Introduction

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Overview

1. Convex functions

2. Basic properties and examples

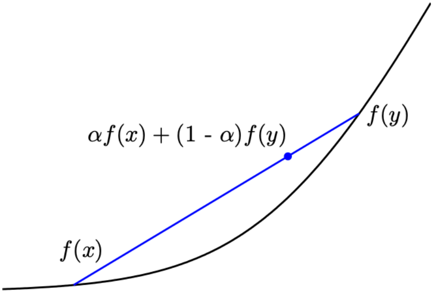
3. Operations preserving convexity

Convex functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is convex, and

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$$

for any $x, y \in \mathrm{dom}(f)$ and any scaler $\alpha \in [0, 1]$.



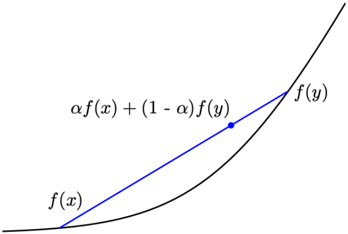
Convex functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is convex, and

$$f\left(\alpha x + (1-\alpha)y\right) \leq \alpha f(x) + (1-\alpha) f(y)$$

for any $x, y \in \mathrm{dom}(f)$ and any scaler $\alpha \in [0,1]$.

- f is strongly convex, if above holds with strict inequality with $x \neq y$ and $\alpha \in (0,1)$
- f is concave if $-f$ is convex
- f is strongly concave if $-f$ is strongly convex



Convex functions – examples

- Linear/affine functions

$$f(x) = b^T x + c$$

Convex functions – examples

- Linear/affine functions

$$f(x) = b^T x + c$$

- Quadratic functions

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

for $A \succeq 0$. For regression

$$\frac{1}{2} \|Xw - y\|_2^2 = \frac{1}{2} w^T X^T X w - y^T X w + \frac{1}{2} y^T y$$

Convex functions – examples

- Norms

$$\begin{aligned}\|\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\|_2 &\le \|\alpha \boldsymbol{x}\|_2 + \|(1-\alpha)\boldsymbol{y}\|_2 \\ &= \alpha \|\boldsymbol{x}\|_2 + (1-\alpha)\|\boldsymbol{y}\|_2\end{aligned}$$

Convex functions – examples

- Norms

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- Composition with an affine function $f(Ax+b)$

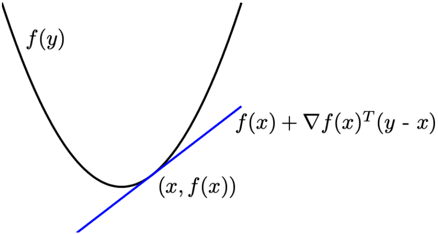
$$\begin{aligned}f(A(\alpha x + (1-\alpha)y) + b) &= f(\alpha(Ax+b) + (1-\alpha)(Ay+b)) \\ &\le \alpha f(Ax+b) + (1-\alpha) f(Ay+b)\end{aligned}$$

First order convexity conditions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then f is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all $x, y \in \mathrm{dom}(f)$.

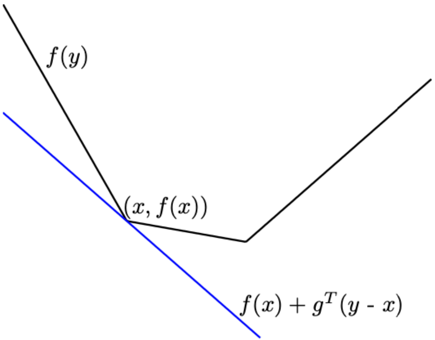


First order convexity conditions

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if there exist non-empty *subdifferential set*

$$\partial f(x) = \left\{ g : f(y) \geq f(x) + \nabla g(x)^T (y - x)\ \forall y \right\}$$

everywhere.

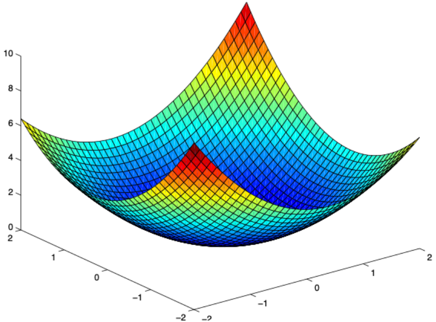


Second order convexity conditions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable. Then f is convex if and only if

$$\nabla^2 f(x) \succeq 0$$

for all $x \in \mathrm{dom}(f)$.



Examples

- Affine function: $f(x) = Ax + b$

$$\nabla f(x) = A, \quad \nabla^2 f(x) = 0$$

  always convex;
- Least square objective: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^T\left(Ax - b\right), \quad \nabla^2 f(x) = 2A^T A$$

  always convex.

Examples

- Quadratic function: $f(x) = \frac{1}{2}x^T Px + q^T x + r$

$$\nabla f(x) = Px + q, \quad \nabla^2 f(x) = P$$

convex if $P \succeq 0$;

- Quadratic-over-linear function: $f(x, y) = x^2/y$

$$\nabla^2 f(x) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T$$

convex for $y > 0$.

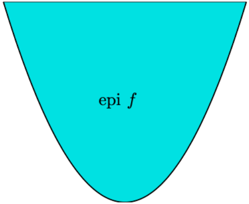
Epigraph and sublevel set

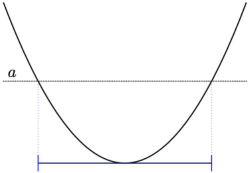
- The epigraph of a function f is the set of points

$$\text{epi}(f) = \{(x, t) : f(x) \leq t\}$$

epi(f) is convex if and only if f is convex

- sublevel set $\{x : f(x) \leq a\}$ are convex for convex function f, converse is false





Jensens inequality

- **Basic:** if f is convex, then for $0 \leq \alpha \leq 1$

$$f\left(\alpha x + (1-\alpha) y\right) \leq \alpha f(x) + (1-\alpha) f(y)$$

- **Extension:** if f is convex, then

$$f\left(\mathbb{E} z\right) \leq \mathbb{E} f(z)$$

- Above are equal when

$$\mathrm{prob}(z = x) = \alpha, \quad \mathrm{prob}(z = y) = 1 - \alpha$$

Convex function

Practical methods for establishing convexity of a function f

- verify definition
- for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
- show that f is obtained from simple convex function by operations that preserve convexity
    - nonnegative weighted sum
    - composition with affine functions
    - point-wise maximum and supremum
    - minimization
    - perspective functions

Nonnegative weighted sum

- nonnegative multiple
  αf is convex if f is convex, $\alpha > 0$
- sum
  $f_1 + f_2$ convex if f_1 and f_2 are convex
- composition with affine $f(Ax + b)$

$$\begin{aligned} f(A(\alpha x + (1-\alpha)y) + b) &= f(\alpha(Ax+b) + (1-\alpha)(Ay+b)) \\ &\leq \alpha f(Ax+b) + (1-\alpha) f(Ay+b) \end{aligned}$$

Pointwise maximum

If $f_1, \cdots, f_m$ are convex, then

$$f(x) = \max\{f_1(x), \cdots, f_m(x)\}$$

is convex.

Proof:

- f_i is convex, $epi(f_i)$ is convex
- $epi(f) = epi(f_1) \cap epi(f_2)$ is convex
- $f = \max(f_1, \cdots, f_m)$ is convex

Pointwise supremum

If $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex.

Examples:

- distance to farthest point in a set S

$$f(X) = \sup_{y \in S} \|x - y\|_2$$

- maximum eigenvalue of a symmetric matrix

$$f(X) = \lambda_{\max}(X) = \sup_{\|v\|_2 = 1} v^T X v$$

Composition

If $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$, then $f(x) = h(g(x))$ is convex if

- g convex, h convex and nondecreasing
- g concave, h convex and nonincreasing

$$\nabla f(x) = \nabla h(g(x)) \nabla g$$
$$\nabla^2 f = \nabla g^T \nabla^2 h \nabla g + \nabla h \nabla^2 g$$

Minimization

If $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex.

- Proof:

$$\begin{aligned} epi(g) &= \left\{ (x, t) \mid \inf_{y \in C} f(x, y) \leq t \right\} \\ &= \{(x, t) \mid (x, y, t) \in epi(f) \text{ for some } y \in C\} \end{aligned}$$

  intersect $epi(f)$ with $\mathbb{R}^n \times C \times \mathbb{R}$ then project to $\mathbb{R}^n \times \varnothing \times \mathbb{R}$
- Example:
  distance to set C $\mathrm{dist}_C(x) = \inf_{y \in C} \|x - y\|_2$ is convex if C is convex

Next Lecture

Convex Problems