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Chapter 1

Fourier Series

The frequency spectrum is a complex-valued function of the frequency variable, and thus it is usually specified in terms of an amplitude spectrum and a phase spectrum [1]. The complex exponential form is given by:

$$\begin{aligned}x(t) &= \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t} \\X_n &= \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt\end{aligned}\tag{1.1}$$

In the next exercises we will also be using the first and/or the second derivative of the previous expressions (??), and therefore we write them here explicitly:

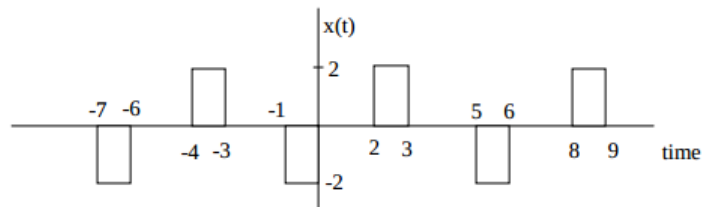
$$\begin{aligned}\dot{x}(t) &= \sum_{-\infty}^{\infty} jn\omega_0 X_n e^{jn\omega_0 t} \\jn\omega_0 X_n &= \int_{-T/2}^{T/2} \dot{x}(t) e^{-jn\omega_0 t} dt\end{aligned}\tag{1.2}$$

$$\ddot{x}(t) = \sum_{-\infty}^{\infty} -n^2 \omega_0^2 X_n e^{jn\omega_0 t}$$

$$-n^2 \omega_0^2 X_n = \int_{-T/2}^{T/2} \dot{x}(t) e^{-jn\omega_0 t} dt \quad (1.3)$$

Problem 1

For the following signal:



- Find the Fourier series.
- Plot the spectra versus frequency, $\omega = n\omega_0$.

Solution

The period of the shown signal is $T = 6$ and therefore $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$.

Taking the derivative of $x(t)$ we get:

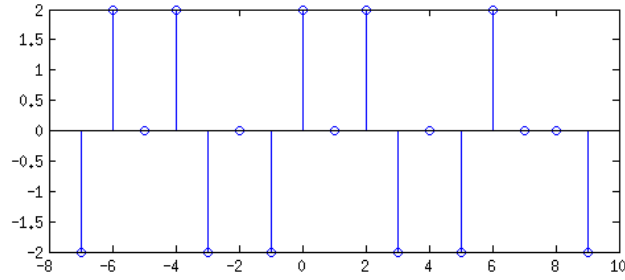


Figure 1.1: Derivative \dot{x}

The range $t = [-3, 3]$ contains one complete period of the signal. Using (1.2) we have:

$$\dot{x}(t) = 2(-\delta(t+1) + \delta(t) + \delta(t-2) - \delta(t-3))$$

The Fourier coefficients are obtained with:

$$\begin{aligned} jn\omega_0 X_n &= \frac{2}{6} \int_{-3}^3 (-\delta(t+1) + \delta(t) + \delta(t-2) - \delta(t-3)) e^{-jn\omega_0 t} dt \\ &= \frac{1}{3} (-e^{jn\omega_0} + 1 + e^{-2jn\omega_0} - e^{-3jn\omega_0}) \\ &= \frac{1}{3} [e^{\frac{jn\omega_0}{2}} (e^{-\frac{jn\omega_0}{2}} - e^{\frac{jn\omega_0}{2}}) - e^{\frac{-5jn\omega_0}{2}} (e^{-\frac{jn\omega_0}{2}} - e^{\frac{jn\omega_0}{2}})] \\ &= \frac{1}{3} [(e^{-\frac{jn\omega_0}{2}} - e^{\frac{jn\omega_0}{2}}) (e^{-jn\omega_0} (e^{\frac{3jn\omega_0}{2}} - e^{\frac{-3jn\omega_0}{2}}))] \\ &= \frac{4j}{3} [\sin \frac{n\omega_0}{2} \sin \frac{3n\omega_0}{2} e^{-jn\omega_0}] \\ X_n &= \frac{-4j}{n\pi} [\sin(n\frac{\pi}{6}) \sin(n\frac{\pi}{2}) e^{-jn\frac{\pi}{3}}] \end{aligned}$$

Next we use Matlab to plot the magnitude and phase of the spectra using the script given in [2]

Listing 1.1: Calculate and plot magnitude and phase of X_n

```
1 n=1:15;
2 Xn=-4*j./n/pi.*sin(pi*n/6).*sin(n*pi/2).*exp(-j*n*pi/3);
3 n=-15:-1;
4 Xn=-4*j./n/pi.*sin(pi*n/6).*sin(n*pi/2).*exp(-j*n*pi/3);
5 Xn=[Xn 0 Xn];
6 n=-15:15;
```

```

7 subplot(211),stem(n,abs(Xn));
8 title('|X_n|')
9 subplot(212),stem(n,angle(Xn))
10 title('angle(X_n) in rad')

```

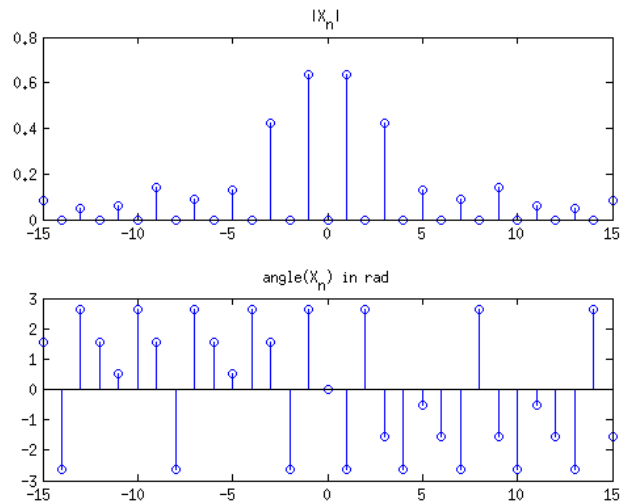


Figure 1.2: Magnitude and Angle X_n

We then plot the approximation of the function using its Fourier coefficients [2].

Listing 1.2: Approximation of $x(t)$ with Fourier coefficients

```

1 function [x,t] = fapprox(N,T)
2     t = -1.5*T:T/1000:1.5*T;
3     w0 = 2*pi/T;
4     X0 = 0;
5     x = X0*ones(1,length(t)); % dc component
6     for n=1:N,
7         Xn = -4*j/n/pi*sin(pi*n/6)*sin(n*pi/2)*exp(-j*n*pi/3);
8         X_n = conj(Xn);
9         x = x + Xn*exp(j*n*w0*t) + X_n*exp(-j*n*w0*t);
10    end
11 end

```

We do this for $N=5$ and $N=50$:

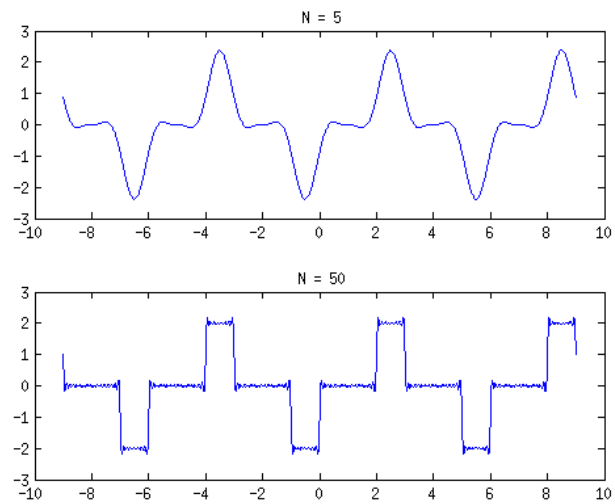
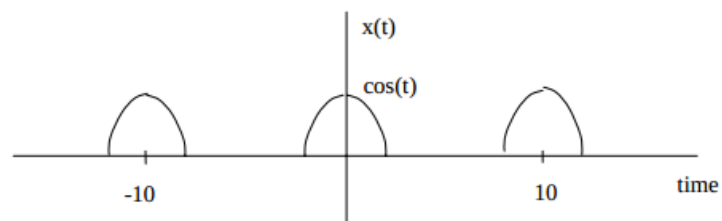


Figure 1.3: Approximation of $x(t)$ by X_n

As we can see, the larger the N the more close to the original function we get. However, as a consequence of the Gibbs effect, we can't say that it'll be equal.

Problem 2

Repeat problem 1 for the following signal:



Solution

The period of the shown signal is $T = 10$ and therefore $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{5}$.

If we take the first and second derivative of $x(t)$ we get:

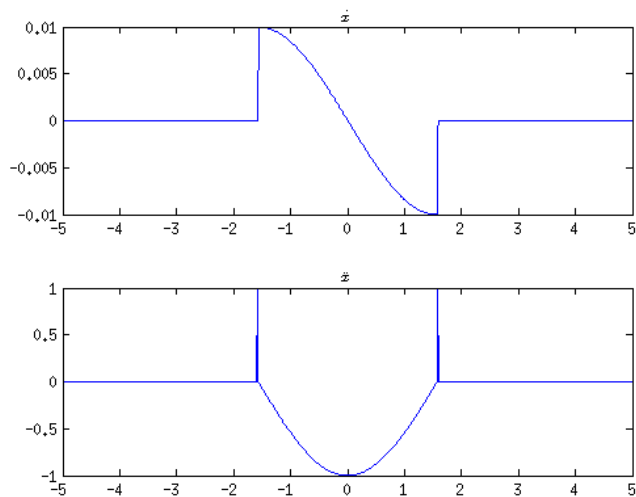


Figure 1.4: Derivative \dot{x}

The range $t = [-5, 5]$ contains one complete period of the signal. Applying (1.2) we have:

$$\begin{aligned}\ddot{x}(t) &= -x(t) + \delta(t + \pi/2) + \delta(t - \pi/2) \\ \sum_{-\infty}^{\infty} -n^2 \omega_0^2 X_n e^{jn\omega_0 t} &= \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t} + \delta(t + \pi/2) + \delta(t - \pi/2) \\ \sum_{-\infty}^{\infty} (1 - n^2 \omega_0^2) X_n e^{jn\omega_0 t} &= \delta(t + \pi/2) + \delta(t - \pi/2)\end{aligned}$$

We can now obtain X_n with:

$$\begin{aligned}
(1 - n^2\omega_0^2)X_n &= \frac{1}{T} \int_{-5}^5 \delta(t + \pi/2) + \delta(t - \pi/2) e^{-jn\omega_0 t} dt \\
&= \frac{1}{T} (e^{jn\frac{\omega_0}{2}} + e^{-jn\frac{\omega_0}{2}}) \\
X_n &= \frac{1}{5(1 - \frac{n^2\pi^2}{25})} \cos(\frac{n\pi^2}{10})
\end{aligned}$$

Next we use Matlab to plot the magnitude and phase of the spectra using the script given in [2]

Listing 1.3: Calculate and plot magnitude and phase of X_n

```

1  n=-10:10;
2  Xn=cos(pi/2*n*w0)/5./(1-(n*w0).^2);
3  subplot(121),stem(n,abs(Xn))
4  title(' |X_n| ')
5  subplot(122),stem(n,angle(Xn))
6  title(' angle(X_n) in rad ')

```

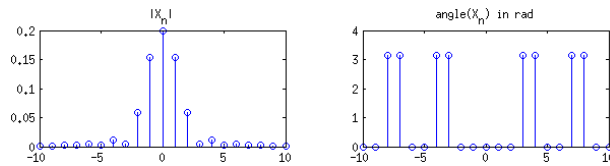


Figure 1.5: Magnitude and Angle X_n

We then plot the approximation of the function using its Fourier coefficients [2].

Listing 1.4: Approximation of $x(t)$ with Fourier coefficients

```

1  function [x,t] = fapprox2(N,T)
2      w0 = 2*pi/T;
3      t = -1.5*T:T/1000:1.5*T;
4      c0 = 1/5;
5      x = c0*ones(1,length(t)); % dc component
6      for n=1:N,
7          cn = cos(pi/2*n*w0)/5/(1-(n*w0)^2);
8          c_n = cn;
9          x = x + cn*exp(j*n*w0*t) + c_n*exp(-j*n*w0*t);
10     end
11     plot(t,x)
12     title([' N = ',num2str(N)])
13 end

```

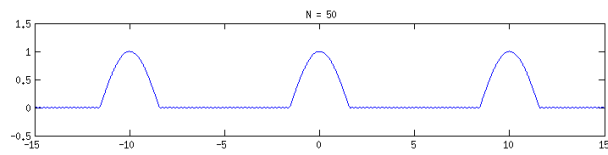
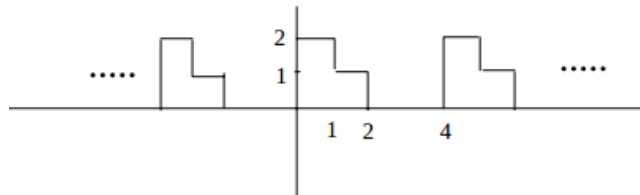


Figure 1.6: Approximation of $x(t)$ by X_n

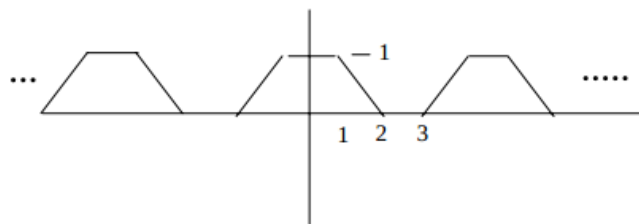
Problem 3

Compute the Fourier series for the following signals:

1. $x(t) = 2 + 4 \cos(50t + \pi/2) + 12 \cos(100t - \pi/3)$
2. $x(t) = 4 \cos(2\pi(100)t) \cos(2\pi(750000)t)$
3. The function:



4. The function:



Solution

Lets recall from [1] that:

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \quad -\infty < t < \infty$$

$$X_0 = a_0$$

$$|X_n| = \frac{1}{2} A_k, k = 1, 2, .. \quad (1.4)$$

$$\angle X_n = \theta_k, k = 1, 2, ..$$

1. From (1.4) we can immediately calculate the X_n coefficients:

Listing 1.5: Plot Magnitude and Angle of X_n

```
1 fe = [-100 -50 0 50 100];
2 Ae = [6 2 2 2 6];
3 pe = [pi/3 -pi/2 0 pi/2 -pi/3];
4
5 figure(1);
6 subplot(2,1,1), stem(fe,Ae);
7 grid on;
8 xlabel('Frec [rad/s]');
9 ylabel('Amplitude');
10 title('|X_n|', 'fontweight', 'bold');
11
12 subplot(2,1,2), stem(fe,pe);
13 grid on;
14 xlabel('Frec [rad/s]');
15 ylabel('Phase');
16 title('$\angle x$', 'interpreter', 'latex')
```

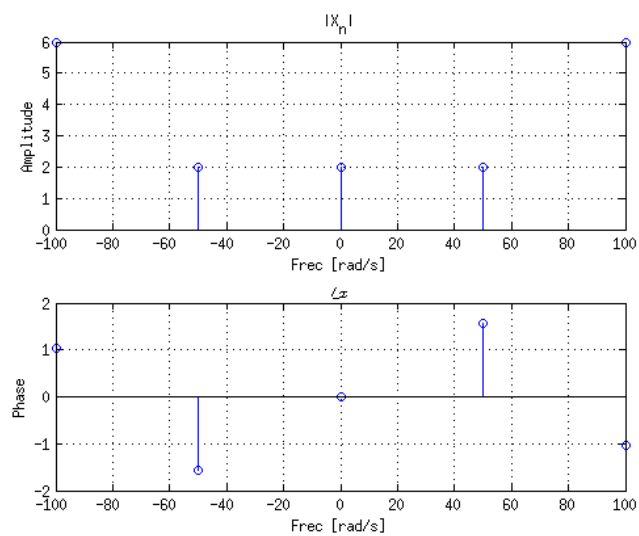


Figure 1.7: Magnitude and Angle of X_n

Chapter 2

Fourier Transform

Chapter 3

Haar Transform

Bibliography

- [1] Edward Kamen and Bonnie Heck. *Fundamentals of Signals and Systems: With MATLAB Examples*. Prentice Hall PTR, 2000.
- [2] Georgia Tech School of Electrical and Computer Engineering. Worked Problems, Chapter 4. http://users.ece.gatech.edu/~bonnie/book/worked_problems/Chap4_Fseries_sol.pdf. [Online; accessed Apr-2013].