# Applied Mathematics : Exam – 04th Nov 2011

Solutions by Prof. D. SIDIBE (4th Nov 2011)

## Problem 1

Let $A$ be a $3\times 4$ matrix

After elimination, yukti gets $U = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1) • From $U$, we see that there are 2 pivots, so the rank of $A$ is 2 : $\text{rank}(A) = 2$.

• There are also 2 free variables : $x_3$ and $x_4$

So the dimension of the nullspace is $\dim N(A) = 2$

To find a basis for $N(A)$

– Set $\begin{cases} x_3 = 1 \\ x_4 = 0 \end{cases}$ and solve for $Ux = 0 \Rightarrow x_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

– Set $\begin{cases} x_3 = 0 \\ x_4 = 1 \end{cases}$ and solve for $Ux = 0 \Rightarrow x_2 = \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

Hence $$N(A) = \left\{ \alpha \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} ; \ \alpha, \beta \in \mathbb{R} \right\}$$

2) The complete solution is given by $y = y_p + y_N$, where $y_p$ is a particular solution and $y_N \in N(A)$.

– First, we find a particular solution $y_p$ setting all free variables to zero : $y_3 = y_4 = 0$, and solve $Uy_p = b$

$\Rightarrow$ we find $y = \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix}$

So the complete set of solution is

$$\boxed{Y = \begin{pmatrix} 33 \\ -6 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad ; \quad \alpha, \beta \in \mathbb{R}}$$

3) $$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{bmatrix}$$

And we perform elimination with $A$ to get $U$:

$$\underbrace{\begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{pmatrix}}_{A} \xrightarrow[r_3 + r_1]{r_2 - 2r_1} \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 6 & 6 & -18 \end{pmatrix} \xrightarrow[r_3 - 3r_2]{} \underbrace{\begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 6 \end{pmatrix}}_{U}$$

elimination steps

4) The complete set of solutions to $Ax = b$ is

$x = x_p + x_N$.

we already know, the nullspace of $A$: $N(A)$ (Question 1)

To find $x_p$, we set all free variables to zero ($x_3 = x_4 = 0$) and solve $Ux = c$ (where $c$ is the vector obtained when applying the same elimination steps to $b$!)

Solving $Ux = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ we get $x_p = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

we know that $N(A) = \{ \alpha u_1 + \beta u_2 \,,\ \alpha, \beta \in \mathbb{R} \}$

with $u_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -15 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

we can see that $\begin{cases} X_C = X_p + u_1 \\ X_R = X_p - u_2 \end{cases}$ with means

that <u>both solutions are correct</u>.

<u>NOTE</u> we can simply answer this question checking

$A X_C = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ and $A X_R = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$, since we

know $A = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 10 & 0 & 0 \\ -1 & 2 & 7 & -21 \end{pmatrix}$.

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## Problem 2

$A$ is a $3 \times 3$ symmetric matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

1) we know $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 0$

so $\lambda_3 = -(\lambda_1 + \lambda_2) = 0 \quad \boxed{\lambda_3 = 0}$

2) For a symmetric matrix, the eigenvectors must be orthogonal to each other.

So we need $v_3 \perp v_1$ and $v_3 \perp v_2$.

If we put $v_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then we have

$$\begin{cases} a + b + c = 0 \\ a - b = 0 \end{cases} \Rightarrow \text{ So we can take } \boxed{v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}.$$

3) To easily find $A^5$, we have to diagonalize $A$.

$A = Q \Lambda Q^T$ ( Since $A$ is symmetric and the matrix $Q$ whose columns are the eigenvectors is orthogonal ).

$Q = \left[ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3} \right]$ : The columns of $Q$ are the eigenvectors of $A$.

So $Q = \begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & -\frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$ and $\Lambda = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

we have $A^5 = Q \Lambda^5 Q^T$

with $\Lambda^5 = \begin{pmatrix} (1)^5 & & \\ & (-1)^5 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

So $A^5 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$

$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -\frac{1}{\sqrt{2}} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\boxed{A^5 = \begin{pmatrix} -1/6 & 5/6 & 1/3 \\ 5/6 & -1/6 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}}$$

# Problem 3

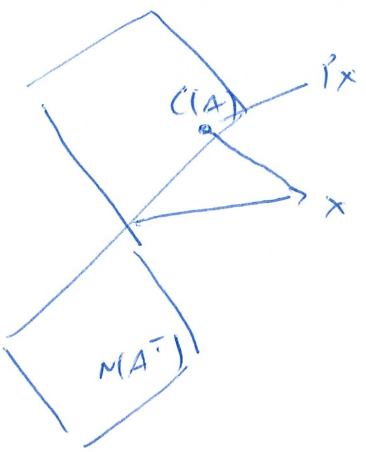
1) $P$ is a projection matrix, then $P^2 = P$ and $P = P^T$

So $(I-P)^T = I - P^T = I - P$

and $(I-P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2$

$$= I - 2P + P = I - P$$

which proves that $(I-P)$ is also a projection matrix.




$C(A)$ and $N(A^T)$ are orthogonal complements.

If $P$ projects onto $C(A)$, then $(I-P)$ projects onto $N(A^T)$

we can also see that $(I-P)x = x - Px \in N(A^T)$.

2) a) $Ax = b$ has a solution for every $b \in \mathbb{R}^3$

which means every $b \in \mathbb{R}^3$ is in $C(A)$.

So $C(A) = \mathbb{R}^3$ and $\boxed{\text{rank}(A) = 3}$

b) Let $B = \begin{bmatrix} A \\ A \end{bmatrix}_{6 \times 5}$

If we perform elimination with $B$, we will get

$B \longrightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}$, so $\boxed{\text{rank}(B) = 3}$

3) $A = \begin{bmatrix} 0,5 & 0,2 & 0,2 \\ 0,1 & 0,5 & 0,5 \\ 0,4 & 0,3 & 0,3 \end{bmatrix}$

- The last two columns are the same $\rightarrow$ A is singular which also means $\underline{\lambda_1 = 0}$ is one eigenvalue

(You can also do elimination and find there are only two pivots).

- A is a Markov matrix $\rightarrow$ $\underline{\lambda_2 = 1}$ is an eigenvalue

- Finally trace (A) = $\lambda_1 + \lambda_2 + \lambda_3 = 1,3$

$\Rightarrow \lambda_3 = \text{trace} - (\lambda_1 + \lambda_2) \Rightarrow \underline{\lambda_3 = 0,3}$