Rank Of Sum Of Positive Semidefinite Matrices

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We want to proof

$$\mathrm{rank}(A + B) \geq \max(\mathrm{rank}(A), \mathrm{rank}(B)) \tag{1}$$

provided that $A, B \in \mathrm{S}_{+}^{N}$, i.e., positive semidefinite matrices of order N.

Lemma 1. *$\forall S \in S_{+}^{N}$*

$$rank(S + I_N) = N$$

Proof. Apply spectral decomposition on S, $\exists U \in \mathrm{U}(N)$ (group of unitary matrices) and diagonal matrices D with nonnegative entries, such that

$$S + I = U(D + I)U^*$$

where the diagonal entries of $D + I$ are strictly positive. It follows from that unitary matrices do not change rank. □

Lemma 2. *$\forall A \in S_{+}^{N}$, $\exists P \in U(r)$ s.t.*

$$\tilde{A} = PP^*$$

where $r = rank(A)$ and $\tilde{A} = A|_{V/\ker A}$.

Proof. First apply spetral decopomsition on $\tilde{A}$

$$\tilde{A} = UDU^*$$

Then let $L = \sqrt{D}$ and $P = UL$ completing the proof. □

Now we are able to prove the inequality of interest.

Proof. (Rank inequality of sum of positive semidefinite matrices)

Suppose A is non-trivial, otherwise the inequality trivially holds. Restrict A and B on $V/\ker A$ to get $\tilde{A}$ and $\tilde{B}$. Then apply lemma 2

$$\tilde{A} + \tilde{B} = P(I + P^* \tilde{B} P)P^* \tag{2}$$

It follows from lemma 1 that

$$\operatorname{rank}(\tilde{A} + \tilde{B}) = \operatorname{rank}(I) = \operatorname{rank}(\tilde{A})$$

Since $A + B$ might not be trivial on $\ker A$, we have

$$\operatorname{rank}(A + B) \geq \operatorname{rank}(\tilde{A} + \tilde{B})$$

Also note that

$$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A})$$

and we are done. □

Equation 2 also provides us a means to study the relationship between the spectral decomposition of (A+B) and of A and B respectively.

We have a more elegant proof though, by introducing a lemma first

Lemma 3. $\forall v \in V,\ v^* A v = 0 \Leftrightarrow v \in \ker A$

Proof. $\Leftarrow$ is obvious.

$\Rightarrow$ follows from lemma 2, if $\tilde{A}$ is non-trivial, then it is positive definite. This completes the proof. □

Now the proof follows

Proof. $\forall A \in \mathrm{S}_+^N$, define $N(A)$ to be

$$N(A) = \{v \in V | v^* A v = 0\}$$

It is easy to verify that

$$N(A + B) = N(A) \cap N(B)$$

Apply lemma 3

$$\ker(A + B) = \ker A \cap \ker B \subseteq \ker A$$

which implies

$$\operatorname{rank}(A + B) \geq \operatorname{rank}(A)$$

which completes the proof. □