Introduction

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General Information

- **Instructor:** Yuqian Zhang
- **Grader:** TBA
- **Office Hour:** Thursday 10am-11am
- **Textbook for Reference:**
  - Convex Optimization, Boyd and Vandenberghe.
  - Convex Analysis. Rockafellar.
  - Introductory Lectures on Convex Optimization: A Basic Course. Nesterov.

Grading Policy

- **Assignments (60%):** approximately biweekly on Fridays, due next Friday. You are encouraged to discuss homework problems with your fellow students, but your final answers should be based on your own understanding. No late submission is acceptable.
- **Project (40%):** The project must be related to optimization, and cannot be your own research with your advisor or your other course projects. Your grade will be evaluated based on the quality of the project and the report.

Grading Policy

Course Project

- Survey
- Simulations
- Theory development
- Application oriented

- Individual project or group of 2.
- Presentation: voluntary, last week of the semester.
- Report: required, due last week of the semester.

Optimization is everywhere

- Electrical engineering:
  signal and image processing, control and robotics, power system
- Computer science:
  machine learning, computer vision
- Finance:
  portfolio selection, asset pricing and arbitrage
- Statistics:
  parametric estimation, Bayesian inference
- Data Science
- ... ...

Mathematical Optimization

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \cdots, m \end{array}$$

- $x = (x_1, \cdots, x_n)$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}$: constraint function

Mathematical Optimization

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \cdots, m \end{array}$$

- feasible x satisfies the constraints
- optimal $x^\star$ has smallest value of f_0 among all feasible x's

Example

Portfolio selection:

- Variables: amounts invested in different assets
- Constraints: budget
- Objective: overall risk or total return

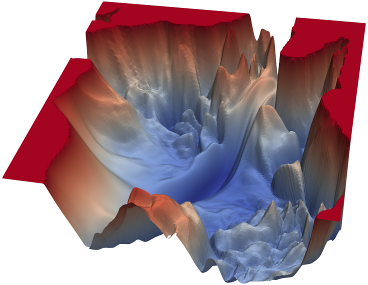
Linear regression

- Variables: coefficients of hyperplane
- Constraints: regularity or sparsity
- Objective: sum of squared residuals

Solving Optimization

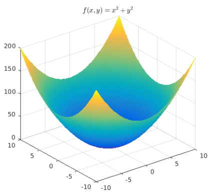
Generally speaking, can be very difficult:

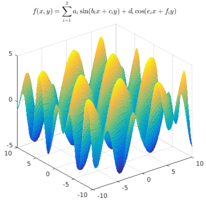
- Many local minimum solutions of the objective
- Constraints can be very complicated
- ... ...



Easy Optimization Problems

- Least square problems
- Linear programming problems
- Convex optimization problems





Least square problems

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $O(n^2 m)$ for generic $A \in \mathbb{R}^{m \times n}$
- easy to recognize

Linear programming problems

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \cdots, m
\end{array}
$$

- no analytical solution
- reliable and efficient algorithms and software

$$x^{(0)} \to x^{(1)} \to x^{(2)} \to x^{(3)} \to \cdots \to x^*$$

- computation time proportional to $O(n^2 m)$
- not as easy to recognize as least square problems

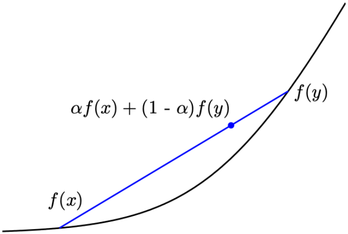
Convex optimization problems

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \le b_i, \quad i = 1, \cdots, m
\end{aligned}
$$

Objective and constraint functions are convex:

For any x, y and any scaler $\alpha \in [0, 1]$,

$f_i\left(\alpha x + (1-\alpha) y\right) \le \alpha f_i(x) + (1-\alpha) f_i(y)$



Convex optimization problems

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq b_i, \quad i = 1, \cdots, m \end{aligned}$$

- no analytical solution
- reliable and efficient algorithms
- computation time proportional to $\max\{n^2 m, n^3, F\}$
- often difficult to recognize

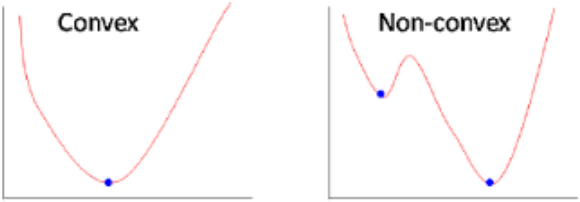
Global optimality in convex function

- Local minimizer

$$f(x^\star) \le f(y), \quad \forall y \in (x^* - \epsilon, x^* + \epsilon)$$

- global minimizer

$$f(x^\star) \le f(y), \quad \forall y \neq x$$



Course Overview

- How to characterize convexity?
- How to recognize whether a problem is convex?
- How to solve a convex problem?
- How to apply and implement the theory and algorithm to address real-world applications?

Brief history

- Fundamental theory: 1900 - 1970
- Algorithms
  - 1947: simplex algorithm for linear programming
  - 1960s: early interior point methods
  - 1970s: ellipsoid method and other sub-gradient method
  - 1980s: polynomial-time interior point methods for linear programming
  - late 1980s-now: polynomial-time interior point methods for nonlinear convex optimization
- Applications
  - before 1990: mostly in operation research; few in engineering
  - after 1990: more applications in engineering; new problem classes

Selective Topics

Nonconvex Optimization Problems

- eigenvalue and eigenspace
- low rank matrix recovery
- dictionary learning
- clustering
- deep neural network
- … …

Next Lecture

Convex Sets