Pbme 1 : VERY NICE PROBLEM ABOUT PROPERTIES OF EIGENVALUES / EIGENVECTORS

1) Just the definition of eigenvalues / eigenvectors

$Ax = x \longrightarrow \lambda = 1$ is an eigenvalue for the eigenvector x.

$Ay = -y \longrightarrow \lambda = -1$ is an eigenvalue for the eigenvector y.

Since A is a 2×2 matrix, A has two eigenvalues only : so the eigenvalues of A are $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$.

2) There are different ways of solving this question.

let $A = \begin{bmatrix} 2 & a \\ 1 & b \end{bmatrix}$

we have to find (a, b) from what we know about the matrix A.

First way

we know the eigenvalues of A satisfy $\det(A - \lambda I) = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & a \\ 1 & b-\lambda \end{vmatrix} = (2-\lambda)(b-\lambda) - a = 0$$

For $\lambda = 1$, we get $b - 1 - a = 0 \Rightarrow b = a + 1$

For $\lambda = -1$, we get $3(b+1) - a = 0 \Rightarrow 3b - a = -3$

we finally solve those 2 equations and find $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

Second way (use properties of eigenvalues)

we know $\lambda_1 + \lambda_2 = \text{trace}(A) \iff 2 + b = 0 \quad (1-1=0$

$$\Rightarrow \quad \underline{b = -2}$$

we know $\lambda_1 \cdot \lambda_2 = \det(A) \iff -1 = 2b - a$

$$\Rightarrow \quad \underline{a = -3}$$

other ways

write $Ax = x$ and $Ay = -y$ and solve for (a, b) using the fact that $x \neq 0$ et $y \neq 0$.

$\Rightarrow$ In all cases, there is only one matrix

$$\boxed{A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}$$

NOTE: Once you have found a matrix A, you should at least verify that it satisfies the given description: $\lambda = 1$, $\lambda = -1$ are eigenvalues.

↳ Just checking your result would have avoided many mistakes.

3) we have to diagonalize A.

– we already know the eigenvalues: $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$

NOTE: If here you calculate the eigenvalue of A and find something different from Question 1, then you should know there is something wrong!

We need to find the eigenvectors. wich are in $N(A-\lambda I)$.

- for $\underline{\lambda_1 = 1}$

$$A - \lambda_1 I = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix},$$ so a vector in $N(A-\lambda I)$ is $x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- For $\underline{\lambda_2 = -1}$

$$A - \lambda_2 I = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix},$$ so a vector in $N(A-\lambda I)$ is $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>NOTE</u> A<u>lways</u> check that $\begin{cases} A x_1 = \lambda_1 x_1 \\ A x_2 = \lambda_2 x_2 \end{cases}$!

Therefore we can diagonalize A as $A = S \Lambda S^{-1}$

with $S = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$

4) <u>NOTE</u>: No much calculations needed to find that $A^{101} = A$.

We know, $A^k = S \Lambda^k S^{-1}$, so $A^{101} = S \Lambda^{101} S^{-1}$,

and $$\Lambda^{101} = \begin{pmatrix} 1^{101} & 0 \\ 0 & (-1)^{101} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda$$

So $A^{101} = S \Lambda S^{-1} = A$.

In fact, every odd power of A is A

every even power of A is Identity.

(This is a very special matrix A !)

<u>NOTE</u>: you can of course do the calculation (<u>correctly</u>)

and find $A^{101} = A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$.

5) NOTE : we want a proof that x and y are independent vectors in the general case.

So :

- taking the two eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and saying there are independent (which is true) is not a general proof. !

- just saying the vectors are independent without showing why (pourquoi ?) is not enough !

Different possible proofs (All use the definition of independence)

If x and y are linearly independent, then the only solution to $\alpha x + \beta y = 0$ is the trivial solution $\alpha = \beta = 0$

let $\alpha x + \beta y = 0$. (1)

Then $A(\alpha x + \beta y) = 0 \iff \alpha Ax + \beta Ay = 0$

$\iff \alpha x - \beta y = 0$ (2)

(Because $Ax = x$ and $Ay = -y$)

we solve the two equations (1) and (2)

$$\begin{cases} \alpha x + \beta y = 0 \\ \alpha x - \beta y = 0 \end{cases} \longrightarrow Mz = 0 \text{ with } M = \begin{bmatrix} \alpha & \beta \\ \alpha & -\beta \end{bmatrix}$$

and $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

Way 1

$$M = \begin{bmatrix} \alpha & \beta \\ \alpha & -\beta \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha & \beta \\ 0 & 2\beta \end{bmatrix} \rightarrow \text{rank}(M) = 2 \text{ . and}$$

the unique solution to $Az = 0$ is $z = 0$ which is impossible since $x = 0$ and $y \neq 0$.

Way 2

$$\begin{array}{l} \alpha x + \beta y = 0 \\ \alpha x - \beta y = 0 \\ \hline 2\alpha x = 0 \end{array}$$

and $2\alpha \cancel{x} = 0 \Rightarrow \alpha = 0$ since $x \neq 0$

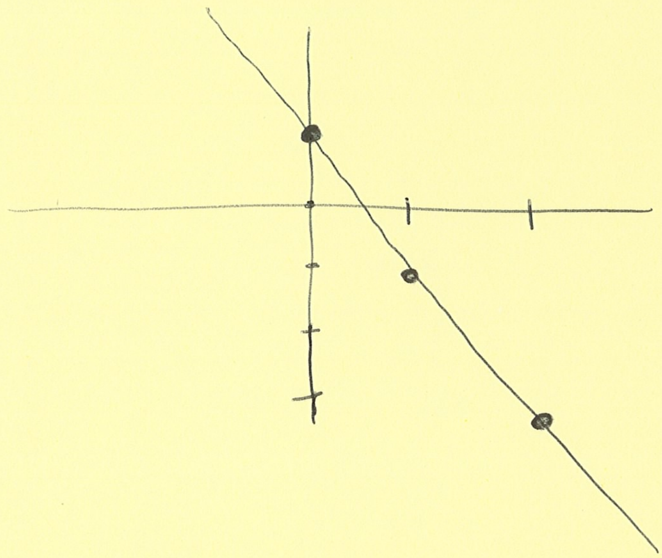
then $\alpha x + \beta y = 0 \Rightarrow \beta = 0$.

VOILA, DONE !

Problem 2

X	0	1	2
Y	1	-1	-3

NOTE : you should always draw the points and see. If you find a line, draw it to check your result !



Oh my God ! it happens here that the 3 points are perfectly aligned, so the fit will be exact, not approximate.

let write the equations :

$$1 = A + B \cdot 0$$
$$-1 = A + B \cdot 1$$
$$-3 = A + B \cdot 2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \quad \text{or} \quad M z = c.$$

you can solve $Mz = c$ in two ways :

a) If you've had the good idea of checking that the point are aligned, then you know you can solve $Mz = c$ $(c \in C(M))$, directly using elimination method.

b) you can use LLS technique : $\hat{z} = (M^T M)^{-1} M^T c$

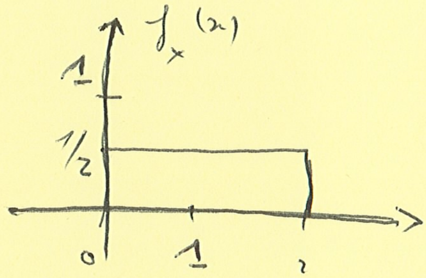
BOTH techniques give the exact same result

$$\boxed{y = 1 - 2x}$$

Problem 3

1) From our course notes, we know the pdf of a uniform random variable.

$$f_X(x) = \begin{cases} 1/2 & \text{if } \; 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



Either from the notes (with $a = 0$ and $b = 2$) or using the definition of mean and variance.

$$E[X] = \frac{b-a}{2} = \frac{2-0}{2} = 1$$

$$Var(X) = \frac{(b-a)^2}{12} = \frac{4}{12} = \frac{1}{3}$$

or $$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^2 \frac{1}{2} x\, dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = \underline{1}$$

and $$Var(X) = E[X^2] - E[X]^2 = \int_0^2 x^2 f_X(x)\, dx - \underline{1}^2$$

$$= \int_0^2 \frac{1}{2} x^2\, dx - 1 = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 - 1$$

$$= \frac{4}{3} - 1 = \frac{1}{3}.$$

2) Mean and variance of XY.

$$E[XY] = \iint xy\, f_{XY}(x,y)\, dx\, dy = \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

(Because X and Y are independent, $f_{XY}(x,y) = f_X(x)\, f_Y(y)$)

So $$E[XY] = \int_0^2 \int_0^2 xy\, \frac{1}{2} \cdot \frac{1}{2}\, dx\, dy$$

$$= \frac{1}{4} \int_0^2 \left[\frac{x^2}{2}\right]_0^2 y \cdot dy = \frac{1}{4} \cdot \frac{4}{2} \left[\frac{y^2}{2}\right]_0^2 = \frac{1}{2} \cdot \frac{4}{2}$$

$$\boxed{E[XY] = 1}$$

Or (easiest way) $E[XY] = E[X]\,E[Y]$ as X and Y are independent.

So $E[XY] = 1 \cdot 1 = 1$.

NOTE For the variance, we cannot use $Var(XY) = Var(X) \times Var(Y)$ even if X and Y are independent ⚠

We need to use the definition $Var(XY) = E[(XY)^2] - E[XY]^2$

where $E[(XY)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)^2 f_{XY}(x,y)\,dx\,dy$

$$= \iint x^2 y^2 f(x) f(y)\,dx\,dy$$

$$= \int_0^2 x^2 f(x)\,dx \int_0^2 y^2 f(y)\,dy$$

$$= \frac{4}{3} \cdot \frac{4}{3} = \frac{16}{9}$$

So $Var(XY) = \frac{16}{9} - 1 = \frac{7}{9}$.

3) "Most difficult" question of all exam :)

NOTE: $P(XY \geq 1) = P(X \geq 1)\,P(Y \geq 1)$ because X and Y are independent is a WRONG answer!

X and Y being independent only tells you that the joint distribution $f_{XY}(x,y) = f_X(x)\,f_Y(y)$ and that all!

Moreover, if $X = \frac{3}{4}$ (which is $X < 1$) and $Y = 2$

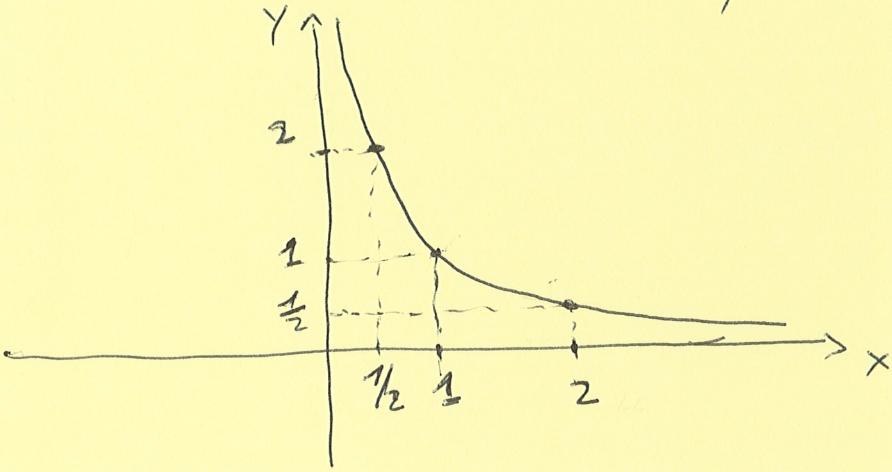
then $XY \geq 1$ even if $X < 1$ ⚠

Correct answer

$$P(xy \geq 1) = P\left(x \geq \frac{1}{y}\right).$$

Now, over which area should we take the integral?

Let draw the curve $x = \frac{1}{y}$



So for $\begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{cases}$

if we want $\frac{1}{y} \leq x$

the the area of interest is

$$\begin{cases} \frac{1}{2} \leq x \leq 2 \\ \frac{1}{2} \leq y \leq 2 \end{cases}$$

Then $P(xy \geq 1) = P\left(x \geq \frac{1}{y}\right)$

$$= \int_{1/2}^{2} \int_{\frac{1}{y}}^{2} f_{XY}(x,y)\, dx\, dy$$

$$= \frac{1}{4} \int_{1/2}^{2} \int_{\frac{1}{y}}^{2} dx\, dy$$

$$= \frac{1}{4} \int_{1/2}^{2} \left(2 - \frac{1}{y}\right) dy$$

$$= \frac{1}{4} \Big[2y - \ln(y) \Big]_{\frac{1}{2}}^{2}$$

$$\boxed{P(xy \geq 1) = \frac{1}{4}(3 - \ln 4)} \simeq 0,4034.$$