Diagnostic Medical Image Processing

Reconstruction

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Solution of Integral Equation

Different ways to approach the reconstruction problem:

- In the analytic approach the integral equation is solved using continuous methods. Discretization is done at the end. An algorithm that results from this approach is filtered back-projection.
- In the algebraic approach discretization is done upfront. The integral is transformed into a sum and the integral equation reduces to a system of linear equations. In the literature these approaches are summarized by the term algebraic reconstruction technique (ART).

Modeling of X-ray Projections

The basic assumptions for the following discussion are:

- X-rays are propagated on straight lines through the volume.
- Projection lines are defined by the acquisition geometry that is known by the mechanical setup or calibration.
- Volume is represented as a piecewise constant function, i.e. for each cell in the volume we have a constant function value.

Reconstruction and Linear Equations

In the discrete version we measure

- for each rotation angle, indexed by θ, and
- detector cell t
- the transformed intensity (logarithm and I_0 corrected) value $p_\theta[t]$ in the projection.

The X-ray particles interact with each voxel the projection ray hits. Due to the X-ray attenuation law, we can approximate the attenuation process by a linear equation for each (θ, t)-projection ray:

$$p_\theta[t] = \sum_{z=0}^{N_z-1} \sum_{y=0}^{N_\mathrm{y}-1} \sum_{x=0}^{N_\mathrm{x}-1} w_{z,y,x,\theta,t} f_{z,y,x} \tag{1}$$

where:

- $f_{z,y,x}$ is the attenuation value of voxel (x, y, z)
- $w_{z,y,x,\theta,t}$ is the weight of voxel (x, y, z) in the t-th projection ray seen under angle θ

A Note on Notation

We now do the following simplifications in our notation:

- We put all pixels $p_{\theta,t}$ of the observed projection images into a single (lengthy) vector

$$\boldsymbol{p} = [p_t]_{t=1,\dots,N_\mathrm{a}} = [p_\theta[t]]_{\theta=1,\dots N_\mathrm{v},t=1,\dots,N_\mathrm{t}-1}$$

  where $N_\mathrm{a} := N_\mathrm{v} \cdot N_\mathrm{t}$.

- We put all volume elements $f_{z,y,x}$ into one long vector

$$\boldsymbol{f} = [f_i]_{i=1,\dots,N_\mathrm{s}}$$

  where $N_\mathrm{s} = N_\mathrm{x} \cdot N_\mathrm{y} \cdot N_z$

- Now we need only two indices for the identification of the proper component of the weight vector $\boldsymbol{w}$: $w_{t,i}$.

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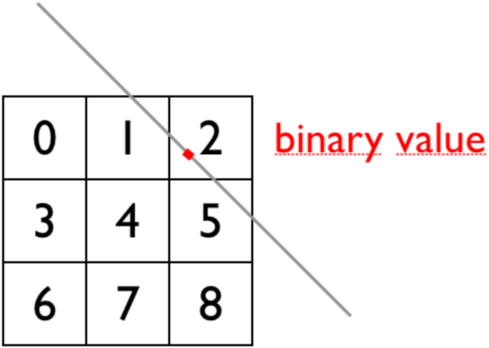
Computation of Weight Factors

The weight factors $w_{z,y,x,\theta,t}$

- are independent from observations,
- are defined by the projection geometry,
- can be pre-computed right after the installation of a system, and practically
- can be defined in various ways.

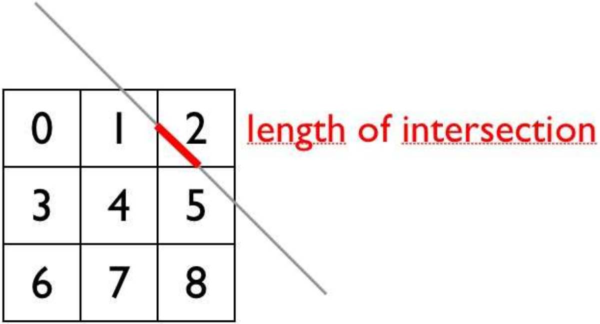
Computation of Weight Factors

The weights can be considered as binary values. The ray hits a voxel, or not.



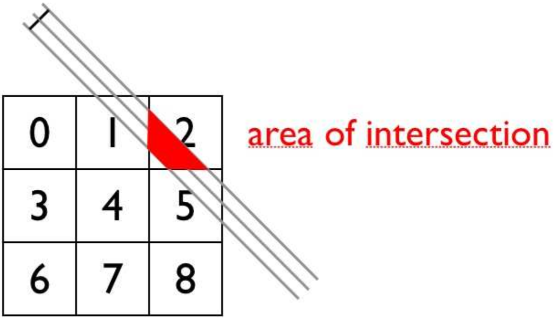
Computation of Weight Factors

The weights can depend on the length of intersection of projection ray and considered voxel.



Computation of Weight Factors

The projection ray can be considered as a stripe, and the area of intersection of a voxel with the projection stripe defines the weight.



Definition of System Matrix

Definition

The matrix $\boldsymbol{W}_\theta$ that defines the mapping of voxel values to projections for a given θ

$$\boldsymbol{W}_\theta \begin{pmatrix} f_0 \\ \vdots \\ f_{N_\mathrm{s}-1} \end{pmatrix} = \begin{pmatrix} p_\theta[0] \\ \vdots \\ p_\theta[N_\mathrm{t} - 1] \end{pmatrix}$$

is called **system matrix**.

Example

Now we compute the system matrix for the following projection from 2-D to 1-D.

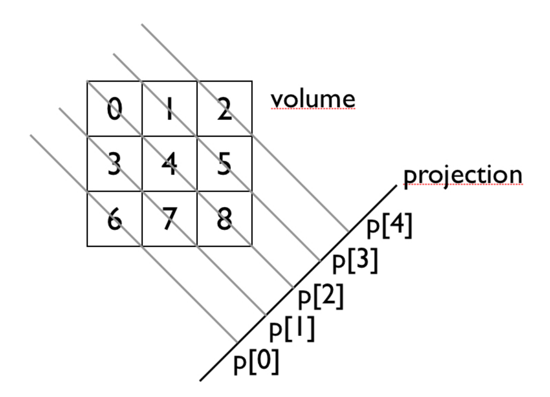


Figure: System matrix for a 3 $\times$ 3 volume

Example

If we consider the projection in diagonal direction we get the following system matrix for a detector with five cells:

$$\boldsymbol{W} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A note on numbers: Assume we have a standard CT volume size of $512 \times 512 \times 512$ and 1024×1024 projections. The system matrix for a single view thus is a $2^{20} \times 2^{27}$ matrix. This is huge.

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Reconstruction Problem

The **reconstruction problem** is now easily stated: Solve the set of linear equations (1).

Size of the problem

Typical C-arm reconstructions, for instance, require

- the computation of $512 \times 512 \times 512$ volumes, i.e. we have 2^{27} ($\approx$ 134 million) unknowns.
- from 512 projections with 1024×1024 pixels. The number of equations thus is 2^{29} ($\approx$ 536 million).

$\rightarrow$ cannot be solved with standard methods (e.g. Gauss elimination)
$\rightarrow$ iterative methods required

Iterative Reconstruction

A simple iterative scheme based on this system of linear equations works intuitively in the following manner:

- Start with an initial guess of the function f t.b. reconstructed.
- Forward-project the guess into the image plane
- Compute the difference between observation and forward-projection image
- Correct the function values to reduce the difference
- Repeat until the difference is below a threshold.

Note: The error is measured in the image plane.

Iterative Reconstruction

A straight forward method to the solution of the system of linear equations is the application of the *method of projections*:

- Each linear equation defines a hyperplane.
- The solution is the intersection of all hyperplanes.

Well-conditioned problem: all hyperplanes are almost orthogonal.
Ill-conditioned problem: small angles between hyperplanes.

Problem: How to choose the right sequence of projections on the hyperplanes?

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Advantages of ART

- Supposed to work better than filtered back–projection for a small number of projections
- ART do not require the uniform sampling of angles.
- No rotation around the object is required but any trajectory can be used for reconstruction.
- No 180^o rotation constraint (as required in Fourier-slice theorem).

Disadvantages of ART

- The definition and computation of the system matrix is crucial.
- ART is computationally hard and still cannot compete with filtered back-projection performance wise.
- ART has not yet proven to outperform filtered back-projection in practical (!) cases.

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Kaczmarz Method

The iterative Kaczmarz method to solve systems of linear equations was developed by the Polish mathematician Stefan Kaczmarz in 1937.

Observations:

- The linear equation associated with each projection ray defines a hyperplane.
- The solution of the system of linear equations is the reconstruction result.
- The intersection of all hyperplanes is the solution of the system of linear equations and thus the reconstruction result.

Kaczmarz Method

Basic idea of the iterative method:

1. Initialize the solution vector (for instance, randomly)
2. Project the initial solution to the first hyperplane
3. Project the intermediate solution to the next hyperplane
4. etc.

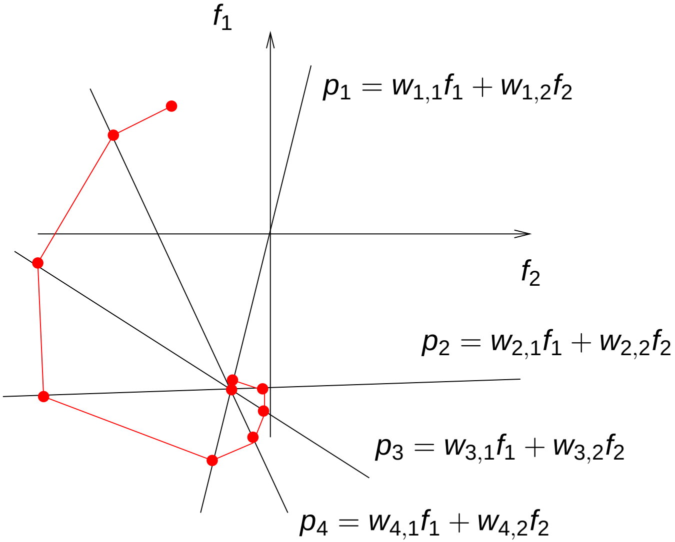

Figure: Kaczmarz method (1937)

Iteration Scheme

1. let $\hat{\boldsymbol{f}}^{(k)}$ be the estimated solution in the k–th iteration
2. let $\boldsymbol{w}_k = [w_{k,i}]_{i=1,\dots,N_s}$ be the weight vector of the hyperplane used in the k–th iteration
3. let p_k be the transformed intensity that corresponds to the considered linear manifold in the k-th iteration

The iteration scheme for volume $\boldsymbol{f} = [f_{ij}]$ is

$$\hat{\boldsymbol{f}}^{(k+1)} = \hat{\boldsymbol{f}}^{(k)} + \frac{p_{k+1} - \boldsymbol{w}_{k+1}{}^{\mathrm{T}}\hat{\boldsymbol{f}}^{(k)}}{\boldsymbol{w}_{k+1}{}^{\mathrm{T}}\boldsymbol{w}_{k+1}}\,\boldsymbol{w}_{k+1} \tag{2}$$

This iteration formula results from very basic linear algebra.

Some Linear Algebra: Orthogonal Projections

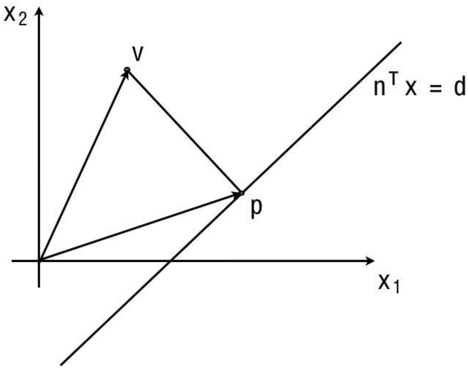

Figure: Orthogonal projection

Some Linear Algebra: Orthogonal Projections

Let us now consider the orthogonal projection of $\boldsymbol{v}$ on the hyperplane defined by

$$\boldsymbol{n}^{\mathsf{T}}\boldsymbol{x} = d \in \mathbb{R} \quad .$$

The orthogonal projection $\boldsymbol{p}$ of $\boldsymbol{v}$ fulfills the properties:

- difference vector $(\boldsymbol{p} - \boldsymbol{v})$ is collinear to $\boldsymbol{n}$

$$\boldsymbol{p} - \boldsymbol{v} = \lambda \boldsymbol{n}$$

- vector $\boldsymbol{p}$ is on the hyperplane

$$\boldsymbol{n}^{\mathsf{T}}\boldsymbol{p} = d \in \mathbb{R} \quad .$$

Obviously we get

$$\lambda = \frac{d - \boldsymbol{n}^{\mathsf{T}}\boldsymbol{v}}{\boldsymbol{n}^{\mathsf{T}}\boldsymbol{n}}$$

and thus

$$\boldsymbol{p} = \boldsymbol{v} + \frac{d - \boldsymbol{n}^{\mathsf{T}}\boldsymbol{v}}{\boldsymbol{n}^{\mathsf{T}}\boldsymbol{n}} \cdot \boldsymbol{n}$$

ART Reconstruction

In our particular iterative scheme, The orthogonal projection leads to:

$$\lambda \boldsymbol{w}_{k+1} = \hat{\boldsymbol{f}}^{(k+1)} - \hat{\boldsymbol{f}}^{(k)} \tag{3}$$

and the new estimate is required to be an element of the straight line:

$$\boldsymbol{w}_{k+1}^T \hat{\boldsymbol{f}}^{(k+1)} = p_{k+1} \tag{4}$$

Combination of both equations yields

$$\lambda = \frac{p_{k+1} - {\boldsymbol{w}_{k+1}}^\mathrm{T} \hat{\boldsymbol{f}}^{(k)}}{{\boldsymbol{w}_{k+1}}^\mathrm{T} \boldsymbol{w}_{k+1}} \tag{5}$$

and thus we get the iteration formula (2).

Convergence Properties of Kaczmarz Method

- Tanabe has shown in 1971 that the iterative scheme converges to the solution, if there exists a unique solution.
- The angle between hyperplanes influences the rate of convergence to the solution.
- If hyperplanes are orthogonal to each other, it is obvious that the method converges rapidly (consider the 2-D case for plausibility)
- Orthogonalization methods applied in advance to iterations will improve convergence (cons: computationally prohibitive, orthogonalization amplifies noise in measurements)
- Alternative to orthogonalization: careful selection of sequence of projections.
- Overdetermined systems and noise: no unique solution and oscillation.

Convergence Properties of Kaczmarz Method

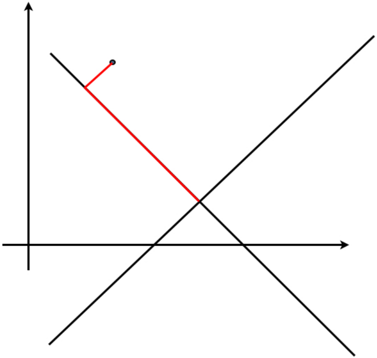

Figure: Orthogonal straight lines: convergence in two iterations

Convergence Properties of Kaczmarz Method

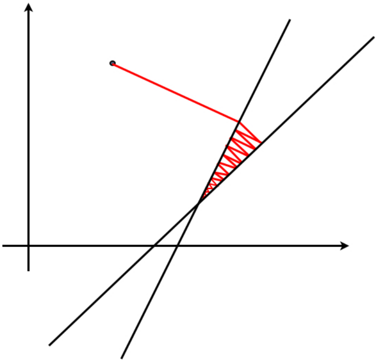

Figure: Small angle between straight lines: slow convergence

Convergence Properties of Kaczmarz Method

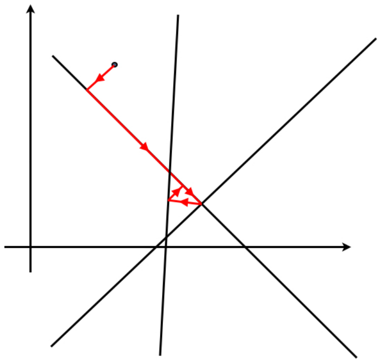

Figure: Oscillating iteration process

Convergence Properties of Kaczmarz Method

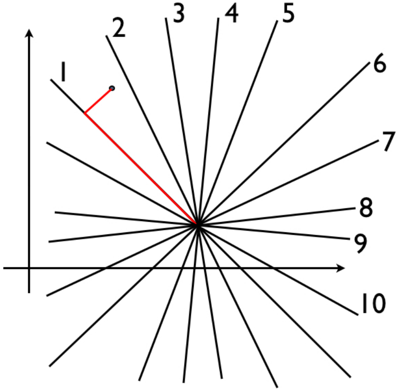

Figure: Projection to line 1, then to line 6 guarantees fast convergence in this particular case.

Convergence Properties of Kaczmarz Method

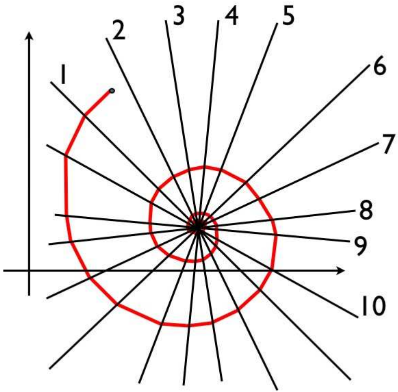

Figure: Overdetermined system with non unique solution: two planes in 3-D intersect in a line.

Convergence Properties of Kaczmarz Method

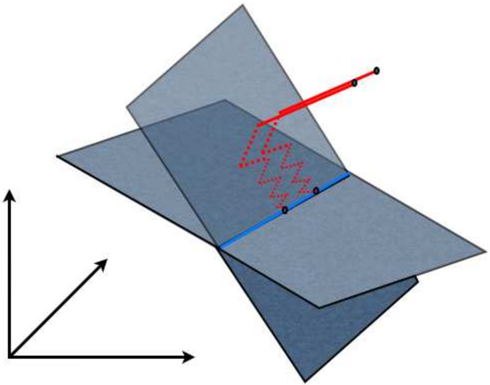

Figure: Projection sequence 1, 10, 9, 8, ... leads to slow convergence.

Simultaneous Algebraic Reconstruction Technique

There exist many variations and modifications of the Kaczmarz method in the literature. The Simultaneous Algebraic Reconstruction Technique is easily parallelized and makes use of the following idea:

1. Estimate an initial solution to the system of linear equations.
2. Compute the orthogonal projections of the current estimate to all hyperplanes.
3. Compute the centroid of all projected points.
4. Use centroid for next iteration.

Simultaneous Algebraic Reconstruction

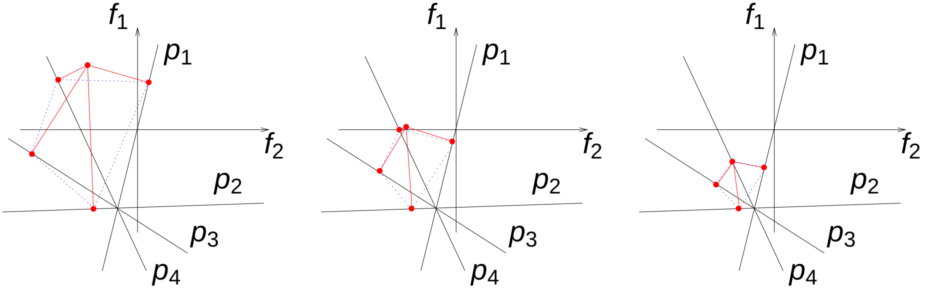

Figure: Simultaneous algebraic reconstruction

Simultaneous Algebraic Reconstruction

Using this geometric characterization it is fairly obvious that the simultaneous algebraic reconstruction algorithm iterates using the following update rule:

$$\hat{\boldsymbol{f}}^{(k+1)} = \hat{\boldsymbol{f}}^{(k)} + \lambda_k \left(\sum_{v=0}^{N_\mathrm{a}-1} u_{k,v} \cdot \frac{p_v - {\boldsymbol{w}_v}^\mathrm{T} \hat{\boldsymbol{f}}^{(k)}}{{\boldsymbol{w}_v}^\mathrm{T} \boldsymbol{w}_v} \boldsymbol{w}_v \right) \tag{6}$$

where we added two additional types of parameters:

- λ_k is a relaxation parameter, and
- $u_{k,v}$ are iteration dependent weights that fulfill the probability constraint

$$\sum_{v=0}^{N_\mathrm{a}-1} u_{k,v} = 1.$$

Simultaneous Algebraic Reconstruction

Remarks:

- For the standard simultaneous algebraic reconstruction, we have obviously $u_{k,v} = 1/N_{\mathrm{v}}$ and $\lambda_k = 1$.
- A commonly used heuristics for the choice of $u_{k,v}$ is a weighting scheme that takes the relative number of voxels into consideration that are hit by this particular projection ray.
- If we set $\lambda_k = 2$ we get the mean of points reflected at the hyperplanes. This leads to the reflection algorithm of Cimmino.

Cimmino's Reflection Method

In 1938 G. Cimmino introduced the reflection method for solving systems of linear equations iteratively:

1. Estimate an initial solution to the system of linear equations.
2. Generate mirror symmetric points regarding each hyperplane.
3. Compute the centroid of the new points.
4. Use centroid for next iteration.

Cimmino's Reflection Method

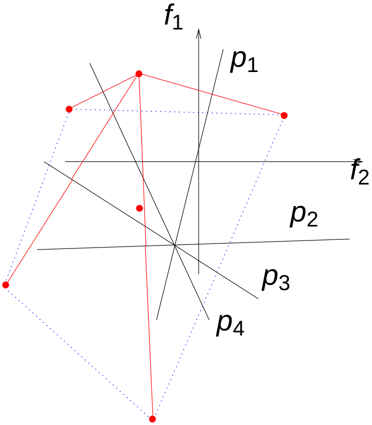

Figure: Cimmino method

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Conditioning of Reconstruction Problem

The SVD does an excellent job in analyzing the reconstruction problem:

- Reconstruction problem can be associated with a system of linear equations (ART).
- The ill–posed reconstruction problem can be analyzed by studying singular values.
- Reconstruction volume can be reduced to a size where the SVD is computationally feasible (down sampling of both volume and projection).
- Condition number allows for stability analysis.

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Take Home Messages

- Tomographic reconstruction can be considered as linear problem: solution of a huge system of linear equations.
- Algebraic reconstruction for CT is not used in today's medical imaging products (with a few exceptions)
- Solution of system of linear equations can be done by using iterative methods with an easy to understand geometric interpretation.
- Algebraic reconstruction results are supposed to be superior (?) to filtered back-projection based methods but still computationally prohibitive.

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Further Readings

- Stefan Kaczmarz: Angenäherte Auflösung von Systemen linearer Gleichungen, Bulletin Internat. Acad. Polon. Sci. Letters A., pages 335-357, 1937.
- G. Cimmino: Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari, Ric. Sci. Prog. Tec. Econ. Naz. I, pages 326-333, 1938.
- As usual, for practitioners we recommend the ART chapter of A.C. Kak and M. Slanely;
  A.C. Kak and M. Slaney: *Principles of Computerized Tomographic Imaging,* Society of Industrial and Applied Mathematics, 2001 (download here)
- A detailed discussion of convergence properties can be found in:
  Ming Jiang, Ge Wang: *Convergence studies on iterative algorithms for image reconstruction*, IEEE Transactions on Medical Imaging, Vol. 22, No. 5, May 2003, pages: 569 - 579