Course: Applied Mathematics (D. Sidibé)

Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1 (40 Points)

Your classmate Yukti performed the usual elimination steps to convert a matrix A into an upper triangular matrix $U = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

1. What is the rank of A? Give a basis for the nullspace $N(A)$.

2. If $Uy = \begin{bmatrix} 9 \\ -12 \\ 0 \end{bmatrix}$, find the complete solution y (i.e. describe all possible solutions y).

3. Your classmate Pablo gives you a matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, and tells you that $A = LU$.

   Describe the complete sequence of elimination steps that Pablo performed (assuming he did elimination the usual way, starting with the first column and eliminating downwards).

4. Now, you are asked to solve the equation $Ax = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$.

   Your classmate Corina tells you that a solution is $x_C = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, while your other classmate Raquel tells you that a correct solution is $x_R = \begin{bmatrix} 11 \\ -2 \\ 0 \\ -1 \end{bmatrix}$.

   Is Corina's solution correct, or Raquel's solution, or both correct?

■ PROBLEM 2 (30 Points)

Let A be a 3×3 real symmetric matrix. The trace of A is equal to zero and two of its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, with eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ respectively.

1. What is the third eigenvalue λ_3?
2. Give an eigenvector v_3 for the eigenvalue λ_3 (Hint: what must be true of v_1, v_2 and v_3?).
3. What is A^5?

■ PROBLEM 3 (30 Points)

1. If P is a projection matrix, show that $(I - P)$ is also a projection matrix.

   If P is the projection onto the column space $C(A)$ of a matrix A, then $(I - P)$ is the projection onto which vector space? Explain your result.

2. Suppose A is a 3×5 matrix and the equation $Ax = b$ has a solution for every b.

   - a) What is the rank of A?
   - b) What is the rank of the 6×5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$?

3. Let $A = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$. What are the eigenvalues of A?

   <u>Hint</u>: very little calculation required. You shouldn't need to compute $det(A - \lambda I)$ unless you really want!