Optimal Wideband Precoder For MIMO

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For a MIMO wireless communication, we have sampled transmit signal $x[n] \in \mathbb{C}^M$, receive signal $y[n] \in \mathbb{C}^N$, and a channel $h[n] \in \operatorname{Hom}\left(\mathbb{C}^M, \mathbb{C}^M\right)$ all with lengths P

$$y[n] = \sum_{n-0}^{P-1} h[n]x[n]$$

And in the frequency domain we have

$$Y[k] = H[k]X[k]$$

where $k = 0, \ldots, P-1$.

We want to find an approximation of $h[n]$ denoted by $\hat{h}$ such that y and $\hat{h}x$ are as close as possible, or, Y and $\hat{h}X$ as close as possible. Later we will show that these two requirements are equivalent.

Note that y and Y are elements of a $\operatorname{Hom}\left(\mathbb{C}^N\right)$-module, h and H are elements of another $\operatorname{Hom}\left(\mathbb{C}^N\right)$-module, and x as well as X are elements of a $\mathbb{C}^{M\times M}$-module. To measure how close to elements of a module are, we can define an "inner product" analogy to the usual inner product defined on Hilbert spaces.

Definition 1 (Inner product on $\operatorname{Hom}\left(\mathbb{C}^N\right)$-modules)**.** An auto-correlation on a P-dimensional free $\operatorname{Hom}\left(\mathbb{C}^N\right)$-module $\mathcal{M}$ is a operation $\langle\cdot,\cdot\rangle\colon \mathcal{M}\times\mathcal{M} \to \operatorname{Hom}\left(\mathbb{C}^N\right)$

$$\langle x, y\rangle = \sum_{n=0}^{P-1} x[n]y[n]^*$$

We also refer C_{xy} to $\langle x, y\rangle$ for convenience under some context.

Definition 2 (Orthogonality on modules)**.** $x, y \in \mathcal{M}$ are orthogonal if

$$\langle x, y \rangle = 0$$

It is easy to verify that $\langle x, x \rangle$ is positive semi-definite. Therefore we can define the norm induced by the inner product

Definition 3 (Euclidean norm on modules)**.** The Euclidean norm of $x \in \mathcal{M}$ is

$$||x|| = \langle x, x \rangle^{1/2}$$

It is obvious that the norm of any element is positive semi-definite. Hence, there is a partial order $\preceq$ induced by the positive semi-definite cone. And we are able to measure the closeness of two elements in a module by this partial order.

The theorem follows tells us that we can study the closeness in either time domain or frequency domain.

Theorem 1 (Extended Parseval's Theorem)**.**

$$||X|| = \sqrt{P}||x||$$

w.r.t. the DFT defined by

$$X[k] = \sum_{n=0}^{P-1} x[n]\omega_P^{nk}$$

where

$$\omega_P^{nk} = e^{-j2\pi nk/P}$$

Proof.

$$\begin{aligned}
||X||^2 &= \sum_{k=0}^{P-1} X[k]X[k]^* \\
&= \sum_{k=0}^{P-1} \left(\sum_{m=0}^{P-1} x[m]\omega_P^{mk}\right)\left(\sum_{n=0}^{P-1} x[n]\omega_P^{nk}\right)^* \\
&= \sum_{k=0}^{P-1}\sum_{m=0}^{P-1}\sum_{n=0}^{P-1} x[m]x[n]^*\omega_P^{(m-n)k}
\end{aligned}$$

Note that the terms being summed are non-zero only if $m = n$, thus

$$||X||^2 = \sum_{k=0}^{P-1} Px[k]x[k]^* = P||x||^2$$

completes the proof. $\square$

To find the optimal $\hat{h}$ in time domain is equivalent to find it in the frequency domain. This amounts to say that we are trying to approximate the frequency-selective $H[k]$ with a constant $\hat{h}$ over all frequencies.

We need to extend the orthogonality principle to modules. Before that we need another kind of product of elements from different modules.

Definition 4 (Cross-correlation)**.** The cross-correlation is a binary operator defined on the product of two modules $\langle \cdot, \cdot \rangle$: $\mathcal{N} \times \mathcal{M} \to \mathrm{Hom}\left(\mathbb{C}^M, \mathbb{C}^N\right)$

$$\langle x, y \rangle = \sum_{n=0}^{P-1} x[n]y[n]^*$$

And we sometimes refer C_{xy} to $\langle x, y \rangle$ for convenience under certain contexts.

When $N = M$, the cross-correlation coincides with auto-correlation.

Lemma 1. *Let $z \in \mathcal{N}$, $x \in \mathcal{M}$ s.t. $C_{zx} = 0$. Then $\forall \mu \in Hom\left(\mathbb{C}^M, \mathbb{C}^N\right)$ we have*

$$||z + \mu x|| \succeq ||z||$$

Proof. It is easy to verify that

$$||z + \mu x||^2 = ||z||^2 + ||\mu x||^2$$

and we are done. $\square$

It follows from lemma 1 that every $y \in \mathcal{N}$ can be decomposed into two parts

$$y = z + \mu x$$

where $C_{zx} = 0$ and $\mu \in \mathrm{Hom}\left(\mathbb{C}^M, \mathbb{C}^N\right)$