Convex problems

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Overview

1. Convex optimization
2. Basic properties
3. Linear optimization
4. Quadratic optimization

Optimization in standard form

$$
\begin{array}{rll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\
 & h_j(x) = 0, & j = 1, \cdots, p
\end{array}
$$

- $x \in \mathbb{R}^n$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}$: inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}$: equality constraint functions

Optimization in standard form

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_j(x) = 0, & j = 1, \cdots, p \end{array}$$

optimal value

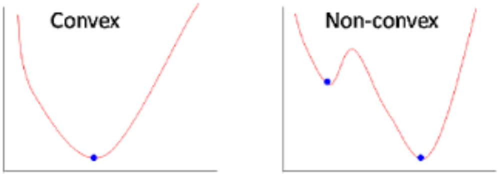
$$p^\star = \inf \{ f_0(x) \mid f_i(x) \leq 0, h_j(x) = 0 \}$$

- $p^\star = \infty$ problem is infeasible
- $p^\star = -\infty$: problem is unbounded below

Optimal and locally optimal points

- x is feasible if $x \in \mathrm{dom}(f_{)})$ and it satisfies the constraints
- a feasible x is optimal if $f_0(x) = p^\star$; X_{opt} is the set of optimal points
- x is locally optimal if there is an $\epsilon > 0$ such that x is optimal for

$$
\begin{array}{rll}
\text{minimize} & f_0(z) & \\
\text{subject to} & f_i(z) \leq 0, & i = 1, \cdots, m \\
& h_j(z) = 0, & j = 1, \cdots, p \\
& \|z - x\|_2 \leq \epsilon &
\end{array}
$$



Examples

- $f_0(x) = 1/x$, $\mathrm{dom}(f_0) = \mathbb{R}_{++}$

$$p^\star = 0, \quad \text{no optimal } x$$

- $f_0(x) = x \log x$, $\mathrm{dom}(f_0) = \mathbb{R}_{++}$

$$p^\star = -1/e, \quad \text{optimal } x = 1/e$$

- $f_0(x) = x^3 - 3x$

$$p^\star = -\infty, \quad \text{local optimal } x = 1$$

Implicit constraints

Above standard form optimization problem has an implicit constraints

$$x \in \mathcal{D} \doteq \cap \operatorname{dom}(f_i) \cap dom(h_j)$$

- $\mathcal{D}$: domain of the problem
- f_i and h_j are the explicit constraints
- a problem is unconstrained if it has no explicit constraints

$$\text{minimize} \quad f_0(x) = -\sum \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraint $a_i^T x \leq b_i$

Feasibility problem

$$\begin{array}{rll} \text{find} & x & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_i(x) = 0, & j = 1, \cdots, p \end{array}$$

can be considered a special case with $f_0(x) = 0$

$$\begin{array}{rll} \text{minimize} & 0 & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_i(x) = 0, & j = 1, \cdots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible, any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

Standard form convex optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \cdots, m \\ & a_j^T x = b_j, \quad j = 1, \cdots, p \end{aligned}$$

- f_0 and f_i are convex,; equality constraints are convex
- feasible set of a convex optimization problem is convex
- equivalent form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \cdots, m \\ & Ax = b \end{aligned}$$

Example

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

- f_0 is convex; f_1 is not convex, h_1 is not affine
- feasible set $\{(x_1, x_2) \mid x_1 = x_2 \leq 0\}$ is convex
- equivalent convex form

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local is global

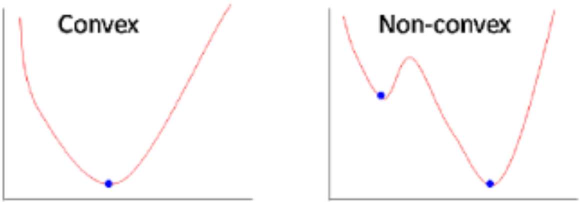
Any local optimal point of a convex problem is globally optimal.

- local minimizer

$$f(x^\star) \le f(y), \quad \forall y \in (x^* - \epsilon, x^* + \epsilon)$$

- global minimizer

$$f(x^\star) \le f(y), \quad \forall y \neq x$$



Local is global — proof

- let x be locally optimal, then there exists ϵ such that $f_0(z) \geq f_0(x)$ holds for any $\|x - z\|_2 \leq \epsilon$
- suppose there exists a feasible y with $f_0(y) < f_0(x)$
- consider $z = \alpha y + (1 - \alpha)x$ with $\alpha = \frac{\epsilon}{2\|y-x\|_2}$
- as $\|y - x\|_2 > \epsilon$, then $0 < \alpha < 1/2 < 1$
- z is convex combination of two feasible points, hence feasible
- then $\|z - x\|_2 = \alpha \|y - x\|_2 = \epsilon/2$ and

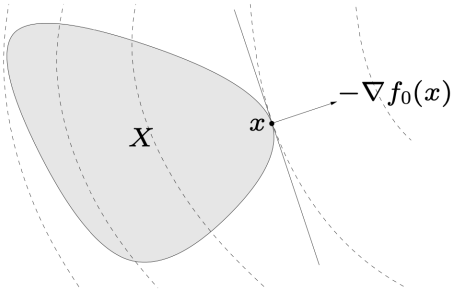
$$f_0(z) \leq \alpha f_0(y) + (1 - \alpha) f_0(0) < f_0(x)$$

- contradicts our assumption that x is locally optimal

Optimality condition for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0, \quad \forall \text{ feasible } y$$

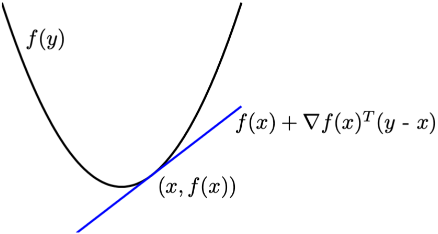


First order convexity conditions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then f is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all $x, y \in \mathrm{dom}(f)$.



Optimality condition for differentiable f_0 — Proof

We need to prove

$$f_0(x) \leq f_0(y) \quad \forall y \in X \iff \nabla f_0(x)^T(y-x) \geq 0 \quad \forall y \in X$$

- $\Longrightarrow$: Suppose x is optimal, but there exists y with $\nabla f_0(x)^T(y-x) < 0$. Consider point $z = \alpha y + (1-\alpha)x$ with $\alpha \in \{0, 1\}$ is also feasible. As

$$\lim_{\alpha \to 0} \frac{f_0(z) - f_0(x)}{t} = \nabla f_0(x)^T(y-x) < 0$$

  then $f_0(z) < f_0(x)$ for sufficiently small α, hence a contradiction.
- $\Longleftarrow$: Suppose $\nabla f_0(x)^T(y-x) \geq 0$ for all y, since f_0 is convex

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x) \geq f_0(x)$$

Examples

- Unconstrained problem: x is optimal if and only if

$$x \in \mathrm{dom}(f_0), \quad \nabla f_0(x) = 0$$

- equality constrained problem

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad \nabla Ax = b$$

  x is optimal if and only if there exists a v such that

$$x \in \mathrm{dom}(f_0), \quad Ax = b, \quad \nabla f_0(x) + Av = 0$$

Linear programming (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron
- number of solutions: zero, one, infinity

Linear programming (LP) — examples

- ℓ_∞ norm

$$\begin{aligned} & \min_x \quad \|Ax+b\|_\infty \\ \Longrightarrow \quad & \min_x \quad \max_i \left|a_i^T x + b_i\right| \\ \Longrightarrow \quad & \min_{x,t} \quad t \quad \text{s.t.} \quad -t \le a_i^T x + b_i \le t \end{aligned}$$

- ℓ_1 norm

$$\begin{aligned} & \min_x \quad \|Ax+b\|_1 \\ \Longrightarrow \quad & \min_x \quad \sum_i \left|a_i^T x + b_i\right| \\ \Longrightarrow \quad & \min_{x,t_i} \quad \sum_i t_i \quad \text{s.t.} \quad -t_i \le a_i^T x + b_i \le t_i \end{aligned}$$

Quadratic programming (QP)

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}x^T Px + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}^n_+$, objective is convex quadratic
- minimize a convex quadratic function over a polyhedron

Quadratic programming (QP) — examples

- Least squares

$$
\begin{aligned}
& \min_x \quad \|Ax - b\|_2 \\
\Longrightarrow \quad & \min_x \quad \|Ax - b\|_2^2 \\
\Longrightarrow \quad & \min_x \quad x^T A^T A x - 2 b^T A x + b^T b
\end{aligned}
$$

- Distance between two polyhedra

$$
\begin{aligned}
& dist\,(P_1, P_2) = \inf \{ \|x_1 - x_2\|_2 \mid x_1 \in P_1, x_2 \in P_2 \} \\
\Longrightarrow \quad & \min_x \quad \|x_1 - x_2\|_2^2 \\
& \text{s.t.} \quad A_1 x_1 \leq b_1, A_2 x_2 \leq b_2
\end{aligned}
$$

Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & \frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0 \\ & Ax = b \end{array}$$

- $P_i \in \boldsymbol{S}_+^n$, objective and constraints are convex quadratic

Second-order cone programming (SOCP)

$$
\begin{array}{ll}
\text{minimize} & h^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i \\
 & Fx = g
\end{array}
$$

with $A \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$

- inequalities are second-order cone (SOC) constraints:

$$\left(A_i x + b_i, c_i^T x + d_i\right) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

- if $n_i = 0$, reduces to an LP;
- if $c_i = 0$, reduces to a QCQP

Next Lecture

Convex problems