Exam, october 23^{rd} 2012 / Solutions

Problem I

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$$

1) we do elimination

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[3r_3 + r_2]{} \begin{bmatrix} \underline{1} & 2 & 1 & 4 \\ 0 & 0 & 0 & \underline{-3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- we have 2 pivots, so $\underline{\text{rank}(A) = 2}$

- The pivot columns of A form a basis for $C(A)$

  so $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$ form a basis for $C(A)$.

2) we have 2 free variables : x_2 and x_3

   so $\underline{\dim N(A) = 2}$

   To find a basis for $N(A)$, we need to find the special solutions to $Ax = 0$, which equivalent to $Ux = 0$

- set $\begin{cases} x_2 = 1 \\ x_3 = 0 \end{cases}$ and solve for $x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- set $\begin{cases} x_2 = 0 \\ x_3 = 1 \end{cases}$ and solve for $x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\Longrightarrow \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for $N(A)$.

The set of all solutions to $Ax = 0$ is given by

$$\boxed{\alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \text{with} \quad \alpha, \beta \in \mathbb{R}.}$$

NOTE we want all solutions ⚠

3) The system $Ax = b$ has a solution if $b \in C(A)$.

We can perform the same row operations with b

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 9 \\ 2 & 4 & 2 & 9 & b_3 \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & b_3 - 6 \end{array}\right]$$

$$\xrightarrow[3r_3 + r_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3b_3 - 18 \end{array}\right]$$

So for $b \in C(A)$, we need $3b_3 - 18 = 0$

$$\Rightarrow \boxed{b_3 = 6}$$

The complete set of solutions is given by $x = x_p + x_N$ where x_p is a particular solution, and $x_N \in N(A)$.

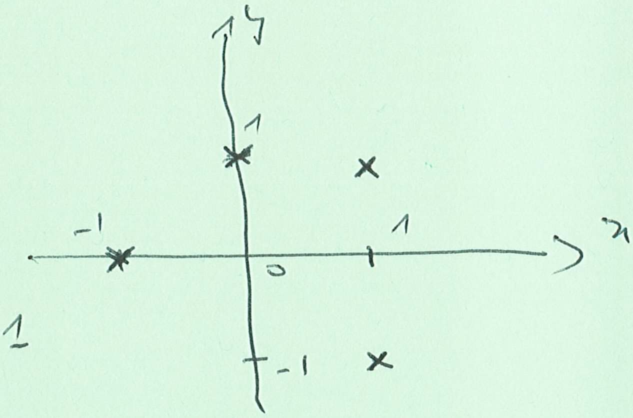
With $b = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$, we see that $x_p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (Sum of first two columns of A)

So the set of solutions is

$$\boxed{x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \alpha, \beta \in \mathbb{R}}$$

Problem 2

x	0	-1	1	1
y	1	0	-1	1



We want the circle of equation $a(x^2+y^2)+b(x+y)=1$

1) The 4 points are ~~th~~ on the circle if their coordinates satisfy:

$$a(0^2+1^2)+b(0+1)=1$$

$$a(-1^2+0^2)+b(-1+0)=1$$

$$a(1^2+(-1)^2)+b(1+(-1))=1$$

$$a(1^2+1^2)+b(1+1)=1$$

$$\Leftrightarrow \quad \begin{aligned} a+b&=1\\ a-b&=1\\ 2a+0&=1\\ 2a+2b&=1 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So, the vector $z = \begin{bmatrix} a \\ b \end{bmatrix}$ must satisfies $Az=b$

with $\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$

2) The system $Az = b$ can be solved only if $b \in C(A)$.

To see that, we can perform elimination with A and B

$$\underset{A\qquad\qquad b}{\left.\begin{matrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{matrix}\ \right|\ \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix}} \longrightarrow \left.\begin{matrix} 1 & 1 \\ 0 & -2 \\ 0 & -2 \\ 0 & 0 \end{matrix}\ \right|\ \begin{matrix} 1 \\ 0 \\ -1 \\ -1 \end{matrix}$$

This tells us that it is not possible to find a linear combination of the columns of A that gives the vector b.

<u>NOTE</u>: If such linear combination exist, let write

$\alpha \, Col_1 + \beta \, Col_2 = b$, then we should have

$0 \cdot \alpha + 0 \cdot \beta = -1$, which is of course impossible

So $\underline{b \notin C(A)}$ and the system cannot be solved.

<u>NOTE</u>:
- Saying $b \notin C(A)$ because $\text{rank}(A) = 2$ is not correct.
- Just saying $b \notin C(A)$ without proof is not enough ⚠

3) To find the LLS, we form the normal equation

$A^T A z = A^T b$.

with
$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Since the matrix A^TA is invertible ($\det \neq 0$)
we can find the LLS by $\boxed{z = (A^TA)^{-1} A^T b}$

First we find $(A^TA)^{-1} = \frac{1}{44}\begin{bmatrix} 6 & -4 \\ -4 & 10 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$

So $z = \frac{1}{22}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\Rightarrow \boxed{z = \begin{bmatrix} 7/11 \\ -1/11 \end{bmatrix}}$

The equation of the circle is

$$\boxed{\frac{7}{11}(x^2+y^2) - \frac{1}{11}(x+y) = 1}.$$

4) $M = A^TA = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$

To find the eigenvalues of M, we solve the equation $\det(M - \lambda I) = 0$

$$\begin{vmatrix} 10-\lambda & 4 \\ 4 & 6-\lambda \end{vmatrix} = 0 \iff (10-\lambda)(6-\lambda) - 16 = 0$$

$$\iff \lambda^2 - 16\lambda + 60 - 16 = 0$$

$$\iff \lambda^2 - 16\lambda + 44 = 0$$

So $\lambda = \dfrac{16 \pm \sqrt{16^2 - 4\times 44}}{2} = \dfrac{16 \pm 4\sqrt{5}}{2}$

So the eigenvalues of M are $\boxed{\begin{cases} \lambda_1 = 8 + 2\sqrt{5} \\ \lambda_2 = 8 - 2\sqrt{5} \end{cases}}$

NOTE
we can check $\begin{cases} \text{trace}(M) = \lambda_1 + \lambda_2 = 16 \\ \det(M) = \lambda_1 \cdot \lambda_2 = 44 \end{cases}$ OK

Problem III

$$A = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \quad \text{has 3 eigenvalues} \quad \begin{cases} \lambda_1 = 18 \\ \lambda_2 = 9 \\ \lambda_3 = -9 \end{cases}$$

1) We know the eigenvalues and want the eigenvectors

We know eigenvectors are vectors in $N(A - \lambda I)$.

For $\lambda_1 = 18$

$$A - \lambda_1 I = \begin{bmatrix} -16 & 10 & -2 \\ 10 & -13 & 8 \\ -2 & 8 & -7 \end{bmatrix}$$

We perform elimination to find a vector in the nullspace.

$$\begin{bmatrix} -16 & 10 & -2 \\ 10 & -13 & 8 \\ -2 & 8 & -7 \end{bmatrix} \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

We solve $Ux = 0$, setting the free variable $x_3 = 1$

A vector in $N(A - \lambda_1 I)$ is $\boxed{v_1 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}}$

NOTE

We can check

$$Av_1 = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \\ 18 \end{pmatrix} = 18 \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} = 18 v_1$$

OK

* You can also continue doing elimination to get the rref of $A - \lambda_1 I$, and see the special solution.

$\longrightarrow$ Same result.

For $\lambda_2 = 9$

$$A - \lambda_2 I = \begin{bmatrix} -7 & 10 & -2 \\ 10 & -4 & 8 \\ -2 & 8 & 2 \end{bmatrix}$$

we perform elimination to find a vector in the nullspace

$$\begin{pmatrix} -7 & 10 & -2 \\ 10 & -4 & 8 \\ -2 & 8 & 2 \end{pmatrix} \longrightarrow \dots \longrightarrow \dots \longrightarrow \underbrace{\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{U}$$

→ the rank is 2, there is one free variable x_3.

→ we solve $Ux = 0$ setting $x_3 = 1$ and get

$$\boxed{v_2 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}}$$

checking

$$Av_2 = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -9/2 \\ 9 \end{bmatrix} = 9 v_2 = \lambda_2 v_2$$

ok

For $\lambda_3 = -9$

$$A - \lambda_3 I = \begin{bmatrix} 11 & 10 & -2 \\ 10 & 14 & 8 \\ -2 & 8 & 20 \end{bmatrix}$$

Elimination

$$\begin{pmatrix} 11 & 10 & -2 \\ 10 & 14 & 8 \\ -2 & 8 & 20 \end{pmatrix} \longrightarrow \dots \longrightarrow \dots \longrightarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we find $\boxed{v_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$

2) Since A is symmetric, the eigenvectors are orthogonal.

<u>Note</u> we can check

$$v_1^T v_2 = \begin{pmatrix} 1/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix} = -\frac{1}{2} - \frac{1}{2} + 1 = 0$$

$$v_1^T v_3 = \begin{pmatrix} \frac{1}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0$$

$$v_2^T v_3 = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -2 + 1 + 1 = 0 .$$

<u>ok</u>

- We normalize the eigenvectors and put them as columns of a matrix Q.

$$q_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$q_2 = \frac{V_2}{\|V_2\|} = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$q_3 = \frac{V_3}{\|V_3\|} = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

So we have $A = Q \Lambda Q^T$ with

$$\Lambda = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

3) The eigenvectors of A are the columns of Q.
And we want a linear combination of the columns of Q that gives $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

So we want $Qy = x \Rightarrow \underline{y = Q^T x}$.

$$y = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

We can check that $\frac{1}{3} q_1 - \frac{2}{3} q_2 + \frac{2}{3} q_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x$.

So we have $x = \frac{1}{3} q_1 - \frac{2}{3} q_2 + \frac{2}{3} q_3$

Therefore $A^{10} x = \frac{1}{3} A^{10} q_1 + \frac{2}{3} A^{10} q_2 + \frac{2}{3} A^{10} q_3$

$$\boxed{A^{10} x = \frac{1}{3} \lambda_1^{10} q_1 - \frac{2}{3} \lambda_2^{10} q_2 + \frac{2}{3} \lambda_3^{10} q_3}$$

Problem 4

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{is positive definitive if}$$

a) All pivots are > 0

b) All eigenvalues are > 0

c) $x^T A x > 0$ for all x.

$\to$ we do elimination and check the pivots are all positive numbers.

$$A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \cdots$$

$$\rightsquigarrow \begin{pmatrix} s & -4 & -4 \\ 0 & s - \frac{16}{s} & -4 - \frac{16}{s} \\ 0 & 0 & \frac{s^2-16}{s} - \left(-4-\frac{16}{s}\right)^2 / \left(s - \frac{16}{s}\right) \end{pmatrix}$$

So we need:

- $s > 0$
- $s - \frac{16}{s} > 0 \to \frac{s^2-16}{s} > 0 \to s^2 > 16 \to s > 4$
- $\frac{s^2-16}{s} - \left(-4 - \frac{16}{s}\right)^2 / \left(s - \frac{16}{s}\right) > 0$

$$\to (s+4)^2 (s-8) > 0 \to s > 8$$

$$\boxed{\text{finally we need } s > 8}$$