Convex problems

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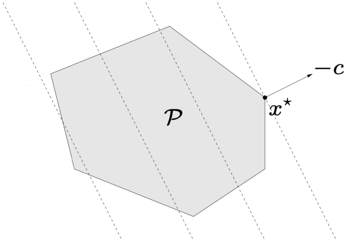
Overview

1. Linear optimization
2. Quadratic optimization
3. Geometric programming
4. Semidefinite programming

Linear programming (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \le h \\ & Ax = b \end{array}$$

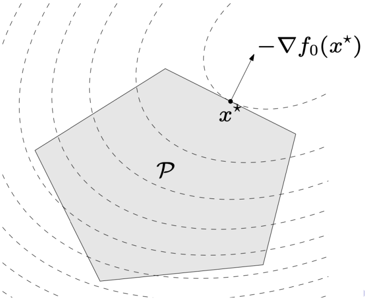
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron
- number of solutions: zero, one, infinity



Quadratic programming (QP)

$$\begin{array}{ll} \text{minimize} & x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}_+^n$, objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Quadratic programming (QP) — examples

- Least squares

$$
\begin{aligned}
& \min_x \quad \|Ax - b\|_2 \\
\Longrightarrow \quad & \min_x \quad \|Ax - b\|_2^2 \\
\Longrightarrow \quad & \min_x \quad x^T A^T A x - 2b^T A x + b^T b
\end{aligned}
$$

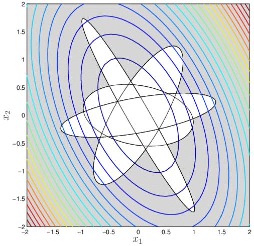
- Distance between two polyhedra

$$
\begin{aligned}
& dist\,(P_1, P_2) = \inf\left\{\|x_1 - x_2\|_2 \mid x_1 \in P_1, x_2 \in P_2\right\} \\
\Longrightarrow \quad & \min_x \quad \|x_1 - x_2\|_2^2 \\
& \text{s.t.} \quad A_1 x_1 \leq b_1, A_2 x_2 \leq b_2
\end{aligned}
$$

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & x^T P_i x + q_i^T x + r_i \le 0 \\
& Ax = b
\end{array}
$$

- $P_0, P_i \in \boldsymbol{S}_+^n$: objective and constraints are convex quadratic
- $P_i \in \boldsymbol{S}_{++}^n$: feasible region is intersection of m ellipsoids and an affine set.



Second-order cone programming (SOCP)

$$\begin{array}{ll} \text{minimize} & h^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i \\ & Fx = g \end{array}$$

with $A \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$

- inequalities are second-order cone (SOC) constraints:

$$\left(A_i x + b_i, c_i^T x + d_i\right) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

- if $n_i = 0$, reduces to an LP;
- if $c_i = 0$, reduces to a QCQP

Geometric programming (QP)

- monomial function

$$f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \mathbf{dom}\, f = \mathbb{R}_{++}^n$$

with $c > 0$; exponent a_i can be any real number

- posynomial function

$$f(x) = \sum_{k=1}^{k} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \mathbf{dom}\, f = \mathbb{R}_{++}^n$$

Geometric programming (QP)

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 1 \\ & h_j(x) = 1 \end{aligned}$$

- f_0, f_i posynomial
- h_j monomial
- some circuit problems can be cast as QP.

Geometric programming (QP) – example

Original optimization formualtion

$$\begin{aligned} \underset{x,y,z>0}{\text{minimize}} \quad & \frac{x}{y} \\ \text{subject to} \quad & 2yz \le x \le 3 \\ & x^2 + 3\frac{y}{z} \le \sqrt{y} \\ & \frac{x}{y} = z^2 \end{aligned}$$

Equivalent QP formulation

$$\begin{aligned} \underset{x,y,z}{\text{minimize}} \quad & xy^{-1} \\ \text{subject to} \quad & \tfrac{1}{3}x \le 1 \\ & 2x^{-1}yz \le 1 \\ & x^2y^{-1/2} + 2y^{1/2}z^{-1} \le 1 \\ & xy^{-1}z^{-2} \le 1 \end{aligned}$$

Geometric programming (QP)

Theorem

GP can be cast as a convex optimization problem,

Change variables $y_i = \log x_i$ and take logarithm for cost / constraints

- monomial function $h(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ transforms to

$$\log h\left(e^{y_1}, e^{y_2}, \cdots, e^{y_n}\right) = a^T y + b, \quad b = \log c$$

- posynomial function $f(x) = \sum_{k=1}^{k} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f\left(e^{y_1}, e^{y_2}, \cdots, e^{y_n}\right) = \log\left(\sum_k e^{a_k^T y + b_k}\right), \quad b_k = \log c_k$$

Geometric programming (QP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1 \\ & h_j(x) = 1 \end{array}$$

- f_0, f_i posynomial
- h_j monomial

Equivalent convex problem

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \leq 0 \\ & Gy + d = 0 \end{array}$$

Semidefinite programming (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with $F_i, G \in \boldsymbol{S}^n$

- linear objective
- linear matrix inequalities / equalities constraints

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ with given $A_i \in \boldsymbol{S}^k$. Above is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI_k \end{array}$$

- I_k: identity matrix of size $k \times k$
- follows from

$$\lambda_{\max}(A) \leq t \iff A \preceq tI_k$$

Eigenvalue minimization

Schur complement

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff C \succ 0, A - BC^{-1}B^T \succ 0$$

$$\begin{aligned} \lambda_{\max}(A) \leq t &\iff A \preceq tI_k \\ &\iff A^T A \preceq t^2 I_k \\ &\iff tI_k - \tfrac{1}{t} A^T I_k A \succeq 0 \end{aligned}$$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ with given $A_i \in \boldsymbol{S}^k$

Equivalent SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI_k & A(x)^T \\ A(x) & tI_k \end{bmatrix} \succeq 0 \end{array}$$

Schur complement

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff C \succ 0, A - BC^{-1}B^T \succ 0$$

Equivalent SDP

$$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP$$

LP and equivalent SDP

- LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \operatorname{diag}(Ax - b) \preceq 0 \end{array}$$

Equivalent SDP

$$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP$$

QCQP and equivalent SDP

- QCQP

$$\begin{array}{ll} \text{minimize} & x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + q_i^T x + r_i \preceq 0, \quad i = 1, \cdots, m \end{array}$$

- equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} -r_i - q_i^T x & x^T P_i^{1/2} \\ P_i^{1/2} x & I_k \end{bmatrix} \succeq 0, \quad i = 0, 1, \cdots, m \end{array}$$

Equivalent SDP

$$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP$$

SOCP and equivalent SDP

- SOCP

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \cdots, m \end{array}$$

- equivalent SDP

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) Id & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \cdots, m \end{array}$$

Next Lecture

Duality