Translation of F$^\circ$ to Bang

October 6, 2009

1 Definitions

$termvar,\ x,\ y,\ z$
$linvar,\ a,\ b,\ c$
α
$index,\ i,\ j,\ n,\ m$

κ	$::=$			kind:
	$\vert$	$\star$		Unrestricted
	$\vert$	$\circ$		Linear
τ	$::=$			type:
	$\vert$	α		type variable
	$\vert$	$\tau_1 \xrightarrow{\kappa} \tau_2$		arrow
	$\vert$	$\forall \alpha{:}\kappa.\ \tau$		all
e	$::=$			expressions:
	$\vert$	x		variable
	$\vert$	$\lambda^\kappa x{:}\tau.\ e$		abstraction
	$\vert$	$e_1\ e_2$		application
	$\vert$	$\Lambda \alpha{:}\kappa.\ e$		type abstaction
	$\vert$	$e\,[\,\tau\,]$		type application
	$\vert$	$(\,e\,)$	S	
v	$::=$			value:
	$\vert$	$\lambda^\kappa x{:}\tau.\ e$		abstraction
	$\vert$	$\Lambda \alpha{:}\kappa.\ e$		type abstaction
Γ	$::=$			
	$\vert$	$.$		
	$\vert$	$\Gamma, x{:}\tau$		
	$\vert$	$\Gamma, \alpha{:}\kappa$		
Δ	$::=$			
	$\vert$	$.$		
	$\vert$	$\Delta, x{:}\tau$		
t	$::=$			term:
	$\vert$	x		variable
	$\vert$	a		linear variable

	$\vert$	$\lambda a : \sigma . t$		abstraction
	$\vert$	$t\, t'$		application
	$\vert$	$\Lambda \alpha . t$		type abstraction
	$\vert$	$t\, [\, \sigma\,]$		type application
	$\vert$	$!\, t$		bang
	$\vert$	$\textbf{let}\, !\, x \;=\; t\, \textbf{in}\, t'$		
	$\vert$	$(\, t\,)$	S	
u	$::=$			value:
	$\vert$	$\lambda a : \sigma . t$		abstraction value
	$\vert$	$\Lambda \alpha . t$		type abstraction value
	$\vert$	$!\, t$		
	$\vert$	$(\, u\,)$	S	
σ	$::=$			types:
	$\vert$	α		type variable
	$\vert$	$\sigma \multimap \sigma'$		type of function
	$\vert$	$!\, \sigma$		type of bang
	$\vert$	$\forall \alpha . \sigma$		universal type
	$\vert$	$(\, \sigma\,)$	S	
Φ	$::=$			contexts:
	$\vert$	$\cdot$		empty context
	$\vert$	$\Phi, x : \sigma$		term variable binding
	$\vert$	Φ, α		type variable binding
	$\vert$	$(\, \Phi\,)$	S	
Ψ	$::=$			linear contexts:
	$\vert$	$\cdot$		empty linear context
	$\vert$	$\Psi, a : \sigma$		linear variable type binding
	$\vert$	$(\, \Psi\,)$	S	

$\boxed{\vdash \Gamma}$ Context well-formed

$\boxed{\Gamma \vdash \tau : \kappa}$ Type well-formed

$\boxed{\Gamma \vdash \Delta}$ Linear context well-formed

$\boxed{\Gamma ; \Delta \vdash e : \tau}$ Term well-typed

$\boxed{t \longrightarrow t'}$ Evaluation

$$\frac{t_1 \longrightarrow t_1'}{t_1\, t_2 \longrightarrow t_1'\, t_2} \quad \text{E\_APP1}$$

$$\frac{}{(\, \lambda a : \sigma . t\,)\, t' \longrightarrow t\, \{\, t' / a\, \}} \quad \text{E\_APPABS}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\, [\, \sigma\,] \longrightarrow t_1'\, [\, \sigma\,]} \quad \text{E\_TAPP}$$

$$\frac{}{(\, \Lambda \alpha . t\,)\, [\, \sigma\,] \longrightarrow t\, \{\, \sigma / \alpha\, \}} \quad \text{E\_TAPPTABS}$$

$$\frac{t_1 \longrightarrow t_1'}{\textbf{let}\, !\, x \;=\; t_1\, \textbf{in}\, t_2 \longrightarrow \textbf{let}\, !\, x \;=\; t_1'\, \textbf{in}\, t_2} \quad \text{E\_LET}$$

$$\frac{}{\mathbf{let}\,!\,x\ =!\,t_1\,\mathbf{in}\,t_2\ \rightarrow\ t_2\,\{\,t_1\,/\,x\,\}}\quad \text{E\_BANG}$$

$\boxed{\vdash\ \Phi}$ Context well-formed

$$\frac{}{\vdash\ \cdot}\quad \text{C\_EMPTY}$$

$$\frac{\vdash\ \Phi\quad \Phi\vdash\sigma\quad x\notin dom(\Phi)}{\vdash\ \Phi, x{:}\sigma}\quad \text{C\_VAR}$$

$$\frac{\vdash\ \Phi\quad \alpha\notin dom(\Phi)}{\vdash\ \Phi, \alpha}\quad \text{C\_TVAR}$$

$\boxed{\Phi\vdash\sigma}$ Types well-formed

$$\frac{\vdash\ \Phi\quad \alpha\in\Phi}{\Phi\vdash\alpha}\quad \text{K\_TVAR}$$

$$\frac{\Phi\vdash\sigma_1\quad \Phi\vdash\sigma_2}{\Phi\vdash\sigma_1\ \multimap\ \sigma_2}\quad \text{K\_ARROW}$$

$$\frac{\Phi\vdash\sigma}{\Phi\vdash!\,\sigma}\quad \text{K\_BANG}$$

$$\frac{\Phi,\alpha\vdash\sigma}{\Phi\vdash\forall\alpha.\sigma}\quad \text{K\_ALL}$$

$\boxed{\Phi\vdash\Psi}$ Linear Context well-formed

$$\frac{\vdash\ \Phi}{\Phi\vdash\cdot}\quad \text{L\_EMPTY}$$

$$\frac{\Phi\vdash\Psi\quad \Phi\vdash\sigma\quad a\notin dom(\Psi)}{\Phi\vdash\Psi, a{:}\sigma}\quad \text{L\_LVAR}$$

$\boxed{\Phi\,;\,\Psi\ \vdash\ t\ :\ \sigma}$ Typing

$$\frac{\vdash\ \Phi\quad x{:}\sigma\in\Phi}{\Phi\,;\,\cdot\ \vdash\ x\ :\ \sigma}\quad \text{T\_VAR}$$

$$\frac{\Phi\vdash a\ :\ \sigma}{\Phi\,;\,a\ :\ \sigma\vdash\ a\ :\ \sigma}\quad \text{T\_LVAR}$$

$$\frac{\Phi\vdash\sigma_1\quad \Phi\,;\,\Psi, a{:}\sigma_1\ \vdash\ t_2\ :\ \sigma_2}{\Phi\,;\,\Psi\ \vdash\ \lambda a{:}\sigma_1.t_2\ :\ \sigma_1\ \multimap\ \sigma_2}\quad \text{T\_ABS}$$

$$\frac{\Phi\,;\,\Psi_1\ \vdash\ t_1\ :\ \sigma_{11}\ \multimap\ \sigma_{12}\quad \Phi\,;\,\Psi_2\ \vdash\ t_2\ :\ \sigma_{11}}{\Phi\,;\,\Psi_1\ \Cup\ \Psi_2\ \vdash\ t_1\,t_2\ :\ \sigma_{12}}\quad \text{T\_APP}$$

$$\frac{\Phi,\alpha\,;\,\Psi\ \vdash\ t_1\ :\ \sigma}{\Phi\,;\,\Psi\ \vdash\ \Lambda\alpha.t_1\ :\ \forall\alpha.\sigma}\quad \text{T\_TABS}$$

$$\frac{\Phi\,;\,\Psi\ \vdash\ t_1\ :\ \forall\alpha.\sigma\quad \Phi\vdash\sigma_1}{\Phi\,;\,\Psi\ \vdash\ t_1\,[\,\sigma_1\,]\ :\ \sigma\,\{\,\sigma_1\,/\,\alpha\,\}}\quad \text{T\_TAPP}$$

$$\frac{\Phi\,;\,\cdot\ \vdash\ t\ :\ \sigma}{\Phi\,;\,\cdot\ \vdash!\,t\ :!\,\sigma}\quad \text{T\_BANG}$$

$$\frac{\Phi\,;\,\Psi_1\ \vdash\ t_1\ :!\,\sigma_1\quad \Phi, x{:}\sigma_1\,;\,\Psi_2\ \vdash\ t_2\ :\ \sigma_2}{\Phi\,;\,\Psi_1\ \Cup\ \Psi_2\vdash\ \mathbf{let}\,!\,x\ =\ t_1\,\mathbf{in}\,t_2\ :\ \sigma_2}\quad \text{T\_LET}$$

$\boxed{\Gamma \vdash \tau : \kappa \Longrightarrow \sigma}$ translate types

$$\frac{\alpha{:}\kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa \Longrightarrow [\![\kappa]\!]\alpha} \quad \text{TR\_TVAR}$$

$$\frac{\begin{array}{c}\Gamma \vdash \tau_1 : \kappa_1 \Longrightarrow \sigma_1 \\ \Gamma \vdash \tau_2 : \kappa_2 \Longrightarrow \sigma_2\end{array}}{\Gamma \vdash \tau_1 \xrightarrow{\kappa} \tau_2 : \kappa \Longrightarrow [\![\kappa]\!](\sigma_1 \multimap \sigma_2)} \quad \text{TR\_ARR}$$

$$\frac{\Gamma, \alpha{:}\kappa_1 \vdash \tau : \kappa \Longrightarrow \sigma}{\Gamma \vdash \forall\alpha{:}\kappa_1.\, \tau : \kappa \Longrightarrow [\![\kappa]\!](\forall\alpha.\sigma)} \quad \text{TR\_ALL}$$

$\boxed{\Gamma \Longrightarrow \Phi}$ translate unrestricted contexts

$$\frac{}{\cdot \Longrightarrow \cdot} \quad \text{TR\_EMPTY}$$

$$\frac{\Gamma \Longrightarrow \Phi \quad \Gamma \vdash \tau : \star \Longrightarrow !\,\sigma}{\Gamma, x{:}\tau \Longrightarrow \Phi, x : \sigma} \quad \text{TR\_CONS\_V}$$

$$\frac{\Gamma \Longrightarrow \Phi}{\Gamma, \alpha{:}\kappa \Longrightarrow \Phi, \alpha} \quad \text{TR\_CONS\_TV}$$

$\boxed{\Gamma; \Delta \Longrightarrow \Psi}$ translate linear contexts

$$\frac{}{\Gamma; \cdot \Longrightarrow \cdot} \quad \text{TR\_LEMPTY}$$

$$\frac{\Gamma; \Delta \Longrightarrow \Psi \quad \Gamma \vdash \tau : \circ \Longrightarrow \sigma \quad \hat{x} = a}{\Gamma; \Delta, x{:}\tau \Longrightarrow \Psi, a : \sigma} \quad \text{TR\_LCONS\_L}$$

$$\frac{\Gamma; \Delta \Longrightarrow \Psi \quad \Gamma \vdash \tau : \star \Longrightarrow \sigma \quad \hat{x} = a}{\Gamma; \Delta, x{:}\tau \Longrightarrow \Psi, a : \sigma} \quad \text{TR\_LCONS\_U}$$

$\boxed{\Gamma; \Delta \vdash e : \tau \Longrightarrow t}$ translate expressions

$$\frac{x : \tau \in \Gamma}{\Gamma; \cdot \vdash x : \tau \Longrightarrow !\, x} \quad \text{TRANS\_UVAR}$$

$$\frac{\hat{x} = a}{\Gamma; x{:}\tau \vdash x : \tau \Longrightarrow a} \quad \text{TRANS\_LVAR}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau_1 : \star \Longrightarrow \sigma_1 \\ \Gamma, x{:}\tau_1; \Delta \vdash e : \tau_2 \Longrightarrow t\end{array}}{\Gamma; \Delta \vdash \lambda^\kappa x{:}\tau_1.\, e : \tau_1 \xrightarrow{\kappa} \tau_2 \Longrightarrow [\![\kappa]\!](\lambda a : \sigma_1.\mathbf{let}\,!\,x = a\,\mathbf{in}\,t)} \quad \text{TRANS\_UFUN1}$$

$$\frac{\begin{array}{l}\hat{x} = a \\ \Gamma \vdash \tau_1 : \kappa_1 \Longrightarrow \sigma_1 \\ \Gamma; \Delta, x{:}\tau_1 \vdash e : \tau_2 \Longrightarrow t\end{array}}{\Gamma; \Delta \vdash \lambda^\kappa x{:}\tau_1.\, e : \tau_1 \xrightarrow{\kappa} \tau_2 \Longrightarrow [\![\kappa]\!](\lambda a : \sigma_1.t)} \quad \text{TRANS\_UFUN2}$$

$$\frac{\begin{array}{l}\Gamma; \Delta_1 \vdash e_1 : \tau_1 \xrightarrow{\star} \tau_2 \Longrightarrow t_1 \\ \Gamma; \Delta_2 \vdash e_2 : \tau_1 \Longrightarrow t_2\end{array}}{\Gamma; \Delta_1 \Cup \Delta_2 \vdash e_1\, e_2 : \tau_2 \Longrightarrow \mathbf{let}\,!\,x = t_1\,\mathbf{in}\,x\,t_2} \quad \text{TRANS\_APP1}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau_1 : \kappa_1 \\ \Gamma; \Delta_1 \vdash e_1 : \tau_1 \xrightarrow{\circ} \tau_2 \Longrightarrow t_1 \\ \Gamma; \Delta_2 \vdash e_2 : \tau_1 \Longrightarrow t_2\end{array}}{\Gamma; \Delta_1 \Cup \Delta_2 \vdash e_1\, e_2 : \tau_2 \Longrightarrow t_1\, t_2} \quad \text{TRANS\_APP2}$$

$$\frac{\Gamma, \alpha{:}\kappa; \Delta \vdash e : \tau \Longrightarrow\ t}{\Gamma; \Delta \vdash \Lambda\alpha{:}\kappa.\, e : \forall\alpha{:}\kappa.\, \tau \Longrightarrow\ [\![\kappa']\!](\,\Lambda\alpha.t\,)} \quad \textsc{trans\_tabs}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau : \star \\ \Gamma \vdash \tau' : \star \Longrightarrow !\,\sigma' \\ \Gamma; \Delta \vdash e : \forall\alpha{:}\star.\, \tau \Longrightarrow\ t\end{array}}{\Gamma; \Delta \vdash e\,[\,\tau'\,] : \tau\,\{\,\tau'\,/\,\alpha\,\} \Longrightarrow\ \mathbf{let}\,!\,x\ =\ t\,\mathbf{in}\,x\,[\,\sigma'\,]} \quad \textsc{trans\_tapp}\mathrm{UU}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau : \circ \\ \Gamma \vdash \tau' : \star \Longrightarrow !\,\sigma' \\ \Gamma; \Delta \vdash e : \forall\alpha{:}\star.\, \tau \Longrightarrow\ t\end{array}}{\Gamma; \Delta \vdash e\,[\,\tau'\,] : \tau\,\{\,\tau'\,/\,\alpha\,\} \Longrightarrow\ t\,[\,\sigma'\,]} \quad \textsc{trans\_tapp}\mathrm{LU}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau : \star \\ \Gamma \vdash \tau' : \kappa \Longrightarrow \sigma' \\ \Gamma; \Delta \vdash e : \forall\alpha{:}\circ.\, \tau \Longrightarrow\ t\end{array}}{\Gamma; \Delta \vdash e\,[\,\tau'\,] : \tau\,\{\,\tau'\,/\,\alpha\,\} \Longrightarrow\ \mathbf{let}\,!\,x\ =\ t\,\mathbf{in}\,x\,[\,\sigma'\,]} \quad \textsc{trans\_tapp}\mathrm{UL}$$

$$\frac{\begin{array}{l}\Gamma \vdash \tau : \circ \\ \Gamma \vdash \tau' : \kappa \Longrightarrow \sigma' \\ \Gamma; \Delta \vdash e : \forall\alpha{:}\circ.\, \tau \Longrightarrow\ t\end{array}}{\Gamma; \Delta \vdash e\,[\,\tau'\,] : \tau\,\{\,\tau'\,/\,\alpha\,\} \Longrightarrow\ t\,[\,\sigma'\,]} \quad \textsc{trans\_tapp}\mathrm{LL}$$

Definition of metafunctions:

$$\begin{array}{rcl}[\![\star]\!]\sigma & = & !\,\sigma \\ [\![\circ]\!]\sigma & = & \sigma \\ [\![\star]\!]t & = & !\,t \\ [\![\circ]\!]t & = & t\end{array}$$

2 Metatheory

Simple Lemmas:

- If $\vdash\ \Gamma$ and $x{:}\tau \in \Gamma$ then $\Gamma\ \vdash\ \tau\ :\ \star$.
- If $\Gamma\ \vdash\ x{:}\tau$ then $\Gamma\ \vdash\ \tau\ :\ \circ$.
- If $\Gamma\ \vdash\ \tau\ :\ \kappa$ and $\Gamma\ \vdash\ \tau\ :\ \kappa\ \Longrightarrow\ \sigma$ and $\Gamma\ \Longrightarrow\ \Phi$ then $\Phi \vdash \sigma$.
- If $\vdash\ \Gamma$ and $\Gamma\ \Longrightarrow\ \Phi$ then $\vdash\ \Phi$.
- If $\Gamma\ \vdash\ \Delta$ and $\Gamma\ \Longrightarrow\ \Phi$ and $\Gamma\,;\,\Delta\ \Longrightarrow\ \Psi$ then $\Phi \vdash \Psi$.
- If $\Gamma\ \vdash\ \tau\ :\ \star\ \Longrightarrow\ \sigma$ then $\sigma\ = !\,\sigma'$ for some σ'.
- If $\Gamma\ \vdash\ \tau_1 \overset{\kappa}{\to} \tau_2\ :\ \circ\ \Longrightarrow\ \sigma$ then $\sigma\ =\ \sigma_1\ \multimap\ \sigma_2$
- If $\Gamma\ \vdash\ \forall\alpha{:}\kappa.\,\tau\ :\ \circ\ \Longrightarrow\ \sigma$ then $\sigma\ =\ \forall\alpha.\sigma'$.
- If $\vdash\ \Gamma$ and $x{:}\tau \in \Gamma$ and $\Gamma\ \Longrightarrow\ \Phi$ then $x{:}\sigma \in \Phi$ where $\Gamma\ \vdash\ \tau\ :\ \star\ \Longrightarrow !\,\sigma$.
- If $\Gamma\,;\,\Delta_1\ \Longrightarrow\ \Psi_1$ and $\Gamma\,;\,\Delta_2\ \Longrightarrow\ \Psi_2$ then $\Gamma\,;\,\Delta_1\ \Cup\ \Delta_2\ \Longrightarrow\ \Psi_1\ \Cup\ \Psi_2$.
- If $\Gamma, \alpha{:}\star\ \vdash\ \tau\ :\ \kappa\ \Longrightarrow\ \sigma$ and $\Gamma\ \vdash\ \tau'\ :\ \star\ \Longrightarrow !\,\sigma'$ then $\Gamma\ \vdash\ \tau\,\{\,\tau'\,/\,\alpha\,\}\ :\ \kappa\ \Longrightarrow\ \sigma\,\{\,\sigma'\,/\,\alpha\,\}$.
- If $\Gamma, \alpha{:}\circ\ \vdash\ \tau\ :\ \kappa\ \Longrightarrow\ \sigma$ and $\Gamma\ \vdash\ \tau'\ :\ \circ\ \Longrightarrow\ \sigma'$ then $\Gamma\ \vdash\ \tau\,\{\,\tau'\,/\,\alpha\,\}\ :\ \kappa\ \Longrightarrow\ \sigma\,\{\,\sigma'\,/\,\alpha\,\}$.

Theorem 1 (Translation is type preserving). *If*

- $\Gamma\,;\,\Delta \vdash e : \tau$
- $\Gamma \vdash \tau : \kappa \Longrightarrow \sigma$
- << no parses (char 3): [G*** ; D |- exp : tau] k ==> t >>
- $\Gamma \Longrightarrow \Phi$
- $\Gamma\,;\,\Delta \Longrightarrow \Psi$

Then:

- $\Phi\,;\,\Psi \vdash t : \sigma$

Proof. By induction on the derivation that $\Gamma\,;\,\Delta \vdash e : \tau$.

□

Notes This translation is actually a relation — there are many possible target translations for a given term. The translation is actually governed by the typing derivation (including resolution of the subkinding and nondeterministic variable binding) found for a source program. The translation is also parameterized by the kind κ, which is an indicator of how the term being translated will be treated by its context—if the context promises to treat the term linearly, then the translation can remove !'s.

The power of using kinds to help with polymorphism is shown by the fact that translating type instantiation takes four forms: the type being instantiated can be linear or unrestricted, and the term it is being supplied to can be linear or unrestricted.