Optimal Eigen Mode of MIMO

October 31, 2021

For MIMO, in the frequency domain for a fixed frequency we have the transmit signal $X \in \mathbb{C}^M$ and recieve signal $Y \in \mathbb{C}^N$ and the channel matrix $H \in \operatorname{Hom}\left(\mathbb{C}^M, \mathbb{C}^N\right)$ related by

$$Y = HX \tag{1}$$

Our aim is to find an X subject to an energy constraint maximizes Y given H.

Assume the rank of H is r, we apply compact SVD on H to get

$$H = UDV^* \tag{2}$$

where $U \in \operatorname{Hom}\left(\mathbb{C}^r, \mathbb{C}^N\right)$, $V \in \operatorname{Hom}\left(\mathbb{C}^r, \mathbb{C}^M\right)$ satisfying $U^*U = I_N$ and $V^*V = I_M$, and

$$D = \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\}$$

is a the diagonal matrix of singular values of H.

With proper pre-processing

$$X = VS$$

and post-processing

$$Z = U^*Y$$

we have

$$Z = U^*UDV^*VS = DS$$

which is a group of r paralell scalar channels without interfering with each other.

Claim 1. Pre-multiplying U^* and V preserve energy.

Proof.

$$X^*X = S^*V^*VS$$

It follows from

$$V^*V = I_M$$

.

The proof for U^* involves a little bit more. We have

$$Z^*Z = Y^*UU^*Y$$

We partition U into column vectors

$$U = \left[u^1, \dots, u^r\right]$$

where $u^i \in \mathbb{C}^N$ being mutually orthogonal. It follows from equation 1 and 2 that $\exists y^i \in \mathbb{C}$ s.t.

$$Y = \sum_{i=1}^{r} y_i u^i$$

Hence

$$\begin{aligned} Y^*UU^*Y &= \left(\sum_{i=1}^{r} y_i^*(u^i)^*\right)\left(\sum_{j=1}^{r} u^j(u^j)^*\right)\left(\sum_{k=1}^{r} y_k u^k\right) \\ &= \sum_{i,j,k} y_i^* y_k \left((u^i)^* u^j\right)\left((u^j)^* u^k\right) \\ &= \sum_{i,j,k} y_i^* y_k \delta_i^j \delta_j^k \\ &= \sum_{i=1}^{r} |y_i|^2 = Y^*Y \end{aligned}$$

which completes the proof. □

Now our goal boils down to finding an S subject to an energy constraint maximizes Z given D, which is a convex optimization problem

$$\begin{array}{ll} \text{maximize} & s^*D^2s \\ \text{subject to} & s^*s \leq 1 \end{array} \qquad (3)$$

However, for $s \in \mathbb{C}^r$, the objective function is not differentiable. To see this, we need to take a detour to discuss the condition of differentiability of operators between complex vector spaces.

Pick any $f : \mathbb{C}^M \to \mathbb{C}^N$, then f is differentiable at $z \in \mathbb{C}^M$ if and only if there exists a linear operator $\nabla f(z) \in \mathrm{Hom}\left(\mathbb{C}^M, \mathbb{C}^N\right)$ such that

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z) - \nabla f(z) \Delta z}{||\Delta z||} = 0 \tag{4}$$

Expand f to be

$$f(z) = u(x, y) + iv(x, y) \tag{5}$$

where $z = x + iy$, and $x, y \in \mathbb{R}^M$, and $u, v : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^N$.

Equation 4 implies the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{||\Delta z||} \tag{6}$$

exists. Choose any $a \neq 0, a \in \mathbb{C}^M$ and a $t \in \mathbb{R}$. Let $\Delta z = ta$, then $\Delta z \to 0$ as $t \to 0$. Substitute 5 into 6. For Δz either being ta or ita, the limit should stay the same, which leads to a "quasi"-Cauchy-Riemann equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

where u_x (at point (x, y)) is defined to be the unique linear operator in $\mathrm{Hom}\left(\mathbb{R}^M, \mathbb{R}^N\right)$ such that $\forall y \in \mathbb{R}^M$

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) - u_x(x, y) \Delta x}{||\Delta x||} = 0$$

and v_y, u_y, v_x are defined in similar ways.

We can see that the objective function of 3 is not differentiable. Indeed, $s^* D^2 s$ is a real-valued funtion which means $v_x = v_y = 0$, but this does not hold for u_x and u_y.

Note that restricting s on ${\mathbb{R}_{\geq 0}}^M$ does not change the optimal value of the objective function. It is not hard to see, by KKT condition, that the quadratic programming 3 restricted on ${\mathbb{R}_{\geq 0}}^M$ can achieve the optimal value

$$\max_s s^* D^2 s = \max_i \sigma_i^2$$

with the optimal point s whose i-th component is

$$s_i = \begin{cases} 1 & , i = \operatorname{argmax}_i \sigma_i \\ 0 & , \text{otherwise} \end{cases}$$

Then we extend the optimal points on $\mathbb{C}^M$. It is obvious that for any real diagonal matrix A, the vector given by

$$w = e^{jA} s$$

is also an optimal point.