Convex Sets

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Overview

1. Convex Sets
2. Operations preserve convexity
3. Separating and supporting hyperplane

Convex optimization problems

Objective and constraint functions are convex

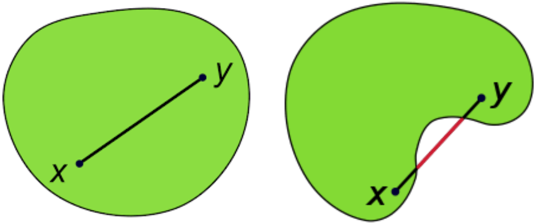
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \cdots, m \end{array}$$

- $x = (x_1, \cdots, x_n)$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}$: constraint function

Convex sets

A set $C \in \mathbb{R}^n$ is convex if for any $x, y \in C$ and any scaler $\alpha \in [0, 1]$

$$\alpha x + (1 - \alpha) y \in C$$



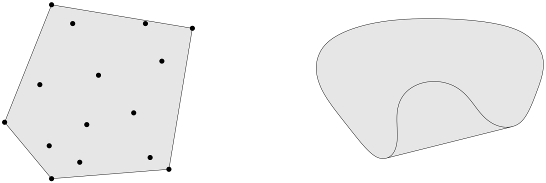
Convex combination and convex hull

- **Convex combination** of $x_1, \cdots, x_k$: any point x of the form

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

with $\alpha_1 + \cdots + \alpha_k = 1$ and $\alpha_i \geq 0$.

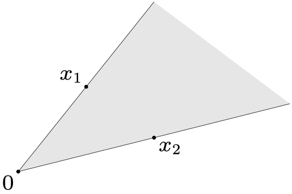
- **Convex hull** conv S: set of all convex combinations of points in S



Convex cone

- **conic (nonnegative) combination** of x_1 and x_2: any point x of the form

$$x = \alpha_1 x_1 + \alpha_2 x_2, \quad \alpha_i \geq 0$$



- **Convex cone** conv S: set that contains all conic combinations of points in the set

Definition

Given any elements $x_1, \cdots, x_k$, then combination $\alpha_1 x_1 + \cdots + \alpha_k x_k$ is called

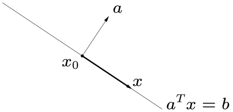
- convex: $\alpha_1 + \cdots + \alpha_k = 1$ and $\alpha_i \geq 0$
- conic: $\alpha_i \geq 0$
- affine: $\alpha_1 + \cdots + \alpha_k = 1$
- linear: $\alpha_i \in \mathbb{R}$

Proposition

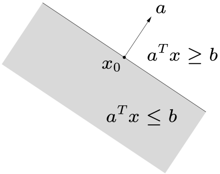
- A convex hull is always convex
- If C is convex, then $\text{conv}(C) = C$
- For any set C, $\text{conv}(C)$ is the smallest convex set containing C.

Hyperplane and halfspaces

- **hyperplane** set of the form $\{x \mid a^T x = b\}$ and $a \neq 0$



- **halfspace** set of the form $\{x \mid a^T x \leq b\}$ and $a \neq 0$



Convex sets – examples

- All of $\mathbb{R}^n$
- Non-negative orthant $\mathbb{R}^n_+$: let $\boldsymbol{x} \geq 0$, $\boldsymbol{y} \geq 0$, then

$$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \geq 0$$

- Affine subspaces: let $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b}$, then

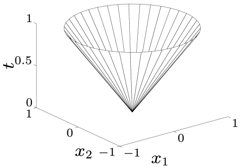
$$\begin{aligned} \boldsymbol{A}\left(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) &= \alpha \boldsymbol{A}\boldsymbol{x} + (1-\alpha)\boldsymbol{A}\boldsymbol{y} \\ &= \alpha \boldsymbol{b} + (1-\alpha)\boldsymbol{b} \\ &= \boldsymbol{b} \end{aligned}$$

Convex sets – examples

- Norm balls: let $\|\boldsymbol{x}\|_2 \leq t$, $\|\boldsymbol{y}\|_2 \leq t$, then

$$\begin{aligned}\|\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\| &\leq \|\alpha \boldsymbol{x}\| + \|(1-\alpha)\boldsymbol{y}\| \\ &= \alpha \|\boldsymbol{x}\| + (1-\alpha)\|\boldsymbol{y}\| \\ &\leq t\end{aligned}$$

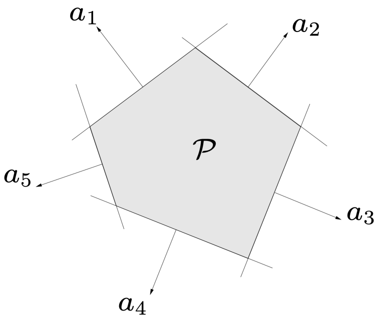
- Norm cones: $\{(x,t) \mid \|x\| \leq t\}$



Convex sets – examples

- Polyhedra: solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$



Convex sets – examples

- $\boldsymbol{S}^n \in \mathbb{R}^{n\times n}$ is the set of symmetric $n \times n$ matrices
- Positive semidefinite matrices $\boldsymbol{S}_+^n \doteq \{X \in \boldsymbol{S}^n \mid X \succeq 0\}$

$$X \in \boldsymbol{S}_+^n \iff v^T X v \geq 0 \quad \forall v \in \mathbb{R}^n$$

Consider the convex combination of X and Y, then

$$v^T \left(\alpha X + (1-\alpha)Y\right) v = \alpha v^T X v + (1-\alpha) v^T Y v \geq 0$$

- Positive definite matrices $\boldsymbol{S}_{++}^n \doteq \{X \in \boldsymbol{S}^n \mid X \succ 0\}$

$$X \in \boldsymbol{S}_+^n \iff v^T X v \ > 0 \quad \forall v \in \mathbb{R}^n$$

Convex sets

Practical methods for establishing convexity of a set C

- apply definition

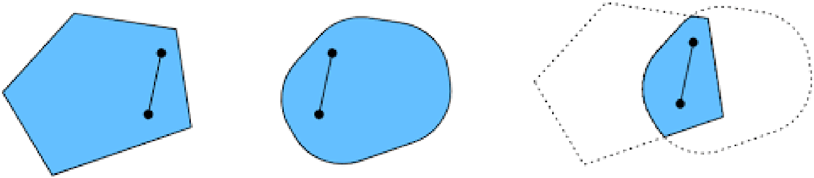
$$x, y \in C, 0 \leq \alpha \leq 1 \quad \Longrightarrow \alpha x + (1-\alpha) y \in C$$

- show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective functions
  - linear-fractional functions

Intersection

- Arbitrary intersections of convex sets is still convex: let C_i be convex for $i \in \mathcal{I}$ and $C = \cap_i C_i$, then

$$\begin{aligned} & x \in C, y \in C \\ & \implies \alpha x + (1-\alpha) y \in C_i, \forall i \in \mathcal{I} \\ & \implies \alpha x + (1-\alpha) y \in C \end{aligned}$$



Affine function

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine, or $f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

- the image of a convex set under f is convex

$$\begin{aligned} & C \in \mathbb{R}^n \text{ convex} \\ & \implies f(C) = \{f(x) \in \mathbb{R}^m \mid x \in C\} \text{ convex} \end{aligned}$$

Let $x, y \in C$, then

$$\begin{aligned} & \alpha f(x) + (1-\alpha) f(y) \\ = & A(\alpha x + (1-\alpha) y) + b \\ \in & f(C) \end{aligned}$$

Affine function

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine, or $f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

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$$C \in \mathbb{R}^n \text{ convex}$$
$$\implies f(C) = \{f(x) \in \mathbb{R}^m \mid x \in C\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \in \mathbb{R}^m \text{ convex}$$
$$\implies f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

Perspective and linear-fractional function

- Perspective function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(x,t) = x/t, \quad \text{dom } P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex set under perspective are convex

- Linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x) = \frac{Ax+b}{c^T x + d}, \quad \text{dom } f = \left\{ x \mid c^T x + d > 0 \right\}$$

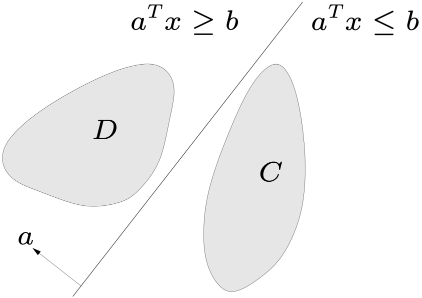
images and inverse images of convex set under perspective are convex

Separating hyperplane

If C and D are nonempty disjoint convex sets, there exist $a \neq 0$ and b such that

$$a^T x \leq b \quad \forall x \in C, \quad a^T x \geq b \quad \forall x \in D$$

the hyperplane $\{x \mid a^T x = b\}$ separates C and D



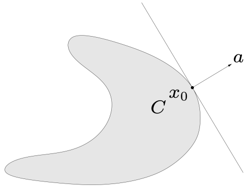
Supporting hyperplane

Supporting hyperplane to set C at boundary point x_0

$$\left\{x \mid a^T x = a^T x_0\right\}, \quad a \neq 0$$

and

$$a^T x \leq a^T x_0 \quad \forall x \in C$$



supporting hyperplane theorem

if C is convex, then there exists a supporting hyperplane at every boundary point of C

Next Lecture

Convex functions