Diagnostic Medical Image Processing

Reconstruction

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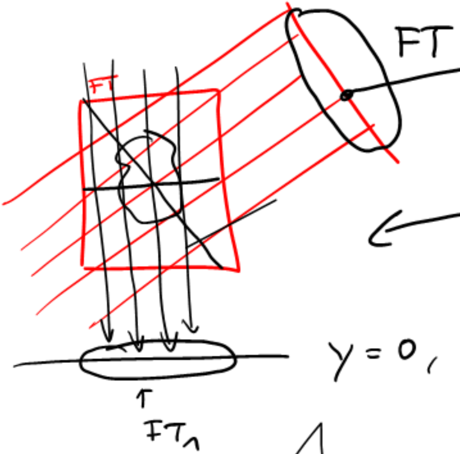
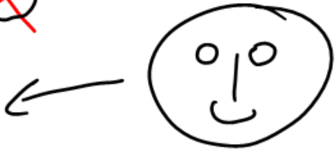
$$-\ln \frac{I}{I_0} = \int f(\vec{x})\, d\vec{x}$$

$\vec{x} \in L$





$y = 0, \theta$

Reconstruction from X-Ray Projections

Computerized Tomography

- $\tau o \mu o \sigma$ = tomos = slice
- reconstruction of functions from line integrals:
  - transmission CT: radiology
    - computer tomography: (mostly) diagnostic radiology
    - 3-D angiography: (mostly) interventional radiology
  - emission CT: nuclear medicine
    - PET: positron emission tomography
    - SPECT: single particle emission tomography
  - ultrasound tomography

X-Ray Attenuation Law

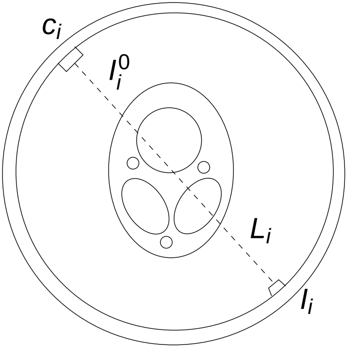


Figure: Acquisition scenario

- X-ray focus c_i of the i-th image,
- projection beam L_i of the i-th view,
- I_i^0 initial intensity, and
- I_i measured intensity

Core problem in CT: Compute the original function from line integrals measured by I_i, where c_i's are usually on a circle.

Fan Beam Acquisition Geometry

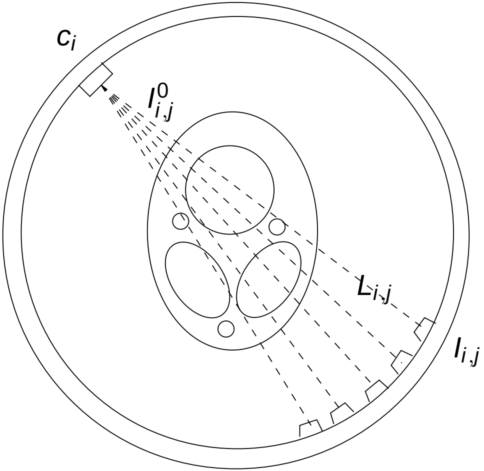


Figure: CT reconstruction: many rays $L_{i,j}$, $j = 1, 2, \ldots$ with single focus c_i

X-Ray Attenuation Law

For the j–th X-ray beam the observed intensity of the i–th view is

$$I_{i,j} = I_{i,j}^{0} \exp\left(-\int_{L_{i,j}} f(x(l),y(l))\, dl\right) \quad ,$$

where

- $I_{i,j}^{0}$ is original intensity (no object),
- $L_{i,j}$ is the j-th line of the X-ray beam of the i-th view, and
- $f(x(l),y(l))$ is the 2–D function value (density) of X-rayed object at $(x(l),y(l))$.

System of Equations

For N_v views ($i = 1, 2, \ldots, N_\mathrm{v}$) and N_t samples ($j = 1, 2, \ldots, N_\mathrm{t}$) in each view we get the following set of integral equations:

$$\log \frac{I_{i,j}}{I^0_{i,j}} = -\int_{L_{i,j}} f(x(l), y(l))\, dl, \text{ for } i = 1, \ldots, N_\mathrm{v} \text{ and } j = 1, \ldots, N_\mathrm{t}.$$

Solution of System of Integral Equations

Different ways to approach this problem:

- discretize at the end: solve integral equation using continuous methods
  (*analytic approach*)
- discretize upfront: discrete version of the integral results in a sum
  (*algebraic approach*)

Two types of algorithms dominate current applications:

- filtered backprojection (FBP)
- algebraic reconstruction technique (ART) and variants

Key Problems in CT

In particular computerized tomography requires the solution of the following **key problems**:

1. *Geometric calibration:*
   Compute the paths $L_{i,j}$, because we are required to know which path the X-ray particles are propagated through the body.
2. *Photometric calibration:*
   Compute the intensities $I^0_{i,j}$ to get the intensity you get if no body is in between the source and the detector.
3. *Numerical solution of the integral equations*
   Compute efficiently and robustly the original function from line integrals.

Key problem 1 and 3 will be considered in this lecture in more detail.

A Note on Key Problems

1. *Geometric calibration:*
   The early CT systems provided an precise acquisition geometry by mechanical construction. More recent 3-D CT system make use of geometric calibration techniques developed in computer vision.
2. *Photometric calibration:*
   The intensities $I^0_{i,j}$ are easily measured while setting up the system.
3. *Numerical solution of integral equations:*
   The Radon inversion formula is known since 1917, but requires further work to come up with a robust and efficient solution to the reconstruction problem. The working horse in practice is still filtered backprojection. In 2002 reconstruction algorithms have experienced a tremendous boost by the ideas of Alexander Katsevich for exact cone beam reconstruction.

Diagnostic Medical Image Processing

Industrial CT

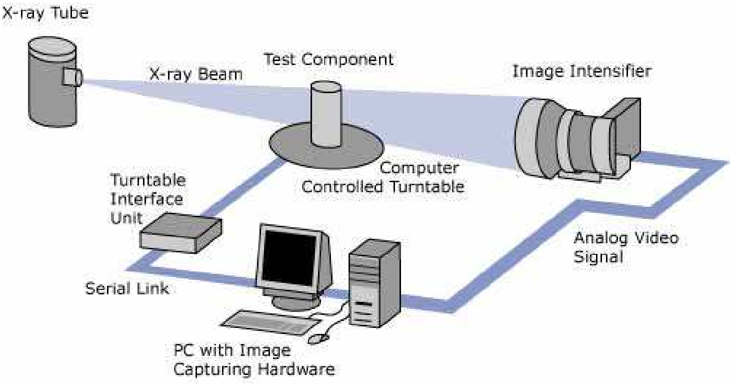


Figure: Principle of Computed Tomography: Rotate Object

Medical CT

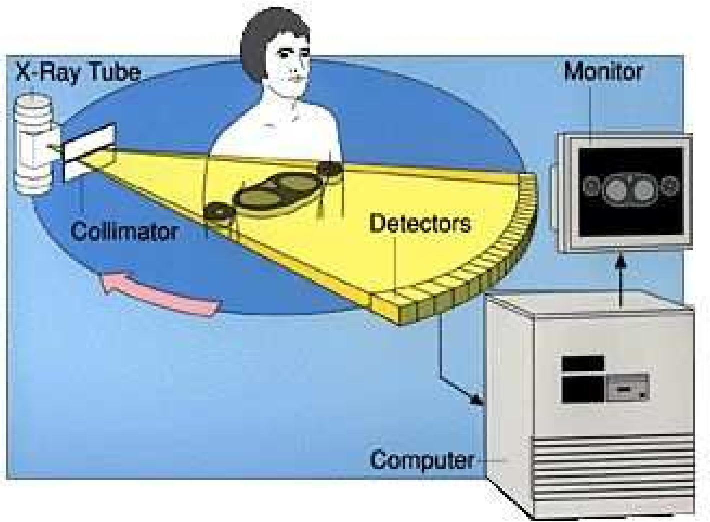


Figure: Principle of Computed Tomography: Rotate Equipment

3-D Computerized Tomography

Recent developments in

- detector technology,
- X-ray tube technology,
- computational power of computers, and
- algorithms for 3-D reconstruction

are the main driving forces in the innovation of 3-D reconstruction systems since the late 90-ies of the last century.

Highlight I: 3-D Reconstruction in dual CT

Figure: Dual source CT introduced in 2005 by Siemens Healthcare (image: Siemens Healthcare)

Highlight II: 3-D Reconstruction using 320 Detector Rows

Figure: November 2007: Toshiba introduces a 320 slice scanner (image: Toshiba)

Highlight III: 3-D Reconstruction in Dental Medicine

Figure: October 2006: Dental 3D reconstruction system (image: http://www.planmeca.com)

Highlight IV: 3-D Reconstruction in Angiography

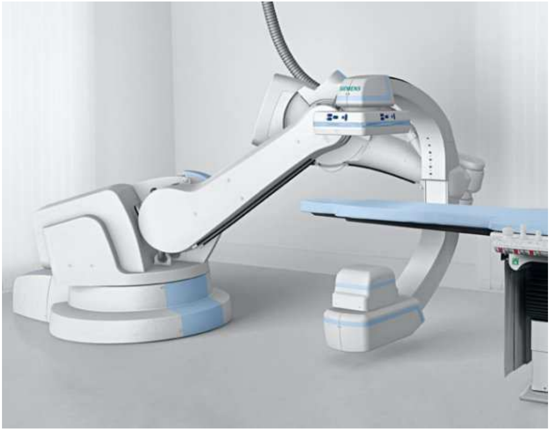


Figure: November 2007: C-arm controlled by a robot arm (image: Siemens Healthcare)

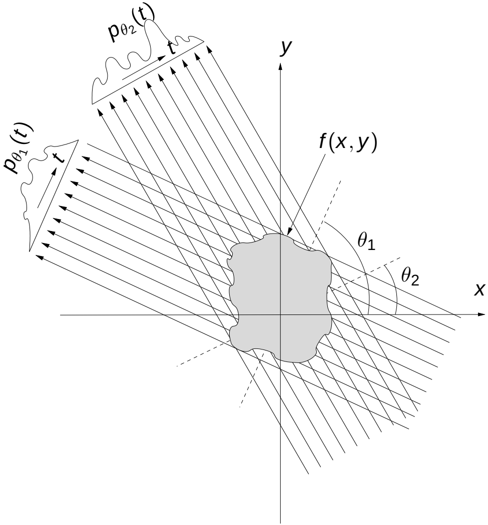
3-D Reconstruction in Angiography

Figure: Standard bi-plane C-arm device (image: Siemens Healthcare)

Computer Tomography System

Figure: Standard CT system (image: Siemens Healthcare)

Parallel X-Ray Projections



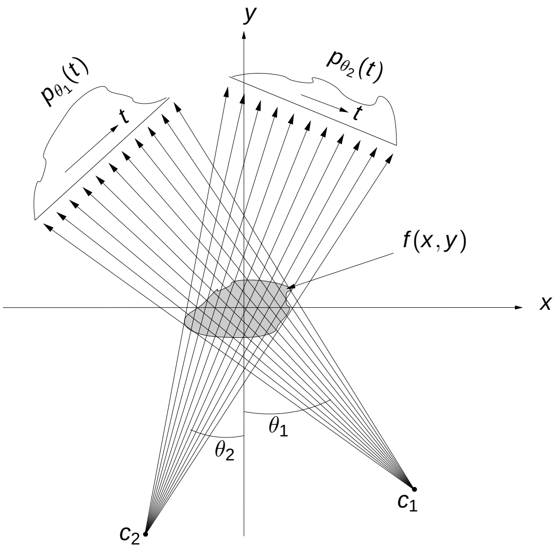
- $p_\theta(t)$: projection under viewing angle θ
- $f(x,y) = f(\boldsymbol{x})$: 2D slice

X- Ray Tube and Perspective Projection

Projection model for X-ray systems:

- X-rays have a focal spot
- all projection rays intersect in the focal spot
- X-ray projection shall be modeled by perspective rather than parallel projection
- perspective projection is a non-linear mapping

Fan Beam Acquisition Geometry



Cone Beam Acquisition Geometry

In recent years the number of detector lines is increasing. Cone beam acquisition geometry became standard.

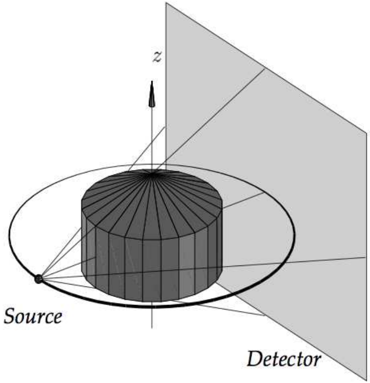


Figure: Circle trajectory: Illustration of cone beam geometry (image: H. Turbell)

Cone Beam Acquisition Geometry

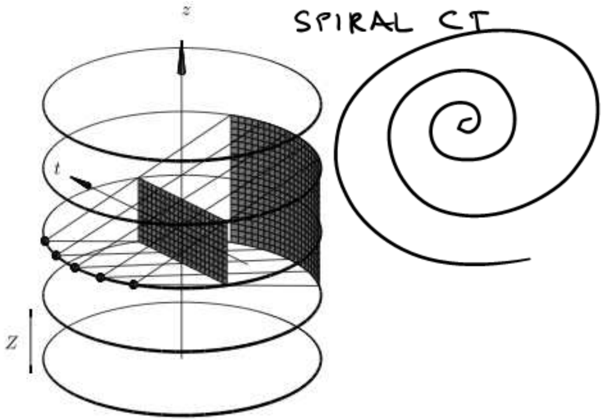
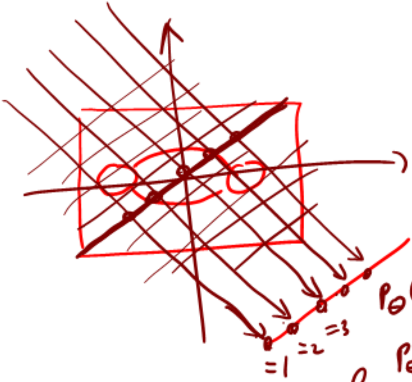


Figure: Helix trajectory: Illustration of cone beam geometry (image: H. Turbell)

Diagnostic Medical Image Processing

Fourier Slice Theorem

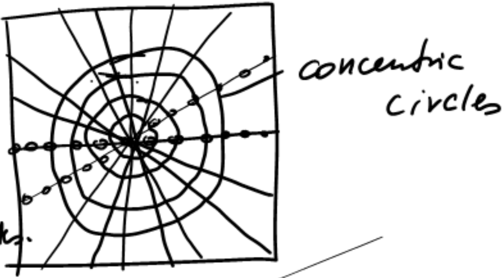
FST



$$p_\theta(t) = I_0 e^{-\int_{L_t} f(\vec{x}(\tau))\,d\tau}$$

$$-\log \frac{p_\theta(t)}{I_0} = \boxed{\int_{L_t} f(\vec{x}(\tau))\,d\tau}$$

We get the 2D FT in polar coordinates.



Sinogram

Definition

The projections $p_\theta(t)$ can be considered as bivariate functions in θ and t. The resulting image in (θ, t) is called *sinogram*.

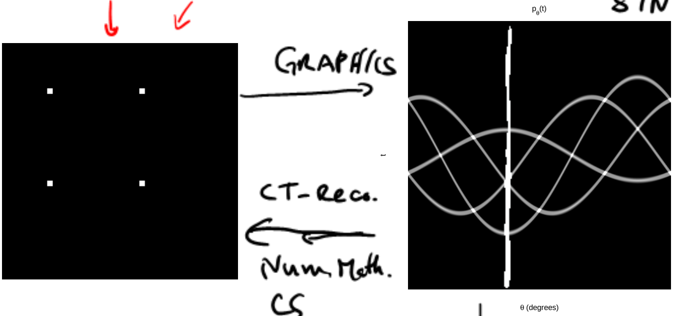


Figure: Intensity image showing four points (left) and the resulting sinogram (right)

Sinogram

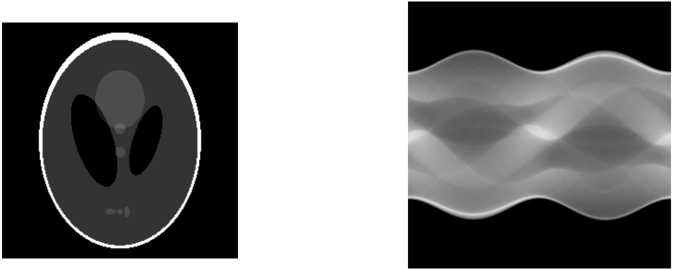


Figure: Shepp Logan Phantom (left) and the resulting sinogram (right)

Sinogram

... and, of course, you also can do some fun stuff:

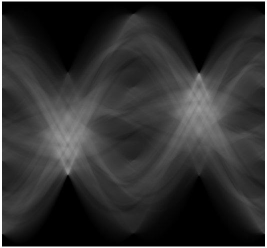


Figure: A portrait of Conrad Wilhelm Röntgen and the computed sinogram

In simple terms we are now looking for an algorithm that allows us to compute the transform from sinogram to the original image.

Fourier Slice Theorem

Definition

The *Radon transform* in 2 dimensions is an integral transform where the function is integrated over the set of all straight lines. If the line is represented by $d = x\cos\theta + y\sin\theta$, then the Radon transform of $f(x, y) \in \mathbb{R}$ is

$$\mathcal{R}[f](\theta, d) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - d)\, dxdy$$

The *inverse Radon transform* maps the line integrals to the original function.

Fourier Slice Theorem

Definition

In the n-dimensional space the integration is not done over straight lines but hyperplanes.

Since we propagate X-rays through material on straight lines we make use of the following definition:

Definition

The integrals of an n-dimensional function along all n-dimensional straight lines is called X-ray transform.

Fourier Slice Theorem

Let $p_0(t)$ be the projection (line integral) of the bivariate function $f(x, y)$ along the y-axis:

$$p_0(t) := \int_{-\infty}^{+\infty} f(t, y)\, dy \qquad (1)$$

Now we define a 2-D rotation by the angle θ around the origin of the coordinate system by the rotation matrix:

$f(x,y) = f(\vec{x})$

$$\boldsymbol{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad (2)$$

We consider the Radon transform $p_\theta(t)$ of the rotated 2-D function $f_\theta = f \circ \boldsymbol{R}_\theta$.

$f(R_\theta \vec{x})$

Radon Transform and Sinogram

The 2-D Fourier transform of $f(x, y)$ is:

$\vec{x} = \binom{x}{y}, \quad \vec{u} = \binom{\xi}{\eta}$

$$FT_2(f)(\xi, \eta) = \int_{\mathbb{R}^2} \underline{f(x, y)}\, \underline{\exp\left(-2\pi i (x\xi + y\eta)\right)}\, dx\, dy \qquad (3)$$

In the first step, we consider the relationship between the Fourier transform of f and $f \circ \mathbf{R}_\theta$. The Fourier transform of the rotated function $f(\mathbf{R}_\theta \mathbf{x})$:

$R^{-1} = R^T$

$\int f(\vec{x})\, e^{-2\pi i\, \vec{x}^T \vec{u}}\, d\vec{x}$

$$\begin{aligned}
FT_2(f \circ \mathbf{R}_\theta)(\mathbf{u}) &= \int_{\mathbb{R}^2} f(\mathbf{R}_\theta \mathbf{x}) \exp\left(-2\pi i \mathbf{x}^\mathsf{T} \mathbf{u}\right) dx\, dy \\
&\overset{\mathbf{x}=\mathbf{R}_{-\theta}\mathbf{x}'}{=} \int_{\mathbb{R}^2} f(\mathbf{R}_\theta \mathbf{R}_{-\theta} \mathbf{x}') \exp\left(-2\pi i \mathbf{u}^\mathsf{T} \mathbf{R}_{-\theta} \mathbf{x}' \rangle\right) \underbrace{|\mathbf{R}_{-\theta}|}_{=1} d\mathbf{x}' \\
&= \int_{\mathbb{R}^2} f(\mathbf{x}') \exp\left(-2\pi i (\mathbf{R}_\theta \mathbf{u})^\mathsf{T} \mathbf{x}'\right) d\mathbf{x}' \\
&= (FT_2(f) \circ \mathbf{R}_\theta)(\mathbf{u})
\end{aligned}$$

$u^T R_\theta^T = u^T R_{-\theta}$

Radon Transform and Sinogram

In these formulas we have used vector notation:

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and $|\boldsymbol{R}_{-\theta}|$ is the determinant of the Jacobian of the coordinate transform.

Next steps:
In the following we consider the Fourier transform of the projections and derive a relationship between 1-D and 2-D Fourier transform that is the key to solve the considered reconstruction problem.

Fourier Slice Theorem

We compute the 1-D Fourier transform $FT_1(p_0)(\xi)$ of the 1-D projection $p_0(t)$:

$$
\begin{aligned}
FT_1(p_0)(\xi) &= \boxed{\int_{-\infty}^{+\infty} p_0(t) \exp\left(-2\pi i \xi t\right) dt} \\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(t,y) dy \right) \exp\left(-2\pi i \xi t\right) dt \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t,y) \exp\left(-2\pi i (\xi t + 0y)\right) dtdy \\
&= FT_2(f)(\xi, 0)
\end{aligned}
$$

Standard FT

Fourier Slice Theorem

$f(\vec{x}) \quad R\vec{x} \longrightarrow f(R\vec{x}), \quad FT(\vec{u})$ for $R_\theta\begin{pmatrix}\xi\\0\end{pmatrix} \to FT\left(R_\theta\begin{pmatrix}\xi\\0\end{pmatrix}\right)$

Using the observation about rotated Fourier transforms, we conclude

$$\begin{aligned} FT_1(p_\theta)(\xi) &= FT_2(f \circ \boldsymbol{R}_\theta)(\xi, 0) \\ &= (FT_2(f) \circ \boldsymbol{R}_\theta)(\xi, 0) \\ &= FT_2(f)(\boldsymbol{R}_\theta(\xi, 0)^\mathsf{T}). \end{aligned}$$

Theorem

Fourier Slice Theorem*: The Fourier transform of the 1-D projection is equal to the 2-D Fourier transform of the original function along the line through the origin that is parallel to the projection resp. orthogonal to the projection direction.*

Fourier Slice Theorem

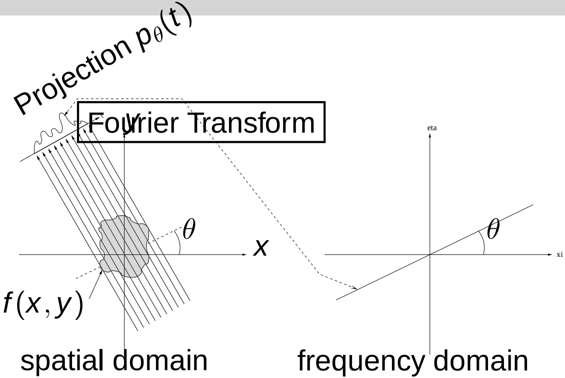


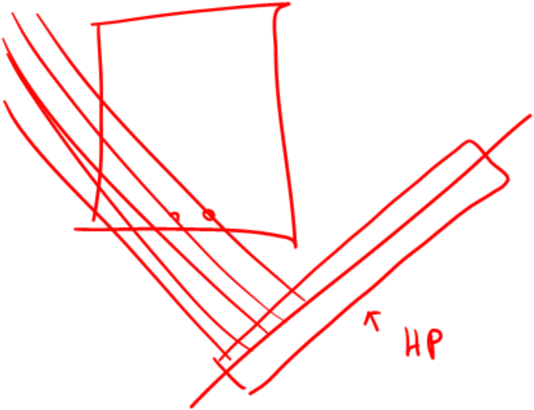
Figure: Illustration of the Fourier Slice Theorem

Note: The projections are nearly independent. The only information shared by all projections is the 2-D Fourier transform at (0,0).

Fourier Slice Theorem

A few comments on the Fourier Slice Theorem:

- It is rather surprising that there is such a remarkably simple relationship between 1-D and 2-D Fourier transform.
- The 1-D Fourier transform of the projections allows for sampling the 2-D Fourier transform of the function to be reconstructed.
- The Fourier Slice Theorem is still the base for mostly all commercially available CT scanners.
- Due to the Fourier Slice Theorem, it is obvious that in the presence of a parallel projection model at least a π-rotation around the object is required for object reconstruction.
- The straightforward application of the Fourier transform is prohibited due to the fact that it results in a 2-D Fourier transform sampled in polar coordinates.



HP

Sampling in Frequency Domain

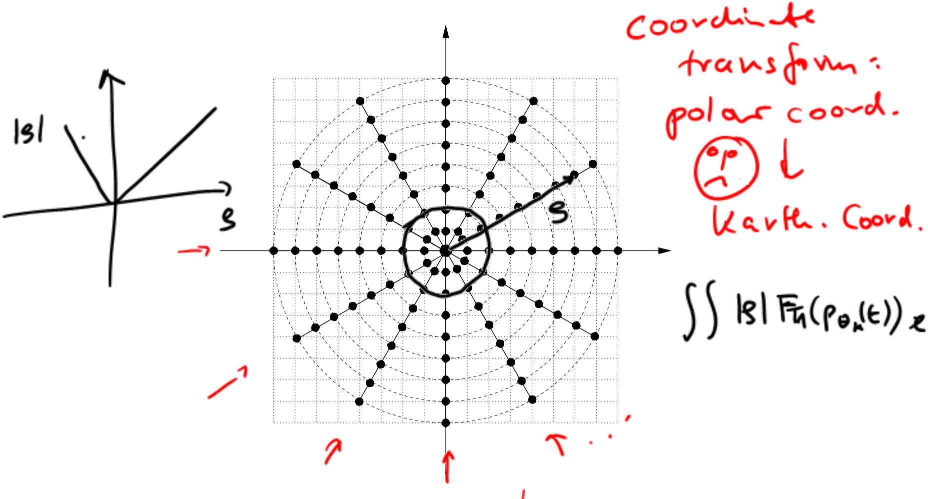


Figure: 2-D Fourier transform is sampled in polar coordinates

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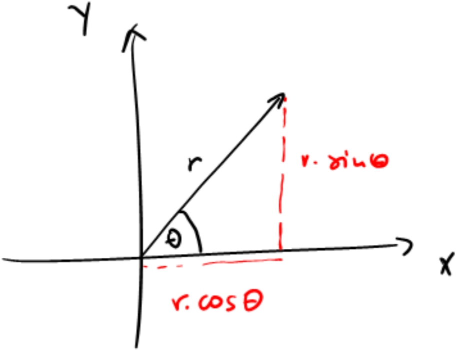

Y
r
θ
r. sinθ
X
r. cosθ

<u>CT Reconstruction</u>

X-ray projection

$L_t \quad p_\theta(t) = I_0 \cdot e^{-\int_{L_t} f(\vec{x})\, d\vec{x}}$

θ, t

DMIP

Modalities

SVD

Pre-Processing

X-ray: <u>Distortion</u>

Defect pixels

MR

Proj. Geometries

→ Homog. Coordinates

→ Projection matrix

Filtered Backprojection

The basic idea of Fourier reconstruction is:

- Sample the 2-D Fourier transform of the function to be reconstructed by Fourier transforms of projections
- Apply the inverse 2-D Fourier transform to compute $f(x,y)$.
- Incorporate transform from polar to Cartesian coordinates.

Let $F(\xi,\eta) = FT_2(f)(\xi,\eta)$. The inverse 2-D Fourier transform and thus $f(x,y)$ is computed by:

CT

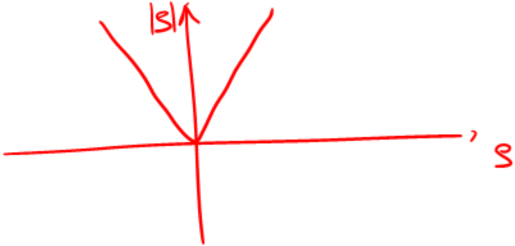
$$
\begin{aligned}
f(x,y) &= FT_2^{-1}(F)(x,y) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\xi,\eta)\exp\left(2\pi i(\xi x + \eta y)\right)\, d\xi\, d\eta .
\end{aligned}
$$

$\xi = \rho\cos\theta \ , \ \eta = \rho\sin\theta$

$$J = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$$

$$|J| = |\rho\cos^2\theta + \rho\sin^2\theta| =$$

$$= |\rho(\cos^2\theta + \sin^2\theta)| = |\rho|$$



Filtered Backprojection

Now we have to incorporate the coordinate transform from polar to Cartesian coordinates. We set:

$$\xi = \varrho \cos\theta \quad \text{and} \quad \eta = \varrho \sin\theta$$

and get

$$\begin{aligned} f(x,y) &= \int_0^{\pi} \int_{-\infty}^{+\infty} |\varrho|\, F(\varrho\cos\theta, \varrho\sin\theta) \exp\left(a(x,y,\varrho,\theta)\right) d\varrho\, d\theta \\ &= \int_0^{2\pi} \int_0^{+\infty} |\varrho|\, F(\varrho\cos\theta, \varrho\sin\theta) \exp\left(a(x,y,\varrho,\theta)\right) d\varrho\, d\theta \end{aligned}$$

where

$$a(x,y,\varrho,\theta) = 2\pi i \, (x\varrho\cos\theta + y\varrho\sin\theta) \quad .$$

Filtered Backprojection

Given the acquisition geometry of parallel projections, the reconstruction problem and the Fourier Slice Theorem, we know that:

$$F(\varrho\cos\theta, \varrho\sin\theta) \quad = \quad FT_1(p_\theta)(\varrho)$$

and thus this yields

$$f(x,y) \quad = \quad \int_0^\pi \int_{-\infty}^{+\infty} |\varrho| \, FT_1(p_\theta)(\varrho) \exp\left(a(x,y,\varrho,\theta)\right) \, d\varrho \, d\theta \quad .$$

The definition of the 1-D Fourier transform now results in:

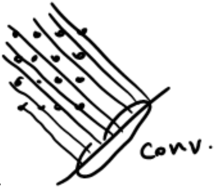
$$f(x,y) \quad = \quad \int_0^\pi \int_{-\infty}^{+\infty} |\varrho| \underbrace{\int_{-\infty}^{+\infty} p_\theta(t) \exp\left(-2\pi i \varrho t\right) \, dt}_{\text{1-D FT}} \cdot$$

$$\exp\left(a(x,y,\varrho,\theta)\right) d\varrho \, d\theta$$

Filtered Backprojection

This integral can be rewritten in form of a convolution followed by an integration:

$$
\begin{aligned}
f(x,y) &= \int_0^\pi \int_{-\infty}^{+\infty} p_\theta(t) \underbrace{\int_{-\infty}^{+\infty} |\varrho| \exp(2\pi i\ \varrho(x\cos\theta + y\sin\theta - t)) d\varrho}_{k(x\cos\theta+y\sin\theta-t)} dt d\theta \\
&= \int_0^\pi (p_\theta \star k)(x\cos\theta + y\sin\theta)\, d\theta
\end{aligned}
$$



where $\star$ denotes the convolution operator, i.e.

$$
(g \star h)(\tau) = \int_t g(t) h(\tau - t)\, dt \quad .
$$

Filtered Backprojection

The function

$$k(x) = \int_{-\infty}^{+\infty} |\varrho| \exp(2\pi i \, \varrho x) d\varrho$$

is called ramp filter due to its shape in Fourier domain.

Using this result, we now have:

Reconstruction = Convolution of input signal with ramp filter kernel followed by numerical integration.

The discrete version of the convolution kernel k is important for the final image quality. In practice there are several choices to approximate the kernel. The kernel basically depends on the tissue class to be reconstructed.

Different Kernels for Convolution

Different discrete convolution kernels ($t = 1 \dots N_\mathrm{t}$) are:

- no filtering (nice try, but never use it):

$$k[t] \quad := \quad \mathrm{id} \tag{4}$$

- high pass filtering using backward differences (better, but no option):

$$k[t] \quad := \quad p[t] - p[t-1] \tag{5}$$

- Shepp–Logan filtering:

$$k[t] \quad := \quad \frac{-2}{\pi(4t^2 - 1)} \tag{6}$$

Different Kernels for Convolution

- RamLak filtering:

$$g[t] := \begin{cases} \frac{\pi}{4}, & \text{if} \quad t = 0 \\ 0, & \text{if} \quad t \text{ even} \\ \frac{-1}{\pi t^2}, & \text{otherwise} \end{cases} \tag{7}$$

Discrete Sampling

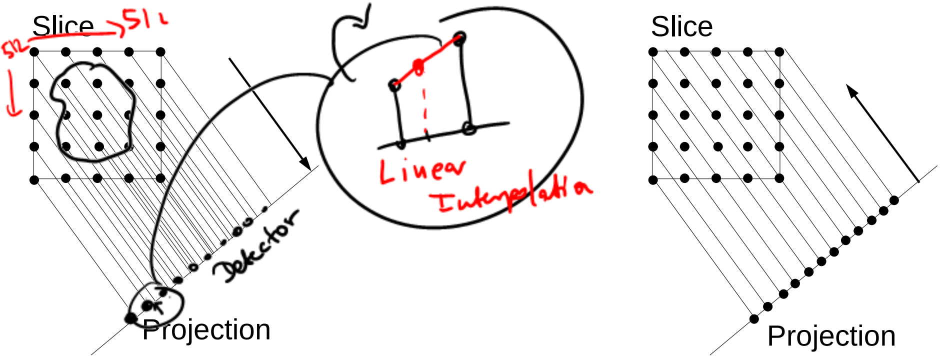


Figure: Different sampling methods for a 2-D slice to be reconstructed: sample the result (left) vs. sample the input projection (right) - here for orthographic projection

Rule of thumb: Always sample in the space where you expect the result.

Discrete Version of Filtered Backprojection

- uniform sampling of the interval of θ:

$$\Delta\theta = \frac{2\pi}{N_{\mathrm{v}}} \tag{8}$$

- the i–th angle is defined by:

$$\theta_i = i \cdot \Delta\theta \quad \text{where} \quad i = 1, \ldots, N_{\mathrm{v}} \tag{9}$$

- let N_t be the number of detector elements and τ the length of the detector

- uniform sampling of the detector results into:

$$\Delta t = \frac{\tau}{N_t} \tag{10}$$

- number of slice columns is N_{x} and number of slice rows is N_{y}

Filtered Backprojection

Initialize slice: $f[y][x] := 0$ for all x, y
FOR $i := 1 \ldots N_\mathrm{v}$ // N_v number of angles
$\theta_i := i \cdot \Delta_\theta$
$p[t] = p_{\theta_i}(t)$
convolution: $h = p \star g$ *filtering* // g e.g. from (**??**)
backprojection (numerical integration)
FOR $y := 0 \ldots N_\mathrm{y} - 1$ // iterate over all slice rows
FOR $x := 0 \ldots N_\mathrm{x} - 1$ // iterate over all slice columns
compute projection of (x, y) in the observed 1–D signal: $t_{x,y}$
compute $h[t_{x,y}]$ (by interpolation)
increment $f[y][x]$ by $h[t_{x,y}]$
Output s

Figure: Filtered backprojection module for one slice

Filtered Backprojection Examples

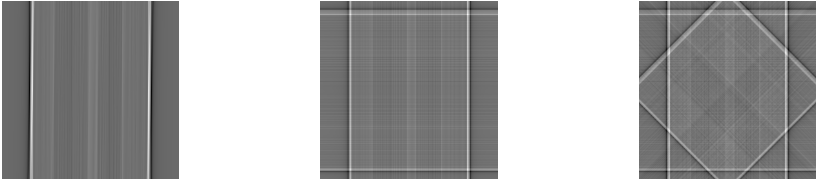


Figure: Filtered backprojection with Shepp-Logan filter (angle increment: 180 degrees (left), 90 degrees (middle), 45 degrees (right))

Filtered Backprojection Examples

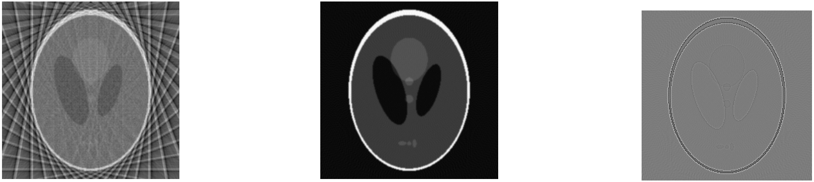


Figure: Filtered backprojection with Shepp-Logan filter (angle increment: 10 degrees (left), 1 degree (middle), difference image of original Shepp-Logan phantom and reconstruction result with 1 degree increment (right))

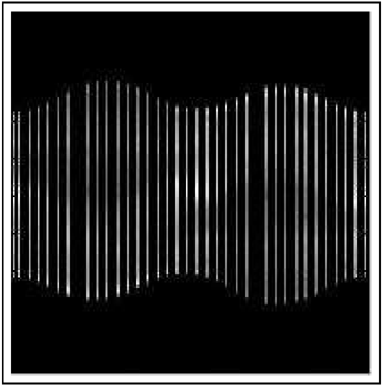
Filtered Backprojection Examples

Figure: RamLak filtering (left) and no filtering (right) followed by backprojection

Missing Projections

Problems:

- Can we reconstruct if some projections are missing?
- To which extend is interpolation working?



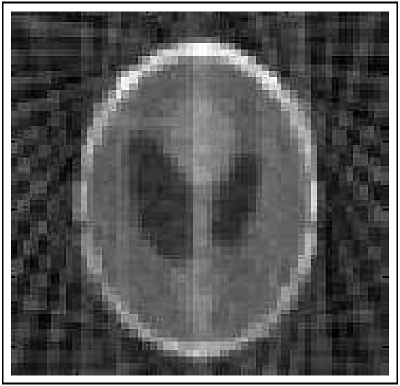


Figure: Incomplete projection data: sinogram (left), FBP reconstruction result (right) with obvious artifacts (stripes)

Missing Projections

Idea: Interpolation of sinogram values

- missing projections lead to missing columns in the sinogram
- observed sinogram is *complete sinogram* multiplied with *defect sinogram* (0/1–values)
- application of **defect interpolation** to sinogram

Missing Projections

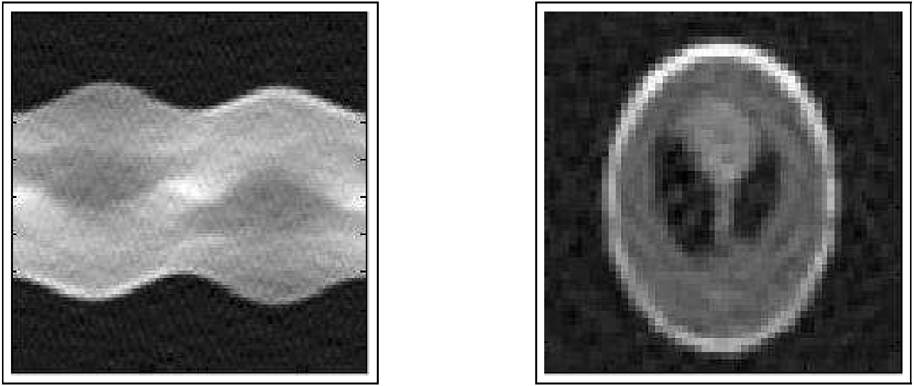


Figure: Interpolated sinogram (left) and reconstruction result (right)

Take Home Messages

Beer's Law

- X-ray attenuation law
- Innovation drivers in CT
- Fourier Slice Theorem
- Empirical result: 5 out of 10 students do **not** know the Fourier Slice Theorem in oral exam.
- Filtered backprojection
- Computational complexity of filtered backprojection

Diagnostic Medical Image Processing

Further Readings

- For practitioners we recommend the book of A.C. Kak and M. Slanely;
  A.C. Kak and Malcolm Slaney: *Principles of Computerized Tomographic Imaging,* Society of Industrial and Applied Mathematics, 2001 (download here)
- A nice overview of CT can be found in the PhD thesis:
  Henrik Turbell: Cone-Beam Reconstruction Using Filtered Backprojection, Linkoping University, Sweden, February 2001.(download here)

Further Readings

- The exact reconstruction method is introduced in Alexander Katsevich: *Theoretically exact filtered backprojection-type algorithm for spiral computed tomography*, SIAM Journal of Applied Mathematics, 62:2012-1026, 2002.
- Radon's original paper where he published his famous inversion formula:
  J. Radon: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, Bericht der Sächsischen Akademie der Wissenschaft, Leipzig Math. Phys. Kl., 69(1917), pp. 262-267.