Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

Introduction and Formulation

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Syllabus

Meeting times

- Lectures (12 hours - 4 three-hour sessions - Jan 18 to Feb 8)
- Exercises (12 hours - 4 three-hour sessions - Jan 26 to Feb 23)
  - Group A: Fatiha Bendali-Mailfert
  - Group B: Hervé Kerivin
- Computer lab (9 hours - 3 three-hour sessions - Feb 3 to Feb 26)
  - Group A: Yves-Jean Daniel
  - Group B: Yves-Jean Daniel
  - Group C: Yves-Jean Daniel

Final grades based on

- final examination on Monday March 8, 2021, 9am-10:30am (books, notes, calculators, phones, laptops: prohibited) totaling 75%
- continuous assessment and computer-lab examination totaling 25%
  - computer-lab examination totaling 60%
  - computer-lab reports totaling 40%

Prerequisite Linear algebra

Objectives and Expectations

Provide an understanding of the principles of linear programming

Focus on

- Formulations
- Simplex method
- Duality theory
- Revised simplex method
- Sensitivity analysis

Each student is expected to

- understand all the material covered during the semester (i.e., definitions, theorems, proofs, geometric intuition, etc.),
- be able to use the concepts and techniques to solve not previously encountered problems.

References

- *Linear Programming* by Vašek Chvátal, W.H. Freeman and Company, New York, 1983
- *A First Course in Linear Optimization - a dynamic book -* by Jon Lee, Third Edition, Reex Press, 2013-19 (`https://sites.google.com/site/jonleewebpage/home/publications`)
- *Combinatorial Optimization* by William J. Cook, William H. Cunningham, William R. Pulleyblank et Alexander Schrijver, Wiley-Interscience in discrete mathematics and optimization, New York, 1998
- *Linear Programming and Network Flows* by Mokhtar S. Bazaaa, John J. Jarvis, and Hanif D. Sherali, Wiley & Sons, NY (1990)

Linear Programming

Mathematical model

Collection of variables and relationships needed to describe pertinent features of real-world problems

Operations research

Study of how to form mathematical models of complex engineering and management problems, and how to analyze them to gain insight about possible solutions

Linear programming

Linear Programming (LP) is concerned with the optimization (minimization or maximization) of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions

Applications

- **Diet Problem**: find the cheapest combination of foods that will satisfy all your nutritional requirements
- **Portfolio optimization**: minimize the risk in your investment portfolio subject to achieving a certain return
- **Airline crew scheduling**: assign crews to flights so that
  - each flight is covered
  - each pilot does not fly more than a certain amount each day
  - minimize costs (e.g., accommodation for crews staying overnight out of town)
  - schedule is robust
- **Manufacturing and transportation**: how should a company supply all its customers? How much of each product should a company produce?
- **Telecommunications**: call routing, network design, Internet traffic

Linearity

Linear function

Let $a_1, a_2, \ldots, a_n, b$ be real numbers (deterministic parameters). Linear function $f : \mathbb{R}^n \to \mathbb{R}$ of real variables $x_1, x_2, \ldots, x_n$:

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = \sum_{j=1}^{n} a_j x_j = \mathbf{a}^T \mathbf{x}$$

where $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ are column vectors of $\mathbb{R}^n$.

Linear constraints

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function and b be a real number. Linear constraints of real variables $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= b \\ f(x_1, x_2, \ldots, x_n) &\geq b \\ f(x_1, x_2, \ldots, x_n) &\leq b \end{aligned}$$

Linear-Programming Problem

Linear programming problem

A LP problem is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints

Example:

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & 0.64x_2 & & \\ \text{subject to} & 50x_1 & + & 31x_2 & \leq & 250 \\ & -3x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

A First Example

A forester has 100 acres of hardwood timber. Felling the hardwood and letting the area regenerate would cost \$10 per acre in immediate resources and bring a subsequent return of \$50 per acre. An alternative course of action is to fell the hardwood and plant the area with pine; that would cost \$50 per acre with a subsequent return of \$120 per acre. Only \$4,000 is available to meet the immediate costs. What is the optimal program the forester should follow to maximize its net profit?

A First Example - Solution

$$
\begin{array}{lrcrcl}
\text{maximize } z = & 40x_1 & + & 70x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \leq & 100 \\
 & 10x_1 & + & 50x_2 & \leq & 4,000 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Formulating a Problem

Translating a (real) problem description into a mathematical model

Three steps

1. define what appear to be necessary variables (what decisions should we make?)
2. use these variables to define a set of (linear) constraints; the feasible points for the constraints correspond to the feasible solutions to the problem, and vice versa (what are the requirements ?)
3. use these variables to define the objective function (what needs to be optimized ?)

Standard Form

Let

- $a_{ij} \in \mathbb{R}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$
- $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$
- $c_j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$

Standard form

$$
\begin{array}{lrll}
\text{maximize } z & = & \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\
 & & x_j \geq 0 & \text{for } j = 1, 2 \ldots, n
\end{array}
$$

Objective function

The linear function that is to be optimized in an LP problem is called the objective function

Nonnegativity constraints

The last n of the $m + n$ linear constraints are called nonnegativity constraints

Converting into Standard Form

Objective function

min w is equivalent to $\max -w$

Inequalities

$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ is equivalent to $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$

Equations

$\sum_{j=1}^{n} a_{ij}x_j = b_i$ is equivalent to $\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ -\sum_{j=1}^{n} a_{ij}x_j \leq -b_i \end{cases}$

Variables

- $x_j \leq 0$ can be replaced by $-x_j \geq 0$
- x_j unrestricted can be replaced by $x_j^+ - x_j^-$ with $x_j^+ \geq 0$ and $x_j^- \geq 0$

Terminology

Feasible solution

Numbers $x_1, x_2, \ldots, x_n$ that satisfy all the constraints of an LP problem are said to constitute a feasible solution

Optimal solution

A feasible solution that maximizes the objective function (or minimizes it, depending on the form of the problem) is called an optimal solution

Optimal value

The value of the objective function corresponding to an optimal solution is called the optimal value

Geometric Solution

- Geometric procedure for solving LP problem

$$
\begin{aligned}
\text{maximize } z &= \sum_{j=1}^{n} c_j x_j \\
\text{subject to } & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2 \ldots, m \\
& x_j \geq 0 \quad \text{for } j = 1, 2 \ldots, n
\end{aligned}
$$

- Method only suitable for very small problems (e.g., two or three variables)

Half-Spaces

Let $a_1, a_2, \ldots, a_n, b$ be real numbers

All the solutions of $a_1x_1 + a_2x_2 = b$ are represented by a line, whereas all the solutions of $a_1x_1 + a_2x_2 \leq b$ are represented by a half-plane bounded by that line

All the solutions of $a_1x_1 + a_2x_2 + a_3x_3 = b$ are represented by a plane, whereas all the solutions of $a_1x_1 + a_2x_2 + a_3x_3 \leq b$ are represented by a half-space bounded by that plane

Half-space

The set of all the solutions of $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$ is called an half-space of the n-dimensional space $\mathbb{R}^n$. (At least one a_j is not zero.)

Feasible Region

Each constraint of an LP problem determines a certain half-space

Feasible region

The feasible region (i.e., the geometric counterpart of the set of all feasible solutions) is the intersection of the half-spaces corresponding to the constraints (including the nonnegativity constraints on the variables).

In general, the intersection of a finite number of half-spaces is called a polyhedron

Polyhedron

A polyhedron $P \in \mathbb{R}^n$ n is the set of vectors which satisfy a finite number of linear inequalities, that is,

$$P = \{x \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$$

where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$

The Graphic Method

- Let $\max c_1x_1 + c_2x_2 + \cdots + c_nx_n$ be the objective function
- Every equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = z$ defines a level set of the objective function
- The gradient (or partial derivative vector) of the objective function corresponds to the direction of greatest increase

Graphic method (LP problems with $n \leq 3$)

1. plot the region of feasibility
2. draw a level set $c_1x_1 + c_2x_2 + \cdots + c_nx_n = d$ passing through the feasible region
3. move the level set by increasing the value of d
4. stop when the level set is about to leave the feasible region
5. the last point of contact represents the optimal solution

Example

Consider the LP

$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 & + & 0.64x_2 & & \\
\text{subject to} & 50x_1 & + & 31x_2 & \leq & 250 \\
 & -3x_1 & + & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0 \\
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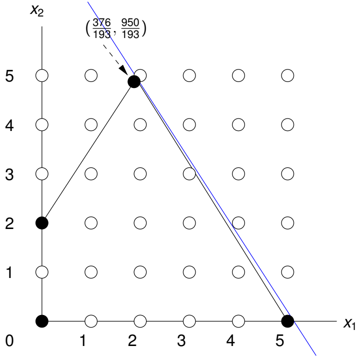
 & -3x_1 & + & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$



Optimality

Uniqueness

Not every LP problem has a unique optimal solution; yet, if there exists an optimal solution, then the optimal value is unique.

- Example:

$$\begin{array}{lrcrcr}
\text{maximize } z = & & & x_2 & & \\
\text{subject to} & 2x_1 & + & x_2 & \geq & 8 \\
& 9x_1 & - & 2x_2 & \leq & 63 \\
& & & x_2 & \leq & 4 \\
& & & x_2 & \geq & 0
\end{array}$$

- some problems have exactly one optimal solution
- some problems have many different optimal solutions
- some problems have no optimal solutions at all

Infeasible LP Problem

Infeasible LP problem

An LP problem that has no feasible solutions at all is called infeasible

Example:

$$
\begin{array}{lrcrcr}
\text{maximize } z = & 3x_1 & - & x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \leq & 2 \\
& -2x_1 & - & x_2 & \leq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}
$$

Infeasibility does not depend on the objective function

Unbounded LP Problem

Unbounded LP problem

An LP problem that has feasible solutions but no optimal solutions is called unbounded

Example:

$$\begin{array}{lrcrcr}
\text{maximize } z = & x_1 & - & x_2 & & \\
\text{subject to} & -2x_1 & + & x_2 & \leq & -1 \\
& -x_1 & - & 2x_2 & \leq & -2 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}$$

Unboundedness does depend on the objective function

Three Categories

Let $S \subseteq \mathbb{R}^n$ denote the feasible region of a linear program

Three categories

A linear program is exactly in one of the three categories

- it is *infeasible* (i.e., $S = \emptyset$)
- it is *unbounded* (i.e., for any $\alpha \in \mathbb{R}$, there exists $\mathbf{x} \in S$ so that $\mathbf{c}^T\mathbf{x} > \alpha$)
- it has an optimal solution