Course: Applied Mathematics
Midterm Exam 1 - Linear Algebra : 1h
SOLUTION (SKETCHES FOR SOLUTION)

■ PROBLEM 1 (35 Points)

Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}$

1. What is the rank of A?
2. Find a basis for the column space of A.
3. Find a basis for the nullspace of A.
4. Find the set of solutions to $Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

SOLUTION

The first thing to do is to find the rref of A by Gauss elimination. This gives

$$R = \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Whe have three pivots, so the rank of A is $rank(A) = 3$.
2. We know that a basis for the column space is given by the pivots columns of A. So, column 1, 2 and 4 of A form a basis of $C(A)$:
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} ; \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} ; \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$
3. The dimension of the nullspace $N(A)$ is equal to $5 - rank(A) = 2$. So we have to find two independent vectors to form a basis for $N(A)$.

   We have to solve $AX = 0$, which is equivalent to solving $RX = 0$. If $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$, then from the rref of A, we know that there are two free variables x_3 and x_5 (the column without pivot).

   - Taking $x_3 = 1$ and $x_5 = 0$, and solving $RX = 0$ gives the first basis vector $X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

- Taking $x_3 = 0$ and $x_5 = 1$, and solving $RX = 0$ gives the first basis vector $X_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

So a basis for $N(A)$ is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

4. We know that the set of solutions to $Ax = b$, is given by $x = x_p + x_N$, where x_p is a particular solution to $Ax = b$ and x_N is any vector in $N(A)$.

   Here, we see that the vector $b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ is the first column of A. So, a solution to $Ax = b$ is $x_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. And the set of all solutions is

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} ; \ \alpha, \beta \in \mathbb{R} \right\}.$$

■ PROBLEM 2 (20 Points)

The coordinates of the three points P, Q and R are respectively $(1, 1, 1)$, $(2, 1, 0)$ and $(3, 2, 3)$.

1. Show that the vectors $\vec{PQ}$ and $\vec{PR}$ are orthogonal.
2. Find a vector $\vec{x}$ such that the three vectors $\{\vec{PQ}, \vec{PR}, \vec{x}\}$ form an orthogonal basis of $\mathbb{R}^3$.

SOLUTION

1. $\overrightarrow{PQ}.\overrightarrow{PR} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0.$

   So $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are orthogonal vectors.

2. We want to find $\vec{x}$ such that $\vec{x}.\overrightarrow{PQ} = 0$ and $\vec{x}.\overrightarrow{PR} = 0$.

   If $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then we have to solve a system of two equations

$$\left\{ \begin{array}{ccccccc} x & & & - & z & = & 0 \\ 2x & + & y & + & 2z & = & 0 \end{array} \right.$$

which gives $\left\{ \begin{array}{c} x = z \\ y = -4x \end{array} \right.$. So we can take $\boxed{\vec{x} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}}$

■ PROBLEM 3 (15 Points)

Use the Cauchy-Schwarz inequality

$$\langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\| \, \|\vec{v}\|$$

to prove the Triangle inequality

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

SOLUTION

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u \rangle + 2 \langle u, v \rangle + \langle v, v \rangle \\ &= \|u\|^2 + 2 \langle u, v \rangle + \|v\|^2 \\ &\leq \|u\|^2 + 2 \|u\| \, \|v\| + \|v\|^2 \text{ (by Cauchy-Schwarz inequality)} \\ &\leq (\|u\| + \|v\|)^2 \end{aligned}$$

By taking the square roots of this inequality, we get the Triangle-inequality:

$$\|u + v\| \leq \|u\| + \|v\|$$

■ PROBLEM 4 (30 Points)

1. Let A be a 4×4 matrix whose eigenvalues are λ_1, λ_2, λ_3 and λ_4.
   - a) Give a condition on the λ's so that A is a non-singular matrix.
   - b) In the case where A is invertible, what is the determinant of A^{-1}?
   - c) Whate is the trace of $A + 2I$?
2. Let P be a 3×3 projection matrix.
   - a) What are the eigenvalues of P?
   - b) Consider the sequence of vectors $U_{k+1} = PU_k$, with $U_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$. Given that $U_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$, what is U_{100}?

SOLUTION

1. Properties of eigenvalues
   - A matrix A is nonsingular if and only if it has no zero eigenvalue. Otherwise, if $\lambda = 0$ is an eigenvalue, then $Ax = 0$ has some nonzero solutions and A is singular.
     So the condition is $\lambda_1 \neq 0, \ldots, \lambda_4 \neq 0$.

- We know that the determinant is the product of the eigenvalues. We also know that the eigenvalues are A^{-1} are the inverse of the eigenvalues of A. Therefore,

$$\boxed{det(A^{-1}) = \frac{1}{\lambda_1}.\frac{1}{\lambda_2}.\frac{1}{\lambda_3}.\frac{1}{\lambda_4}}$$

.

- The trace of a matrix is the sum of its eigenvalues. And the eigenvalues of $A + 2I$ are $\lambda_1 + 2$, $\lambda_2 + 2$, $\lambda_3 + 2$ and $\lambda_4 + 2$. So, $trace(A + 2I) = trace(A) + 8$.

$$\boxed{trace(A + 2I) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 8}$$

2. Properties of projection matrices

- If P is a projection matrix, then P takes any vector of $\mathbb{R}^3$ to a subspace $S \subset \mathbb{R}^3$ (that is P is the projection onto S). Moreover, $\mathbb{R}^3$ can be decomposed as an orthognal sum:

$$\mathbb{R}^3 = S \oplus S^\perp$$

Let $x \in \mathbb{R}^3$. x can be decomposed as $x = x_1 + x_2$, where $x_1 \in S$ and $x_2 \in S^\perp$.
So $Px = Px_1 + Px_2 = x_1$.
If $x \in S$, then $Px = x$. So $\lambda = 1$ is an eigenvalue of P.
If $x \in S^\perp$, then $Px = 0$. $\lambda = 0$ is an eigenvalue of P.
A projection matrix has only two eigenvalues: 0 and 1.

- We know that P is a projection matrix. So $P^2 = P$.
Therefore, $\forall k > 1, P^k = P$.
Since $U_{k+1} = PU_k$, we have $\forall k > 1, U_k = P^k U_0 = PU_0 = U_1$.

$$\boxed{U_{100} = U_1 = \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}}$$