Course: Applied Mathematics (D. Sidibé)

Mid-Term Exam: 2h

You must show all work and all reasoning - Full credit will be given only for clearly explained results!

■ PROBLEM 1 (30 Points)

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$.

1. What is the rank of A? Give a basis for the column sapce $C(A)$.
2. Find the dimension of the nullspace $N(A)$ and give a basis for $N(A)$.

   What is the set of **all solutions** to $Ax = 0$?
3. For which number b_3 does the system $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

   Find the complete set of solutions for that value of b_3.

■ PROBLEM 2 (30 Points)

We want to find the circle of equation $a(x^2 + y^2) + b(x + y) = 1$ which best fits the following data:

x	0	-1	1	1
y	1	0	-1	1

1. Let $z = \begin{bmatrix} a \\ b \end{bmatrix}$. What is the system $Az = b$ that the vector z must satisfies for the points to be on the circle? In other words, give the matrix A and the vector b.
2. Is the previous system solvable? Justify your answer.
3. Find the linear least squares solution and draw the data points and the obtained circle on a figure.
4. Let $M = A^T A$, where A is the matrix in Question 1. Find the eigenvalues of M.

■ PROBLEM 3 (30 Points)

The matrix $A = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix}$ has three eigenvalues $\lambda_1 = 18$, $\lambda_2 = 9$ and $\lambda_3 = -9$.

1. Find the eigenvectors corresponding to those three eigenvalues.
2. Find an orthogonal matrix Q such that $A = Q\Lambda Q^T$. What is the matrix Λ?
3. Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Write x as a linear combination of the three eigenvectors and compute $A^{10}x$.

■ PROBLEM 4 (10 Points)

For which number s is the following matrix positive definite?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$$