Applied to Image

- Let
  - $p_r(r)$ denote the PDF of random variable r
  - $p_s(s)$ denote the PDF of random variable s
- If $p_r(r)$ and $T(r)$ are known and $T^{-1}(s)$ satisfies condition (a) then $p_s(s)$ can be obtained using a formula :

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Applied to Image

The PDF of the transformed variable s

is determined by

the gray-level PDF of the input image

and by

the chosen transformation function

Transformation function

- A transformation function is a cumulative distribution function (CDF) of random variable r :

$$s = T(r) = \int_0^r p_r(w)\,dw$$

where w is a dummy variable of integration

Note: $T(r)$ depends on $p_r(r)$

Cumulative Distribution function

- CDF is an integral of a probability function (always positive) is the area under the function
- Thus, CDF is always single valued and monotonically increasing
- Thus, CDF satisfies the condition (a)
- We can use CDF as a transformation function

Finding $p_s(s)$ from given $T(r)$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

$$= \frac{d}{dr}\left[\int_0^r p_r(w)dw\right]$$

$$= p_r(r)$$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

$$= p_r(r)\left|\frac{1}{p_r(r)}\right|$$

$$= 1 \quad \text{where } 0 \le s \le 1$$

Substitute and yield

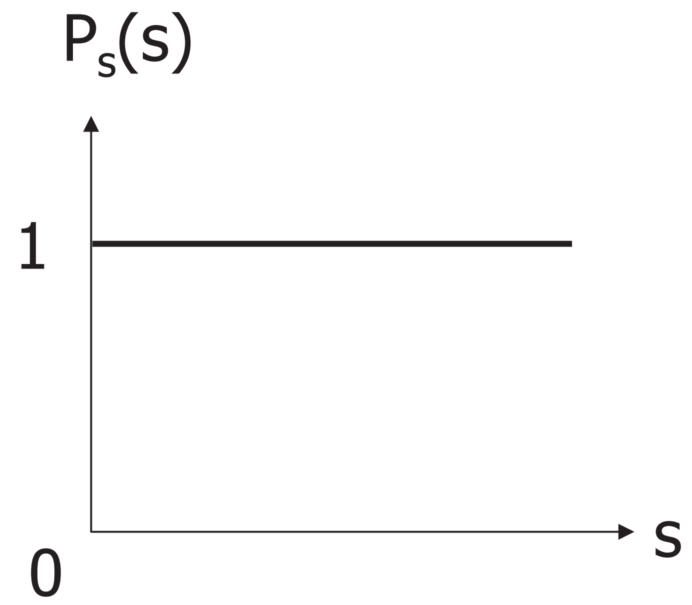
$p_s(s)$

- As $p_s(s)$ is a probability function, it must be zero outside the interval [0,1] in this case because its integral over all values of s must equal 1.
- Called $p_s(s)$ as **a uniform probability density function**
- $p_s(s)$ is always a uniform, independent of the form of $p_r(r)$

$$s = T(r) = \int_0^r p_r(w)\,dw$$

yields

a random variable s characterized by a uniform probability function



Discrete transformation function

- The probability of occurrence of gray level in an image is approximated by

$$p_r(r_k) = \frac{n_k}{n} \qquad where\ k = 0, 1, \ldots, L\text{-}1$$

- The discrete version of transformation

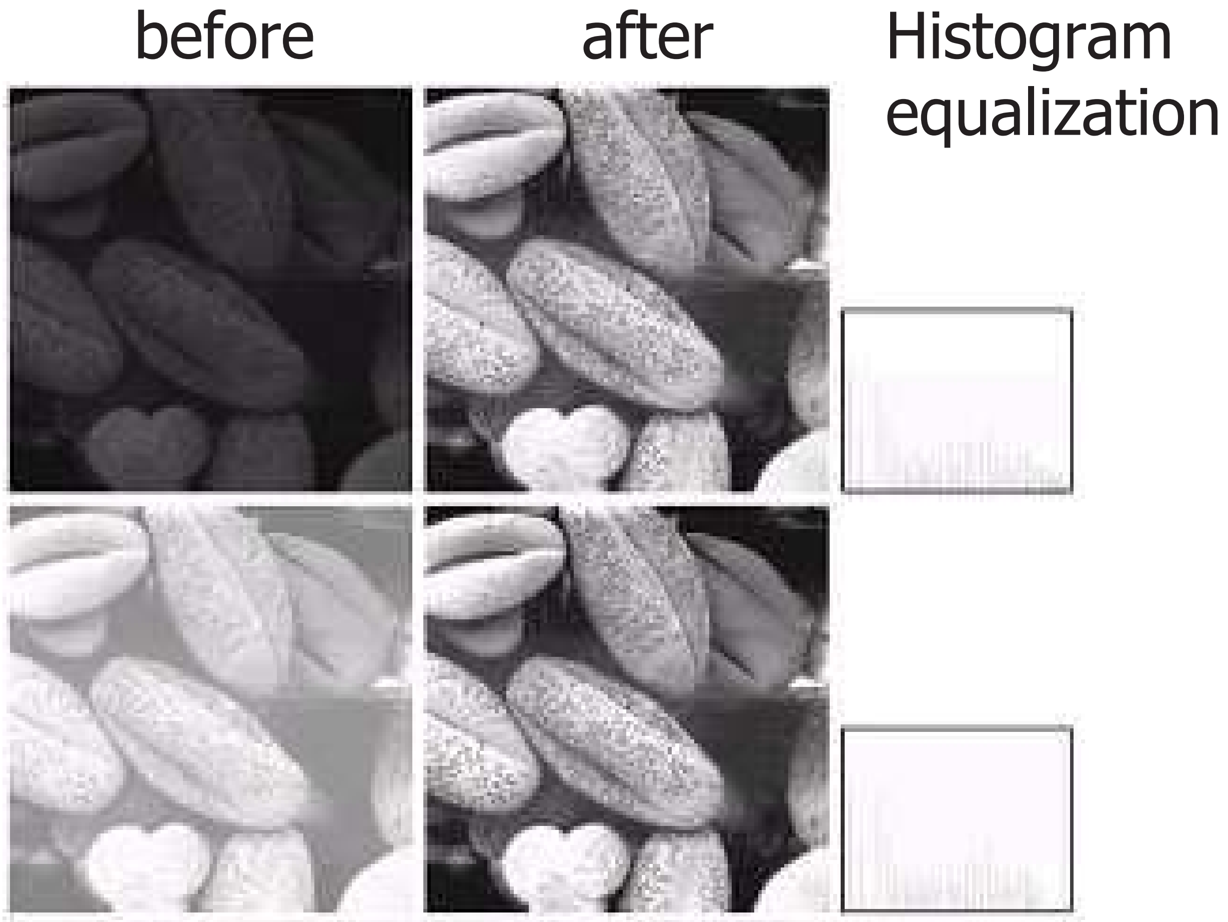
$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j)$$

$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad where\ k = 0, 1, \ldots, L\text{-}1$$

Histogram Equalization

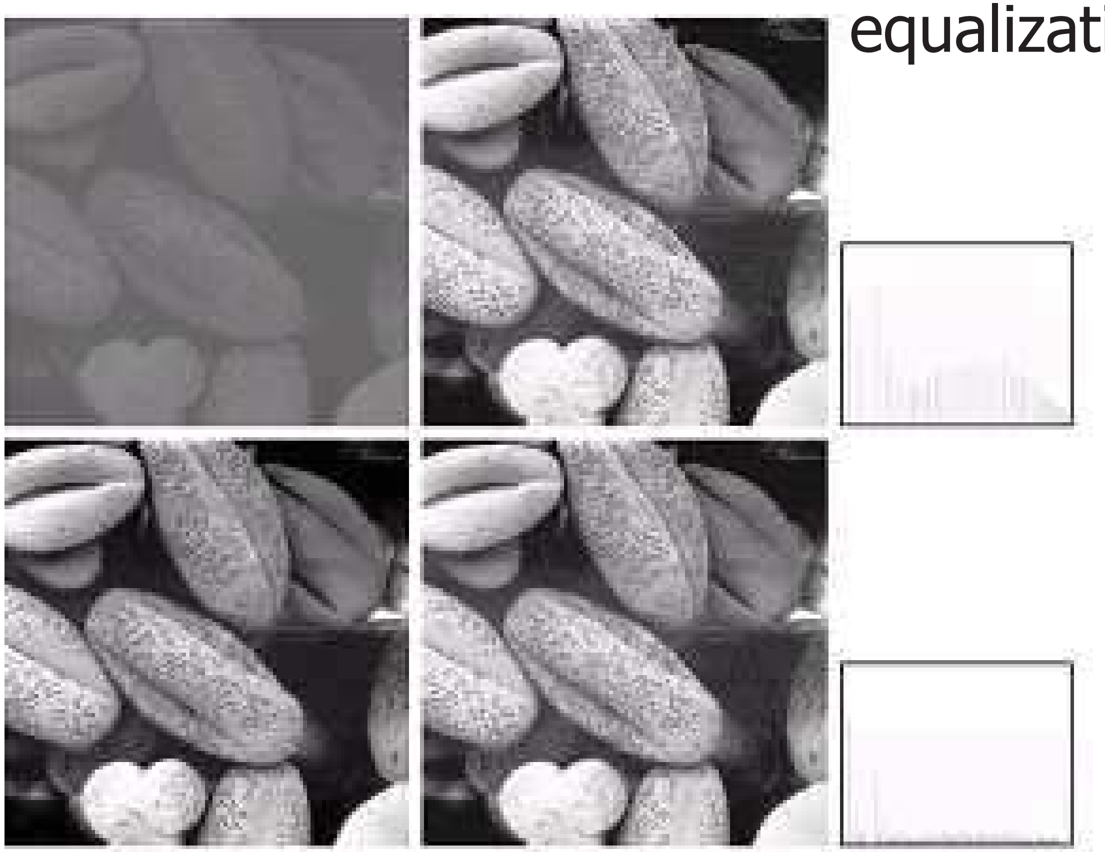
- Thus, an output image is obtained by mapping each pixel with level r_k in the input image into a corresponding pixel with level s_k in the output image
- In discrete space, it cannot be proved in general that this discrete transformation will produce the discrete equivalent of a uniform probability density function, which would be a uniform histogram

Example



Example

before after Histogram equalization



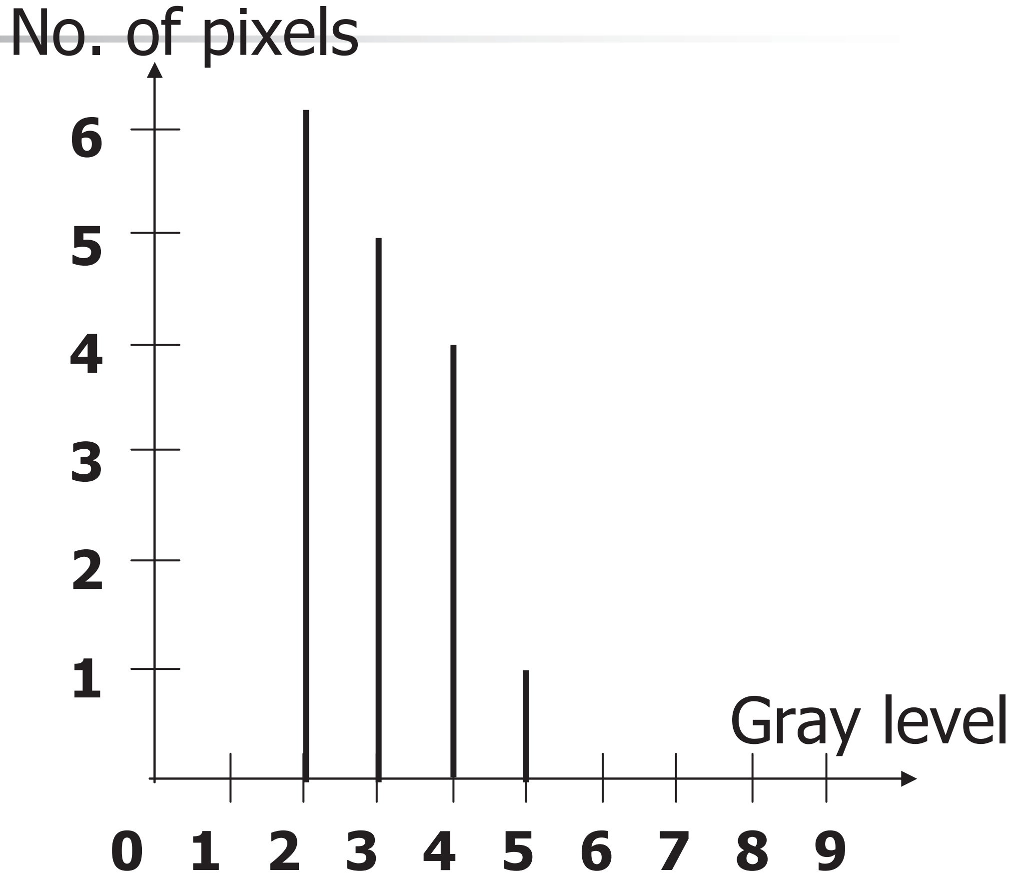
The quality is not improved much because the original image already has a broaden gray-level scale

Example

2	3	3	2
4	2	4	3
3	2	3	5
2	4	2	4

4x4 image

Gray scale = [0,9]



histogram

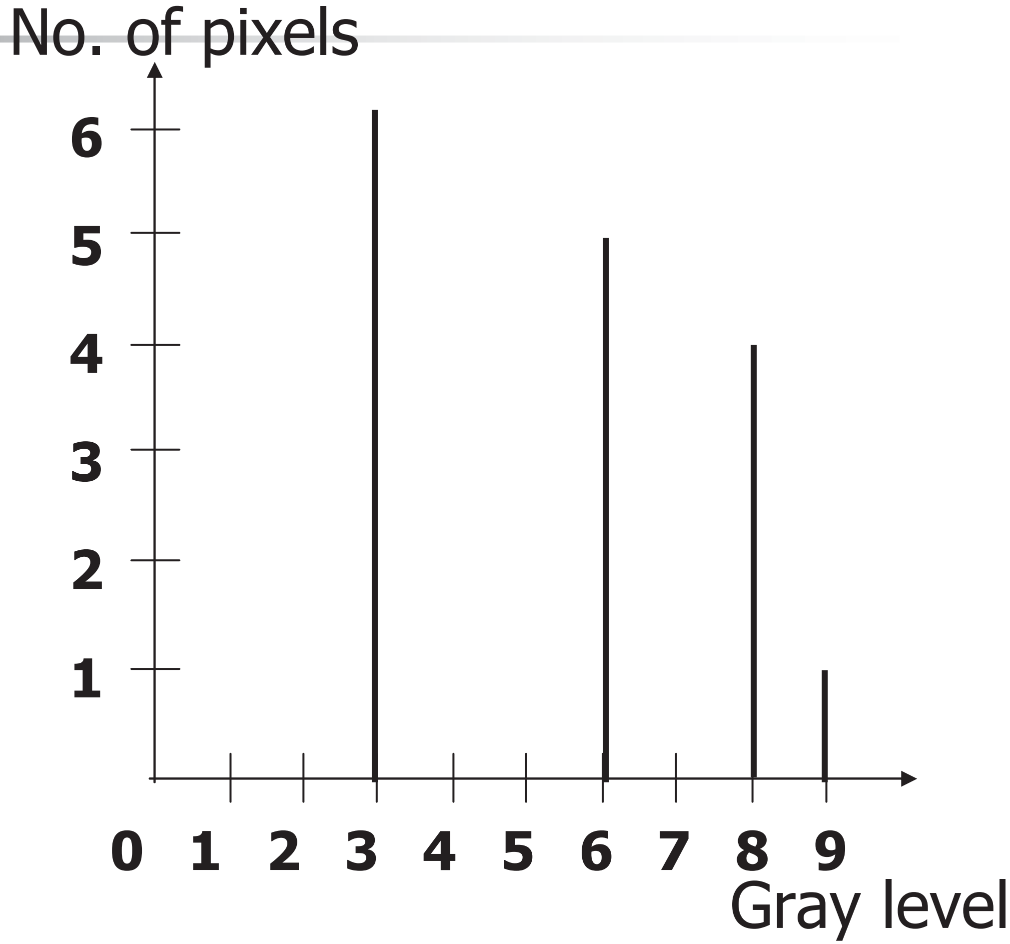
Gray Level(j)	0	1	2	3	4	5	6	7	8	9
No. of pixels	0	0	6	5	4	1	0	0	0	0
$\sum_{j=0}^{k} n_j$	0	0	6	11	15	16	16	16	16	16
$s = \sum_{j=0}^{k} \frac{n_j}{n}$	0	0	6 / 16	11 / 16	15 / 16	16 / 16	16 / 16	16 / 16	16 / 16	16 / 16
$s \times 9$	0	0	3.3 ≈3	6.1 ≈6	8.4 ≈8	9	9	9	9	9

Example

3	6	6	3
8	3	8	6
6	3	6	9
3	8	3	8

Output image

Gray scale = [0,9]



Histogram equalization

Note

- It is clearly seen that
  - Histogram equalization distributes the gray level to reach the maximum gray level (white) because the cumulative distribution function equals 1 when $0 \le r \le L-1$
  - If the cumulative numbers of gray levels are slightly different, they will be mapped to little different or same gray levels as we may have to approximate the processed gray level of the output image to integer number
  - Thus the discrete transformation function can't guarantee the one to one mapping relationship

Histogram Matching (Specification)

- Histogram equalization has a disadvantage which is that it can generate only one type of output image.
- With Histogram Specification, we can specify the shape of the histogram that we wish the output image to have.
- It doesn't have to be a uniform histogram

Consider the continuous domain

Let $p_r(r)$ denote continuous probability density function of gray-level of input image, r

Let $p_z(z)$ denote desired (specified) continuous probability density function of gray-level of output image, z

Let s be a random variable with the property

$$s = T(r) = \int_0^r p_r(w)dw$$ Histogram equalization

Where w is a dummy variable of integration

Next, we define a random variable z with the property

$$g(z) = \int_0^z p_z(t)\,dt = s$$

Histogram equalization

Where t is a dummy variable of integration

thus

$$s = T(r) = G(z)$$

Therefore, z must satisfy the condition

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

Assume G^{-1} exists and satisfies the condition (a) and (b)

We can map an input gray level r to output gray level z

Procedure Conclusion

1. Obtain the transformation function T(r) by calculating the histogram equalization of the input image

$$s = T(r) = \int_0^r p_r(w)dw$$

2. Obtain the transformation function G(z) by calculating histogram equalization of the desired density function

$$G(z) = \int_0^z p_z(t)dt = s$$

Procedure Conclusion

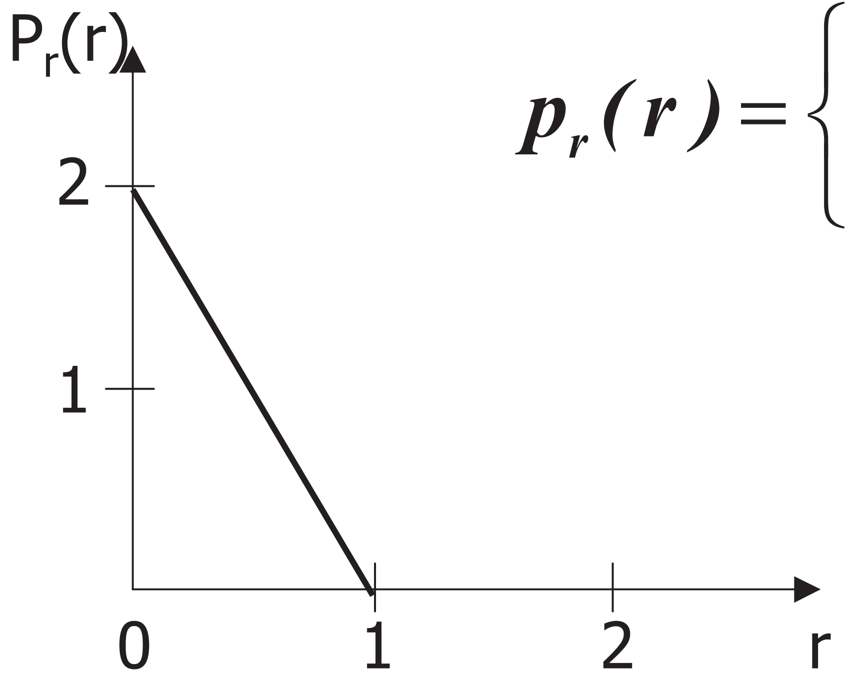
3. Obtain the inversed transformation function G^{-1}

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

4. Obtain the output image by applying the processed gray-level from the inversed transformation function to all the pixels in the input image

Example

Assume an image has a gray level probability density function $p_r(r)$ as shown.

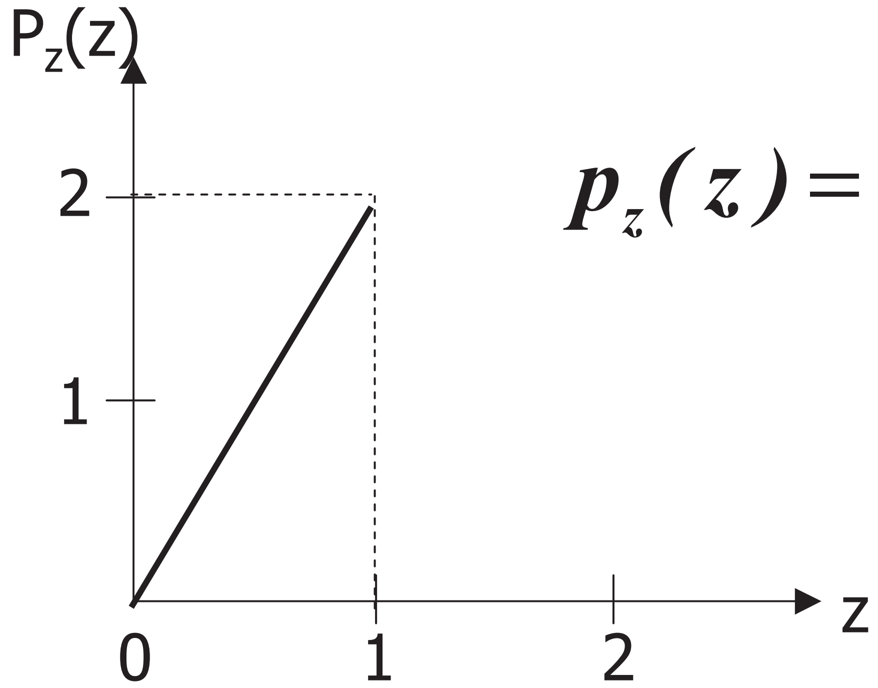


$$p_r(r) = \begin{cases} -2r + 2 & ;0 \le r \le 1 \\ 0 & ;\text{elsewhere} \end{cases}$$

$$\int_0^r p_r(w)\,dw = 1$$

Example

We would like to apply the histogram specification with the desired probability density function $p_z(z)$ as shown.

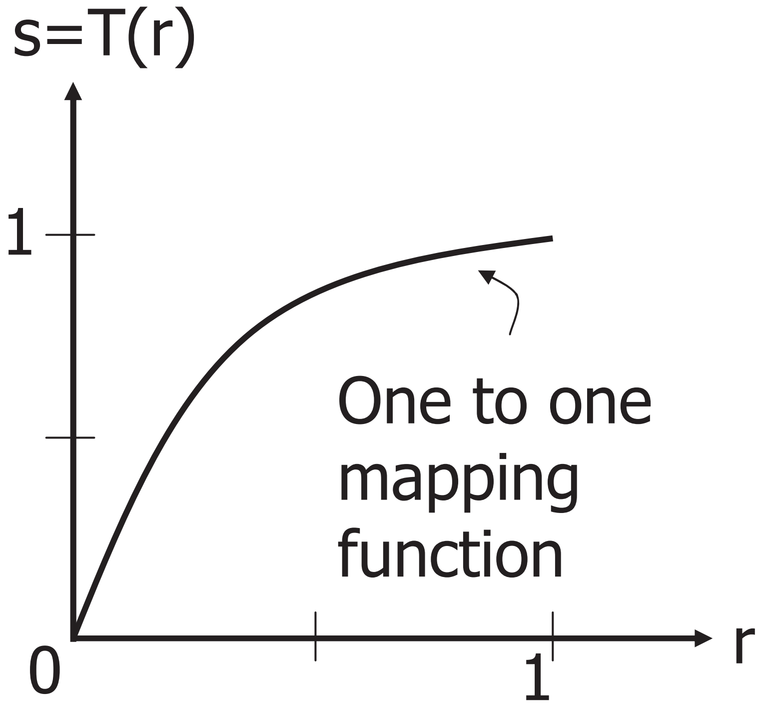


$$p_z(z) = \begin{cases} 2z & ;0 \le z \le 1 \\ 0 & ;\text{elsewhere} \end{cases}$$

$$\int_0^z p_z(w)\,dw = 1$$

Step 1:

Obtain the transformation function T(r)



$$s = T(r) = \int_0^r p_r(w)dw$$

$$= \int_0^r (-2w + 2)dw$$

$$= -w^2 + 2w\Big|_0^r$$

$$= -r^2 + 2r$$

Step 2:

Obtain the transformation function G(z)

$$G(z) = \int_0^z (2w)dw \quad = z^2\Big|_0^z \quad = z^2$$

Step 3:

Obtain the inversed transformation function G^{-1}

$$G(z) = T(r)$$

$$z^2 = -r^2 + 2r$$

$$z = \sqrt{2r - r^2}$$

We can guarantee that $0 \le z \le 1$ when $0 \le r \le 1$

Discrete formulation

$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j)$$

$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad k = 0,1,2,\ldots,L-1$$

$$G(z_k) = \sum_{i=0}^{k} p_z(z_i) = s_k \qquad k = 0,1,2,\ldots,L-1$$

$$z_k = G^{-1}\left[T(r_k)\right]$$

$$= G^{-1}\left[s_k\right] \qquad k = 0,1,2,\ldots,L-1$$