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An unbiased method of measuring the ratio of two data sets: take the lensing ratio as an example

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Problem we want to solve:

- The meaningful physical quantity we want is the ***ratio*** of two data sets

$$D_1 = \bar{D}_1(1+n_1) \quad D_2 = \bar{D}_2(1+n_2) \quad \bar{D} \text{ is the true value} \quad \langle n \rangle = 0$$

$$\text{the theory predicts} \quad \bar{D}_2/\bar{D}_1 = R$$

$$\text{naive estimator} \quad \langle \hat{R} \rangle = \left\langle \frac{D_2}{D_1} \right\rangle = R\left(1 + \langle n_1^2 \rangle - \langle n_1 n_2 \rangle + \cdots \right) \neq R$$

- Simply taking the ratio of two data sets is sub-optimal / biased, as long as the D1 (denominator) has measurement errors

Many applications in astronomical data analysis

- **Lensing (shear) ratio** (Jain&Taylor 2003, Zhang+2005, DES, CFHTLens, Planck, SPT)
  - The ratio of galaxy-galaxy lensing of different sources (e.g. cosmic shear at various redshifts and CMB lensing) but identical lenses
  - The ratio provides a clean measure on the geometry of the universe
  - Do not require modeling galaxy bias or matter power spectrum
- **The decay rate of gravitational potential.** The ratio between galaxy-ISW cross-correlation and galaxy-lensing cross-correlation (Zhang 2006, Dong+ 2022)
- **Interloper rate due to line confusion in spectroscopic redshift surveys.** This particular error in redshift measurement can be approximated as the ratio between the cross-correlation of two target galaxy samples and the auto-correlation (CSST, Gong+ 2021)
- E_G **as a probe of gravity at cosmological scales.** Consistency test of GR. Ratio of galaxy-velocity correlation and galaxy-lensing correlation (Zhang+ 2007)

Methodology: when $\mathbf{d}_{1,2}$ are uncorrelated

Theory prediction is fixed

the dependence on R

$$\mathbf{d}_1 = \mathbf{A}_1\lambda + \mathbf{n}_1 \qquad \mathbf{d}_2 = \mathbf{A}_2(R)\lambda + \mathbf{n}_2$$

Based on Bayesian analysis:

$$\boxed{P(R|\mathbf{d}_1,\mathbf{d}_2) \propto \int P(\mathbf{d}_1,\mathbf{d}_2|R,\lambda)P(\lambda)P_{\rm prior}(R)d\lambda \quad \textbf{(1)}}$$

$P(\lambda) \propto$ const.

Jeffreys $1/R$ prior

Joint probability distribution

$$P(\mathbf{d}_1,\mathbf{d}_2|R,\lambda) = P(\mathbf{d}_1|R,\lambda)P(\mathbf{d}_2|R,\lambda)$$

$$\begin{cases} P(\mathbf{d}_1|R,\lambda) = \dfrac{1}{\sqrt{(2\pi)^N \det\mathbf{C}_1}}\exp\left[-\dfrac{1}{2}(\mathbf{d}_1-\mathbf{A}_1\lambda)^T\mathbf{C}_1^{-1}(\mathbf{d}_1-\mathbf{A}_1\lambda)\right] \\ P(\mathbf{d}_2|R,\lambda) = \dfrac{1}{\sqrt{(2\pi)^N \det\mathbf{C}_2}}\exp\left[-\dfrac{1}{2}(\mathbf{d}_2-\mathbf{A}_2\lambda)^T\mathbf{C}_2^{-1}(\mathbf{d}_2-\mathbf{A}_2\lambda)\right] \end{cases}$$

$$E = -\frac{1}{2}(\lambda^T\mathbf{Q}\lambda - \lambda^T\mathbf{T} - \mathbf{T}^T\lambda)$$

Quadratic **Linear**

Trick

$$\mathbf{Q} \equiv \mathbf{Q}_1+\mathbf{Q}_2, \mathbf{Q}_i \equiv \mathbf{A}_i^T\mathbf{C}_i^{-1}\mathbf{A}_i$$

$$\mathbf{T} \equiv \mathbf{T}_1+\mathbf{T}_2, \mathbf{T}_i \equiv \mathbf{A}_i^T\mathbf{C}_i^{-1}\mathbf{d}_i$$

$$E = -\frac{1}{2}\left[(\lambda-\mathbf{Q}^{-1}\mathbf{T})^T\mathbf{Q}(\lambda-\mathbf{Q}^{-1}\mathbf{T}) - \mathbf{T}^T\mathbf{Q}^{-1}\mathbf{T}\right]$$

$$\boxed{P(R|\mathbf{d}_1,\mathbf{d}_2) \propto \int \exp(E)P_{\rm prior}(R)d\lambda}$$

Methodology: when $\mathbf{d}_{1,2}$ are uncorrelated

$$P(R|\mathbf{d}_1, \mathbf{d}_2) \propto P_{\mathrm{prior}}(R)$$

$$\cdot \int \exp\left[-\frac{1}{2}\left[\underbrace{(\lambda - \mathbf{Q}^{-1}\mathbf{T})^T\mathbf{Q}(\lambda - \mathbf{Q}^{-1}\mathbf{T})} - \underbrace{\mathbf{T}^T\mathbf{Q}^{-1}\mathbf{T}}\right]\right] d\lambda \quad \text{(2)}$$

The generating function of Gaussian function is

replace $\mathbf{z}$ with $\lambda - \mathbf{Q}^{-1}\mathbf{T}$

$$G(\mathbf{Q}) \equiv \int \exp\left[-\frac{1}{2}\mathbf{z}^T\mathbf{Q}\mathbf{z}\right] \prod_{i=0}^{N_\lambda} dz_i = (2\pi)^{N/2}(\det\mathbf{Q})^{-1/2} \quad \text{(3)}$$

$$\mathbf{Q} \equiv \mathbf{Q}_1 + \mathbf{Q}_2, \mathbf{Q}_i \equiv \mathbf{A}_i^T\mathbf{C}_i^{-1}\mathbf{A}_i$$

$$\mathbf{T} \equiv \mathbf{T}_1 + \mathbf{T}_2, \mathbf{T}_i \equiv \mathbf{A}_i^T\mathbf{C}_i^{-1}\mathbf{d}_i$$

$$\boxed{P(R|\mathbf{d}_1, \mathbf{d}_2) \propto (\det\mathbf{Q})^{-1/2} \exp\left[\frac{1}{2}\mathbf{T}^T\mathbf{Q}^{-1}\mathbf{T}\right] P_{\mathrm{prior}}(R)} \quad \text{(4)}$$

Analytical expression of PDF P(R)

When $\mathbf{d}_{1,2}$ are correlated

$$\mathrm{d}^T = (\mathrm{d}_1, \mathrm{d}_2)$$

$$\Delta \equiv \mathbf{d} - (\mathbf{A}_1\lambda, \mathbf{A}_2\lambda)$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}, \mathbf{C}^{-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

$$\mathbf{Q}' \equiv \mathbf{A}_1^T\mathbf{B}_{11}\mathbf{A}_1 + \mathbf{A}_1^T\mathbf{B}_{12}\mathbf{A}_2 + \mathbf{A}_2^T\mathbf{B}_{21}\mathbf{A}_1 + \mathbf{A}_2^T\mathbf{B}_{22}\mathbf{A}_2$$

$$\mathbf{T}' \equiv \mathbf{A}_1^T\mathbf{B}_{11}\mathbf{d}_1 + \mathbf{A}_1^T\mathbf{B}_{12}\mathbf{d}_2 + \mathbf{A}_2^T\mathbf{B}_{21}\mathbf{d}_1 + \mathbf{A}_2^T\mathbf{B}_{22}\mathbf{d}_2$$

Application: measuring the lensing ratio

Lensing ratio,
depends only on geometry

$$\boxed{w^{g\kappa_{\rm CMB}}(\theta, z_L) = R(z_L, z_\gamma) w^{g\kappa_\gamma}(\theta, z_L, z_\gamma)}$$

Galaxy - CMB convergence cross-correlation

Galaxy - convergence (shear) cross-correlation

Narrow tracer bin approximation

$$C^{g\kappa}(\ell) = \frac{2\pi^2}{\ell^3} \int d\chi n_L(z) W(\chi) d_A(\chi) \boxed{\Delta^2_{\rm gm}\left(k = \frac{\ell + 1/2}{d_A(\chi)}, z\right)}$$

$$R = \frac{C^{g\kappa_{\rm CMB}}}{C^{g\kappa_\gamma}} = \frac{W^{\rm CMB}(\bar{\chi}_L)}{W^{\gamma}(\bar{\chi}_L)}$$

Modeling the lensing ratio

The **key** is the cancellation of the galaxy-matter power spectrum

$$R = \frac{\int_0^\infty (1+z) \frac{d_L d_{L\rm CMB}}{d_{\rm CMB}} n_L(z) \chi^{-2+\beta} dz}{\int_0^\infty (1+z) d_L n_L(z) \chi^{-2+\beta} dz \frac{\int_z^\infty \frac{d_{L\gamma}}{d_\gamma} n_\gamma(z_\gamma) dz_\gamma}{\int_0^\infty n_\gamma(z_\gamma) dz_\gamma}}$$

Application: measuring the lensing ratio

Observable:

$w^{g\gamma_t}$: galaxy - tangential shear cross-correlation

$w^{g\kappa_{\rm CMB}}$: galaxy - CMB lensing cross-correlation

Rescale the correlation functions

$$w \longrightarrow \tilde{w} \equiv w/w_{\rm tem}$$

$$w_{\rm tem}^{g\gamma_t}(\theta) = \int_0^{\ell_{\max}} \frac{\ell d\ell}{2\pi} J_2(\ell\theta) C_{\rm tem}^{g\kappa_{\rm CMB}}(\ell)\,,$$

$$w_{\rm tem}^{g\kappa_{\rm CMB}}(\theta) = \int_0^{\ell_{\max}} \frac{\ell d\ell}{2\pi} J_0(\ell\theta) C_{\rm tem}^{g\kappa_{\rm CMB}}(\ell)\,.$$

Proportionality relation

$$\tilde{w}^{g\kappa_{\rm CMB}} = R\tilde{w}^{g\gamma_t}$$

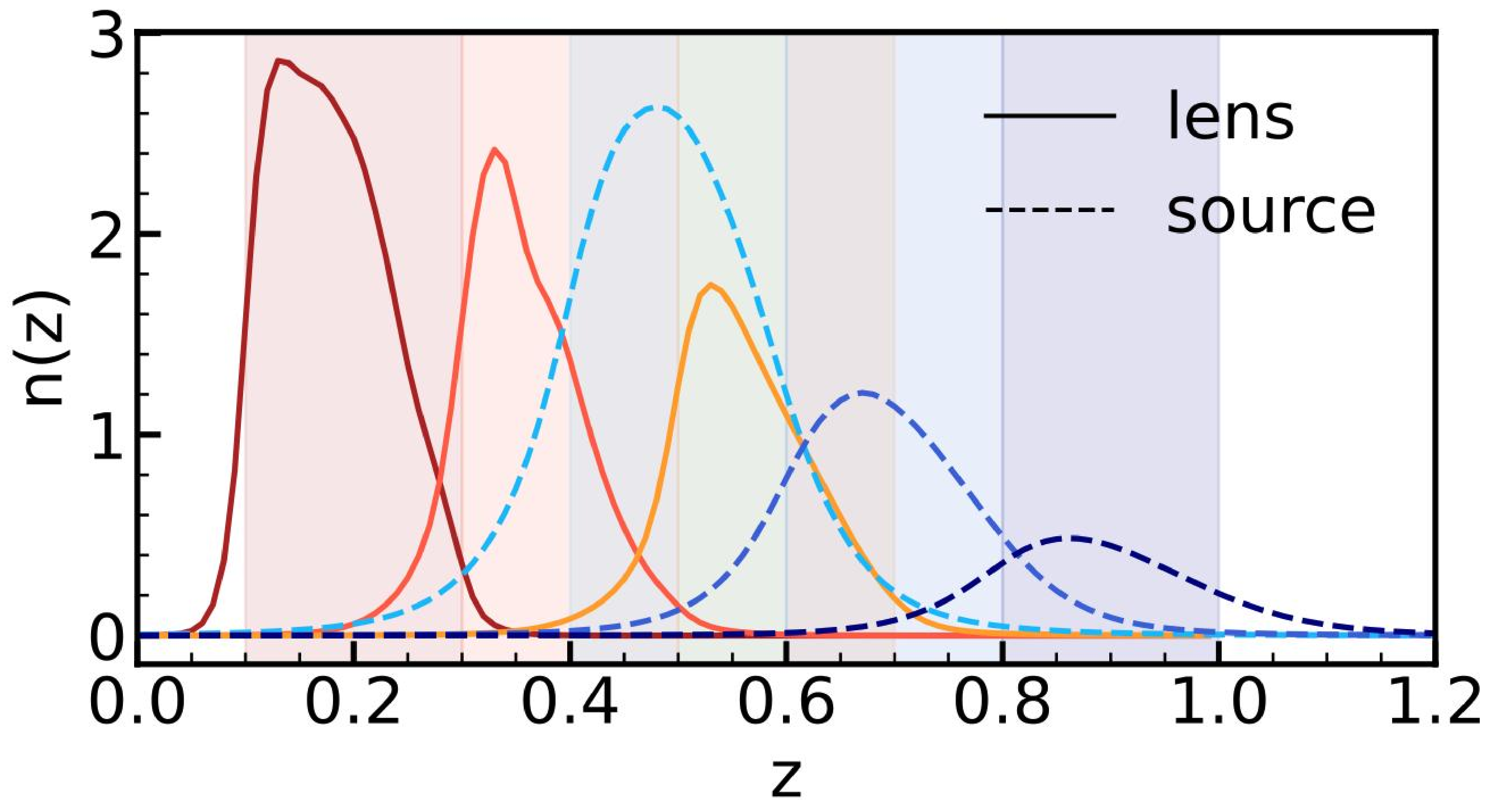
$$\mathbf{d}_1 = \tilde{w}^{g\gamma_t}\,, \quad \mathbf{d}_2 = \tilde{w}^{g\kappa_{\rm CMB}}\,,$$

$$\lambda = \langle \mathbf{d}_1 \rangle\,,\ \mathbf{A}_1 = \mathbf{I}\,,\ \mathbf{A}_2 = R\mathbf{I}\,.$$

Theoretical template $w_{\rm tem}$ converted from the same theoretical $C_{\rm tem}^{g\kappa_{\rm CMB}}$ by FFTLog.
It absorbs the J2/J0 Bessel function dependences.

Data

- **Lens**: DESI DR8 galaxies (Zou+ 2019)
  - 14,000 deg^2
  - [0.1,0.3], [0.3,0.5], [0.5,0.7], 8M in total
- **Source1**: DECaLS DR8 shear catalog
  - 15,000 deg^2 $\gamma_j^{\text{obs}} = (1 + m_j)\gamma_j^{\text{true}} + c_j$
  - [0.4,0.6],[0.6,0.8],[0.8,1.0], 43M in total
- **Source2**: Planck 2018 CMB lensing map
  - Filter to remove modes ℓ>1536, N_{nside}=512



$R_i^\alpha \ (i \leq \alpha)$

$i = 1, 2, 3$: lens

$\alpha = 1, 2, 3$: shear

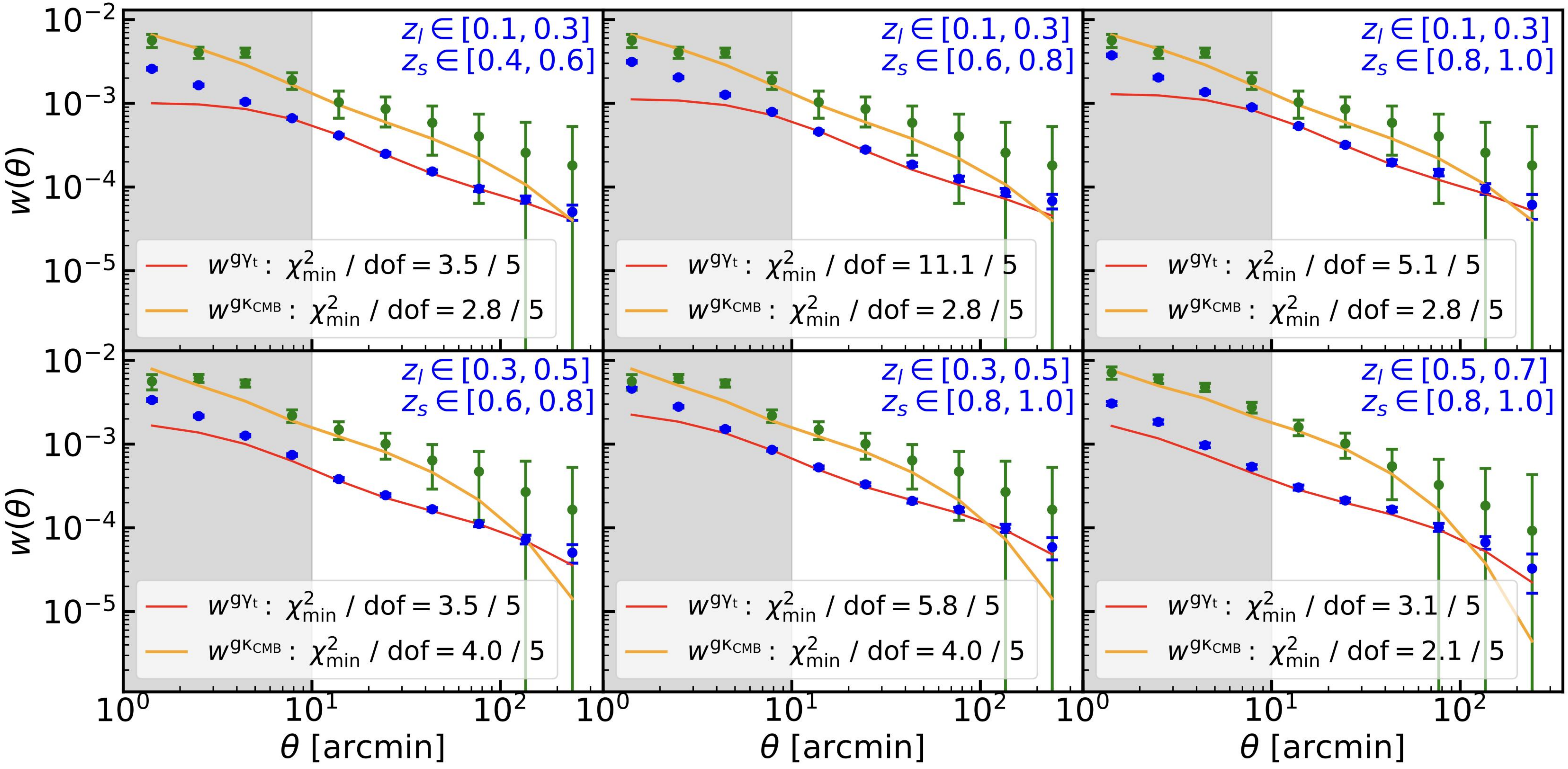
6 ratios:

R_1^1, R_1^2, R_1^3

R_2^2, R_2^3

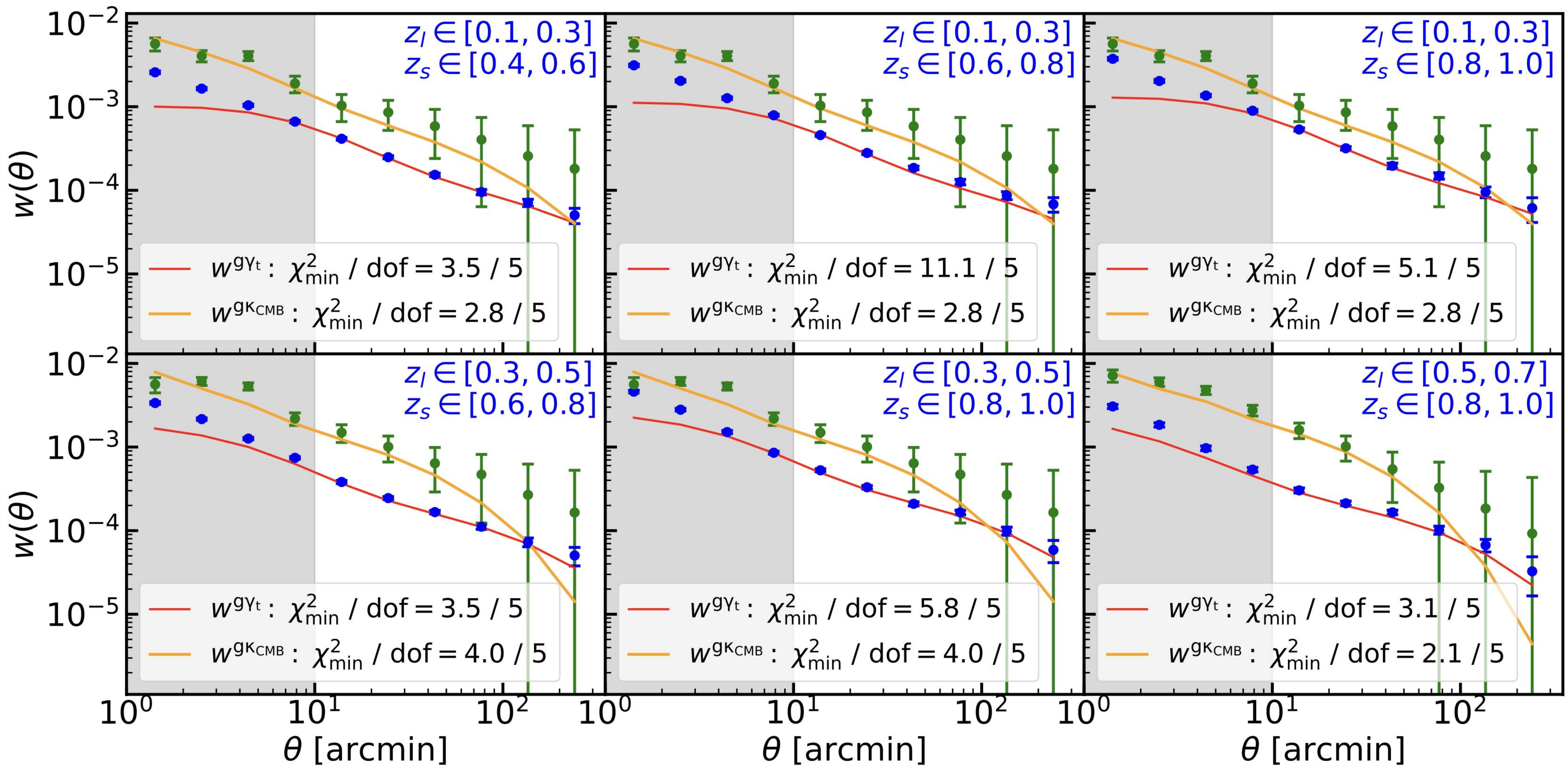
R_3^3

2PCF: gal-shear & gal-CMB lensing



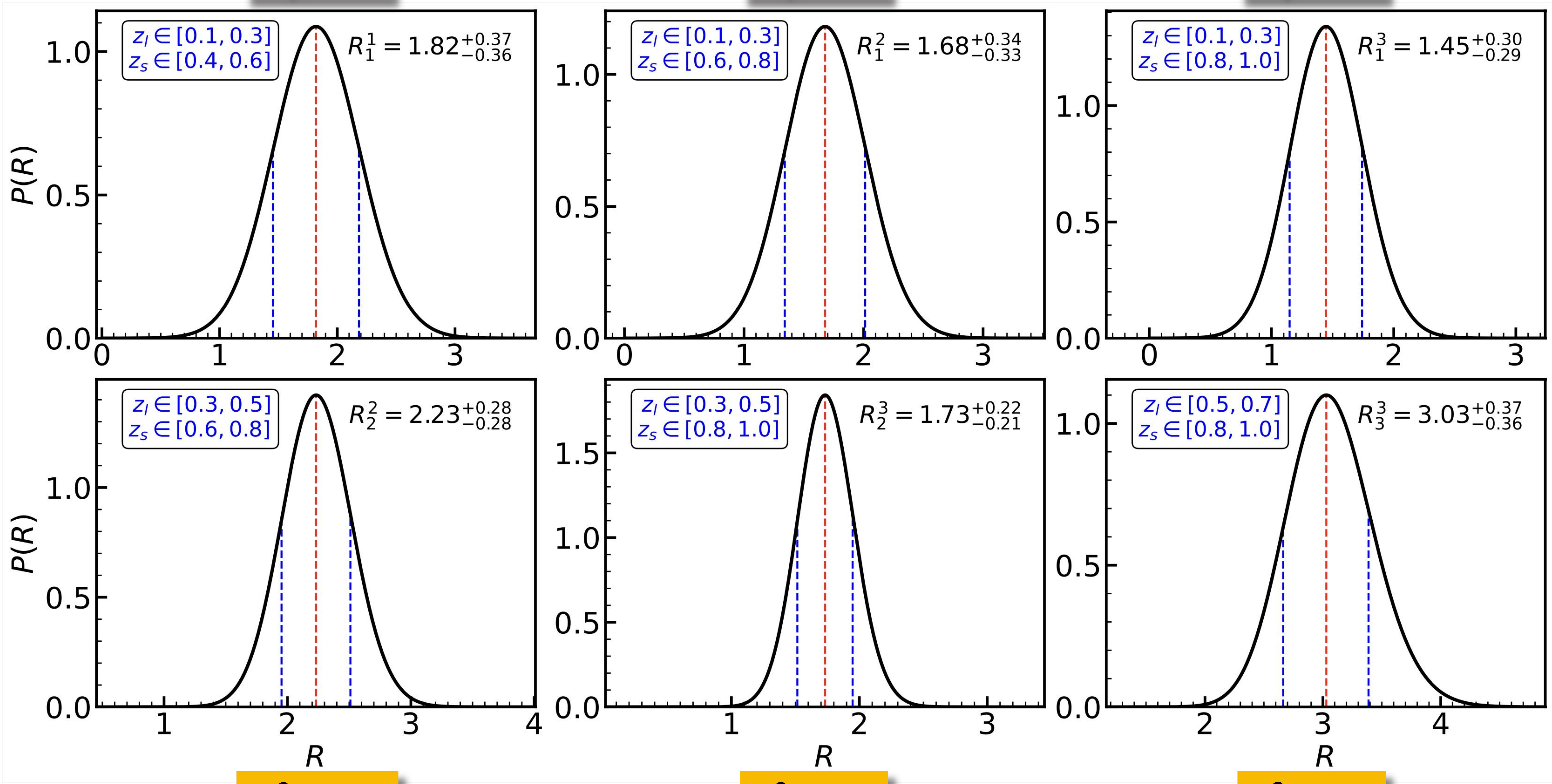
Cross-correlat l-CMB lensing

Each gal-shear pair ~ 28σ
Each gal-CMB pair ~ 7σ



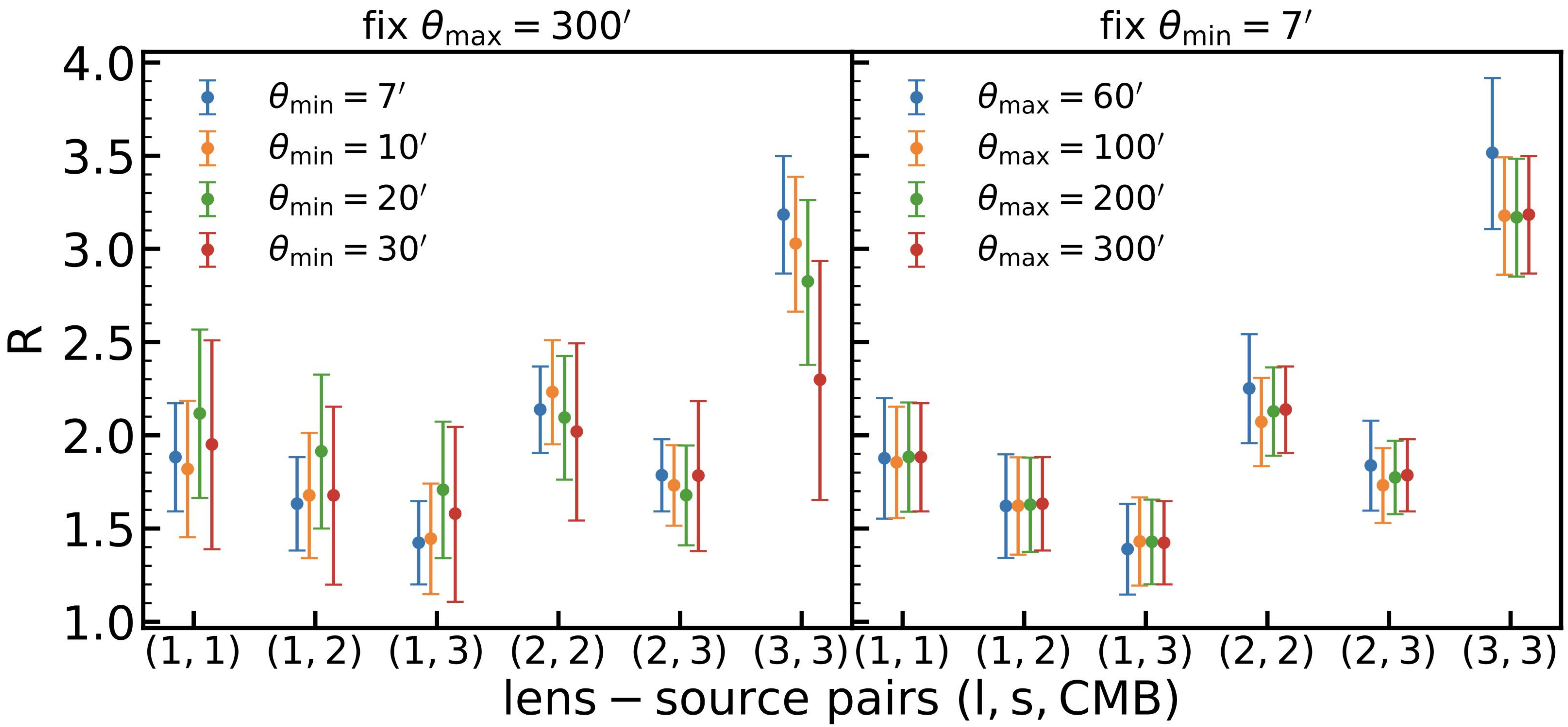
Ratio PDF P(R): nearly Gaussian

$R_1^1, 5.0\sigma$ $R_1^2, 5.0\sigma$ $R_1^3, 4.9\sigma$



$R_2^2, 8.0\sigma$ $R_2^3, 8.0\sigma$ $R_3^3, 8.4\sigma$

Consistency test 1: scale cut $\theta_{\rm min} < \theta < \theta_{\rm max}$



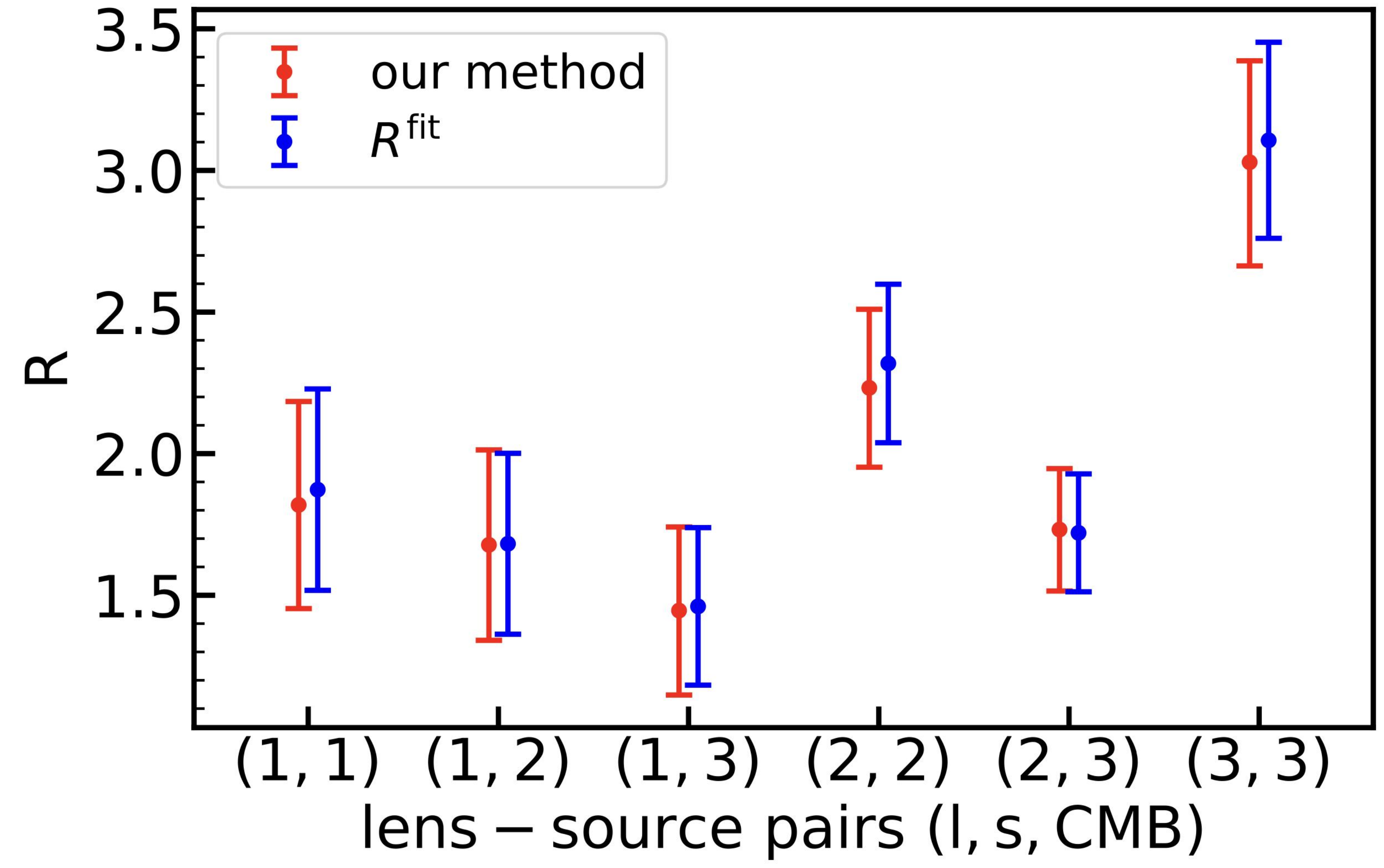
$$10 < \theta < 300 \text{ arcmin}$$

$$A = 1.1 \pm 0.1$$

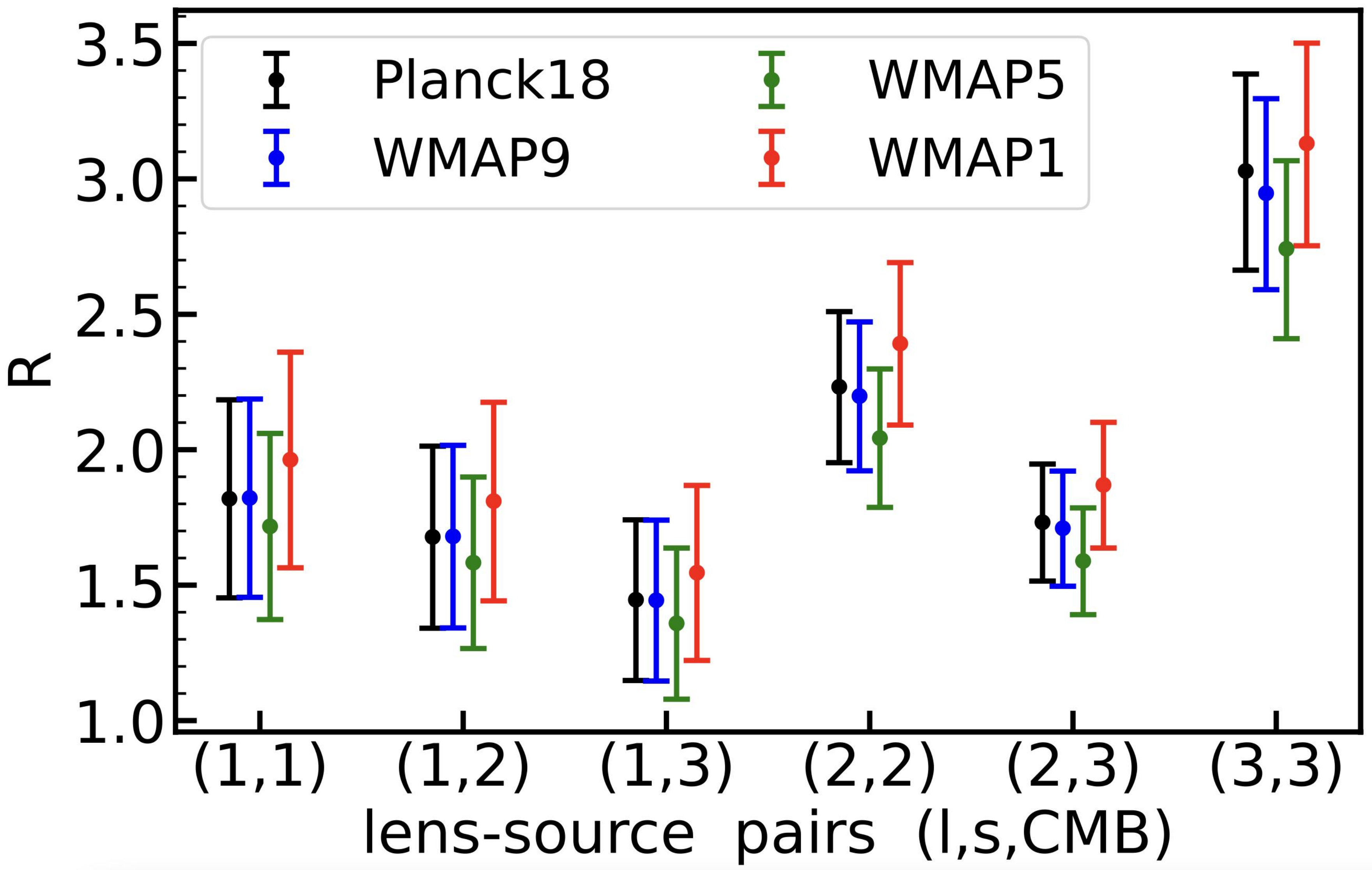
$$R = A R_{\rm Planck},\ A = 1.08 \pm 0.09$$

DES+Planck+SPT Prat et al. (2019)

Consistency test 2: direct model fitting
Consistency test 3: potential cosmological dependence



$$w \longrightarrow \tilde{w} \equiv w/w_{\rm tem}$$



$$w = b w_{\rm tem} \begin{cases} b = \dfrac{\sum_{\theta\theta'} w_{\rm tem}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\rm obs}(\theta')}{\sum_{\theta\theta'} w_{\rm tem}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\rm tem}(\theta')} \\ \sigma_b = \left[\dfrac{1}{\sum_{\theta\theta'} w_{\rm tem}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\rm tem}(\theta')}\right]^{1/2} \end{cases}$$

Parameter	$\Omega_{\rm c}$	$\Omega_{\rm b}$	n_s	H_0	σ_8
WMAP1	0.224	0.0463	0.99	72	0.9
WMAP5	0.206	0.0432	0.961	72.4	0.787
WMAP9	0.235	0.0464	0.9710	69.7	0.820
*Planck*18	0.265	0.04887	0.9649	67.36	0.8111

$$R^{\rm fit} = b^{g\kappa_{\rm CMB}}_{\rm fit} / b^{g\gamma_t}_{\rm fit} \qquad \Delta R^{\rm fit} = \frac{b^{g\kappa_{\rm CMB}}_{\rm fit}}{b^{g\gamma_t}_{\rm fit}} \sqrt{\left(\frac{\sigma_b^{g\kappa_{\rm CMB}}}{b^{g\kappa_{\rm CMB}}_{\rm fit}}\right)^2 + \left(\frac{\sigma_b^{g\gamma_t}}{b^{g\gamma_t}_{\rm fit}}\right)^2}$$

Summary

- Based on the Bayesian analysis and the usual assumption of Gaussian data error distribution, we derive an analytical expression of the PDF $P(R)$. This result enables fast and unbiased R measurement, with minimal statistical errors.

- We take the lensing ratios as an example to demonstrate our method. We take lenses as DESI imaging survey galaxies, and sources as DECaLS cosmic shear and *Planck* CMB lensing. We measure the ratio with S/N ranging from 5 to 8.

Thank you

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Our unbiased method of measuring the ratio of two data sets

- Based on the **Bayesian** analysis $P(R|\mathbf{d}_1,\mathbf{d}_2) \propto \int P(\mathbf{d}_1,\mathbf{d}_2|R,\lambda)P(\lambda)P_{\mathrm{prior}}(R)d\lambda$
- Under the usual assumption of **Gaussian** data $P(\mathbf{d}_i|R,\lambda) = \frac{1}{\sqrt{(2\pi)^N \det \mathbf{C}_i}} \exp\left[-\frac{1}{2}\mathbf{\Delta}_i^T \mathbf{C}_i^{-1}\mathbf{\Delta}_i\right]$
- **Analytical expression of the PDF P(R)** $P(R|\mathbf{d}_1,\mathbf{d}_2) \propto (\det \mathbf{Q})^{-1/2} \exp\left[\frac{1}{2}\mathbf{T}^T\mathbf{Q}^{-1}\mathbf{T}\right] P_{\mathrm{prior}}(R)$
- Enable unbiased R measurement with **minimal statistical errors**
- More generally, it applies to the cases where the proportionality relation holds for the underlying physics/statistics instead of the two data sets directly

Lensing ratio formalism

Lensing convergence

$$\kappa(\boldsymbol{\theta}) = \int_0^{\chi_s} d\chi W^{\kappa}(\chi)\delta(\boldsymbol{\theta},\chi) \quad \textbf{(5)}$$

Kappa kernel

$$\begin{cases} W^{\kappa}(\chi) = \dfrac{3\Omega_{m0}H_0^2}{2}(1+z)\displaystyle\int_{\chi}^{\infty} d\chi' n_s(\chi')\dfrac{d_A(\chi)d_A(\chi,\chi')}{d_A(\chi')} & \textbf{(6)} \\ W^{\kappa_{\rm CMB}}(\chi) = \dfrac{3}{2}\Omega_{m0}H_0^2(1+z)\dfrac{d_A(\chi)d_A(\chi,\chi^*)}{d_A(\chi^*)} & \textbf{(7)} \end{cases}$$

Projected density of tracer galaxies

$$\delta_g(\boldsymbol{\theta}) = \int d\chi W^{l}(\chi)\delta_g^{\rm 3D}(\boldsymbol{\theta},\chi) \quad \textbf{(8)}$$

Galaxy-convergence cross-correlation

$$C^{g_i\kappa}(\ell) = \frac{2\pi^2}{\ell^3}\int d\chi n_g^i(z)W^{\kappa}(\chi)d_A(\chi)\Delta^2_{\rm gm}\left(k=\frac{\ell+1/2}{d_A(\chi)};z\right) \quad \textbf{(9)}$$

$$\Delta^2_{\rm gm} = \frac{k^3}{2\pi^2}P_{\rm gm}\left(k=\frac{\ell+1/2}{d_A(\chi)};z\right) \quad \textbf{(10)}$$

Harmonic space to real space

$$w^{g_i\gamma_t}(\theta) = \int_0^{\infty}\frac{\ell d\ell}{2\pi}J_2(\ell\theta)C^{g_i\kappa_\gamma}(\ell) \quad \textbf{(11)}$$ **2nd order Bessel function**

$$w^{g_i\kappa_{\rm CMB}}(\theta) = \int_0^{\ell_{\max}}\frac{\ell d\ell}{2\pi}J_0(\ell\theta)C^{g_i\kappa_{\rm CMB}}(\ell) \quad \textbf{(12)}$$ **zero order Bessel function**

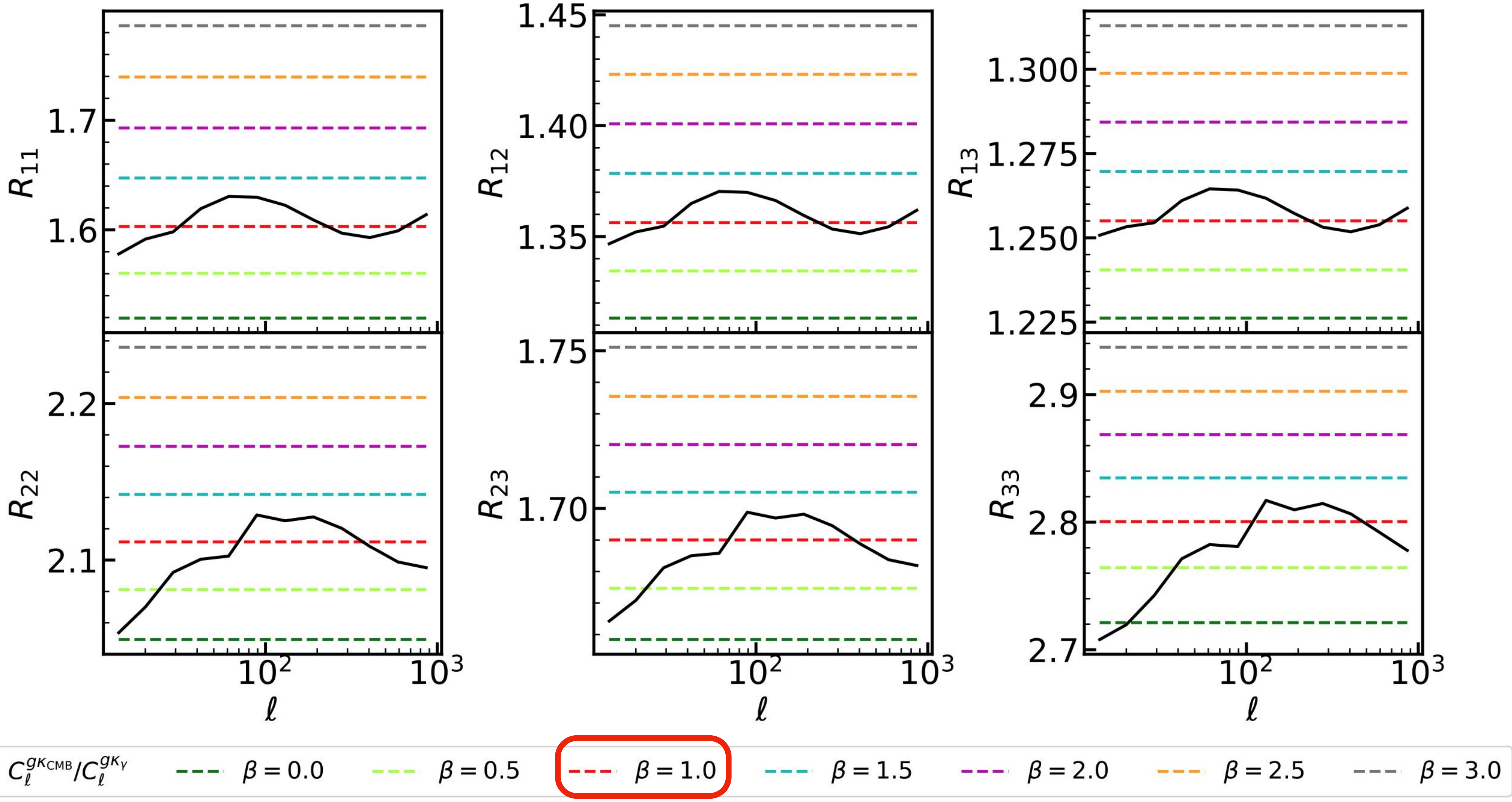
Modeling the lensing ratios

Test narrow tracer bin approximation

To cancel the galaxy-matter power spectrum

$$\frac{\ell^2 C_\ell}{2\pi} = \frac{\pi}{\ell}\int_{z_{l_1}}^{z_{l_2}} \Delta^2_{\rm gm}\left(k=\frac{\ell+1/2}{\chi(z)}, z\right) W_l(\chi) n_l(z)\chi dz$$
$$\doteq \frac{\pi}{\ell}\Delta^2_{\rm gm}\left(k=\frac{\ell+1/2}{\chi(\bar z)}, \bar z\right)\int_{z_{l_1}}^{z_{l_2}} W_l(\chi) n_l(z)\chi dz \quad \textbf{(13)}$$

$$C_\ell = \int_{z_{l_1}}^{z_{l_2}} P\left(k=\frac{\ell+1/2}{\chi(z)}, z\right) W_l(\chi) n_l(z)\chi^{-2} dz$$
$$\doteq P_{\rm gm}\left(k=\frac{\ell+1/2}{\chi(\bar z)}, \bar z\right)\int_{z_{l_1}}^{z_{l_2}} W_l(\chi) n_l(z)\chi^{-2} dz \quad \textbf{(14)}$$



$$R = \frac{C^{g\kappa_{\rm CMB}}}{C^{g\kappa_\gamma}} = \frac{W^{\rm CMB}(\bar\chi_l)}{W^\gamma(\bar\chi_l)} = \frac{\int_0^\infty (1+z)\frac{D_l D_{l\rm CMB}}{D_{\rm CMB}} n_l^i(z)\chi^{-2+\beta} dz}{\int_0^\infty (1+z) D_l n_l^i(z)\chi^{-2+\beta} dz \frac{\int_z^\infty \frac{D_{ls}}{D_s} n_s^j(z_\gamma)dz_\gamma}{\int_0^\infty n_s^j(z_\gamma)dz_\gamma}} \quad \textbf{(15)}$$

Measurement of the 2PCF

Galaxy-shear

$$w^{g\gamma_t} = \frac{\Sigma_{\mathrm{ED}}\, w_j \gamma_j^{+}}{\Sigma_{\mathrm{ER}}(1+m_j) w_j}$$

Σ_{ED} : tangential ellipticity - data pair

Σ_{ER}: tangential ellipticity - random pair

Galaxy-CMB convergence

$$w^{g\kappa_{\mathrm{CMB}}}(\theta) = \frac{\sum_g \omega_g \kappa_g}{\sum_g \omega_g}(\theta)$$

Best-fit bias and associated error

$$b = \frac{\sum_{\theta\theta'} w_{\mathrm{tem}}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\mathrm{obs}}(\theta')}{\sum_{\theta\theta'} w_{\mathrm{tem}}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\mathrm{tem}}(\theta')} \qquad \sigma_b = \left[\frac{1}{\sum_{\theta\theta'} w_{\mathrm{tem}}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\mathrm{tem}}(\theta')}\right]^{1/2}$$

$$\mathrm{S/N} \equiv \sqrt{\chi^2_{\mathrm{null}}} \qquad \chi^2_{\mathrm{null}} = \sum_{\theta\theta'} w_{\mathrm{obs}}(\theta)\mathbf{Cov}^{-1}_{\theta\theta'} w_{\mathrm{obs}}(\theta')$$

$$\mathrm{S/N} \equiv \sqrt{\chi^2_{\mathrm{null}} - \chi^2_{\mathrm{min}}} \qquad \chi^2_{\mathrm{min}} = \sum_{\theta\theta'} (w_{\mathrm{obs}}(\theta) - b^{\mathrm{fit}} w_{\mathrm{tem}}(\theta))\mathbf{C}^{-1}_{\theta\theta'}(w_{\mathrm{obs}}(\theta') - b^{\mathrm{fit}} w_{\mathrm{tem}}(\theta'))$$

Implications: photo-z uncertainty

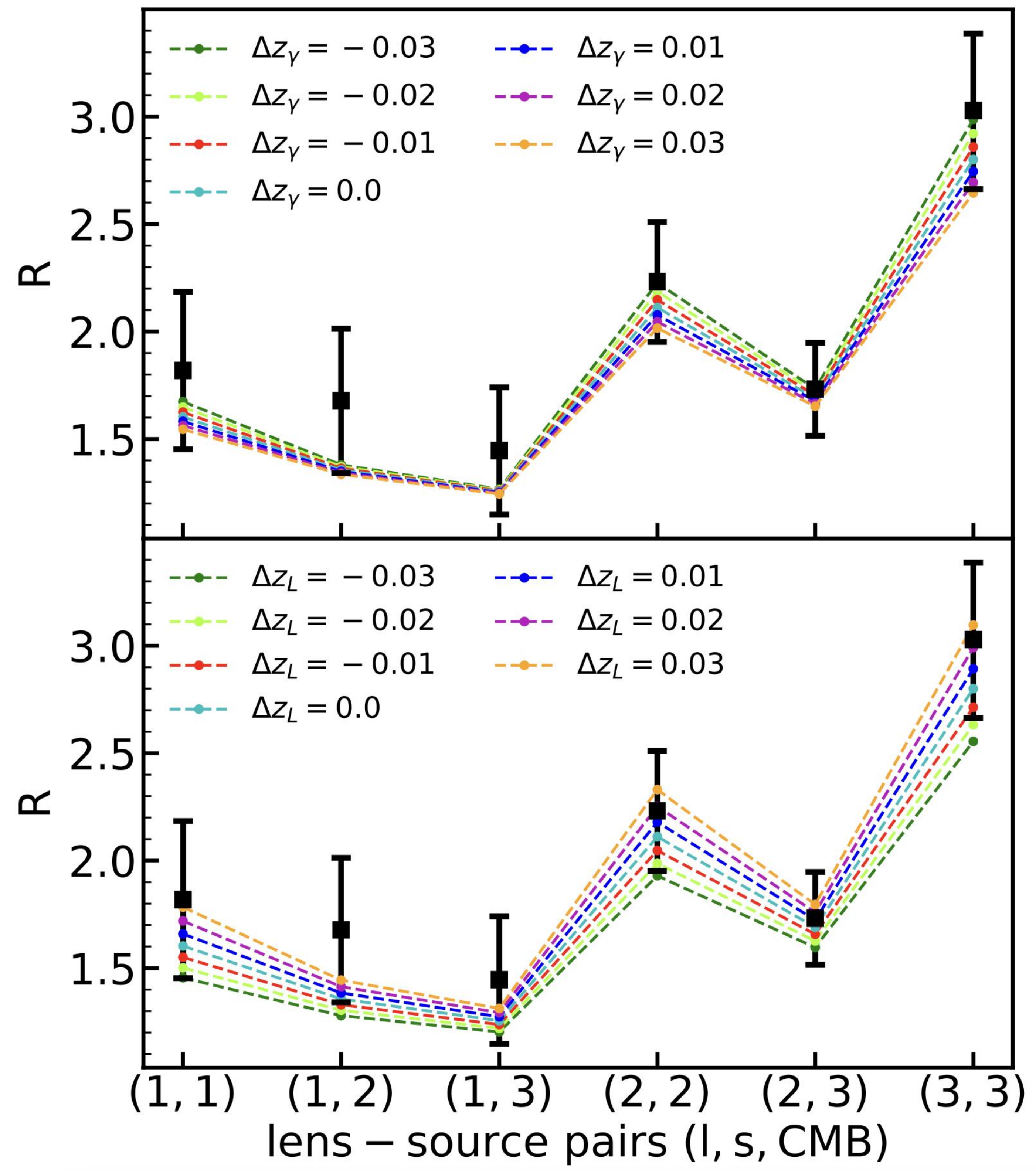
Both for lens and source galaxies

$$p(z|z^P) = \frac{1}{\sqrt{2\pi}\sigma_z(1+z)}\exp\left[-\frac{(z-z^P)^2}{2[\sigma_z(1+z)]^2}\right]$$

σ_z **become twice as much as before**

$C_\ell^{g\kappa_{\mathrm{CMB}}}/C_\ell^{g\kappa_\gamma}$ **become higher less than 4%**

The error of photo-z σ_z is not the major reason of the overestimation of ratios ($\sim$ 16%)



$$n^i(z) \rightarrow n^i(z+\Delta z)$$

Dominate possibility of the overestimation of ratios

Other materials

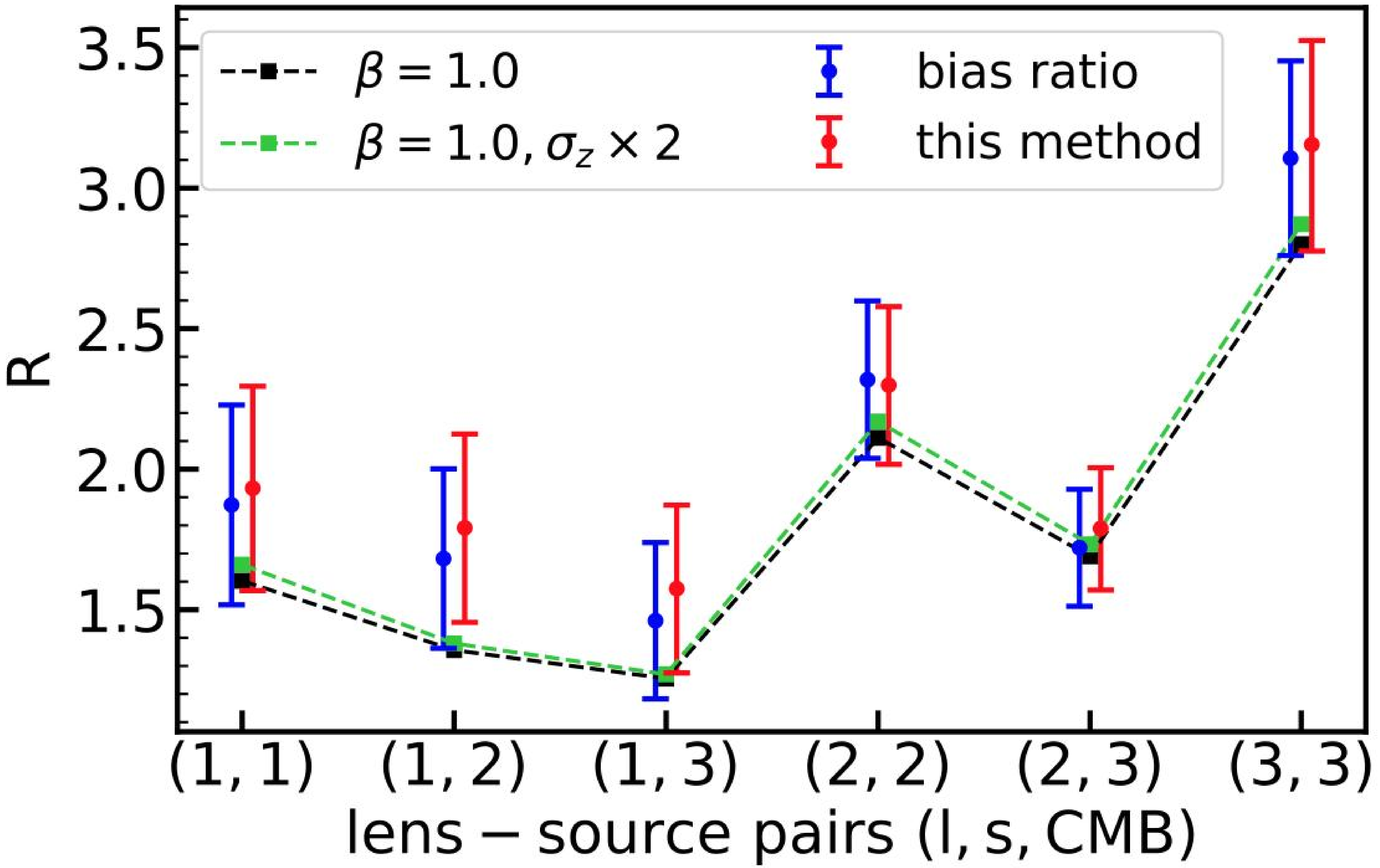
$$R = \frac{C_\ell^{g\kappa_{\rm CMB}}}{C_\ell^{g\kappa_\gamma}} = \frac{w^{g\kappa_{\rm CMB}}(\theta)}{w^{g\kappa_\gamma}(\theta)} = \frac{w^{g\gamma_{\rm CMB}}(\theta)}{w^{g\gamma_t}(\theta)}$$

$$R = \frac{\hat{w}^{g\kappa_{\rm CMB}}/w_{\rm tem}^{g\kappa_{\rm CMB}}}{\hat{w}^{g\gamma_t}/w_{\rm tem}^{g\gamma_{\rm CMB}}}$$

$$\Delta R^{\rm fit} = \frac{b_{\rm fit}^{g\kappa_{\rm CMB}}}{b_{\rm fit}^{g\gamma_t}} \sqrt{\left(\frac{\sigma_b^{g\kappa_{\rm CMB}}}{b_{\rm fit}^{g\kappa_{\rm CMB}}}\right)^2 + \left(\frac{\sigma_b^{g\gamma_t}}{b_{\rm fit}^{g\gamma_t}}\right)^2}$$

$$w^{g_i \boxed{\kappa_{\rm CMB}}}(\theta) = \int_0^{\ell_{\rm max}} \frac{\ell d\ell}{2\pi} J_0(\ell\theta) \boxed{C}^{g_i \kappa_{\rm CMB}}(\ell)$$

$$w^{g_i \gamma_{\rm CMB}}(\theta) = \int_0^{\ell_{\rm max}} \frac{\ell d\ell}{2\pi} J_2(\ell\theta) C^{g_i \kappa_{\rm CMB}}(\ell)\ ,$$



$$\langle \hat{R}_i \rangle = \langle R_{i,\mathrm{Planck}} \rangle (1 + \delta_i)$$

$\delta_i = 0.17, 0.04, 0.08$, for $i = 1, 2, 3$.