(\phi^{\wedge}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^{\wedge})^n.$$

  - Directly computing this Taylor expansion is intractable

4. Lie Group and Lie Algebra

- Take the length and direction of ϕ , then $\phi = \theta a$
- For a unit-length vector, we have:

$$\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} = \boldsymbol{a}\boldsymbol{a}^{T} - \boldsymbol{I},$$

$$\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} = -\boldsymbol{a}^{\wedge}.$$

This will be useful when handling the high-order Taylor expansion items

4. Lie Group and Lie Algebra

- Compute the Taylor expansion:

$$
\begin{aligned}
\exp\left(\phi^{\wedge}\right) = \exp\left(\theta \boldsymbol{a}^{\wedge}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\theta \boldsymbol{a}^{\wedge}\right)^{n} \\
&= \boldsymbol{I} + \theta \boldsymbol{a}^{\wedge} + \frac{1}{2!}\theta^{2}\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} + \frac{1}{3!}\theta^{3}\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} + \frac{1}{4!}\theta^{4}\left(\boldsymbol{a}^{\wedge}\right)^{4} + \ldots \\
&= \boldsymbol{a}\boldsymbol{a}^{T} - \boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} + \theta \boldsymbol{a}^{\wedge} + \frac{1}{2!}\theta^{2}\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} - \frac{1}{3!}\theta^{3}\boldsymbol{a}^{\wedge} - \frac{1}{4!}\theta^{4}\left(\boldsymbol{a}^{\wedge}\right)^{2} + \ldots \\
&= \boldsymbol{a}\boldsymbol{a}^{T} + \left(\theta - \frac{1}{3!}\theta^{3} + \frac{1}{5!}\theta^{5} - \ldots\right)\boldsymbol{a}^{\wedge} - \left(1 - \frac{1}{2!}\theta^{2} + \frac{1}{4!}\theta^{4} - \ldots\right)\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} \\
&= \boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} + \boldsymbol{I} + \sin\theta \boldsymbol{a}^{\wedge} - \cos\theta \boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} \\
&= (1 - \cos\theta)\boldsymbol{a}^{\wedge}\boldsymbol{a}^{\wedge} + \boldsymbol{I} + \sin\theta \boldsymbol{a}^{\wedge} \\
&= \cos\theta \boldsymbol{I} + (1 - \cos\theta)\boldsymbol{a}\boldsymbol{a}^{T} + \sin\theta \boldsymbol{a}^{\wedge}.
\end{aligned}
$$

- Finally we get:

$$
\exp(\theta \boldsymbol{a}^{\wedge}) = \cos\theta \boldsymbol{I} + (1 - \cos\theta)\boldsymbol{a}\boldsymbol{a}^{T} + \sin\theta \boldsymbol{a}^{\wedge}.
$$

- Which is exactly the Rodrigues' formula!

4. Lie Group and Lie Algebra

- So so(3) is just the rotation vector
- Same as exponential map, we can also define logarithm map as:

$$\boxed{\phi = \ln(\boldsymbol{R})^{\vee} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (\boldsymbol{R} - \boldsymbol{I})^{n+1} \right)^{\vee}.}$$

- And also don't need to actually compute this stuff, we take the conversion equations from rotation matrix to rotation vector:

$$\theta = \arccos\left(\frac{\operatorname{tr}(\boldsymbol{R}) - 1}{2}\right). \qquad \boldsymbol{R}\boldsymbol{n} = \boldsymbol{n}.$$

4. Lie Group and Lie Algebra

- For SE(3), the exponential map is:

$$\exp\left(\xi^{\wedge}\right) = \begin{bmatrix} \sum\limits_{n=0}^{\infty} \frac{1}{n!}\left(\phi^{\wedge}\right)^{n} & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\phi^{\wedge}\right)^{n}\boldsymbol{\rho} \\ \mathbf{0}^{T} & 1 \end{bmatrix}$$

$$\triangleq \begin{bmatrix} \boldsymbol{R} & \boldsymbol{J}\boldsymbol{\rho} \\ \mathbf{0}^{T} & 1 \end{bmatrix} = \boldsymbol{T}.$$

- The rotation part is just a SO(3), but the translation part has a Jacobian matrix: (left as an assignment)

$$\boldsymbol{J} = \frac{\sin\theta}{\theta}\boldsymbol{I} + \left(1 - \frac{\sin\theta}{\theta}\right)\boldsymbol{a}\boldsymbol{a}^{T} + \frac{1-\cos\theta}{\theta}\boldsymbol{a}^{\wedge}.$$

4. Lie Group and Lie Algebra

Rotation matrix

Lie group

$SO(3)$

$R \in \mathbb{R}^{3\times3}$

$RR^T = I$

$\det(R) = 1$

Exponential

$$\exp(\theta a^\wedge) = \cos\theta I + (1-\cos\theta) a a^T + \sin\theta a^\wedge$$

Logarithm

$$\theta = \arccos\frac{tr(R)-1}{2} \qquad Ra = a$$

Lie algebra

$\mathfrak{so}(3)$

$\phi \in \mathbb{R}^3$

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

Transform matrix

Lie group

$SE(3)$

$T \in \mathbb{R}^{4\times4}$

$$T = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$

Exponential

$$\exp(\xi^\wedge) = \begin{bmatrix} \exp(\phi^\wedge) & J\rho \\ 0^T & 1 \end{bmatrix}$$

$$J = \frac{\sin\theta}{\theta} I + \left(1 - \frac{\sin\theta}{\theta}\right) a a^T + \frac{1-\cos\theta}{\theta} a^\wedge$$

Logarithm

$$\theta = \arccos\frac{tr(R)-1}{2} \qquad Ra = a \qquad t = J\rho$$

Lie algebra

$\mathfrak{se}(3)$

$$\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6$$

$$\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix}$$

4. Lie Group and Lie Algebra

- Next question
  - We still don’t have plus operation for Lie group
  - Then we can’t define derivatives

- Solution
  - Take advantage of the plus in the Lie algebra, and convert it back to Lie group

- A primal question:
  - Plus in Lie algebra is equal to multiplication in Lie group?

$$\exp\left(\phi_1^{\wedge}\right)\exp\left(\phi_2^{\wedge}\right) = \exp\left(\left(\phi_1 + \phi_2\right)^{\wedge}\right). \qquad ?$$

4. Lie Group and Lie Algebra

- Unfortunately, this does not work for matrices
- Baker-Campbell-Hausdorff formula gives the full version of this multiplication:

$$\ln(\exp(\boldsymbol{A})\exp(\boldsymbol{B})) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0,\\ 1\leqslant i\leqslant n}} \frac{\left(\sum_{i=1}^{n}(r_i+s_i)\right)^{-1}}{\prod_{i=1}^{n} r_i! s_i!} [\boldsymbol{A}^{r_1}\boldsymbol{B}^{s_1}\boldsymbol{A}^{r_2}\boldsymbol{B}^{s_2}\cdots\boldsymbol{A}^{r_n}\boldsymbol{B}^{s_n}]$$

- where

$$[\boldsymbol{A}^{r_1}\boldsymbol{B}^{s_1}\boldsymbol{A}^{r_2}\boldsymbol{B}^{s_2}\cdots\boldsymbol{A}^{r_n}\boldsymbol{B}^{s_n}] = [\underbrace{\boldsymbol{A},\cdots[\boldsymbol{A},}_{r_1}\underbrace{[\boldsymbol{B},\cdots[\boldsymbol{B},}_{s_1}\cdots\underbrace{[\boldsymbol{A},\cdots[\boldsymbol{A},}_{r_n}\underbrace{[\boldsymbol{B},\cdots[\boldsymbol{B},\boldsymbol{B}]\cdots]}_{s_n}]\cdots]\cdots]\cdots]]\cdots]$$

4. Lie Group and Lie Algebra

- First part of BCH formula:

$$\ln\left(\exp\left(\boldsymbol{A}\right)\exp\left(\boldsymbol{B}\right)\right) = \boldsymbol{A} + \boldsymbol{B} + \frac{1}{2}\left[\boldsymbol{A}, \boldsymbol{B}\right] + \frac{1}{12}\left[\boldsymbol{A}, \left[\boldsymbol{A}, \boldsymbol{B}\right]\right] - \frac{1}{12}\left[\boldsymbol{B}, \left[\boldsymbol{A}, \boldsymbol{B}\right]\right] + \cdots$$

- If A or B is small enough we can keep the linear item only, the BCH can be approximately written as:

$$\ln\left(\exp\left(\phi_1^\wedge\right)\exp\left(\phi_2^\wedge\right)\right)^\vee \approx \begin{cases} \boldsymbol{J}_l(\phi_2)^{-1}\phi_1 + \phi_2 & \text{if } \phi_1 \text{ is small,} \\ \boldsymbol{J}_r(\phi_1)^{-1}\phi_2 + \phi_1 & \text{if } \phi_2 \text{ is small.} \end{cases}$$

- where

$$\boldsymbol{J}_l = \boldsymbol{J} = \frac{\sin\theta}{\theta}\boldsymbol{I} + \left(1 - \frac{\sin\theta}{\theta}\right)\boldsymbol{a}\boldsymbol{a}^T + \frac{1-\cos\theta}{\theta}\boldsymbol{a}^\wedge.$$

Left Jacobian

$$\boldsymbol{J}_l^{-1} = \frac{\theta}{2}\cot\frac{\theta}{2}\boldsymbol{I} + \left(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\right)\boldsymbol{a}\boldsymbol{a}^T - \frac{\theta}{2}\boldsymbol{a}^\wedge.$$

Right Jacobian

$$\boldsymbol{J}_r(\phi) = \boldsymbol{J}_l(-\phi).$$

4. Lie Group and Lie Algebra

- Rewrite it (we take left multiplication as an example)

$$\exp\left(\Delta\phi^{\wedge}\right)\exp\left(\phi^{\wedge}\right) = \exp\left(\left(\phi + \boldsymbol{J}_l^{-1}\left(\phi\right)\Delta\phi\right)^{\wedge}\right).$$

- Left multiplication in Lie group means an addition in Lie algebra with an Jacobian
- Inversely, if we do addition in Lie algebra, the in Lie group:

$$\exp\left(\left(\phi + \Delta\phi\right)^{\wedge}\right) = \exp\left(\left(\boldsymbol{J}_l\Delta\phi\right)^{\wedge}\right)\exp\left(\phi^{\wedge}\right) = \exp\left(\phi^{\wedge}\right)\exp\left(\left(\boldsymbol{J}_r\Delta\phi\right)^{\wedge}\right).$$

4. Lie Group and Lie Algebra

- Similar in SE(3)'s case:

$$\exp\left(\Delta\boldsymbol{\xi}^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right) \approx \exp\left(\left(\mathcal{J}_l^{-1}\Delta\boldsymbol{\xi} + \boldsymbol{\xi}\right)^\wedge\right),$$

$$\exp\left(\boldsymbol{\xi}^\wedge\right)\exp\left(\Delta\boldsymbol{\xi}^\wedge\right) \approx \exp\left(\left(\mathcal{J}_r^{-1}\Delta\boldsymbol{\xi} + \boldsymbol{\xi}\right)^\wedge\right).$$

- Where:

$$\mathcal{J}_r(\boldsymbol{\xi}) = \begin{bmatrix} \boldsymbol{J}_r & \boldsymbol{Q}_r \\ \boldsymbol{0} & \boldsymbol{J}_r \end{bmatrix}$$

$$\mathcal{J}_\ell(\boldsymbol{\xi}) = \begin{bmatrix} \boldsymbol{J}_\ell & \boldsymbol{Q}_\ell \\ \boldsymbol{0} & \boldsymbol{J}_\ell \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{Q}_\ell(\boldsymbol{\xi}) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+m+2)!}(\boldsymbol{\phi}^\wedge)^n\boldsymbol{\rho}^\wedge(\boldsymbol{\phi}^\wedge)^m \\
&= \frac{1}{2}\boldsymbol{\rho}^\wedge + \left(\frac{\phi - \sin\phi}{\phi^3}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\
&\quad + \left(\frac{\phi^2 + 2\cos\phi - 2}{2\phi^4}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge - 3\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\
&\quad + \left(\frac{2\phi - 3\sin\phi + \phi\cos\phi}{2\phi^5}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)
\end{aligned}$$

$$\boldsymbol{Q}_r(\boldsymbol{\xi}) = \boldsymbol{Q}_\ell(-\boldsymbol{\xi}) = \boldsymbol{C}\boldsymbol{Q}_\ell(\boldsymbol{\xi}) + (\boldsymbol{J}_\ell\boldsymbol{\rho})^\wedge\boldsymbol{C}\boldsymbol{J}_\ell$$

4. Lie Group and Lie Algebra

- With BCH formula, we can define the derivate of a function of a rotation or transform matrix
- Example: rotating a point p
- We want to know the derivative: $\frac{\partial (\boldsymbol{R}\boldsymbol{p})}{\partial \boldsymbol{R}}.$
- We have two solutions:
  - Add a small item in the Lie algebra, and set its limit to zero (Derivative model)
  - (Left) Multiply a small item in the Lie group, and set its Lie algebra's limit to zero (Disturb model)

4. Lie Group and Lie Algebra

- Derivative model:

$$
\begin{aligned}
\frac{\partial\left(\exp\left(\phi^{\wedge}\right)\boldsymbol{p}\right)}{\partial\phi} &= \lim_{\delta\phi\to 0}\frac{\exp\left(\left(\phi+\delta\phi\right)^{\wedge}\right)\boldsymbol{p}-\exp\left(\phi^{\wedge}\right)\boldsymbol{p}}{\delta\phi}\\
&= \lim_{\delta\phi\to 0}\frac{\exp\left(\left(\boldsymbol{J}_l\delta\phi\right)^{\wedge}\right)\exp\left(\phi^{\wedge}\right)\boldsymbol{p}-\exp\left(\phi^{\wedge}\right)\boldsymbol{p}}{\delta\phi}\\
&\approx \lim_{\delta\phi\to 0}\frac{\left(\boldsymbol{I}+\left(\boldsymbol{J}_l\delta\phi\right)^{\wedge}\right)\exp\left(\phi^{\wedge}\right)\boldsymbol{p}-\exp\left(\phi^{\wedge}\right)\boldsymbol{p}}{\delta\phi}\\
&= \lim_{\delta\phi\to 0}\frac{\left(\boldsymbol{J}_l\delta\phi\right)^{\wedge}\exp\left(\phi^{\wedge}\right)\boldsymbol{p}}{\delta\phi}\\
&= \lim_{\delta\phi\to 0}\frac{-\left(\exp\left(\phi^{\wedge}\right)\boldsymbol{p}\right)^{\wedge}\boldsymbol{J}_l\delta\phi}{\delta\phi} = -(\boldsymbol{R}\boldsymbol{p})^{\wedge}\boldsymbol{J}_l.
\end{aligned}
$$

4. Lie Group and Lie Algebra

- Disturb model:

$$\begin{aligned}
\frac{\partial(\boldsymbol{R}\boldsymbol{p})}{\partial\boldsymbol{\varphi}} &= \lim_{\boldsymbol{\varphi}\to 0}\frac{\exp(\boldsymbol{\varphi}^\wedge)\exp(\boldsymbol{\phi}^\wedge)\boldsymbol{p}-\exp(\boldsymbol{\phi}^\wedge)\boldsymbol{p}}{\boldsymbol{\varphi}} \\
&\approx \lim_{\boldsymbol{\varphi}\to 0}\frac{(1+\boldsymbol{\varphi}^\wedge)\exp(\boldsymbol{\phi}^\wedge)\boldsymbol{p}-\exp(\boldsymbol{\phi}^\wedge)\boldsymbol{p}}{\boldsymbol{\varphi}} \\
&= \lim_{\boldsymbol{\varphi}\to 0}\frac{\boldsymbol{\varphi}^\wedge\boldsymbol{R}\boldsymbol{p}}{\boldsymbol{\varphi}} = \lim_{\boldsymbol{\varphi}\to 0}\frac{-(\boldsymbol{R}\boldsymbol{p})^\wedge\boldsymbol{\varphi}}{\boldsymbol{\varphi}} = -(\boldsymbol{R}\boldsymbol{p})^\wedge.
\end{aligned}$$

- More simple and clear
- In some literature we use operator $\oplus$ to denote this disturb model

$$\Delta R \oplus R = \exp(\delta\phi^\wedge)R$$

4. Lie Group and Lie Algebra

- Disturb model in SE(3):

$$
\begin{aligned}
\frac{\partial (\boldsymbol{T}\boldsymbol{p})}{\partial \delta \boldsymbol{\xi}} &= \lim_{\delta \boldsymbol{\xi} \to 0} \frac{\exp\left(\delta \boldsymbol{\xi}^{\wedge}\right) \exp\left(\boldsymbol{\xi}^{\wedge}\right) \boldsymbol{p} - \exp\left(\boldsymbol{\xi}^{\wedge}\right) \boldsymbol{p}}{\delta \boldsymbol{\xi}} \\
&\approx \lim_{\delta \boldsymbol{\xi} \to 0} \frac{\left(\boldsymbol{I} + \delta \boldsymbol{\xi}^{\wedge}\right) \exp\left(\boldsymbol{\xi}^{\wedge}\right) \boldsymbol{p} - \exp\left(\boldsymbol{\xi}^{\wedge}\right) \boldsymbol{p}}{\delta \boldsymbol{\xi}} \\
&= \lim_{\delta \boldsymbol{\xi} \to 0} \frac{\delta \boldsymbol{\xi}^{\wedge} \exp\left(\boldsymbol{\xi}^{\wedge}\right) \boldsymbol{p}}{\delta \boldsymbol{\xi}} \\
&= \lim_{\delta \boldsymbol{\xi} \to 0} \frac{\begin{bmatrix} \delta \boldsymbol{\phi}^{\wedge} & \delta \boldsymbol{\rho} \\ \boldsymbol{0}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}\boldsymbol{p} + \boldsymbol{t} \\ 1 \end{bmatrix}}{\delta \boldsymbol{\xi}} \\
&= \lim_{\delta \boldsymbol{\xi} \to 0} \frac{\begin{bmatrix} \delta \boldsymbol{\phi}^{\wedge} \left(\boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}\right) + \delta \boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta \boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{I} & -\left(\boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}\right)^{\wedge} \\ \boldsymbol{0}^T & \boldsymbol{0}^T \end{bmatrix} \triangleq (\boldsymbol{T}\boldsymbol{p})^{\odot}.
\end{aligned}
$$

Questions?