

# **BALLISTICS**

**THEORY AND DESIGN OF  
GUNS AND AMMUNITION**



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## THEORY AND DESIGN OF GUNS AND AMMUNITION

DONALD E. CARLUCCI  
SIDNEY S. JACOBSON



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*To Peg C., Sandy J., and our families,  
without whose patience and support  
we could not have brought  
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## *Preface*

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This book is an outgrowth of a graduate course taught by the authors for the Stevens Institute of Technology at the Picatinny Arsenal in New Jersey. Engineers and scientists at the arsenal have long felt the need for an armature of the basic physics, chemistry, electronics, and practice on which to flesh out their design tasks as they go about fulfilling the needs and requirements of the military services for armaments. The Stevens Institute has had a close association with the arsenal for several decades, providing graduate programs and advanced degrees to many of the engineers and scientists employed there. It is intended that this book be used as a text for future courses and as a reference work in the day-to-day business of weapons development.

Ballistics as a human endeavor has a very long history. From the earliest developments of gunpowder in China more than a millennium ago, there has been an intense need felt by weapon developers to know how and why a gun works, how to predict its output in terms of the velocity and range of the projectiles it launched, how best to design these projectiles to survive the launch, fly to the target and perform the functions of lethality, and the destructions intended.

The discipline over the centuries has divided itself into three natural regimes: Interior Ballistics or what happens when the propellant is ignited behind the projectile until the surprisingly short time later when the projectile emerges from the gun; Exterior Ballistics or what happens to the projectile after it emerges and flies to the target and how to get it to fly there reproducibly shot after shot; and Terminal Ballistics or once it is in the vicinity of the target, how to extract the performance from the projectile for which the entire process was intended, usually lethality or destruction.

Ballisticians, those deeply involved in the science of ballistics, tend to specialize in only one of the regimes. Gun and projectile designers, however, must become proficient in all the regimes if they are to successfully field weapons that satisfy the military needs and requirements. The plan of this book is bilateral: first, an unfolding of the theory of each regime in a graduated ascent of complexity, so that a novice engineer gets an early feeling for the subject and its nomenclature and is then brought into a deeper understanding of the material; second, an explanation of the design practice in each regime. Most knowledge of weapon design has been transmitted by a type of apprenticeship with experienced designers sharing their learning with newer engineers. It is for these engineers that this work is intended, with the hope that it will make their jobs easier and their designs superior.



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## ***Authors***

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**Donald E. Carlucci** has been an engineer at the U.S. Army Armament, Research, Development and Engineering Center, Picatinny Arsenal, since May 1989. He is currently chief of the Analysis and Evaluation Technology Division, Fuze and Precision Munitions Technology Directorate responsible for the modeling and evaluation of cannon-launched munitions programs at Picatinny, and chief scientist for the XM982 Excalibur guided projectile. Dr. Carlucci has formerly held the position of development program officer (chief engineer) for sense and destroy armor (SADARM). Before his employment at Picatinny, he was a design engineer for Titanium Industries, located in Fairfield, New Jersey.

Dr. Carlucci has held positions as chief engineer, quality assurance manager, and purchasing manager for Hoyt Corporation, located in Englewood, New Jersey. He is a licensed professional engineer in the states of New Jersey and New York and holds a doctor of philosophy in mechanical engineering (2002) and a master of engineering (mechanical) (1995) degree from the Stevens Institute of Technology, Hoboken, New Jersey. In 1987, he received his bachelor of science degree in mechanical engineering from the New Jersey Institute of Technology, Newark, New Jersey.

Dr. Carlucci is an adjunct professor of mechanical engineering at the Stevens Institute of Technology where he teaches graduate classes on interior, exterior, and terminal ballistics as well as undergraduate classes on engineering design.

**Sidney S. Jacobson** was a researcher, designer, and developer of ammunition and weapons at the U.S. Army's Picatinny Arsenal in New Jersey for 35 years. He rose from junior engineer through eight professional levels in research and development laboratories to become associate director for R&D at the arsenal. His specialty for most of his career was in the development of large caliber tank munitions and cannons. Many of these weapons, such as the long rod, kinetic energy penetrators (APFSDS rounds), and the shaped charge, cannon-fired munitions (HEAT rounds), have become standard equipment in the U.S. Army. For these efforts and successes he earned several awards from the army including, in 1983, the Department of the Army Meritorious Civilian Service Medal. In 1972, he was awarded an Arsenal Educational Fellowship to study continuum mechanics at Princeton University where he received his second MS degree (1974). He earned a master of science in applied mechanics from Stevens Institute of Technology (1958) and a bachelor of arts in mathematics from Brooklyn College (1951).

He retired in 1986 but maintains his interest in the field through teaching, consulting, and lecturing. He holds two patents and was a licensed professional engineer in New Jersey.



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In the four years of development of this book, many people have contributed in a variety of ways. This has probably been the most difficult section that we had to write for fear of neglecting key people.

We are a generation apart in our careers, mentors, and experience, but not in our enthusiasm for the subject. As the elder, Sidney S. Jacobson must acknowledge several people who inspired his enthusiasm beginning in the 1950s: Robert Schwartz, designer of the 280 mm atomic shell who also wrote the first set of ammunition design notes; Alfred A. Loeb, aeroballistician and friend; and Ralph F. Campoli, a peerless ammunition designer and longtime friend.

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Both of us must acknowledge as mentor, role model, and friend Victor Lindner, an acknowledged leader in the world of weaponry, whose career spanned both of ours.

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# **Part I**

# **Interior Ballistics**



# 1

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## *Introductory Concepts*

---

The subject of ballistics has been studied for centuries by people at every level of academic achievement. Some of the world's greatest mathematicians and physicists such as Newton, Lagrange, Bernoulli, and others solved problems in mathematics and mechanics that either directly or indirectly were applied to the various ballistic disciplines. At the other end of the academic scale, there are individuals such as James Paris Lee (inventor of the Lee-Enfield rifle) who developed his first weapon (not the famous Lee-Enfield) at age 12 with no formal education.

The dominant characteristic of any of the ballistic disciplines is the "push-pull" relationship of experiment and analysis. It is a rare event, even as of this writing, when an individual can design a ballistic component or device, either digitally or on paper, and have it function "as designed" in the field. Some form of testing is always required and consequent tweaking of the design. This inseparable linkage between design and test is due to three things: the stochastic nature of ballistic events, the infinite number of conditions into which a gun-projectile-charge combination can be introduced, and the lack of understanding of the phenomena.

The stochastic behavior that dominates all of the ballistic disciplines stems from the tremendous number of parameters that affect muzzle velocity, initial yaw, flight behavior, etc. These parameters can be as basic as how or when the propellant was produced to what was the actual diameter of the projectile measured to 0.0001 in. Even though, individually, we believe that we understand the effect of each parameter, when all parameters are brought together the problem becomes intractable. Because of this parameter overload condition, the behavior is assumed to be stochastic.

The number of battlefield and test conditions that a gun-projectile-charge combination can be subjected to is truly infinite. For safety and performance estimates, the U.S. Army is often criticized for demanding test conditions which could not possibly occur. While this may be true, it is simply a means of over-testing a design to assure that the weapon system is safe and reliable when the time comes to use it. This philosophy stems from the fact that you cannot test every condition and also because soldiers are an ingenious bunch and will invent new ways to employ a system beyond its design envelope.

Lack of understanding of the phenomena may seem rather strong wording even though there are instances where this is literally true. In most cases, we know that parameters are present which affect the design. We also know how they should affect the design. Some of these parameters cannot be tested because there is some other, more fundamental variable that affects the test setup to a far greater degree.

The overall effect of ballistic uncertainty, as described above, is that it will be very unusual for you to see the words "always" or "never" when describing ballistic phenomena in this work.

---

## 1.1 Ballistic Disciplines

The field of ballistics can be broadly classified into three major disciplines: interior ballistics, exterior ballistics, and terminal ballistics. In some instances, a fourth category named intermediate ballistics has been used.

Interior ballistics deals with the interaction of the gun, projectile, and propelling charge before emergence of the projectile from the muzzle of the gun. This category would include the ignition process of the propellant, the burning of propellant in the chamber, pressurization of the chamber, the first-motion event of the projectile, engraving of any rotating band and obturation of the chamber, in-bore dynamics of the projectile, and tube dynamics during the firing cycle.

Intermediate ballistics is sometimes lumped together with interior ballistics, but has come into its own category of late. Intermediate ballistics deals with the initial motion of the projectile as it is exiting the muzzle of the tube. This generally includes initial tip-off, tube and projectile jump, muzzle device effects (such as flash suppression and muzzle brake venting), and sabot discard.

Exterior ballistics encompasses the period from when the projectile has left the muzzle until impact with the target. One can see the overlap here with intermediate ballistics. In general, all that the exterior ballistian is required to know is the muzzle velocity and tip-off and spin rates from the interior ballistian, and the physical properties (shape and mass distribution) from the projectile designer. In exterior ballistics, one generally is concerned with projectile dynamics and stability, the predicted flight path and time of flight, and angle, velocity and location of impact. More often, now than in previous years, the exterior ballistian (usually called an aero-ballistian) is also responsible for designing or analyzing guidance algorithms carried onboard the projectiles.

Terminal ballistics covers all aspects of events that occur when the projectile reaches the target. This means penetration mechanics, behind armor effects, fragment spray patterns and associated lethality, blast overpressure, nonlethal effects, and effects on living tissue. This last topic is becoming more and more important because of the great interest in less-than-lethal armaments and, indeed, it has been categorized into its own discipline known as wound ballistics.

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## 1.2 Terminology

Throughout this work we will be using the word “gun” in its generic sense. A gun can be loosely defined as a one-stroke internal combustion engine. In this case, the projectile is the piston and the propellant is the air-fuel mixture. Guns themselves can be classified in four broad categories: a “true” gun, a howitzer, a mortar, and a recoilless rifle.

A true gun is a direct-fire weapon that predominantly fires a projectile along a relatively flat trajectory. Later on we will decide what is truly flat and what is not. Notice the word “predominantly” crept in here. A gun, say on a battleship, can fire at a high trajectory sometimes. It is just usually used in the direct-fire mode. A gun can be further classified as rifled or smooth bore, depending upon its primary ammunition. Guns exhibit a relatively high muzzle velocity commensurate with their direct-fire mission. Examples of guns include tank cannon, machine guns, and rifles.

A howitzer is an indirect-fire weapon that predominantly fires projectiles along a curved trajectory in an attempt to obtain improved lethal effects at well-emplaced targets.

Again, howitzers can and have been used in a direct-fire role; it is simply not one at which they normally excel.

A mortar is a tube that is usually man-portable used to fire at extremely high trajectories to provide direct and indirect support to the infantry. Mortars generally have much shorter ranges than howitzers and cannot fire a flat trajectory at all.

A recoilless rifle is a gun designed with very little weight. They are usually mounted on light vehicles or man emplaced. They are used where there is insufficient mass to counteract the recoil forces of a projectile firing. This is accomplished by venting the high-pressure gas out of a rear nozzle in the breech of the weapon in such a way as to counter the normal recoil force.

A large listing of terminology unique to the field of ballistics is included in the glossary in [Appendix A](#).

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### 1.3 Units and Symbols

The equations included in the text may be used with any system of units. That being said, one must be careful of the units chosen. The literature that encompasses the ballistic field uses every possible system and is very confusing for the initiate engineer. The U.S. practice of mixing the International System of Units (SI), United States Customary System (USCS), and Centimeter–Gram–Seconds (CGS) units is extremely challenging for even the most seasoned veteran of these calculations. Because of this an emphasis has been placed on the units in the worked-out examples and cautions are placed liberally in the text.

Intensive and extensive properties (where applicable) are denoted by lowercase and uppercase symbols, respectively. In some instances, it is required to use the intensive properties on a molar basis. These will be denoted by an overscore tilde. In all cases, the reader is advised to always be sure of the units.



# 2

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## *Physical Foundation of Interior Ballistics*

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### 2.1 The Ideal Gas Law

The fundamental means of exchanging the stored chemical energy of a propellant into the kinetic energy of the projectile is through the generation of gas and the accompanying pressure rise. We shall proceed in a disciplined approach, whereby, we introduce concepts at their simplest level and then add the complications associated with the real world.

Every material exists in some physical state of either solid, liquid, or gas. There are several variables that we can directly measure and some that we cannot but which are related to one another through some functional relationship. This functional relationship varies from substance to substance and is known as an equation of state.

Thermodynamically, the number of independent properties required to define the state of a substance is given by the so-called state postulate, which is described in Ref. [1]. For all of the substances examined in this text we shall assume they behave in a simple manner. This essentially means that the equilibrium state of all of our substances can be defined by specification of two independent, intrinsic properties. In this sense, an intrinsic property is a property that is characteristic of (in other words, governed by) molecular behavior.

The ideal gas law is essentially a combination of three relationships [2]. Charles's law states that volume of a gas is directly proportional to its temperature. Avogadro's principle states that the volume of a gas is directly proportional to the number of moles of gas present. Boyle's law states that volume is inversely proportional to pressure. If we combine these three relationships, we arrive at the famous ideal gas law, which states in extensive form.

$$p\tilde{v} = N\mathfrak{R}T \quad (2.1)$$

Here  $p$  is the pressure of the gas,  $\tilde{v}$  is the molar specific volume,  $N$  is the number of moles of the gas,  $\mathfrak{R}$  is the universal gas constant, and  $T$  is the absolute temperature.

The units of Equation 2.1 are not always convenient to work with. For this reason, the form of the ideal gas law that we shall use most often in this text is

$$pv = RT \quad (2.2)$$

In this case,  $p$  is the pressure of the gas,  $v$  is the specific volume (in mass units as we are used to),  $R$  is the specific gas constant, unique to each gas, and  $T$  is again the absolute temperature. The specific gas constant can be determined from the universal gas constant by dividing the latter by the molar mass.

$$R = \frac{\mathfrak{R}}{M} \quad (2.3)$$

Here  $M$  is the molar mass of the gas (e.g., 15.994 lb<sub>m</sub>/lb-mol for oxygen). There are many other variants of the ideal gas law, which differ only in units. The other two versions that we occasionally utilize are

$$pV = m_g RT \quad (2.4)$$

and

$$p = \rho RT \quad (2.5)$$

In these equations,  $V$  (non-italicized) is the volume the gas occupies,  $m_g$  is the mass of the gas, and  $\rho$  is the gas density. One should always check units when using these equations.

The pressure in a vessel filled with gas is caused by innumerable collisions of the gas molecules on the walls of the vessel [2]. The more tightly packed the molecules are, the more collisions occur—the higher the pressure is. Similarly, temperature excites the gas molecules so that they move faster, collide more—thus also increasing pressure. It is these collisions, among other things, that must be handled somehow by our equation of state.

The ideal gas law relies upon the fact that the gas molecules are very far apart relative to one another [3]. If the molecules linger in the neighborhood of one another they will be influenced by strong intermolecular forces, which can either attract or repel them from one another. Thus, the ideal gas law ignores this effect. The ideal gas law further assumes intermolecular collisions occur completely elastically (i.e., like billiard balls). These assumptions must be kept in mind when using the ideal gas law. We shall soon see that under the pressures and temperatures in a gun that these assumptions are invalid. Nevertheless, they provide us with a point of departure and a useful stepping-stone for our studies.

To use the ideal gas law to determine the state of the gas in a gun, we need to invoke classic thermodynamic relationships. The second law of thermodynamics can be stated as follows:

$$Q = \Delta U + W + \text{losses} \quad (2.6)$$

In Equation 2.6,  $Q$  is the energy added to the system,  $\Delta U$  is the change in internal energy,  $W$  is the work done on the system, and the losses term contains all of the energy that cannot be recovered if, say, we pushed the projectile back to its starting position in the gun tube. Our sign convention shall be that a  $Q$  will be positive when energy is added to the system,  $\Delta U$  will be positive if the internal energy of the system is increased, and  $W$  will be positive if work is done on the system. Losses are always negative.

If we tailor Equation 2.6 to a gun launch situation, then  $Q$  would be the energy released by burning our propellant,  $\Delta U$  would be the change in internal energy of the propellant, and  $W$  would be the work done on the projectile.

Let us further define the work term in the classical sense. It is typical of a first year engineering curriculum to define the work as follows:

$$W = \int \mathbf{F} \cdot d\mathbf{x} \quad (2.7)$$

In Equation 2.7, work is defined as a scalar that results from the vector dot product of force,  $\mathbf{F}$  with the distance over which the force acts, also a vector,  $d\mathbf{x}$  (note that all vectors are characterized by bold type in this book). If we restrict our analysis to a gun system, we can see that, given pressure acting on the base of a projectile, it only has one direction to travel

due to the constraints of the gun tube. If we imagine that this gun tube is perfectly straight (it never is) and we align a coordinate system with the axis of the tube, then the displacement vector,  $\mathbf{dx}$ , must be aligned with force vector,  $\mathbf{F}$  (i.e., the cosine of the angle between them is zero); and our relation for a dot, or more formally, the scalar product of these two vectors gives us

$$\mathbf{F} \cdot \mathbf{dx} = |\mathbf{F}| \cdot |\mathbf{dx}| \cdot \cos(0) = Fdx \quad (2.8)$$

Our work definition for this case is then

$$W = \int Fdx \quad (2.9)$$

This relationship for work has to be refined somewhat to fulfill our needs. We will need to put the force acting on the projectile in terms of the pressure and sometimes would like the volume to be included in the equation. If we look at the ideal gas equation of state in the form of [Equation 2.4](#), we do not see a force in there but we do see a pressure term and a volume term.

We know from the mechanics of materials [4] that

$$F = pA \quad (2.10)$$

This has not been written in vector form so as to keep things simple (we will write it differently later). Equation 2.10 states that the resultant force,  $F$ , on a body is equal to the average pressure,  $p$ , on that body times the area,  $A$ , over which the pressure acts. So we can rewrite Equation 2.9 using this result as

$$W = \int pAdx \quad (2.11)$$

We now need to get volume in there somehow. We shall use the fact that, except for the chamber of a gun (and a few notable exceptions with the bore), the area over which the pressure acts is constant and equal to the bore cross-sectional area which we have defined as  $A$  above. The area of the rifling grooves does contribute here if the tube is rifled, but let us assume a nice smooth cylindrical bore for now. If  $A$  is the cross-sectional area and  $dx$  is a differential element of length, then the differential element of volume,  $dV$ , can be defined as

$$dV = Adx \quad (2.12)$$

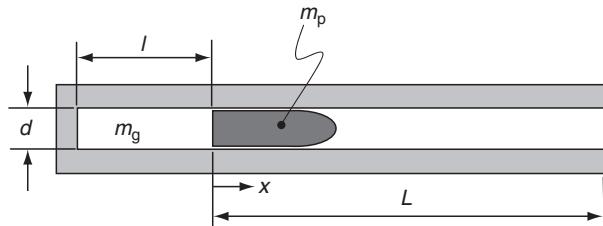
We can now write Equation 2.11 in terms of pressure and volume as

$$W = \int pdV \quad (2.13)$$

You may recall this form of the definition of work from thermodynamics [5].

We now have two equations and a definition at our disposal as a pedagogical device that can help illustrate the energy exchange mechanism in a gun. The equations are an ideal gas equation of state [Equation 2.4](#) and the second law of thermodynamics, [Equation 2.6](#); and the definition of how we defined work in [Equation 2.13](#).

Let us imagine that we have a simple gun as depicted in [Figure 2.1](#).



**FIGURE 2.1**  
Simple gun system.

We shall assume that we have somehow placed a mass,  $m_g$ , of a gas that behaves according to the ideal gas equation of state in the tube and compressed it, adiabatically, using the projectile and no leakage has occurred. We shall further assume that there is no friction between the projectile and the tube wall. Thus, in the situation depicted by Figure 2.1, we have an ideal gas trapped between the projectile and the breech, compressed to some pressure,  $p$ , at some absolute temperature,  $T$ . We shall further assume that the projectile of mass,  $m_p$ , is somehow held at position  $x = 0$  and no gas or energy can escape. In this situation, the volume the gas occupies, which we shall call the chamber volume,  $V_c$ , is given by

$$V_c = \frac{\pi d^2}{4} l \quad (2.14)$$

What we have done essentially is compressed the projectile against an imaginary spring (the gas), which now has a potential energy associated with it. From a thermodynamic standpoint, we can reduce [Equation 2.6](#) to

$$0 = \Delta U + W \quad (2.15)$$

Recapping, we note that  $Q = 0$  because there was no heat lost through the tube wall (adiabatic compression) and there is no propellant per se that will burn to generate heat. The losses were zero because we have no friction.

Now that everything is set, we need to release our projectile and see what happens. If we substitute [Equation 2.4](#) into [Equation 2.13](#), we can write

$$W = \int m_g RT \frac{dV}{V} \quad (2.16)$$

This equation now shows how much work is being done on the projectile as a function of the volume. It is noteworthy here that we are assuming the gas that is actually pushing on the projectile is massless. By this we mean that no energy is being applied to accelerate the mass of the gas. We will remove this assumption later in our studies. What we do not like about [Equation 2.16](#) is that temperature still appears as a variable.

By our earlier assumptions, we stated that the process was frictionless and adiabatic. Recall, again from thermodynamics, that this actually defines an isentropic process [1]. For a closed system (one with constant mass), it can be shown [6] that the absolute temperature,  $T$ , of our system is related to the initial temperature of the gas,  $T_i$ , through

$$T = T_i \left( \frac{V_c}{V} \right)^{(\gamma-1)} \quad (2.17)$$

In [Equation 2.17](#),  $V$  is the volume at a given time  $t$ ,  $V_c$  is the initial chamber volume, and  $\gamma$  is the specific heat ratio of the gas (defined later). If we substitute [Equation 2.17](#) into [Equation 2.16](#), we can write

$$W = m_g R T_i V_c^{\gamma-1} \int_{V_c}^V V^{-\gamma} dV \quad (2.18)$$

This equation is easy to work with because we know most of the terms on the RHS (right-hand side) when we set up our pedagogical gun. We know the mass,  $m_g$ , of the gas. We know  $R$  and  $\gamma$  because we picked which gas it was. We know the initial temperature of the gas and we know the chamber volume.

Now that we did all of this work with volumes, we want to convert these back to distances. A typical output desired by ballisticians is the pressure versus travel (i.e., distance) curve. This plot helps the gun designer determine where to make his tube thick and where he can get away with thinning the wall. If we again recognize that our gun has a constant inner diameter, we can use [Equation 2.14](#) to write [Equation 2.18](#) as

$$W = m_g R T_i l^{(\gamma-1)} \int_0^L (l+x)^{-\gamma} dx \quad (2.19)$$

If we perform this integration, we obtain

$$W = \frac{m_g R T_i l^{(\gamma-1)}}{(1-\gamma)} \left[ (l+L)^{(1-\gamma)} - l^{(1-\gamma)} \right] \quad (2.20)$$

We need to recall from dynamics that the kinetic energy of the projectile can be written as

$$\text{K.E.}_{\text{projectile}} = \frac{1}{2} m_p V_m^2 \quad (2.21)$$

If we assume that all of the energy of the gas is converted with no losses into kinetic energy of the projectile, then we can use [Equation 2.15](#) to state that

$$\text{K.E.}_{\text{projectile}} = W \quad (2.22)$$

We can make use of Equations 2.20 and 2.21 to write this as

$$\frac{1}{2} m_p V_m^2 = \frac{m_g R T_i l^{(\gamma-1)}}{(1-\gamma)} \left[ (l+L)^{(1-\gamma)} - l^{(1-\gamma)} \right] \quad (2.23)$$

This is an important result as it relates muzzle velocity to the properties and amount of the gas used, the mass of the projectile, and includes the effect of tube length. We can use this equation to estimate muzzle velocity. So a convenient form of this equation is

$$V_m = \sqrt{2 \frac{m_g}{m_p} \frac{R T_i l^{(\gamma-1)}}{(1-\gamma)} \left[ (l+L)^{(1-\gamma)} - l^{(1-\gamma)} \right]} \quad (2.24)$$

In some instances, we would like to use these relationships to determine the state of the gas or velocity of the projectile at some point in the tube other than the muzzle. If this is the case, the procedure would be as follows:

- (1) Solve for the work term up to the position of interest,  $x_{\text{proj}}$ , using

$$W(x_{\text{proj}}) = m_g R T_i l^{(\gamma-1)} \int_0^{x_{\text{proj}}} (l+x)^{-\gamma} dx \quad (2.25)$$

- (2) Determine the volume at the position of interest using

$$V(x_{\text{proj}}) = \frac{\pi d^2}{4} (l + x_{\text{proj}}) \quad (2.26)$$

- (3) Determine the gas temperature at this position from [Equation 2.17](#).

- (4) Determine pressure from the ideal gas [Equation 2.4](#).

This procedure is relatively straightforward.

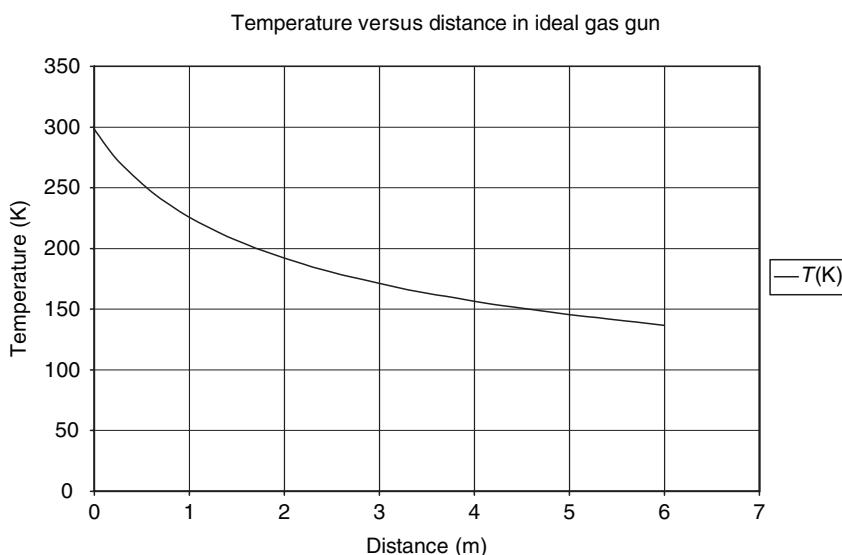
If, as an example, we look at an idealized 155-mm compressed air gun and assume the following parameters

Projectile weight = 100 lb<sub>m</sub>

Initial pressure = 45 MPa (approximately 6500 psi)

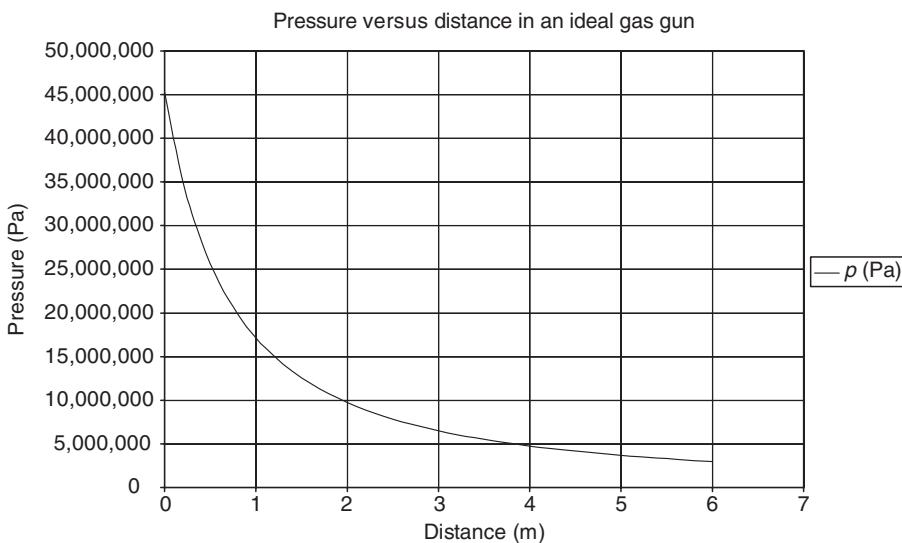
Tube length = 6 m

From Figures 2.2 through [2.4](#), we can depict the results of a calculation for temperature, pressure, and velocity versus distance for this idealized situation.



**FIGURE 2.2**

Temperature versus distance in an ideal gas gun.

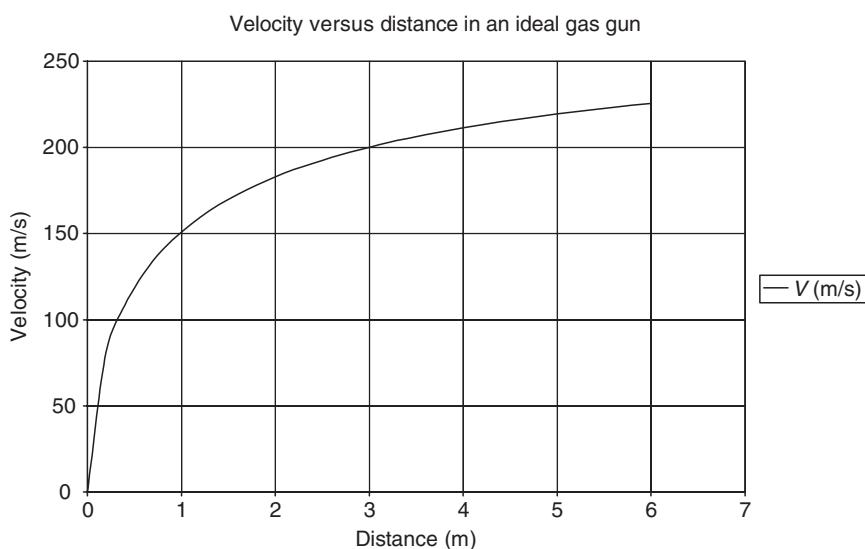


**FIGURE 2.3**  
Pressure versus distance in an ideal gas gun.

### Problem 1

Assume we have a quantity of 10 g of 11.1% nitrated nitrocellulose ( $C_6H_8N_2O_9$ ) and it is heated to a temperature of 1000 K assuming it changes from solid to gas somehow without changing chemical composition. If the process takes place in an expulsion cup with a volume of 10 in.<sup>3</sup>, assuming ideal gas behavior, what will the final pressure be in pounds per square inch?

$$\text{Answer: } p = 292 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$



**FIGURE 2.4**  
Velocity versus distance in an ideal gas gun.

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## 2.2 Other Gas Laws

There are many times when ideal gas behavior is insufficient to model real gases. This is certainly true under the pressures and temperatures of gun launch. Although there are many models that attempt to account for the deviation of real gases from ideal or perfect behavior [2,3], we shall examine only two, the simplest of which we shall use.

Ideal gas behavior is approached when the distance between molecules (known as the mean free path) is large. Thus, molecules do not collide or interact with one another very often. Temperature is a measure of the internal energy of the gas. Thus, when the temperature is high, the molecules are moving around faster and have more of an opportunity to interact with one another. Pressure is a result of how closely the molecules are packed together, thus a higher pressure tends to put the molecules in close proximity. It is for these reasons that we cannot normally use the ideal gas law in gun launch applications.

The Noble–Abel equation of state is given by

$$p(V - m_g b) = m_g RT \quad (2.27)$$

Here  $p$  is the pressure of the gas,  $V$  is the volume the gas occupies,  $m_g$  is the mass of the gas,  $R$  is the specific gas constant,  $T$  is the absolute temperature, and  $b$  is the co-volume of the gas.

The co-volume of the gas has been described as a parameter which takes into account the physical size of the molecules and any intermolecular forces created by their proximity to one another. Think of it as not having physical meaning but as simply a number which allows for a better fit to observed experimental data. The units of the co-volume are cubic length per mass unit. Usually, the gas co-volume is provided in the literature but an estimation tool has been provided by Corner [7] which will not be repeated here since actual data exists.

Occasionally, the Noble–Abel equation of state is insufficient to suit our needs. At these times, it is typical to use a Van der Waals equation of state given by

$$p = \frac{\tilde{R}T}{\tilde{v} - b'} - \frac{a'}{\tilde{v}^2} \quad (2.28)$$

In this case,  $p$  is again the pressure of the gas,  $\tilde{v}$  is the molar specific volume,  $\tilde{R}$  is the molar specific gas constant, unique to each gas,  $T$  is again the absolute temperature, and  $a'$  and  $b'$  are constants particular to the gas.

The Noble–Abel equation of state is the basis for nearly all of our work in this text, therefore Equation 2.27 is very important. At times, we may write it a little differently but you will always be reminded of where it originated.

### Problem 2

Perform the same calculation as in [Problem 1](#), but use the Noble–Abel equation of state and assume the co-volume to be 32.0 in.<sup>3</sup>/lbm

$$\text{Answer: } p = 296.3 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

---

## 2.3 Thermophysics and Thermochemistry

The main energy exchange process of conventional interior ballistics is through combustion. Once ignited, the chemical energy of the propellant is released through an

oxidation reaction. This energy release will be in the form of heat, which, in turn, increases the pressure in the volume behind the projectile (i.e., in a combustion chamber). The pressure exerts a force on the projectile, which accelerates it to the desired velocity.

In general, combustion requires three main ingredients to commence: a fuel, an oxygen source, and heat. In a common combustion reaction, such as an internal combustion engine like the one in your car, oxygen is supplied to the reaction independently of the fuel. The heat in this case is generated by a spark ignition and the burning of the air-fuel combination that ensues.

A gun chamber has very little room for oxygen once it is stuffed with propellant. It is important to note that, for other reasons, there is always free volume in the chamber (called ullage)—we will explain this later. For now, we should understand that although there is some oxygen in the chamber, the amount is insufficient to completely combust the propellant. It is for this reason that propellants are formulated to contain both the fuel and the oxidizer. In general, the propellant burning is an under-oxidized reaction. This has some implications as the propellant gases leave the muzzle—again, we shall discuss this in more detail later.

This brief introduction should make clear the reason to examine thermochemistry, thermophysics, and combustion phenomena. To proceed, we shall first define each field of study. The definitions of Ref. [8] shall be used here to describe the first two topics as they are extremely straightforward and clear. Thermophysics is defined as the quantification of changes in a substance's energy state caused by changes in the physical state of the material. An example of this would be the determination of the amount of energy required to vaporize water in your teapot. Thermochemistry is then the quantification of changes in a substance's energy state caused by changes in the chemical composition of the material's molecules. An example of this would be the energy required to dissociate (break up) water molecules into hydrogen and oxygen. Combustion is defined in Ref. [1] as the quantification of the energy associated with oxidizer-fuel reactions. Thus, combustion is a natural outgrowth of thermophysics and thermochemistry.

Now that we have categorized these three fields of study, we shall attack them in a somewhat jumbled order. The reason for this is that, from our perspective, we really need not distinguish between any of them and all of them appear in our gun launch physics. It is also important to realize that whether the energy change comes from a chemical reaction or a phase change from solid to gas, as long as we can calculate the extent of the energy change, we can perform a valuable analysis.

Energy to all intents and purposes consists of two types: potential and kinetic. Potential energy can be considered as stored energy. There are many ways to store energy. We can store energy by compressing a steel bar or spring, by lifting a mass to a higher elevation in the earth's gravitational field, and by chemically preparing a compound that, whether by combustion or chemical reaction, will release energy. Each of these forms of potential energy elastic strain, gravitational potential and chemical potential energy, has a different method of storing and releasing the energy but they are all potential energies. There are other forms of potential energy but we need not deal with them in this context.

Kinetic energy is the energy of a mass in motion. It can be observed in objects that are in translational motion or in rotational motion. To extract some or all of this energy, it is necessary to slow or stop the moving mass that has the kinetic energy. The energy in a spinning flywheel is an example of rotational kinetic energy.

The field of thermodynamics is the study of energy transformations. It quantifies the balance of energy between kinetic and potential. In thermodynamics, it is common to see two energy transformation mechanisms: heat and work.

Heat transfer is essentially an exchange of energy through molecular motion. As we shall soon see, molecules of a substance are always in motion. The faster they are in motion, the

hotter the substance is. These molecules can influence other molecules when they are placed in contact with them, thus giving up some of their energy and increasing the energy of the contacted substance. Temperature is a sensible measure of an object's internal energy.

Work is a means of increasing an object's energy by application of a force through a distance. This method of energy transfer can create either potential energy, as in compressing a spring, or kinetic energy as applied to a free, rigid mass. While the equations for heat transfer can be the subject of entire texts (e.g., [9]), work can be defined through the vector equation

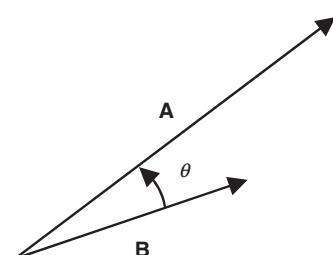
$$W = \mathbf{F} \cdot d\mathbf{x} \quad (2.29)$$

Here  $W$  is the work done on or by the system,  $\mathbf{F}$  is the force vector, and  $d\mathbf{x}$  is the vector distance through which the force acts, known as the displacement vector. We must note that this is a *vector* equation. The work term is a scalar because the dot product of two vectors results in a scalar. Because of the dot product term, the sign of  $W$  is dependent upon the cosine of the angle between  $\mathbf{F}$  and  $d\mathbf{x}$ . Recall the definition of a dot product as

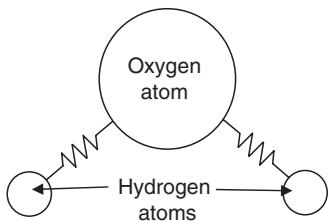
$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \quad (2.30)$$

Here  $A$  and  $B$  are the scalar magnitudes of the vectors  $\mathbf{A}$  and  $\mathbf{B}$  (Figure 2.5). If we use Equation 2.30 with the variables of Equation 2.29, this tells us that if the angle between the force vector and the displacement vector is between  $0^\circ$  and  $90^\circ$  or  $270^\circ$  and  $0^\circ$ , the work is positive, i.e., it is work performed on the system. If, however the angle is between  $90^\circ$  and  $270^\circ$ , the work is negative, and therefore work performed by the system.

Internal energy,  $U$ , of a substance can be considered a form of potential energy. Some authors [5] categorize the internal energy separately from potential and kinetic energies. This can clearly be done in general, but for the application of gun launch it seems proper to group it as a potential energy. The internal energy of a substance is manifested in the molecular motions within that substance. These motions generally are translational or vibrational in nature. The molecules of a substance are attracted to and repelled by one another and are in some degree of translational motion. Additionally, the attractive or repulsive forces within a molecule itself allow us to use an analogy of springs holding the atoms together. Imagine a structure of a water molecule, for instance as depicted in Figure 2.6. If the oxygen and hydrogen atoms are assumed to be steel balls and the molecular bond springs, we could pick this molecule up, hold the oxygen atom, and shake it. If the springs were really stiff in bending and much less so in tension or compression, we would see the hydrogen atoms oscillating in and out at some frequency. The greater the frequency, the more energy we would need to put into the system. Even though the springs are stiff in bending, it does not mean that they cannot bend. This just takes more energy. Like springs, we can store energy in the molecules this way.



**FIGURE 2.5**  
Depiction of two vectors for scalar product definition.



**FIGURE 2.6**  
Model of a water molecule.

This simple model of a molecule is a crude but useful approximation. Imagine now that we put our model on a frictionless surface, like an ice hockey rink. If we hit the molecule in a random way, we will excite these vibrational modes as well as create translational and rotational motion. Now, if we fill the ice hockey rink with models...well, you get the idea. As stated previously, the level of this interaction (collisions) must be represented somehow. The metric used is internal energy with the level of activity defined as zero at the temperature known as absolute zero ( $0^\circ$  on the Kelvin or Rankine scales).

The internal energy also includes the energy required to maintain a particular phase of the material such as solid, liquid, or gas. Additionally, certain phases associated with molecular structure such as face-centered cubic (FCC), body-centered cubic (BCC), etc. are accounted for in the internal energy.

Quite often we shall see internal energy and what is commonly known as “ $p\text{d}V$ ” work terms together in our energy balance equations. The term is called  $p\text{d}V$  work because it is special and separate from work generated by, say, a paddle wheel moving fluid around. This work term arises from pressure pushing on a given volume. If the volume changes by an infinitesimal amount,  $\text{d}V$ , we essentially have force acting through a distance. To prove this to yourself, look at the units. Because we see these terms together so often, it is convenient for us to group them into one term, which we will call enthalpy,  $H$ . Mathematically, the enthalpy is defined as

$$H = U + pV \quad (2.31)$$

Notice here that we have removed the differential from the work term. The reason for this is that, considering both enthalpy and internal energy, we are concerned with changes in  $H$  and  $U$ . Therefore, the differential appears when we write the entire equation in differential form as

$$\text{d}H = \text{d}U + p\text{d}V \quad (2.32)$$

For proof of this result, refer to any thermodynamics text (e.g., [1,5]). An example of the difference between internal energy and enthalpy is the rigid container or piston container. Consider a rigid container that has some amount of gas in it. Assume the container is sealed so that matter cannot enter or leave. Let us also assume that the container will allow energy to be transferred to and from the gas. If we transfer heat (energy) to the gas, the temperature will rise as will the pressure. Since the volume of the container is fixed, no work can be done; thus all of the energy added to the gas is internal energy. From Equation 2.32, we see that in this case the change in enthalpy would be exactly equal to the change of internal energy.

Now we assume that, instead of our container being rigid, the roof of the container is a sealed yet moveable piston. In this case, once again matter cannot escape, however, the volume is able to change. Now the only thing holding up the roof is the pressure of the gas acting to just counteract the weight of the roof itself. Let us add the same amount of heat

that we added to the original, rigid, container. In this case, the temperature of the gas will increase (but less than before) and the volume will increase because the piston is moveable and the pressure must remain constant and just sufficient to counteract the weight of the roof. In this instance, the enthalpy would be greater than the internal energy because it includes the work done in lifting the piston.

When a substance changes form, chemically or physically, energy is either absorbed or released. The method that we use to quantify this energy change is through heats of formation and the like. Though called a “heat,” what is really implied is an enthalpy change. We shall proceed through these different enthalpy changes, attempting to list some of the more common ones. For greater detail, the reader is encouraged to consult thermodynamics texts in addition to the descriptions provided in Ref. [8]. Specific values for text problems will be given as needed. It is not the intent of the authors to tabulate the different energy parameters of different materials.

When a substance is formed, atomic bonds in the constituent molecules are destroyed and then recreated (at least this is a clean way to think of it from a bookkeeping perspective). The energy absorbed or generated by this process is commonly called the heat of reaction,  $\Delta H_r^0$ . The  $\Delta$  reminds us that we always are concerned with changes in enthalpy from a particular reference state (usually standardized as 25°C and 1 atm). The “0” superscript is a convenient reminder that this is from a reference state of 1 atm. As the subscript, sometimes we see “298” meaning 298 K. Though 298 K and 25°C are the same value, one must always be wary of the reference state chosen by a particular author.

The heat of formation,  $\Delta H_f^0$ , is the energy required to form a particular substance from its individual component atoms. The heats of formation are the building blocks that determine the heat of reaction. Any elemental substance in its stable configuration at standard conditions has a heat of formation equal to zero at that state. For instance, diatomic nitrogen, N<sub>2</sub>, has  $\Delta H_f^0 = 0$  at 25°C and 1 atm. We will provide an example of the heat of formation calculation in a later section.

Now that with the above quantities defined, we can write an equation for the heat of reaction

$$\Delta H_r^0 = \sum_{\text{products}} \Delta H_f^0 - \sum_{\text{reactants}} \Delta H_f^0 \quad (2.33)$$

Equation 2.33 states that the heat of reaction for a given substance is equal to the sum of the heats of formation of the final products of the reaction that created the substance minus the sum of the heats of formation of the materials that had to be reacted together to create the new substance. This is further reinforcement of the definition of the heat of reaction. Recall that we stated the atomic bonds of the molecules were destroyed and then remade. This is essentially what Equation 2.33 is saying. The energy it took to create each of the reactants has to be accounted for and then the energy it takes to create the new substances from the constituents is calculated—energy is conserved. If the heat of reaction is a negative number, heat is liberated by the reaction otherwise it is absorbed.

When a compound is specifically combusted with sufficient oxygen to attain its most oxidized state, the heat of reaction has a special name the heat of combustion. The heat of combustion is identified by the symbol  $\Delta H_c^0$ . The heat of combustion is typically what is obtained when propellant is burned in a closed bomb. The equation for the heat of combustion mirrors that of the heat of reaction, the only difference being as noted above.

$$\Delta H_c^0 = \sum_{\substack{\text{fully oxidized} \\ \text{products}}} \Delta H_f^0 - \sum_{\text{reactants}} \Delta H_f^0 \quad (2.34)$$

The heats of detonation and explosion have meanings which seem to be reversed. The heat of detonation is the heat of reaction taken when detonation products are formed from an explosive compound during a detonation event. The formula for the heat of detonation is given by

$$\Delta H_d^0 = \sum_{\text{detonation products}} \Delta H_f^0 - \sum_{\text{original explosive}} \Delta H_f^0 \quad (2.35)$$

What is termed the heat of explosion is the amount of energy released when a propellant or explosive is burned (not detonated) and is given by

$$\Delta H_{\text{exp}}^0 = \sum_{\text{burning products}} \Delta H_f^0 - \sum_{\text{original propellants}} \Delta H_f^0 \quad (2.36)$$

The heat of afterburn is another type of heat of reaction that occurs often in propellants and explosives. Because the composition of propellants and explosives usually force an under-oxidized reaction, the reaction products will often combine with the oxygen present in the air outside the gun or explosive device, given sufficient temperature and pressure. This secondary reaction results in a second pressure wave or blast and a fireball. The heat of afterburn can be described mathematically as

$$\Delta H_{\text{AB}}^0 = \sum_{\text{fully oxidized products}} \Delta H_c^0 - \sum_{\text{remaining detonation products}} \Delta H_d^0 \quad (2.37)$$

Not all energy changes involve chemical reactions. We mentioned earlier that changes in physical state and structure require energy. When a solid melts to form a liquid or a liquid solidifies, we call the energy required, the latent heat of fusion,  $\lambda_f$ . These values are tabulated in any chemistry book or thermodynamics text. Some authors use different symbols so one must, as always, be careful.

In a similar vein, the energy required to vaporize a liquid to a gas or condense a gas to a liquid is known as the latent heat of vaporization and given by the symbol  $\lambda_{fg}$ .

If a material changes the structure of its atoms, say from BCC to FCC, the energy is known as the heat of transition,  $\lambda_t$ .

There are many other types of material transitions that require energy. The types described above cover the needs of this work.

## 2.4 Thermodynamics

The combustion process that occurs in a gun is a thermodynamic process. The term thermodynamics is a bit misleading because it implies that the dynamics of the combustion process is examined. This is not quite true. Classical thermodynamics is based on the examination of the various processes through equilibrium states. This is somewhat akin to frames of a motion picture. We examine the state of the system before some event and we usually examine it at some point, later in time, we are interested in.

Some of the concepts of thermodynamics were introduced in earlier sections, work and energy being the major ones. Here we shall look in detail at two ways of describing thermodynamic systems to proceed with our study.

We shall define energy for an arbitrary system as

$$E = U + \frac{1}{2}mV^2 + mgz \quad (2.38)$$

Equation 2.38 is our extensive form of the definition of the system energy,  $E$ . In this equation,  $U$  is the internal energy,  $m$  is the system mass,  $V$  is the system velocity,  $g$  is a gravitational constant, and  $z$  is some height above a reference datum. The second and third terms on the RHS of the equation are the kinetic and potential energies, respectively. If we examine this equation, it is easy to see why some authors group the internal energy as a separate energy type. However, in the case of a gun launch, the potential energy term is insignificant. This focuses us on the transfer of energy between internal and kinetic.

We sometimes write Equation 2.38 in its intensive form as

$$e = u + \frac{1}{2}V^2 + gz \quad (2.39)$$

Recall from our earlier discussions that an intensive property is the associated extensive property divided by mass.

We shall now examine the first law of thermodynamics as it is applied to two different types of systems: a fixed mass of material and a fixed volume of space through which material flows. The first type of analysis, where the material is a fixed mass, is known as a Lagrangian approach, while the fixed or control volume (CV) approach is known as Eulerian. Both are important from a ballistic analysis standpoint and are prevalent in interior, exterior, and terminal ballistic studies.

For a fixed mass of material, undergoing some thermodynamic process, the first law of thermodynamics can be written as

$$Q_{1-2} + W_{1-2} = \Delta E_{1-2} \quad (2.40)$$

In this equation,  $Q$  is the heat or energy added to the system,  $W$  is the work performed on or by the system, and  $\Delta E$  is the change in the energy state of the material. The subscript 1–2 simply lets us know that the process began at some state 1 and ends at some state 2. The signs on the terms are very important. We assume a positive change in energy comes about through adding heat to the system and doing work on the system. Thus, work performed on the system is positive and heat added is also positive. Different thermodynamics texts write the first law slightly different, but if you understand that the net result of work on the system or heat transfer to the system is to increase its energy, then few mistakes will be made.

An interesting observation of Equation 2.40 is that the energy state change has an infinite number of paths that lead to the same result. For instance, if we wanted to add 24 kJ of energy to some arbitrary system, we could do it by adding 12 kJ of heat and performing 12 kJ of work on the system. We could obtain the same result by adding 36 kJ of heat and extracting 12 kJ of work from the system. The possibilities are limitless. This reinforces our assertion that thermodynamics is really only concerned with end states.

Caution is warranted at this point. Equation 2.40 does not say how the energy, once added to the system, is partitioned between potential (internal) energy or kinetic energy. This reveals something. Heat and work are added to or removed from the system at the system boundaries while the distribution of energy between internal or kinetic energy is done within the system.

We shall now write out Equation 2.40 explicitly for a Lagrangian system

$$Q_{1-2} + W_{1-2} = m \left[ \left( u_2 + \frac{1}{2} V_2^2 \right) - \left( u_1 + \frac{1}{2} V_1^2 \right) \right] \quad (2.41)$$

Here we have neglected the gravitational potential energy terms and used the intensive form of the energy, multiplied by the system mass. As previously stated, many times we would like to use enthalpies instead of internal energies. If this is the case, we can rewrite Equation 2.41 using our relationship between the two from Equation 2.40. We shall use the intensive form of Equation 2.40 to yield

$$Q_{1-2} + W_{1-2} = m \left[ \left( h_2 - pv_2 + \frac{1}{2} V_2^2 \right) - \left( h_1 - pv_1 + \frac{1}{2} V_1^2 \right) \right] \quad (2.42)$$

Here we note that  $h$  is the specific enthalpy and  $v$  is the specific volume.

We shall now examine the first law of thermodynamics in the Eulerian frame of reference. Recall that in the Eulerian frame, we chose a CV (real or imaginary) and observed how the energy within the volume changes based upon the energy carried into or out of it by any entering or exiting substance as well as any heat or work done at the system boundaries. It is convenient for us to write the first law in terms of the time rate of change of energy, heat, and work. We start by writing Equation 2.40 as a rate equation

$$\frac{dQ}{dt} + \frac{dW}{dt} = \frac{dE}{dt} \quad (2.43)$$

or

$$\dot{Q} + \dot{W} = \dot{E} \quad (2.44)$$

Here the dots over the heat and work terms indicate the time rate of change of the variable. Proper thermodynamics terminology would require us to use the “ $\delta$ ” instead of “ $d$ ” in Equation 2.43 because of path dependency considerations, but for our purposes we shall ignore this fact. The reader is advised to consult any thermodynamics text for a better understanding of the difference.

The substitutions that were performed to arrive at Equation 2.41 are not as straightforward in this case. Because we have material entering and leaving the CV, we can imagine that this material can enter or leave with a different pressure and density as it interacts with our fixed CV. Because of this, we must account for the energy used to make these changes. Alternatively, one can envision the material coming in at a higher pressure or density and wanting to push our imaginary CV outward, but since we fixed our CV it cannot. The energy from this must go somewhere so it works on the fluid in and around our CV. Mathematically, this results in the energy term in Equation 2.44 having to include a  $pv$  term. This is sometimes known as flow work [10]. With this in mind, Equation 2.44 can be written as

$$\dot{Q} + \dot{W} = \dot{m}_{\text{out}}(e_{\text{out}} + p_{\text{out}}v_{\text{out}}) - \dot{m}_{\text{in}}(e_{\text{in}} + p_{\text{in}}v_{\text{in}}) \quad (2.45)$$

Here by multiplying the intensive properties by the mass flow rate,  $\dot{m}$ , we have the rate of change of the energy terms. We have also arbitrarily assumed one inlet and one outlet. If more inlets or outlets in our CV were present and they had different mass flow rates

or pressures, we would have to consider each with a term identical to our outlet or inlet terms above. We now can make the substitution for our energy terms to yield

$$\dot{Q} + \dot{W} = \dot{m}_{\text{out}} \left( u_{\text{out}} + \frac{1}{2} V_{\text{out}}^2 + p_{\text{out}} v_{\text{out}} \right) - \dot{m}_{\text{in}} \left( u_{\text{in}} + \frac{1}{2} V_{\text{out}}^2 + p_{\text{in}} v_{\text{in}} \right) \quad (2.46)$$

In this case, we have also assumed a uniform velocity over the inlets and outlets. With one inlet and outlet, the mass flow in must equal the mass flow out so we can write Equation 2.46 as

$$\dot{Q} + \dot{W} = \dot{m} \left[ \left( u_{\text{out}} + \frac{1}{2} V_{\text{out}}^2 + p_{\text{out}} v_{\text{out}} \right) - \left( u_{\text{in}} + \frac{1}{2} V_{\text{out}}^2 + p_{\text{in}} v_{\text{in}} \right) \right] \quad (2.47)$$

Substitution of enthalpy into the above equation puts it into a compact form:

$$\dot{Q} + \dot{W} = \dot{m} \left[ \left( h_{\text{out}} + \frac{1}{2} V_{\text{out}}^2 \right) - \left( h_{\text{in}} + \frac{1}{2} V_{\text{out}}^2 \right) \right] \quad (2.48)$$

In many fluid dynamics texts, there are wonderful examples of how these equations are used with multiple inlets and outlets [11]. You may be asking yourself how useful are these equations if we only use one inlet or outlet? The answer is that they are very useful. Except for flows through muzzle devices or through internal ports like bore evacuators and ports for automatic weapons, a gun is a right circular tube that contains the propellant gas. Any flow field analysis we perform on the moving gases will have just one inlet (toward the breech) and one outlet (toward the projectile). Thus, as we develop our equations later for in-bore motion, we can use these simple equations in the above form.

As a review, we have two equations that state the first law of thermodynamics. For a fixed mass of material (Lagrangian frame), we have

$$Q_{1-2} + W_{1-2} = m \left[ \left( h_2 - p v_2 + \frac{1}{2} V_2^2 \right) - \left( h_1 - p v_1 + \frac{1}{2} V_1^2 \right) \right] \quad (2.42)$$

and for a fixed volume that material can flow in and out of (Eulerian frame)

$$\dot{Q} + \dot{W} = \dot{m} \left[ \left( h_{\text{out}} + \frac{1}{2} V_{\text{out}}^2 \right) - \left( h_{\text{in}} + \frac{1}{2} V_{\text{out}}^2 \right) \right] \quad (2.48)$$

These equations have been repeated here because of their critical importance to our work.

In many instances, we will find that we require a relationship between internal energy or enthalpy and temperature. If we have a gas that is not reacting and intermolecular forces are small enough to ignore, we can consider the gas to be thermally perfect [12]. The implications of this are that internal energy and enthalpy are functions of the temperature alone. With this model, we can write expressions for internal energy and enthalpy as follows:

$$du = c_v dT \quad (2.49)$$

$$dh = c_p dT \quad (2.50)$$

Here  $c_v$  is the specific heat at constant volume and  $c_p$  is the specific heat at constant pressure. Normally,  $c_p$  and  $c_v$  vary with temperature. In many practical cases, this variation

is small and we can further assume that the gas is calorically perfect which results in the above equations being written as

$$u = c_v T \quad (2.51)$$

$$h = c_p T \quad (2.52)$$

For a thermally or calorically perfect gas (not a reacting gas), there is a relationship between  $c_p$ ,  $c_v$ , and  $R$ . If we define  $\gamma$  as the ratio of specific heats where

$$\gamma = \frac{c_p}{c_v} \quad (2.53)$$

then we can write the aforementioned relationships as

$$c_p - c_v = R \quad (2.54)$$

$$c_p = \frac{\gamma R}{\gamma - 1} \quad (2.55)$$

$$c_v = \frac{R}{\gamma - 1} \quad (2.56)$$

The second law of thermodynamics defines the concept of entropy for us [1]. We know from the second law of thermodynamics that

$$Tds = du + pdv \quad (2.57)$$

or, if we insert the definition of enthalpy

$$Tds = dh - vdp \quad (2.58)$$

If we evaluate Equations 2.57 and 2.58 under the assumptions of a calorically perfect gas, we obtain

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right) \quad (2.59)$$

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{v_2}{v_1}\right) \quad (2.60)$$

In these expressions, the subscripts "1" and "2" indicate the initial and final states of the substance, respectively. An isentropic process is a process in which there is no entropy change. This is also known as a reversible process. In a real system, entropy must always increase or, at best, stay constant. Many processes have slight enough entropy increases as to be considered isentropic. Isentropic processes also are excellent to examine as theoretical limits on real processes. If we examine Equations 2.59 and 2.60 under an isentropic assumption, we see that the left-hand side (LHS) is zero in both. This has implications that allow us to write (for an isentropic process)

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma = \left(\frac{v_2}{v_1}\right)^{-\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \quad (2.61)$$

### Problem 3

The M898 SADARM projectile weighs 102.5 lb. The projectile was fired from a 56 caliber, 155-mm weapon and a pressure-time trace was obtained. The area under the pressure-time curve was (after converting the time to distance) calculated to be 231,482 psi-m. Calculate the muzzle energy of the projectile in megajoules. Assume the bore area to be 29.83 in.<sup>2</sup>

Answer:  $E = 30.7[\text{MJ}]$ .

### Problem 4

An 8-in. Mk. 14 Mod. 2 Navy cannon is used at NSWC Dahlgren, VA for “canister” firings. These firings are used to gun harden electronics which are carried in an 8-in. projectile. The projectile used weighs 260 lb. The measured muzzle velocity is around 2800 ft/s. Calculate the muzzle energy of the projectile in megajoules. Assume the bore area to be 51.53 in.<sup>2</sup> The rifled length of the tube (distance of projectile travel) is 373.65 in.

Answer:  $E \approx 43[\text{MJ}]$ .

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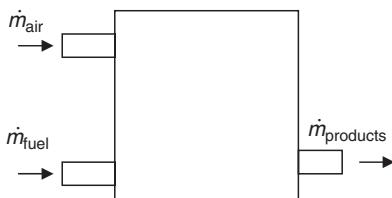
## 2.5 Combustion

As stated in the previous two sections, combustion is the process through which the energy of the solid propellant is converted to useful work. The purpose of this section is to quantify the oxidation reaction. The tactic we shall employ is to examine the more common, everyday combustion processes which combine (relatively) simple fuels with air to produce work. In this way, we shall, hopefully, bring to mind the combustion thermodynamics that has been taught at an undergraduate level and perhaps has been forgotten or not exercised since it was first learned.

If we utilize the concept of a fixed CV, we can imagine a combustion chamber as depicted in Figure 2.7. In this CV, we can envision a mass of fuel entering as well as some mass of air. The two are then combusted with one another and the gaseous products leave as a mixture. We can write the first law of thermodynamics for this system then as in [Equation 2.79](#), which we shall repeat here with subscripts that reflect Figure 2.7

$$\dot{Q} + \dot{W} = \dot{m}_{\text{products}} \left( h_{\text{products}} + \frac{1}{2} V_{\text{products}}^2 \right) - \dot{m}_{\text{air}} \left( h_{\text{air}} + \frac{1}{2} V_{\text{air}}^2 \right) - \dot{m}_{\text{fuel}} \left( h_{\text{fuel}} + \frac{1}{2} V_{\text{fuel}}^2 \right) \quad (2.62)$$

In [Equation 2.62](#), we can see how the heat and energy generated are affected by the amount of mass flow, the enthalpies, and the velocities of the fuel, the oxidizer (air in this case), and the product gases. We require some means of determining the energy converted through the chemical reaction. We achieve this through the balancing of the chemical reaction. We shall return to [Equation 2.62](#) once we have discussed chemical reactions.



**FIGURE 2.7**

Fixed control volume (CV) combustion chamber.

One of the most important compounds in the study of combustion is air. We shall adopt a convention that is standard in many thermodynamics texts [5,13,14] that models air as 21% diatomic oxygen ( $O_2$ ) and 79% diatomic nitrogen ( $N_2$ ). This means that every mole of oxygen carries with it 3.76 moles of nitrogen. This relationship comes about because

$$\frac{0.79 \left[ \frac{\text{moles } N_2}{\text{mole air}} \right]}{0.21 \left[ \frac{\text{moles } O_2}{\text{mole air}} \right]} = 3.76 \left[ \frac{\text{moles } N_2}{\text{mole } O_2} \right] \quad (2.63)$$

As can be seen in [Appendix B.1](#), the molecular weight for our simple model of air is 28.97 kg/kg-mol.

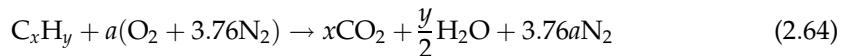
The balancing of a chemical reaction determines what the molecular composition of the combustion products will be and furthermore helps us to quantify the amount of energy absorbed or released. If energy is absorbed in a chemical reaction, in other words, if we had to add energy to force the reaction to completion, the reaction is said to be endothermic. If heat is liberated, the reaction is said to be exothermic [15].

A reaction can be said to be theoretically or stoichiometrically balanced if the reaction goes to completion and there is no excess oxygen in the products [1]. We shall define a complete reaction as one in which all of the oxygen combines first with all of the hydrogen to form steam and then with all the carbon to form carbon dioxide. Oxygen has a greater affinity for combining with hydrogen than with carbon [1]. The only time that carbon monoxide ( $CO$ ) will be formed is if there is insufficient oxygen. We must keep in mind that in any real reaction there will usually be some amounts of carbon monoxide and other compounds such as nitric oxide ( $NO$ ) in the combustion products. We shall return to this issue later. For the time being, we shall assume that the only reaction products in the stoichiometric reaction are  $CO_2$  and  $H_2O$ . The balancing of these chemical reactions is an important part of our study of the combustion process which we shall now examine.

We shall use two convenient forms of chemical equations: a molar-based equation and a mass-based equation. In the molar-based equation, we shall usually combust one mole of fuel with some amount of air. The result may be multiplied by the number of moles of fuel actually burned to obtain a final answer. When the mass-based equation is employed, we generally use one mass unit of fuel (lbm or kg) and some amount of air, again multiplying the solution by whatever the actual mass of fuel happens to be. The techniques just described are applicable to a system where the mass is fixed. The same equations can be used with mass or molar flow rates if the system happens to be a steady flow or open system.

It is informative to balance the chemical reactions in the context of everyday systems that combust a fuel with air. Usually, this fuel is a hydrocarbon composition. The stoichiometric amount of air required would be enough so that all of the carbon combusts with sufficient oxygen to form  $CO_2$  and all of the hydrogen combusts to form water or steam.

If we had a hydrocarbon fuel of chemical composition  $C_xH_y$ , we would like to find the number of moles,  $a$ , of air required to completely combust the fuel and we would write the balanced chemical reaction as



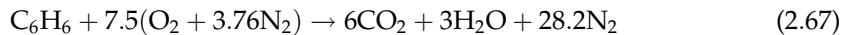
We could solve for  $a$  to yield

$$a = x + \frac{y}{4} \quad (2.65)$$

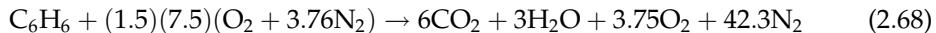
As an example, let us say we have one mole of Benzene ( $C_6H_6$ ) that we would like to burn in air. The balanced, stoichiometric equation would be found by first determining  $a$  from [Equation 2.65](#)

$$a = 6 + \frac{6}{4} = 7.5 \quad (2.66)$$

Now the balanced equation is found using [Equation 2.64](#)



This is an example of a stoichiometrically balanced equation using a molar basis. There are times when a particular fuel is burned with too much air (over oxidized) or too little air (under oxidized). The latter is usually the case with propellants in the chamber of a gun. When a fuel is over oxidized, we usually categorize it by stating how much excess air is included in the reaction. For instance, 50% excess air used in the reaction of Equation 2.67 would alter the balanced equation to be written as



If the fuel were burned with 50% deficient air we would have



In this case, we have used the rules set forth earlier where steam is formed first then carbon monoxide. At this point, all of the oxygen has been used up so solid carbon is formed. From this simple example, you can see that the amount of air used in the combustion is critical to determination of the products.

We can now define an air-fuel ratio as the ratio mass of air combusted to the mass of fuel combusted. This is given mathematically by

$$A - F = \frac{m_{air}}{m_{fuel}} = \frac{\dot{m}_{air}}{\dot{m}_{fuel}} \quad (2.70)$$

If we continue using our three examples, we could find the mass fuel ratio for each of the reactions defined in Equations 2.67 through 2.69. If we note here that the molar mass of Benzene is 78.11 lbm/lb-mol and the molar mass of air is 28.97 lbm/lb-mol, we have for the stoichiometric reaction

$$A - F_{\text{Stoich}} = \frac{(7.5)[\text{mol}_{air}](4.76)(28.97) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]}{(1)[\text{mol}_{C_6H_6}](78.11) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]} = 13.24 \left[ \frac{\text{lbm}_{air}}{\text{lbm}_{C_6H_6}} \right] = 13.24 \quad (2.71)$$

For the reaction with 50% excess air, we have

$$A - F_{50\%\text{excess}} = \frac{(1.5)(7.5)[\text{mol}_{air}](4.76)(28.97) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]}{(1)[\text{mol}_{C_6H_6}](78.11) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]} = 19.85 \left[ \frac{\text{lbm}_{air}}{\text{lbm}_{C_6H_6}} \right] = 19.85 \quad (2.72)$$

For the reaction with 50% deficient air, we have

$$A - F_{50\text{deficient}} = \frac{(0.5)(7.5)[\text{mol}_{\text{air}}](4.76)(28.97) \left[ \frac{\text{lbf}}{\text{lb-mol}} \right]}{(1)[\text{mol}_{\text{C}_6\text{H}_6}](78.11) \left[ \frac{\text{lbf}}{\text{lb-mol}} \right]} = 6.61 \left[ \frac{\text{lbf}_{\text{air}}}{\text{lbm}_{\text{C}_6\text{H}_6}} \right] = 6.61 \quad (2.73)$$

Now that we have introduced the process of chemical equation balancing and some of the mathematics required, we must quantify the energy released (or absorbed) by the chemical reaction. We have already introduced the concept of enthalpy as well as defined the enthalpy of formation. We shall pause here to examine how a heat of formation is obtained.

We shall consider carbon dioxide for our example. If we have a combustion chamber in which we react pure oxygen with solid carbon, we can put the two substances into the container at 25°C and start the reaction somehow. The balanced equation on a molar basis would be



The first law of thermodynamics states that

$$Q + W = N_{\text{products}} \bar{h}_{\text{products}} - N_{\text{reactants}} \bar{h}_{\text{reactants}} \quad (2.75)$$

Here we have used specific values so that everything is on a molar basis. Since the container is rigid, there is no work performed on or by the system, thus Equation 2.75 reduces to

$$Q = N_{\text{products}} \bar{h}_{\text{products}} - N_{\text{reactants}} \bar{h}_{\text{reactants}} \quad (2.76)$$

If we were to perform this experiment, we would find that the container would get hot. Theoretically, we could extract this heat from the container until the temperature returned to 25°C; if we were to do this, we would find that 393,546 kJ/kg-mol of energy would have been produced. Examination of [Appendix B.1](#) reveals that this is exactly the value of the heat of formation of carbon dioxide recalling that a negative value denotes heat given off by the reaction.

The enthalpy of a substance allows us to quantify the energy state of a material. The enthalpy of formation was defined as the energy required to form a particular composition from its basic elements resulting in the compound as a product at some reference temperature and pressure (we shall use 25°C or 298 K and 1 atm as this reference condition). If we were to take this compound and arbitrarily increase its temperature or pressure by some amount and if there were no phase change or change in composition, we will have increased its enthalpy. If we restrict our analysis to an ideal gas, it can be shown [1] that the enthalpy is a function of temperature only. With this, we can write for a composition

$$\bar{h}_T = \bar{h}_f^0 + \Delta \bar{h}_{298 \rightarrow T} \quad (2.77)$$

Here  $\bar{h}_T$  is the enthalpy of the material at temperature,  $T$ ,  $\bar{h}_f^0$  is the enthalpy of formation, and  $\Delta \bar{h}_{298 \rightarrow T}$  is the change in enthalpy from the reference state to the temperature,  $T$ . We define  $\Delta \bar{h}_{298 \rightarrow T}$  as

$$\Delta \bar{h}_{298 \rightarrow T} = \bar{h}(T) - (\bar{h}_{298}^0) \quad (2.78)$$

Tables of enthalpies are located in [Appendix B](#) at the end of the book. As an example, consider carbon monoxide at 2000 K. The enthalpy of this compound using Appendices B.1 and B.2 would be

$$h_{\text{CO}_{2000\text{K}}} = -110,541 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] + 56,737 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] = -53,804 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] \quad (2.79)$$

Now that we have worked with enthalpies a bit, we can begin to apply what we have learned. We shall look at an example of these principles applied first to a closed bomb where there is no work performed and then to a gun where there is.

For a closed bomb, we shall tailor [Equation 2.52](#) to our needs. If we consider a closed vessel, we realize that there is no velocity into or out of the CV, and there is no work performed on or by the system. This allows us to write Equation 2.52 as

$$Q_{1-2} = m[(h_2 - pv_2) - (h_1 - pv_1)] = m(u_2 - u_1) \quad (2.80)$$

If we write this equation on a molar basis as limit to ideal gas behavior, we can state that

$$Q = \sum_i N_i (\bar{h}_{\text{prod}} - R_u T_{\text{prod}}) - \sum_i N_i (\bar{h}_{\text{reac}} - R_u T_{\text{reac}}) \quad (2.81)$$

This relationship is important because it tells us that the heat given off by the closed bomb is affected by the enthalpy change of the chemical reaction and the temperature of the products.

We shall examine a pressure vessel containing 0.001 kg of methane ( $\text{CH}_4$ ) and 0.002 kg of air. The enthalpy of formation for methane is  $-74,850 \text{ kJ/kg-mol}$  and its molecular weight is  $16.04 \text{ kg/kg-mol}$ . The reaction will begin at 298 K and we shall remove enough heat from the vessel that the final temperature becomes 1500 K. We would like to determine how much heat is given off.

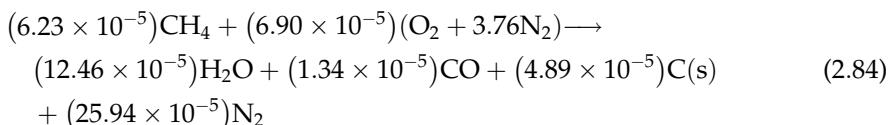
We need to balance the chemical reaction on a molar basis, so we shall determine how many moles of methane and air we have in the container. For methane, we have

$$N_{\text{CH}_4} = \frac{(0.001) \left[ \frac{\text{kg}_{\text{CH}_4}}{\text{kg}} \right]}{(16.04) \left[ \frac{\text{kg}}{\text{kg-mol}} \right]} = 6.23 \times 10^{-5} \left[ \text{kg-mol}_{\text{CH}_4} \right] \quad (2.82)$$

For the air, we have

$$N_{\text{air}} = \frac{(0.002) \left[ \frac{\text{kg}_{\text{air}}}{\text{kg}} \right]}{(28.97) \left[ \frac{\text{kg}}{\text{kg-mol}} \right]} = 6.90 \times 10^{-5} \left[ \text{kg-mol}_{\text{air}} \right] \quad (2.83)$$

Our balanced reaction is then



We shall examine the reactants first. For methane, we have

$$N_{\text{CH}_4} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{CH}_4} \right) = (6.23 \times 10^{-5}) [\text{kg-mol}] \left\{ -74,850 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] + 0 - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (298) [\text{K}] \right\}$$

$$N_{\text{CH}_4} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{CH}_4} \right) = -4.82 [\text{kJ}]$$

For oxygen and nitrogen, we have

$$N_{\text{O}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{O}_2} \right) = (6.90 \times 10^{-5}) [\text{kg-mol}] \left\{ 0 + 0 - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (298) [\text{K}] \right\}$$

$$N_{\text{O}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{O}_2} \right) = -0.17 [\text{kJ}]$$

$$N_{\text{N}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{N}_2} \right) = (3.76) (6.90 \times 10^{-5}) [\text{kg-mol}] \left\{ 0 + 0 - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (298) [\text{K}] \right\}$$

$$N_{\text{N}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{N}_2} \right) = -0.64 [\text{kJ}]$$

The enthalpies of the reactants are therefore

$$\sum_i N_i (\bar{h}_{\text{reac}} - R_u T_{\text{reac}}) = -4.82 [\text{kJ}] - 0.17 [\text{kJ}] - 0.64 [\text{kJ}] = -5.63 [\text{kJ}]$$

For the products, we have (using the tables in the appendix)

$$N_{\text{H}_2\text{O}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{H}_2\text{O}} \right) = (12.46 \times 10^{-5}) [\text{kg-mol}] \left\{ -241,845 + 48,181 - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (1500) [\text{K}] \right\}$$

$$N_{\text{H}_2\text{O}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{H}_2\text{O}} \right) = -25.69 [\text{kJ}]$$

$$N_{\text{CO}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{CO}} \right) = (1.34 \times 10^{-5}) [\text{kg-mol}] \left\{ -110,541 + 38,847 - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (1500) [\text{K}] \right\}$$

$$N_{\text{CO}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{CO}} \right) = -1.13 [\text{kJ}]$$

$$N_{\text{C}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{C}} \right) = (4.89 \times 10^{-5}) [\text{kg-mol}] \left\{ 0 + 23,253 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (1500) [\text{K}] \right\}$$

$$N_{\text{C}} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{C}} \right) = 0.53 [\text{kJ}]$$

$$\begin{aligned}
& N_{\text{N}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{N}_2} \right) \\
& = (3.76)(6.90 \times 10^{-5}) [\text{kg-mol}] \left\{ 0 + 38,404 \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] - (8.314) \left[ \frac{\text{kJ}}{\text{kg-mol} \cdot \text{K}} \right] (1500) [\text{K}] \right\} \\
& N_{\text{N}_2} \left( \bar{h}_{\text{f}}^0 + \Delta \bar{h}_{298 \rightarrow T} - R_u T_{\text{N}_2} \right) = 6.73[\text{kJ}]
\end{aligned}$$

The enthalpies of the products are then given by

$$\sum_i N_i (\bar{h}_{\text{prod}} - R_u T_{\text{prod}}) = -25.69[\text{kJ}] - 1.13[\text{kJ}] + 0.53[\text{kJ}] + 6.73[\text{kJ}] = -19.56[\text{kJ}]$$

The heat given off by the reaction is then calculated through [Equation 2.81](#) as

$$Q = (-19.56)[\text{kJ}] - (-5.63)[\text{kJ}] = -13.93[\text{kJ}] \quad (2.85)$$

This illustrates the process of calculating the amount of energy given off by a closed-bomb reaction as well as the effect of temperature on the reaction products. It must be noted that had we decided to lower the temperature of the products, even more energy would have been removed. This will be examined as a problem at the end of the chapter.

If we apply the same principles to a gun launch, we can determine the amount of energy imparted to the projectile and in so doing, obtain a feeling for the process of energy conversion between propellant chemical energy and projectile kinetic energy.

Unlike the fixed boundary examined in the closed-bomb problem, above, a gun launch involves a boundary that is moving (the base of the projectile). This problem is similar to a piston of an internal combustion engine that undergoes one stroke. We have defined work earlier as a form of energy and if we assume all of the energy of the propellant goes into heating of the gaseous products, kinetic energy of the projectile, and a loss term (including friction, swelling of the gun tube, etc.), we can write the first law of thermodynamics as given in [Equation 2.75](#). Rewriting this by assuming the velocity of the seated projectile is zero, we obtain our thermodynamic equation for a gun launch as

$$Q + \frac{1}{2} m V^2 = \sum_i N_i (\bar{h}_{\text{prod}}) - \sum_i N_i (\bar{h}_{\text{reac}}) + \text{losses} \quad (2.86)$$

We have neglected potential energy changes here because they are usually quite small relative to the other terms. We shall examine an example in the form of a potato gun to illustrate the use of [Equation 2.86](#) and the other methods of this chapter.

A potato gun is a device that people use to project potatoes at targets. These devices can be very dangerous to the operator as well as the target. We would like to calculate the muzzle velocity of a half-pound potato projectile used in a particular gun. This gun is made of 2-in. diameter PVC pipe (a very good insulator). The projectile rests on a stop when loaded through the muzzle so that there is a 6-in. long chamber. The device in question was injected with 0.005 oz (mass) of lighter fluid as a gas (*n*-butane—C<sub>4</sub>H<sub>10</sub> (g))  $\bar{h}_{\text{f}}^0 = -124,733 \text{ kJ/kg-mol}$ ,  $n = 58.123 \text{ kg/kg-mol}$ ) to fire the potato. We shall assume the potato obturates perfectly and that there is no bore friction. The travel of the potato in the gun tube is 24 in. The weapon is fired under standard conditions of 77°F and 14.7 psi. Assume the reactants and the products both exist at these conditions. We would like

to determine the velocity of the potato at the completion of combustion in feet per second assuming no losses.

The chamber was 6-in. long and 2 in. in diameter, so our chamber volume is

$$V_i = Al = \pi \frac{(2)^2}{4} [\text{in.}^2](6)[\text{in.}] = 18.85[\text{in.}^3] \quad (2.87)$$

The air weighs 28.97 lbm/lb-mol and if we assume ideal gas behavior, the density of air is calculated from

$$pv = RT \rightarrow \rho = \frac{p}{RT} \quad (2.88)$$

$$\rho = \frac{(14.7) \left[ \frac{\text{lbf}}{\text{in.}^2} \right] (28.97) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]}{(1545) \left[ \frac{\text{ft-lbf}}{\text{lb-mol} - R} \right] (12) \left[ \frac{\text{in.}}{\text{ft}} \right] (537)[R]} = 0.0000428 \left[ \frac{\text{lbm}}{\text{in.}^3} \right]$$

So the amount of air we actually have is

$$m_{\text{air}} = \rho V_i = (0.0000428) \left[ \frac{\text{lbm}}{\text{in.}^3} \right] (18.85) [\text{in.}^3] = 0.0008068[\text{lbm}] \quad (2.89)$$

The amount of fuel was given in ounces

$$m_{\text{fuel}} = (0.005)[\text{oz}](0.0625) \left[ \frac{\text{lbm}}{\text{oz}} \right] = 0.0003125[\text{lbm}]$$

For the actual combustion, we need to use our mass information and convert it to molar values, recognizing that the molar mass is the same whether it is kg/kg-mol or lbm/lb-mol. For the fuel and air, we have

$$N_{\text{fuel}} = \frac{m_{\text{fuel}}}{n_{\text{fuel}}} = (0.0003125)[\text{lbm}] \frac{1}{(58.123) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]} = 0.0000054[\text{lb-mol}] \quad (2.90)$$

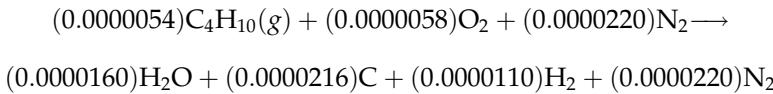
$$N_{\text{air}} = \frac{m_{\text{air}}}{n_{\text{air}}} = (0.0008068)[\text{lbm}] \frac{1}{(28.97) \left[ \frac{\text{lbm}}{\text{lb-mol}} \right]} = 0.0000278[\text{lb-mol}] \quad (2.91)$$

For each lb-mol of air, we know that 1/4.76 lb-mol of it is oxygen so we have

$$N_{\text{O}_2} = \frac{1}{4.76}(0.0000278)[\text{lb-mol}] = 0.0000058[\text{lb-mol}]$$

$$N_{\text{N}_2} = \frac{3.76}{4.76}(0.0000278)[\text{lb-mol}] = 0.0000220[\text{lb-mol}]$$

Now we can write our combustion equation as



To determine the muzzle velocity, we start with our first law of thermodynamics equation, simplified by the fact that there is no heat transfer and no shaft work. Then the energy of the fuel-air mixture equals the work done on the projectile plus the energy of the products of combustion.

$$H_R = H_p + W_p \quad (2.92)$$

Let us look at the internal energies for each of the reactants

Reactant	Enthalpy of Formation (kJ/kg-mol)	Enthalpy of Formation (in.-lbf/lb-mol)
C <sub>4</sub> H <sub>10</sub> (g)	-124,733	-500,728,155
O <sub>2</sub>	0	0
N <sub>2</sub>	0	0

The conversion used here is as follows:

$$(x) \left[ \frac{\text{kJ}}{\text{kg-mol}} \right] (0.4299) \left[ \frac{\text{BTU}}{\frac{\text{lb-mol}}{\text{kJ}}} \right] (778.16) \left[ \frac{\text{ft-lbf}}{\text{BTU}} \right] (12) \left[ \frac{\text{in.}}{\text{ft}} \right] \rightarrow 4014.4x \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \quad (2.93)$$

For the products, we have

Product	Enthalpy of Formation (kJ/kg-mol)	Enthalpy of Formation (in.-lbf/lb-mol)
H <sub>2</sub> O(g)	-241,845	-970,862,568
N <sub>2</sub>	0	0
C <sub>2</sub>	0	0
H <sub>2</sub>	0	0

We will rearrange our first law equation as follows:

$$W_p = H_R - H_p$$

We calculate  $H_R$  first

$$H_R = N_{\text{C}_4\text{H}_{10}} (\bar{h}_f^0 + \Delta\bar{h}_{298 \rightarrow T}) + N_{\text{O}_2} (\bar{h}_f^0 + \Delta\bar{h}_{298 \rightarrow T}) + N_{\text{N}_2} (\bar{h}_f^0 + \Delta\bar{h}_{298 \rightarrow T})$$

Plugging in the numbers we have, we get

$$\begin{aligned}
 H_R &= (0.0000054)[\text{lb-mol}](-500,728,155 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 &\quad + (0.0000058)[\text{lb-mol}](0 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 &\quad + (0.0000220)[\text{lb-mol}](0 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 H_R &= -2704[\text{in.-lbf}]
 \end{aligned}$$

We calculate  $H_p$  in a similar manner

$$\begin{aligned}
 H_p &= N_{H_2O} \left( \bar{h}_f^0 + \Delta \bar{h}_{298 \rightarrow T} \right) + N_{H_2} \left( \bar{h}_f^0 + \Delta \bar{h}_{298 \rightarrow T} \right) \\
 &\quad + N_{N_2} \left( \bar{h}_f^0 + \Delta \bar{h}_{298 \rightarrow T} \right) + N_C \left( \bar{h}_f^0 + \Delta \bar{h}_{298 \rightarrow T} \right) \\
 H_p &= (0.0000160)[\text{lb-mol}](-970,862,568 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 &\quad + (0.0000110)[\text{lb-mol}](0 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 &\quad + (0.0000220)[\text{lb-mol}](0 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right] \\
 &\quad + (0.0000216)[\text{lb-mol}](0 + 0) \left[ \frac{\text{in.-lbf}}{\text{lb-mol}} \right]
 \end{aligned}$$

$$H_p = -15,534[\text{in.-lbf}]$$

Then the work done on the projectile is

$$W_p = -2,704[\text{in.-lbf}] - (-15,534)[\text{in.-lbf}] = 12,830[\text{in.-lbf}]$$

Since this work equals the muzzle energy of the projectile

$$W_p = \frac{1}{2} m V^2 = 12,830[\text{in.-lbf}]$$

Therefore,

$$V = \sqrt{\frac{2W_p}{m}} = \sqrt{\frac{(2)(12,830)[\text{in.-lbf}](32.2) \left[ \frac{\text{lbm-ft}}{\text{lbf-s}^2} \right]}{(0.5)[\text{lbm}](12) \left[ \frac{\text{in.}}{\text{ft}} \right]}} = 371 \left[ \frac{\text{ft}}{\text{s}} \right]$$

Wow! That's pretty fast but we used a lot of butane, assumed the products return to ambient conditions quickly, and neglected things. Also note that the length of the tube did not come into play. We would definitely have to account for this as we shall later see.

One important parameter in determining the amount of energy transferred to the projectile is the temperature of the product gases. As you can see from our example, an increase in the temperature of the product gases will result in a decrease in the projectile

velocity because  $H_p$  goes up. Typically, we can assume the product gases exit at a temperature between  $0.6T_0$  and  $0.7T_0$ , where  $T_0$  is the adiabatic flame temperature of the product gases [7]. The adiabatic flame temperature of a gas is the temperature that is achieved if the gases burn to completion in the absence of any heat transfer or work being performed [1]. The calculation of the adiabatic flame temperature is relatively straightforward but requires iteration. This is beyond the scope of this text but the reader is referred to the references at the end of this chapter for a complete description of the procedure. In addition, there are several commercially available codes (including some that come with the purchase of textbooks now, for instance [13]). To achieve our objectives, the temperature of the reaction products will always be given.

### Problem 5

Calculate the A-F ratio for the combustion of the following fuels. Calculate the ratio with both theoretical air and 10% excess air.

1. Benzene ( $C_6H_6$ )  
*Answer:* 13.24 and 14.56
2. *n*-Butane ( $C_4H_{10}$ )  
*Answer:* 15.42 and 12.5.96
3. Ethyl alcohol ( $C_2H_5OH$ )  
*Answer:* 8.98 and 9.88

### Problem 6

Let us examine a pressure vessel identical to the example problem in the text containing 0.001 kg of methane ( $CH_4$ ) and 0.002 kg of air. The enthalpy of formation for methane is  $-74,850 \text{ kJ/kg-mol}$  and its molecular weight is  $16.04 \text{ kg/kg-mol}$ . The reaction will begin at 298 K and we shall remove enough heat from the vessel that the final temperature becomes 1000 K.

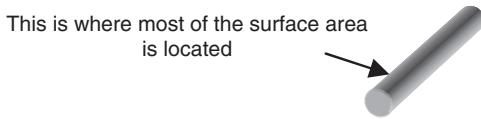
1. Determine the maximum heat given off.  
*Answer:*  $Q = -20.02[\text{kJ}]$
2. Compare the result in (1) above with the example problem in this chapter.  
*Answer:* This situation removes 6.09 kJ more energy than the example.

---

## 2.6 Solid Propellant Combustion

Now that we have examined the background of the thermochemistry and thermodynamics of combustion, we shall see how this applies to the behavior of a burning solid propellant. We shall endeavor, in this section, to come up with definitions and relationships that will allow us to define the state of the propellant behind a projectile at any given time. The process we will use is somewhat simplified because the real situation behind a moving projectile is generally a two phase, reacting flow field. Some of our assumptions, though not necessarily valid in the purest sense, are good enough to predict bulk behavior of the propelling gas.

In the previous sections, we have discussed how energy is evolved by the propellant. We saw that thermodynamic properties were not dynamic at all, merely means of accounting



**FIGURE 2.8**  
Long cylindrical propellant grain.

for energy knowing the initial and end states and making assumptions on the process between them. This section will allow us to add in some time dependency to the equations to somewhat understand the rates at which combustion is occurring.

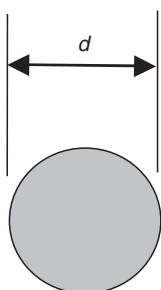
Solid propellants are generally nitrocellulose compounds that are manufactured by nitrating through immersion in acid. The details of this process for various materials can be examined in detail in Refs. [7,8,16–18]. This material is then chopped and worked into a doughy substance and pushed through dies to form various shapes. The material then has solvents removed and it is dried. When this process is complete, the propellant has the consistency of uncooked (i.e., hard and somewhat brittle) pasta. Though this statement is general, there are, as always, exceptions.

The burning of solid propellant is a surface phenomenon. The rate of gas evolution is dependent upon the amount of surface area of the propellant. Because of this, the shape that the propellant takes is extremely important. Burning is the mechanism of transforming the solid propellant to a gas. The burn rate of a propellant is highly dependent upon the pressure at which the burning reaction takes place. Essentially, the greater the pressure, the faster the propellant burns. These two behavioral observations tell us that if we can control the geometry and confinement of a given propellant, we can, to a large degree, control the rate of gas evolution.

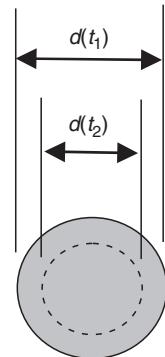
We shall examine a single propellant grain to gain an understanding of how the geometry affects the rate of evolution of gas. Consider a long cylinder of solid propellant which is commonly referred to as a grain. If the cylinder were long enough, we could see that most of the surface area would be located along the circumference and length. In other words, we can neglect the two small surface areas that comprise the ends. This is illustrated in Figure 2.8. If we neglect the burning of the end surfaces, it allows us to examine the geometry through simple mathematical relationships.

As our grain begins to burn, solid material will be evolved into gas. Thus, we can imagine the solid surfaces shrinking toward the centerline of the grain. If we examine our grain from the end looking down its axis, we would see a circular section as depicted in Figure 2.9. We could then write an expression for the surface area of our grain as a function of its diameter and length.

$$A(t) = \pi d(t)l \quad (2.94)$$



**FIGURE 2.9**  
Propellant grain cross-section.



**FIGURE 2.10**

Propellant grain cross-section at two times.

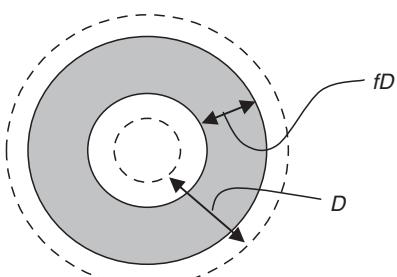
In this expression,  $A(t)$  is the surface area of the grain,  $d(t)$  is the diameter, and  $l$  is the length. We have denoted the surface area and diameter as functions of time to remind us of our assumption of no burning at the ends of the grain. After some time,  $t$ , the grain surface will have regressed such that our diameter has decreased. This is depicted in Figure 2.10. This graphically shows us that at time  $t_1$  the grain clearly has more surface area than at time  $t_2$ ; therefore, as burning progresses, the rate of evolution of gas slows down. This is commonly called regressive burning.

Propellant geometry is characterized by a quantity known as the web thickness or simply the web. The symbol use for the web is  $D$ . The web is the smallest thickness of the initial propellant grain. In the case of our cylindrical grain, it would be the initial diameter.

In the interior ballistics analysis of a gun system, we need to track how much gas is evolved and also how much solid is remaining. This is important because we have seen that all of our equations of state are dependent upon volume as well as pressure and temperature, and these, in turn, affect the burning rate. The amount of solid propellant remaining is tracked through use of the web fraction,  $f$ . The web fraction is the fraction of web remaining at a given time,  $t$ . Through use of this web fraction, we can write an expression for the amount of propellant remaining at any time as a function of the web.

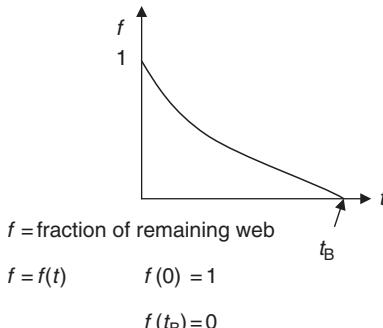
$$d(t) = fD \quad (2.95)$$

This is illustrated for a grain with a single perforation (known colloquially as a perf) in Figure 2.11. It is important to note here that for a single perf grain, the web is defined as the outside radius minus the inside radius. This sometimes is confusing for new ballisticians since we use  $D$  as the web thickness. Also one can see from the figure that an advantage of a single perf grain is that it burns from both the inside out and the outside in, thus



**FIGURE 2.11**

Burning of a single perforated propellant grain.



**FIGURE 2.12**  
Fraction of remaining web versus time.

decreasing the surface on the outside while increasing the surface on the inside—known as neutral burning behavior.

Use of the web fraction is convenient because, mathematically, it is a function that varies from unity to zero. The manner in which it varies may be somewhat complex, but at least the end states are well defined. An example plot of web fraction versus time is shown in Figure 2.12. In this figure,  $t_B$  is the time at which all of the propellants have evolved into gas—the burnout time.

Many times, we are interested more in the volume of the propellant that has evolved into gas rather than the fraction of the web remaining. It should be clear that the two quantities are related since the gas had to come from the solid material and conservation of mass states that we can neither destroy nor create mass. This is handled through use of the fraction of propellant burnt,  $\phi$ . Since  $\phi$  is a function of  $f$  and  $f$  is a function of time, we see that  $\phi$  must be also a function of time. Since propellant geometries can be fairly complicated,  $\phi$  can be a rather complicated function of  $f$ . For simple shapes, this relationship is straightforward. For instance, a single perforated grain has the functional relationship that

$$\phi(t) = 1 - f(t) \quad (2.96)$$

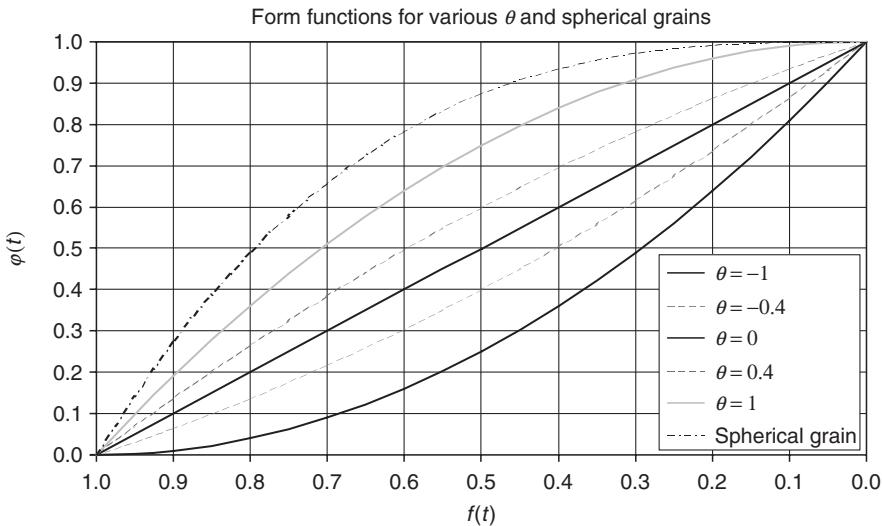
Most shapes can be simplified to express  $\phi$  as a quadratic function of  $f$  through use of a shape function,  $\theta$ .

$$\phi(t) = [1 - f(t)][1 + \theta f(t)] \quad (2.97)$$

This expression allows us to cover almost any simple geometry, the most notable exception being a sphere. Figure 2.13 depicts how variation in the shape function affects the relationship between  $f$  and  $\phi$ .

With the formulations above, we have been able to mathematically define the effect of propellant geometry on the rate of gas evolution. The second important parameter in this generation of gas was stated to be the effect of pressure on burning. Whenever a propellant burns, say in a fixed volume, two competing processes are happening: the volume into which the gaseous propellant is moving is increasing because there is less solid material—this decreases the pressure, and the more and more propellant gas is being pushed into a confined space—this increases the pressure. The rate at which the surface area decreases affects this relationship. The simplest model for the relationship between burn rate and pressure is given by

$$D \frac{df}{dt} = -\beta p_B(t) \quad (2.98)$$



**FIGURE 2.13**  
Effect of different values of  $\theta$  on  $\varphi$  and  $f$ .

In this equation,  $D df/dt$  is the time rate of change of the web (i.e., the burning rate),  $\beta$  is a burn rate coefficient, and  $p_B$  is the pressure (we will discuss the subscript later). The negative sign comes about because the amount of propellant would be increasing if  $D df/dt$  were to result in a positive number. This simple relationship facilitates our analysis of propellant behavior in a gun. Other relationships can more accurately describe propellant behavior, but their complexity is such that computer codes must be used to obtain answers with them. Two very common burn relationships are

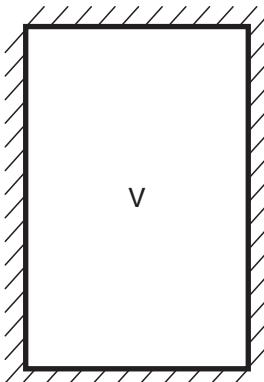
$$D \frac{df}{dt} = -\beta [p_B(t)]^\alpha \quad (2.99)$$

$$D \frac{df}{dt} = -\beta [p_B(t) - P_1] \quad (2.100)$$

Equation 2.99 is by far the most commonly used in computer codes. Caution must be exercised when using burn rate data from the literature as the units will be an indicator of the proper burn rate form of the governing equation. If we examine the units of  $D df/dt$ , we see that they are in terms of [length]/[time]. This type of data is usually obtained from a strand burner. A strand burner is a device that can accurately measure the rate of linear burning in a propellant. Reference [19] contains an excellent diagram of a strand burner.

If we consider a pressure vessel so thick as to be rigid and the amount of propellant so small such that we can neglect its contribution to the volume, we can describe the burning of the propellant as a constant volume process. This is the essence of closed-bomb testing. We can further assume that this pressure vessel can be isolated thermally and the gas behavior is ideal. In this case, our closed bomb, with internal volume,  $V$ , would resemble Figure 2.14. Since we assumed ideal gas behavior, we can write an expression for pressure as a function of volume and temperature

$$p_B V = m_g R T \quad (2.101)$$



**FIGURE 2.14**  
Diagram of a closed bomb.

Here  $p_B$  is the pressure,  $V$  is the volume,  $m_g$  is the mass of the gas,  $R$  is the specific gas constant, and  $T$  is the temperature. When we place our solid propellant into the closed bomb, it has an initial weight that we would call  $c$ . So initially, we can write

$$c = \rho V_{\text{solid}} \quad (2.102)$$

In this equation,  $\rho$  is the density of the solid propellant and  $V_{\text{solid}}$  is its volume. If we now assume that the propellant is cylindrical, we can write its volume as the product of its cross-sectional area and its length. The initial diameter is the web for a cylindrical grain, so

$$V_{\text{cyl.grain}} = \pi \frac{D^2}{4} l \quad (2.103)$$

This volume at any time,  $t$ , can be expressed as

$$V(t)_{\text{cyl.grain}} = \pi \frac{[d(t)]^2}{4} l \quad (2.104)$$

Because mass is conserved, the amount of solid propellant burned is equal to the amount of gas generated. This is an important concept. If we started with 1 lbm of propellant and completely burned it, we would be left with 1 lbm of gas. Based on this, we can write for the mass of the gas as

$$m_g(t) = \rho [V_{\text{cyl.grain}} - V(t)_{\text{cyl.grain}}] = \rho \frac{\pi}{4} l \{ D^2 - [d(t)]^2 \} \quad (2.105)$$

We discussed the fraction of propellant burnt,  $\phi$ , earlier. We are now in a position to formally define it as

$$\phi(t) \equiv \frac{m_g(t)}{c} \quad (2.106)$$

If we substitute Equations 2.102, 2.103, and 2.105 into Equation 2.106, we obtain

$$\phi(t) = \frac{\rho \frac{\pi}{4} l \left\{ D^2 - [d(t)]^2 \right\}}{\rho \frac{\pi}{4} l D^2} = \left\{ 1 - \frac{[d(t)]^2}{D^2} \right\} \quad (2.107)$$

Now we insert Equation 2.95 into Equation 2.107 to yield

$$\phi(t) = 1 - \left( \frac{fD}{D} \right)^2 = 1 - f^2 = (1 - f)(1 + f) \quad (2.108)$$

Comparing this expression (derived for a cylindrical grain) to Equation 2.97 shows that the shape factor  $\theta = 1$  for a cylindrical grain. Also, by comparison to Equation 2.96 we see that the shape factor  $\theta = 0$  for a single perforated grain. Essentially, any shape factor can be derived using this same procedure. So up to this point, we have determined that the shape factor

$$\begin{aligned} \theta &= 0 && \text{for single perforated grains} \\ \theta &= 1 && \text{for cylindrical grains} \end{aligned}$$

An interesting thing has happened. We started this section attempting to find a relationship for the mass of gas evolved from the solid propellant and we have come around to finding the relationship between  $\phi$  and  $f$  again. The key procedure here is now to rearrange Equation 2.106.

$$m_g(t) = c\phi(t) \quad (2.109)$$

This is the relationship that governs the amount of gas evolved from the burning propellant. It looks rather simple, but consider that  $\phi$  is a function of  $f$  and  $t$ , and  $f$  is a function of  $p_B$  and  $t$ . We shall return to this later.

The burning propellant in our closed bomb must generate pressure. To take this further, we need to rearrange Equation 2.98 into

$$p_B(t) = -\frac{D}{\beta} \frac{df}{dt} \quad (2.110)$$

In this expression, we know that  $D$  is the initial web and therefore a constant, and we shall assume that  $\beta$  is a constant ( $\beta$  actually increases somewhat with pressure).

Because we want to work with masses of substances,  $f$  is not a convenient variable. We shall use a relationship to express it in terms of  $\phi$ . At this point, caution must be exercised. Recall that the relationship between  $\phi$  and  $f$  varies with propellant geometry. We shall proceed using our cylindrical grain relationship (Equation 2.108). Rearranging Equation 2.108, we obtain

$$f(t) = \sqrt{1 - \phi(t)} \quad (2.111)$$

if we differentiate this relationship with respect to time, we obtain

$$\frac{df}{dt} = -\frac{1}{2\sqrt{1 - \phi(t)}} \frac{d\phi}{dt} \quad (2.112)$$

This form allows us to rewrite [Equation 2.110](#) as

$$p_B(t) = -\frac{D}{2\beta\sqrt{1-\phi(t)}} \frac{d\phi}{dt} \quad (2.113)$$

We now have all the expressions we need to bring this together. We have an equation of state

$$p_B(t)V = m_g(t)RT(t) \quad (2.114)$$

We have an expression for conservation of mass (relationship between  $m_g$  and  $\phi$ )

$$m_g(t) = c\phi(t) \quad (2.115)$$

and we have an expression that relates the amount of pressure generated to the amount of propellant burnt (burn rate equation)

$$p_B(t) = -\frac{D}{2\beta\sqrt{1-\phi(t)}} \frac{d\phi}{dt} \quad (2.116)$$

All these expressions are in terms of constants we know beforehand or  $f$ ,  $\phi$ , and  $T$ .

To describe the temperature of gas, we need to define a parameter used often in interior ballistics, the propellant force,  $\lambda$ . Propellant force is a constant that is defined as the amount of energy released from a propellant under adiabatic conditions. In other words, it is the most energy one can obtain by burning a propellant. Mathematically, we express it as

$$\lambda \equiv RT_0 \quad (2.117)$$

In this equation,  $R$  is the specific gas constant and  $T_0$  is the adiabatic flame temperature of the gas. This constant has units of energy per unit mass. Sometimes,  $T_0$  is referred to as the uncooled explosion temperature. In our development, we shall assume that all gases are evolved at the adiabatic flame temperature. There are many theories that describe combustion. Introductory treatments are provided in Refs. [20,21], but all of the references in the end of this section cover the topic to some degree. References [22–24] treat the topic in great detail. If we utilize this reactive assumption, we can rewrite [Equation 2.114](#) using [Equation 2.117](#) to give us

$$p_B(t)V = \lambda m_g(t) \quad (2.118)$$

Now we can combine [Equations 2.118](#) and [2.116](#) to yield (for a cylindrical grain)

$$\frac{\lambda m_g(t)}{V} = -\frac{D}{2\beta\sqrt{1-\phi(t)}} \frac{d\phi}{dt} \quad (2.119)$$

We then substitute [Equation 2.115](#) into the expression above, resulting in

$$\frac{\lambda c\phi(t)}{V} = -\frac{D}{2\beta\sqrt{1-\phi(t)}} \frac{d\phi}{dt} \quad (2.120)$$

This can be rearranged to yield

$$\frac{1}{\phi(t)\sqrt{1-\phi(t)}} \frac{d\phi}{dt} = -\frac{2\beta\lambda c}{DV} \quad (2.121)$$

This is a separable, first order, nonlinear, differential equation which can be written in integral form as

$$\int_0^1 \frac{d\phi}{\phi(t)\sqrt{1-\phi(t)}} = -\frac{2\beta\lambda c}{DV} \int_0^{t_B} dt \quad (2.122)$$

The solution of which is

$$\ln\left(\frac{\sqrt{1-\phi}-1}{\sqrt{1-\phi}+1}\right) \Big|_0^1 = -\frac{2\beta\lambda c}{DV} t \Big|_0^{t_B} \quad (2.123)$$

This expression is somewhat problematic because of its singular behavior at  $\phi=1$  and  $\phi=0$ . The equation was approximated numerically to yield

$$t_B \approx 350 \frac{DV}{\beta\lambda c} \quad (2.124)$$

In this case, the solution to this expression was problematic; however, in many cases, it can be evaluated more readily. The techniques that will follow are much simpler from a hand calculation standpoint.

Even though the closed bomb may seem academic, it is actually quite a useful device for determining propellant parameters. If we consider Equations 2.110 and 2.113, we see that since we know the initial web,  $D$ , and we can measure the pressure, the only thing missing is  $\beta$  and  $\phi$  or  $f$ . Equation 2.114 tells us that if we measure  $p_B$  and  $T$  and know  $V$ , we can get  $\phi$  or  $f$ . Thus, the closed bomb is useful for determining the burn rate coefficient,  $\beta$ .

### Problem 7

M1 propellant is measured in a closed bomb. Its adiabatic flame temperature is 3906°F. Its molar mass is 22.065 lbm/lb-mol. What is the effective mean force constant in ft-lbf/lbm?

*Answer:*  $\lambda = 305,709 \left[ \frac{\text{ft-lbf}}{\text{lbfm}} \right]$

### Problem 8

M15 propellant was tested in a strand burner to determine the linear burning rate. The average pressure evolved was 10,000 psi. If the burning exponent,  $\alpha$ , was known to be 0.693 and the pressure coefficient,  $\beta$ , was known to be 0.00330 in./s/psi<sup>0.693</sup>. Determine the average linear burning rate,  $B$  in inch per second.

*Answer:*  $B(p) = 1.952 \left[ \frac{\text{in.}}{\text{s}} \right]$

### Problem 9

Derive the functional form of  $\phi$  in terms of  $f$  for a flake propellant. Assume cylindrical geometry.

*Hint:* Flake propellant consists of grains that have thicknesses much smaller than any other characteristic dimension.

*Answer:*  $\phi(t) = 1 - f$

### Problem 10

An M60 projectile is to be fired from a 105-mm M204 Howitzer. The propellant used in this semi-fixed piece of ammunition is 5.5 lbm of M1 propellant. M1 propellant consists of

single perforated grains ( $\theta = 0$ ) with a web thickness of 0.0165 in., if the average pressure (over the launch of this projectile) developed in the weapon is 20,455 psi. Calculate the average burning rate coefficient in  $\text{in.}^3/\text{lbf}\cdot\text{s}$  if the burn rate is (we use a negative sign in the burn rate to make the form come out right later)

$$\frac{df}{dt} = -185.9 [\text{s}^{-1}]$$

*Answer:*  $\beta = 1.50 \times 10^{-4} \left[ \frac{\text{in}^3}{\text{lbf}\cdot\text{s}} \right]$

### Problem 11

$\beta$  is actually a function of pressure and temperature (it is really given in tables at 25°F at this value). For simplification (and illustration), we will assume it is constant. Given this assumption, calculate the functional form of the web fraction,  $f$  from [Problem 10](#), above.

*Answer:*  $f = 1 - \frac{\beta p_{\text{avg}}}{D} t$

### Problem 12

Given the data provided in Problems 10 and 11, above, determine the proper form of the fraction of charge burnt.

*Answer:*  $\phi(t) = 185.9t$

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## 2.7 Fluid Mechanics

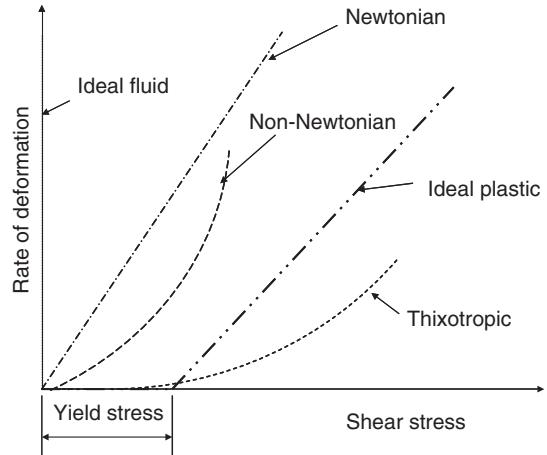
The entire field of ballistics is steeped in the principles of fluid mechanics. The flow of propellant gases in the gun tube, the flow of the propellant gases through a muzzle brake upon shot exit, the flow of the air around the projectile in flight, and even, as we shall see, the flow of target material during a penetration event can many times be modeled as a fluid. This section is devoted to a basic treatment of fluid mechanics principles. Some of these we will use very soon, others will be used at a later time. All of them are important in the study of ballistics.

A fluid differs from a solid in its behavior when placed in shear. In general, fluids can support little or no shear loads or tensile stress. Fluids are generally characterized by their behavior under shear stress. Because a fluid will, in general, flow readily under a shear stress, this behavior is normally plotted in a graph of rate of deformation versus shear stress as depicted in [Figure 2.15](#).

A fluid is considered to exhibit Newtonian behavior if there is a linear relationship between shear stress and rate of deformation. A fluid is non-Newtonian otherwise. Some fluids such as an ideal plastic or a thixotropic material actually do exhibit a yield stress. In the case of an ideal plastic, after a certain yield stress is achieved, the material exhibits a linear relationship between stress and deformation rate. A thixotropic material exhibits a nonlinear relationship after yield stress is reached. An ideal fluid is one where the material will flow and continue to accelerate regardless of the amount of shear stress applied.

Many of the fluids we will deal with are Newtonian. Mathematically, the relationship between applied shear stress and deformation rate is given by

$$\tau = \mu \frac{\partial u}{\partial y} \quad (2.125)$$



**FIGURE 2.15**  
Rate of deformation versus shear stress.

Here  $\tau$  is the applied shear stress,  $\mu$  is the dynamic viscosity of the fluid, and  $\partial u / \partial y$  is the deformation gradient (change in velocity with respect to a spatial coordinate). The ratio of the dynamic viscosity to the fluid density occurs so often that it is customary to define a kinematic viscosity as

$$\nu = \frac{\mu}{\rho} \quad (2.126)$$

Here  $\nu$  is the kinematic viscosity and  $\rho$  is the density of the fluid.

In the section on thermodynamics, we introduced the concept of a Lagrangian or control mass approach and an Eulerian or control volume approach to solving transport problems. In examination of a fluid's behavior, we need to develop both of these techniques. Our plan of attack will be to develop these equations in a CV and provide equations to change the reference frame afterwards. For a more complete treatment, the reader is referred to Refs. [11,12,25–28].

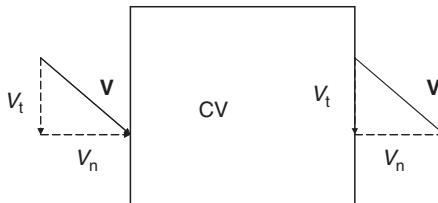
The basis for our development of the following equations are the tenets that (a) mass must be conserved and (b) Newton's second law must hold true. Newton's second law can be written as

$$\sum \mathbf{F} = \frac{d}{dt} (m \mathbf{V}) \quad (2.127)$$

In the above equation,  $\Sigma \mathbf{F}$  is the vector sum of all the forces acting on a body (or blob of fluid or CV),  $m$  is the mass of the body, and  $\mathbf{V}$  is the vector velocity of the body. It is important to note that throughout this work,  $V$  is volume (a scalar),  $v$  is velocity (as a scalar quantity), and  $\mathbf{V}$  is velocity (as a vector quantity).

Since CVs can be oriented in an arbitrary manner, it is important to understand that only that component of velocity normal to the control surface (CS) (i.e., the boundary of the CV) transports material or energy into the CV. If we examine Figure 2.16 where we have broken the velocity vectors into normal and tangential components (denoted  $V_n$  and  $V_t$ , respectively), we can clearly see why this is so.

Consider an arbitrary property,  $N$ , of a substance. We would like to see how this property is transported into and out of a CV. If we define an intensive property,  $\eta$ , such that



**FIGURE 2.16**

Depiction of normal and tangential velocity components with respect to an arbitrary CV.

$$\eta = \frac{N}{m} \text{ or } N = \eta m \quad (2.128)$$

Then we can write

$$\frac{dN}{dt} = \frac{\partial}{\partial t} \int_{\text{CV}} \eta \rho dV + \int_{\text{outflow area}} \eta \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \eta \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.129)$$

This equation defines how a property of interest is transported into and out of the CV. If we look at each of the terms, we see that this is an intuitively satisfying equation. The term on the LHS is the time rate of change (decrease) of any property of the CV over a time of interest. The first term on the RHS tells us how much of that property is stored in the CV over this time. The second term on the RHS tells us how much material has left the CV, while the third term tells us how much material has entered.

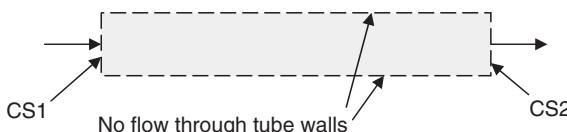
Now wait a minute! If we look at the signs on the second and third terms, they seem to be incorrect—should not the stuff leaving have a negative sign and the stuff entering have a positive sign? The answer to this is yes, but Equation 2.129 is written correctly. The key to this seemingly inconsistent sign convention lies in the fact that the dot product in the second term is positive when we define the area as a vector which points outward and is normal to the surface. Similarly, the inflow term will always lead to a negative number since the velocity vector points inward and the area vector points outward.

We shall now examine the flow of propellant gases in a suitable CV located somewhere behind a projectile at an instant in time. This will serve to foster understanding of the CV approach.

Consider a CV in a gun tube located somewhere behind a moving projectile as depicted in Figure 2.17. There will be a velocity associated with the propelling gases (we will see this later) such that the gases are flowing in one side and out the other, but no gases flow through the walls.

The ends of this cylindrical CV are designated as CSs. The inlet side is CS1 and the outlet side is CS2. If we would like to write an equation for how mass is transferred into or out of this CV, we set  $N$ , the flow variable in Equation 2.129, equal to  $m$ , the property of interest. When this is done, Equation 2.128 tells us that

$$\eta = \frac{N}{m} = \frac{m}{m} = 1 \quad (2.130)$$



**FIGURE 2.17**

Typical gun tube CV.

So for this case, we can write

$$\frac{dm}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{\text{outflow area}} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.131)$$

We know mass can neither be created nor destroyed, so  $dm/dt = 0$ , then we arrive at what is commonly called the equation of conservation of mass, or the continuity equation. In a general form, it is given as

$$\frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{\text{outflow area}} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \rho \mathbf{V} \cdot d\mathbf{A} = 0 \quad (2.132)$$

The first term on the LHS states how the mass in the CV is changing with time. The second term is the amount of mass exiting the CV and the third term is the amount of mass entering the CV.

The flow inside a gun tube is never steady or uniform. Nevertheless, it is informative to look at this expression using these two assumptions to gain some physical insight into the nature of the terms. The steady flow assumption means that there is no increase or decrease in material flow into or out of our CV. This implies that the first term is zero. So for the special case of steady flow, we have

$$\int_{\text{outflow area}} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \rho \mathbf{V} \cdot d\mathbf{A} = 0 \quad (2.133)$$

Simply put, this equation states that what comes into the CV equals what goes out of the CV.

Uniform flow is a special case where fluid viscosity effects are neglected. This results in a constant velocity across the CSs. In essence, the velocity at the wall of the gun tube is the same as the velocity on the centerline of the tube. We will discuss this and its implications in more detail later.

When we apply this assumption to Equation 2.133 and note that  $\mathbf{V} \cdot d\mathbf{A}$  is negative at CS1 (because the vectors have opposite directions) and positive at CS2, we obtain the following simple relationship:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m} \quad (2.134)$$

Thus, under the steady flow assumption, the mass flow rate,  $\dot{m}$ , is constant.

We shall now examine the use of momentum,  $m\mathbf{V}$ , as our flow variable. Use of [Equation 2.128](#) with this flow variable yields

$$\eta = \frac{N}{m} = \frac{m\mathbf{V}}{m} = \mathbf{V} \quad (2.135)$$

Now we can include this into [Equation 2.129](#) to obtain

$$\frac{d(m\mathbf{V})}{dt} = \frac{\partial}{\partial t} \int_{CV} \mathbf{V} \rho dV + \int_{\text{outflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.136)$$

Through Newton's second law, we know that the term on the LHS (time rate of change of momentum) equals the forces on the system. The first term on the RHS is the change in the systems momentum through storage in the CV. The second and third terms are the momentum leaving and momentum entering the CV, respectively. It is again informative to examine the steady flow case which reduces our equation to

$$\mathbf{F} = \int_{\text{outflow area}} \mathbf{V}\rho\mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \mathbf{V}\rho\mathbf{V} \cdot d\mathbf{A} \quad (2.137)$$

Here we have replaced the time rate of change of momentum term with the force. Once again we shall use the uniform flow assumption to facilitate our understanding of this equation. Consider the same gun tube CV as earlier, drawn slightly differently in Figure 2.18.

As discussed earlier, the velocity and area scalar products result in a negative sign on the inflow and a positive sign on the outflow side. With this uniform flow assumption (recall we also included a steady flow assumption to reduce the equation to the form of Equation 2.137), our Equation 2.137 would become

$$\mathbf{F} = \rho_2 V_2 \mathbf{V}_2 A_2 - \rho_1 V_1 \mathbf{V}_1 A_1 \quad (2.138)$$

Note that this is still a vector equation with the vectors  $\mathbf{V}_1$  and  $\mathbf{V}_2$  determining the direction of  $\mathbf{F}$ . If we had already worked out or it was obvious what direction the resultant force would be in, then we could write

$$F = \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 \quad (2.139)$$

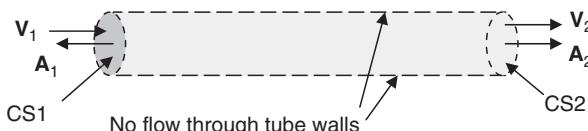
Equation 2.138 only tells us part of the story. It tells us the inertial reaction of the CV to the forces arising from a fluid passing through it. There are two types of forces that occur on the LHS in response to or independent of this, body forces and surface tractions.

Body forces are those that act through the bulk of the material (i.e., directly affecting every molecule). Examples of this are gravitational loads, electromagnetic loads, etc. It is customary to write these loads on a unit mass basis to be consistent with the rest of the equation. In many cases, these are small and are neglected.

Surface tractions are forces which act on the CS. These forces tend to be large and can be categorized into normal forces and shear forces. As the name implies, normal forces act normal to the CS. Pressure is the most common normal force. Because pressure cannot be negative, it always acts opposite to the surface area vector.

Shear stresses are a result of the fluid's propensity to stick to a solid (or other fluid) surface. The fluid viscosity, as defined earlier, is a measure of the intensity of these stresses. Shear stresses always act opposite to the direction of flow and along the CS. If a fluid is modeled as inviscid, there can be no shear stresses.

Picking up from Equation 2.138, if we model a flow as steady with no viscosity, there will still be pressure forces present. This is depicted in Figure 2.19.



**FIGURE 2.18**  
Typical gun tube CV.



**FIGURE 2.19**  
CV with no viscous forces acting.

Since pressure forces always act opposite to the area vector, it is customary to define the pressure forces as

$$\mathbf{F}_p = - \int_{\text{outflow area}} p d\mathbf{A} - \int_{\text{inflow area}} p d\mathbf{A} \quad (2.140)$$

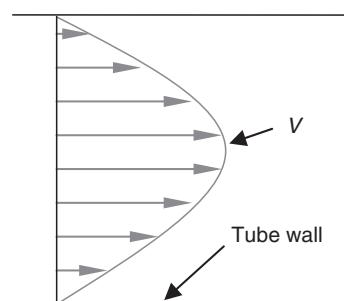
In Equation 2.140, the signs of the area vectors would define the direction of the force.

Before we establish a CV with viscous forces acting, it is instructive to describe these viscous forces and their effect on the flow field. As previously established, viscosity is a property of a fluid. The greater the viscosity of a fluid is, the more difficult it is to shear the material. If the viscosity is high enough or the flow velocity low enough, a fluid will exhibit what is known as laminar flow. Laminar flow is a very orderly shearing of the fluid from a solid surface where the fluid sticks to the boundary. In a tube or pipe, after some entrance length required for the flow to establish itself, the fluid will achieve a parabolic velocity distribution as depicted in Figure 2.20.

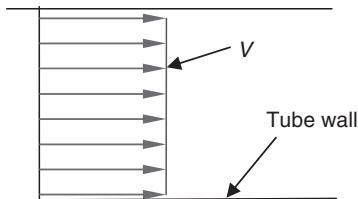
The laminar profile in Figure 2.20 is in stark contrast to the uniform profile that we had assumed in our previous discussions depicted in Figure 2.21. If the flow velocity is high enough or the viscosity low enough, the flow will transition from laminar flow to what is known as turbulent flow. Turbulent flow is characterized by a large number of eddies which swirl around in the flow. These eddies are important in that they tend to distribute momentum, energy, and matter throughout the fluid resulting in better mixing and very different transport properties. Many more flows are turbulent than laminar. The dimensionless parameter which governs this behavior is known as the Reynolds number and is given by

$$Re = \frac{\rho V d}{\mu} = \frac{V d}{\nu} \quad (2.141)$$

In Equation 2.141,  $Re$  is the Reynolds number and is dimensionless,  $\rho$  is the fluid density,  $V$  is the fluid velocity,  $d$  is a relevant characteristic length of the system (an internal diameter of a pipe, a length of a projectile, etc.), and  $\mu$  and  $\nu$  are the dynamic and kinematic viscosities of the fluid, respectively. If the Reynolds number is high enough, the flow will be turbulent. This demarcation is, in general, a range of values that also depends whether



**FIGURE 2.20**  
Laminar velocity profile in a tube.



**FIGURE 2.21**  
Uniform velocity profile in a tube.

the flow is an internal one (such as the gas flow in a gun tube) or an external one (such as the flow about a projectile). The velocity profile of a turbulent flow is depicted in Figure 2.22. Here we can see that the effect of the eddies is to distribute the momentum, resulting in a profile that is flatter and more akin to our inviscid flow model of Figure 2.21.

If we now return to our discussion on the surface tractions, we can discern that the effect of fluid viscosity is to create a shear stress at the boundary between the fluid inside a gun tube and the solid tube itself (i.e., on our CS). If we consider the diagram of Figure 2.19, we can redraw this figure to include the effect of shear stresses as depicted in Figure 2.23. Since the shear stress,  $\tau_w$ , acts all over the area of our CV, we can add a term in for this into Equation 2.140 to obtain an expression for all of the surface forces as follows:

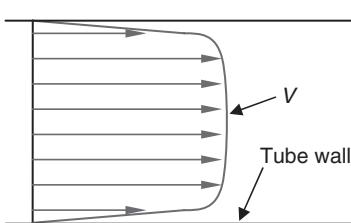
$$\mathbf{F}_{\text{surface}} = - \int_{\text{outflow area}} p d\mathbf{A} - \int_{\text{inflow area}} p d\mathbf{A} - \int_{\text{surface area}} \tau_w d\mathbf{A} \quad (2.142)$$

We can insert this into our expression for the conservation of momentum Equation 2.137 to obtain, for steady flow

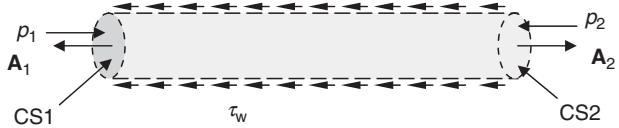
$$- \int_{\text{outflow area}} p d\mathbf{A} - \int_{\text{inflow area}} p d\mathbf{A} - \int_{\text{surface area}} \tau_w d\mathbf{A} = \int_{\text{outflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.143)$$

or, in a more general sense

$$\begin{aligned} & - \int_{\text{outflow area}} p d\mathbf{A} - \int_{\text{inflow area}} p d\mathbf{A} - \int_{\text{surface area}} \tau_w d\mathbf{A} \\ &= \frac{\partial}{\partial t} \int_{\text{CV}} \mathbf{V} \rho dV + \int_{\text{outflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \mathbf{V} \rho \mathbf{V} \cdot d\mathbf{A} \end{aligned} \quad (2.144)$$



**FIGURE 2.22**  
Turbulent velocity profile in a tube.



**FIGURE 2.23**

Surface tractions on a gun tube CV.

The next transport property we shall examine is that of energy. In Sections 2.4 and 2.5, this was discussed to a degree. The objective of this section is to demonstrate that we can use the same transport Equation 2.129 to come up with the energy equations we have used earlier. We start by recognizing that our transport variable is energy,  $E$ . With this in mind, Equation 2.128 can be rewritten as

$$\eta = \frac{E}{m} = e \quad (2.145)$$

Recall that lower case letters are intensive properties. Then we can write

$$\frac{dE}{dt} = \frac{\partial}{\partial t} \int_{CV} e\rho dV + \int_{\text{outflow area}} e\rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} e\rho \mathbf{V} \cdot d\mathbf{A} \quad (2.146)$$

This states that the change in energy of a system is equal to the change in energy stored in the system minus that which is advected away plus that which is advected into the system. Recall from Equation 5.6 that

$$\frac{dQ}{dt} + \frac{dW}{dt} = \frac{dE}{dt} \quad (2.147)$$

From our definition of work, we know that

$$W = \int p dV \quad (2.148)$$

But volume is nothing more than a length times an area. This allows us to write

$$W = \int p \mathbf{x} \cdot d\mathbf{A} \quad (2.149)$$

If we take the derivative of this expression with respect to time assuming pressure is an average value over the time increment, we can write

$$\frac{dW}{dt} = \int p \frac{d\mathbf{x}}{dt} \cdot d\mathbf{A} = \int p \mathbf{V} \cdot d\mathbf{A} \quad (2.150)$$

There are many types of work terms. The term above happens to be called  $pdV$  work or pressure work. The other types of work, such as shaft work, are usually not present in a gun launch so we shall neglect them. Insertion of Equation 2.150 into Equation 2.146 and rearranging yields

$$\frac{dQ}{dt} = \frac{\partial}{\partial t} \int_{CV} e\rho dV + \int_{\text{outflow area}} \left( e + \frac{p}{\rho} \right) \rho \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inflow area}} \left( e + \frac{p}{\rho} \right) \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.151)$$

In the section on thermodynamics, we defined the specific energy through [Equation 2.39](#). If we insert this definition into the above expression, we obtain

$$\frac{dQ}{dt} = \frac{\partial}{\partial t} \int_{CV} e\rho dV + \int_{CS} \left( gz + \frac{V^2}{2} + u + \frac{p}{\rho} \right) \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.152)$$

Here we have combined the last two terms on the RHS of Equation 2.151 with the understanding that the integral of the last term in Equation 2.152, being an integral over the entire CS, accounts for the difference between inflow and outflow. It is informative to look at this equation with respect to a gun launch. The term on the LHS represents the transfer of heat to or from the system. The first term on the RHS represents the change in stored energy of the system (such as energy released by propellant combustion). The last term on the RHS is the change in energy of the system. Since gravitational potential energy, the product  $gz$ , is small relative to the other energy terms, it is usually neglected allowing us to rewrite the expression as

$$\frac{dQ}{dt} = \frac{\partial}{\partial t} \int_{CV} e\rho dV + \int_{CS} \left( \frac{V^2}{2} + u + \frac{p}{\rho} \right) \rho \mathbf{V} \cdot d\mathbf{A} \quad (2.153)$$

Earlier in this section, we introduced the common practice of characterizing a fluid based on its behavior under shear stress. This allowed us to come up with a relationship between applied shear stress and deformation rate. Another distinction has to be made between fluids with respect to the density. If the density is considered constant in a fluid or solid that we model, we call this material incompressible. If the density varies, we must analyze the problem with the assumption of compressible material. This has many ramifications. The most significant ramification is that if the material is incompressible, then the energy equation is decoupled from the momentum equation and we can solve them independently [25]. This makes problem solving much simpler. We do not have this luxury when the density varies significantly.

In fluid flows, such as those which we shall study later, a dimensionless parameter known as the Mach number is used as a measure to determine the effect of compressibility, among its other uses. The Mach number is given by

$$Ma = \frac{V}{a} \quad (2.154)$$

Here  $V$  is some characteristic velocity in the material and  $a$  is the speed of sound in the material. In general, if the Mach number is below 0.3, the deviation from incompressible flow is small so the assumption of incompressibility leads to an acceptably small error [16]. In an ideal gas, the speed of sound is given by the relation

$$a = \sqrt{\gamma RT} \quad (2.155)$$

In this equation,  $\gamma$  is the specific heat ratio,  $R$  is the specific gas constant, and  $T$  is the absolute temperature (i.e., in degrees Rankine or Kelvin). The speed of sound in any material is formally defined as

$$a = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_s} \quad (2.156)$$

That is to say that the speed of sound in a material is equal to the square root of the partial derivative of pressure with respect to density evaluated with constant entropy. The interested reader is referred to any of Refs. [15,16,25] for the detailed proof of this equation.

The speed of sound is essentially the fastest speed at which a disturbance can be propagated by molecular interaction. If a disturbance is created that is strong enough, a shock will form. This shock must always move faster than the speed of sound in the material. We will discuss this in detail later.

In the study of compressible flows, it is common practice to utilize stagnation values in many of our calculations. Stagnation values are the values of the enthalpy, pressure, temperature, and density that are achieved by adiabatically slowing a flow down to zero velocity. The assumption of adiabatic behavior is warranted in many of the situations we will examine, particularly in exterior ballistics. The stagnation enthalpy is given by

$$h_0 = h + \frac{1}{2} V^2 \quad (2.157)$$

In this and the following equations, the subscript “0” indicates the stagnation value,  $V$  is the velocity of the flowing fluid, and the values without the subscript are the static value, in the case of Equation 2.157,  $h$  is the static enthalpy. Equation 2.157 holds for any material. If the material is an ideal gas, we can define the stagnation temperature, pressure, and density as

$$T_0 = h + \frac{1}{2} \frac{V^2}{c_p} \quad \text{or} \quad \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} Ma^2 \quad (2.158)$$

$$\frac{p_0}{p} = \left( 1 + \frac{\gamma - 1}{2} Ma^2 \right)^{\frac{1}{\gamma-1}} \quad (2.159)$$

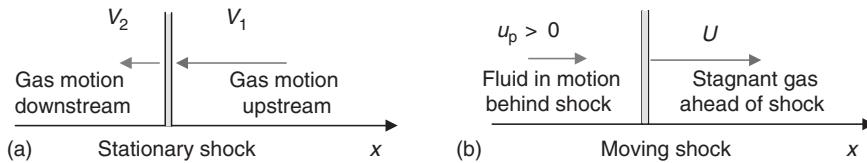
$$\frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma - 1}{2} Ma^2 \right)^{\frac{1}{\gamma-1}} \quad (2.160)$$

In each of these cases, thermodynamic relations have been used for an ideal gas ([Equation 2.61](#)).

Shock waves are formed in materials when disturbances of sufficient strength propagate through the medium. “Sufficient” strength is a term that we throw about rather loosely to describe conditions where shocks are formed—it can be cast in terms of flow velocities or pressures (the two are linked as we shall see). Shocks can be classified as normal or oblique, depending upon the direction of material flow into them. They can also be analyzed as steady or transient. In general, shocks can take curved and rather complex shapes, but the simple analytical tools we have allow us to look at them only under simplified geometries. More complex geometries require the assistance of a computer.

We shall only examine normal shocks in this brief review and direct the reader to Ref. [16] for the handling of oblique shocks. The best way to examine the behavior of a shock is to look at a shock tube. This simple device will allow us to introduce all of the material necessary for the introductory study of ballistics and set the stage for later work when we discuss stress waves in solids.

Before we look at a shock tube, we need to discuss the principle of superposition as applied to shock waves. Consider two shocks as depicted in [Figure 2.24](#). One of these cases



**FIGURE 2.24**  
Stationary and moving shock waves.

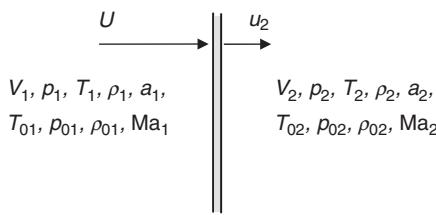
is a stationary shock where we could consider ourselves “riding on the wave,” while in the other case, we can consider ourselves to be sitting on the ground watching the shock pass by. If, in both cases, the shock were passing into a stagnant medium, we would see some important correlations. The passage of a shock wave always induces motion that follows the wave. Consider the situation where we are sitting on the ground, the air about us is stagnant and all of a sudden a shock passed by us just as is shown in Figure 2.24b. If the shock were moving at velocity,  $U$ , we would feel an induced motion, a wind, immediately afterwards moving at velocity  $u_p$  in the same direction that the shock was moving. If we experienced this same situation but instead were riding on the shock, we would feel a wind of velocity  $U$  coming toward our face. This would be analogous to the situation in Figure 2.24a. In this situation, the velocity  $V_1$  would be equal to  $U$ . Note the direction of the velocity vectors in the figure. The velocity vector of magnitude  $V_2$  is moving away from the wave. The figure is drawn correctly, but in the case that was just described, based on superposition, since  $U$  is larger than  $u_p$  (and it always is). If we were riding on the wave, we would see material leaving us at velocity  $(U - u_p)$ . When we examine a shock wave in the frame of Figure 2.24b, we are said to be using an Eulerian frame of reference. If we analyze the very same situation as shown in Figure 2.24a, we are using a Lagrangian reference frame.

The difference between Lagrangian and Eulerian reference frames is important because we sometimes prefer to solve a problem in one frame or the other because the mathematics are simpler. As long as the reference frame motion is accounted for, solving in one frame or the other leads to the same answer.

We shall now use the Lagrangian approach to examine the governing equations for a stationary normal shock wave. Consider the situation in Figure 2.25 where a shock wave is moving to the left at velocity,  $U$ . Since we would like to examine the behavior of this shock, we will put ourselves in a reference frame attached to the shock itself. We form a CV enclosing the shock only. We observe, while riding on this shock, that fluid enters the CV at velocity  $U$  and leaves at velocity  $u_2$ . We can write the conservation of mass, momentum, and energy equations for this system as follows:

Conservation of mass (continuity equation)

$$\rho_1 U = \rho_2 u_2 \quad (2.161)$$



**FIGURE 2.25**  
Stationary shock wave.

## Conservation of momentum

$$p_1 + \rho_1 U^2 = p_2 + \rho_2 u_2^2 \quad (2.162)$$

## Conservation of energy

$$h_1 + \frac{1}{2} U^2 = h_2 + \frac{1}{2} u_2^2 = h_{01} = h_{02} = h_0 = \text{constant} \quad (2.163)$$

We see from the last equation that across a shock wave the stagnation enthalpy must remain constant. This falls out directly from the fact that we assumed the shock wave was adiabatic. These equations are coupled through a material model such as the ideal gas equation of state (relates  $p$ ,  $V$ , and  $T$ ) and the calorically perfect assumption (relates  $h$  to  $T$ ). If we consider the special case where the shock under examination is moving into a stagnant fluid as depicted in Figure 2.26, we can write the above three equations as

$$\rho_1 U = \rho_2 (U - u_p) \quad (2.164)$$

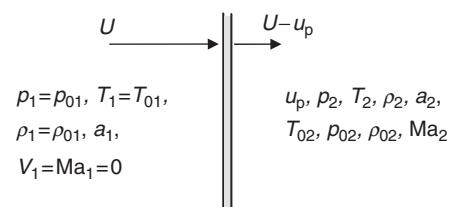
$$p_1 + \rho_1 U^2 = p_2 + \rho_2 (U - u_p)^2 \quad (2.165)$$

$$h_1 + \frac{1}{2} U^2 = h_2 + \frac{1}{2} (U - u_p)^2 = h_0 = \text{constant} \quad (2.166)$$

The conservation of mass, momentum, and energy equations can be combined as detailed in Refs. [10,16] to yield the Rankine–Hugoniot relationship. This relationship determines how the energy changes across a normal shock wave. It is very important and will appear again when the terminal ballistics material is discussed. It can be written in terms of total specific energy,  $e$ , or if some of the energy components are negligible, it can be written in terms of enthalpy,  $h$ . At this stage, we will use the latter expression, but we shall switch when we discuss shock in the terminal ballistics section. Writing the Rankine–Hugoniot relationship in terms of enthalpy, we have

$$h_2 - h_1 = \frac{1}{2} (p_2 - p_1) \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \quad (2.167)$$

The strength of a shock is normally assessed by the change in pressure across it. In other words, its strength is given by the ratio  $p_2/p_1$ . If we assume the material through which this shock is propagating is an ideal gas, Equations 2.164 through 2.166 can be combined with the relationships provided in [Equation 2.61](#) to yield expressions that relate all of the values ahead of the shock to values after the passage of the shock. The details of this are available in Ref. [16]. These expressions are as follows:



**FIGURE 2.26**  
Stationary shock wave moving into a stagnant fluid.

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \left[ \frac{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}{1 + \frac{\gamma+1}{\gamma-1} \left( \frac{p_2}{p_1} \right)} \right] \quad (2.168)$$

$$\frac{\rho_1}{\rho_2} = \frac{1 + \frac{\gamma+1}{\gamma-1} \left( \frac{p_2}{p_1} \right)}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}} \quad (2.169)$$

The real power of these equations lies in the fact that with just the strength of the shock known we can determine all of the other items of interest. In the above equations, we have seen that given the pressure ratio (i.e., the strength) of the shock, we know the temperature behind the wave and the increase in density across the wave. We can also determine the wave speed,  $U$ , and induced velocity,  $u_p$ , through

$$U = a_1 \sqrt{\frac{\gamma+1}{2\gamma} \left( \frac{p_2}{p_1} - 1 \right) + 1} \quad (2.170)$$

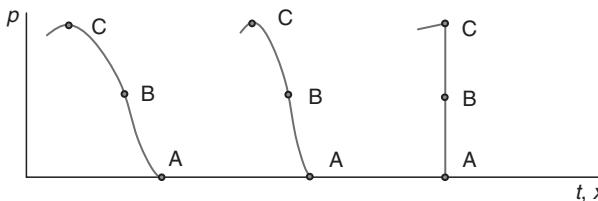
$$u_p = U \left( 1 - \frac{\rho_1}{\rho_2} \right) = \frac{a_1}{\gamma} \left( \frac{p_2}{p_1} - 1 \right) \sqrt{\frac{\frac{2\gamma}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} + \frac{p_2}{p_1}}} \quad (2.171)$$

If we change reference frames to one in which we are stationary and the shock is moving, then the assumption of constant stagnation enthalpy,  $h_0$ , is no longer valid. The reason is best illustrated by an example. Consider the gas ahead of the shock wave. It was initially motionless so  $h_1 = h_{01}$ . After the wave passes, we know that the temperature must increase so  $h_2 > h_1$ . Additionally, the gas is now moving at velocity,  $u_p$ , so that we can see

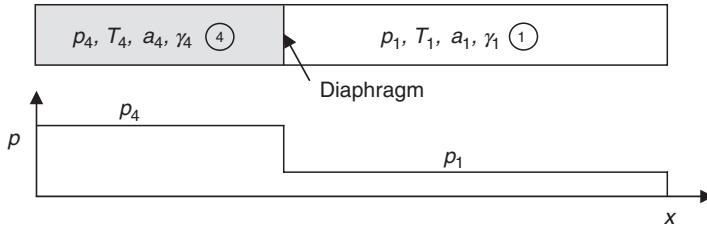
$$h_1 = h_{01} < h_{02} = h_2 + \frac{1}{2} u_p^2 \quad (2.172)$$

It is by this very same logic that the stagnation pressure, temperature, and density must also increase.

We have discussed some governing equations but let us break for a moment to discuss why a gas shocks up. If we examine Equation 2.170 closely, we see that a higher pressure causes a faster motion of the wave. If we imagine a shock wave as depicted in Figure 2.27 moving to the right, we can pick out three points that we shall follow for some time. Point A is essentially the beginning of the pressure increase and at the un-shocked initial pressure. Point B is at some pressure in between the peak pressure of the shock and the initial pressure of the material into which the shock is propagating. Point C is at the peak



**FIGURE 2.27**  
Formation of a shock wave.



**FIGURE 2.28**

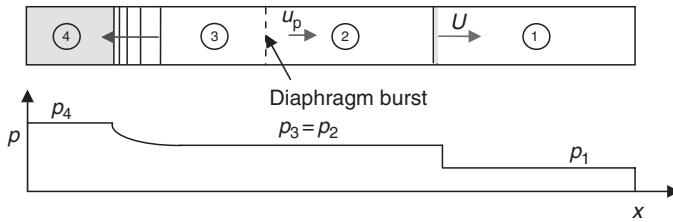
The shock tube in its initial state. (From Anderson, J.D., *Modern Compressible Flow with Historical Perspective*, 3rd ed., McGraw-Hill, New York, NY, 2003. With permission.)

shock pressure. From [Equation 2.170](#), we see that the local velocity of point B must be greater than point A and also that the local velocity of point C is greater still. This means that at some time,  $t$ , these points must converge thereby forming a step discontinuity in pressure. This step discontinuity is the way we model the shock—there is actually a very small distance over which a shock will develop so that the pressure increase is rapid, but continuous. With this information, we see that compression shocks are the only admissible shocks. Later we will introduce rarefactions that are the converse of shocks. Since the pressure decreases in a rarefaction wave, the wave will tend to spread out over time and distance.

Now that we have the governing equations, we shall examine the behavior of a shock wave in a shock tube. A shock tube is a device as depicted in Figure 2.28 that contains two regions of gas. These regions are separated by a diaphragm which can be burst very quickly and contain one gas at high pressure and another at lower pressure. The gases could be different (thus all of their properties as well) as can their temperatures. Below the graphic of the shock tube is a pressure versus distance plot showing that the pressure in region 4 (the high-pressure region) is greater than that of region 1 and the diaphragm divides the two regions. If the diaphragm is burst, then a shock will propagate into the lower pressure region, increasing the pressure, and a rarefaction wave (to be discussed later) will propagate into the high-pressure region, decreasing the pressure. If we examine the shock tube after some very short time,  $t$ , the situation will appear as shown in [Figure 2.29](#) with the corresponding pressure-distance profile. One of the most interesting aspects of compressible fluid flow is that if we know what the initial states of the ideal gases in the shock tube are, we can predict the pressures and temperatures of the unsteady motion afterwards by [Equations 2.161](#) through [2.171](#). In fact, we can predict the pressure behind the initial shock from

$$\frac{p_4}{p_1} = \frac{p_2}{p_1} \left\{ 1 - \frac{(\gamma_4 - 1) \left( \frac{a_1}{a_4} \right) \left( \frac{p_2}{p_1} - 1 \right)}{\sqrt{2\gamma_1 \left[ 2\gamma_1 + (\gamma_1 + 1) \left( \frac{p_2}{p_1} - 1 \right) \right]}} \right\}^{\frac{-2\gamma_4}{(\gamma_4 - 1)}} \quad (2.173)$$

[Equation 2.173](#) needs to be solved for the initial shock strength,  $p_2/p_1$ , but afterwards [Equations 2.161](#) through [2.171](#) can be used directly to calculate the parameters of interest. The details of this derivation can be found in Ref. [16].



**FIGURE 2.29**

The shock tube after some short time,  $t$ . (From Anderson, J.D., *Modern Compressible Flow with Historical Perspective*, 3rd ed., McGraw-Hill, New York, NY, 2003. With permission.)

You can see from Figures 2.28 and 2.29 that the shock tube is not infinite in extent. At some point, the shock produced by the bursting of the diaphragm will reach the right end of the tube. When this occurs, the condition at the wall is such that no flow through it is possible. Consider that all the fluid behind the shock wave is moving with induced velocity,  $u_p$ , toward the wall. Clearly, this situation is at odds with the wall-imposed boundary condition of zero velocity. Nature handles this issue by creating a shock wave of strength  $U_R$  that propagates back into the fluid that is heading toward the wall at velocity  $u_p$ . Notice that we have used a velocity here to define the strength of the shock—it should be clear by now that if we know either the velocity of the shock or the pressure ratio, we can find the other. The net effect of this reflected shock is that it stagnates the fluid between it and the fixed end of the shock tube as depicted in Figure 2.30. In this figure, we shall assume the tube is extremely long on the rarefaction side so we do not have to discuss rarefaction reflections, yet. If we look at our conservation Equations 2.161 through 2.163 and consider that the shock wave sees material coming into it at velocity  $U_R + u_p$ , we can write equations for the reflected shock that are analogous to Equations 2.164 through 2.166 for the incident wave. These are

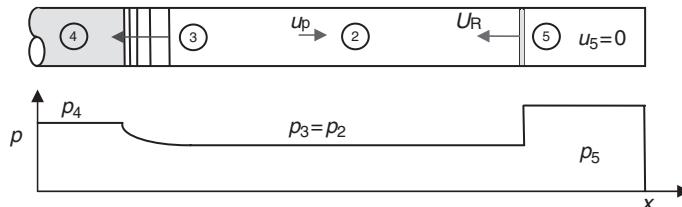
$$\rho_2(U_R + u_p) = \rho_5 U_R \quad (2.174)$$

$$p_2 + \rho_2(U_R + u_p)^2 = p_5 + \rho_5 U_R^2 \quad (2.175)$$

$$h_2 + \frac{1}{2}(U_R + u_p)^2 = h_5 + \frac{1}{2}U_R^2 \quad (2.176)$$

A simple method for determination of the speed of the reflected shock is to first determine the Mach number of the incident pulse,  $\text{Ma}_s$ , through

$$\text{Ma}_s = \frac{U}{a_1} \quad (2.177)$$



**FIGURE 2.30**

The shock tube after a reflection of the incident wave.

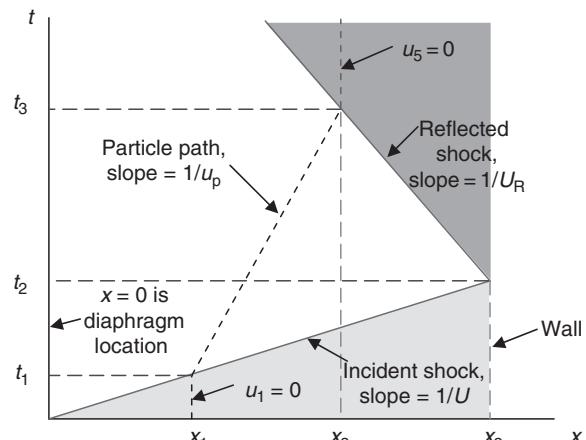
The relationship between the incident shock velocity and the reflected velocity is derived in Ref. [16] and given by

$$\frac{Ma_R}{Ma_R^2 - 1} = \frac{Ma_s}{Ma_s^2 - 1} \sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2} (Ma_s^2 - 1) \left( \gamma + \frac{1}{Ma_s^2} \right)} \quad (2.178)$$

Here  $Ma_R$  is the Mach number of the reflected shock which can be converted to a velocity through use of

$$Ma_R = \frac{U_R + u_p}{a_2} \quad (2.179)$$

In our discussions on shock waves throughout the terminal ballistics sections, we will make use of time-distance diagrams, so-called  $x-t$  plots. It is prudent to introduce them here as reinforcement of the shock wave discussion. An  $x-t$  plot places distance on the abscissa and time on the ordinate. Because of this placement, which is opposite to normal function versus time plots, we need to adjust some of our logic that we are used to. For instance, slopes of straight lines on these diagrams are reciprocal velocities. If we consider the situation in Figure 2.30 and draw an  $x-t$  plot for it with the origin starting from the initial diaphragm location, we would have a plot as depicted in Figure 2.31. We shall examine the shocks in this diagram first. If we assume that the incident shock forms immediately (this is not really true, as we learned earlier, but close enough for our purposes), it propagates toward the wall which is located at point  $x_2$  in our figure. If we wanted to determine what the velocity distribution was in this device at any time,  $t$ , we would examine a horizontal line in the figure. For instance, if we examined the situation at time,  $t_1$ , we would see that the material in the unshaded region up to point  $x_1$  would have a velocity  $u_p$  and everything between  $x_1$  and  $x_2$  (the wall) would have zero velocity. Once the incident shock reflects off the wall a new shock of velocity  $U_R$  propagates back into the fluid. This is depicted by the upper line in the diagram. Note that the slope is greater on this reflected shock, indicative of a lower velocity than the incident wave. The material in the shaded region behind this wave has been stagnated to zero velocity. We can use an  $x-t$  diagram to determine how a particle moves over time. Consider a particle located initially at location  $x_1$ . It remains stationary until the shock wave passes by at time  $t_1$ , as indicated by a vertical line. At time  $t_1$ , the incident shock passes it and induces a



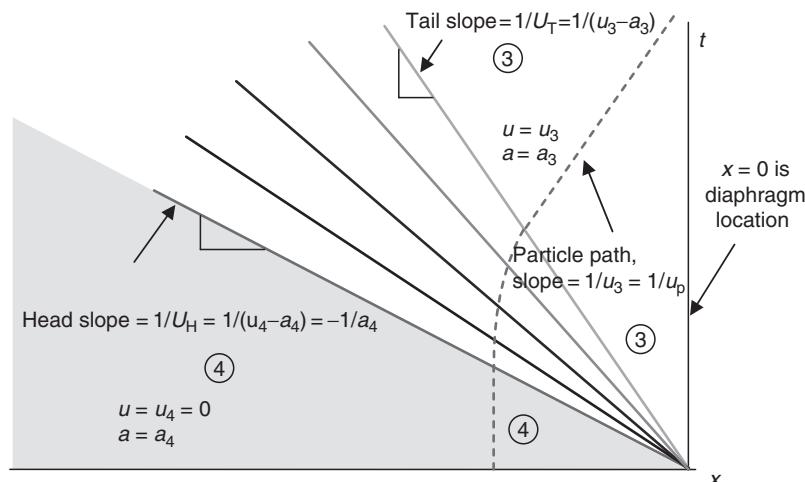
**FIGURE 2.31**

$x-t$  Plot for the reflection of a shock wave. (From Anderson, J.D., *Modern Compressible Flow with Historical Perspective*, 3rd ed., McGraw-Hill, New York, NY, 2003. With permission.)

velocity,  $u_p$  to the particle. When the particle moves at velocity,  $u_p$ , it will trace out a line on the diagram that has a slope of  $1/u_p$ . While this particle is moving at velocity  $u_p$ , the shock interacts with the wall and reflects at time  $t_2$ . While the reflected shock is approaching, the observed particle has no idea anything is about to happen and continues to move at velocity  $u_p$  until the reflected shock passes by at time  $t_3$ . This passage of the reflected shock stagnates the particle to zero velocity and its motion (or lack thereof) traces out a vertical line. A final point of interest regarding  $x-t$  plots is that we can actually see the compression of the material. If we consider all of the material initially between points  $x_1$  and  $x_2$ , we see that, after the passage of the shock and its reflection, it has all been compressed to the region between  $x_3$  and  $x_2$ . With this information, the basis for our future discussions using  $x-t$  plots is established.

A rarefaction wave, sometimes known as an expansion or relief wave, is the means by which nature handles a sudden drop in pressure. As we stated earlier, compression waves (also known as condensations) eventually coalesce into shocks which are analyzed as step discontinuities in pressure. This coalescence was brought about by the fact that the local velocity increases with increasing pressure. In a rarefaction, the opposite is true. A rarefaction increases over time because the pressure at the head of the wave is greater than that at the tail of the wave. In the case of our shock tube, the head of the rarefaction will propagate at the local speed of sound in the material ( $a_4$  in Figure 2.29), while the tail will propagate at velocity  $(u_3 - a_3)$  which is equal to  $(u_p - a_2)$ . This is depicted schematically in Figure 2.32. Throughout the rarefaction wave, the velocity continuously decreases between these two values. Because of this continuous decrease in velocity, it is common to model the decrease as a series of wavelets. The more wavelets we include, the smoother the curve. If we use Figure 2.32 to trace a particle path after the bursting of the diaphragm, we see that the particle would not move until the head of the rarefaction wave passed by it. After the passage of the head of the wave, the velocity would continuously increase until passage of the tail of the wave, after which it would be moving at velocity  $u_p$ . The length of the rarefaction can be determined at any time by scribing a horizontal line through the diagram. If we do this at two points in time on the diagram, we can see how the length of the wave increases.

What is depicted in Figure 2.32 is a simple, centered rarefaction wave. A wave is considered simple if all of the characteristics (the rays emanating from the origin) are



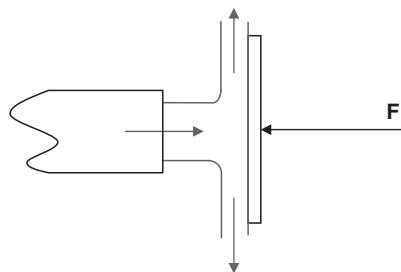
**FIGURE 2.32**  
 $x-t$  Plot for a rarefaction wave.

straight. Reflections of a rarefaction are somewhat more complicated than that of a shock. The reflection of the head of the rarefaction wave must pass through the characteristics of the rest of the wave being both affected by as well as affecting them. The result is that the characteristics tend to bend making the calculations somewhat more complex. We will handle this in a simplified fashion later, but the interested reader is directed to Ref. [16] for an outstanding treatment for handling these situations.

We now have sufficient information to handle the fluid mechanics of interior and exterior ballistics. We shall treat the formation of shocks and rarefactions as necessary in the terminal ballistics section.

### Problem 13

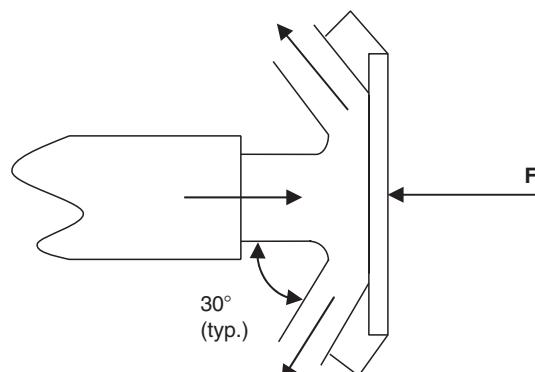
The principle behind a muzzle brake on a gun is to utilize some of the forward momentum of the propelling gases to reduce the recoil on the carriage. In the simple model below, the brake is assumed to be a flat plate with the jet of gases impinging upon it. If the jet diameter is 105 mm and the velocity and density of the gas (assume air) are 750 m/s and 0.457 kg/m<sup>3</sup>, find the force on the weapon in Newtons assuming the gases are directed 90° to the tube and the flow is steady.



*Answer:* -2225.9 [N]

### Problem 14

Some engineer gets the idea that if deflecting the muzzle gases to the side is a good idea, then deflecting it rearward would be better (until of course an angry gun crew gets hold of him). If the jet diameter is again 105 mm and the velocity and density of the gas (again assume air) are 750 m/s and 0.457 kg/m<sup>3</sup>, find the force on the weapon in Newtons assuming the gases are directed 150° to the tube and the flow is steady.



*Answer:* -4153.5 [N]

### Problem 15

Consider a shock tube that is 6-ft long with a diaphragm at the center. Air is contained in both sections ( $\gamma = 1.4$ ). The pressure in the high-pressure region is 2000 psi. The pressure in the low pressure region is 14.7 psi. The temperature in both sections is initially 68°F. When the diaphragm is burst, determine the following:

1. The velocity that the shock wave propagates into the low pressure region.  
*Answer: 2798 [ft/s]*
2. The induced velocity behind the wave.  
*Answer: 1946 [ft/s]*
3. The velocity of a wave reflected normally off the wall (relative to the laboratory).  
*Answer: 1232 [ft/s]*
4. The temperature behind the incident wave.  
*Answer: 657 [°F]*
5. Draw an  $x-t$  diagram of the event. Include the path of a particle located 2 ft from the diaphragm.

### Problem 16

An explosion generates a shock wave in still air. Assume we are far enough from the initial explosion that we can model the wave as a one-dimensional shock. Assume that the pressure generated by the explosion was 10,000 psi and the ambient atmospheric pressure, density, and temperature are 14.7 psi, 0.06 lbm/ft<sup>3</sup> and 68°F, respectively. Determine

1. The static pressure behind the wave (assume  $\gamma = 1.4$  and since we are far away from the effects of the explosion assume  $a_1/a_4 \approx 0.5$ ).  
*Answer:  $p_2 = 376.6$  [psi]*
2. The velocity that the wave propagates in still air.  
*Answer:  $U = 5294$  [ft/s]*
3. The induced velocity that a building would see after the wave passes.  
*Answer:  $u_p = 4212$  [ft/s]*
4. The velocity of a wave reflected normally off a building.  
*Answer:  $U_R = 1921$  [ft/s]*

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# 3

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## *Analytic and Computational Ballistics*

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[Chapter 2](#) has provided us with the necessary background to discuss procedures that calculate the behavior of projectiles and propellant in the gun tube. The chapter had to be brief because detailed treatment of any one of the subjects could be (and are) collected into complete texts in their own right. The reader is directed to the references at the end of the chapter if a more complete background in the individual subject is felt to be necessary.

Much like other introductory texts on difficult subjects, this chapter shall begin with fundamental treatments that will allow the reader to perform meaningful calculations of interior ballistic problems. This simplified treatment will, by its very nature, not provide exact answers but answers which are reasonable from an engineering viewpoint. As will be discussed, more exact methods require a varying degree of computer assets.

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### **3.1 Computational Goal**

The interior ballistian is charged with devising a propellant charge that will deliver the projectile of interest to the gun muzzle intact, with the desired muzzle velocity, with no damage to the weapon from excess pressure, and with high probability that successive charges propelling the same projectiles will produce the same results. To do this, the ballistian must be able to predict *a priori* what the charge will do, i.e., what pressures will both the gun and the projectile experience during travel down the bore and what the velocity and acceleration profile would be during the travel to the muzzle. Over the centuries, ballisticians, including some quite eminent mathematicians and physicists, have devised computational schemes that can be used to make such predictions. We intend to explore a few of these analytic tools in sufficient depth so that the physics and mathematics become clear to the user, who would then also be able to discern reasonable answers from patently erroneous ones.

It is important to understand how predictions of pressure and velocity are verified experimentally in real guns. Such understanding has led to the development of pressure ratios that allow the gun and projectile designers to know what pressures are acting on the gun and on the projectile at locations that practical instrumentation has some difficulty capturing. Pressure is most readily measured at the base of the gun chamber, where the gas flow is minimal or nonexistent. When pressure taps are introduced along the bore to take measurements while the projectile is traveling and the gases are flowing, it has been found that turbulent flow and shock waves make such measurements difficult to interpret. Copper crusher gauges are used in which small copper cylinders are crushed to a barrel shape in the gauge by the applied pressure and the distortion of the cylinders measured. These gauges are placed in the base of the charge and recovered after firing. Distortion

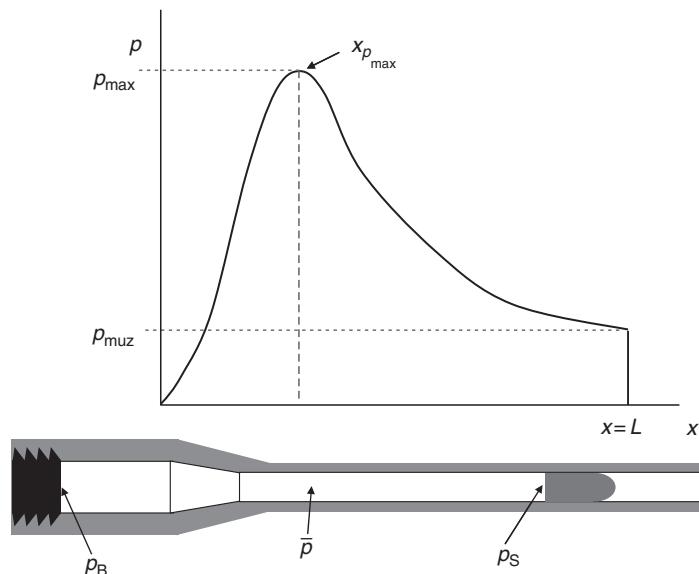
is checked against a calibration chart and the pressure is quickly read. Of course, pressure measured in this way is representative only of the maximum pressure sensed by the gauge, which gives no indication of its profile in time or in travel.

Even such a primitive measurement was and still is of use; because the designer would know the maximum pressure, the projectile and gun would have to contend with an indication that piezo type pressure gauges are functioning properly. These gauges are still widely used to check the pressure consistency of already developed charges. Knowledge of how that copper pressure was related to pressures at other locations during the travel was a great advance. When the pressure ratios were devised that related chamber pressure to the pressure at the base of the projectile during its travel down the bore, these were greatly appreciated by the designers. Even better was the introduction of electronic piezo gauges installed through the breech that allowed the measurement of pressure over time so that a pressure-time profile could be available. The study of a few of the computational theories that develop these ratios follows in succeeding sections.

### 3.2 Lagrange Gradient

To determine the time-dependent motion of the projectile, we need to make some assumptions about the behavior of the gas pushing it out of the gun. These assumptions will involve the pressure, mass, and density distribution of the gas. We shall refer to the sketch in Figure 3.1 in the text that follows. We shall continue to use  $x$  as the distance from the projectile base position at the seating location to its position at all later times with the time derivative defined as

$$\frac{dx}{dt} = \dot{x} = V \quad (3.1)$$



**FIGURE 3.1**  
Pressure-distance relationship in a typical gun firing.

We will first assume that the gas density is uniform in the volume behind the projectile at time  $t$ . We can then write, for any time,  $t$ , that

$$\rho = \rho(x_g, t) \quad (3.2)$$

In this equation,  $x_g$  is the  $x$ -location of the gas mass center behind the projectile. We shall also assume that there is no spatial gradient in density at any time, thus

$$\frac{\partial \rho}{\partial x_g} \Big|_t = 0 \quad (3.3)$$

We can also write the continuity equation for a compressible fluid as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_g} (\rho V_{x_g}) = 0 \quad (3.4)$$

We can expand the continuity Equation 3.4 as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_g} V_{x_g} + \rho \frac{\partial V_{x_g}}{\partial x_g} = 0 \quad (3.5)$$

Inserting our assumption of the absence of a spatial density gradient allows us to simplify this expression to

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial V_{x_g}}{\partial x_g} = 0 \quad (3.6)$$

Now because we stated that the density was not a function of  $x$ , we can remove the partial derivative notation from the temporal term and rearrange to yield

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \frac{\partial V_{x_g}}{\partial x_g} \quad (3.7)$$

Assume at this point that the solid propellant in the charge has all turned to gas, then what was initially a solid propellant of charge weight,  $c$ , is now a gas of identical weight,  $c$ . So the gas density is this weight divided by the volume the gas occupies, or

$$\rho(t)|_c = \frac{c}{V(t)} \quad (3.8)$$

Here, the subscript “ $c$ ” refers to conditions after the charge has burned out, i.e., all the solid has evolved into gas. If the base of the projectile has moved a distance,  $x$ , and the bore area is  $A$ , then the volume behind the projectile containing gas is

$$V(t) = Ax(t) \quad (3.9)$$

If we insert Equation 3.9 into Equation 3.8 and then take the derivative with respect to time, the result can be simplified to Equation 3.10.

$$\frac{1}{x} \frac{dx}{dt} = \frac{\partial V_{x_g}}{\partial x_g} \quad (3.10)$$

Note that there is a difference here between  $x$  and  $x_g$ :

$x$  is the location of the base of the projectile

$x_g$  is the location of the mass center of the gas

If we integrate [Equation 3.10](#) with respect to  $x_g$  and use the boundary conditions of  $V_{x_g} = 0$  when  $x_g = 0$ , then we get

$$\frac{x_g}{x} \frac{dx}{dt} = V_{x_g}(x_g) \quad (3.11)$$

Now, since  $x$  is the position of the base of the projectile at time  $t$  we see that  $dx/dt$  is the velocity of the projectile at time  $t$ , so we can write

$$\frac{V}{x} = \frac{V_{x_g}}{x_g} \quad (3.12)$$

This implies that the gas particle velocity varies linearly from the breech face to the projectile base, and is a fundamental tenet of the Lagrange\* approximation. We can describe the kinetic energy of the gas stream as

$$KE_g = \frac{1}{2} m_g V_{x_g}^2 \quad (3.13)$$

But, as described earlier, the mass of the gas is its density times the volume it occupies at time  $t$ , therefore

$$KE_g = \int_0^x \frac{1}{2} \rho A V_{x_g}^2 dx_g \quad (3.14)$$

Moving the spatially constant terms,  $\rho A / 2$ , outside the integral and performing the integration gives us

$$KE_g = \frac{\rho A}{2} \frac{V^2}{x^2} \frac{x_g^3}{3} \Big|_0^x = \frac{1}{6} \rho A x V \quad (3.15)$$

But we know from our earlier work that

$$\rho A x = c \quad (3.16)$$

So we can write

$$KE_g = \frac{1}{6} c V^2 \quad (3.17)$$

The total kinetic energy of the system (neglecting recoil) is

$$KE_g = \frac{\rho A}{2} \frac{V^2}{x^2} \frac{x_g^3}{3} \Big|_0^x = \frac{1}{6} \rho A x V \quad (3.18)$$

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\* Joseph-Louis Lagrange, 1736–1813, Italian/French mathematician.

But the kinetic energy of the projectile is

$$KE_{\text{shot}} = \frac{1}{2} w_p V^2 \quad (3.19)$$

where  $w_p$  is the projectile mass.

So the Lagrange approximation for kinetic energy is

$$KE_{\text{tot}} = \frac{1}{2} w_p V^2 + \frac{1}{6} c V^2 = \frac{1}{2} \left( w_p + \frac{c}{3} \right) V^2 \quad (3.20)$$

In this development the volume of gas is assumed to be a cylinder of cross-sectional area  $A$ . In reality, it is not; while the bore is cylindrical, the chamber is not. Chamber diameters can be much greater than bore diameters. To account for this, an effort to modify the Lagrange gradient approximations has been performed [1]. This will be explored subsequently. The changes from the Lagrange gradient will be found to be small but not insignificant and the so-called chambrage gradient will be explained in Section 3.3 and incorporated in the discussion of numerical methods in Section 3.4.

We can describe the linear momentum of the gas stream as

$$MOM_g = m_g V_{x_g} \quad (3.21)$$

But, again, the mass of the gas is its density times the volume it occupies at time  $t$ , therefore

$$MOM_g = \int_0^x \rho A V_{x_g} dx_g \quad (3.22)$$

We can use our continuity relationship in Equation 3.11 to write

$$MOM_g = \rho A \int_0^x \left( \frac{x_g}{x} \frac{dx}{dt} \right) dx_g = \rho A \int_0^x \left( \frac{x_g}{x} V \right) dx_g \quad (3.23)$$

Performing the integration gives us

$$MOM_g = \rho A \frac{V}{x} \frac{x_g^2}{2} \Big|_0^x = \frac{1}{2} \rho A x V \quad (3.24)$$

If we recall Equation 3.16, we can write

$$MOM_g = \frac{1}{2} c V \quad (3.25)$$

The total linear momentum of the system (neglecting the weapon) is

$$MOM_{\text{tot}} = MOM_{\text{shot}} + MOM_g \quad (3.26)$$

The linear momentum of the projectile is

$$MOM_{\text{shot}} = w_p V \quad (3.27)$$

So the Lagrange approximation for linear momentum is

$$\text{MOM}_{\text{tot}} = w_p V + \frac{1}{2} c V = \left( w_p + \frac{c}{2} \right) V \quad (3.28)$$

Because we are looking for the parameters, we can readily measure breech pressure and muzzle velocity, and we must develop predictive equations for them, i.e., equations for pressure in terms of charge parameters and equations of motion of the projectile. To do this, we adopt a Lagrangian approach to track the motion of a particle of gas. What follows is a derivation for the equation of motion for an element of gas. For a rigorous, complete treatment, see any text on fluid mechanics, for example [2].

For differentiation that tracks a fluid element (the Lagrangian approach), the following differential operator (called the substantial derivative or material derivative) is used:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \quad (3.29)$$

where  $u$ ,  $v$ , and  $w$  are the velocity components in the  $x$ ,  $y$ , and  $z$  directions, respectively.

If we consider a one-dimensional flow operating on the velocity  $V_{x_g}(x)$  (here  $V_{x_g}$  is the axial velocity and replaces  $u$  above)

$$\frac{DV_{x_g}}{Dt} = \frac{\partial V_{x_g}}{\partial t} + V_{x_g} \frac{\partial V_{x_g}}{\partial x} \quad (3.30)$$

In vector notation, the gradient of a function is

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (3.31)$$

Force is the time rate of change of momentum

$$\mathbf{F} = \frac{\partial}{\partial t} (m\mathbf{v}) \quad (3.32)$$

It can be shown using Gauss's theorem [3] that the rate of change of linear momentum of the fluid inside a surface  $S$  in changing to surface  $S'$  in time,  $dt$ , is

$$\int_V \rho \frac{d\mathbf{v}}{dt} dV \quad (3.33)$$

From the equations of motion for an inviscid fluid we know that the total force equals the pressure on the boundary element integrated over the boundary plus the body force  $\mathbf{F}$  integrated over the mass in  $S$ , or

$$\int_S p \mathbf{n} dS + \int_V \mathbf{F} \rho dV = - \int_V \nabla p dV + \int_V \mathbf{F} \rho dV \quad (3.34)$$

Because by Gauss's theorem

$$\int_S p \mathbf{n} dS = - \int_V \nabla p dV \quad (3.35)$$

Setting the RHS of [Equation 3.35](#) equal to [Equation 3.33](#) we get

$$\int_V \left[ \mathbf{F}\rho - \nabla p - \rho \frac{d\mathbf{v}}{dt} \right] dV = 0 \quad (3.36)$$

Since V is chosen arbitrarily, the sum in brackets must equal zero

$$\mathbf{F}\rho - \nabla p - \rho \frac{d\mathbf{v}}{dt} = 0 \quad (3.37)$$

In the absence of a body force  $\mathbf{F}$ , we can rewrite this as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \nabla p \quad (3.38)$$

We can write Equation 3.38 as follows for one-dimensional flow and negligible body forces

$$\frac{dp}{dx_g} = -\rho \left[ \frac{\partial V_{x_g}}{\partial t} + V_{x_g} \frac{\partial V_{x_g}}{\partial x_g} \right] \quad (3.39)$$

Note here that we have used the substantial derivative for the velocity of the gas stream.

If we insert the relationship for the gas stream velocity we obtained through the continuity [Equation 3.11](#) into Equation 3.39, we can write

$$\frac{dp}{dx_g} = -\rho \left[ \frac{\partial}{\partial t} \left( \frac{x_g}{x} \frac{dx}{dt} \right) + V_{x_g} \frac{\partial V_{x_g}}{\partial x_g} \right] \quad (3.40)$$

or

$$\frac{dp}{dx_g} = -\rho \left[ \frac{\partial}{\partial t} \left( \frac{x_g}{x} \frac{dx}{dt} \right) + \left( \frac{x_g}{x} \frac{dx}{dt} \right) \frac{\partial}{\partial x_g} \left( \frac{x_g}{x} \frac{dx}{dt} \right) \right] \quad (3.41)$$

We can combine terms in Equation 3.41 as follows:

$$\frac{dp}{dx_g} = -\rho \left[ -\frac{x_g}{x^2} \left( \frac{dx}{dt} \right)^2 + \frac{x_g}{x} \frac{d^2x}{dt^2} + \frac{x_g}{x^2} \left( \frac{dx}{dt} \right)^2 \right] \quad (3.42)$$

Simplifying the expression gives us

$$\frac{dp}{dx_g} = -\rho \frac{x_g}{x} \frac{d^2x}{dt^2} \quad \text{or} \quad \frac{dp}{dx_g} = -\rho \frac{x_g}{x} \ddot{x} \quad (3.43)$$

If we use our relationship between density and charge weight in [Equation 3.8](#), we can write

$$\frac{dp}{dx_g} = -\frac{cx_g}{Ax^2} \ddot{x} \quad (3.44)$$

We can integrate this expression with respect to the gas mass center as

$$\int_0^{x_g} \frac{dp}{dx_g} dx_g = -\frac{c}{Ax^2} \ddot{x} \int_0^{x_g} x_g dx_g \quad (3.45)$$

Performing the integration yields

$$p = -\frac{cx_g^2}{2Ax^2} \ddot{x} + \text{constant} \quad (3.46)$$

Let us now define

$p_S$  = pressure at the projectile base

$p_B$  = pressure at the breech

$\bar{p}$  = mean pressure in volume behind projectile

$p_R$  = pressure resisting projectile motion (force/bore area)

We will develop the equations of motion both with a resistive force in the bore (such as friction and the air being compressed in front of the projectile) and neglecting the resistance. If we write Newton's second law for a projectile being acted upon by propellant gases, we have

$$w\ddot{x} = Ap_S \quad (3.47)$$

Writing this in terms of the acceleration we get

$$\ddot{x} = \frac{A}{w} p_S \quad (3.48)$$

where  $w$  is the projectile mass. Since the base of our projectile is at location  $x$  and the local pressure on the base is  $p_S$ , we can substitute these values into Equation 3.46 for  $x_g$  and  $p$  to obtain

$$p_S = -\frac{c}{2A} \ddot{x} + \text{constant} \quad (3.49)$$

Keep in mind that this is a local condition that we applied to the gas in the vicinity of the base (that gas's mass center is approximately at  $x$ ). We can rearrange Equation 3.49 to yield our constant of integration.

$$\text{constant} = p_S + \frac{c}{2A} \ddot{x} \quad (3.50)$$

If we use Equation 3.48, we obtain

$$\text{constant} = p_S + \frac{c}{2A} \frac{A}{w} p_S = \left(1 + \frac{c}{2w}\right) p_S \quad (3.51)$$

Inserting this constant back into our [Equation 3.46](#) gives us

$$p = -\frac{cx_g^2}{2Ax^2} \ddot{x} + \left(1 + \frac{c}{2w}\right) p_S = -\frac{cx_g^2}{2Ax^2} \left(\frac{A}{w}\right) p_S + \left(1 + \frac{c}{2w}\right) p_S \quad (3.52)$$

or

$$p = p_S + p_S \left(1 - \frac{x_g^2}{x^2}\right) \frac{c}{2w} \quad (3.53)$$

This equation relates the pressure at the base of the projectile to that at the location of the gas mass center. By similar logic, at the breech,  $x_g = 0$ , and the pressure,  $p = p_B$ , so we can substitute the values into Equation 3.53 to obtain a relationship between the breech pressure and the pressure at the projectile base

$$p_B = p_S + p_S \frac{c}{2w} = p_S \left(1 + \frac{c}{2w}\right) \quad (3.54)$$

The space-mean pressure is formally defined as

$$\bar{p} = \frac{1}{x} \int_0^x p dx_g \quad (3.55)$$

If we insert Equation 3.53 into this equation, we get

$$\bar{p} = \frac{1}{x} \int_0^x \left[ p_S + p_S \left(1 - \frac{x_g^2}{x^2}\right) \frac{c}{2w} \right] dx_g \quad (3.56)$$

Solving this integral, inserting the limits of integration, and simplifying yields

$$\bar{p} = \frac{1}{x} \left[ p_S x_g + p_S \frac{c}{2w} x_g - p_S \frac{c}{2w} \frac{x_g^3}{3x^2} \right]_0^x \quad (3.57)$$

Inserting the limits of integration gives us

$$\bar{p} = p_S + p_S \frac{c}{2w} - \frac{1}{3} p_S \frac{c}{2w} \quad (3.58)$$

Simplifying we get

$$\bar{p} = p_S \left(1 + \frac{c}{3w}\right) \quad (3.59)$$

This equation relates the space-mean pressure to the base pressure acting on the projectile. We now have equations that relate breech pressure to base pressure (Equation 3.54) and space-mean pressure to base pressure (Equation 3.59). What is missing is a relationship between breech pressure and space-mean pressure. We can arrive at the desired result by dividing Equation 3.59 by Equation 3.54, simplifying to yield

$$\frac{\bar{p}}{p_B} = \frac{p_S \left(1 + \frac{c}{3w}\right)}{p_S \left(1 + \frac{c}{2w}\right)} \quad (3.60)$$

For easier manipulation, it is sometimes desirable to expand Equation 3.60 in a Taylor series, which, neglecting higher order terms, would be

$$\frac{\bar{p}}{p_B} = 1 - \frac{c}{6w} + \dots \quad (3.61)$$

To account for the effects of bore resistance, we again write Newton's second law for a projectile being acted upon by propellant gases and bore friction as

$$w_1 \ddot{x} = A(p_S - p_R) \quad (3.62)$$

Here we have used  $w_1$  to represent the mass of the projectile (you will see why later) and have included a resistive pressure,  $p_R$ , that fights the gas pressure. Note that the resistive pressure is simply the resistive force divided by the bore cross-sectional area so that the terms in the above equation can be conveniently grouped—it is not actually a pressure at all. Writing this in terms of the acceleration we get

$$\ddot{x} = \frac{A}{w_1} (p_S - p_R) \quad (3.63)$$

Again, since the base of our projectile is at location  $x$  and the local pressure on the base is  $p_S$ , we can substitute these values into [Equation 3.46](#) for  $x_g$  and  $p$  to obtain

$$p_S = -\frac{c}{2A} \ddot{x} + \text{constant} \quad (3.64)$$

Remember that this is a local condition that we applied to the gas in the vicinity of the base where the gas's mass center is approximately at  $x$ .

Following the same procedure that we used to arrive at a general expression for pressure, but now with bore resistance, we rearrange Equation 3.64 to find the constant of integration, and with simplification arrive at

$$\text{constant} = p_S + \frac{c}{2A} \ddot{x} \quad (3.65)$$

If we use Equation 3.63, we obtain

$$\text{constant} = p_S + \frac{c}{2A} \left[ \frac{A}{w_1} (p_S - p_R) \right] = \left( 1 + \frac{c}{2w_1} \right) p_S - \frac{c}{2w_1} p_R \quad (3.66)$$

Inserting this constant back into Equation 3.64 gives us

$$p = -\frac{cx_g^2}{2Ax^2} \ddot{x} + \left( 1 + \frac{c}{2w_1} \right) p_S - \frac{c}{2w_1} p_R \quad (3.67)$$

or

$$p = -\frac{cx_g^2}{2Ax^2} \left( \frac{A}{w_1} \right) (p_S - p_R) + \left( 1 + \frac{c}{2w} \right) p_S - \frac{c}{2w_1} p_R \quad (3.68)$$

or

$$p = p_S + (p_S - p_R) \frac{c}{2w_1} \left( 1 - \frac{x_g^2}{x^2} \right) \quad (3.69)$$

which relates the pressure at the base of the projectile to the pressure at the gas mass center, but with the effect of bore friction included. Having this general equation we can again proceed as we did earlier to find equations that relate breech to base pressure, space-mean to base pressure, and space-mean to breech pressure for the bore friction case. These are

$$p_B = p_S + \frac{c}{2w_1} p_S - \frac{c}{2w_1} p_R \quad (3.70)$$

$$\bar{p} = p_S \left( 1 + \frac{c}{3w_1} \right) - p_R \frac{c}{3w_1} \quad (3.71)$$

$$\frac{\bar{p}}{p_B} = \frac{1 + \left( 1 - \frac{p_R}{p_S} \right) \frac{c}{3w_1}}{1 + \left( 1 - \frac{p_R}{p_S} \right) \frac{c}{2w_1}} \quad (3.72)$$

If we plot breech, space-mean, and base pressure versus  $x$ , the position of the projectile base, we shall see that a gradient of pressure exists in which the breech pressure is always the greatest and the base pressure is always the smallest. This is the so-called Lagrange gradient and is fundamental to our modeling of the propellant gas. There are instances where this gradient is reversed and this usually means that we have a problem—a so-called negative delta- $p$ . This is indicative of a fragmented propellant charge caused by poor ignition. A charge designed to move with the accelerating projectile, the traveling charge, is a notable exception.

We are essentially prepared now to treat the  $\mathbf{F}$  in the equation  $\mathbf{F} = ma$  which is in its simplest form, the base pressure times the base area. We now need to determine what generates the pressure, what the acceleration of the projectile will be, and how the acceleration and the ever-increasing volume behind the projectile affect the pressure. To do this, we shall review the equations from our initial discussions of propellant burning as well as revisiting our notation before moving on to combining everything into the equations of motion of the projectile.

We have previously defined the following quantities and shall simply list them here for ease of reference. The first quantity is the projectile's acceleration,  $\ddot{x}$ . The pressure acting on the base of the projectile is the stimulus that causes the acceleration

$p_S(t)$  = pressure at the base of the projectile at time  $t$ .

We usually measure pressure at the breech of the weapon and it is this pressure that we are determining when we examine the burning of the propellant. We need to constantly refer this breech pressure to the base pressure. We do this by invoking the Lagrange gradient assumption, keeping in mind that we begin by neglecting bore resistance

$$p_B = p_S \left(1 + \frac{c}{2w}\right) \quad (3.73)$$

We can write Newton's second law for the force on the projectile base as

$$w\ddot{x} = p_S A \quad (3.74)$$

If we substitute our Lagrange gradient into this equation to put it in terms of the breech pressure and the projectile velocity, we can write

$$w \frac{dV}{dt} = \frac{p_B}{\left(1 + \frac{c}{2w}\right)} A \quad (3.75)$$

If we want to include losses,  $w$  can be replaced by  $w_1$ , an effective projectile mass that can be thought of as an added mass due to the combination of resistance of bore friction, engraving by the rifling, resistance due to compression of the air ahead of the shot, etc. Then we have

$$\left(w_1 + \frac{c}{2}\right) \frac{dV}{dt} = p_B A \quad (3.76)$$

The burning of the propellant generates the pressure that pushes on the projectile. Let us now recall the equation that relates the amount of propellant turned to gas

$$\phi = (1 - f)(1 + \theta f) \quad (3.77)$$

Also recall that the rate of gas evolution (burning) is a function of the pressure

$$D \frac{df}{dt} = -\beta \bar{p} \approx \beta p_B \quad (3.78)$$

In our earlier study of solid propellant combustion, we developed an equation of state for the gas that related  $\phi$  to the pressure and the distance the projectile traveled

$$p_B(x + l) = \frac{c\lambda\phi}{A} \left[ \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right] \quad (3.79)$$

Finally, we have our equation of motion for the projectile

$$\left(w_1 + \frac{c}{2}\right) \frac{dV}{dt} = p_B A \quad (3.80)$$

whose initial conditions are  $x = 0, V = 0, f = 1$  at  $t = 0$ .

These equations may be manipulated to determine the parameters of interest as functions of the fraction of the remaining web  $f = f(t)$

$x$  = projectile travel

$V$  = projectile velocity

$p_B$  = breech pressure

If we combine Equations 3.78 and 3.79, eliminating the breech pressure between them, we can write

$$-\frac{D}{\beta} \frac{df}{dt} = \frac{w_1}{A} \left( 1 + \frac{c}{2w_1} \right) \frac{dV}{dt} \quad (3.81)$$

We can rearrange this to get the equation in terms of the projectile acceleration

$$-\frac{DA}{\beta w_1} \left[ \frac{1}{1 + \frac{c}{2w_1}} \right] \frac{df}{dt} = \frac{dV}{dt} \quad (3.82)$$

This can be integrated resulting in

$$V = -\frac{AD}{\beta w_1 \left( 1 + \frac{c}{2w_1} \right)} f + \text{constant} \quad (3.83)$$

If we insert the initial conditions that  $V=0$  when  $f=1$ , Equation 3.83 yields

$$\text{constant} = \frac{AD}{\beta w_1 \left( 1 + \frac{c}{2w_1} \right)} \quad (3.84)$$

This gives us

$$V(t) = \frac{AD}{\beta w_1 \left( 1 + \frac{c}{2w_1} \right)} (1 - f(t)) \quad (3.85)$$

From above we can rearrange Equation 3.82 as follows:

$$\frac{dV}{dt} = -\frac{AD}{\beta w_1 \left( 1 + \frac{c}{2w_1} \right)} \frac{df}{dt} \quad (3.86)$$

We can now substitute our relationship between velocity and fraction of web remaining (Equation 3.86) into our projectile equation of motion (Equation 3.80), algebraically simplifying it and inserting the relationship for base pressure (Equation 3.79) to yield

$$-\frac{D}{\beta} \frac{df}{dt} = \frac{c\lambda\phi}{A(x+l)} \left[ \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right] \quad (3.87)$$

This may be rearranged to obtain

$$\frac{df}{dt} = -\frac{c\lambda\phi\beta}{AD(x+l)} \left[ \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right] \quad (3.88)$$

Using the chain rule transformation between distance and time

$$\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} = V \frac{df}{dx} \quad (3.89)$$

This can be written as

$$\frac{df}{dx} = \frac{1}{V} \frac{df}{dt} \quad (3.90)$$

Now let us substitute Equations 3.85 and 3.88 into Equation 3.90, simplify the result and yield

$$\frac{df}{dx} = -\frac{w_1 c \lambda \phi \beta^2}{A^2 D^2 (x + l)(1 - f)} \left[ \frac{\left(1 + \frac{c}{2w_1}\right)^2}{1 + \frac{c}{3w_1}} \right] \quad (3.91)$$

To examine the rate of change of  $f$ , the fraction of web remaining, with the travel distance,  $x$ , we take the reciprocal of Equation 3.91

$$\frac{dx}{df} = -\frac{A^2 D^2 (1 - f)}{w_1 c \lambda \phi \beta^2} \left[ \frac{1 + \frac{c}{3w_1}}{\left(1 + \frac{c}{2w_1}\right)^2} \right] (x + l) \quad (3.92)$$

Here  $l$  is an initial chamber length, to be described subsequently.

By inserting the relationship between  $\phi$  and  $f$ , from Equation 3.77 we get

$$\frac{dx}{df} = -\frac{A^2 D^2}{w_1 c \lambda \beta^2} \left[ \frac{1 + \frac{c}{3w_1}}{\left(1 + \frac{c}{2w_1}\right)^2} \right] \frac{(x + l)}{(1 + \theta f)} \quad (3.93)$$

Equation 3.93 is cumbersome and following Corner [4] we find that we can define a dimensionless central ballistic parameter,  $M$ , that is a function of the gun, the charge, and the projectile, i.e., the system

$$M = \frac{A^2 D^2}{w_1 c \lambda \beta^2} \left[ \frac{1 + \frac{c}{3w_1}}{\left(1 + \frac{c}{2w_1}\right)^2} \right] \quad (3.94)$$

This simplifies our distance–web fraction relationship to

$$\frac{dx}{df} = -M \frac{(x + l)}{(1 + \theta f)} \quad (3.95)$$

The dimensionless nature of  $M$  can be shown if we note that  $c$  and  $w_1$  are mass units. We can also write the units of the burning rate coefficient as

$$[\beta] = \left[ \frac{D}{p_B} \frac{df}{dt} \right] \Rightarrow [\beta] = \left[ \frac{L^2 T}{M} \right] \quad (3.96)$$

The units of the propellant force,  $\lambda$ , are

$$[\lambda] = \left[ \frac{\text{energy}}{\text{mass}} \right] = \left[ \frac{ML}{T^2} \times \frac{L}{M} \right] = \left[ \frac{L}{T} \right]^2 = [\text{velocity}]^2 \quad (3.97)$$

Using these in our definition of the central ballistic parameter, we can show

$$[M] = \left[ \frac{L^6}{\left( \frac{L^2 T}{M} \right)^2 M \times M \left( \frac{L}{T} \right)^2} \right] = [0] \quad (3.98)$$

thereby demonstrating that  $M$  is dimensionless. Equation 3.95, repeated here,

$$\frac{dx}{df} = -M \frac{(x+l)}{(1+\theta f)} \quad (3.95)$$

shows how  $M$  relates the burning of the propellant,  $f$ , with the expansion of the volume represented by  $x$ , the travel. A similar concept appears in all interior ballistic theories.

We are now in a position to compute the parameters that the interior ballistian really seeks, the projectile's velocity and the concomitant instantaneous breech pressure for each point along its travel down the tube. If we wish to know the pressure on the base of the projectile or the space-mean pressure in the volume behind the projectile, we need only apply the appropriate Lagrange approximation to the breech pressure. This is an extraordinary result. By simply understanding the amount of propellant burnt and some gun or propellant or projectile data, we have determined everything we need to know about the interior ballistics.

We can now take the distance–web fraction relationship and integrate it directly. But we must examine two distinct cases for  $\theta$ , the form factor of the grain. One where  $\theta \neq 0$  and one where  $\theta = 0$ . Let us separate the variables in Equation 3.95 to obtain

$$\frac{dx}{(x+l)} = -M \frac{df}{(1+\theta f)} \quad (3.99)$$

Then we can write for  $\theta \neq 0$

$$\int_0^x \frac{dx}{(x+l)} = -M \int_0^f \frac{df}{(1+\theta f)} \quad (3.100)$$

or, for  $\theta = 0$

$$\int_0^x \frac{dx}{(x+l)} = -M \int_0^f df \quad (3.101)$$

Evaluation of the integral [Equation 3.100](#) for  $\theta \neq 0$  gives us

$$\ln(x + l) = -\frac{M}{\theta} \ln(1 + \theta f) + \ln(K) = \ln \left[ K(1 + \theta f)^{-\frac{M}{\theta}} \right] \quad (3.102)$$

Solving for  $K$  with the initial conditions,  $f=1$  at  $x=0$  we get

$$K = l(1 + \theta)^{\frac{M}{\theta}} \quad (3.103)$$

This constant, when inserted in the original [Equation 3.102](#), gives us

$$x + l = l \left( \frac{1 + \theta}{1 + \theta f} \right)^{\frac{M}{\theta}} \quad (3.104)$$

In a similar fashion, we can evaluate [Equation 3.101](#) to give us the distance-remaining web fraction relation for  $\theta = 0$

$$x + l = l e^{M(1-f)} \quad (3.105)$$

We now know how the web fraction,  $f$ , varies with distance, and have, incidentally, shown the algebraic simplification inherent in the central ballistic parameter,  $M$ . We can now pursue a relationship between pressure and web fraction. If we look at [Equation 3.88](#), we see the quotient on the RHS and note that this occurs frequently. We define it as our Lagrange ratio,  $R_L$ , another simplification.

$$R_L = \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \quad (3.106)$$

This will allow us to rewrite [Equation 3.79](#) in simpler form as

$$p_B(x + l) = \frac{c\lambda\phi}{A} R_L \quad (3.107)$$

We will make an assumption that the chamber and bore diameters are the same and relate the volume behind the projectile to a fictitious chamber length,  $l$ . (We will correct this subsequently when we examine the chambrage gradient.)

$$V_i = Al = U - \frac{c}{\delta} \quad (3.108)$$

In this expression,  $U$  is the empty chamber volume and  $c/\delta$  is the volume occupied by the solid propellant charge.

We continue by substituting [Equations 3.108](#) and [3.77](#) into [Equation 3.107](#) and rearranging to give our relationship between the breech pressure and the fraction of remaining web for  $\theta \neq 0$ .

$$p_B = \frac{\lambda c R_L}{V_i} (1 - f)(1 + \theta f) \left( \frac{1 + \theta f}{1 + \theta} \right)^{\frac{M}{\theta}} \quad \text{for } \theta \neq 0 \quad (3.109)$$

We can also proceed in similar fashion for  $\theta = 0$  by substituting Equations 3.105 and 3.77 into Equation 3.107 to find the relationship between the breech pressure and the fraction of remaining web.

$$p_B = \frac{\lambda c R_L}{V_i} (1-f)(1+\theta f) \exp[-M(1-f)] \quad \text{for } \theta \neq 0 \quad (3.110)$$

Summarizing, we now have the definition of the central ballistic parameter (Equation 3.94) and equations that relate velocity as a function of remaining web (Equation 3.85) and travel as a function of remaining web for different form functions (Equations 3.104 and 3.105) as well as breech pressure as a function of remaining web for different form functions (Equations 3.109 and 3.110). With these we can now integrate the governing equations and find solutions for velocity at peak pressure, at all-burnt point of travel, and at muzzle exit.

Equations 3.109 and 3.110 are somewhat cumbersome to work with, so we shall define a parameter,  $Q$ , as follows:

$$Q = \frac{\lambda c}{V_i} \frac{\left(1 + \frac{c}{2w_1}\right)}{\left(1 + \frac{c}{3w_1}\right)} = \frac{\lambda c}{V_i} R_L \quad (3.111)$$

Then we can rewrite Equation 3.109 in a more compact way

$$p_B = Q(1-f)(1+\theta f) \left(\frac{1+\theta f}{1+\theta}\right)^{\frac{M}{\theta}} \quad (3.112)$$

The maximum or peak pressure attained is then found by taking the first derivative of  $p_B$  with respect to  $f$  and setting it equal to zero

$$\frac{dp_B}{df} = Q \left[ (1-f) \left( \frac{M}{\theta} + 1 \right) \theta (1+\theta f)^{\frac{M}{\theta}} - (1+\theta f)^{\frac{M+1}{\theta}} \right] = 0 \quad (3.113)$$

Let us solve Equation 3.113 for  $f$ . By introducing the subscript "m" to denote maximum, we obtain the product of two terms

$$(1+\theta f_m)[(M+\theta)(1-f_m) - (1+\theta f_m)] = 0 \quad (3.114)$$

Solving this we have two choices here, either

$$(1+\theta f_m) = 0 \quad \text{or} \quad [(M+\theta)(1-f_m) - (1+\theta f_m)] = 0 \quad (3.115)$$

The first would only be admitted for the special case of  $\theta = -M$ , thus, our criteria for determination of  $f_m$  is

$$(M+\theta)(1-f_m) - (1+\theta f_m) = 0 \quad (3.116)$$

and

$$f_m = \frac{M + \theta - 1}{M + 2\theta} \quad (3.117)$$

Equation 3.117 works for all values of  $\theta$ . If we want to determine  $\phi_m$ , the fraction of propellant burnt at peak pressure, we call on our relationship between  $f$  and  $\phi$ , [Equation 3.77](#). Here we have denoted peak values with the subscript  $m$ .

$$\phi = (1 - f)(1 + \theta f) \quad (3.118)$$

Substitution of Equation 3.117 into the above yields

$$\phi_m = 1 - \left( \frac{M + \theta - 1}{M + 2\theta} \right) \left[ 1 + \theta \left( \frac{M + \theta - 1}{M + 2\theta} \right) \right] \quad (3.119)$$

This, when simplified, gives

$$\phi_m = \frac{(1 + \theta)[M + \theta + \theta(M + \theta)]}{[M + 2\theta]^2} = \frac{(1 + \theta)[(M + \theta)(1 + \theta)]}{[M + 2\theta]^2} \quad (3.120)$$

or the following (valid for all  $\theta$ ):

$$\phi_m = \frac{(M + \theta)(1 + \theta)^2}{[M + 2\theta]^2} \quad (3.121)$$

In designing a gun (and for other reasons), it is desirable to know where a projectile is in its travel down bore when the pressure is at a maximum. This involves substitution of Equation 3.117 into [Equation 3.104](#) and for the case where  $\theta \neq 0$  this yields

$$x_m + l = l \left[ \frac{1 + \theta}{1 + \theta \left( \frac{M + \theta + 1}{M + 2\theta} \right)} \right]^{\frac{M}{\theta}} \quad (3.122)$$

Simplifying, we finally get

$$x_m + l = l \left[ \frac{(M + 2\theta)}{(M + \theta)} \right]^{\frac{M}{\theta}} \quad \text{for } \theta \neq 0 \quad (3.123)$$

For the case where  $\theta = 0$ , we substitute Equation 3.117 into [Equation 3.105](#), which we rewrite as follows:

$$x_m + l = l \exp[M(1 - f_m)] \quad (3.124)$$

On substitution we get

$$x_m + l = l \exp \left[ M \left( 1 - \frac{M + \theta - 1}{M + 2\theta} \right) \right] \quad (3.125)$$

Simplifying this result and substituting  $\theta = 0$  into it gives us

$$x_m + l = le \quad \text{for } \theta = 0 \quad (3.126)$$

for a zero form factor.

Knowing now the position of the peak pressure in the bore, we can then ask what the breech pressure would be at this point. We can insert the value we have for the fraction of remaining web at peak pressure,  $f_m$ , back into the breech pressure equation for  $\theta \neq 0$  ([Equation 3.109](#))

$$p_{B_m} = Q(1 - f_m)(1 + \theta f_m) \left( \frac{1 + \theta f_m}{1 + \theta} \right)^{\frac{M}{\theta}} \quad (3.127)$$

With considerable algebraic simplification including substituting the values for  $Q$ , and the Lagrange ratio,  $R_L$ , for the case  $\theta \neq 0$ , we finally arrive at

$$p_{B_m} = \frac{\lambda c}{V_i} \left( \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right) \left( \frac{(1 + \theta)^2 (M + \theta)^{\frac{M}{\theta}+1}}{(M + 2\theta)^{\frac{M}{\theta}+2}} \right) \quad (3.128)$$

Following a similar procedure we now insert the value we have for the fraction of remaining web at peak pressure,  $f_m$ , into the breech pressure equation for  $\theta = 0$  ([Equation 3.110](#))

$$p_{B_m} = Q(1 - f_m) \exp[-M(1 - f_m)] \quad (3.129)$$

Then substituting for  $Q$  and  $R_L$  and simplifying, we see that we have characterized the breech pressure at the instant peak pressure is achieved down bore.

$$p_{B_m} = \frac{\lambda c}{V_i} \left( \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right) \left( \frac{1}{Me} \right) \quad (3.130)$$

Determining the breech pressure and travel when the solid grains have been completely consumed is also of considerable interest. We shall use the subscript  $c$  to represent charge burnout. If the charge is designed properly, it will burnout somewhere in the bore which allows us to extract most energy from the propellant and reduces the muzzle blast. Recall from our previous discussions that at  $t = 0$ ,  $x = 0$ ,  $f = 1$ , and  $\phi = 0$  but at all burnt (subscript  $c$ ),  $t = t_c$ ,  $x = x_c$ ,  $f = 0$ , and  $\phi = 1$ . If we substitute  $f = 0$  in Equations 3.109 and 3.110, we obtain the breech pressure at the instant of charge burnout

$$p_{B_c} = \frac{\lambda c}{V_i} \frac{\left( 1 + \frac{c}{2w_1} \right)}{\left( 1 + \frac{c}{3w_1} \right)} \left( \frac{1}{1 + \theta} \right)^{\frac{M}{\theta}} \quad \text{for } \theta \neq 0 \quad (3.131)$$

and

$$p_{B_c} = \frac{\lambda c}{V_i} \frac{\left(1 + \frac{c}{2w_1}\right)}{\left(1 + \frac{c}{3w_1}\right)} e^{-M} \quad \text{for } \theta = 0 \quad (3.132)$$

The travel of the projectile at burnout is a data point we usually want to know because if this distance turns out to be longer than the barrel length, then the charge is not completely burnt when the projectile exits. If we substitute  $f=0$  in Equations 3.104 and 3.105, we obtain the position of the projectile at the instant of charge burnout

$$x_c + l = l(1 + \theta)^{\frac{M}{\theta}} \quad \text{for } \theta \neq 0 \quad (3.133)$$

and

$$x_c + l = le^M \quad \text{for } \theta = 0 \quad (3.134)$$

It is a good idea to use these equations first to see whether the propellant burns out in the tube with the parameters we have designed into the grain. Still-burning grains leaving the tube signify a poorly designed charge. For completeness, however, if charge burnout happens outside the bore, the pressure at the breech location when the projectile leaves the muzzle may be calculated by evaluating  $f$  at the muzzle through Equation 3.104 or 3.105 and using this value to calculate  $p_B$  from Equation 3.109 or 3.110. The muzzle velocity could then be obtained from Equation 3.85.

If charge burnout is, as desired, in the bore, recall that there is still a net force (pressure) pushing on the projectile. A simple means of calculating this pressure is to assume that the process occurs so quickly that it is essentially adiabatic and that the gas behaves as an ideal gas. With these assumptions and the initial conditions that the pressure is  $p_{B_c}$  and the distance is  $x_c$ , we have a closed form solution to the problem. It is vitally important to note that the expansion of the gas after charge burnout is neither adiabatic nor isentropic, however the result is usually within about 5% with respect to pressure. The isentropic relationships for an ideal gas are

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma = \left(\frac{\frac{1}{v}}{\frac{1}{v_0}}\right)^\gamma = \left(\frac{v_0}{v}\right)^\gamma = \left(\frac{v}{v_0}\right)^{-\gamma} \quad (3.135)$$

This equation relates pressure to specific volume in a general way, but we need to involve the projectile travel as well. We can express the volume behind the projectile as a function of distance as

$$V(x) = (x + l)A \quad (3.136)$$

Then the specific volume of the gas is this value divided by the mass of the gas, which we know is still  $c$  after burnout. Thus, we can write the point at which the charge burns out to be

$$v(x) = (x + l) \frac{A}{c} \quad (3.137)$$

Furthermore, we can specialize this to

$$v(x_c) = (x_c + l) \frac{A}{c} \quad (3.138)$$

We can now tailor [Equation 3.135](#) to our needs by substituting the conditions at burnout as our reference conditions

$$\frac{\bar{p}(x)}{\bar{p}_c} = \left( \frac{v(x)}{v(x_c)} \right)^{-\gamma} = \left( \frac{x + l}{x_c + l} \right)^{-\gamma} \quad (3.139)$$

This condition occurs often so we define

$$r(x) = \frac{x + l}{x_c + l} \quad (3.140)$$

which can be written in more compact form as

$$\frac{\bar{p}(x)}{\bar{p}_c} = r(x)^{-\gamma} \quad (3.141)$$

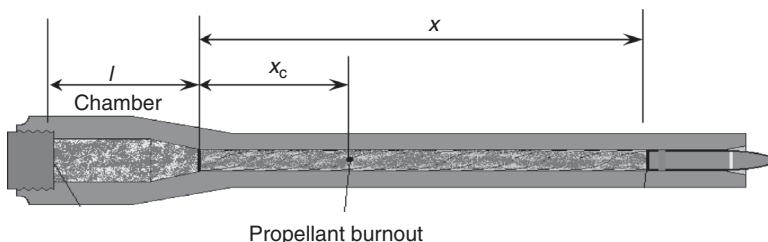
A sketch of this situation is depicted in Figure 3.2.

These extensive preparations have finally brought us to the goal of interior ballistics and the design of a gun system—imparting a desired velocity to a projectile and being able to repeat that process at will. We have developed the means for predicting how the propellant burns over time, how the breech, space-mean, and base pressures vary with time, and where the projectile moves to in relation to these pressures. Now we will focus on the velocity of the projectile during this ballistic cycle. Recall that the kinetic energy of the projectile plus the gas losses was written as

$$KE_{\text{tot}} = \frac{1}{2} w_p V^2 + \frac{1}{6} c V^2 = \frac{1}{2} \left( w_p + \frac{c}{3} \right) V^2 \quad (3.20)$$

The work done on the projectile and the gas from charge burnout to the point of interest (usually muzzle exit) is

$$W = A \int_{x_c}^x \bar{p} dx \quad (3.142)$$



**FIGURE 3.2**  
Position of projectile at charge burnout.

Combining Equations 3.20 and 3.142 and inserting 3.139 yields

$$\frac{1}{2} \left( w_1 + \frac{c}{3} \right) [V^2(x) - V^2(x_c)] = A \bar{p}_c \int_{x_c}^x \left( \frac{x+l}{x_c+l} \right)^{-\gamma} dx \quad (3.143)$$

We must keep in mind that we are using space-mean pressure here because the work is being done on both the projectile and the gas. We can use any of breech, space-mean, and base pressure (with the appropriate relationship) because we know each in terms of the others. Integrating and rearranging we get

$$\frac{1}{2} \left( w_1 + \frac{c}{3} \right) [V^2(x) - V^2(x_c)] = A \bar{p}_c \frac{1}{(1-\gamma)(x_c+l)^{-\gamma}} (x+l)^{-\gamma+1} \Big|_{x_c}^x \quad (3.144)$$

Evaluation of the limits of integration yields

$$\frac{1}{2} \left( w_1 + \frac{c}{3} \right) [V^2(x) - V^2(x_c)] = A \bar{p}_c \frac{1}{(1-\gamma)(x_c+l)^{-\gamma}} \left[ (x+l)^{1-\gamma} - (x_c+l)^{1-\gamma} \right] \quad (3.145)$$

Rearranging and inserting Equation 3.132 into the above we get a velocity relationship after burnout for  $\theta=0$

$$V^2(x) - V^2(x_c) = \frac{2A\lambda ce^{-M}(x_c+l)^{1-\gamma}}{V_i(1-\gamma)(x_c+l)^{-\gamma}(w_1+\frac{c}{3})} \left[ \frac{(x+l)^{1-\gamma}}{(x_c+l)^{1-\gamma}} - 1 \right] \quad (3.146)$$

But recall that the volume  $V_i = Al$  and if we define

$$\Phi = \frac{2}{(1-\gamma)} \left[ \left( \frac{x+l}{x_c+l} \right)^{1-\gamma} - 1 \right] \quad (3.147)$$

We can write

$$V^2(x) - V^2(x_c) = \frac{\lambda c(x_c+l)e^{-M}}{l(w_1+\frac{c}{3})} \Phi \quad (3.148)$$

Here again we revert to the cases of the form factor being zero or not zero and examine the former first. Recall Equation 3.105. For conditions after charge burnout there is no remaining web ( $f=0$ ), so we can write

$$x_c + l = le^M \quad \text{for } \theta = 0 \text{ and } f = 0 \quad (3.149)$$

Rearranging this and substituting it into Equation 3.149 yields

$$V^2(x) - V^2(x_c) = \frac{\lambda c}{\left( w_1 + \frac{c}{3} \right)} \Phi \quad (3.150)$$

This allows us to calculate the velocity of a projectile after burnout of the web for  $\theta = 0$ . The case for nonzero  $\theta$  requires further examination and manipulation. In [Equation 3.85](#) we had a general expression for velocity as a function of remaining web. After burnout this becomes

$$V(x_c) = \frac{AD}{\beta \left( w_1 + \frac{c}{2} \right)} \quad (3.151)$$

Since we are working with kinetic energy, squaring this gives

$$V^2(x_c) = \frac{A^2 D^2}{\beta^2 \left( w_1 + \frac{c}{2} \right)^2} \quad (3.152)$$

We defined the central ballistic parameter,  $M$ , in [Equation 3.94](#), and can rearrange it for our purposes into the form

$$\frac{c\lambda M}{\left( w_1 + \frac{c}{3} \right)} = \frac{A^2 D^2}{\beta^2 \left( w_1 + \frac{c}{2} \right)^2} \quad (3.153)$$

When this is compared with Equation 3.152, we conclude that

$$V^2(x_c) = \frac{\lambda c M}{\left( w_1 + \frac{c}{3} \right)} \quad (3.154)$$

This is an important result—it says that just by knowing the physical parameters of the weapon, projectile, and charge one can predict the projectile velocity at charge burnout. With this result and continuing the examination for nonzero  $\theta$ , we can then say that [Equation 3.148](#) is valid for any  $\theta$ . If we solve [Equation 3.150](#) for the velocity at any point,  $V(x)$ , insert Equation 3.154, and rearrange the terms, we get

$$V^2(x) = \frac{c\lambda}{\left( w_1 + \frac{c}{3} \right)} (M + \Phi) \quad \text{for } \theta \neq 0 \quad (3.155)$$

This result along with our earlier work allows us now to determine projectile velocity at all points in the gun for charge grains of all form factors both before and after burnout.

We have been through many derivations that have led us to the essentials of interior ballistics—breech pressure and velocity in terms of projectile travel. These results, furthermore, are in closed form, accessible to computation by hand calculator. Specialized pressures, space-mean pressure and projectile base pressure, may be computed from the breech pressure data using the Lagrange approximations. Projectile design and gun design proceed from these equations. In the following sections, we shall discuss refinements to the Lagrange formulation with an emphasis on the use of modern computer programs that take the drudgery out of hand calculation and provide the ability to iterate solutions for small changes in the parameters.

### Problem 1

You are asked to analyze the pressure of a charge zero (igniter) firing in an M31 boom for a 120-mm mortar projectile. You decide to examine it as a closed bomb first. Assume we have 59 g of M48 propellant (properties given below). The volume of the closed bomb is 5.822 in.<sup>3</sup> The propellant grains are balls (roughly spherical) with a diameter (web) of 0.049 in.

M48 propellant properties

$$\text{Density } \rho = 0.056 \left[ \frac{\text{lbf}}{\text{in.}^3} \right]$$

Ratio of specific heats  $\gamma = 1.21$

$$\text{Co-volume } b = 26.72 \left[ \frac{\text{in.}^3}{\text{lbfm}} \right]$$

Isochoric flame temperature  $T_0 = 3720^\circ\text{F}$

Burn rate exponent  $\alpha = 0.9145$

$$\text{Average burn rate coefficient } \beta = 0.0095 \left[ \frac{\text{in.}}{\text{s-psi}} \right]$$

$$\text{Burn rate } D \frac{df}{dt} = 40.341 \left[ \frac{\text{in.}}{\text{s}} \right]$$

$$\text{Force constant } \lambda = 391,000 \left[ \frac{\text{ft-lbf}}{\text{lbfm}} \right]$$

1. Come up with the equation for the web fraction,  $f$ , as a function of time.

*Answer:*  $f = (1 - 823.29t)[\%]$

2. For a sphere, the fraction of propellant burnt has the functional form  $\phi = 1 - f^3$ , write this in terms of time and  $\phi$ .

*Answer:*  $\phi = 2470t - 2,033,419t^2 + 558,031,251t^3$

3. Determine how long it will take the propellant to burn halfway through and all the way through.

*Answer:* Time to burn through halfway is 0.6 ms

4. Using the Noble–Abel equation of state, determine the pressure in the vessel when half of the propellant is burnt and when all of the propellant is burnt. Note that this cannot usually occur as the propellant is a charge zero firing that is vented into the main ullage volume behind the mortar bomb (significantly greater volume).

*Answer:* At all burnt  $p = 73,881 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$

### Problem 2

If we use the Lagrange approximation in examination of a 155-mm projectile launch, what is the average pressure in the volume behind a 102-lbm projectile if the breech pressure is 55,000 psi? The propelling charge weighs 28 lbm.

*Answer:*  $\bar{p} = 52,787 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$

### Problem 3

A 120-mm projectile is to be examined while in the bore of a tank cannon at a time 4 ms from shot start. Over this time period, the projectile has acquired an average velocity of 1000 ft/s. The propellant grain (M15) is single perf ( $\theta = 0$ ) with a 0.034-in. initial web. The co-volume of the propellant is 31.17 in.<sup>3</sup>/lbm. The density of the propellant is 0.06 lbm/in.<sup>3</sup>. If the projectile weighs 50.4 lbm, the propellant weighs 12.25 lbm and the chamber volume

is 330 in.<sup>3</sup> At this time, 0.02 in. of the web remains. The propellant force is 337,000 ft-lbf/lbm. Determine the breech pressure in the weapon. Be careful with the units!

$$\text{Answer: } p_B = 21,784 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

#### Problem 4

The Paris gun was a monstrous 210 mm weapon designed by Germany during the First World War to bombard Paris from some 70 miles away. It was unique in that it fired the first exo-atmospheric projectile ever designed. The weapon had a chamber volume of 15,866 in.<sup>3</sup> Very little of the projectile protrudes into the chamber after it seats (so ignore the volume the base occupies). The length of travel for the projectile from shot start to shot exit is 1182 in. The projectile weighs 234 lb. The propelling charge weighs 430.2 lb. The propellant used was specially designed and was similar to U.S. M26 propellant. It consisted of 64%–68% NC, 25%–29% NG with 7% Centralite (symmetrical diethyl diphenylurea C<sub>17</sub>H<sub>20</sub>N<sub>2</sub>O), and some other additives. The propellant was single perforated with a web thickness described below. Assume the propellant has the following properties. (Note that these are the authors guesses—a better estimate of the properties can be found in Ref. [5].)

Adiabatic flame temperature  $T_0 = 2881 \text{ K}$

Specific heat ratio  $\gamma = 1.237$

Co-volume  $b = 1.06 \text{ cm}^3/\text{g}$

Density of solid propellant  $\rho = 1.62 \text{ g/cm}^3$

Propellant burn rate coefficient  $\beta = 0.0707 \text{ (cm/s)/(MPa)}$

Web thickness  $D = 0.217 \text{ in.}$

Propellant force  $\lambda = 1019 \text{ J/g}$

- Using the above data determine (a) the projectile base pressure in psi, (b) velocity in ft/s, and (c) distance down the bore of the weapon in inches for peak pressure.

$$\text{Answers: (a) } p_{s_{\max}} = 31,548 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$(b) V_{p_{\max}} = 2880 \left[ \frac{\text{ft}}{\text{s}} \right]$$

$$(c) x_{p_{\max}} = 270.1[\text{in.}]$$

- Determine the pressure in psi at a point 3 in. behind the projectile base when the charge burns out.

$$\text{Answer: } p_{x=3} = 28,527 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

- Assuming the gas behaves according to the Noble–Abel equation of state, determine the muzzle velocity of the projectile in ft/s.

$$\text{Answer: } V = 5791 \left[ \frac{\text{ft}}{\text{s}} \right]$$

#### Problem 5

A British 14-in. Mark VII gun has a chamber volume of 22,000 in.<sup>3</sup> A 5 in. of the projectile protrude into the chamber after it seats. The length of travel for the projectile from shot start to shot exit is 515.68 in. The weapon has a uniform twist of 1 in 30. The projectile weighs 1590 lb. The propelling charge weighs 338.25 lb. The propellant used is called “SC” and consists of 49.5% NC (12.2% nitrated), 41.5% NG with 9% Centralite. Assume SC propellant has the following properties:

Adiabatic flame temperature  $T_0 = 3090$  K

Specific heat ratio  $\gamma = 1.248$

Co-volume  $b = 26.5$  in.<sup>3</sup>/lbm

Density of solid propellant  $\rho = 0.0567$  lbm/in.<sup>3</sup>

Propellant burn rate  $\beta = 0.000331$  (in./s)/(psi)

Web thickness  $D = 0.25$  in.

Specific molecular weight  $n = 0.04262$  lb-mol/lbm

1. Determine the force constant,  $\lambda$  in ft-lbf/lbm.
2. Determine the central ballistic parameter for this gun-projectile combination.
3. Using the above data, determine the projectile base pressure, velocity, and distance down the bore of the weapon for both peak pressure and charge burnout assuming the grain is a cylindrical propellant ( $\theta = 1$ ).

*Answers:*

$$\lambda = 366,246 \left[ \frac{\text{ft-lbf}}{\text{lbm}} \right]$$

$$M = 1.933$$

$$p_{B_{\max}} = 41,200 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$p_{B_c} = 25,080 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$p_{S_{\max}} = 37,240 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$p_{S_c} = 22,670 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$V_{p_{\max}} = 1082 \left[ \frac{\text{ft}}{\text{s}} \right]$$

$$V_c = 2128 \left[ \frac{\text{ft}}{\text{s}} \right]$$

$$x_{p_{\max}} = 79.5[\text{in.}]$$

$$x_c = 293.7[\text{in.}]$$

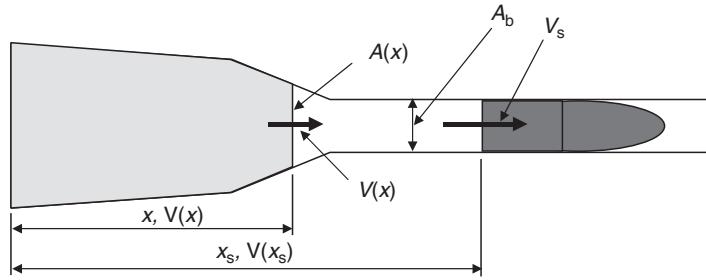
### Problem 6

Verify Equation 3.148 is valid for any  $\theta$ .

---

### 3.3 Chamber Gradient

In our derivation of the Lagrange gradient approximations we assumed that the chamber of the gun was simply an extension of the bore. The volume of the chamber was converted to a cylinder of bore diameter and the tube was lengthened appropriately behind the projectile. In doing this, we neglected the effects of short, larger diameter chambers



**FIGURE 3.3**

Chamber with large chambrage.

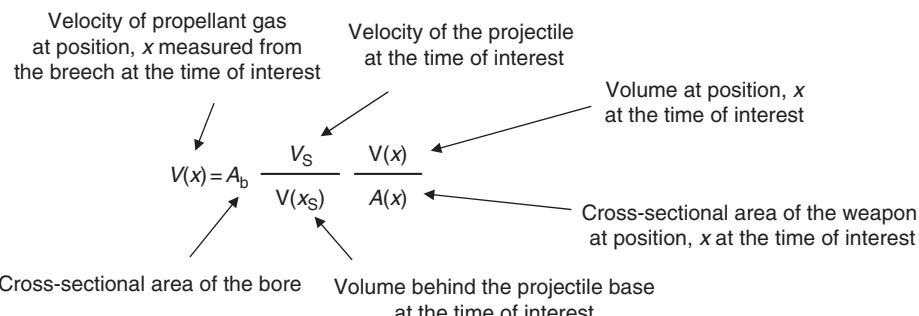
(the definition of chambrage is the ratio of the diameter of the chamber to the bore inner diameter) and all calculations that are functions of distance from the breech,  $x - x_s$ , are inaccurate in the distance term. If we account for these differences by deriving a chambrage gradient, we find that the two methods yield similar but close answers. Nevertheless, one should understand how the answers relate to each other and to the real problem. Fredrick W. Robbins of the Army Research Laboratory, who has allowed us to base this section on his excellent work, derived the chambrage gradient formulation that follows.

The formulation of the chambrage gradient follows much the same pattern that was used in the development of the Lagrange gradient. It leads, however, to an algorithm that is best applied with the aid of a computer. Small increments of time (hence distance) are chosen and computations of pressure (breech, mean, and base), velocity, acceleration, and distance traveled are made for the end point of the interval. The calculation is then repeated for the next increment of time. This is done until the projectile exits the bore. A representation of the situation is shown in Figure 3.3 for a chosen time step.

The definitions of the terms used in Figure 3.3 are shown in Figure 3.4.

In Robbins' derivation, certain integrals called  $J$  integral factors are developed and must be computed. They are

$$J_1(x_0) = \int_0^{x_0} \frac{V(x)}{A(x)} dx \quad (3.156)$$



**FIGURE 3.4**

Definitions of terms used in chambrage gradient development.

$$J_1(x_s) = J_1(x_0) + \frac{1}{A_b} \left[ V(x_0)(x_s - x_0) + \frac{A_b}{2}(x_s - x_0)^2 \right] \quad (3.157)$$

$$J_2(x_s) = \frac{[V(x_0) + A_b(x_s - x_0)]^2}{A_b^2} \quad (3.158)$$

$$J_3(x_s) = J_3(x_0) + A_b J_1(x_0)(x_s - x_0) + \frac{V(x_0)}{2}(x_s - x_0)^2 + \frac{A_b}{6}(x_s - x_0)^3 \quad (3.159)$$

$$J_4(x_s) = J_4(x_0) + \frac{[V(x_0) + A_b(x_s - x_0)]^3 - [V(x_0)]^3}{3A_b^2} \quad (3.160)$$

The acceleration at any point,  $a_s$ , appears in one of our algorithm factors explicitly

$$a(t) = a_1(t) + a_2(t)p_s \quad (3.161)$$

where

$$a_1(t) = \frac{cA_b}{[V(x_s)]^2} \left[ \frac{A_b V_s^2}{V(x_s)} + \frac{cA_b p_{\text{resist}}}{m_p} \right] \quad (3.162)$$

$$a_2(t) = -\frac{cA_b^2}{m_p[V(x_s)]^2} \quad (3.163)$$

Another factor required in the algorithm is  $b(t)$  derived as

$$b(t) = -\frac{cA_b^2 V_s^2}{2[V(x_s)]^3} \quad (3.164)$$

The way the algorithm is used is (roughly) as follows:

At each time step

- The breech pressure is calculated from the burning rate equations.
- $J_1$  through  $J_4$  are calculated.
- $a(t)$  and  $b(t)$  are calculated.
- The projectile acceleration, velocity, and distance down the bore are calculated.
- The volume behind the projectile is updated.
- The process moves to the next time step.

This gradient, while only slightly more accurate than the Lagrange gradient in the computed distance from the breech, is used in some modern interior ballistic computer codes.

### 3.4 Numerical Methods in Interior Ballistics

In this section, we shall briefly discuss methods for solving the interior ballistics problem through use of computational tools. In recent decades, computational capabilities have increased at an astronomical rate. One of the most famous early uses of the computer to solve the exterior ballistics problem (firing tables) was the use of the ENIAC (Electronic Numerical

Integrator and Computer) machine during and immediately after the Second World War. In this case, the computer was used to solve tedious exterior ballistics problems in rapid order.

In the field of interior ballistics, the computer revolution has given the individual ballistician the tools (although some commercial packages can be expensive) to solve extremely complicated interior ballistics problems and optimize a system quickly. The complexity of these tools is driven by the physics that are incorporated in the particular code. We shall discuss some general categories of software, their uses, and their limitations.

Many interior ballistics codes are of the zero-dimensional variety. In these types of codes, the density of the propellant gas (as stipulated by the Lagrange approximation) is considered constant in the volume between the breech and the projectile. The Lagrange pressure gradient is assumed to be in effect and results in a nice, always well behaved launch. These codes are extremely useful for predictive applications because they run fast. One of the features of these codes that make them so useful is that we can easily include and track burn characteristics of multiple propellant types (both geometry and chemical composition). This allows us to tailor the burn characteristics so that a particular pressure-distance distribution is achieved while maintaining a particular muzzle velocity.

Another excellent feature of this type of code is that heat transfer to the weapon can be accounted for in the energy balance. This provides a more realistic muzzle velocity than if it is neglected and can be of great value to the gun designer. Friction and blow-by effects can be fudged in and burn rate parameters varied to replicate actual tests. Additionally, the effects of the regression of all surfaces (recall that we neglected end effects in our hand calculation methods) can be simulated and accounted for. Zero-dimensional codes can also include the effects of inhibitors on the propellant grains as well as highly nonlinear pressure–burn rate relationships. Since zero-dimensional codes track the pressure, it is simple enough to use them to develop recoil models as well. All in all, zero-dimensional codes are probably the most effective tools at the disposal of the interior ballistician for basic ballistics design work. Once a set of experiments have been conducted to validate these codes, their accuracy is excellent.

A quasi-one-dimensional code is one in which the density of the propellant gas behind the projectile is a known function of some other variable. An example of this would be a zero-dimensional code that incorporated the chambrage gradient. Essentially, beyond the ability to track the effect of variable chamber or bore area on the density, the limitations and benefits of this type of code are the same as discussed in the zero-dimensional section.

A one-dimensional interior ballistics code allows density to vary based on the physical equations and conservation laws in the axial direction only. Thus, at a given cross-section, the density is considered constant throughout the radial direction. These codes are very good at predicting pressure waves and therefore can estimate the pressure differential along the volume behind the projectile. The benefit of this is that, since propellant generally burns faster under higher pressure, the local burn rate and therefore the amount of gas evolved can be tracked. This allows the user to see pressure waves develop and propagate. The disadvantage is that, since the code can only track pressure waves in the axial direction, unless the charge fully fills the volume behind the projectile, it is difficult to completely match the physics of the firing. This occurs because the presence of solids and gases in the chamber is generally not uniform—the solids are usually at the bottom of the chamber. This affects the gas dynamics. Solids will also be entrained by the gas flow down the bore and some modeling of their motion has to be accomplished (or ignored). In most cases, the propellant bed is assumed to be a monolithic mass that regresses and stretches as the propellant is burned. These codes are usually very good but the user should completely understand the assumptions on how the propellant is allowed to move before using them.

A two-dimensional model is one where the density can vary in the radial direction as well. These models are better at predicting pressure waves but take somewhat longer

time to run than one-dimensional models. Pressure can be tracked in the radial direction and the propellant motion included. The same issues with propellant motion are present as they were in the one-dimensional models though it is possible to track propellant motion.

A three-dimensional model has it all. Because of this they usually take an excruciatingly long time to set up and run. This time constraint makes them generally reserved for failure investigations rather than predictive simulations. Individual propellant grains can regress and be tracked and one can imagine the difficulty with this in the sense of model validation. With suitable stress and failure models, grain fracture can also be examined. If erosion models are incorporated, the effect of gas wash on propellant burn rate can even be included. One has to ask oneself if all of this is really necessary. In some cases, these models are crucial, in other cases, they are certainly overkill. The usefulness of this type of model is still somewhat limited by computer speed, but as computers become faster the limitation will change to a lack of accurate physical models for motion, surface regression, propellant and gun tube erosion, grain fracture, etc. These issues are certainly solvable, but finding a proponent who will fund the research is difficult.

Now that we have described the general types of models, it is important to explain their use further. In general, all of them are used in a similar manner. We shall use the zero-dimensional model as an example and leave the rest to the readers imagination (and budget restrictions). Typically, a propellant formulation and geometry is chosen as a point of departure given that we have a preliminary gun design and a projectile to work with. This propellant is then further developed in terms of geometry or chemical composition. Some zero-dimensional codes are provided with optimization subroutines so that particular characteristics of the ballistic cycle can be achieved. The pressure–time, acceleration–time, and pressure–distance curves are examined and, if suitable, some experimental charges are made up. The configuration is then fired and the results checked against the code. These results then can be used to adjust burn rates and resistive characteristics, and the model can be used to predict all future firings and design iterations.

A particular example of the power of these codes is their usefulness in assessing the interior ballistics of systems that vary widely in matters of scale, for example, in mass of projectile, diameter of bore, and muzzle velocity. In the 1960s, ballisticians J. Frankle and M. Baer at the Ballistics Research Laboratories at Aberdeen, Maryland [6] and others elsewhere devised codes largely based on Corner's zero-dimensional analysis that we described in detail in [Section 3.2](#). Among these the Frankle–Baer simulation, still in use today, which examined and expanded on the basic energy equation,

$$\begin{aligned} \text{Energy released by burning propellant} \\ = \text{internal energy of gases} + \text{work done on the projectile} + \text{secondary losses} \end{aligned}$$

or

$$Q = U + W + \text{losses} \quad (3.165)$$

developed equations of state of the propellant gases based on more recent thermodynamic theories and refined the losses term from new experimental data. This led to more refined ratios for breech, mean, and shot base pressures, and more accurate equations of motion for the projectile.

To examine the effects of scale we computed the relevant pressure ratios for three widely different gun–projectile combinations. We show these combinations and the resultant ratio values in [Tables 3.1](#) through [3.5](#). What is noteworthy is the applicability of the theory over the range of size, projectile mass, and propellant type and volume. Notice also the closeness of the pressure ratios for each projectile between the Corner and the Frankle–Baer simulations.

**TABLE 3.1**

Inputs for Numerical Comparison of Corner and Frankle–Baer

Parameter	Expression or Value (J. Corner)	$\begin{bmatrix} \text{M735} \\ \text{M1} \\ \text{M193} \end{bmatrix}$	Expression or Value (Frankle–Baer)	$\begin{bmatrix} \text{M735} \\ \text{M1} \\ \text{M193} \end{bmatrix}$
Charge weight	$c$	$\begin{bmatrix} 13.125 \\ 9.000 \\ 4.020 \times 10^{-3} \end{bmatrix}$	$c$	$\begin{bmatrix} 13.125 \\ 9.000 \\ 4.020 \times 10^{-3} \end{bmatrix}$
Projectile weight	$w$	$\begin{bmatrix} 12.78 \\ 31.97 \\ 7.86 \times 10^{-3} \end{bmatrix}$	$w_p$	$\begin{bmatrix} 12.78 \\ 31.97 \\ 7.86 \times 10^{-3} \end{bmatrix}$
Propellant type	–	$\begin{bmatrix} \text{M30} \\ \text{M1} \\ \text{Ball} \end{bmatrix}$	–	$\begin{bmatrix} \text{M30} \\ \text{M1} \\ \text{Ball} \end{bmatrix}$

**TABLE 3.2**

Burn Characteristic Inputs for Numerical Comparison of Corner and Frankle–Baer

Parameter	Expression or Value (J. Corner)	$\begin{bmatrix} \text{M735} \\ \text{M1} \\ \text{M193} \end{bmatrix}$	Expression or Value (Frankle–Baer)	$\begin{bmatrix} \text{M735} \\ \text{M1} \\ \text{M193} \end{bmatrix}$
Propellant impetus (force)	$\lambda$	$\begin{bmatrix} 3.64 \times 10^5 \\ 3.05 \times 10^5 \\ 3.32 \times 10^5 \end{bmatrix}$	$\lambda$	$\begin{bmatrix} 3.64 \times 10^5 \\ 3.05 \times 10^5 \\ 3.32 \times 10^5 \end{bmatrix}$
Specific heat ratio	$\gamma$	$\begin{bmatrix} 1.2385 \\ 1.2592 \\ 1.26 \end{bmatrix}$	$\gamma$	$\begin{bmatrix} 1.2385 \\ 1.2592 \\ 1.26 \end{bmatrix}$
Polytropic index	$\frac{1}{\gamma - 1}$		$n$	

**TABLE 3.3**

Pressure Gradient Calculations for Numerical Comparison of Corner and Frankle–Baer

Parameter	Expression or Value (J. Corner)	Expression or Value (Frankle–Baer)
$\frac{\bar{p}}{p_s}$	$1 + \frac{c}{3w}$	$1 + \frac{1}{\delta} \frac{c}{w_p}$
$\frac{1}{\delta}$	n/a	$\frac{1}{2n+3} \left[ 1 + \alpha n \left( \frac{1 + c_1 \beta n}{1 + c_1 n} \right) \right]$
$\frac{1}{a_b}$	n/a	$\left[ \frac{2n+3}{\delta} + \frac{2(n+1)}{c/w_p} \right]$
$\frac{\bar{p}}{p_B}$	$\left[ 1 - \frac{1}{6} \frac{c}{w} \right]$	$\left[ 1 - \frac{1}{\delta} \left( \frac{c}{w_p} \right) \right] (1 - a_b)^{n+1}$
$\frac{p_B}{p_s}$	$\left[ 1 + \frac{1}{2} \frac{c}{w} \right]$	$\left[ (1 - a_b)^{-(n+1)} \right]$

**TABLE 3.4**

Specific Frankle–Baer Computations for the M735, M1, and M193 Projectiles

Projectile	$\alpha$	$\beta$	$c_1$	$\frac{1}{\delta}$	$\epsilon = \frac{c}{w_p}$	$\frac{1}{a_b}$	$(1 - a_b)^{n+1}$
M735 (105-mm KE)	0.56	1.07	1.05	0.333	1.027	11.002	0.683
M1 (105-mm HE)	0.63	1.01	1.02	0.322	0.282	37.811	0.876
M193 (5.56-mm ball)	0.60	1.03	1.04	0.315	0.511	22.325	0.800

**TABLE 3.5**

Specific Gradient Comparison of Corner and Frankle–Baer for the M735, M1, and M193 Projectiles

Projectile	$\frac{\bar{p}}{p_s}$		$\frac{\bar{p}}{p_B}$		$1 + \frac{1}{2} \frac{c}{w}$	
	Corner	Frankle–Baer	Corner	Frankle–Baer	Corner	Frankle–Baer
M735 (105-mm KE)	1.342	1.342	0.829	0.917	1.514 (1.619)	1.464 (1.463)
M1 (105-mm HE)	1.094	1.091	0.953	0.956	1.141 (1.148)	1.142 (1.141)
M193 (5.56-mm ball)	1.170	1.161	0.915	0.929	1.256 (1.259)	1.250 (1.250)

### 3.5 Sensitivities and Efficiencies

Having explored the detailed development of theories of interior ballistic events, we will now probe the outcome of varying some of the parameters that are under the control of the charge designer. To do this, we will be referring back to definitions and equations developed under [Section 3.2](#).

A useful quantity for our analysis is the dimensionless central ballistic parameter,  $M$

$$M = \frac{A^2 D^2}{w_1 c \lambda \beta^2} \left[ \frac{1 + \frac{c}{3w_1}}{\left(1 + \frac{c}{2w_1}\right)^2} \right] \quad (3.166)$$

Of particular importance in this are the variables  $D$  and  $\beta$ , the original web dimension and the burning rate coefficient, respectively. If we examine Equation 3.167

$$p_{B_m} = \frac{\lambda c}{V_i} \left( \frac{1 + \frac{c}{2w_1}}{1 + \frac{c}{3w_1}} \right) \left( \frac{1}{Me} \right) \quad (3.167)$$

We can see that, at least for the case of  $\theta = 0$ ,  $M$  is in the denominator and as the ratio  $D/\beta$  decreases,  $M$  decreases and from Equation 3.167, the peak pressure,  $p_B$ , increases. That is, if the original web size is decreased, the peak pressure will increase. This is a parameter much under the control of the designer.

Referring again to Equation 3.166, we see that if the charge mass (weight),  $c$ , is increased, then  $M$  decreases (the  $c$ 's in the gradient term largely cancel out and  $c$  in the first term

denominator governs). In [Equation 3.167](#),  $c$  appears in the numerator and  $M$  in the denominator causing  $p_{B_m}$ , the peak pressure, to again rise.

Let us now examine the shift in location of  $x_m$ , the point in travel where the peak pressure exists.

$$x_m + l = l \left[ \frac{(M + 2\theta)}{(M + \theta)} \right]^{\frac{M}{\theta}} \quad (3.168)$$

Equation 3.168 relates  $x_m$  to  $M$ . If the ratio  $D/\beta$  or the charge mass,  $c$ , decreases, then  $M$  decreases and consequently  $x_m$  is reduced (it moves toward the breech). This kind of shift is important in gun design since wall thickness and center of mass are important considerations for weapon mounting.

The sensitivity of muzzle velocity,  $V$ , to changes in web size or charge weight can be seen in [Equation 3.169](#).

$$V^2(x) - V^2(x_c) = \frac{\lambda c(x_c + l)e^{-M}}{l\left(w_1 + \frac{c}{3}\right)\Phi} \quad (3.169)$$

Here  $M$  is the governing term and is decreased as we showed earlier, if charge mass is increased or web size reduced. Because  $M$  has a negative exponent in the equation, its reduction drives an increase in  $V$ .

Finally, the influence of travel on muzzle velocity can be shown to be quite weak. The computation is complex and will not be shown here. But, for example, by doubling the travel, velocity increases only by a factor of about a tenth, hardly worth the effort in the real world.

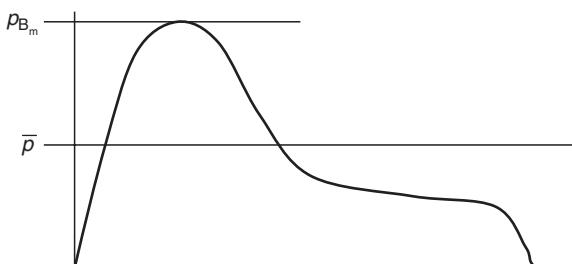
There are two measures of efficiency that are of interest to the interior ballistician: piezometric or pressure efficiency and ballistic or energy efficiency.

Piezometric efficiency,  $\varepsilon_p$ , is the ratio of the average pressure during the entire ballistic cycle to the peak pressure during the cycle.

$$\varepsilon_p = \frac{\bar{p}}{p_{B_m}} \quad (3.170)$$

An illustration of the space-mean pressure and maximum breech pressure is provided as [Figure 3.5](#).

Increasing  $\varepsilon_p$  implies that the muzzle pressure will be high (usually an undesirable trait), and that the charge burnout point will move toward the muzzle (hopefully never outside the muzzle). High piezometric efficiency usually means poor regularity, i.e., round-to-round muzzle velocity repeatability is poor (an undesirable trait). For powerful,



**FIGURE 3.5**

Average and maximum breech pressure for a typical gun firing.

high-velocity cannons, this efficiency is usually in the 50%–60% range. Other cannons are lower. High piezometric efficiency also implies that the expansion ratio, the ratio of total gun volume to chamber volume, will be low: powerful guns have large chambers and consume lots of propellant.

Ballistic efficiency,  $\varepsilon_b$ , is defined as the ratio of the kinetic energy of the projectile as it exits the muzzle to the total potential energy of the propellant charge.

$$\varepsilon_b = \frac{\text{muzzle KE}}{\text{propellant PE}} = \frac{\frac{1}{2}wV^2}{\frac{\lambda c}{\gamma - 1}} = \frac{(\gamma - 1)wV^2}{2\lambda c} \quad (3.171)$$

because the potential energy is defined as

$$\text{Propellant PE} = \frac{RT_0}{(\gamma - 1)} \quad \text{and } \lambda = RT_0 \quad (3.172)$$

Increasing  $\varepsilon_b$  tends to shift the all-burnt position toward the breech and increases the expansion ratio. Reducing the central ballistic parameter,  $M$ , by going to a smaller web will also increase  $\varepsilon_b$ . The ballistic efficiency of most guns is approximately 0.33.

## References

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# 4

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## Ammunition Design Practice

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Chapter 3 provided us with the information necessary to determine the forces acting on the projectile and gun. This chapter endeavors to describe techniques necessary for the projectile or weapon designer to be successful. Sections 4.1 and 4.2 describe topics in the field of mechanics of materials. This material will form the basis by which we will evaluate designs. Sections 4.3 through 4.8 apply these concepts to the design of projectiles and guns. This chapter ends with practices and techniques used to design modern ammunition that must be fired from a gun.

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### 4.1 Stress and Strain

Before proceeding with our examination of design practices, a discussion of the fundamentals of the general state of stress in materials is in order. Consider an arbitrary cube of material under load as depicted in Figure 4.1. The state of stress can be completely defined by six stress components  $\sigma_x$ ,  $\sigma_y$ ,  $\sigma_z$ ,  $\tau_{xy}$ ,  $\tau_{yz}$  and  $\tau_{zx}$ . Here we have used a Cartesian coordinate system where the normal stresses are denoted by  $\sigma$  and the shear stresses are denoted by  $\tau$ .

The first subscript represents the plane in which the stress acts (defined by its normal vector) while the second subscript indicates the direction of action. These components form the stress tensor which is actually a  $3 \times 3$  matrix of nine elements; except that we have assumed that  $\tau_{xy} = \tau_{yx}$ ,  $\tau_{zy} = \tau_{yz}$ , and  $\tau_{zx} = \tau_{xz}$ . When written as a tensor, the state of stress in a material is defined as

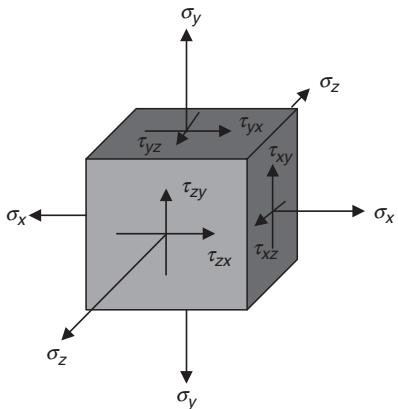
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \quad (4.1)$$

It can be shown that the coordinate system in which we measure the stresses can be rotated so that the shear stresses vanish. The three remaining stresses are normal stresses, known as the principal stresses and are denoted as  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ .

These stresses are important because, regardless of what coordinate system we view the component in, the stress state is uniquely determined. Also, in some materials, these stresses are associated with failure and fracture.

These points are sometimes shown graphically through use of Mohr's circle. The determination of the principle stresses will be discussed later in this section.

It is also very important to understand this when we try to examine the stress levels in a part experimentally with a strain gage. Stress is a point function defined by force per unit area expressed as



**FIGURE 4.1**  
Cartesian stress components.

$$\sigma = \frac{F}{A} \quad (4.2)$$

Here  $\sigma$  is the stress,  $F$  is a force, and  $A$  is the cross-sectional area of the component. The same equation also holds if we use the symbol  $\tau$  signifying a shear stress.

When we examine a structure, we normally are given the loads that are imposed on it. We then either choose a material or evaluate a given material to see how it will behave under the applied loads. This process requires us to convert the external loads to stress. These stresses will cause movement of the material in the form of either stretching (tension) or compression. This movement is the actual displacement of the material. There is an intermediate analytical step between these two where we need to define the strain of the material. The strain in the material is defined as the change in length of a part over its initial, unstressed length. Mathematically, this is expressed as

$$\varepsilon = \frac{\Delta l}{l} \quad (4.3)$$

We require a relationship between stress and strain to evaluate material behavior under a load. The link between stress and strain is called a stress-strain relationship. The most common and simplest stress-strain relationship is that for a linear-elastic material. This is known as Hooke's law and is given for small deformations and uniaxial loading by

$$\varepsilon = \frac{\sigma}{E} \quad (4.4)$$

Here  $E$  is the modulus of elasticity, sometimes known as Young's modulus. In a linear-elastic material, any loading and unloading of the structure occurs along a curve in stress-strain space that has a slope equal to the modulus of elasticity. Under the assumption of general loading, material will be "pulled in" in the transverse directions as it is stretched longitudinally. The ratio of lateral strain to axial strain is denoted as  $\nu$  and called Poisson's ratio and is given for an isotropic material as

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x} \quad (4.5)$$

This assumption of general loading changes our Hooke's law relation as follows:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \quad (4.6)$$

$$\varepsilon_y = -\frac{\nu \sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu \sigma_z}{E} \quad (4.7)$$

$$\varepsilon_z = -\frac{\nu \sigma_x}{E} - \frac{\nu \sigma_y}{E} + \frac{\sigma_z}{E} \quad (4.8)$$

While we have defined  $\varepsilon$  to represent longitudinal strain in a material, a different type of strain can be examined—shear strain. Shear strain,  $\gamma$ , is defined as the angular deviation of a material from its original, undeformed shape. Shear strain is given by its own version of Hooke's law as

$$\gamma = \frac{\tau}{G} \quad (4.9)$$

In this equation,  $G$  is known as the shear modulus of the material.

In an isotropic material  $E$ ,  $\nu$ , and  $G$  are not independent. The relationship that links them is

$$G = \frac{E}{2(1 + \nu)} \quad (4.10)$$

When we perform hand calculations, it is customary to convert the loads to stresses, then the stresses to strains, and finally strains to deformations. The process is somewhat different (i.e., reversed) in a finite element analysis.

The determination of the principle stresses is important in several failure criteria. When a part is being examined experimentally during a gun launch it is customary to utilize a strain gage. A strain gage measures the change in a parts length using the fact that resistance increases in a conductor as it is stretched. Strain gages are not always placed along the directions in which it is desired to compute stress however. Since strain gages only measure in-plane stress, it is common to transform this two-dimensional measurement into a desired in-plane direction. To transform stress from the strain gage coordinate system to the desired coordinate system, we use the following equations:

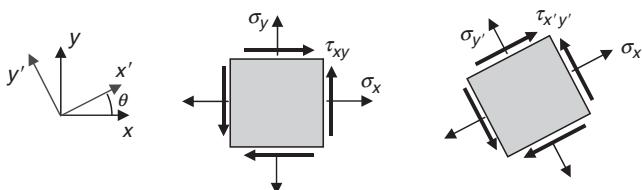
$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta \quad (4.11)$$

$$\sigma_{y'} = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta \quad (4.12)$$

$$\tau_{x'y'} = \tau_{xy}(\cos^2 \theta - \sin^2 \theta) + (\sigma_y - \sigma_x) \sin \theta \cos \theta \quad (4.13)$$

In each of these equations, the primed variables are those in the desired direction and the unprimed variables are those measured by the strain gages. This is depicted in Figure 4.2.

Rotation of coordinate systems in three dimensions is covered in excellent detail in Ref. [1]. It was stated earlier that a rotation can be made such that the shear stresses vanish and this results in what are known as principle stresses [2]. To determine the values of the principle stresses, we determine the stress invariants through solution of the eigenvalue problem. The three stress invariants are given by



**FIGURE 4.2**  
Transformation of stress components.

$$I_1 = \sigma_x + \sigma_y + \sigma_z \quad (4.14)$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \quad (4.15)$$

$$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx} \quad (4.16)$$

Once these invariants are obtained, the principle stresses are obtained through

$$\sigma_1 = \frac{I_1}{3} + 2\sqrt{I_1^2 - 3I_2} \cos \phi \quad (4.17)$$

$$\sigma_2 = \frac{I_1}{3} + 2\sqrt{I_1^2 - 3I_2} \cos \left( \phi + \frac{2\pi}{3} \right) \quad (4.18)$$

$$\sigma_3 = \frac{I_1}{3} + 2\sqrt{I_1^2 - 3I_2} \cos \left( \phi + \frac{4\pi}{3} \right) \quad (4.19)$$

In Equations 4.17 through 4.19, the quantity  $\phi$  is calculated through

$$\phi = \frac{1}{3} \cos^{-1} \left[ \frac{2I_1^3 - 9I_1I_2 + 27I_3}{2(I_1^2 - 3I_2)^{\frac{3}{2}}} \right] \quad (4.20)$$

Now we have all of the basic information necessary to discuss failure criteria. Limits of space prevent a more in-depth treatment of this topic. The reader is referred to the references at the conclusion of this chapter for a more detailed treatment.

### Problem 1

For the state of stress below, find the principal stresses and the maximum shear stress.

$$[\sigma] = \begin{bmatrix} 20 & 15 & 0 \\ 15 & 4 & 0 \\ 0 & 0 & -9 \end{bmatrix} [\text{MPa}]$$

*Answer:*  $\tau_{\max} = 57[\text{MPa}]$

## 4.2 Failure Criteria

When embarking on the design of a particular projectile component, we must initially determine certain characteristics of the material contemplated for the design: Will we use a metal or a plastic? Does it have a distinct yield point? Is it brittle or very ductile? Such determinations will govern which criteria we use when we calculate the stresses that will cause failure of the component. There are three commonly used criteria for yield or failure: von Mises, which is also known as the maximum distortion energy criterion; Tresca, which is known as the maximum shear stress criterion; and Coulomb, which uses a maximum normal stress criterion. Other materials may require unique failure criteria, e.g., composites or non-isotropic metals may require Tsai-Wu or Tsai-Hill criteria.

The von Mises or maximum distortion energy criterion is used typically, when the component is to be made of metal. It assumes that the energy required to change the shape of the material is what causes yielding and that a hydrostatic state of stress will not result in failure. The materials for which it is used should have a distinct yield point. Our convention shall follow that of structural engineers in which we shall assume tensile stress to be positive. By this criterion, we assume that the distortion of the material will precipitate the failure. We shall order the stresses with 1 as largest to 3 being smallest and state the following:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \text{constant} \quad (4.21)$$

We set this constant equal to  $2\sigma_Y^2$  or  $6K^2$ . Here  $\sigma_Y$  is the yield stress in simple tension and  $K$  is the yield stress in pure shear. This implies that

$$\frac{1}{3}\sigma_Y^2 = K^2 \quad (4.22)$$

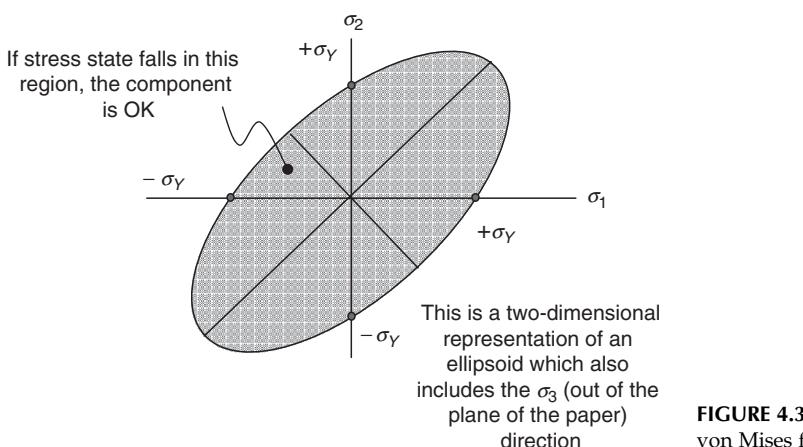
or

$$K = \frac{2}{\sqrt{3}} \left( \frac{\sigma_Y}{2} \right) = 1.155 \left( \frac{\sigma_Y}{2} \right) \quad (4.23)$$

$\sigma_Y$  is also known as the equivalent stress and either  $\sigma_Y$  or  $K$  can be found experimentally. In  $\sigma_1-\sigma_2-\sigma_3$  space, the criterion is represented by an ellipsoidal surface whose inner region symbolizes stress states that are safe (non-distorting). This is shown two-dimensionally in Figure 4.3.

The Tresca or maximum shear stress criterion is used when the material is known to have great ductility. It assumes the failure mechanism is by slippage along shear planes generated by the shear stress in the material. This assumption says that the material will not fail unless the shear stress it is experiencing is greater than that exhibited by a tensile test specimen of the same material at its failure point. Again we assume that tensile stress is positive and order the stresses with 1 the largest to 3 the smallest, and state the following:

$$\frac{(\sigma_1 - \sigma_3)}{2} = \text{constant} \quad (4.24)$$



**FIGURE 4.3**  
von Mises failure surface.

We set this constant equal to  $\sigma_Y^2$  or  $K$ . Here  $\sigma_Y$  is the yield stress in simple tension and  $K$  is the yield stress in pure shear. This implies that, for a component not to exhibit failure

$$(\sigma_1 - \sigma_3) < \sigma_Y \quad (4.25)$$

and

$$\sigma_Y = 2K \quad (4.26)$$

as well as

$$(\sigma_1 - \sigma_2) < \sigma_Y, \quad (\sigma_1 - \sigma_3) < \sigma_Y, \quad \text{and} \quad (\sigma_2 - \sigma_3) < \sigma_Y \quad (4.27)$$

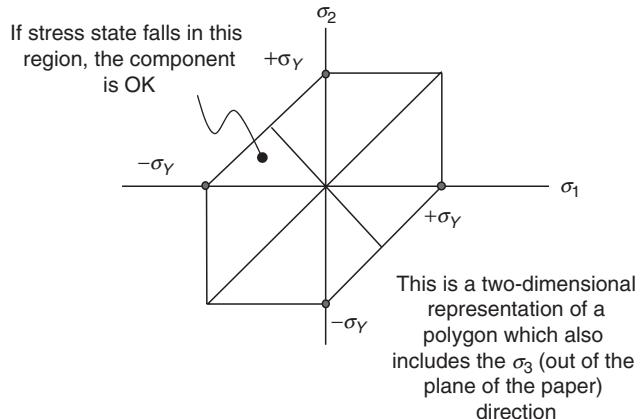
Once again either  $\sigma_Y$  or  $K$  can be found experimentally. In  $\sigma_1-\sigma_2-\sigma_3$  space, this is represented by a polyhedral surface whose inner region includes all stress states that are safe (non-failing). This is shown as a two-dimensional sketch in Figure 4.4.

The Tresca criterion is slightly more conservative if used for metals than von Mises. The Tresca polyhedron is contained within (circumscribed by) the ellipsoid of von Mises.

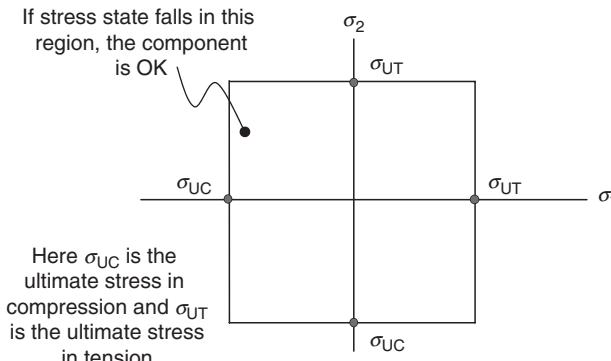
The third failure criterion we will examine is the Coulomb or maximum normal stress criterion. Here we assume that the normal stress in the material will precipitate the failure. Tensile stress is again assumed to be in the positive direction and stresses from 1 to 3 are again in order of decreasing magnitude. In this criterion, we require that, for a material that does not exhibit failure

$$\sigma_1, \sigma_2, \sigma_3 < \sigma_U \quad (4.28)$$

That is, all of the principal stresses must be less than the ultimate stress,  $\sigma_U$ , in the material in that particular direction. Recall that we use this for brittle materials where there is no yield point or yielding behavior. The failure surface is a rectangular polyhedron whose edges are the ultimate stresses in each principal direction. Stress levels within the polyhedron will not cause failure. A two-dimensional representation is depicted as in Figure 4.5. Even though this figure is shown as a square, in many materials the compressive strength is much greater than the tensile strength resulting in different limits and thereby changing the appearance (and sometimes resulting in a name change as well to a Mohr–Coulomb criteria) of the failure surface. In this instance, the failure surface would look like Figure 4.6. In the Mohr–Coulomb failure criteria, a greater compressive normal stress allows the material to carry more load. This is caused by the locking of slip planes akin to the sliding friction of a block causing greater resistance when the block gets heavier (i.e., an increase in



**FIGURE 4.4**  
Tresca failure surface.



**FIGURE 4.5**  
Coulomb failure surface.

normal stress on the slip plane). When this is applicable, our criteria results in an equation for the failure surface as follows:

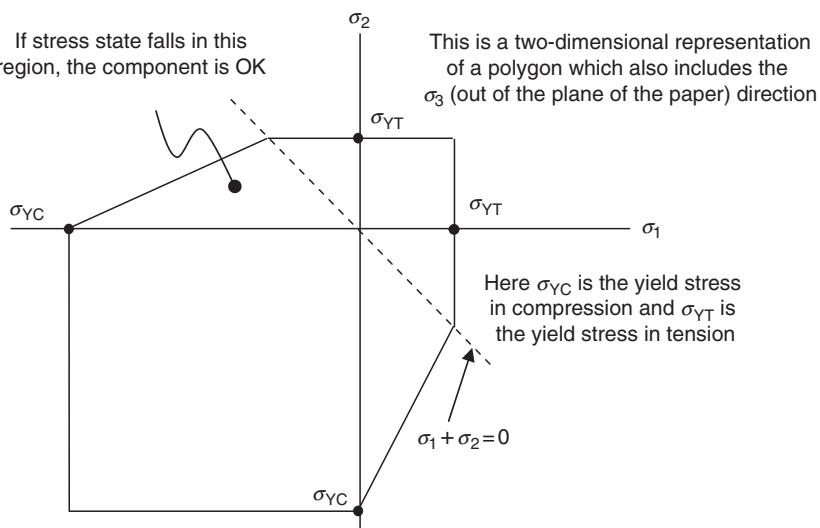
$$\max [|\tau| - \lambda \cdot (\sigma)] = \sigma_E \quad (4.29)$$

This equation results in a greater stress to failure due to the internal friction coefficient,  $\lambda$ . Since compressive strength is negative and  $\lambda$  is a positive quantity, the equivalent failure stress,  $\sigma_E$ , is greater with greater normal stress,  $\sigma$ .

Occasionally, it will be essential that we combine two or more of these criteria due to a change in material behavior. We shall describe this in due course.

### Problem 2

A component has principal stress values of 20,000, 56,000, and -220,000 psi (note that negative means compressive stress), if the yield strength in a simple tension test of the



**FIGURE 4.6**  
Mohr-Coulomb failure surface.

material was found to be 180,000 psi, will the part survive based on the von Mises failure criteria?

*Answer:* No the part will fail.

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### 4.3 Ammunition Types

Just as weapons are categorized by their usage as guns (low angle, line-of-sight, direct-fire), howitzers (high angle, beyond-line-of-sight, indirect-fire), or mortars (very high angle, short range, indirect-fire), the munitions for them are also categorized, not by use, but by their construction or assembly methods. Ammunition can be fixed, separable, or separate loaded.

Fixed ammunition, usually called a cartridge, consists of a container for the propellant charge, called the cartridge case, that is firmly attached to the projectile by crimping or cement and which remains in the weapon after firing and is ejected near it or is consumed during firing, and the projectile which flies downrange to the target. The charge, priming, and ignition system are assembled inside the case and are not alterable. This type of ammunition is characteristically used in tank, antiaircraft, aircraft weapons, and in most small arms (rifles and pistols).

Separable ammunition (also called semi-fixed ammunition) also consists of the cartridge case and projectile, but the case is not attached firmly to the projectile and can be removed in the field to adjust the charge, which can be changed incrementally. This type of ammunition was used in older howitzers and is still used in shotguns.

Separate-loaded ammunition (sometimes called separated ammunition) consists of the projectile, which is loaded first into the weapon, the propellant charge loaded next, and finally the primer and igniter loaded last. The charge, which is supplied to the weapon site, is in bagged increments and is altered, along with the quadrant elevation of the weapon, to vary the range. The primer is usually loaded into the weapon's breechblock. The block is self-sealing and assumes this function, which in fixed ammunition is done by the cartridge case. Ammunition of this type is used in howitzers and large naval guns.

Mortar ammunition is essentially of the separated type. The charge is incremental to help vary the range by altering the muzzle velocity. The charge increments are held in place on the projectile body by clips or holders. Increments may be added or deleted in the field by the gunner. Priming is done through an integral attachment to the projectile (a boom). Primer initiation is by a firing pin in the weapon that strikes an initiator in the boom at the termination of the fall of the projectile as it is dropped down the tube from the muzzle end. Trigger firing is also possible in some weapon designs.

The practical design of fixed ammunition cartridges, which is what we will mainly dwell on, encompasses the design of the propellant charge ignition system, the construction of the main body of the propellant charge, the design of the projectile body itself, including its shape and mass distribution, and its obturation and stabilization components. The design must also incorporate into the projectile the ancillary systems necessary for its intended functioning, e.g., fuzes, expulsion charges, explosive trains, and in modern projectiles, guidance and control.

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### 4.4 Propellant Ignition

Energetic devices that are combined in a specific manner into the ignition train accomplish the initiation of combustion of the propellant charge. The first of these elements is the

highly sensitive, detonator cup filled with material such as lead azide which itself receives an energy pulse from a trigger mechanism that delivers the pulse in the form of a mechanical (spring actuated) or electrical (hot wire or laser) impulse. The sensitive mix is detonated by the impulse and flashes into the main, less sensitive, ignition charge. This material is contained in the primer head.

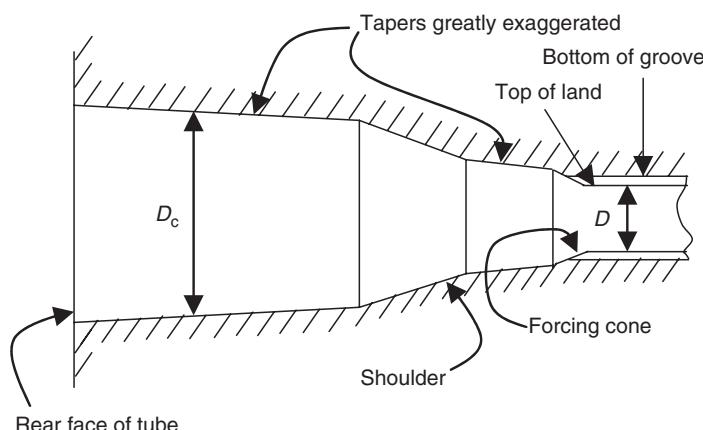
Secondary ignition takes place in the main primer body, where the ignition material known as the primer charge is stored. This material has traditionally been fine-grained black powder, which is known to have certain undesirable properties such as hygroscopicity. Attempts have been made to replace black powder, but it still remains the chief secondary ignition material. Two basic forms of primer charge are used in large caliber munitions: flat base-pad igniter charges are used with separate-loaded bagged propellant charges and in fixed, stick propellant charges; central core or bayonet-type primer bodies are used in most fixed, loose, granular propellant charges.

The design goal of all ignition systems is to provide rapid but smooth ignition of the main propellant charge avoiding at all cost pressure surges or spikes. Such surges can crush individual grains or sticks causing large, uncontrolled increases in burning surfaces and uncontrolled burning of the main charge. Symptoms of such burning are negative delta pressure ( $-\Delta p$ ) waves, i.e., negative gradients of pressure along the length of the chamber. One cause of pressure surges are the so-called blind primers, where vent holes are missing along the length of the primer body tube. The pressure build-up in the tube can rupture it causing asymmetric ignition and a  $-\Delta p$ .

Other caveats are to avoid overly sensitive detonator mixes and to provide gas flow space in the main propellant charge. Ignition and burning are surface phenomena and too tightly packed charges do not provide the necessary surfaces.

## 4.5 The Gun Chamber

To the rear of the long cylindrical portion of the gun (the bore) is the chamber, shown in Figure 4.7. The tapers shown facilitate the removal rearward of the spent cartridge case that hugs the chamber wall. During the firing cycle, the case swells because of the internal pressure and firmly contacts the chamber wall sealing the gases from exiting rearward.



**FIGURE 4.7**  
Chamber geometry.

When the pressure decays, a properly designed case comes away from the wall and the tapers insure that it does not stick in the chamber. When a fixed round of ammunition is loaded into the chamber, the rear face of the tube provides the stop and seat for the rim of the case. During the expansion of the pressurized case, the forcing cone of the chamber forms the seal for the hot gases by the extrusion and engraving of the rotating band in a rifled bore or the extrusion of the obturating band in a smooth bore. The ratio  $D_c/D$  is known as the chambrage, an important characteristic of the design. Large values of the chambrage tend to cause turbulent flow of the gases as they enter the bore. Such turbulence contributes to the erosion of the bore surfaces.

The gun designer is caught in a curious bind: for a desired volume of propellant, a large chambrage provides a shorter cartridge length, frequently a highly desirable parameter in the tight confines of a turret, for example; on the other hand, large chambrage values subject the bore to more erosion. Some of this difficulty has been overcome by the use of erosion reducing coolants. It has been found that much of the erosive wear in high performance guns and howitzers can be ameliorated by the introduction of a cool liquid, gaseous, or particulate layer between the hot propellant gases and the bore. Materials such as titanium dioxide, wax, talc, or silicone oil have proven efficacious. If these materials are assembled in the body of the propellant charge so that the gas flow keeps the coolant at the bore wall, a substantial decrease of erosion results. This is called laminar flow and is observed in low chambrage guns. Thus, a compromise may have to be made in the chambrage to reduce the turbulent flow.

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## 4.6 Propellant Charge Construction

In fixed cartridges, the most common practice is to fill a metallic cartridge case with perforated granular propellant grains around a bayonet-type primer that has already been inserted in the case. The grains commonly have seven perforations for progressive burning. In high performance rounds, vibrating the case to help settle the grains maximizes the loading density of the charge. Tank munitions are often loaded with perforated stick propellant. The sticks are bundled and carefully laid up around the boom and fin components that intrude into the depth of the case. Supplementary granular propellant is occasionally added to the stick bundles to further boost the charge mass and increase the progressivity of burning. Rocket grain configurations with complex star and slit perforations have been tried as well as 19-perf grains to raise the burning rate, but these are difficult to make and are not standard.

Howitzer (separate-loaded) charges are made up of bagged increments that are ignited by the last increment loaded, the base-pad igniter. A primer in the breechblock sets off the igniter. In these and in the fixed ammunition charges, coolants are strategically emplaced to promote erosion resistance. With bagged propelling charges, since there is no cartridge case present, it is extremely important that all of the material be combusted. Great care is taken in selecting materials—silk was used for many years in the Navy—to assure that there are no burning embers left in the weapon after it was fired. It is typical for a howitzer crewman to look down the bore and shout “bore clear” during firing operations. If burning materials are present and a fresh charge is inserted into the bore, the propellant may ignite and cause serious injury to the gun crew. This has been termed “cook-off.”

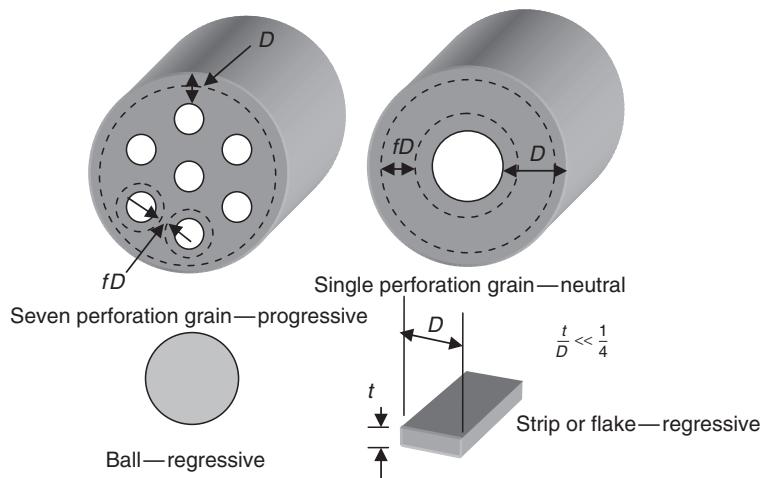
There have been extensive efforts to take advantage of the convenience of stowage, low cost, and inherent safety of liquid bipropellants (LP). However, severe operational and performance problems have prevented their adoption. These problems have centered on combustion instability that manifests itself in destructive, unpredictable pressure peaks,

particularly in bulk-loaded systems. Attempts to get around these so-called Taylor instabilities have had some success with regenerative pressurized systems that atomize the pumped-in liquids, ignite this cloud, and avoid the pressure wave unpredictability of an ignited bulk of liquid. This concept, even though it has shown promise, still may not be able to overcome the poor low temperature properties of the liquid propellants. They show marked increases in viscosity at low temperatures causing severe flow and pumping problems.

Two other concepts of gun propulsion should be mentioned. These are the use of electromagnetically generated force to propel a projectile down a gun and the idea of using a low molecular weight gas to propel the projectile—the light gas gun. At the time of this writing neither concept has shown the ability to progress beyond the laboratory stage to a fieldable weapon, although light gas guns are in common use in laboratories to reach velocities with small projectiles approaching meteorite entry speeds.

## 4.7 Propellant Geometry

The geometry of the propellant grain is one of the parameters available to the interior ballistician to tailor the pressure curve in the gun. Production of gas from a grain depends on the evolution of the total surface of the grain as the burning proceeds. If the surface area increases with time, the grain is considered progressive. If the total surface remains constant over time the grain is neutral, and if the surface decreases with time the grain is considered regressive. The perforations in the grain affect the surface area and therefore the burning characteristics. In cylindrical grains, the number of perforations are usually one of the numbers in the sequence: 1, 7, 19, and 37. The largest number in use in the United States is 19, and this is rarely found because of the difficulty of manufacture. The various types of grains are shown in Figure 4.8. The web,  $D$ , that is the smallest thickness of propellant between any two surfaces is one of the major parameters in interior ballistic computations.



**FIGURE 4.8**  
Typical propellant grain geometries.

## 4.8 Cartridge Case Design

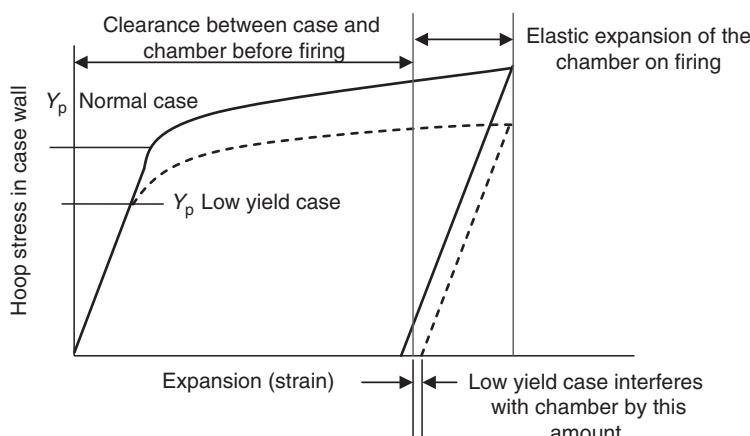
The design of a metallic cartridge case must fulfill four basic roles: the case must seal or obturate the gun breech so that gases do not stream backward out of the gun; it must serve as a protective container for the propellant charge; it must act as a structural member of the cartridge assembly to allow for vigorous handling during shipping, stowage, and loading into the chamber; and it must be easily extractable from the chamber after the round is fired. Metallic cases have been used for much more than a hundred years and the design practices are well established to fulfill these roles. Yet difficulties still arise in the extraction of the case after firing—it can stick in the chamber, rendering the weapon useless until it is removed. The case by itself cannot sustain the gun pressure and is intended to be supported by the chamber walls. Yet the case must be designed with sufficient clearance to permit loading and ramming. The analysis of sticking that follows must be part of the design engineer's overall task before a new weapon can be fielded.

It is possible, through use of some relatively simple equations, to determine if a cartridge case will expand enough to stick in the chamber of the weapon after firing. Graphically, we can depict this as shown in Figure 4.9. In this figure, we see the effect when a case with a low yield strength is loaded to the same levels as a good case. The expansion and contraction of the gun tube itself must be taken into account when the cartridge case is designed. This condition can be approximated using a bilinear, kinematic hardening model where the stress-strain curves of the case material are modeled as depicted in Figure 4.10.

The first step in this procedure is to model the gun tube. In this case, we assume that the material is perfectly elastic—which will be the case for any properly designed tube—and we can determine the radial expansion through [2]

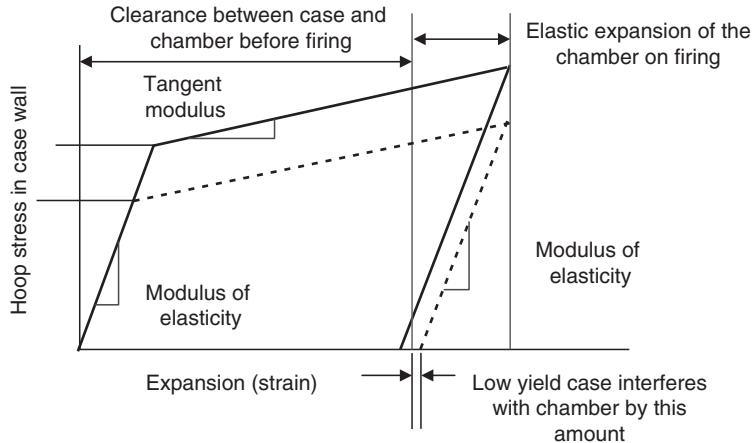
$$u_{\text{tube}} = \frac{a'}{E_{\text{tube}}(b^2 - a'^2)} [(1 - \nu)(p_1 a'^2 - p_2 b^2) + (1 + \nu)b^2(p_1 - p_2)] \quad (4.30)$$

In this equation (which has been tailored from a previous formula for a thick-walled cylinder because the point we are interested in is on the inside radius of the tube wall),  $a'$  is the inner radius of the chamber,  $b$  is the outer radius of the gun tube,  $p_1$  is the internal



**FIGURE 4.9**

Stress-strain diagram of a normal case and one with low yield strength.



**FIGURE 4.10**

Stress-strain diagram of a normal case and one with low yield strength modeled as bilinear kinematic hardening materials.

pressure,  $p_2$  is the external pressure (usually conservatively taken as 0),  $\nu$  is Poisson's ratio for the tube material, and  $E_{\text{tube}}$  is the modulus of elasticity.

We now calculate the stress, strain, and displacement of the case through use of the thin-wall cylinder equations [3]

$$u_{\text{case}} = \frac{a^2 p_1}{E_{\text{case}} h} \quad (4.31)$$

$$\sigma_{\theta\theta} = \frac{ap_1}{h} \quad (4.32)$$

$$\varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E_{\text{case}}} \quad (4.33)$$

In these equations,  $u_{\text{case}}$  is the radial expansion of the case,  $\sigma_{\theta\theta}$  is the hoop stress in the case,  $\varepsilon_{\theta\theta}$  is the hoop strain,  $a$  is the outside radius of the case, and  $h$  is the case wall thickness. Now the gun tube will stop the case from expanding further once contact is made so the maximum expansion of the case will be as follows:

$$u_{\text{case}_{\max}} = u_{\text{tube}} = a\varepsilon_{\theta\theta_{\max}} \quad (4.34)$$

Because we know the pressure and the tube dimensions and therefore the value of  $u_{\text{tube}}$ , we can calculate  $\varepsilon_{\theta\theta_{\max}}$ . We can then use this value to calculate the stress in the case at the maximum expansion.

$$\varepsilon_{\theta\theta_{\max}} - \varepsilon_Y = \frac{\sigma_{\theta\theta_{\max}} - \sigma_Y}{E_{\text{case-tangent}}} \quad (4.35)$$

In this expression, the subscript  $Y$  indicates yield values and  $E_{\text{case-tangent}}$  is the tangent modulus of the cartridge case material. Once we determine the stress at the maximum expansion, we need to recall that a material which has yielded will retract along its original elastic modulus. Thus, we can write

$$\varepsilon_{\text{return}} = \frac{\sigma_{\theta\theta_{\max}}}{E_{\text{case}}} \quad (4.36)$$

Now the residual strain in the case is given by

$$\varepsilon_{\text{residual}} = \varepsilon_{\theta\theta_{\max}} - \varepsilon_{\text{return}} \quad (4.37)$$

We can then find the permanent radial displacement through

$$u_{\text{residual}} = a\varepsilon_{\text{residual}} \quad (4.38)$$

If we now add  $u_{\text{residual}}$  to the original radius of the case,  $a$ , we can see that if

$$u_{\text{residual}} + a \geq a', \quad \text{the case will stick} \quad (4.39)$$

or if

$$u_{\text{residual}} + a < a', \quad \text{the case will not stick} \quad (4.40)$$

Over the last 20 years, the metallic case of drawn brass, extruded steel, or spirally wrapped steel has been replaced in certain systems by a fully combustible or consumable case. These cases are manufactured of felted nitrocellulose and usually consist of a base and sidewall that are assembled with cement, filled with granular or stick propellant, and attached to the projectile with clamps and cement. Since the cases are consumed completely, they do not seal the breech. With these munitions, a self-sealing breech must be designed for the weapon. For guns with non-sealing breeches that are already fielded for use with conventional metallic cartridge cases, a case that has a metallic stub and a combustible sidewall has been devised to take advantage of the small volume of the ejected stub in the confines of a tank turret, for example, and the overall reduction in cost and weight of the round. While systems with the combustible case have been fielded, the success of this development has not been complete. Occasional problems with incomplete combustion of a case that leaves smoldering residue capable of igniting the next loaded round (cook-off) have required scavenging systems for the chamber to be installed. The inherent structural weakness of nitrocellulose has also posed problems of case attachment and handling. Yet the obvious advantages of the combustible case have kept the concept in the weapon designer's toolbox for possible use.

### Problem 3

A design for a 105-mm weapon is being considered. The chamber is stated to withstand the desired 35,000 psi and is essentially a steel cylinder of 4.5-in. ID and 7-in. OD ( $E_{\text{tube}} = 30 \times 10^6$  psi,  $\nu = 0.3$ ). We have decided to use brass with an OD of 4.490 in. If we use a bilinear, kinematic hardening model where the brass has a modulus of elasticity of  $15 \times 10^6$  psi, a local tangent modulus of  $12.5 \times 10^6$  psi, a yield stress of 15,000 psi (yield occurs in this material at  $\varepsilon = 0.001$ ), and an ultimate tensile strength of 45,000 psi, with the information given, what is the radial clearance between the case and the chamber after firing neglecting thermal effects?

*Answer:* Approximately 0.004-in. radial clearance

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## 4.9 Projectile Design

While propulsion systems are fairly straightforward in design because their intended use is simple, projectiles vary widely in use and as a consequence their designs are complex and

demanding. The propulsion system must get the projectile through the launch environment with consistent muzzle velocities, but without undue stress to the gun or the projectile. The projectile, on the other hand, must withstand the forces of launch, be efficient, consistent and precise in its flight environment, and deliver its intended utility at the target. We will explore only projectile design for launch in this section, reserving design for flight and terminal effects until later.

Projectiles may be classified into two general types: cargo carriers and pure kinetic energy deliverers. The cargo carriers include shells that deliver high explosives (HE), submunitions and mines, pyrotechnics, smart munitions, and other specialized lethal systems, e.g., shaped charges (HEAT) and explosively formed penetrators (EFP) shells. The kinetic energy delivery systems, used chiefly for the attack of armor, are monobloc steel shot (AP), sabotized, long-rod, heavy metal penetrators (APFSDS), and older types of spin-stabilized, sabotized (APDS) projectiles.

The stresses induced into a projectile during launch are chiefly due to the acceleration that the gases impart to it. The cargo carriers are shells whose stresses are due to relatively low accelerations and which, except for the tank cannon fired HEAT shell, achieve only moderate muzzle velocities. We will therefore explore the kinds of stresses and failures inherent in shell-like structures under load in Section 4.10. Kinetic energy munitions, on the other hand, are subject to extremely high accelerations and have high muzzle velocities. For these types, we will explore the driving mechanism stresses and other aspects of these designs.

The gamut of topics in projectile design is almost unlimited. However, several suggest themselves because of their general applicability or timely interest. Shell design is a ubiquitous problem and will be explored in depth in Section 4.10. The use of buttress threads is so common in projectile and gun design that it warrants its own [in Section 4.11](#). Sabot design is more specialized as are the problems of kinetic energy rods and their buttress driving grooves. These will be explored [in Section 4.12](#).

Modern projectiles employ a variety of electronic and electromechanical devices for fuzing, target detection, and guidance and control. This relatively new engineering discipline called “gun hardening” deals with designing these devices to survive the harsh environment of gun launch.

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## 4.10 Shell Structural Analysis

Most cargo-carrying projectiles, whether fin- or spin-stabilized, are designed with cargo bodies in the shape of an axisymmetric cylindrical shell. Because the loads on these cylinders are the result of spin and acceleration of the shells and their contents, the stresses encountered are highly variable along and through their walls. These stresses will be examined as will the consequences of failure criteria.

The symbols and definitions of the constants and variables of shell loading are tabulated below:

$A$ —Bore area of the gun

$a$ —Linear acceleration

$d$ —Diameter of bore (across lands), diameter of assumed shear circle in base of shell

$d_i$ —Inside diameter (ID) of projectile

$d_o$ —Outside diameter (OD) of projectile

$F_b$ —Maximum force on base of projectile and rotating band  
 $F_T$ —Maximum tangential force on projectile wall  
 $F_{TR}$ —Hoop tension (force) in wall of projectile resulting from rotation of the shell  
 $F'_T$ —Tangential force at section of shell  
 $f'$ —Setback force  
 $g$ —Acceleration due to gravity  
 $h$ —Total depth of filler from nose  
 $h'$ —Total depth of filler from nose end of cavity to section under consideration  
 $I_{zz}$ —Polar moment of inertia  
 $I'_{zz}$ —Polar moment of inertia of metal parts forward of section when section is ahead of rotating band and aft of it when section is aft of the rotating band  
 $n$ —Twist of the rifling  
 $p_b$ —Maximum propellant pressure  
 $p_h$ —Filler pressure due to setback  
 $p_{rot}$ —Filler equivalent pressure due to rotation, includes wall inertia  
 $r_i$ —Inside radius of projectile  
 $r_o$ —Outside radius of projectile  
 $S$ —Compressive strength of the rotating band  
 $S_1$ —Longitudinal stress  
 $S_2$ —Tangential stress  
 $S_3$ —Radial stress  
 $\tau$ —Shear stress  
 $\sigma_Y$ —Static yield stress in tension  
 $T$ —Torque applied to the projectile  
 $t$ —Base thickness, wall thickness  
 $V$ —Muzzle velocity  
 $w$ —Total projectile weight  
 $w'$ —Weight of metal parts forward of section under consideration  
 $w'_f$ —Weight of filler forward of section under consideration  
 $\alpha$ —Angular acceleration  
 $\rho_m$ —Density of projectile material  
 $\rho_f$ —Density of filler material  
 $\omega$ —Angular velocity  
 $r_b$ —Radius of band seat  
 $p_{band}$ —Band pressure

We distinguish between thin-walled and thick-walled cylinders in this analysis so that the designer may run quick, ballpark estimates of the stress levels encountered. In practice, finite element analysis (FEA) is usually conducted on the components, but as emphasized earlier, the designer should have a good idea of the bounds of the answer before beginning the FEA.

We begin with a review of basic mechanics of materials as applied to cylinders. If a cylinder is subjected to an axial load and does not buckle, the axial stress can be determined from

$$S_1 = -\frac{F_{\text{Axial}}}{A} = -\frac{F_{\text{Axial}}}{\pi(r_o^2 - r_i^2)} \quad (4.41)$$

The stress-strain relationships for a cylinder are as follows:

$$\varepsilon_{rr} = \frac{1}{E} [\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})] \quad (4.42)$$

$$\varepsilon_{\theta\theta} = \frac{1}{E} [\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})] \quad (4.43)$$

$$\varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{\theta\theta} + \sigma_{rr})] \quad (4.44)$$

Here  $\nu$  is Poisson's ratio,  $\sigma_{rr}$  is the radial stress,  $\sigma_{\theta\theta}$  is the transverse (hoop) stress,  $\sigma_{zz}$  is the axial stress, and  $E$  is Young's modulus.

If a cylinder is subjected to a torsional load, it will twist. We typically assume that this deformation is small and plane sections remain plane. Thus, when we apply a torque,  $T$ , to a cylinder of length,  $L$ , with shear modulus,  $G$ , and polar moment of inertia,  $J$ , the structure will rotate through an angle  $\phi$  (in radians).

$$\phi = \frac{TL}{JG} \quad (4.45)$$

For a hollow cylinder,

$$J = \frac{1}{2}\pi(r_o^4 - r_i^4) \quad (4.46)$$

For a material which behaves according to Hooke's law,

$$G = \frac{E}{2(1 + \nu)} \quad (4.47)$$

Such a material under pure torsion will only exhibit shear stress according to

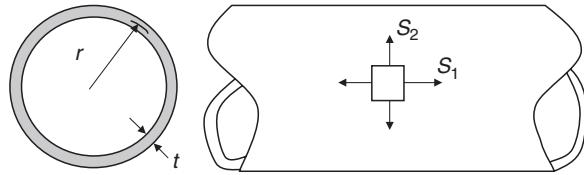
$$\tau_{\theta z} = \frac{Tr}{J} \quad (4.48)$$

While the thick-wall cylinder analysis, which we describe below, is an exact solution, a quick way to assess the major stresses if the wall thickness is less than 10% of the cylinder radius is to assume that the stresses in the radial direction,  $S_3$ , are negligible.

Thus, we examine only the meridional or longitudinal and the circumferential or hoop stresses. We define  $S_1$  as the longitudinal stress,  $S_2$  as the hoop stress, and  $p$  as the pressure depicted in [Figure 4.11](#).

If the cylinder has closed ends, then internal pressure can cause a longitudinal stress

$$S_1 = \sigma_{zz} = \frac{pr}{2t} \quad (4.49)$$



**FIGURE 4.11**  
Thin-wall cylinder geometry.

otherwise  $S_1 = 0$ . Internal (or external) pressure always causes hoop stress

$$S_2 = \sigma_{\theta\theta} = \frac{pr}{t} \quad (4.50)$$

In practical shell design, we always perform a thick-wall cylinder analysis assuming that the stresses in the radial direction are significant enough to be considered. Thus, we must examine longitudinal, hoop, and radial stresses. We again define  $S_1 = \sigma_{zz}$  = longitudinal stress,  $S_2 = \sigma_{\theta\theta}$  = hoop stress,  $S_3 = \sigma_{rr}$  = radial stress, and  $p$  = pressure. This is depicted in Figure 4.12.

The following solutions are known as the Lamé formulas and assume open ends which implies  $S_1 = 0$  if no axial loads are present. If axial loads are present, they must be accounted for. Internal (or external) pressure always causes hoop stress. (Note that the subscripts "o" and "i" refer to the outer and inner surfaces, respectively.)

$$S_2 = \sigma_{\theta\theta} = \frac{1}{(r_o^2 - r_i^2)} \left[ p_i r_i^2 - p_o r_o^2 - \frac{r_i^2 r_o^2 (p_o - p_i)}{r^2} \right] \quad (4.51)$$

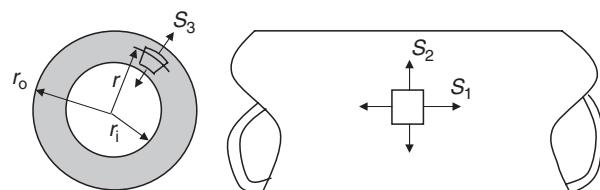
with a maximum at  $r = r_i$ . The radial stress can be calculated from

$$S_3 = \sigma_{rr} = \frac{1}{(r_o^2 - r_i^2)} \left[ p_i r_i^2 - p_o r_o^2 + \frac{r_i^2 r_o^2 (p_o - p_i)}{r^2} \right] \quad (4.52)$$

with a maximum again at the inner surface  $r = r_i$ , and equal to  $S_3 = -p_i$ .

Initially, we will analyze the state of stress caused by the centrifugal loading induced by the rotation of a projectile in a rifled gun tube. In a spin-stabilized projectile, besides the longitudinal loads induced by the acceleration through the tube, the rotation of the projectile, which is dependent upon the axial velocity and the twist of the rifling in the tube, induces stresses in the walls. The twist of the rifling is usually measured in revolutions per caliber of travel (i.e., a twist of 1 in 20 means the projectile makes one revolution in 20 calibers of travel [ $n = 20$ ]). The units of  $n$  are calibers per revolution. If we multiply  $n$  by the diameter,  $d$ , we get units of length per revolution.

$$n \left[ \frac{\text{caliber}}{\text{revolution}} \right] \times d \left[ \frac{\text{length}}{\text{caliber}} \right] = nd \left[ \frac{\text{length}}{\text{revolution}} \right] \quad (4.53)$$



**FIGURE 4.12**  
Thick-wall cylinder geometry.

Since there are  $2\pi$  radians per revolution, the angular velocity a projectile has attained is defined as

$$\omega = \frac{2\pi V}{nd} = \frac{\left[ \frac{\text{radians}}{\text{revolution}} \right] \left[ \frac{\text{length}}{\text{time}} \right]}{\left[ \frac{\text{length}}{\text{revolution}} \right]} = \left[ \frac{\text{radians}}{\text{time}} \right] = [t^{-1}] \quad (4.54)$$

The centrifugal force directed radially outward on an element of material at radius  $r$  is

$$F_c = ma_r = \frac{w}{g} r \omega^2 \quad (4.55)$$

In the tangential direction, the inertial forces on an element of material can be determined from

$$F_t = ma_t = \frac{w}{g} r \alpha \quad (4.56)$$

We can determine the centrifugal force on the cylinder wall caused by spinning the cylinder in the absence of other loads by integrating Equation 4.55 from the inner diameter to the outer diameter. To do this, we consider the differential element as depicted in Figure 4.13. From this diagram, we see that the mass of an infinitesimal annular ring of material is

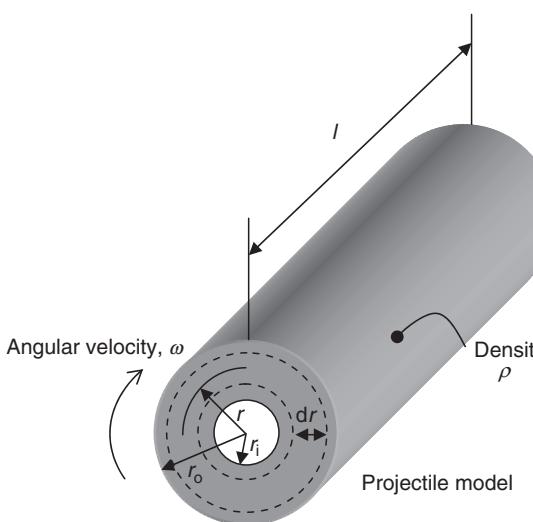
$$dm = \frac{dw}{g} = \rho dV = \rho l 2\pi r dr \quad (4.57)$$

Inserting Equation 4.57 into Equation 4.55 yields

$$dF_c = a_r dm = \rho l 2\pi r^2 \omega^2 dr \quad (4.58)$$

which, when integrated from the inner to the outer radius, gives

$$dF_{cWALL} = 2\pi \rho l \omega^2 \int_{r_i}^{r_o} r^2 dr = \frac{2\pi \rho l \omega^2}{3} (r_o^3 - r_i^3) \quad (4.59)$$



**FIGURE 4.13**  
Differential thickness element geometry.

This is the radial force on the wall due to the inertia of the wall material only. If the projectile is filled with material, we need to account for this filler as well. Thus, if we integrate from the centerline to the inner radius of the projectile wall, we obtain

$$dF_{\text{cFILL}} = 2\pi\rho_{\text{FILL}}l\omega^2 \int_0^{r_i} r^2 dr = \frac{2\pi\rho_{\text{FILL}}l\omega^2}{3}(r_i^3) \quad (4.60)$$

The total force acting on the projectile wall due to spin is then

$$F_c = F_{\text{cWALL}} + F_{\text{cFILL}} = \frac{2\pi l\omega^2}{3} [\rho(r_o^3 - r_i^3) + \rho_{\text{FILL}}(r_i^3)] \quad (4.61)$$

For stress computations, we require an internal pressure; thus, we need to convert the centrifugal forces to an equivalent internal pressure. If we assume that our centrifugal forces are acting on the interior of the shell, pushing radially outward, the area for our equivalent pressure is

$$A_{\text{rad}} = 2\pi r_i l \quad (4.62)$$

Thus, our equivalent pressure can be written as

$$p_{\text{rot}} = \frac{F_c}{A_{\text{rad}}} = \frac{\omega^3}{3r_i} [\rho(r_o^3 - r_i^3) + \rho_{\text{FILL}}(r_i^3)] \quad (4.63)$$

In [Equation 4.56](#), we determined the tangential force arising from the angular acceleration. If we perform a similar analysis to that which developed [Equation 4.61](#), we will obtain an expression for the torque as follows:

$$T = M_{\text{WALL}} + M_{\text{FILL}} = \frac{1}{2}\pi\alpha l[\rho(r_o^4 - r_i^4) + \rho_{\text{FILL}}(r_i^4)] \quad (4.64)$$

The derivation of this is left as an exercise for the interested reader and is included as a problem at the end of the chapter.

The formulas for calculating the tangential and radial stresses at radial location,  $r$ , in a rotating cylinder where  $r_o > 10(r_o - r_i)$  can be given as

$$\sigma_{\theta\theta} = \rho\omega^2 \left( \frac{3+\nu}{8} \right) \left( r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right) \quad (4.65)$$

$$\sigma_{rr} = \rho\omega^2 \left( \frac{3+\nu}{8} \right) \left( r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - r^2 \right) \quad (4.66)$$

We are reminded that the longitudinal stress (assuming the structure does not buckle) is simply the axial acceleration multiplied by the weight of all of the material forward of the location of interest divided by the shell cross-sectional area—we will discuss this presently. These formulas were developed for the centrifugal loading of a spinning projectile by forces that act during both in-gun setback and flight. The axial load on a projectile, however, is for the most part only present during acceleration in the tube, is a function of time, and occurs whether the projectile is spinning or not. Beyond this, there is also an applied torque due to the angular acceleration, which is applied through the rotating band

or slip obturator. The setback load and (if spinning) the centrifugal and torsional loads must all be superimposed on the projectile to determine its state of stress.

The axial force on the projectile during firing is given by

$$F = p_s A \quad (4.67)$$

Here  $p_s$  is the pressure acting on the base of the projectile defined by the Lagrange approximation

$$p_s = p_B \left( \frac{1}{1 + \frac{c}{2w}} \right) \quad (4.68)$$

The D'Alembert force is the force due to acceleration that exactly equals this pressure force

$$a = \frac{p_s A g}{w} \quad (4.69)$$

At any axial position, the force on the cross-sectional area can be shown to be proportional to the weight of material forward of the section.

$$f' = \frac{w'}{w} p_s A \quad (4.70)$$

To calculate the force (or really the pressure) in the filler material, we usually resort to a hydrostatic model

$$p_h = \rho h a = \rho h \frac{p_s A g}{w} \quad (4.71)$$

Here  $\rho$  is the density of the filler,  $h$  is the filler head height, and  $p_h$  is the hydrostatic pressure that is developed.

In a spin-stabilized projectile, the angular acceleration,  $\alpha$ , is proportional to the linear acceleration,  $a$ , where

$$\alpha = K a \quad (4.72)$$

Then

$$\alpha = K \frac{p_s A g}{w} \quad (4.73)$$

Here  $K$  has units of length<sup>-1</sup> and is dependent upon the twist,  $n$  (in calibers of travel per turn), and the bore diameter,  $d$ , thus

$$K = \tan \theta = \frac{2\pi}{nd} \quad (4.74)$$

Here  $\theta$  is the angle between the circumferential twist distance and the axial distance traveled. From Equations 4.73 and 4.74, we get

$$\alpha = \frac{2\pi}{nd} \frac{p_s A g}{w} \quad (4.75)$$

If we define a tangential force applied to the rotating band of the projectile as  $F_T$ , then the torque on the projectile is

$$T = F_T \frac{d}{2} \quad (4.76)$$

We know that the torque is equal to the product of the polar moment of inertia of the projectile and its angular acceleration

$$T = I_{zz}\alpha \quad (4.77)$$

Solving for the angular acceleration in terms of the tangential force, we get

$$\alpha = \frac{F_T}{I_{zz}} \frac{d}{2} \quad (4.78)$$

Inserting this into [Equation 4.56](#) and solving for  $F_T$  yields

$$F_T = \frac{\pi^2 I_{zz} p_s}{n w} \quad (4.79)$$

Since

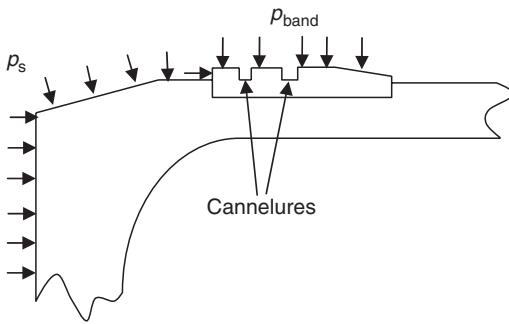
$$A = \pi \frac{d^2}{4} \quad (4.80)$$

The force that is applied by the rifling to the rotating band is transmitted through the structure to regions both forward and aft of the rotating band. These forces are proportional to the moment of inertia of the sections ahead of or behind the application of the torque load,  $I'_{zz}$ . We assume that this force acts over a mean diameter of the outer and inner wall surfaces of the shell and then we get

$$F'_T = \frac{16\pi I'_{zz}}{n(d_o + d_i)^2} \frac{p_s A}{w} \quad (4.81)$$

Because the rotating band is intended to act as a gas seal (obturator) as well as the rotational driver, designs typically exhibit a diameter over the band that is slightly larger than the groove diameter of the weapon. The engraving action of the gun lands and the interference fit in the grooves causes a plastic flow of the band resulting in a pressure on the band seat as well as a developed reaction in the gun wall. This pressure can be greater than the gas base pressure on the projectile. Measurements of this pressure have been obtained by strain gaging of the gun tube and computing the stress at the weapon's inner diameter. The pressure required to cause this stress is called the interface pressure. It has been shown that cannelures or circumferential grooves cut into the band surface reduce this pressure substantially by allowing room for band material to flow rather than being loaded in a quasi-hydrostatic condition. This is depicted in [Figure 4.14](#). The composition/material of the rotating band can have a dramatic effect upon the behavior of the projectile in the tube as well as tube wear. An excellent example of this relationship is contained in Ref. [4].

We have the forces on the projectile structure but now must translate these into stresses that allow us to determine how much design margin is present. Once determined, these



**FIGURE 4.14**  
Rotating band pressure.

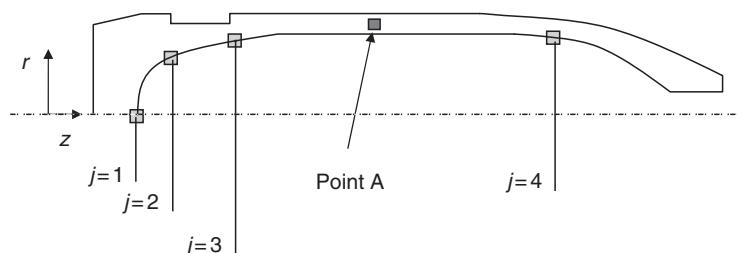
stresses are then linked to well-established failure criteria to determine the failure point of the material. Since projectiles may be made of a variety of materials, specialized criteria may have to be used on each material. This full procedure is somewhat complicated and beyond the scope of this book, but we will attempt to describe the basics through an examination of a simple M1, high explosive projectile structure depicted in Figure 4.15.

Assume a thick-walled cylinder as shown for stress calculations where

- $S_{1j}$ —Longitudinal stress at the  $j$ th location
- $S_{2j}$ —Hoop stress at the  $j$ th location
- $S_{3j}$ —Radial stress at the  $j$ th location
- $\tau_{11}$ —Longitudinal shear at the base
- $\tau_{2j}$ —Torsional (shear) stress at the  $j$ th location

It is helpful to recap here all of the loads on an element of projectile wall material at a generalized location (such as point A) in the diagram. This element of material is

- Compressed in the axial direction due to the axial acceleration
- Loaded in tension in the hoop direction because of the wall mass being pulled radially outward due to the spin
- Loaded in tension in the hoop direction because of the filler material moving outward due to the setback load and the spin
- Loaded in shear due to the rotating band accelerating the projectile in an angular direction
- Loaded in shear due to the greater stress in the outer wall than on the inside wall



**FIGURE 4.15**  
Stress locations in an M1 high explosive (HE) projectile.



**FIGURE 4.16**  
Load conditions for an M1 HE projectile.

Note that when including mass forward of a particular section, we must include all mass transmitting loads to the section, e.g., fuze, bushings, cups, etc. The pressures applied to our model of the M1 projectile are shown in Figure 4.16.

Now let us examine specific locations of interest along the shell where experience tells us failures might occur. For convenience, these have been tabulated in Table 4.1 and tailored to each individual location with the symbol, source load, and type of stress noted.

At location 1, these are the formulas used to calculate stresses due to the setback of filler on base, the moments caused thereby, and by gas pressure on base:

$$S_{11} = -p_h \quad (4.82)$$

$$S_{21} = 0 \quad (4.83)$$

$$S_{31} = \frac{r_o^2}{t^2} (p_s - p_h) \quad (4.84)$$

$$S'_{31} = -p_s \left[ \frac{3r_o^3}{2(r_o^3 - r_i^3)} \right] + p_h \left[ \frac{r_o^3 + 2r_i^3}{2(r_o^3 - r_i^3)} \right] \quad (4.85)$$

Equation 4.82 is the axial component stress. We can see that it is just driven by the reaction of the fill and shell to the axial acceleration. Since this is a centerline location, by definition

**TABLE 4.1**

Typical Stresses in an High Explosive (HE) Projectile and Their Sources

Type of Stress	Symbol	Source of Load
Compressive load on base	$S_{11}$	Setback of filler
Radial stress on base at centerline	$S_{31}$	Moments of filler setback and base pressures (flat base)
Radial stress on base at centerline	$S'_{31}$	Moments of filler setback and base pressures (round base)
Hoop stress at rear of the band	$S_{22}$	Setback of filler, rotation, and external pressure (band and gas)
Radial stress at ends of band and maximum ID	$S_{32}, S_{33}, S_{34}$	Rotation of projectile, filler setback, and filler rotation
Longitudinal stress at ends of band and maximum ID	$S_{12}, S_{13}, S_{14}$	Setback of metal parts in wall (filler contribution usually neglected)
Hoop stress at forward end of band and maximum ID	$S_{23}, S_{24}$	Filler pressure and rotation of wall
Shear stress through thickness $t$	$\tau_{11}$	Moments of filler setback and base pressures (round base)
Torsional shear in projectile wall	$\tau_{22}, \tau_{23}, \tau_{24}$	Setback of filler, rotation, and external pressure (band and gas)

there is no hoop stress which is defined by [Equation 4.83](#). [Equation 4.84](#) specifies the radial stress assuming the base is flat faced. This comes about from the difference in the base pressure reacting against the internal forces and attempting to push the center of the base into the fill. [Equation 4.85](#) is the radial stress equation assuming the base is a rounded bottom (i.e., with the concave portion enclosing the fill). We can see from this equation that the stresses are much lower as it carries the load more efficiently than a flat bottom shell. The drawback is that a base of this type requires a skirted boat tail which is more expensive to manufacture but saves considerable weight.

Moving to location 2, these are the stresses due to setback of filler, filler rotation, wall rotation, and band pressure:

$$S_{12} = -\frac{w' + w'_f}{w} \left[ \frac{p_s A}{\pi(r_o^2 - r_i^2)} \right] + \left[ \frac{(p_h + p_{rot})r_o^2}{(r_o^2 - r_i^2)} \right] \quad (4.86)$$

$$S_{22} = (p_h + p_{rot}) \left( \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \right) - p_{band} \left( \frac{r_b^2}{r_b^2 - r_i^2} \right) \quad (4.87)$$

$$S_{32} = -(p_h + p_{rot}) \quad (4.88)$$

At this location, we see that the axial stress defined by [Equation 4.86](#) has two parts. The first term on the RHS is the inertia of all the fill and shell material ahead of this location. The second term is the axial stress caused by the internal pressure of the fill expanding. In [Equation 4.87](#), the first term on the RHS is the contribution of spin to the hoop stress and the second term is the restoring force caused by the gun tube pushing in on the rotating band. [Equation 4.88](#) is simply the radial stress caused by the rotation and compression of the fill and wall.

Further forward on the shell at location 3, the stresses due to setback of filler, filler rotation, wall rotation, and band pressure have identical formulas to location 2 but with, of course, different values of the variables due to the lower hydrostatic pressure component.

$$S_{13} = -\frac{w' + w'_f}{w} \left[ \frac{p_s A}{\pi(r_o^2 - r_i^2)} \right] + \left[ \frac{(p_h + p_{rot})r_o^2}{(r_o^2 - r_i^2)} \right] \quad (4.89)$$

$$S_{23} = (p_h + p_{rot}) \left( \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \right) - p_{band} \left( \frac{r_b^2}{r_b^2 - r_i^2} \right) \quad (4.90)$$

$$S_{33} = -(p_h + p_{rot}) \quad (4.91)$$

Finally at location 4, near the nose of the shell, the stresses due to setback of filler, filler rotation, and wall rotation are as follows:

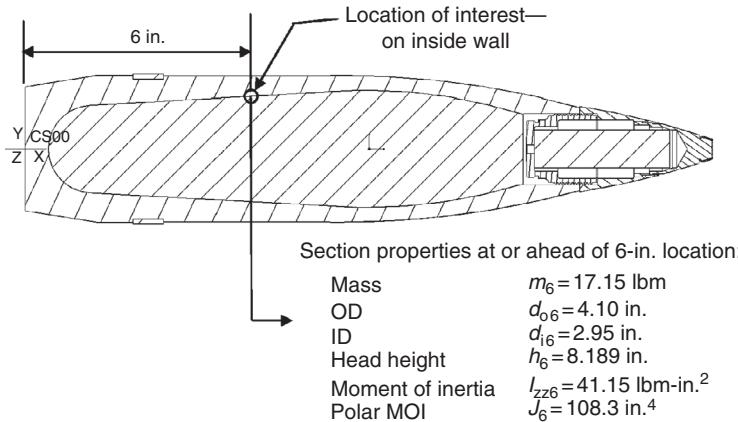
$$S_{14} = -\frac{w' + w'_f}{w} \left[ \frac{p_s A}{\pi(r_o^2 - r_i^2)} \right] + \left[ \frac{(p_h + p_{rot})r_o^2}{(r_o^2 - r_i^2)} \right] \quad (4.92)$$

$$S_{24} = (p_h + p_{rot}) \left( \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \right) \quad (4.93)$$

$$S_{34} = -(p_h + p_{rot}) \quad (4.94)$$

At each location, one must be certain to use the proper head height of filler and the proper inner and outer radii of the shell.

We must also account for the shear stresses which are most severe at location 1. For simplicity, we will assume a flat base and calculate the shear stress due to wall torsion.



**FIGURE 4.17**

Location of interest on a 105-mm M1 projectile.

Wherever these calculations are done on the shell, the proper  $I_{zz}$  and the proper inner and outer diameters must be used.

$$\tau_{11} = \frac{(p_s - p_h)\pi r_i^2}{2\pi r_i t} = \frac{(p_s - p_h)r_i}{2t} \quad (4.95)$$

$$\tau_{22}, \tau_{23}, \tau_{24} = \frac{F'_T}{\frac{\pi}{4}(d_o^2 - d_i^2)} = \frac{64I'_{zz}}{n(d_o + d_i)^3(d_o - d_i)} \frac{p_s A}{w} \quad (4.96)$$

A typical loading of the shell using known weights, pressures, and acceleration is shown in Figure 4.17 and Table 4.2.

The common practice currently used in projectile design is to dispense with the hand calculations and go right to a finite element analysis. While this is usually very accurate and saves a good deal of time, there are instances when one would like to check the answers through a hand calculation. Let us examine one location on this 105-mm M1 HE projectile fired from an M2A2 cannon at 145°F.

Projectile Data:

Shell material: HF-1 Steel

- Density—0.283 lbm/in.<sup>3</sup>
- Projectile OD—4.10 in.
- Projectile ID (average)—2.95 in.
- Projectile effective (including friction) mass (fuzed)—42 lb

**TABLE 4.2**

Typical Values for Use in an HE Projectile Design

Component	Weight (lbm)	Loads
Fuze	2.1	Breech pressure (psi)
Body	34.0	Spin rate—maximum $p$ (Hz)
Rotating band	0.4	Base pressure (psi)
Filler (TNT)	5.5	Acceleration (g/s)
Total	42.0	Angular acceleration (rad/s <sup>2</sup> )

- Projectile base intrusion into cartridge case—85.43 in.<sup>3</sup>
- $I_{zz}$ —80.24 lbm-in.<sup>2</sup>

Fill material: TNT

- Density—0.036 lbm/in.<sup>3</sup>
- Total length of explosive column—13.44 in.
- $I_{zz}$ —5.17 lbm-in.<sup>2</sup>
- Average fill cross-sectional area—6.49 in.<sup>2</sup>
- Fill surface area—124 in.<sup>2</sup>

The M1 projectile fired from our cannon is depicted in Figure 4.17. The properties of the section ahead of the location of interest are provided in Figure 4.17. We shall determine the stress tensor at the location shown. We shall assume the projectile obturates perfectly and that there is no friction between the projectile and the tube.

To begin, we should always draw a free-body diagram of an infinitesimal element at the point of interest.

Let us look at the hoop direction first. We shall use Equation 4.51.

$$\sigma_{\theta\theta} = \frac{1}{(r_o^2 - r_i^2)} \left[ p_i r_i^2 - p_o r_o^2 - \frac{r_i^2 r_o^2 (p_o - p_i)}{r^2} \right] \quad (4.97)$$



In this case,  $r = r_i$  and  $p_o = 0$  so we can write

$$\sigma_{\theta\theta} = \frac{1}{(r_o^2 - r_i^2)} [p_i(r_i^2 + r_o^2)] \quad (4.98)$$

The internal pressure is found through our equivalent pressure technique above.

$$p_{\text{rot}} = \frac{\omega_{P_{\text{max}}}^2}{3r_i} [\rho(r_o^3 - r_i^3) + \rho_{\text{fill}}r_i^3]$$

$$p_{\text{rot}} = \frac{(82.4) \left[ \frac{\text{rev}}{\text{s}} \right]^2 (2\pi)^2 \left[ \frac{\text{rad}}{\text{rev}} \right]^2}{(3)(1.475)[\text{in.}](12) \left[ \frac{\text{in.}}{\text{ft}} \right] (32.2) \left[ \frac{\text{lbf-ft}}{\text{lbf-s}^2} \right]} \times \left\{ (0.283) \left[ \frac{\text{lbf}}{\text{in.}^3} \right] [(2.05)^3 - (1.475)^3][\text{in.}^3] + (0.036) \left[ \frac{\text{lbf}}{\text{in.}^3} \right] (1.475)^3[\text{in.}^3] \right\}$$

$$p_{\text{rot}} = 258 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

For the hydrostatic component of the equivalent pressure, we know that

$$p_h = \rho_{\text{fill}} a_{P_{\text{max}}} h_6 \quad (4.100)$$

$$p_h = \frac{(0.036) \left[ \frac{\text{lbm}}{\text{in.}^3} \right] (382,300) \left[ \frac{\text{ft}}{\text{s}^2} \right] (8.189) [\text{in.}]}{(32.2) \left[ \frac{\text{lbm-ft}}{\text{lbf-s}^2} \right]}$$

$$p_h = 3500 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

The equivalent internal pressure is then

$$p_{\text{eq}} = p_{\text{rot}} + p_h \quad (4.101)$$

$$p_{\text{eq}} = p_i = 3758 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

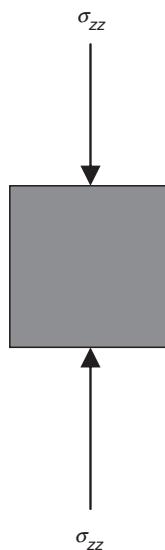
The hoop stress is then

$$\sigma_{\theta\theta} = \frac{1}{\left[ \left( \frac{4.10}{2} \right)^2 - \left( \frac{2.95}{2} \right)^2 \right] [\text{in.}^2]} \left\{ (3758) \left[ \frac{\text{lbf}}{\text{in.}^2} \right] \left[ \left( \frac{4.10}{2} \right)^2 + \left( \frac{2.95}{2} \right)^2 \right] [\text{in.}^2] \right\}$$

$$\sigma_{\theta\theta} = 11,830 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

Now let us look at the axial stress. This is the stress at the point due to two things: the axial inertia of all the material ahead of the cut setting back and the effective internal pressure caused by the rotation of the projectile and the hydrostatic compression of the fill material.

$$\sigma_{zz} = \frac{(p_i r_i^2 - p_o r_o^2)}{(r_o^2 - r_i^2)} - \frac{F_{\text{Axial}}}{\pi(r_o^2 - r_i^2)} \quad (4.102)$$



We shall use the radii given in the problem statement. We put negative sign in the above equation to denote compressive stress because only the axial component loads the inner wall in compression. The force acting on the section of interest due to setback is given by

$$F_{\text{Axial}} = m_6 a_{p_{\max}} \quad (4.103)$$

$$F_{\text{Axial}} = \frac{(382,300) \left[ \frac{\text{ft}}{\text{s}^2} \right] (17.15) [\text{lbfm}]}{(32.2) \left[ \frac{\text{lbf}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} \right]} = 203,600 [\text{lbf}]$$

Using this result, we have

$$\sigma_{zz} = \frac{(3758) \left[ \frac{\text{lbf}}{\text{in.}^2} \right] \left( \frac{2.95}{2} \right)^2 [\text{in.}^2]}{\left[ \left( \frac{4.10}{2} \right)^2 - \left( \frac{2.95}{2} \right)^2 \right] [\text{in.}^2]} - \frac{(203,000) [\text{lbf}]}{\frac{\pi}{4} [(4.10)^2 - (2.95)^2] [\text{in.}^2]}$$

$$\sigma_{zz} = -27,940 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

Many times we neglect the first term in equation above for conservatism. In the radial direction, we only have our equivalent pressure pushing radially outward and our location of interest is on the ID, so

$$\sigma_{rr} = -p_{\text{eq}} \quad (4.104)$$

$$\sigma_{rr} = -3758 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$



The angular acceleration will generate a torque through the rotating band that results in a shear stress in the plane normal to the axis of the projectile.



The torque on the projectile is also the opposite of the torque on the gun tube and comes directly from [Equation 4.77](#).

$$T_6 = I_{zz6} \alpha_{p_{\max}} \quad (4.105)$$

The moments of inertia were provided and we must use the angular acceleration calculated at peak pressure provided above. Now the torque comes about through

$$T_6 = (41.15)[\text{lbm-in.}^2](348,600) \left[ \frac{\text{rad}}{\text{s}^2} \right] \left( \frac{1}{32.2} \right) \left[ \frac{\text{lbf-s}^2}{\text{lbm-ft}} \right] \left( \frac{1}{12} \right) \left[ \frac{\text{ft}}{\text{in.}} \right]$$

$$T_6 = 37,130[\text{lbf-in.}]$$

The in-plane shear stress is given by

$$\tau = \frac{Tr}{J} \quad (4.106)$$

Then we have

$$\tau_{z\theta} = \frac{(37,130)[\text{lbf-in.}] \left( \frac{2.95}{2} \right) [\text{in.}]}{(108.3)[\text{in.}^4]} = 506 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

The shear stress caused by the rotation is generated by the shell trying to spin up the explosive fill. The torque on the explosive fill is determined through

$$T_{\text{fill}} = I_{zz\text{fill}} \alpha_{p_{\max}} \quad (4.107)$$

$$T_{\text{fill}} = (5.17)[\text{lbm-in.}^2](348,600) \left[ \frac{\text{rad}}{\text{s}^2} \right] \left( \frac{1}{32.2} \right) \left[ \frac{\text{lbf-s}^2}{\text{lbm-ft}} \right] \left( \frac{1}{12} \right) \left[ \frac{\text{ft}}{\text{in.}} \right]$$

$$T_{\text{fill}} = 4644[\text{lbf-in.}]$$

This generates a force at the internal radius of

$$F_{\text{fill}} = \frac{T_{\text{fill}}}{r_i} \quad (4.108)$$

$$F_{\text{fill}} = \frac{(4644)[\text{lbf-in.}]}{\left( \frac{2.95}{2} \right) [\text{in.}]} = 3162[\text{lbf}]$$

Smearing this over the entire internal surface area gives us

$$\tau_{r\theta} = \frac{(3162)[\text{lbf}]}{(124)[\text{in.}^2]} = 25.5 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

The axial shear is approximated as a worst case by calculating the hydrostatic pressure at the bottom of the explosive column, transforming it into a force, and smearing that force over the entire internal cavity area. We know the entire explosive column height is

$$h = 13.44[\text{in.}]$$

Then the peak hydrostatic pressure of the fill is

$$p_h = \rho_{\text{fill}} a_{p_{\max}} h \quad (4.109)$$

$$p_h = \frac{(0.036) \left[ \frac{\text{lbm}}{\text{in.}^3} \right] (382,300) \left[ \frac{\text{ft}}{\text{s}^2} \right] (13.44)[\text{in.}]}{(32.2) \left[ \frac{\text{lbm-ft}}{\text{lbf-s}^2} \right]}$$

$$p_h = 185,000 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

Calculating this pressure over the average cross-sectional area of the projectile, we obtain

$$F_{\text{base}} = \frac{p_h}{A_{\text{avg}_{\text{fill}}}}$$

$$F_{\text{Axial}} = \frac{(185,000) \left[ \frac{\text{lbf}}{\text{in.}^2} \right]}{(6.49) [\text{in.}^2]} = 28,500 [\text{lbf}]$$

Now this force smeared over the interior surface area will yield the stress

$$\tau_{rz} = \frac{-F_{\text{Axial}}}{A_{\text{fill}}} = -\frac{(28,500) [\text{lbf}]}{(124) [\text{in.}^2]} = -230 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

We can now write our stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -3758 & 25.5 & -230 \\ 25.5 & 11,830 & 506 \\ -230 & 506 & -36,010 \end{bmatrix} \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

It must be noted that these equations assumed that there were no other forces acting on the projectile. For instance, in some projectiles with poorly designed rotating bands, leaking of the propellant gases (known as blow-by) causes the exterior of the projectile to be pressurized. This load must be considered because it has been known to collapse projectiles in development. Another point is that, while it is common to check a projectile at peak acceleration, the spin rate at this location is not a maximum. Maximum spin occurs at the exit of the muzzle of the weapon where the velocity is the highest. It is always good practice to check a projectile for maximum spin with no axial acceleration to simulate this.

#### Problem 4

A high explosive projectile is to be designed for a 155-mm cannon using a  $\frac{1}{2}$  in. thick steel wall with TNT as the filler material. Assume the shell and filler are a cylinder 0.75 m in length. It is to be capable of surviving a worn-tube torsional impulse (angular acceleration) of 440,000 rad/s<sup>2</sup>.

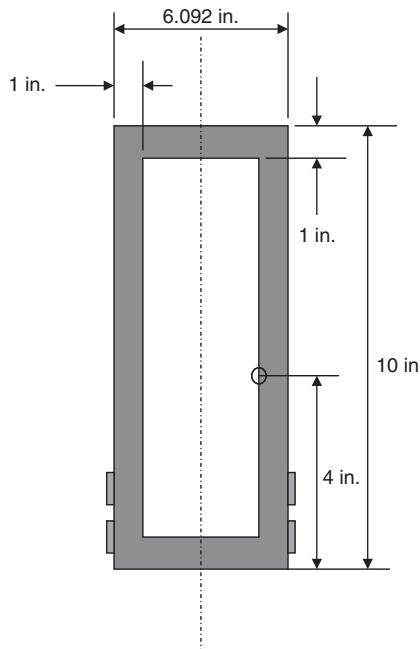
1. Derive the expression to calculate the torque on the projectile that achieves this acceleration if the torque is applied at the OD of the shell.
2. Calculate the value of the torque assuming the density of steel is 0.283 lbm/in.<sup>3</sup> and TNT is 0.060 lbm/in.<sup>3</sup>

*Hint:* Start from  $F_T = ma_T$

*Answer:* 1.  $T = M_{\text{WALL}} + M_{\text{FILL}} = \frac{1}{2}\pi\alpha l[\rho(r_o^4 - r_i^4) + \rho_{\text{FILL}}r_i^4]$ , 2.  $T = 796,600$  [lbf-in.]

### Problem 5

To participate in a failure investigation of an explosive, someone asks you to look at their design of a cylinder that was supposed to hold the explosive during a 155-mm Howitzer launch. Assume the explosive sticks completely to the interior wall. The firing conditions at the time of the failure were as follows:



$$\text{Axial acceleration} = 10,000 \text{ g}$$

$$\text{Angular acceleration} = 300,000 \text{ rad/s}^2$$

$$\text{Angular velocity} = 100 \text{ Hz}$$

The projectile was as shown below:

The wall is AISI 4140

$$E = 30 \times 10^6 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$\nu = 0.29$$

$$\rho = 0.283 \left[ \frac{\text{lbfm}}{\text{in.}^3} \right]$$

The explosive is Composition B

$$\rho_{\text{fill}} = 0.71 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

Write the stress tensor for a point on the inside diameter, 4 in. from the base

*Answer:*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -2265 & 177 & -266 \\ 177 & 5564 & -9620 \\ -266 & -9620 & -19,307 \end{bmatrix} \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

### Problem 6

A 155-mm projectile is fired from a tube with a 1 in 20 twist. Its muzzle velocity is 1000 m/s. What is the spin rate at the muzzle in Hz?

Answer: 322.6[Hz]

### Problem 7

It is requested that a brass slip ring be constructed for a spin test fixture to allow electrical signals to be passed (although real noisy) to some instrumentation. The design requirements are for the ring to have an ID of 4 in., a length of 2 in., and be capable of supporting itself during a 150-Hz spin test. How thick does the ring have to be? The properties of brass are as follows: Yield strength of 15,000 psi and density of approximately 0.32 lbm/in.<sup>3</sup>

Answer: 1/4 in. thickness will work but it can be thinner

## 4.11 Buttress Thread Design

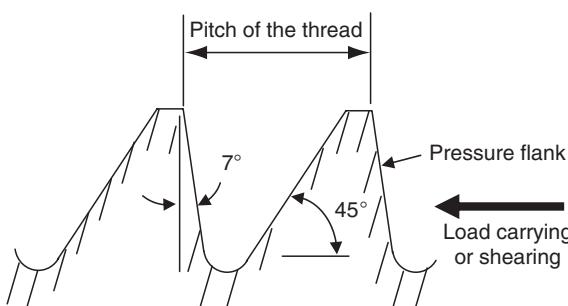
There are a variety of instances where a buttress thread form is the desired means of transmitting loads between mating components. In some instances, the thread form is not the usual continuous spiral associated with a normal thread, but a series of discontinuous grooves that exhibit the cross-sectional form of the buttress. In this section, we will discuss a true thread with lead-ins and partial thread shapes, but we will assume that the basic analysis will apply to buttress grooves as well.

Buttress threads are designed to maximize the load carrying capability in one direction of a threaded joint. There are many variations on such threads but on ammunition components we predominantly use threads with a pressure flank angle (described later) of 7° as shown in Figure 4.18. Thread callouts on drawings usually appear, for example

2.750-4UNC-2A LH Buttress

The meanings of these callouts are as follows:

- First number is the major diameter of the thread (here it is in inches).
- Second number is how many threads per inch.
- The letters are the thread form callout (UNC = Unified National Coarse).
- The last number is the class of fit of the thread related to clearances in the engagement (3 is the tightest fit, 1 the loosest).
- The last letter determines whether the thread is male (A) or female (B) (mnemonic – A = Adam = male).
- LH means left handed (there will be no callout if the threads are right-hand twist or if the thread is a groove and not a continuous spiral).



**FIGURE 4.18**  
Depiction of a standard buttress thread.

Thread nomenclature of relevance is as follows:

- Major diameter is the largest diameter of the thread form.
- Minor diameter is the smallest diameter of the thread form.
- Pitch diameter is the diameter where there is 1/2 metal and 1/2 air.

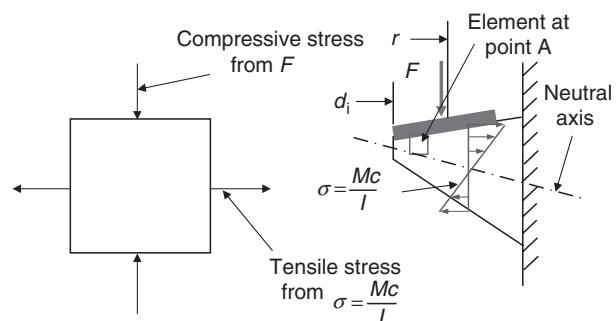
We use buttress threads for several reasons: most important is to improve the directional loading characteristics of the thread; also to allow for a more repeatable, controllable shear during an expulsion event, i.e., if we want the thread to intentionally and controllably fail allowing separation of the components; and to prevent thread slip in joints with fine threads or threads on thin shell walls. If thread slip occurs, the threads can either dilate or contract elastically and the joint can pop apart with little or no apparent damage to the threads.

When we design for strength, we typically calculate the strength based on the shear area at the pitch diameter in the weaker material. This, of course, translates to half the length of engagement of the threads. This is acceptable because we usually use conservative properties and add a safety factor to account for material variations and tolerances. We must always base our calculations on the weaker material if the design is to be robust. When designing to actually fail the threads, however, we need to be more exact in our analysis and take everything such as actual material property variation and tolerancing into account or our answers will be wrong.

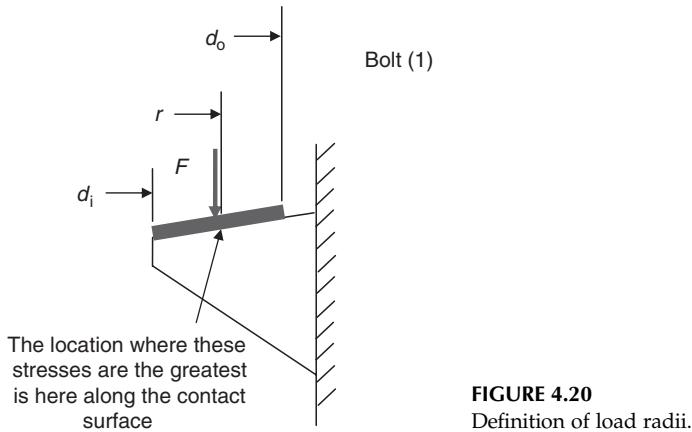
We will proceed in this analysis in meticulous detail, initially, as a cantilevered beam subjected to compressive and tensile stresses caused by contact forces and bending moments. This technique was first developed during the U.S. Army's sense and destroy armor (SADARM) program by Dan Pangburn of Aerojet Corporation [5] and has been used by the U.S. Army.

We consider the thread form as a short, tapered, cantilever beam and assume that failure will occur as a result of a combination of stresses and that combined bending and compressive stress precipitate the failure. This is depicted in Figure 4.19. If we examine this figure, we see that the distributed force,  $F$ , causes our beam to bend in the classical sense with the loaded flank in tension and the unloaded flank in compression about the neutral axis. We have separated an element of material out from point A in the figure. The free-body diagram of this element shows that the bending of the beam puts it in tension, while the loading on the pressure flank puts it in compression. It is this combined load that will cause failure of the material.

If we were analyzing this in a finite element code, the bending and compression would cause combined stresses and the part would fail by one of the failure criteria that were discussed earlier. However, in this case, we will use the maximum shear criteria to check for failure at some radius in the thread and will also check the load at which failure occurs



**FIGURE 4.19**  
Depiction of a standard buttress thread.



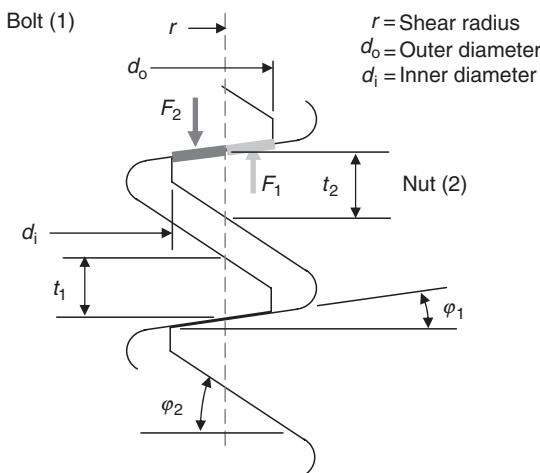
**FIGURE 4.20**  
Definition of load radii.

with the von Mises criteria at the thread roots,  $d_i$  on the male thread and  $d_o$  on the female thread. These are the diameters of the loading (i.e., the mating thread contact areas) as depicted in Figure 4.20.

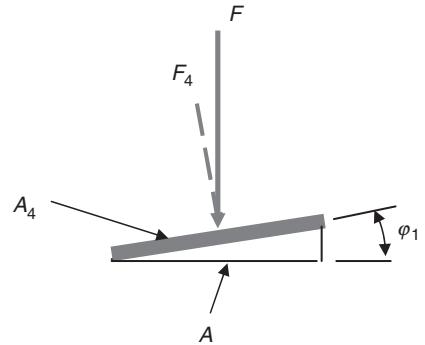
For simplicity, we shall call the male thread the “bolt” (subscript 1) and the female thread the “nut” (subscript 2). The loading is further described by Figure 4.21. In this figure, the radius,  $r$ , is the plane at which the threads will shear.

If we assume the contact is frictionless, the average normal stress is simply the total axial force,  $F$ , divided by the projected area,  $A$ . We have assumed that the normal stress is constant over the contact area. This gives us a negative value because the stress is compressive. Figure 4.22 shows the configuration where the normal force has been termed  $F_4$  and the thread area is  $A_4$ . Since an axial loading is what shears the threads, we need to project the components of this force along the axis of the projectile (i.e., rotate through the angle,  $\phi_1$ ). This allows us to express the stress as

$$\sigma_N = \frac{-F_4}{A_4} = \frac{-\frac{F}{\cos \phi_1}}{\frac{A}{\cos \phi_1}} = -\frac{F}{A} = \sigma_v \quad (4.112)$$



**FIGURE 4.21**  
Loading diagram of buttress threads.



**FIGURE 4.22**  
Loading of a thread surface.

Here  $\sigma_N$  and  $\sigma_v$  are the normal and axial stresses, respectively. By substituting the area,  $A$ , we get

$$\sigma_v = -\frac{F}{\frac{\pi}{4}(d_o^2 - d_i^2)} \quad (4.113)$$

If we assume that failure takes place at a radius,  $r$ , yet to be determined, the bearing force on the external thread (bolt) that produces bending in the thread is

$$F_1 = -\pi\sigma_v \left[ \left(\frac{d_o}{2}\right)^2 - r^2 \right] \quad (4.114)$$

Similarly, the force that produces bending in the internal thread (nut) is

$$F_2 = -\pi\sigma_v \left[ r^2 - \left(\frac{d_i}{2}\right)^2 \right] \quad (4.115)$$

Now the pitch diameter is defined as the location where the thickness of the thread is one-half the thread pitch. Since thread failure occurs at an assumed radius,  $r$ , we need to define the thicknesses of both the male and female threads at this location.

First, recall that the thread pitch is  $p$  and then define  $d_{pf}$  as the internal (female) thread pitch diameter and  $d_{pm}$  as the external (male) thread pitch diameter. Then  $t_1$  and  $t_2$  from our earlier diagram can be expressed as follows:

$$t_1 = \frac{p}{2} - \left(r - \frac{d_{pm}}{2}\right)(\tan \phi_1 + \tan \phi_2) \quad (4.116)$$

$$t_2 = \frac{p}{2} - \left(\frac{d_{pf}}{2} - r\right)(\tan \phi_1 + \tan \phi_2) \quad (4.117)$$

Then the bending stress can be calculated from simple beam theory as

$$\sigma = \frac{Mc}{I} = \frac{M\frac{t}{2}}{\frac{1}{12}(2\pi r)t^3} = \frac{3M}{\pi r t^2} \quad (4.118)$$

Here  $c$  is the distance from the point of interest,  $r$ , to the neutral (bending) axis and  $I$  is the area moment of inertia of the cross section. The bending stress in the external (male) thread is then

$$\sigma_1 = \frac{3F_1 \left( \frac{d_o}{2} - r \right)}{2\pi r t_1^2} \quad (4.119)$$

Similarly, we can show that the bending stress in the internal (female) thread is

$$\sigma_2 = \frac{3F_2 \left( r - \frac{d_i}{2} \right)}{2\pi r t_2^2} \quad (4.120)$$

In considering the failure criteria, we shall assume that the maximum shear stress in the material must not exceed 0.6 times the material strength in a tensile test. We will use the yield strength as this material strength because at that point in failure the geometry of the part is changing. Experience has shown that once this begins to happen the part is in the process of failing anyway and will not recover.

In a state of combined loading, the maximum shear stress can be found from

$$\tau_{\max} = \frac{1}{2} |\sigma_{\max} + \sigma_{\min}| \quad (4.121)$$

This averaging can be shown to be

$$\tau_{\max} = \frac{\sigma - \sigma_N}{2} = 0.6Y \quad (4.122)$$

Here we are reminded that  $\sigma_N$  and  $\sigma_v$  are compressive therefore negative numbers and  $Y$  is the yield stress in tension. The equivalent stress at failure in the male thread is then

$$Y_1 = \frac{\sigma_1 - \sigma_v}{1.2} \quad (4.123)$$

and in the female thread it is

$$Y_2 = \frac{\sigma_2 - \sigma_v}{1.2} \quad (4.124)$$

In these equations,  $Y_1$  and  $Y_2$  are the yield stress in the male and female threads, respectively.

We will now combine Equations 4.123 and 4.119 as well as Equations 4.124 and 4.120 to eliminate  $\sigma_1$  and  $\sigma_2$ , respectively. This yields

$$Y_1 = 1.25F_1 \frac{\frac{1}{2}d_o - r}{\pi r t_1^2} - \frac{\sigma_v}{1.2} \quad (4.125)$$

and

$$Y_2 = 1.25F_2 \frac{r - \frac{1}{2}d_i}{\pi r t_2^2} - \frac{\sigma_v}{1.2} \quad (4.126)$$

We now combine Equations 4.125 and 4.116 as well as Equations 4.126 and 4.117 to eliminate the tractions,  $t_1$  and  $t_2$ , respectively. This yields

$$Y_1 = 1.25F_1 \frac{\frac{1}{2}d_o - r}{\pi r \left[ \frac{1}{2}p - \left( r - \frac{1}{2}d_{pm} \right) (\tan \phi_1 + \tan \phi_2) \right]^2} - \frac{\sigma_v}{1.2} \quad (4.127)$$

and

$$Y_2 = 1.25F_2 \frac{r - \frac{1}{2}d_i}{\pi r \left[ \frac{1}{2}p - \left( \frac{1}{2}d_{pf} - r \right) (\tan \phi_1 + \tan \phi_2) \right]^2} - \frac{\sigma_v}{1.2} \quad (4.128)$$

We will now insert Equation 4.114 into Equation 4.127 and Equation 4.115 into Equation 4.128 to eliminate  $F_1$  and  $F_2$ , respectively. This yields

$$Y_1 = -0.3125\sigma_v(d_o^2 - 4r^2) \frac{\frac{1}{2}d_o - r}{r \left[ \frac{1}{2}p - \left( r - \frac{1}{2}d_{pm} \right) (\tan \phi_1 + \tan \phi_2) \right]^2} - \frac{\sigma_v}{1.2} \quad (4.129)$$

and

$$Y_2 = -0.3125\sigma_v(4r^2 - d_i) \frac{r - \frac{1}{2}d_i}{r \left[ \frac{1}{2}p - \left( \frac{1}{2}d_{pf} - r \right) (\tan \phi_1 + \tan \phi_2) \right]^2} - \frac{\sigma_v}{1.2} \quad (4.130)$$

Now we must solve Equations 4.129 and 4.130 in terms of  $\sigma_v$ . The first of these is

$$\sigma_v = \frac{-Y_1}{G_3 + G_2 + G_1 + G_0 + \frac{1}{1.2}} \quad (4.131)$$

where

$$G_3 = \frac{0.15625d_o^3}{r(0.5p - r \tan \phi_1 - r \tan \phi_2 + 0.5d_{pm} \tan \phi_1 + 0.5d_{pm} \tan \phi_2)^2} \quad (4.132)$$

$$G_2 = \frac{-0.3125d_o^2}{(0.5p - r \tan \phi_1 - r \tan \phi_2 + 0.5d_{pm} \tan \phi_1 + 0.5d_{pm} \tan \phi_2)^2} \quad (4.133)$$

$$G_1 = \frac{-0.625rd_o}{(0.5p - r \tan \phi_1 - r \tan \phi_2 + 0.5d_{pm} \tan \phi_1 + 0.5d_{pm} \tan \phi_2)^2} \quad (4.134)$$

$$G_0 = \frac{1.25r^2}{(0.5p - r \tan \phi_1 - r \tan \phi_2 + 0.5d_{\text{pm}} \tan \phi_1 + 0.5d_{\text{pm}} \tan \phi_2)^2} \quad (4.135)$$

The second equation is

$$\sigma_v = \frac{-Y_2}{H_3 + H_2 + H_1 + H_0 + \frac{1}{1.2}} \quad (4.136)$$

where

$$H_3 = \frac{0.15625d_i^3}{r(0.5p + r \tan \phi_1 + r \tan \phi_2 - 0.5d_{\text{pf}} \tan \phi_1 - 0.5d_{\text{pf}} \tan \phi_2)^2} \quad (4.137)$$

$$H_2 = \frac{-0.3125d_i^2}{(0.5p + r \tan \phi_1 + r \tan \phi_2 - 0.5d_{\text{pf}} \tan \phi_1 - 0.5d_{\text{pf}} \tan \phi_2)^2} \quad (4.138)$$

$$H_1 = \frac{-0.625rd_i}{(0.5p + r \tan \phi_1 + r \tan \phi_2 - 0.5d_{\text{pf}} \tan \phi_1 - 0.5d_{\text{pf}} \tan \phi_2)^2} \quad (4.139)$$

$$H_0 = \frac{1.25r^2}{(0.5p + r \tan \phi_1 + r \tan \phi_2 - 0.5d_{\text{pf}} \tan \phi_1 - 0.5d_{\text{pf}} \tan \phi_2)^2} \quad (4.140)$$

We now solve [Equation 4.113](#) for  $F$  and we get

$$F = \frac{\pi}{4}\sigma_v(d_o^2 - d_i^2) \quad (4.141)$$

Substitution of [Equation 4.131](#) for  $\sigma_v$  yields (for a full thread on the bolt)

$$F = \frac{\pi}{4}(d_o^2 - d_i^2) \frac{-Y_1}{G_3 + G_2 + G_1 + G_0 + \frac{1}{1.2}} \quad (4.142)$$

We perform a similar operation with [Equation 4.136](#) giving us (for a full thread on the nut)

$$F = \frac{\pi}{4}(d_o^2 - d_i^2) \frac{-Y_2}{H_3 + H_2 + H_1 + H_0 + \frac{1}{1.2}} \quad (4.143)$$

[Equations 4.142](#) and [4.143](#) now contain only two unknowns,  $r$  and  $F$ . The procedure now involves solving both [Equations 4.142](#) and [4.143](#) and plotting the force,  $F$  versus  $r$ . The lowest value in either equation is then the force (and location) at which the joint will fail. It is recommended that these solutions be performed with the aid of a computerized numerical calculation program such as MathCAD.

Partial threads can have a significant effect on the failure strength of a joint. If the joint were designed to survive, it is generally best to ignore the additional strength afforded by partial threads and base the design margin on the calculation method above. When a joint is designed to fail, however, they must be accounted for unless sufficient margin is available in the expulsion system such that two additional threads may be added to the calculation, yet still be overcome with ease.

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## 4.12 Sabot Design

Sabots (French for wooden shoe) are used in both rifled and smoothbore guns to allow a standard weapon to fire a high density, streamlined sub-projectile whose diameter is much smaller than the bore, at a velocity higher than would normally be possible if the gun were sized to the sub-projectile's diameter. Discarding sabots have been in general use since the Second World War and are still popular. They are called "discarding sabots" since they are shed from the sub-projectile at the muzzle allowing it to fly unencumbered to the target.

As stated previously, velocity is proportional to the square root of the pressure achieved in the tube, the area of the bore, the length of travel, and inversely proportional to the square root of the projectile weight. In mathematical terms,

$$V \sim \sqrt{\frac{pAL}{w_p}} \quad (4.144)$$

We can see that if the area over which the pressure is applied is much greater than the area presented at the rear of the sub-projectile, a larger force would be applied to accelerate it than if it were fired at the same pressure from a bore of its own diameter. Furthermore, decreasing the launch weight of the as-fired assembly also increases the velocity. Therefore, we must design as light a sabot as feasible so that we can maintain a very dense, small diameter sub-projectile (usually an armor penetrator). The combination of the full bore area, a dense, streamlined sub-projectile, and a lightweight sabot has the overall effect of generating unusually high velocities, a characteristic essential for kinetic energy armor penetration.

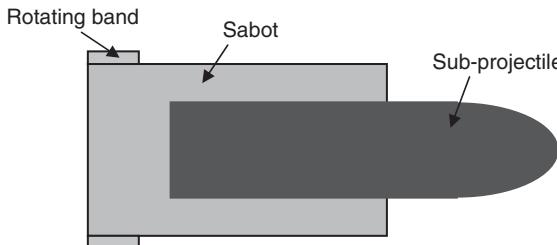
There are many requirements for a successful sabot:

- It must seal the propellant gases behind the projectile (obturate).
- It must support the sub-projectile during travel in the bore to provide stable motion (called providing a suitable wheelbase).
- It must transfer the pressure load from the propellant gases to the sub-projectile.
- It must completely discard at the muzzle of the weapon without interfering with the flight of the sub-projectile.
- The discarded sabot parts must also fall reliably within a danger area in front of the weapon so as not to injure troops nearby.
- It must be minimally parasitic, i.e., it must be as light as possible and remove as little energy from the sub-projectile as possible.

These are formidable requirements that necessitate great ingenuity on the part of the designers.

The problem has been solved in a variety of ways. In the 1950s, designers, chiefly British, used cup- or pot-type sabots to launch armor-penetrating, discarding-sabot (APDS) sub-projectiles ([Figure 4.23](#)). The guns from which these munitions were fired were rifled to launch conventional full caliber, spin-stabilized rounds and so the sub-projectiles of the APDS rounds were spin-stabilized too. Such armor defeating munitions were highly effective against the tank armor of the times and pot-type, sabotized, kinetic energy penetrators were adopted in tank cannon around the world.

Tank armor changed in the 1960s and became more difficult to penetrate with the tungsten carbide cores of the sub-projectiles in use. Initially, incremental changes were



**FIGURE 4.23**

Simplified diagram of an armor-piercing, discarding-sabot (APDS) projectile.

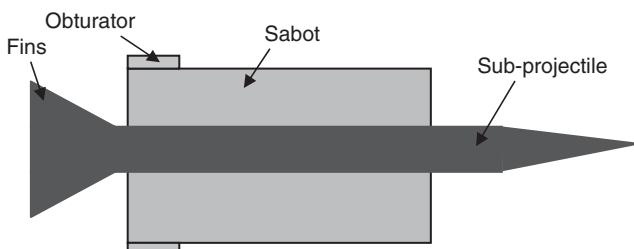
made in the material of the core (sintered tungsten was used instead of sintered tungsten carbide), but it was eventually realized that longer, smaller diameter, high-density penetrators were the answer. There are physical limits to the degree of sub-calibering practical in spin-stabilized projectiles: the spin required for flight stability increases as the square of the ratio of bore to sub-projectile for conventionally shaped projectiles and it becomes nearly impossible to spin-stabilize very long projectiles. Rifling twists were increased to attempt to accommodate the APDS rounds; in one case, a 1:12 twist was tried when the normal twist would have been 1:40. In the end, APDS designs were abandoned in favor of very long, fin-stabilized penetrators (APFSDS) that used a radically different type of sabot (Figure 4.24). The guns too were changed to smoothbores although to preserve older weapons in use, designers learned how to make fin-stabilized munitions firable in rifled guns as well.

The basic type of sabot used with long-rod, fin-stabilized penetrators is the ring with its subvarieties: base pull, double ramp, and saddle sabots. Whereas, pot sabots were essentially discarded rearward as a unit, ring sabots are segmented into three or more sections and discard radially outward at the muzzle to clear the fins that are larger in diameter than the rod. The finned sub-projectile is frequently imparted with a slow spin to average out unavoidable manufacturing asymmetries during flight that could cause trajectory drift. This type of munition is now in the arsenals of all nations.

The design of the ring sabot begins with the stress analysis of the shear traction between the sabot inner diameter and the penetrator outer diameter. This analysis is crucial for determining the mass of the ring and thus the parasitic weight of the sabot. We will follow the work of Drysdale [6] throughout this development. The essential parameters of the computation are shown in Figure 4.25.

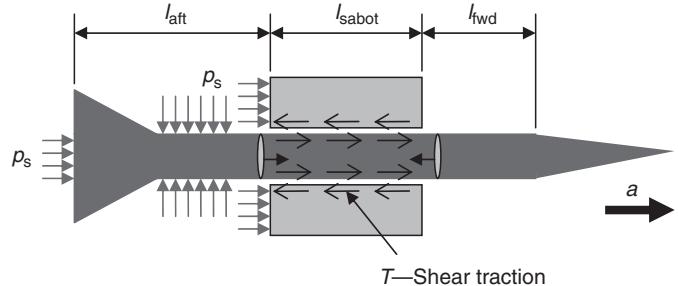
From this free-body diagram, we can infer that

$$T = p_s(A - A_p) - m_{\text{sabot}} a \quad (4.145)$$



**FIGURE 4.24**

Simplified diagram of an armor-piercing, fin-stabilized, discarding-sabot (APFSDS) projectile.



**FIGURE 4.25**  
Free-body diagram for rings and rods.

$T$ —Shear traction

A reasonable estimate for the masses where the symbols are as follows:

$$m_{\text{sabot}} = \frac{1}{2} m_{\text{sub-projectile}} \quad (4.146)$$

where

$T$ —Total shear traction force

$A$ —Bore area

$A_p$ —Area of the penetrator cross section

$m_{\text{sabot}}$ —Mass of the sabot

$m_{\text{sub-projectile}}$ —Mass of the sub-projectile

$a$ —Projectile acceleration

$p_s$ —Pressure on the base of the shot (note that the net pressure on the fins is zero)

$\sigma_1$ —Axial stress on the penetrator

Because the sabot needs to be as light as possible, the material is usually much weaker than the penetrator; thus, the sabot length depends mostly on the sabot material. If the penetrator were weaker for some reason, the sabot length would depend upon that material. Thus, we can write for the surface traction

$$T_{\text{allow}} = \frac{\pi}{2} d_p l_{\text{sabot}} \tau_{\text{allow}} \quad (4.147)$$

where

$d_p$ —Diameter of the penetrator or sub-projectile

$T_{\text{allow}}$ —Allowable traction force

$\tau_{\text{allow}}$ —Maximum shear stress allowed in the weaker material

The shear traction is usually transmitted through matching grooves or threads. Analysis of these surfaces can be rather complicated but is similar to standard or buttress thread design practice. Given no actual data on the allowable shear stress in the material, we can use the following formulas based on the Tresca or the von Mises yield criteria:

$$\tau_{\text{allow}} = \frac{\sigma_e}{2} \quad (4.148)$$

By the Tresca criteria or

$$\tau_{\text{allow}} = \frac{1.155 \sigma_e}{2} = 0.577 \sigma_e \quad (4.149)$$

by the von Mises criteria. In both of these expressions,  $\sigma_e$  is the equivalent stress as discussed in [Section 4.2](#). Thus, the allowable surface traction can be stated as

$$T_{\text{allow}} = K\pi d_p l_{\text{sabot}} \sigma_e \quad (4.150)$$

where  $K$  is either 0.25 or 0.2885 dependent upon the failure criteria.

If we substitute Equation 4.150 into [Equation 4.145](#), we can solve for the proper sabot length

$$K\pi d_p l_{\text{sabot}} \sigma_e = p_s(A - A_p) - m_{\text{sabot}} a \quad (4.151)$$

and

$$l_{\text{sabot}} = \frac{p_s}{K\pi d_p \sigma_e} \left( \frac{A}{A_p} - 1 \right) A_p - \frac{m_{\text{sabot}} a}{K\pi d_p \sigma_e} \quad (4.152)$$

But

$$A_p = \frac{\pi}{4} d_p^2 \quad (4.153)$$

Then

$$l_{\text{sabot}} = \frac{p_s d_p}{4K\sigma_e} \left( \frac{A}{A_p} - 1 \right) - \frac{m_{\text{sabot}} a}{K\pi d_p \sigma_e} \quad (4.154)$$

Now, by our earlier assumption ([Equation 4.146](#))

$$p_s A = ma = (m_{\text{sabot}} + m_{\text{sub-projectile}})a = 3m_{\text{sabot}} a \quad (4.155)$$

Then

$$l_{\text{sabot}} = \frac{p_s d_p}{4K\sigma_e} \left( \frac{A}{A_p} - 1 \right) - \frac{p_s a}{3K\pi d_p \sigma_e} \quad (4.156)$$

Multiplying and dividing the second RHS term by  $A_p$  and simplifying, we get

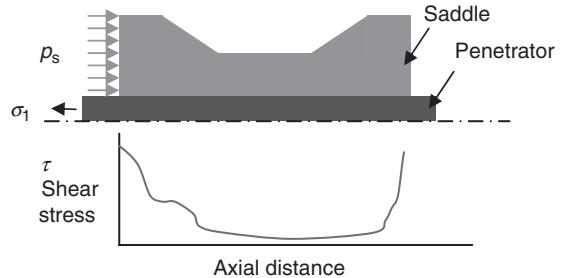
$$l_{\text{sabot}} = \frac{p_s d_p}{4K\sigma_e} \left( \frac{A}{A_p} - 1 \right) - \frac{p_s d_p}{12K\sigma_e} \frac{A}{A_p} \quad (4.157)$$

More generally, if the mass of the sabot is not half of the sub-projectile mass, then we must use Equation 4.154 to determine the proper length.

The shape of ring sabots evolved over time from quite heavy designs to highly efficient ones. Early sabots were saddle shaped ([Figure 4.26](#)). These had points of high shear stress concentrations near the ends.

These sabots had an excellent wheel base (the distance between the forward and aft bourrelets) which prevented balloting in the tube and provided good accuracy. The parasitic weight, however, was high and sufficiently high muzzle velocities were not attained.

Single- and double-ramp sabots have come into use because of the favorable weight reduction that can be obtained with this design. They utilize gun pressure to help clamp



**FIGURE 4.26**

Shear stress variation in a saddle-type sabot.

the sabot to the penetrator and have the added advantage of maintaining an almost constant shear stress between the sabot and the penetrator. The double-ramp sabot is shown in Figure 4.27.

Detailed studies have shown that a higher order (nonlinear) curved ramp yields a constant shear stress under load. The method of solution for finding the best shape of the sabot taper depends on a free-body analysis of the sabot and the penetrator. Figures 4.28 and 4.29 show differential elements of the sub-projectile and the sabot, respectively.

If we examine Figure 4.28, we see that the axial forces consist of the net internal stress,  $(d\sigma_{zp}/dz)\Delta z$ ; the inertial resistance to acceleration,  $\rho_p V_p a$ ; and the shear stress imparted by the sabot,  $\tau$ . Similarly, on the sabot, we have the net internal stress,  $(d\sigma_{zs}/dz)\Delta z$ ; the inertial resistance to acceleration,  $\rho_s V_s a$ ; the shear stress imparted by the sub-projectile,  $\tau$ ; and the component of pressure in the axial ( $z$ ) direction. We proceed by initially finding an expression for the volume of the sabot free body. Details of this derivation are found in Ref. [6]. The incremental volume of the sabot can be shown as follows:

$$V_s = \pi[R_s^2(z) - R_p^2]\Delta z \quad (4.158)$$

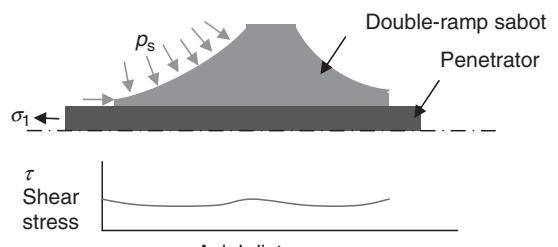
We then sum the forces on the sabot in the axial direction

$$\begin{aligned} & p_s \pi[R_s^2(z + \Delta z) - R_p^2]\Delta z - \sigma_{zs} \pi[R_s^2(z) - R_p^2] \\ & + \left( \sigma_{zs} + \frac{d\sigma_{zs}}{dz} \Delta z \right) \pi[R_s^2(z + \Delta z) R_p^2] - \rho_s V_s a - 2\pi R_p \tau \Delta z = 0 \end{aligned} \quad (4.159)$$

After collection of terms and simplification, we get

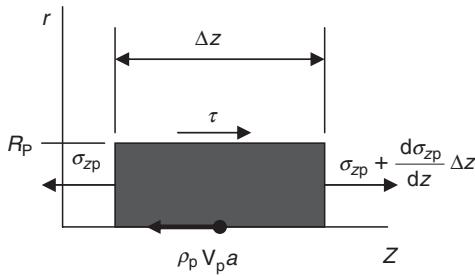
$$(p_s + \sigma_{zs}) \frac{dR_s^2}{dz} + \left( \frac{d\sigma_{zs}}{dz} - \rho_s a \right) [R_s^2(z) - R_p^2] - 2R_p \tau = 0 \quad (4.160)$$

Note here that  $R_s$  and  $\sigma_{zs}$  are functions of  $z$ .



**FIGURE 4.27**

Shear stress variation in a double ramp-type sabot.



**FIGURE 4.28**

Differential element in a sub-projectile showing forces acting.

Next we find  $\sigma_{zp}$  assuming it is linear in  $z$  through the expression

$$\sigma_{zp} = \frac{F}{A} = \frac{1}{\pi R_p^2} (\rho_p V_p a - 2\pi R_p \tau \Delta z) + \sigma_1 = \frac{1}{\pi R_p^2} (\rho_p \pi R_p^2 a - 2\pi R_p \tau) \Delta z + \sigma_1 \quad (4.161)$$

or

$$\sigma_{zp} = \left( \rho_p a - \frac{2\tau}{R_p} \right) \Delta z + \sigma_1 \quad (4.162)$$

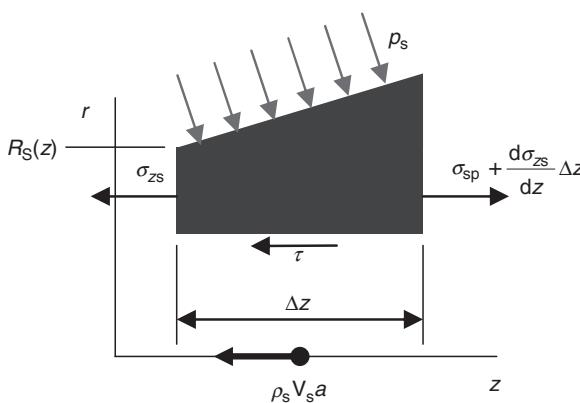
Here  $\sigma_1$  is the axial stress in the penetrator as depicted earlier. Now we need to relate  $\sigma_{zp}$  to  $\sigma_{zs}$  by applying the assumption of strain compatibility, i.e., the strain in the sabot equals the strain in the penetrator.

We then use the appropriate elastic moduli and Poisson's ratio in Hooke's law to relate the penetrator stresses to those in the sabot

$$\varepsilon_{zs} = \frac{1}{E_s} [\sigma_{zs} - \nu_s (\sigma_{rs} + \sigma_{\theta s})] = \varepsilon_{zp} = \frac{1}{E_p} [\sigma_{zp} - \nu_p (\sigma_{rp} + \sigma_{\theta p})] \quad (4.163)$$

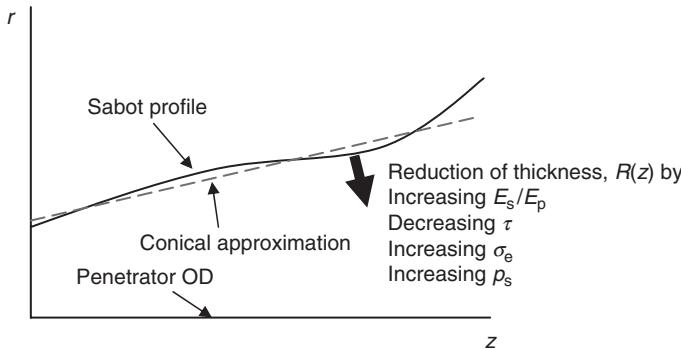
Thus,

$$\sigma_{zs} = \frac{E_s}{E_p} [\sigma_{zp} - \nu_p (\sigma_{rp} + \sigma_{\theta p})] + \nu_s (\sigma_{rs} + \sigma_{\theta s}) \quad (4.164)$$



**FIGURE 4.29**

Differential element in a sabot showing forces acting.



**FIGURE 4.30**

Sabot radial profile. (Source: Based on analysis from Drysdale, W.H., *Design of Kinetic Energy Projectiles for Structural Integrity*, Technical Report ARBRL-TR-02365, U.S. Army Ballistic Research Laboratory, Aberdeen, MD, September 1981.)

If we ignore the bimetallic nature of the components and assume that

$$\sigma_{rp} + \sigma_{\theta p} = \sigma_{rs} + \sigma_{\theta s} = -2p_s \quad (4.165)$$

Then Equation 4.164 becomes

$$\sigma_{zs} = \frac{E_s}{E_p} (\sigma_{zp} + 2\nu_p p_s) - 2\nu_s p_s \quad (4.166)$$

These assumptions allow integration of the differential equation for  $R(z)$  producing the profile in Figure 4.30 (solid curve).

Two of the basic types of sabots have been shown in Figures 4.26 and 4.27. The double ramp also incorporates a front air scoop to facilitate discard in the air stream as well as providing additional bourrelets riding surface in the tube.

A great deal of work on the effect of sabot design parameters has been accomplished at the U.S. Army research laboratory. A treatment of the effect of sabot stiffness on how clean a projectile launch is can be found in Ref. [7].

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## *Weapon Design Practice*

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This section discusses weapon design practice as it directly applies to interior ballistics. The design of gun systems is so complex that it is best dealt with as a text in its own right. We begin with an introduction to fatigue so that some understanding of the basic principles in gun design can be developed. We then proceed to discuss some introductory concepts in tube design, gun dynamics, and muzzle devices. The reader is directed to the references for a more in-depth treatment of these topics.

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### **5.1 Fatigue and Endurance**

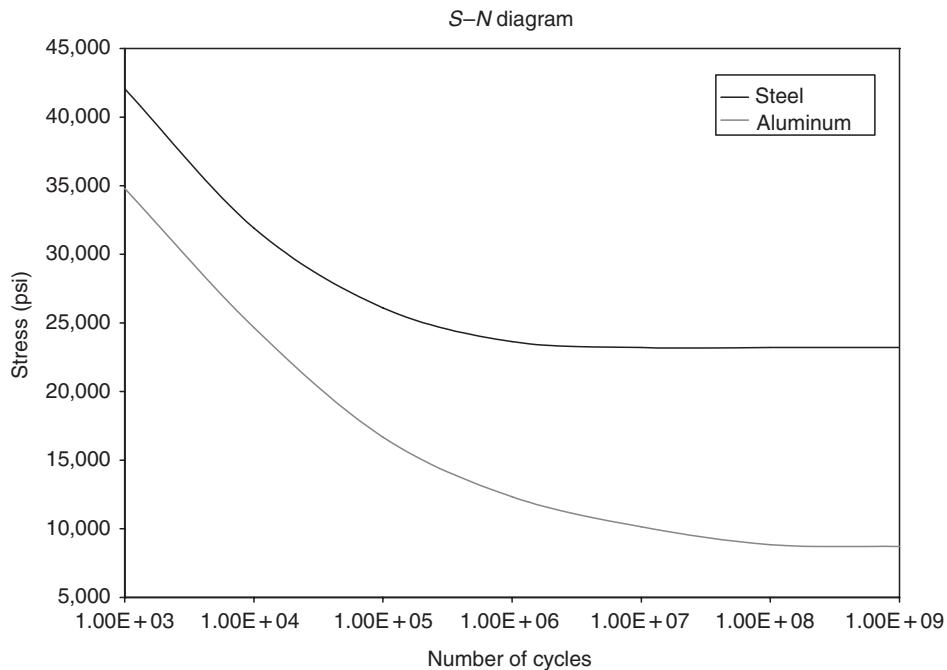
Many parts in civil and military service are subject to fatigue. Fatigue is the term used for a mechanical part that undergoes cyclic loading and fails suddenly. Unlike a component that is simply overstressed and fails because the yield or ultimate strength is exceeded, a part that is subject to fatigue failure has been subjected to many small loads that stress the component below the yield strength. Damage begins to accumulate through various mechanisms such as micro-crack growth or slipping along macroscopic boundaries. A simple example of fatigue is one where you take a metal paper clip and bend it 90°. After this first bend, the paperclip is still in one piece so the ultimate strength of the material was not exceeded (though it certainly has yielded). If one repeats this multiple times with the same paper clip, it will eventually break.\* This failure can occur even without yielding the material.

A projectile usually undergoes one cycle of loading so fatigue is normally not an issue. Gun tubes, however, undergo thousands of cycles and fatigue is a major consideration in their design. The U.S. design practice is to assure that a weapon shoots out before it fatigues out. What this means is that the weapon will become inaccurate because of wearing away of the rifling or the bore itself well before it fails in a sudden manner because of fatigue. This is determined by every maintenance crew by periodically checking the internal condition of the bore of the weapon. If the bore has worn away sufficiently, the tube is condemned. This condemnation is known to occur statistically after a certain number of rounds have been fired. The limit to the number of firings is compared to the design fatigue life of the weapon and, if the design was done correctly, there is sufficient margin remaining before a fatigue failure will develop.

The endurance of a material is the ability of the material to survive multiple cycles of loading. This ability of a material is depicted graphically in [Figure 5.1](#). This figure is called an *S-N* diagram. An *S-N* diagram plots the allowable stress in the material against the number of cycles required by the designer. For example, if the designer required 10,000

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\* This example was chosen by the author because it has been used so frequently by Dr. Jennifer Cordes of Picatinny Arsenal when she explains the nature of fatigue to new engineers or visitors.



**FIGURE 5.1**  
S-N diagram for steel and aluminum.

cycles for a particular design using steel, it would be necessary to keep the stress below approximately 31,000 psi.

Some materials have an endurance limit. An endurance limit is the stress below which the material can withstand an infinite number of load cycles. Figure 5.1 shows that for this particular steel, the endurance limit is around 24,000 psi. Aluminums are notorious for not having an endurance limit. This means that aluminum components always have a finite fatigue life expectancy.

There are many contributing factors to the endurance of a component. Three of these factors which we have already touched upon are the material of the part, the number of loading cycles, and the stress level of each load cycle. Others are the rate of loading, rate of load reversal, the surface finish of the component, and even confidence in the endurance data used to generate the S-N diagram. Every reference that deals with this subject has a different twist (no pun intended) to the governing equation. References [1] and [2] are excellent treatments of this behavior. A particularly simple approach is to define the fatigue strength of a material (i.e., the load that cannot be exceeded by any one cycle) as

$$S_n = S'_n C_R C_G C_S \quad (5.1)$$

Here  $S'_n$  is the stress in psi read from an S-N diagram for the desired number of cycles,  $C_R$  is a factor that is chosen by the designer based on the reliability required in the design,  $C_G$  is a factor that is based on the rapidity of load reversal and steepness of stress gradients in the component, and  $C_S$  is a factor that accounts for the surface finish. These factors effectively reduce the allowable stress in the part (they all should be  $\leq 1$ ). Unfortunately, they are all subject to interpretation and vary with each material and even from reference to reference.

Some references use additional factors as well. The best advice in the case of fatigue is for you to find a reference that has calculated fatigue in a component similar to the one you are designing and base your design on that data.

### Problem 1

It is desired to construct a 75-mm gun for a pressure of 43,000 psi. The chamber diameter has been chosen to be 3.1 in. If we use AISI 4340 steel with a yield strength ( $S_Y$ ) of 100,000 psi, determine the outer diameter (OD) of the weapon over the chamber. Assume that the tube is not autofrettaged and the endurance limit ( $S'_h$ ) for 4340 is  $0.875S_Y$  for the amount of cycles desired. Assume the following factors from our cyclic loading discussion:  $C_R = 0.93$ ,  $C_G = 0.95$ , and  $C_S = 0.99$ . Assume the chamber is open ended as a conservative measure.

*Answer:* 20-in. OD will just work

### Problem 2

A shotgun is to be modified so that it can be rigidly mounted to a vehicle. The recoil force is estimated to be 800 lbf. There are two failure points: a weld on the barrel and two 10-32 screws connecting the receiver to the barrel. If we assume that each point of failure (the two screws act together) must individually take the full load, determine how many firings can be achieved using the curve for steel provided in the text and the data below:

Both materials:  $C_R = 0.8$ ;  $C_G = 0.85$

Screws:  $C_S = 0.78$ ; shear area = 0.019 in.<sup>2</sup> each

Welds (1/8 in. fillet):  $C_S = 0.5$ ; shear area = 0.247 in.<sup>2</sup>

*Answer:* Screws will survive approximately 2000 cycles, welds will last an infinite number of cycles.

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## 5.2 Tube Design

In the discussion of the design of conventional projectile bodies that we completed earlier, many of the concepts we introduced are now applicable, with particular modifications, to the design of gun tubes. For example, the idea of safety margins has counterparts in the design of a gun tube, but where a projectile has to withstand a single cycle of applied stress, the gun tube must remain serviceable for many cycles at stress levels very much comparable to the fired projectile.

The gun tube designer is interested in determining the structure which has the minimum weight, which usually translates to a minimum radial dimension, consistent with safely firing a projectile. The projectile designer is usually interested in determining the projectile structure of minimum weight sufficient to meet safety, reliability, and, especially, effectiveness requirements. The projectile designer needs to know the maximum pressure on the base of the moving projectile during its time in the tube, known as the single base maximum pressure. Once this single pressure induced stress is accommodated, the designer can move on to other considerations. The tube designer, on the other hand, must know the maximum pressure exerted on the tube at every axial location in the bore as the projectile transits the tube. These are known as the station maximum pressures in tube design. We use the projectile and charge combination which applies the most stress to the weapon (usually this is the heaviest projectile and the biggest charge). These pressures

are applied over and over again as the tube is cycled with each shot fired, leading to the necessity to account for and predict the fatigue failure of the design.

Finite element analysis (FEA) methods are used less frequently in gun design than projectile design because FEA is a much more difficult method when used to predict fatigue failures. The reasons for this are that the gun launch phenomenon is highly transient, erosion of the weapon is impossible to predict at the present time, boundary conditions of a firing position change the dynamic response of the weapon, and in overall gun design, there are many different parts to consider. The “tried and true” hand calculation processes developed at the Watervliet, Frankford, and Picatinny Arsenals still yield excellent, reliable weapons. But FEA will become more important as the codes develop and weight of the weapon becomes more of an issue.

Another major consideration in tube design is the degradation of material strength with temperature. The repetitive firing of a weapon with propellants burning in the chamber and in the bore generates a large amount of heat. In tube artillery or tank cannons, the temperatures developed can become high enough to begin to affect the material properties in an adverse way. In rapid fire weapons particularly, it is absolutely critical that the degraded material strength properties be accounted for in tube and in chamber stress calculations.

There are several types of tube designs that may be encountered in service weapons: the monobloc tube is made from one piece of metal which is not the most efficient way to construct a tube; the jacketed tube which consists of separate layers or jackets built up as a composite structure; this type is mostly obsolete now, and is being replaced by a process called autofrettaging or self-jacketing; the quasi-two piece tube is formed by inserting a liner into an otherwise monobloc, pressure containing tube; this allows for a more resilient material for the projectile to ride against and helps with the wear of the tube; British warships used a now obsolete, wire-wrapped tube construction that was cheap to make, but quite inaccurate in use.

When we begin a design of a new tube, the interior ballistian computes the space-mean pressure-travel and pressure-time curves for the most stressful projectile expected to be fired at a temperature of 70°F. The maximum pressure of this curve gives the computed maximum pressure (CMP), which is the nominal pressure for the gun. However, because of the stochastic nature of a gun launch, the designer will add 2400 psi to the CMP. This is the rated maximum pressure (RMP) for the weapon. This pressure is one which cannot be exceeded by the average of the maximum pressures of a group of projectiles fired at 70°F.

$$\text{RMP} \left[ \frac{\text{lbf}}{\text{in.}^2} \right] = \text{CMP} \left[ \frac{\text{lbf}}{\text{in.}^2} \right] + 2400 \left[ \frac{\text{lbf}}{\text{in.}^2} \right] \quad (5.2)$$

After a statistically significant number of projectiles are fired out of the weapon, data is taken to validate the CMP. This experimentally determined number is the normal operating pressure (NOP) for the weapon and should replace the CMP as soon as it is available and accepted.

Under service conditions, many rounds will be fired at many different operating temperatures. We define the permissible individual maximum pressure (PIMP) as the pressure which cannot be exceeded by any individual round under any service condition.

In design terms, it is calculated as 15% over the RMP.

$$\text{PIMP} \left[ \frac{\text{lbf}}{\text{in.}^2} \right] = (1.15)\text{RMP} \left[ \frac{\text{lbf}}{\text{in.}^2} \right] \quad (5.3)$$

The permissible mean maximum pressure (PMMP) is the pressure that cannot be exceeded by the average of all rounds fired under any service condition.

From an analysis standpoint, we need to define a pressure at which enough stress is developed (assuming tube material at 70°F) at some point in the tube so that yielding occurs, i.e., the elastic limit of the material is reached. This is the elastic strength pressure (ESP) for the tube. At higher temperatures, we must also define an  $ESP_{hot}$  to account for material strength loss at temperature. A good example of how these concepts are applied can be found in Ref. [3].

When we examine the travel of the most stressful projectile down the tube, a point is reached,  $x_{max}$ , where the breech pressure is at maximum,  $p_{B\ max}$ . At this same instant, the pressure on the base of the projectile is also at a maximum (but, as we saw in the section on the Lagrange gradient, lower than the breech pressure) and will never increase beyond this value ( $p_{s\ max} < p_{B\ max}$ ). There is a pressure gradient at every point,  $x$ , between the breech of the weapon and the base of the projectile. With this in mind, it is worthy to note that the pressure at any location forward of  $x_{max}$  will never “see” a pressure higher than that acting on the base of the projectile at this point. Nevertheless, as a measure of the inbred conservatism of gun designers, we design the tube to the pressure experienced at the breech while the projectile traverses the gun. These various pressures and the gradients are shown in Figure 5.2.

Designing a gun tube requires a knowledge of the stress state of the tube and a judgment of what constitutes a failure when it is under stress. For this, we turn to the von Mises criterion for failure under stress

$$\sigma_Y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \quad (5.4)$$

where

$\sigma_1$  = Axial stress

$\sigma_2$  = Tangential stress

$\sigma_3$  = Radial stress

$\sigma_Y$  = Equivalent stress

For an open-ended tube  $\sigma_1 = 0$  and Equation 5.4 becomes

$$\sigma_Y^2 = \sigma_2^2 - \sigma_2\sigma_3 + \sigma_3^2 \quad (5.5)$$

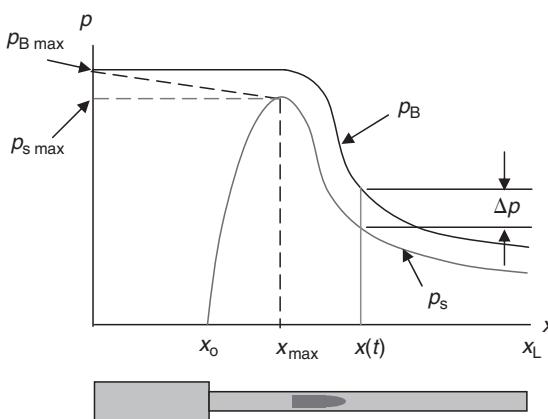


FIGURE 5.2

Pressure-distance curve for a gun tube.

Recalling Lamé's formulas for stress in a thick-walled tube

$$\sigma_{\theta\theta} = \sigma_2 = p_i \frac{r_1^2}{r^2} \left[ \frac{(r_o^2 + r^2)}{(r_o^2 - r_i^2)} \right], \quad \text{maximum at } r = r_i \quad (5.6)$$

$$\sigma_{rr} = \sigma_3 = p_i \frac{r_1^2}{r^2} \left[ \frac{(r_o^2 - r^2)}{(r_o^2 - r_i^2)} \right], \quad \text{maximum at } r = r_i \quad (5.7)$$

Let us put Lamé's formulas into a more useful form by letting

$$\zeta = \frac{r_o}{r_i} > 1 \quad \text{and} \quad \zeta_x = \frac{r}{r_i} > 1 \quad (5.8)$$

then

$$\sigma_{\theta\theta} = \sigma_2 = p_i \frac{1}{\zeta^2 - 1} \left[ \frac{(\zeta_x^2 + \zeta^2)}{\zeta_x^2} \right] \quad (5.9)$$

$$\sigma_{rr} = \sigma_3 = p_i \frac{1}{\zeta^2 - 1} \left[ \frac{(\zeta_x^2 - \zeta^2)}{\zeta_x^2} \right] \quad (5.10)$$

and

$$\sigma_{\theta\theta \max} = \sigma_{2 \max} = p_i \frac{\zeta^2 + 1}{\zeta^2 - 1} \quad \text{at } r = r_i \quad (5.11)$$

$$\sigma_{rr \max} = \sigma_{3 \max} = -p_i \quad \text{at } r = r_i \quad (5.12)$$

Failure is considered to have occurred when the equivalent stress,  $\sigma_Y$ , is greater than the yield strength,  $Y$ , of the material. If we substitute Equations 5.11 and 5.12 into [Equation 5.5](#) and substitute  $Y$  in for  $\sigma_Y$ , we get a solution for the ratio of internal pressure to yield strength.

$$Y^2 = \left[ p_i \frac{\zeta^2 + 1}{\zeta^2 - 1} \right]^2 + \left[ p_i^2 \frac{\zeta^2 + 1}{\zeta^2 - 1} \right] + p_i^2 \quad (5.13)$$

which by manipulation and expansion yields

$$\frac{Y^2}{p_i^2} = \frac{3\zeta^4 + 1}{(\zeta^2 - 1)^2} \quad \text{or} \quad (5.14)$$

$$\frac{p_i}{Y} = \frac{\zeta^2 - 1}{\sqrt{3\zeta^4 + 1}} \quad (5.15)$$

If the relationship in Equation 5.15 is plotted on a semilog plot, we see that for a monobloc tube (one which is made out of one piece) of yield strength  $Y$  and an internal pressure of  $1/2 Y$ , we obtain a wall thickness ratio  $\zeta = 2.75$ . This ratio rapidly becomes infinite as  $p_i/Y \rightarrow 0.58$ . This is depicted in [Figure 5.3](#). Thus, pressure levels are restricted below this

value. If we consider that a good gun steel of 180,000 psi yield strength is used, the allowable internal pressure should be kept lower than 100,000 psi. For modern, high velocity cannons, this restriction had to be overcome and the autofrettaging process described below has been used with marked success.

Jackets improve the efficiency of the gun tube by utilizing more of the metals load carrying capacity. The design concept began around 1870 and has been in use since, but is now considered obsolete. The idea is that one can shrink fit one or more cylinders over the inner cylinder or liner so that a compressive stress is induced in the inner layers. When an internal pressure is applied, the stresses on the inner cylinders are relieved by the pressure and then put into tension as the pressure is increased. Autofrettaging (self-jacketing) rather than shrink-fitting is now the process in use.

Autofrettage is a method of prestressing a tube to improve its load carrying capability as well as its fatigue life. The procedure consists of plastically deforming the interior of the gun tube toward the outside diameter. Regions of the interior wall will now exceed the yield point, but the exterior will not have yielded. When the load is removed, the outer layers of the material attempt to return to their unstressed state but cannot because of the plastically deformed portion of the wall. Thus, an equilibrium condition is attained where the outer wall regions remain in tension and the inner wall regions are in compression. The process is physically accomplished by either pressurizing the interior of the tube with water above its elastic limit or by pulling an oversized mandrel through the tube to force the yielding.

The pressure induced to autofrettage is on the order of

$$p_f = Y \ln \zeta \quad (5.16)$$

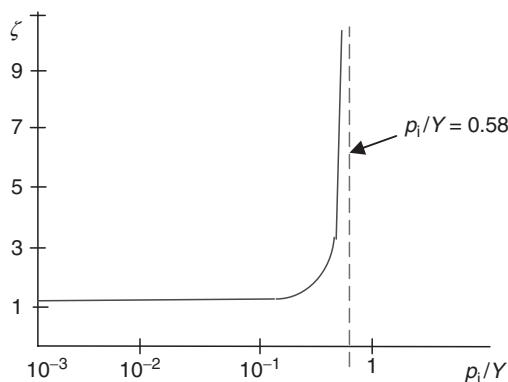
This is the pressure required to barely stress the outer wall during the process. The current practice is to keep this value below the elastic strength pressure by at least 8% in a finished tube. To further insure that the OD never goes plastic, tubes are sometimes autofrettaged in containers that act as an outer jacket during manufacture. The figures below illustrate the process (Figures 5.3 through 5.7).

### Problem 3

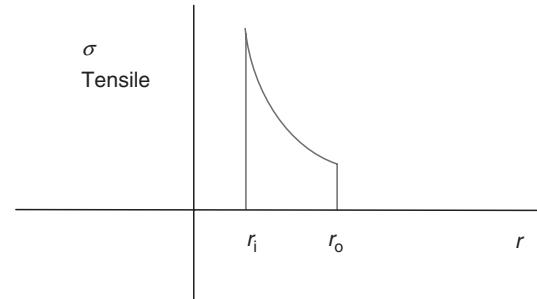
The gun in [Problem 1](#) is sized to a 20-in. OD. The manufacturer decides to autofrettage the weapon with 75,000 psi of hydraulic fluid. Assuming the material behaves elastic-perfectly-plastic:

1. Approximately to what radial distance does the compressive layer extend into the tube wall?

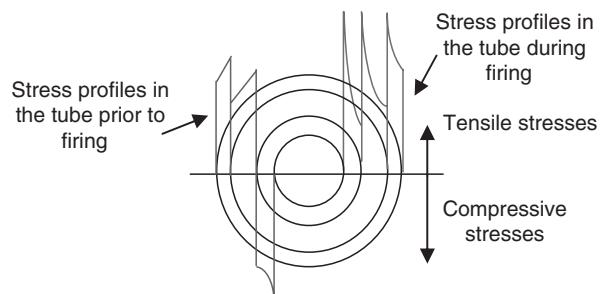
*Answer:* Approximately 0.57 [in.]



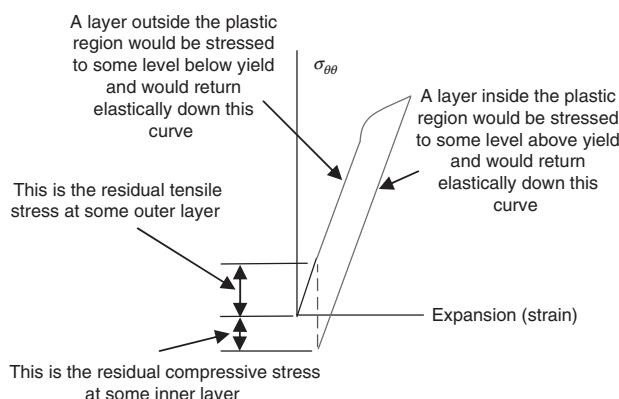
**FIGURE 5.3**  
Wall-thickness ratio as a function of internal pressure to yield stress ratio.



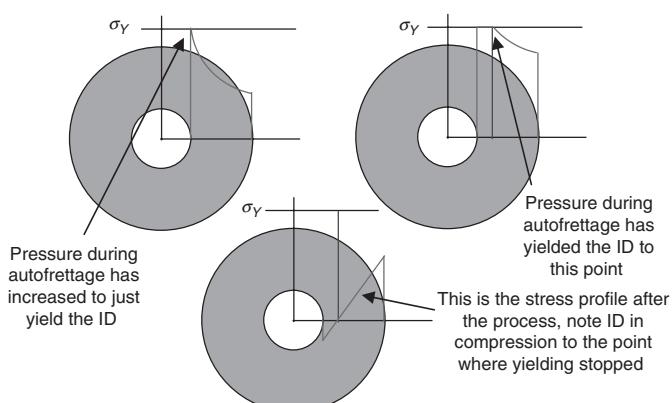
**FIGURE 5.4**  
Stress profiles in a monolithic tube.



**FIGURE 5.5**  
Stress profiles in an autofrettaged tube.



**FIGURE 5.6**  
Hoop stress versus strain in an autofrettaged tube.



**FIGURE 5.7**  
The autofrettage process.

### 5.3 Gun Dynamics

In this study, we intend to discuss how a gun behaves as a dynamic entity, during a projectile firing and immediately after the projectile exits the muzzle. We will discuss the recoil response in terms of the forces and motions and the response known as gun jump. We will not attempt to discuss recoil abatement or mounting techniques.

Recoil is generated on the gun by the reaction of its moveable parts to the impulse of the gas pressure both while the projectile is in the tube and while the propelling gases are being exhausted after the projectile exits. After projectile exits, we usually assume that the pressure decays linearly with time. This period is called the gas exhaust aftereffect and is shown in Figure 5.8.

We show the forces on the gun during the time the projectile is moving through the tube (including the forces attributable to the rifling) in Figure 5.9.

During and after firing the unbalanced forces on the gun can be categorized as the gas force,  $F_R$ ; the projectile resistance force,  $F_{P_r}$ ; and the rifling force,  $F_T$ .

$$F_R = p \frac{\pi}{4} d^2 = F_P \quad (5.17)$$

Note here that  $F_P$  only acts on the bore diameter, whereas  $F_R$  acts on the breech face diameter, normally larger than the bore. The resistance pressure is estimated as follows:

$$\text{For smooth bores: } F_{P_r} \approx 0.01 F_R \quad (5.18)$$

$$\text{For rifled bores: } F_{P_r} \approx (\mu + \tan \alpha) F_T$$

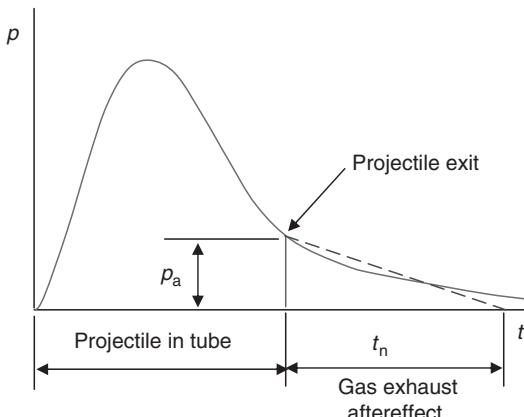
The rifling force is

$$F_T = \left( \frac{k}{d/2} \right)^2 F_P \tan \alpha \quad (5.19)$$

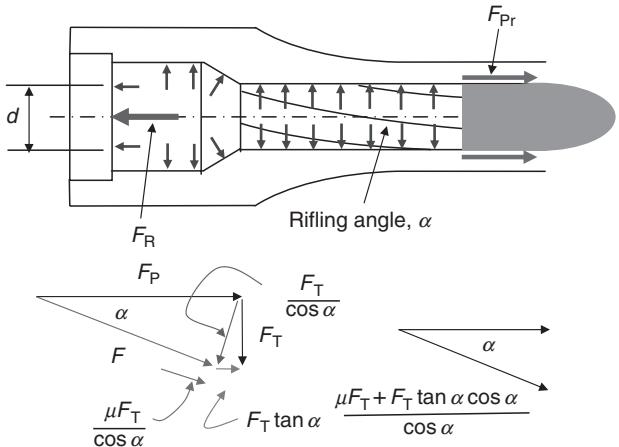
Here  $k$  is the projectile's radius of gyration ( $I_{zz} = wk^2$  in terms of axial moment of inertia and mass),  $\mu$  is the coefficient of friction, and  $\alpha$  is the rifling angle. Our object is to find  $F_P$ .

From the Lagrange approximation for the pressure gradient we know that

$$p_B = p_S \left( 1 + \frac{c}{2w} \right) \quad (5.20)$$



**FIGURE 5.8**  
Pressure-time curve for a typical gun firing.



**FIGURE 5.9**

Forces acting on a gun tube and projectile reactions.

In terms of forces, this may be written as

$$F_R = F_{Ps} \left( \frac{w + c/2}{w} \right) \quad (5.21)$$

or

$$F_{Ps} = F_P \left( \frac{w}{w + c/2} \right) \quad (5.22)$$

Therefore, for rifled guns, from [Equation 5.18](#)

$$F_{Pr} = \left( \frac{k}{d/2} \right)^2 (\mu + \tan \alpha) \tan \alpha \cdot F_R \left( \frac{w}{w + c/2} \right) \quad (5.23)$$

While these are the defined forces, now let us examine the motion of the gun during recoil. This depends essentially on the balance of momentum between the projectile and its propelling gases and the mass of the gun. Let us first write a momentum balance in the direction of fire

$$M_{\text{recoil}} = M_{\text{proj}} + M_{\text{prop.gas}} \quad (5.24)$$

We recall from the Lagrange approximation for the projectile and its propelling gas that

$$\text{Total momentum} = \left( w + \frac{c}{2} \right) V_{\text{muzzle}} \quad (5.25)$$

Thus at projectile exit,

$$V_{\text{recoil}} = \left( \frac{w + c/2}{w_{\text{recoil}}} \right) V_{\text{muzzle}} \quad (5.26)$$

In this expression,  $w_{\text{recoil}}$  is the recoil mass of the weapon. This quantity includes all mass attached to the tube that must be moved rearward when the weapon fires such as breech closing mechanisms, sighting devices, etc.

After the projectile has left the muzzle, the propellant gases exit at a continually decreasing velocity whose mean value,  $\bar{V}$ , can be approximated by

$$\bar{V} = \sqrt{V_{\text{muzzle}}^2 + \bar{c}^2} \quad (5.27)$$

where  $\bar{c}$  is the average speed of sound in air at standard temperature and pressure ( $\approx 330 \text{ m/s}$  or  $1080 \text{ ft/s}$ ). Thus,  $\bar{V}$  is roughly 4000–4600 fps (1200–1400 m/s). As an example, consider

$$\bar{V} \approx \sqrt{4000^2 + 1080^2} \approx 4143 \text{ fps}$$

We can alternately define an aftereffect coefficient by

$$\bar{V} = \beta V_{\text{muzzle}} \quad (5.28)$$

Here we can estimate  $\beta$  from the graph of Figure 5.10.

Using  $\beta$ , we can write a new equation for the momentum balance

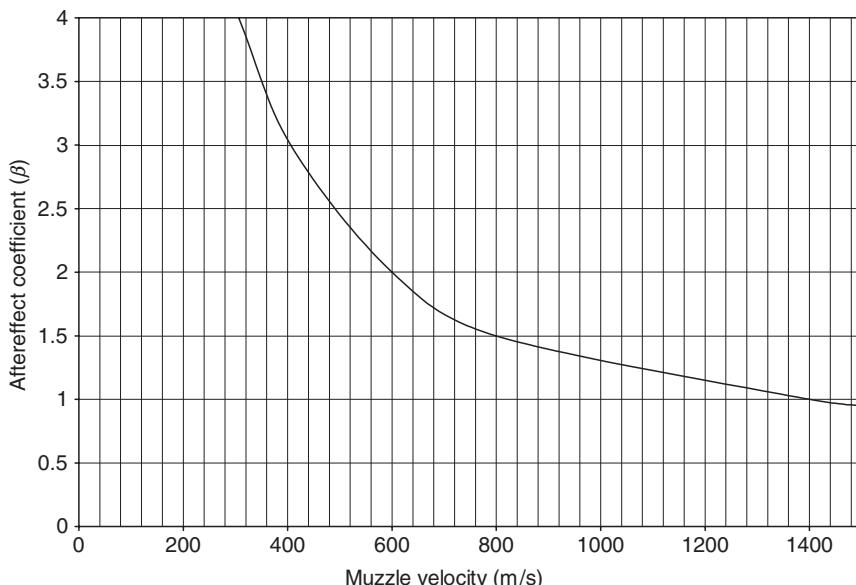
$$w_{\text{recoil}} V_{\text{final}} = \left( w + \frac{c}{2} \right) V_{\text{muzzle}} + c\beta V_{\text{muzzle}} \quad (5.29)$$

or

$$w_{\text{recoil}} V_{\text{final}} = \left[ w + c \left( \frac{1}{2} + \beta \right) \right] V_{\text{muzzle}} \quad (5.30)$$

This allows us to solve for the final velocity of the recoiling parts

$$V_{\text{final}} = \left[ \frac{w + c(1/2 + \beta)}{w_{\text{recoil}}} \right] V_{\text{muzzle}} \quad (5.31)$$



**FIGURE 5.10**  
Aftereffect coefficient.

In free recoil, the total distance traveled,  $S_{Re}$ , is the sum of the distance traveled while the projectile is in the gun,  $S_{Ra}$ , and the distance traveled during the gas ejection phase,  $S_{Rn}$

$$S_{Re} = S_{Ra} + S_{Rn} \quad (5.32)$$

With no external forces acting, the common center of mass stays at rest, with half of the charge mass lumped with the gun and half with the projectile. Once again, we write the momentum balance

$$\left(w_{\text{recoil}} + \frac{c}{2}\right)V_{\text{recoil}} = \left(w + \frac{c}{2}\right)V_{\text{muzzle}} \quad (5.33)$$

By considering that the velocities, on average, are distance divided by the time, we can rewrite Equation 5.33 as a distance equation

$$\left(w_{\text{recoil}} + \frac{c}{2}\right)S_{Ra} = \left(w + \frac{c}{2}\right)S_{Pa} \quad (5.34)$$

where we can see from Figure 5.11 that

$$S_0 = S_{Ra} + S_{Pa} = S_{Ra} \left(1 + \frac{S_{Pa}}{S_{Ra}}\right) \quad (5.35)$$

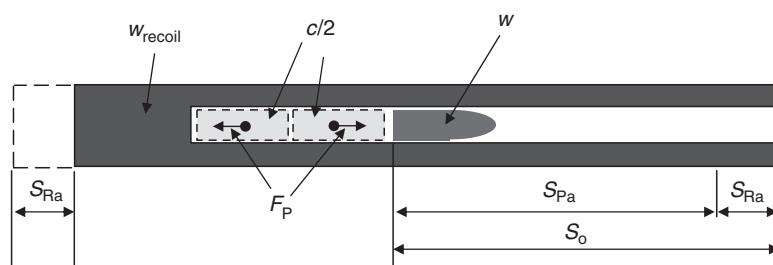
We can then see that the free recoil motion of the gun while the projectile is in the tube,  $S_{Ra}$ , may be found from Equation 5.34 as

$$S_{Ra} = S_{Pa} \left(\frac{w + c/2}{w_{\text{recoil}} + c/2}\right) \quad (5.36)$$

And from Equation 5.35 we can show that

$$S_{Ra} = S_0 \left(\frac{w + c/2}{w_{\text{recoil}} + w + c}\right) \quad (5.37)$$

As was mentioned in Equation 5.32, further motion of the gun in free or unconstrained recoil occurs after the projectile has left the tube. It is caused by the momentum exchange of the mass of gas still exhausting from the tube after the projectile is long gone. We look for an estimate of the length of this motion,  $R_{an}$ , by examining the impulse of the gas. The



**FIGURE 5.11**  
Diagram of gun displacements.

duration of the tube-emptying phase,  $t_n$ , can be computed from the aftereffect impulse,  $I_n$ , by assuming that the gas force,  $F_R$ , decreases linearly with time (see Figure 5.8).

$$I_n = \frac{1}{2} F_a t_n \quad (5.38)$$

But impulse may also be defined as the change of momentum over time

$$I_n = w_{\text{recoil}}(V_{\text{Re}} - V_{\text{Ra}}) \quad (5.39)$$

By application of Equations 5.30 and 5.31, we can show that

$$I_n = \beta c V_{\text{muzzle}} \quad (5.40)$$

If we solve for  $t_n$  by inserting Equation 5.40 into Equation 5.38, we get

$$I_n = \frac{2\beta c V_{\text{muzzle}}}{F_a} \quad (5.41)$$

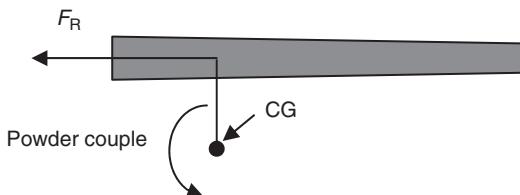
By assuming a linear velocity change between  $V_{\text{Re}}$  and  $V_{\text{Ra}}$ , and integrating the acceleration twice as calculated from the gas force,  $F_a$ , we get an approximation for the remaining travel,  $S_{\text{Rn}}$

$$S_{\text{Rn}} = \left( \frac{V_{\text{Ra}} + V_{\text{Re}}}{2} + \frac{V_{\text{Re}} - V_{\text{Ra}}}{2} \right) t_n \quad (5.42)$$

In this analysis, we have assumed that the weapon was in free recoil. In real weapons, this never occurs. We normally have recoil mechanisms that rely on pneumatic or hydraulic systems to slow and finally stop the recoil within a relatively short distance. These forces need to be added to the above analysis to make it more accurate. The effect of a muzzle brake should be added as well.

Let us now consider the phenomenon known as "gun jump." The axis of the gun bore, which is where the gas forces are applied, is usually not collinear with the mass center of the recoiling parts. This creates a moment couple often referred to as the "powder couple," which acts upon firing (Figure 5.12). This couple causes a rotation of the gun that usually results in muzzle rise. This contributes to projectile jump but is by no means the sole cause of it.

There are other dynamic reactions of the gun during firing. The gun is an elastic body, so that when the propelling charge is ignited many complicated structural reactions take place. Stress and pressure waves are set up in the chamber and in the unpressurized portion of the bore, loading the tube in a highly transient fashion. Swelling and elongation occur due to pressure, the rotating band is engraved by the rifling (if present) causing local



**FIGURE 5.12**  
Powder couple illustrated.

stressing of the tube, and a thermal gradient is set up. These phenomena are highly complicated and we will not discuss them further here.

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## 5.4 Muzzle Devices and Associated Phenomena

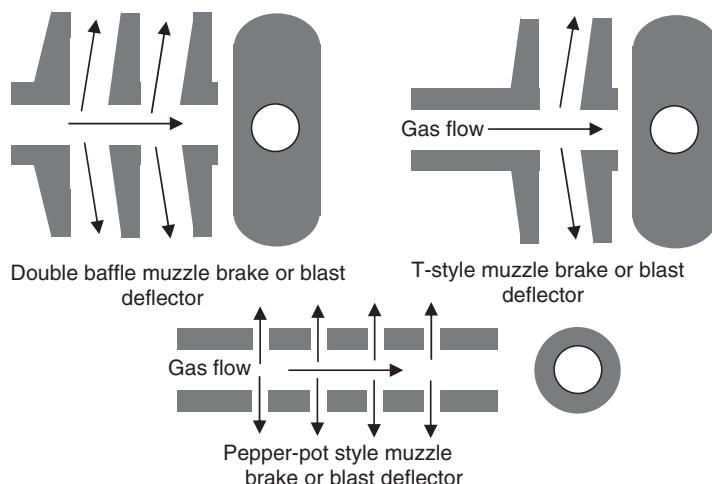
We use muzzle devices for three main reasons: reduce recoil, suppress flash, and decrease report. Sometimes increased accuracy results from shot to shot because of reduced weapon movement. Muzzle devices have also been devised to limit muzzle climb.

Muzzle brakes consist of surfaces placed perpendicular to the bore axis such that impinging gases exert a net forward thrust on the weapon. This thrust is accomplished through conservation of momentum principles. Best design practice is to divert gases to the sides of the weapon because rearward diversion could affect an exposed gun crew. Downward diversion could kick up excessive debris and without a balancing upward diversion would strain operating gun mechanisms.

There are generally two types of muzzle brakes: closed and open. Closed brakes channel the exiting gases through fixed openings and usually have multiple baffles or ports. Open muzzle brakes generally have only one baffle and direct the gas flow to a lesser extent than closed brakes. The chief purpose of these brakes is to mitigate the recoil.

A blast deflector is similar in concept to a muzzle brake although not designed to assist recoil as much. The purposes of the blast deflector are to direct blast away from the gun crew, minimize obscuration of the battlefield by limiting the amount of dust kicked up during the discharge of the weapon, and in the case of small arms, limit muzzle climb. A detriment of a blast deflector is that to reduce dust one usually needs to vent the gases upward which tends to load the tube and to support structure of the weapon. If the weapon is already horizontal and the venting thrust has a large vertical component, this can be a substantial loading.

There are four basic types of blast deflectors (Figure 5.13). The baffle type is identical to a baffled muzzle brake with the gases vented to the sides of the piece. The perforated type, sometimes called a “pepper-pot” brake, has multiple side ports in a tubular section (none



**FIGURE 5.13**  
Typical muzzle brake or blast deflector geometry.

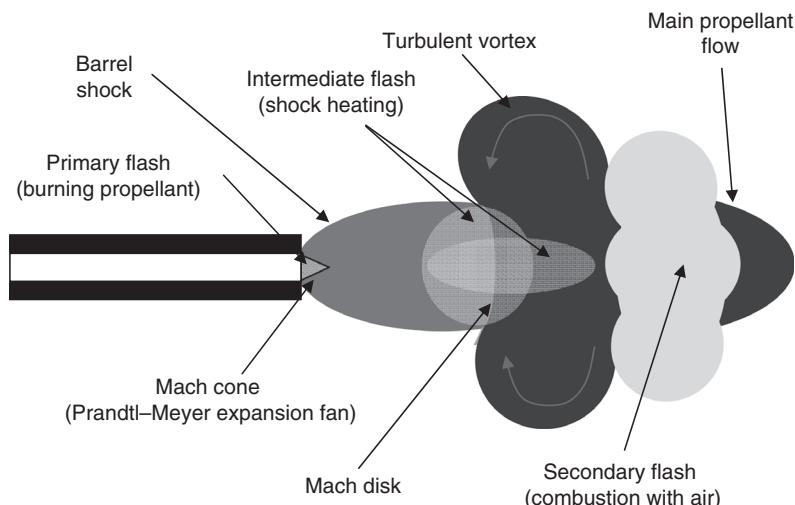
of the ports venting straight down). The T-type is the same as a single-baffle muzzle brake and lastly, the ducted type. This latter device has a complicated array of ducting to divert the flow back and upward near the mounting trunnions. It diverts the blast load so that it is carried by the trunnions. Unfortunately, at high quadrant elevations, it ducts the blast toward the crew, which is not good.

Muzzle flash was noticed as a problem during First World War when significant night actions were commonplace and suppression of muzzle flash became highly desirable. The study of flash has used high-speed photography and other recording devices to distinguish five types of flashes (Figure 5.14):

1. Pre-flash—this is flash caused by blow-by, a condition where propellant gas leaks around the projectile's rotating band or obturator and exits before the projectile.
2. Primary flash—this is the flash caused by any propellant solids or gases that are still burning upon muzzle exit of the projectile.
3. Muzzle glow—this is the illumination caused by gas inside the shock bottle (defined later).
4. Intermediate flash—this is the illumination caused by gas that managed to get ahead of the normal shock of muzzle gas ejection and is caused by the increased pressure and temperature of the gas as it passes through the shock front.
5. Secondary flash—this is the flash caused by the reaction of the combustion products when they enter the air (really, another, secondary oxidation reaction transpires).

Propellant additives are often used, but do not suppress flash sufficiently and besides add smoke. It was observed early on that muzzle brakes and blast deflectors actually suppressed flash somewhat. This has led to the development of mechanical flash suppressors.

However, the only types of muzzle flashes that can be controlled by the attachment of a mechanical flash suppressor are muzzle glow, intermediate flash, and secondary flash.



**FIGURE 5.14**  
Muzzle blast structure.



Bar-style flash suppressor



Simple silencer

**FIGURE 5.15**  
Typical muzzle devices.

These all are affected by the presence of the expanding gas shock wave. Secondary flash is the least controllable from a mechanical standpoint because the temperature of the propellant gas mixture may be so high that the shock is only an amplifying factor.

Various designs of suppressors have been developed and they fall into two basic types (Figure 5.15):

1. Conical flash suppressors appear similar to the bell end of a trumpet and are sometimes called flash hiders.
2. Bar-type flash suppressors resemble a cage around the muzzle of the weapon. They are difficult to clean and if they are of an open-end design, then they can get caught on objects such as clothing and vegetation during combat.

Smoke on a battlefield is disadvantageous if not generated where and when it is desired as an obscurant. In the days of black powder, it was a real problem as the battlefield became obscured for friend and foe alike. When nitrocellulose propellants were introduced, they were called "smokeless powders" because they generated much less smoke than black powder. Even with smokeless powders, large volumes of fire still produced significant quantities of smoke. An alternative would be to add chemicals to the propellant to reduce smoke, but this usually increases flash and devices that suppress flash usually increase smoke.

Smoke generated from a weapon is usually made up of a solid–liquid–gas mixture and is composed of metal or metal oxide particles from the cartridge case and its components, the projectile and the tube. Also present are water vapor or condensate liquid and chemical elements such as carbon, copper, lead, zinc, antimony, iron, titanium, aluminum, potassium, chlorine, sodium, sulfur, and other particulates. These components in themselves obscure vision, but they may also combine with the atmosphere to allow water vapor there to condense on the particles. Air temperature and relative humidity affect the density and longevity of the smoke as well.

Smoke suppressors are really filters that capture the solid particulates, yet allow the gaseous composition to pass through them. They are either electrostatic in nature or mechanical filters. Electrostatic types are primarily used in a laboratory environment. Mechanical types work by robbing momentum from the particles. The pores of these suppressors must be quite large so that the gas flows through them without difficulty and that only the particulates are removed.

These suppressors work by forcing the propellant gas to pass through non-straight channels similar to pores. The impingement of the particles robs them of momentum.

When the gas pressure in the suppressor (which is super caliber) becomes higher than the muzzle pressure, the gas evacuates in the opposite direction to entry, leaving the solids and liquids behind. The downside to this is that the suppressor adds weight to the tube at the muzzle end, adds cost, and requires frequent maintenance.

There are three basic types of mechanical smoke suppressors:

1. Uniform perforation spacing where no attempt to control the flow is made.
2. Increasing perforation density toward the muzzle which allows particles that would normally build up closer to the muzzle to be spread more evenly in the device because pressure drops in the axial direction.
3. A tapered bore type which is similar to the above but includes a taper that becomes smaller as one approaches the exit with a larger inner diameter (ID) near the end at the rifling. This allows some axial impingement and also helps spread out the heavier particles.

Noise on the battlefield is also the subject of mitigation. Devices used to reduce noise, which are sometimes referred to as silencers, attempt to reduce the report of the weapon.

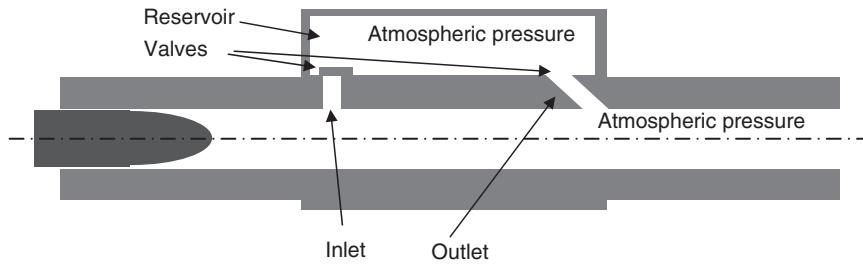
Removal of noise is important on the battlefield for several reasons. Noise affects communications, is harmful to soldier's hearing, can reveal position, and makes covert operations difficult. Noise is related to flash and blast, and usually reducing one of these reduces noise as well. The filters used in smoke suppression generally also work well to reduce noise.

In a closed-land vehicle, ship or aircraft, there is frequently a differential in air pressure between the interior and the exterior environment. When, after a round is fired, the breech of the weapon is opened, there is a tendency for residual propellant gas in the bore to enter the closed fighting compartment. This impairs sight and breathing or burning particles introduced into the compartment could ignite ready ammunition. Flashback could occur when un-reacted propellant gas combines with the air in the compartment similar to the events at the muzzle. Removal of residual propellant gases is a major consideration in the design of fighting vehicles' crew compartments. In a ship mounting, ventilators are usually installed which mechanically push the muzzle gases out after shot exit. This equipment is rather large and is not practical in a land vehicle or aircraft. We design bore evacuators or bore scavengers to deal with this problem in land vehicles and aircraft.

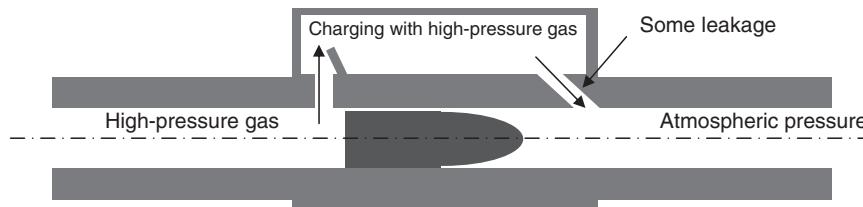
This method is simply to mount a chamber on the outside of the tube with ports that connect directly into the tube bore. These ports are designed so that they discharge in the direction of the muzzle. When the projectile passes the open ports, gas pressure builds up in the evacuation chamber. Once shot exit occurs, the pressure in the tube eventually drops below the evacuator chamber pressure. When this occurs, the gas trapped in the evacuator rushes out of the muzzle, dragging with it the majority of the residual gas in the tube. This generates a partial vacuum so that when the breech is opened fresh air is pulled in from the compartment. If the breech is not opened for a while after firing, the vacuum dissipates, but by then the propellant gases should have been removed. These actions are shown in [Figures 5.16 through 5.19](#).

The phenomena of muzzle flows for which the variety of devices we have described are meant to mask or mitigate are complex and are still under active study. We will examine these flows in some detail at this point. We shall step through the muzzle exit process in the order in which the events occur.

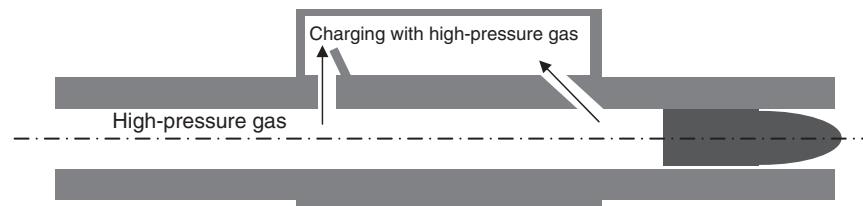
As a projectile begins to move down the gun tube, it compresses the air ahead of it. The gun tube acts like a shock tube in which a near-planar shock forms. When this shock exits the muzzle, it forms a spherical shock wave as seen in [Figure 5.20](#).



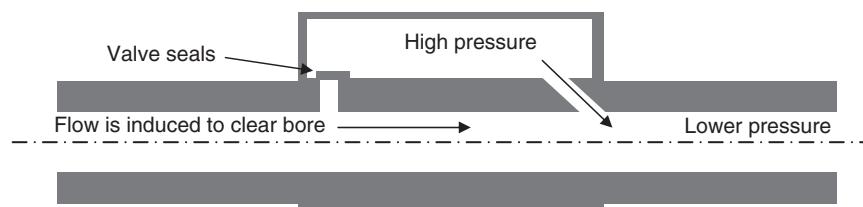
**FIGURE 5.16**  
Projectile approaching bore evacuator.



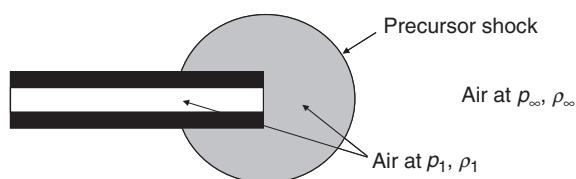
**FIGURE 5.17**  
Bore evacuator charges with gas.



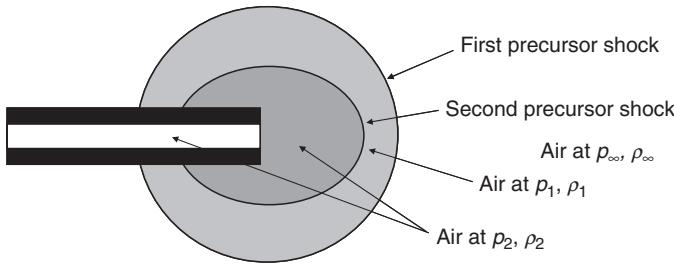
**FIGURE 5.18**  
Bore evacuator still charges with gas.



**FIGURE 5.19**  
Projectile has exited, bore evacuator discharges inducing outflow.



**FIGURE 5.20**  
Precursor shock geometry.

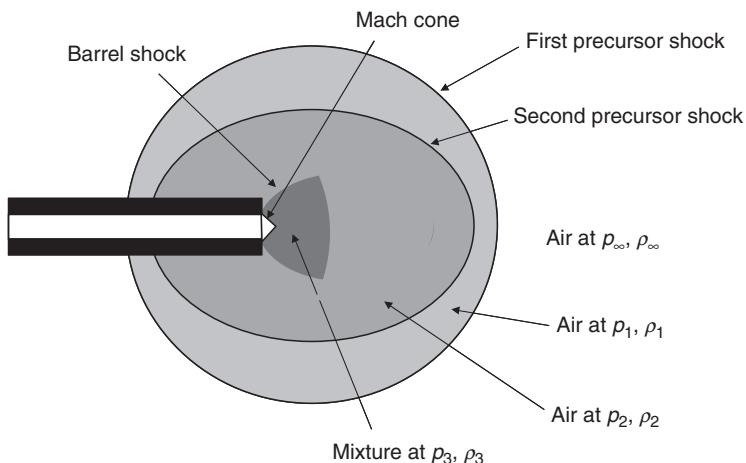


**FIGURE 5.21**  
Second precursor shock formation.

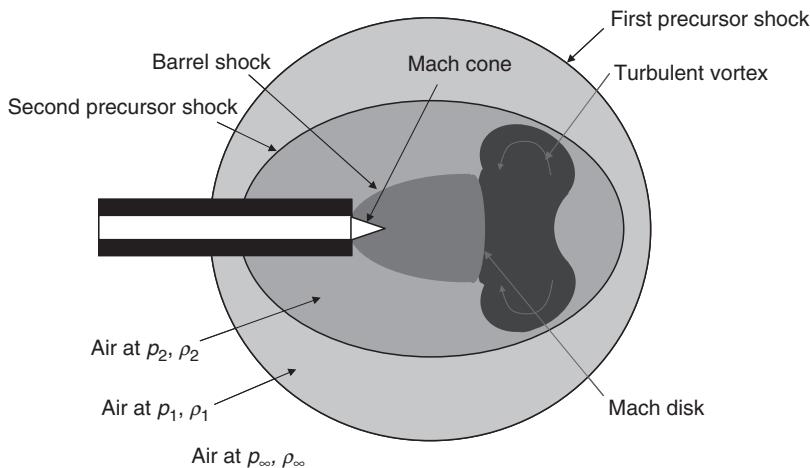
As a projectile moves faster and faster in the tube, if the velocity is low enough (that is correct, “low enough”), a second precursor will form. This precursor moves faster than the first one because it is moving into the higher density fluid bounded by the first precursor as seen in Figure 5.21.

No projectile ever obturates perfectly because gun wear occurs; rotating bands and obturators erode; and in high-firing-rate weapons, barrel heating occurs, swelling the bore. As we know, propellants are under-oxidized and because of this, any propellant gas blow-by will combine with the oxygen in the precursor flow fields and, when the temperature is high enough, react. Because this occurs before projectile exit, it is known as pre-flash. It can occur regardless of the presence of the precursors.

Several microseconds after the precursor shock appears, but before the projectile emerges, the so-called barrel shock and Mach cone form. This bottle-shaped structure is referred to as the shock bottle. The barrel shock is created as the higher pressure gases being compressed by the onrushing projectile attempt to push their way into the high-pressure precursor flow field. One important concept to keep in mind is that pressure acts in all directions—it is a point function. Think of the precursor flow field as “pushing in” on anything that is trying to come out of the muzzle. Thus, the precursor flow field actually constrains the flow exiting the muzzle. The Mach cone is generated by the fact that the fluid jet of the gas ahead of the projectile suddenly sees that there is no more wall constraining it and it tries to turn the 90° corner but cannot, so an expansion fan forms. This is shown in Figure 5.22.



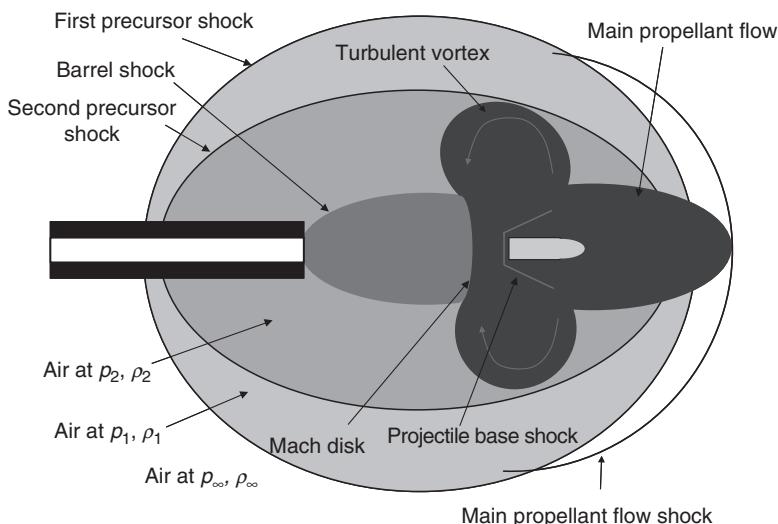
**FIGURE 5.22**  
Generation of the Mach disk.



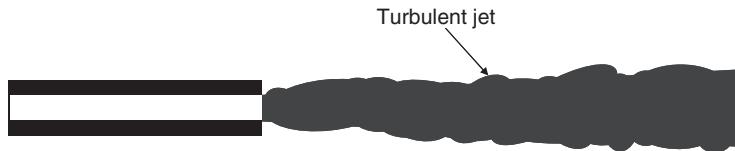
**FIGURE 5.23**  
Formation of the turbulent vortex.

After formation of the shock bottle but still before shot ejection, gases are still jetting out of the muzzle. An annular vortex is formed as the gas at the center of the jet continues to rush out while gas near the outer boundary is being robbed of momentum forming a vortex. This is depicted in Figure 5.23. This vortex progresses downrange and will eventually approach the precursor shock.

When the projectile obturator uncorks from the muzzle, there is more room for high-pressure gases to escape. These gases may still be reacting and expand at a rate which results in them moving faster than the projectile. In many instances, they are supersonic with respect to the projectile and a base shock forms. The projectile may be flying backwards in this flow. This propellant plume is constrained by the precursor flow field



**FIGURE 5.24**  
Shock structure at shot exit.



**FIGURE 5.25**  
Turbulent jet formation.

and rapidly overtakes it, since it is at a higher temperature and pressure. The result is a bulge of the propellant gases through the precursor shock both preceding and following the projectile. This is depicted in Figure 5.24.

For some time after shot exit the flow field remains as depicted in Figure 5.24. The turbulent vortex and length of the main propellant flow increase but the shock bottle remains fairly constant. After this phase, the propellant flow leaving the muzzle diminishes. The Mach disk retreats toward the muzzle and the shock bottle recedes. Upon completion of this process, the situation is reminiscent of effluents from a smokestack as illustrated in Figure 5.25.

Reference [5] describes the influence of the muzzle exit event on accuracy and general motion of the projectile. This motion can be critical in direct fire applications.

We have examined the phenomena of muzzle exit flows and the types of muzzle devices commonly used on weapons. The purpose of these devices is to affect the muzzle flow so that certain physical phenomena are altered. Research in this field is still in its infancy and the literature abounds with theories and simulations.

## Gun Dynamics Nomenclature

<i>F</i>	Force
<i>S</i>	Distance
<i>V</i>	Velocity
<i>M</i>	Momentum
<i>I</i>	Impulse
<i>d</i>	Bore diameter
$\alpha$	Rifling angle
$\beta$	Atereffect coefficient
$\mu$	Coefficient of friction
<i>w</i>	Projectile mass
<i>c</i>	Charge mass
<i>w</i> <sub>recoil</sub>	Mass of recoiling parts
<i>I</i> <sub>zz</sub>	Polar moment of inertia of projectile
<i>k</i>	Radius of gyration of projectile
<i>p</i> <sub>B</sub>	Breech pressure
<i>p</i> <sub>S</sub>	Base pressure on projectile

## Subscripts on *F*, *S*, *V*, *M*, and *I*

<i>P</i>	Projectile base
<i>R</i>	Reaction at breech
<i>a</i>	Time when projectile exits the muzzle
<i>t</i>	Tangential direction

n	Aftereffect phase
o	Total distance in tube
Re	Reaction at end of recoil
Ra	Reaction at breech until projectile exits muzzle
Rn	Reaction at breech from when projectile exits to the end of the aftereffect
Pr	Reaction on gun caused by projectile rotating band
Ps	Reaction at the projectile base (essentially the same as P)
Pa	Reaction at the base when the projectile exits the muzzle

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  2. Norton, R.L., *Machine Design, an Integrated Approach*, 3rd ed., Pearson-Prentice Hall, Upper Saddle River, NJ, 2006.
  3. Smith, D.C. and Coppola, E.E., *Safe Maximum Pressure Determination for the M829E3/M256 Cannon Qualification Program*, U.S. Army ARDEC Technical Report ARCCB-TR-02013, Benet Laboratories, Watervliet, New York, NY, September 2002.
  4. Germershausen, R. et al., *Handbook on Weaponry*, 2nd English ed., Rheinmetall GmbH, Düsseldorf, 1982.
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## Further Reading

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- Headquarters, U.S. Army Materiel Command, *Muzzle Devices*, U.S. Army Engineering Design Handbook AMCP-706-251, U.S. Army Research Office, Washington, D.C., May 1968.
- Klingenber, G. and Heimerl, J.M., "Gun muzzle blast and flash," *Progress in Astronautics and Aeronautics*, Vol. 139, American Institute of Astronautics and Aeronautics, Washington, D.C., 1992.

## **Part II**

# **Exterior Ballistics**



# 6

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## *Introductory Concepts*

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When the projectile has left the environment of the gun, including the effects of the exiting propellant gases that momentarily surround it, it enters the realm of the exterior ballistian. Here, it is subject to the force of the pressure of the atmosphere that it is flying through, the force induced by its spin, and the force due to the acceleration of gravity. The projectile in flight is no longer constrained in lateral motions by the walls of the gun and, as a free body, can develop motions that are complex and occasionally inimical to the intent of its user and embarrassingly, to its designer. The study of these motions and the progress of the projectile to its target are the subject of this part of the book.

We will begin with the simplest case, consideration of the projectile as a point mass flying in a vacuum with only the force due to gravity acting on it. Then we will proceed to introduce the force due to the pressure of the air, but still considering the projectile as a mass concentrated at a point. Finally, we will consider the projectile as a three-dimensional body acted upon by the air, its spin, and gravity. In the final sections of this part of the text, we shall examine the complex motions arising from the coupling of projectile dynamics and aero-mechanical forces. Our object will be to examine the conditions necessary for a precise, predictable, satisfactory trajectory enabling the projectile to fulfill its terminal ballistic utility.

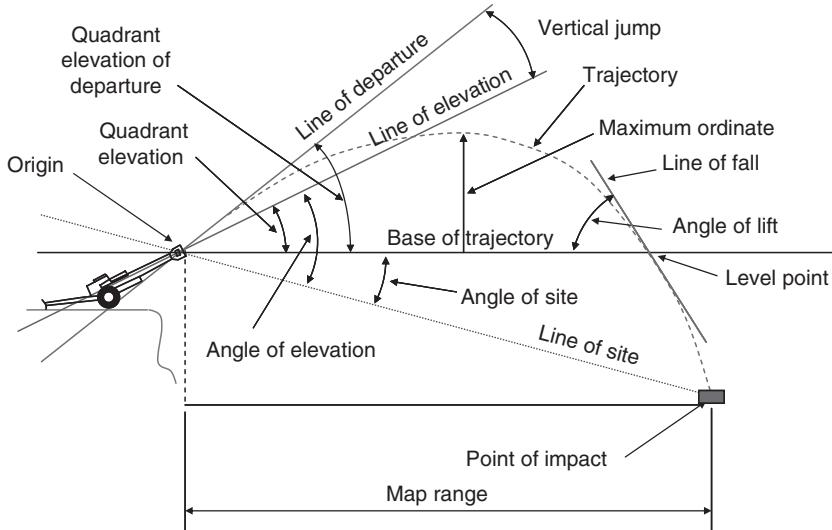
Since this text is intended to have a broad scope, some of the material is not derived in detail. The reader is encouraged to seek the more detailed treatments in the references noted throughout each section.

Many of the principles and terms concerned with fluid mechanics required for the understanding of interior ballistics were introduced [in Section 2.7](#). These principles will be extended in this section with a view toward an exterior ballistian—commonly called an aero-ballistian.

We shall first examine the elements of a trajectory as depicted in [Figure 6.1](#). These terms are commonly used in the military by gunners and researchers alike. Although most of the symbols and terms in this figure are self-explanatory, some require comment. First is the so-called map range. This is the range to the target that the gunner would see if he or she were to plan firing using a map. The base of the trajectory is quite important and is defined as being level in a plane with the firing point. Gunners of large caliber weapons and mortars take great care in assuring that the sights on the weapon are leveled in the direction depicted as well as the plane out of the paper.

Since larger ordnance fires over extensive ranges, it is common to assume that the origin of the trajectory is coincident with the ground beneath the artillery piece. The line of site and angle of site (yes, they are spelled that way in much of the literature) are what the gunner uses to aim at the target. As you can see, they only assist in determination of the pointing of the weapon and the relative height of the target.

An important feature of this diagram is the line of departure. You have probably noticed that it is not collinear with the elevation of the weapon (i.e., where the bore is pointed).

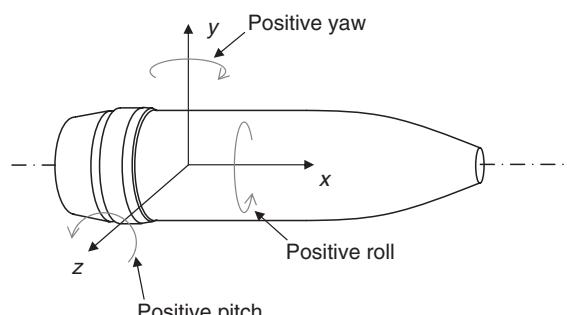


**FIGURE 6.1**  
Elements of a trajectory.

The reality is that a projectile almost never leaves the bore of a gun aligned with the bore—we shall discuss this in detail later. For now, we will simply state that this is due to the dynamics of the projectile and gun as well as aerodynamic effects. It should be noted that Figure 6.1 is drawn as two dimensional. The out-of-plane angular position of the projectile at muzzle exit is known as lateral or azimuthal jump. This will combine vectorially with the vertical jump that is depicted to give a resultant jump vector.

The angle of lift and line of fall are defined for the level point; however, it is common to see these used at the target even though, officially these quantities at the target are called angle of impact and line of impact (sometimes shot line).

The aerodynamics and ballistics literature are quite diverse and terminology is far from consistent. This has particular significance in the coordinate systems used to define the equations of motion. In this text, we shall use the coordinate system of Ref. [1] as depicted in Figure 6.2. The primary difference between this scheme and those of, say, Refs. [2–5] is that the  $y$ -axis is deemed to be positive pointing up, with the  $z$ -axis as positive to the right as opposed to the  $z$ -axis down and  $y$ -axis to the right. This makes sense to the authors with up being a more intuitive positive direction. The only issues (and some people consider them significant) with this scheme are that first, the nice right handed naming convention of the aerodynamic coefficients is disturbed (as we shall see later  $x$ - $y$ - $z$  corresponds to  $l$ - $n$ - $m$ )



**FIGURE 6.2**  
Definition of projectile coordinates.

not  $l-m-n$  as one would normally like); second, what would normally be a positive rotation in the  $y$ -direction (i.e., nose left) is defined as negative—we shall handle this when we define the associated equations.

We shall now define some terminology and, more importantly, the forces, moments, and associated coefficients that are used throughout this part of the text. It is important for the reader to recognize that these force, moment, and coefficient definitions are by no means an all-inclusive collection. Occurrences of additional forces or moments at times require additional definitions—e.g., control deflections. We shall adhere to the broad scope of this text by including only what is necessary for a basic understanding of ballistics.

We mentioned the yaw and pitch of the projectile earlier in this section. The projectile geometry in an arbitrary state of yaw is depicted in Figure 6.3. This illustration shows the projectile yawed and pitched to some angle,  $\alpha_t$ , relative to the velocity vector. The illustration also shows the trajectory which is defined as the curve traced out by the velocity vector. Thus, the velocity vector is everywhere tangent to the trajectory curve. The inset shows the decomposition of the angle between the projectile axis of symmetry,  $x$  ( $\overline{OB}$ ), and the velocity vector,  $V$  ( $\overline{OA}$ ). We first measure the sideslip angle,  $\beta$  ( $\angle AOC$ ) and then measure the yaw angle,  $\alpha$  ( $\angle COB$ ), from the axis of symmetry,  $x$ , to side  $\overline{OC} = V\cos\beta$ . The side  $\overline{BC}$  of right triangle OBC then has a value of  $V\cos\beta \sin\alpha$ . The resulting angle  $\angle AOB$  is defined as the total yaw angle,  $\alpha_t$  and in triangle AOB where side  $\overline{AB} = V\sin\alpha_t$ . It should be noted that triangle ABC with sides  $V\sin\alpha_t$ ,  $V\cos\beta \sin\alpha$ , and  $V\sin\beta$  is not a right triangle.

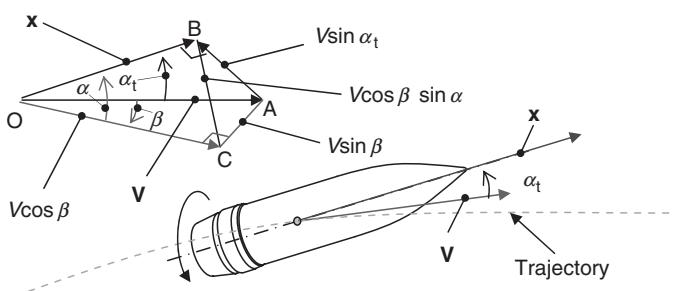
Most projectiles have at least trigonal symmetry. This is symmetry about three planes through the projectile long axis,  $120^\circ$  apart. Because of symmetry, it is common to vectorially combine the yaw and pitch of the projectile into one term which we simply call total yaw,  $\alpha_t$ . All of our coefficients will be based on this total yaw. Later, when we discuss advanced topics it will be necessary to once again separate them.

An examination of Figure 6.3 shows that we can relate the total yaw to  $\alpha$  and  $\beta$  through

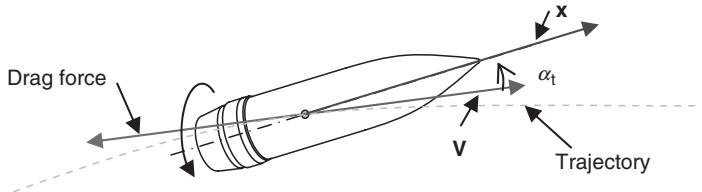
$$\sin \alpha_t = \sqrt{\sin^2 \beta + \cos^2 \beta \sin^2 \alpha} \quad (6.1)$$

The drag on a projectile is the force exerted on it by the medium through which it is moving, usually air. Since the drag is generated by the motion of the projectile through the air, it is naturally directed opposite to the velocity vector as illustrated in Figure 6.4.

There are, in general, two types of drag: pressure drag and skin friction drag. This is because nature can only act on the surface area of the projectile in two ways: normal to the surface and along it. A third type of drag called wave drag is a form of pressure drag generated by a shock wave formed when the local velocity along the surface of the projectile reaches Mach 1. We will discuss drag in further detail later but in all cases it is convenient to lump the effect of the drag into one coefficient called the drag coefficient. The drag force is defined in terms of this drag coefficient as



**FIGURE 6.3**  
Generalized yaw of a projectile.



**FIGURE 6.4**  
Drag of a projectile.

$$\text{Drag force} = F_D = \frac{1}{2} \rho S C_D V V = \frac{1}{2} \rho V^2 S C_D \quad (6.2)$$

Equation 6.2 shows two forms of the defining expression for drag force, vector and scalar. We shall define all of our forces and moments in this way because, although we will initially examine the scalar forms, it will be necessary later to use the vector forms. For now, knowing that the drag force is opposite to the velocity vector and that its scalar magnitude, as depicted on the far RHS in Equation 6.2, is sufficient.

Like many of the coefficients we shall discuss, the drag coefficient can be a complicated function of the yaw angle. In a more general form, we can write the drag coefficient as the sum of a linear part and a cubic term.

$$C_D = C_{D_0} + C_{D_{\delta^2}} \delta^2 \quad (6.3)$$

Here  $\delta$  is the total yaw defined as

$$\delta = \sin \alpha_t \quad (6.4)$$

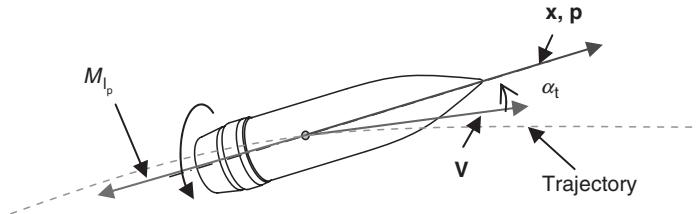
The first term on the RHS is the linear part of the drag coefficient, known as the zero-yaw drag coefficient, while the second part is the cubic term known as the yaw drag coefficient. The reason that there is no quadratic term is that for a symmetric body, the drag has to be the same whether the body is angled at, say,  $+5$  or  $-5^\circ$ . This is discussed more elegantly in Ref. [2]. The reason the nonlinear term is called a cubic term is that the drag coefficient is multiplied by the total yaw to yield

$$C_D \delta = C_{D_0} \sin \alpha_t + C_{D_{\delta^2}} \sin^3 \alpha_t \quad (6.5)$$

We shall see later that the drag coefficient varies with Mach number in a complex manner.

Dynamic pressure is a quantity defined as  $1/2\rho V^2$ , where  $\rho$  is the density of a fluid that an object is immersed in and  $V$  is the velocity of the fluid relative to the object. It is simply the physical reaction of the fluid when trying to force an object through it and occurs so often that it has been given its own name. This dynamic pressure is multiplied by a reference area,  $S$ . It is always important to know what reference area is used in the definition of the coefficients. In every case we shall examine, this reference area is based on the projectile circular cross-section. Also, as we shall soon see, moments require a length scale as well. In all of these instances, we shall use the projectile diameter as the reference length.

When a projectile spins in a medium, the viscous interaction of the medium and the projectile surface is such that the projectile will spin down throughout the flight. This phenomenon is accounted for by a moment applied to the projectile called the spin-damping moment. It is defined as



**FIGURE 6.5**  
Spin damping of a projectile.

$$\text{Spin-damping moment} = M_{l_p} = \frac{1}{2} \rho V^2 S d \left( \frac{pd}{V} \right) C_{l_p} \quad (6.6)$$

This moment is directed opposite to the spin vector,  $\mathbf{p}$ , of the projectile as depicted in Figure 6.5, and the tendency is for the projectile to spin down thus there is no negative sign in Equation 6.6, because the vector handles the decay. One needs to note that the figure is drawn for a right-hand twist. If a left-hand twist were involved, the spin vector,  $\mathbf{p}$  and the spin-damping moment vector would be reversed.

Some projectiles have fins or jets which impart a roll torque to the projectile such that the spin rate increases. This rolling moment is depicted in Figure 6.6 and defined through

$$\text{Rolling moment} = M_{l_\delta} = \frac{1}{2} \rho V^2 S d \delta_F C_{l_\delta} \quad (6.7)$$

In this expression,  $\delta_F$  is a cant angle provided to the fins to generate the lift required to sustain rotation.

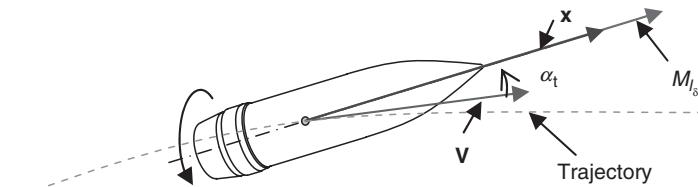
Lift is defined as the aerodynamic force which acts orthogonal to the velocity vector. This is depicted in Figure 6.7. The lift force can be defined in both scalar and vector notations as

$$\text{Lift force} = F_L = \frac{1}{2} \rho S C_{L_\alpha} [\mathbf{V} \times (\mathbf{x} \times \mathbf{V})] = \frac{1}{2} \rho V^2 S C_{L_\alpha} \delta \quad (6.8)$$

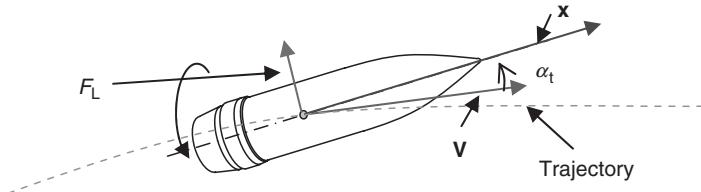
The lift force coefficient can be written in its nonlinear form as

$$C_{L_\alpha} = C_{L_{\alpha_0}} + C_{L_{\alpha_2}} \delta^2 \quad (6.9)$$

With a symmetric projectile, we must note that if there is no angle of attack (i.e.,  $\delta = 0$ ) there is no lift. This is obvious even for the linear case since  $\delta$  appears in Equation 6.8. Some



**FIGURE 6.6**  
Roll moment of a projectile.



**FIGURE 6.7**  
Lift vector of a projectile.

authors prefer to work in coordinates other than those we are utilizing here. In those cases, expressions such as  $C_x$  and  $C_N$  are used for drag and lift, respectively. In these cases, it is important that proper transformations are used to change the coefficients. An example of this is provided in Ref. [1].

At this point, we must discuss two quantities known as center of pressure (CP) and center of gravity (CG) (sometimes called center of mass). The CG is the location on the projectile where all of the mass can be concentrated so that for an analysis, the gravitational vector will operate at this point. The CP is the point through which a vector can be drawn, i.e., the resultant of all infinitesimal pressure forces acting on the projectile. For most projectiles that are spin-stabilized, the CP is ahead of the CG and the reverse is true with fin-or drag-stabilized projectiles. Figure 6.8 is an illustration of this.

The separation of the CP and CG gives rise to an overturning moment in all projectiles (Figure 6.9). As we shall see later, this moment is destabilizing for spin-stabilized projectiles (which is why they must be spun) and stabilizing for fin-stabilized projectiles. The overturning moment (sometimes called the pitching moment) is defined as

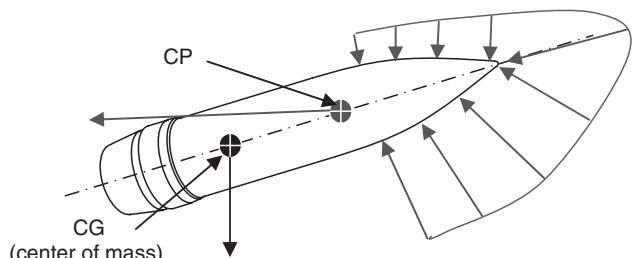
$$\text{Overturning moment} = M_\alpha = \frac{1}{2}\rho S d V C_{M_\alpha} (\mathbf{V} \times \mathbf{x}) = \frac{1}{2}\rho V^2 S d C_{M_\alpha} \delta \quad (6.10)$$

We can see from Equation 6.10 that this moment is a function of the angle of attack and because of the cross product, a positive overturning moment (nose up) is oriented along the positive z-axis.

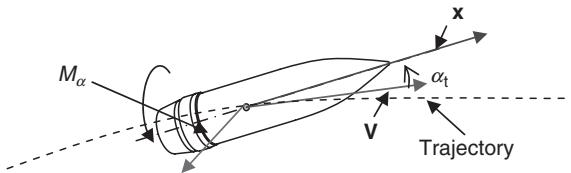
The overturning moment coefficient can be written in a nonlinear form similar to the lift and drag forces as

$$C_{M_\alpha} = C_{M_{\alpha_0}} + C_{M_{\alpha_2}} \delta^2 \quad (6.11)$$

When a body of circular cross-section is immersed in a flow-field perpendicular to its axis and is spun about its axis, a force known as the Magnus force is developed [6]. This



**FIGURE 6.8**  
Center of gravity (CG) and center of pressure (CP) illustrated.



**FIGURE 6.9**  
Overturning moment vector of a projectile.

force comes about because on one side of the body the free stream velocity of the flow is added to the velocity of the surface, while on the other side the free stream velocity is reduced by the surface velocity. On the basis of Bernoulli's equation (Equation 6.12), we see that along the body surface streamline, the pressure must be higher on the side with the lower velocity [7].

$$\frac{p}{\rho} + \frac{1}{2} V^2 + z = \text{constant} \quad (6.12)$$

This results in a side force on the body as illustrated in Figure 6.10.

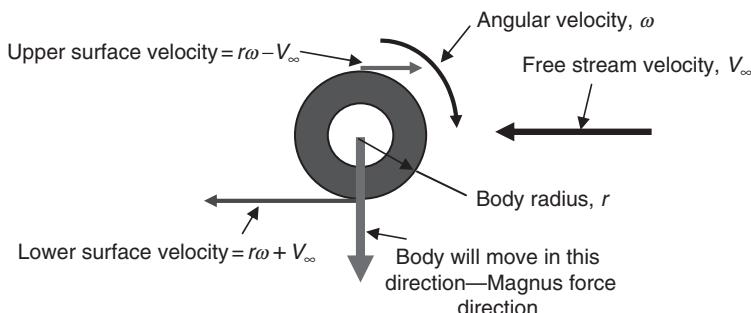
This might not seem like a big deal because a projectile almost never flies sideways, but if we consider a projectile in a crosswind, or, more importantly, one that is yawed, we see that this side component can contribute somewhat to the aerodynamic loading. For all practical purposes, however, if a projectile is not yawed in flight then there is no Magnus force. The Magnus force is defined for our purposes as

$$\text{Magnus force} = F_{N_{pa}} = \frac{1}{2} \rho S V \left( \frac{\omega d}{V} \right) C_{N_{pa}} (\mathbf{V} \times \mathbf{x}) = \frac{1}{2} \rho V^2 S \left( \frac{\omega d}{V} \right) C_{N_{pa}} \delta \quad (6.13)$$

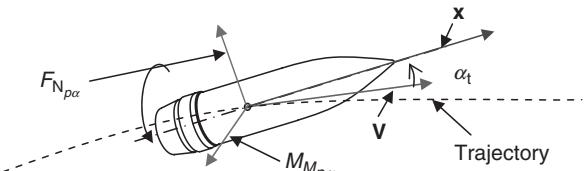
The Magnus force coefficient can be written in a nonlinear form in the same manner as [Equation 6.11](#) which we will not repeat ([Figure 6.11](#)).

In many cases, the Magnus force is small and is usually neglected with respect to the other forces acting on the projectile. In contrast, the moment developed because of this force is considerable. We define the Magnus moment as

$$\text{Magnus moment} = M_{M_{pa}} = \frac{1}{2} \rho V S d \left( \frac{\omega d}{V} \right) C_{M_{pa}} [\mathbf{x} \times (\mathbf{V} \times \mathbf{x})] = \frac{1}{2} \rho V^2 S d \left( \frac{\omega d}{V} \right) C_{M_{pa}} \delta \quad (6.14)$$



**FIGURE 6.10**  
Magnus effect on a projectile.



**FIGURE 6.11**

Magnus force and moment on a projectile.

The Magnus moment contributes significantly to the stability of the projectile and will be discussed in detail later. The Magnus moment coefficient can be written as a nonlinear term in the same way as all our other coefficients.

The CP for the lift force and the CP for the Magnus force are usually not the same, thus the moments will act through differing moment arms. The reason for this is the different physics that give rise to the different phenomena. These change during flight as well since a projectile's yaw changes as it moves downrange.

Pitch damping is the tendency of a projectile to cease its pitching motion due to air resistance. It is usually more difficult to visualize for someone new to the field. It is relatively simple to think about a right circular cylinder mounted in a fixture with its spin axis held by a frictionless bearing on each end. If we spin the projectile, it will slow down because of the sticking of the fluid to the surface and the resultant viscous action (remember the bearings are magically frictionless). If we mount the projectile such that the bearing is transverse to the long axis and spin it, we will still have the viscous action slowing the projectile down; however, this will be overwhelmed by the pressure forces that retard the motion and the projectile will spin down much faster. This combination of forces is called pitch damping. For projectiles, we can define the pitch damping force as

$$\text{Pitch damping force} = F_{N_{q+\dot{\alpha}}} = \frac{1}{2}\rho VSd \left( \frac{dx}{dt} \right) C_{N_q} + \frac{1}{2}\rho VSd C_{N_{\dot{\alpha}}} \left( \frac{dx}{dt} - \frac{dl}{dt} \right) \quad (6.15)$$

or in scalar terms,

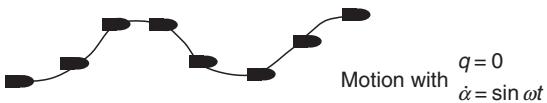
$$\text{Pitch damping force} = F_{N_{q+\dot{\alpha}}} = \frac{1}{2}\rho V^2 S \left[ \left( \frac{q_t d}{V} \right) C_{N_q} + \left( \frac{\dot{\alpha}_t d}{V} \right) C_{N_{\dot{\alpha}}} \right] \quad (6.16)$$

In Equation 6.16, we have defined the total pitching motion,  $q_t$ , and the total rate of change of angle of attack,  $\dot{\alpha}_t$ , as

$$q_t = \sqrt{q^2 + r^2} \text{ and } \dot{\alpha}_t = \frac{d\alpha_t}{dt} \quad (6.17)$$

We note here that this pitch damping comes about through two motions. The first motion is brought about through the pitching rate  $q$ , while the second is developed because of the resistance to the changing angle of attack. This is described in eloquent detail in Ref. [5]. The simplest way of depicting this is to assume a sinusoidal motion of a projectile along its flight path. With this assumption, Figure 6.12 shows what motions would result if  $q$  only was present and contrasts this with motion if  $\dot{\alpha}$  only were present.

It is generally difficult to separate  $q$  and  $\dot{\alpha}$  in experimental flight data. For this reason, the two coefficients are almost always written as a sum and recorded in the literature as such.



**FIGURE 6.12**  
Pictorial description of  $q$  and  $\dot{\alpha}$ .

With assumptions on the yawing motion of the projectile and the practice of combining coefficients, as described previously in Equations 6.15 and 6.16, they can be combined as detailed in Ref. [1] into

$$\begin{aligned}\text{Pitch damping force } F_{N_{q+\dot{\alpha}}} &= \frac{1}{2} \rho V S d \left( C_{N_q} + C_{N_{\dot{\alpha}}} \right) \frac{dx}{dt} \\ &= \frac{1}{2} \rho V^2 S d \left( \frac{q_t d}{V} \right) \left( C_{N_q} + C_{N_{\dot{\alpha}}} \right)\end{aligned}\quad (6.18)$$

The pitch damping force is, like the Magnus force, generally neglected because it is small with respect to the other forces such as lift and drag. The moment caused by this pitch damping is frequently significant (Figure 6.13). It can be described mathematically as follows:

Pitch damping moment

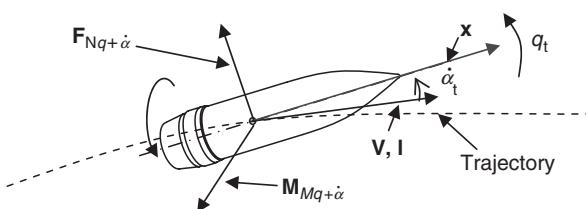
$$= M_{M_{q+\dot{\alpha}}} = \frac{1}{2} \rho V S d^2 \left( \mathbf{x} \times \frac{dx}{dt} \right) C_{M_q} + \frac{1}{2} \rho V S d^2 C_{M_{\dot{\alpha}}} \left[ \left( \mathbf{x} \times \frac{dx}{dt} \right) - \left( \mathbf{x} \times \frac{d\mathbf{l}}{dt} \right) \right]\quad (6.19)$$

In scalar form, we can write

$$\text{Pitch damping moment } M_{M_{q+\dot{\alpha}}} = M_{M_{q+\dot{\alpha}}} = \frac{1}{2} \rho V^2 S d \left[ \left( \frac{q_t d}{V} \right) C_{M_q} + \left( \frac{\dot{\alpha}_t d}{V} \right) C_{M_{\dot{\alpha}}} \right]\quad (6.20)$$

These can be simplified as per Ref. [1] into

$$\text{Pitch damping moment } M_{M_{q+\dot{\alpha}}} = M_{M_{q+\dot{\alpha}}} = \frac{1}{2} \rho V S d^2 \left( C_{M_q} + C_{M_{\dot{\alpha}}} \right) \left( \mathbf{x} \times \frac{dx}{dt} \right)\quad (6.21)$$



**FIGURE 6.13**  
Pitch damping force and moment on a projectile.

$$\text{Pitch damping moment} = M_{M_{q+\dot{\alpha}}} = \frac{1}{2} \rho V^2 S d \left( \frac{q_t d}{V} \right) [C_{M_q} + C_{M_{\dot{\alpha}}}] \quad (6.22)$$

At certain times and in some special cases, there are other combinations of forces and moments and therefore additional coefficients require attention. We will not go any further here as this text is meant to be most general.

We now have the basic terms defined that we shall use in our study of exterior ballistics.

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## Further Reading

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## Dynamics Review

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Throughout the study of exterior ballistics, dynamics play a great role in the flight of the projectile. The Coriolis effect in long-range trajectories or the drag changes due to the precessional and nutational motion of the projectile are just two examples of the effect of projectile body dynamics on flight. We will find that at least a cursory review of dynamics is essential to the understanding of projectile motion. Analyzing dynamics of projectile flight is best approached through the use of vectors and we will begin our review with their study.

A vector is defined as a quantity having a magnitude and a direction. Two vectors are considered equal if both their magnitude and direction are identical. However, this does not mean that they have to originate at the same point, i.e., a translation has no effect on whether vectors are equal. A scalar is simply a numerical quantity (a magnitude). When a scalar and a vector are multiplied (in any order) they form a vector. Thus, we can define any vector as a scalar magnitude multiplied by a vector of unit length (a unit vector) in the proper direction (Figure 7.1).

$$\mathbf{A} = A \mathbf{e}_A \quad (7.1)$$

A vector can be written as the sum of its scalar magnitude in each individual coordinate direction times a unit vector in that particular direction.

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \quad (7.2)$$

The magnitude of the vector is defined as

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (7.3)$$

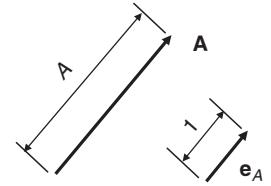
Vectors may be added together in any order by summing the individual components in each direction. This is the commutative property:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = (A_x + B_x) \mathbf{i} + (A_y + B_y) \mathbf{j} + (A_z + B_z) \mathbf{k} \quad (7.4)$$

The following is also true when adding more than one vector together:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad (7.5)$$

Equation 7.5 represents the associative property of vectors. In all of the above expressions, note that  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are the unit vectors in the  $x$ ,  $y$ , and  $z$  coordinate directions, respectively.



**FIGURE 7.1**  
Vector and associated unit vector.

Multiplication of vectors can occur in two different ways—each applicable to particular situations. Consider two vectors  $\mathbf{A}$  and  $\mathbf{B}$  shown in Figure 7.2, we define the scalar product or dot product as

$$\mathbf{A} \cdot \mathbf{B} = A \cdot B \cdot \cos \theta \quad (7.6)$$

Both the commutative and associative laws of multiplication apply to the dot product.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad (7.7)$$

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \quad (7.8)$$

The dot product of two vectors is given by

$$\mathbf{A} \cdot \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) \quad (7.9)$$

This equation when expanded is

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_x B_x \mathbf{i} \cdot \mathbf{i} + A_x B_y \mathbf{i} \cdot \mathbf{j} + A_x B_z \mathbf{i} \cdot \mathbf{k} + A_y B_x \mathbf{j} \cdot \mathbf{i} + A_y B_y \mathbf{j} \cdot \mathbf{j} + A_y B_z \mathbf{j} \cdot \mathbf{k} + A_z B_x \mathbf{k} \cdot \mathbf{i} \\ &\quad + A_z B_y \mathbf{k} \cdot \mathbf{j} + A_z B_z \mathbf{k} \cdot \mathbf{k} \end{aligned} \quad (7.10)$$

However, since the unit vectors are orthogonal, and the dot product of two orthogonal vectors is identically zero while the dot product of parallel vectors is unity as follows from

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \cdot 1 \cdot \cos(0^\circ) = 1$$

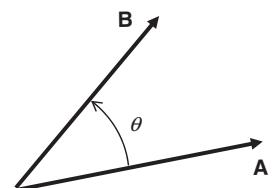
$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 1 \cdot 1 \cdot \cos(90^\circ) = 0$$

Therefore, we can write

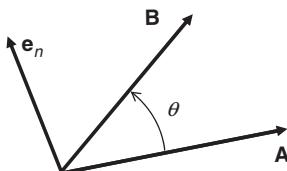
$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad (7.11)$$

The second type of vector multiplication is the vector or cross product, which is defined as

$$\mathbf{A} \times \mathbf{B} = A \cdot B \cdot \sin \theta \mathbf{e}_n \quad (7.12)$$



**FIGURE 7.2**  
Vector pair illustrated.



**FIGURE 7.3**  
Vector cross product normal unit vector.

Here  $\mathbf{e}_n$  is a unit vector normal to the plane made by vectors  $\mathbf{A}$  and  $\mathbf{B}$ . This is depicted in Figure 7.3. The cross product does not obey the commutative property because

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (7.13)$$

The distributive property, however, does apply to the cross product. Thus,

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \quad (7.14)$$

The cross product of two vectors is given by

$$\mathbf{A} \times \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) \quad (7.15)$$

When expanded, Equation 7.15 can be written as

$$\begin{aligned} \mathbf{A} \times \mathbf{B} = & A_x B_x \mathbf{i} \times \mathbf{i} + A_x B_y \mathbf{i} \times \mathbf{j} + A_x B_z \mathbf{i} \times \mathbf{k} + A_y B_x \mathbf{j} \times \mathbf{i} + A_y B_y \mathbf{j} \times \mathbf{j} + A_y B_z \mathbf{j} \times \mathbf{k} \\ & + A_z B_x \mathbf{k} \times \mathbf{i} + A_z B_y \mathbf{k} \times \mathbf{j} + A_z B_z \mathbf{k} \times \mathbf{k} \end{aligned} \quad (7.16)$$

Since the unit vectors are orthogonal,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 1 \cdot 1 \cdot \sin(0^\circ) \mathbf{e}_n = 0 \text{ and } \mathbf{i} \times \mathbf{j} = 1 \cdot 1 \cdot \sin(90^\circ) = \mathbf{e}_n$$

But, since we have a right-handed coordinate system, by the right-hand rule, the normal to  $\mathbf{i}$  and  $\mathbf{j}$  is the unit vector  $\mathbf{k}$ , thus  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ . We can also invoke Equation 7.13 to get  $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$ . We can carry this logic further to show that  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$  or  $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$  and  $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$  or  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ . Thus, we can rewrite Equation 7.16 as

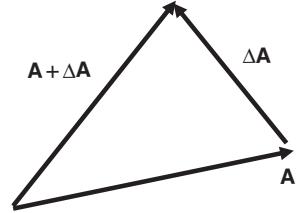
$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \mathbf{i} + (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k} \quad (7.17)$$

This Equation 7.17 is the following determinant expanded by its minors:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (7.18)$$

We will proceed next to the calculus of vectors. Let us consider a vector,  $\mathbf{A}$ , dependent upon a scalar variable,  $u$ , as shown in Figure 7.4. Then  $\mathbf{A} + \Delta\mathbf{A}$  corresponds to  $u + \Delta u$  and we can write for its derivative

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta\mathbf{A}}{\Delta u} \quad (7.19)$$



**FIGURE 7.4**  
Vector sum illustrated.

Differentiation is distributive so that

$$\frac{d(\mathbf{A} + \mathbf{B})}{du} = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \quad (7.20)$$

The chain rule also applies for scalar and vector products so that

$$\frac{d}{du}(g\mathbf{A}) = \frac{dg}{du}\mathbf{A} + g\frac{d\mathbf{A}}{du} \quad (7.21)$$

$$\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{du} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \quad (7.22)$$

$$\frac{d}{du}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{du} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{du} \quad (7.23)$$

Consider a vector  $\mathbf{A}$  dependent upon time,  $t$ . If we take its derivative with respect to time, we get

$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j} + \frac{dA_z}{dt}\mathbf{k} + A_x \frac{d\mathbf{i}}{dt} + A_y \frac{d\mathbf{j}}{dt} + A_z \frac{d\mathbf{k}}{dt} \quad (7.24)$$

If the coordinate system is inertial (i.e., it does not move), we can write

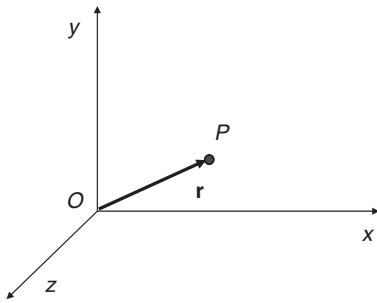
$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j} + \frac{dA_z}{dt}\mathbf{k} \quad (7.25)$$

If the coordinate system is moving (like on a rotating earth), the rate of change terms for the unit vectors cannot be neglected. This gives rise to what we call “Coriolis terms” as we shall discuss later.

We will now examine the kinematics of a particle. Kinematics is the study of the motion of particles and rigid bodies without regard to the forces which generate the motion. Particle kinematics assumes that a point can represent the body. The rotations of the particle itself are neglected making this a three degree of freedom (DOF) model. If we have the inertial reference frames  $x$ ,  $y$ , and  $z$ , the position of a particle,  $P$ , is defined by a position vector,  $\mathbf{r}$ , drawn from the origin to the particle as is shown in [Figure 7.5](#).

If the particle,  $P$ , moves along a trajectory,  $T$ , its instantaneous velocity is always in a direction tangent to the trajectory and its magnitude is the speed at which it moves along the curve. Thus, the tip of this vector,  $\mathbf{r}$ , traces out the trajectory ([Figure 7.6](#)) and the velocity,  $\mathbf{v}$ , is defined as the time rate of change of  $\mathbf{r}$ , written as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad (7.26)$$



**FIGURE 7.5**  
Position vector.

If we were to take the velocity vector at every instant of time and fix its tail to the origin of an inertial coordinate system, then the curve traced out by its tip would be called a hodograph (Figure 7.7) and the velocity of the tip would be the time rate of change of velocity or the acceleration,  $\mathbf{a}$ . Thus, we can write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad (7.27)$$

Now, if we examine the particle as moving in two dimensions only, we can break its motion up into two components, one parallel to and one perpendicular to the position vector,  $\mathbf{r}$  (Figure 7.8). The position vector written in this coordinate system is given by

$$\mathbf{r} = r\mathbf{e}_r \quad (7.28)$$

So from our definition for the velocity in Equation 7.26, we get

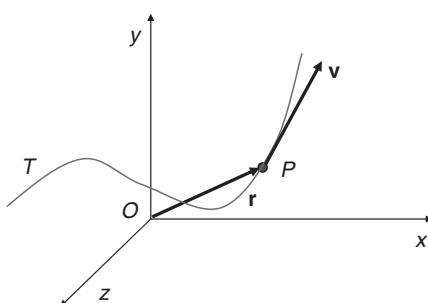
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \quad (7.29)$$

Since  $\mathbf{e}_r$  is a unit vector (its magnitude is a constant = 1) the only thing that changes with time is its direction.

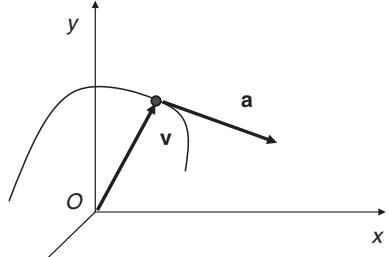
This introduces the concept of curvilinear motion with radial coordinates,  $(r, \theta)$ . The direction is defined by the angle,  $\theta$ . For a small change in the angle,  $\theta$ , we can write

$$\frac{d\mathbf{e}_r}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{e}_r}{\Delta t} \quad (7.30)$$

However, we can observe that for small angles



**FIGURE 7.6**  
Trajectory curve.



**FIGURE 7.7**  
Hodograph.

$$\Delta \mathbf{e}_r = |\mathbf{e}_r| \sin(\Delta\theta) = (1) \sin(\Delta\theta) \approx \Delta\theta \quad (7.31)$$

We also see from [Figure 7.9](#) that  $\Delta \mathbf{e}_r$  acts in the  $\mathbf{e}_\theta$  direction thus,

$$\Delta \mathbf{e}_r \approx \Delta\theta \mathbf{e}_\theta \quad (7.32)$$

Then, returning to [Equation 7.30](#) we can write

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \mathbf{e}_\theta \quad (7.33)$$

Now, we can insert [Equation 7.33](#) into [Equation 7.29](#) to get the desired relation for the velocity.

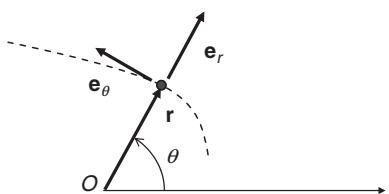
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta \quad (7.34)$$

The first term on the RHS of [Equation 7.34](#) is the radial velocity, the second term is the tangential velocity sometimes denoted as  $v_r$  and  $v_\theta$ , respectively. The magnitude of the velocity is given by

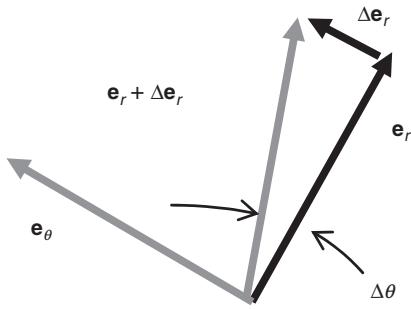
$$v = |\mathbf{v}| = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2} \quad (7.35)$$

To obtain the acceleration in curvilinear coordinates, we need to take the time derivative of [Equation 7.34](#) as follows:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta \right) = \frac{d^2r}{dt^2} \mathbf{e}_r + \frac{dr}{dt} \frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta + r \frac{d^2\theta}{dt^2} \mathbf{e}_\theta + r \frac{d\theta}{dt} \frac{d\mathbf{e}_\theta}{dt} \quad (7.36)$$



**FIGURE 7.8**  
Differentiation of a vector through use of tangential and radial unit vectors.



**FIGURE 7.9**  
Rotation of the radial unit vector.

We have already solved for the derivative of  $\mathbf{e}_r$  with respect to time; now, in a similar manner, we will find the derivative of the tangential component. Again, since  $\mathbf{e}_\theta$  is a unit vector, the only thing that changes with time is its direction. This direction is again defined by the angle,  $\theta$ , so for a small change in the angle,  $\theta$ , we can write

$$\frac{d\mathbf{e}_\theta}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{e}_\theta}{\Delta t} \quad (7.37)$$

But we see again that for small angles

$$\Delta \mathbf{e}_\theta = |\mathbf{e}_\theta| \sin(\Delta\theta) = (1) \sin(\Delta\theta) \approx \Delta\theta \quad (7.38)$$

which acts in the negative  $\mathbf{e}_r$  direction as depicted in Figure 7.10. If

$$\Delta \mathbf{e}_\theta \approx -\Delta\theta \mathbf{e}_r \quad (7.39)$$

then we can write,

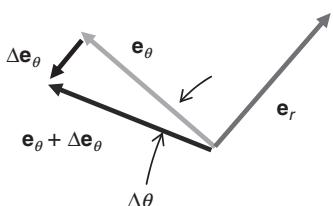
$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = -\frac{d\theta}{dt} \mathbf{e}_r \quad (7.40)$$

Insertion of Equations 7.33 and 7.40 into Equation 7.36 yields

$$\mathbf{a} = \frac{d^2 r}{dt^2} \mathbf{e}_r + \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta + \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta + r \frac{d^2 \theta}{dt^2} \mathbf{e}_\theta - r \left( \frac{d\theta}{dt} \right)^2 \mathbf{e}_r \quad (7.41)$$

Rearranging and combining like terms gives us

$$\mathbf{a} = \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{e}_r + \left( r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta \quad (7.42)$$



**FIGURE 7.10**  
Rotation of the tangential unit vector.

Each of these terms has a specific name and meaning in the dynamics of a body

$$\frac{d^2r}{dt^2} = \text{Radial acceleration}$$

$$r\left(\frac{d\theta}{dt}\right)^2 = \text{Centripetal acceleration}$$

$$\frac{d^2\theta}{dt^2} = \text{Angular acceleration}$$

$$2\frac{dr}{dt}\frac{d\theta}{dt} = \text{Coriolis acceleration}$$

To move on in our study, we need to examine the planar kinematics of a rigid body. First, we will examine a pure translation where we have a body-fixed coordinate system moving relative to our inertial coordinate system. Here we note from Figure 7.11 that by vector addition, we obtain

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \quad (7.43)$$

To determine the velocity of point  $B$ , which is under a pure translation, we have to differentiate Equation 7.43 to get

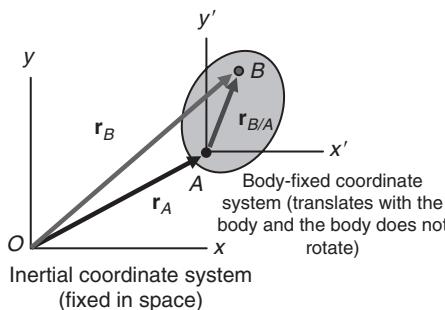
$$\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt} \quad (7.44)$$

We know, however, that since this is a pure translation (no rotation)  $\frac{d\mathbf{r}_{B/A}}{dt} = 0$  and  $\frac{d\mathbf{r}_A}{dt} = \mathbf{v}_A$ . Thus, for a pure translation,

$$\mathbf{v}_B = \mathbf{v}_A \quad (7.45)$$

If we differentiate Equation 7.45, we get the acceleration of a point during a pure translation as

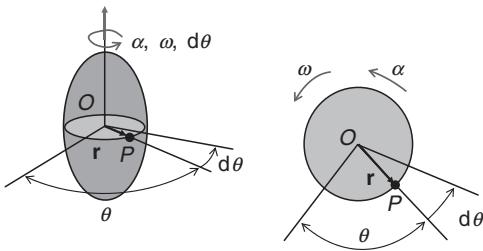
$$\mathbf{a}_B = \frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} = \mathbf{a}_A \quad (7.46)$$



- $\mathbf{r}_A$ =Position vector of point  $A$
- $\mathbf{r}_B$ =Position vector of point  $B$
- $\mathbf{r}_{B/A}$ =Relative position vector of point  $B$  with respect to point  $A$
- $\mathbf{v}_A$ =Velocity of point  $A$
- $\mathbf{v}_B$ =Velocity of point  $B$
- $\mathbf{v}_{B/A}$ =Relative velocity of point  $B$  with respect to point  $A$
- $\mathbf{a}_A$ =Acceleration of point  $A$
- $\mathbf{a}_B$ =Acceleration of point  $B$
- $\mathbf{a}_{B/A}$ =Acceleration of point  $B$  with respect to point  $A$

**FIGURE 7.11**

Definition of vectors associated with rigid body translational motion.



**FIGURE 7.12**

Example of rigid body rotation. On the left is the body rotating in space. On the right is a view of this same body looking down the axis toward  $O$ .

We will now examine the rotation of a body fixed in space. Let us define the angular velocity,  $\omega$ , as the time rate of change of angular position,  $\theta$ , thus

$$\omega = \frac{d\theta}{dt} \quad (7.47)$$

Let us further define the angular acceleration,  $\alpha$ , as the time rate of change of angular velocity, or

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad (7.48)$$

The angular velocity,  $\omega$ , and the angular acceleration,  $\alpha$ , are depicted in Figure 7.12. We will now look at the rotation in terms of the vector kinematic equations for point,  $P$ . We first examine the velocity whose direction we specify by the right-hand rule. Then let us define the position of point,  $P$ , by the position vector,  $\mathbf{r}$ , as shown in Figure 7.13. Now we can write the velocity,  $\mathbf{v}$ , in terms of radial and circumferential components as we discussed earlier.

Thus, from [Equation 7.34](#) we have

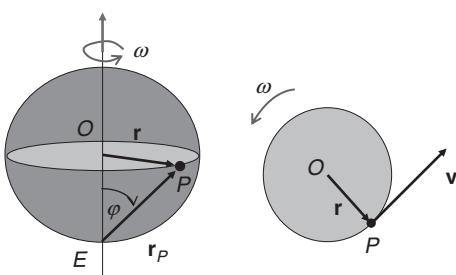
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta \quad (7.49)$$

But, since this is a rigid body,  $dr/dt = 0$ , so we get

$$\mathbf{v} = r\frac{d\theta}{dt}\mathbf{e}_\theta = r\omega\mathbf{e}_\theta \quad (7.50)$$

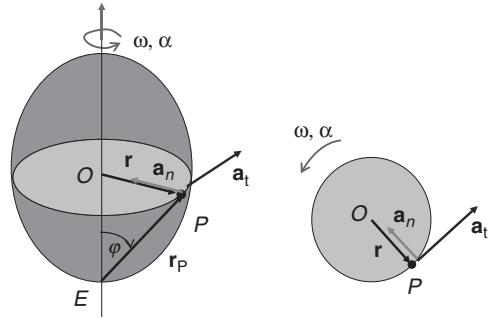
If we were instead to draw the position vector from a more general location such as point  $E$  in Figure 7.13, we see that

$$r = r_P \sin \phi \quad (7.51)$$



**FIGURE 7.13**

Rigid body rotation. On the left is the body rotating at angular velocity,  $\omega$ . On the right is a view of this same body looking down the axis toward  $O$ .



**FIGURE 7.14**

Rigid body rotation with acceleration. On the left is the body rotating at angular velocity,  $\omega$ , and accelerating with angular acceleration,  $\alpha$ . On the right is a view of this same body looking down the axis toward  $O$ .

If we substitute Equation 7.51 into Equation 7.50, we see a form we have derived earlier.

$$\mathbf{v} = r_P \boldsymbol{\omega} \sin \phi \mathbf{e}_\theta \quad (7.52)$$

This can be written in vector form if we invoke Equation 7.12. Thus, we have

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P \quad (7.53)$$

We now will examine the acceleration whose direction is once more specified by the right-hand rule. We shall define the position of point,  $P$ , by the position vector,  $\mathbf{r}$ , as shown in Figure 7.14. We can then write the acceleration,  $\mathbf{a}$ , in terms of radial and circumferential components as discussed earlier.

We need to recall Equation 7.42 and note that  $d\mathbf{r}/dt = 0$ . This leaves us with

$$\mathbf{a} = \left[ -r \left( \frac{d\theta}{dt} \right)^2 \right] \mathbf{e}_r + \left( r \frac{d^2\theta}{dt^2} \right) \mathbf{e}_\theta \quad (7.54)$$

Here we need to note that the first term is negative because it acts in the negative radial direction. Now, from our previous definitions we can rewrite Equation 7.54 as

$$\mathbf{a} = (-r\boldsymbol{\omega}^2) \mathbf{e}_r + (r\alpha) \mathbf{e}_\theta \quad (7.55)$$

We can further state that

$$\mathbf{a}_n = -r\boldsymbol{\omega}^2 \text{ and } \mathbf{a}_t = r\alpha \quad (7.56)$$

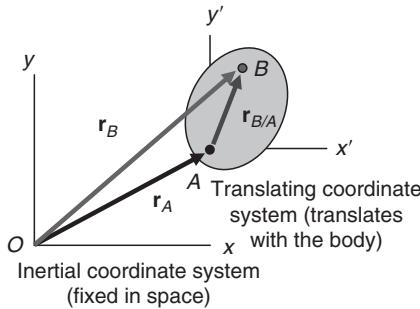
We can differentiate Equation 7.53 to obtain the more general result

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt} \quad (7.57)$$

If we insert Equations 7.48 and 7.53 into Equation 7.57, we get the general vector form for the acceleration of a rigid body rotating in an inertial coordinate system.

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) \quad (7.58)$$

To move closer toward a more general treatment, we shall now derive the kinematic equations for plane motion of a rigid body using a translating coordinate system (the



$\mathbf{r}_A$	= Position vector of point A
$\mathbf{r}_B$	= Position vector of point B
$\mathbf{r}_{B/A}$	= Relative position vector of point B with respect to point A
$\mathbf{v}_A$	= Velocity of point A
$\mathbf{v}_B$	= Velocity of point B
$\mathbf{v}_{B/A}$	= Relative velocity of point B with respect to point A
$\mathbf{a}_A$	= Acceleration of point A
$\mathbf{a}_B$	= Acceleration of point B
$\mathbf{a}_{B/A}$	= Acceleration of point B with respect to point A

FIGURE 7.15

Definition of vectors associated with rigid body translational and rotational motion.

body is free to rotate). We can break down any planar motion of the rigid body into a translation and a rotation about some point. Let us choose point  $A$  in Figure 7.15 to be a location about which the body rotates. [Equation 7.43](#) is still valid, but  $d\mathbf{r}/dt$  no longer equals zero.

Thus, from [Equation 7.44](#), we have

$$\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt} \quad (7.59)$$

Not only is  $d\mathbf{r}_{B/A}/dt \neq 0$  but since we chose point  $A$  as one about which rotation will take place,  $d\mathbf{r}_A/dt$  is a pure translation and  $d\mathbf{r}_{B/A}/dt$  is a pure rotation about point  $A$ . Thus, we can write [Equation 7.59](#) in terms of the velocities as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad (7.60)$$

We saw earlier that for a pure rotation, we can write the velocity in the form of [Equation 7.53](#). Thus, we have

$$\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (7.61)$$

If we substitute [Equation 7.61](#) into [Equation 7.60](#), we obtain the vector equation for planar motion of a rigid body in which our coordinate system translates with a point in the body but does not rotate with the body.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (7.62)$$

To obtain the acceleration of point  $B$ , we need to differentiate [Equation 7.60](#), thus

$$\mathbf{a}_B = \frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt} \quad (7.63)$$

Let us examine this equation term by term. The first term is straightforward, showing that the acceleration of the translation is simply the linear acceleration of our chosen reference point.

$$\frac{d\mathbf{v}_A}{dt} = \mathbf{a}_A \quad (7.64)$$

The second term is differentiated as follows:

$$\frac{d\mathbf{v}_{B/A}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_{B/A}) = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times \frac{d\mathbf{r}_{B/A}}{dt} \quad (7.65)$$

We again need to call upon Equations 7.48 and 7.53 to get Equation 7.65 into a more general form.

$$\frac{d\mathbf{v}_{B/A}}{dt} = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \quad (7.66)$$

Inserting Equations 7.64 and 7.66 into Equation 7.63 yields the kinematic equation for the acceleration of a rigid body in a coordinate system that translates with the body. The coordinate system in this case moves with the body but does not rotate allowing the body to rotate relative to the moving coordinate system.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \quad (7.67)$$

We shall now derive the kinematic equations for plane motion of a rigid body using a translating and rotating coordinate system (the body is capable of movement in both coordinate systems). We choose point  $A$  in Figure 7.16 to be a location from which we want to measure the motion of the body. At the instant considered, point  $A$  has a position,  $\mathbf{r}_A$ , a velocity,  $\mathbf{v}_A$ , and an acceleration  $\mathbf{a}_A$ , while the  $x$ - $y$  axes (and the body) are rotating with angular velocity,  $\boldsymbol{\omega}$ , and accelerating with angular acceleration,  $\boldsymbol{\alpha}$ .

Equation 7.43 is still valid for determination of the position vectors. For the velocity of point  $B$ , we shall use the form

$$\mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt} \quad (7.68)$$

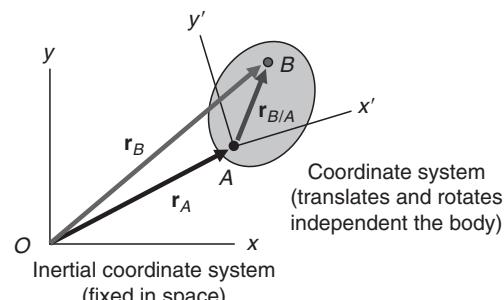
If we examine our earlier derivations under the curvilinear motion and replace  $\mathbf{e}_r$  with  $\mathbf{i}$  and  $\mathbf{e}_\theta$  with  $\mathbf{j}$  ( $\mathbf{i}$  and  $\mathbf{j}$  represent the unit vectors in our  $x$ - $y$  coordinate system), we obtain the following relations:

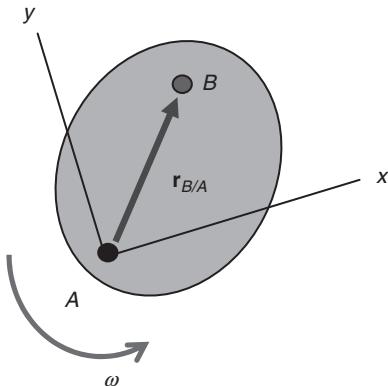
$$\frac{d\mathbf{i}}{dt} = \frac{d\theta}{dt} \mathbf{j} = \boldsymbol{\omega} \mathbf{j} \quad (7.69)$$

$$\frac{d\mathbf{j}}{dt} = \frac{d\theta}{dt} (-\mathbf{i}) = -\boldsymbol{\omega} \mathbf{i} \quad (7.70)$$

**FIGURE 7.16**

Definition of vectors associated with rigid body translational and rotational motion including a rotating local coordinate system.





**FIGURE 7.17**  
Body rotating in moving coordinate system.

Using the definition of the cross product and noting they are orthogonal, Equations 7.69 and 7.70 can be rewritten as

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i} \quad (7.71)$$

$$\frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j} \quad (7.72)$$

If we look at the body and our moving coordinate system as shown in Figure 7.17, we see that if the body translates and rotates relative to our  $x$ - $y$  axes we can write, in light of Equation 7.62

$$\frac{d\mathbf{r}_{B/A}}{dt} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (7.73)$$

Substituting this into Equation 7.68 yields the general relation for the velocity of a point in a rigid body as seen from an arbitrary coordinate system.

$$\mathbf{v}_B = \mathbf{v}_A + (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (7.74)$$

To obtain the acceleration of our point  $B$ , we need to differentiate Equation 7.74 with respect to time, thus

$$\mathbf{a}_B = \frac{d\mathbf{v}_A}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times \frac{d\mathbf{r}_{B/A}}{dt} \quad (7.75)$$

Let us again move term by term through Equation 7.75. The first term is

$$\frac{d\mathbf{v}_A}{dt} = \mathbf{a}_A \quad (7.76)$$

Since our point  $A$  does not rotate, the acceleration of its translation is simply this linear acceleration.

The second term is differentiated by first breaking up  $\mathbf{v}_{B/A}$  into its components along the  $x$  and  $y$  axes of our moving frame. Since we are looking only at motion in the  $x$ - $y$  plane, the  $z$  component is nonexistent.

$$(\mathbf{v}_{B/A})_{xyz} = (\mathbf{v}_{B/A})_x \mathbf{i} + (\mathbf{v}_{B/A})_y \mathbf{j} \quad (7.77)$$

This allows us to write the second term as

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = \frac{d(\mathbf{v}_{B/A})_x}{dt} \mathbf{i} + \frac{d(\mathbf{v}_{B/A})_y}{dt} \mathbf{j} + (\mathbf{v}_{B/A})_x \frac{d\mathbf{i}}{dt} + (\mathbf{v}_{B/A})_y \frac{d\mathbf{j}}{dt} \quad (7.78)$$

The first pair of terms in Equation 7.78 are the acceleration components of point  $B$  relative to point  $A$  as seen by an observer moving with the coordinate system at point  $A$ . The second pair of terms of Equation 7.78 can be rewritten as the cross product of the angular velocity of the  $x$ - $y$  coordinate system and the velocity vector of point  $B$  relative to point  $A$ . So we can write

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = (\mathbf{a}_{B/A})_{xyz} + \boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{xyz} \quad (7.79)$$

Returning now to [Equation 7.75](#), in the third term, we simply rewrite the term  $d\boldsymbol{\omega}/dt$  as  $\boldsymbol{\alpha}$ . Finally, for the last term of Equation 7.75, we use [Equation 7.73](#) to obtain

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_{xyz} + \boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \quad (7.80)$$

After a slight rearrangement and combination of like terms, we have the general kinematic equation for acceleration of point  $B$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \quad (7.81)$$

It is important to review each of the terms in [Equations 7.74](#) and 7.81. First, let us review the generalized velocity and acceleration equations we derived.

$$\mathbf{v}_B = \mathbf{v}_A + (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (7.74)$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \quad (7.81)$$

The terms in these equations have meanings as tabulated in Table 7.1.

As one can imagine the addition of the third dimension in these equations adds significant complexity to the expressions although the basic principles remain the same. The

**TABLE 7.1**

Vector Terms Used in Equations 7.74 and 7.81

Variable	Definition
$\mathbf{r}_{B/A}$	Relative position vector of point $B$ with respect to point $A$
$\mathbf{v}_A$	Velocity of point $A$ in the inertial coordinate system
$\mathbf{v}_B$	Velocity of point $B$ in the inertial coordinate system
$(\mathbf{v}_{B/A})_{xyz}$	Relative velocity of point $B$ with respect to point $A$ in the $xyz$ coordinate system
$\mathbf{a}_A$	Acceleration of point $A$ in the inertial coordinate system
$\mathbf{a}_B$	Acceleration of point $B$ in the inertial coordinate system
$(\mathbf{a}_{B/A})_{xyz}$	Acceleration of point $B$ with respect to point $A$ in the $xyz$ coordinate system
$\boldsymbol{\omega}$	Angular velocity of the $xyz$ coordinate system measured from the inertial coordinate system
$\boldsymbol{\alpha}$	Angular acceleration of the $xyz$ coordinate system measured from the inertial coordinate system

interested reader is referred to Ref. [1] or any similar text on dynamics to familiarize themselves with the three-dimensional uses of these equations.

### Problem 1

A 155-mm projectile is in flight at its maximum ordinate. At this instant in time, the nose of the projectile is pointing along (and spinning about) the unit vector:

$$\mathbf{x} = (0.998\mathbf{e}_1 + 0.030\mathbf{e}_2 + 0.056\mathbf{e}_3)$$

The projectile velocity vector is

$$\mathbf{V} = (1199\mathbf{e}_1 + 0\mathbf{e}_2 + 49\mathbf{e}_3) \left[ \frac{\text{ft}}{\text{s}} \right]$$

In both of these cases,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are unit vectors in the  $x$ ,  $y$ , and  $z$  planes, respectively. Also at this location the air density, spin rate, and projectile mass are as follows:

$$\rho = 0.052 \left[ \frac{\text{lbm}}{\text{ft}^3} \right], p = \omega = 150 \left[ \frac{\text{rev}}{\text{s}} \right], \text{ and } m = 100[\text{lbm}]$$

The projectile characteristics are assumed to be

$$C_D = 0.29$$

$$C_{M_\alpha} = 3.0 \quad (C_{M_q} + C_{M_{\dot{\alpha}}}) = -10.2$$

$$C_{L_\alpha} = 2.12 \quad (C_{N_q} + C_{N_{\dot{\alpha}}}) = 0.002$$

$$C_{N_{p\alpha}} = -0.010 \quad C_{M_{p\alpha}} = 0.51$$

$$C_l_p = -0.015$$

Please answer the following questions:

1. Draw the situation.

2. Determine the drag force vector.

*Answer:  $F_D = (-68.39\mathbf{e}_1 - 2.80\mathbf{e}_3) [\text{lbf}]$*

3. Determine the lift force vector.

*Answer:  $F_L = (-0.310\mathbf{e}_1 + 14.980\mathbf{e}_2 + 7.600\mathbf{e}_3) [\text{lbf}]$*

4. Determine the overturning moment vector.

*Answer:  $M_M = (-0.441\mathbf{e}_1 - 5.468\mathbf{e}_2 + 10.783\mathbf{e}_3) [\text{ft-lbf}]$*

5. Determine the magnus moment vector.

*Answer:  $M_{M_{pa}} = (0.043\mathbf{e}_1 - 0.732\mathbf{e}_2 - 0.369\mathbf{e}_3) [\text{ft-lbf}]$*

### Reference

- Greenwood, D.T., *Principles of Dynamics*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1988.

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## Further Reading

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- Colley, S.J., *Vector Calculus*, Prentice Hall, Upper Saddle River, NJ, 1998.
- Hibbeler, R.C., *Engineering Mechanics—Statics and Dynamics*, 7th ed., Prentice Hall, Englewood Cliffs, NJ, 1995.
- O'Neil, P.V., *Advanced Engineering Mathematics*, 2nd ed., Wadsworth Publishing Co., Belmont, CA, 1986.

# 8

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## Trajectories

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Now that the basics of the terminology and the dynamic equations have been presented, we shall begin to look at their uses in the form of prediction of trajectories. The aero-ballistician is usually faced with one of two problems: "If I want to hit a target at position  $x$ , to what elevation (and perhaps with how much propelling charge) do I have to elevate the weapon?" or "My weapon is elevated to elevation  $x$  and I expect muzzle velocity  $y$ —where is the projectile going to end up?"

To approach this in a logical and easily understandable fashion, we shall begin with a great many simplifying assumptions, relieving these as we progress. Each section builds upon the previous one so that we recommend even seasoned veterans progress in numerical order.

Initially, we will only look at the effect that gravity imposes on the projectile, a vacuum trajectory, so that even the air is removed from our area of concern thus neutralizing the fluid mechanics for a while. As we progress, we shall add in the atmosphere but neglect dynamics, atmospheric perturbations, and earth rotation. One by one we shall continually step up the complexity until finally we shall introduce the full six degree-of-freedom (6 DOF) equations.

One might initially think that these simplified models have no practical use, but are merely educational stepping stones. Nothing could be further from the truth. In many instances, some of the complications only slightly affect the solution and a ballistian is well placed to assume them away. Some of these common situations will be pointed out as they arise.

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### 8.1 Vacuum Trajectory

In this section, we will make two broad assumptions: First, that the projectile mass is concentrated at a point (which allows us to neglect body dynamics affected by mass distribution) and second, that the only force acting on the projectile is that due to the acceleration of gravity (this allows us to neglect the rather complicated fluid dynamic effects when a solid body moves through a fluid). With these assumptions, the two governing differential equations of motion are

$$m\ddot{x} = 0 \quad (8.1)$$

$$\ddot{y} = -g \quad (8.2)$$

The solutions to these equations, found by integrating with respect to time, are

$$x = V_0 t \cos \phi_0 \quad (8.3)$$

$$y = V_0 t \sin \phi_0 - \frac{1}{2} g t^2 \quad (8.4)$$

A sketch of this simplified trajectory is seen in Figure 8.1 below. Notice that unlike the generalized trajectory shown previously in [Figure 6.1](#), the terminal point is on the line  $y=0$  and the entire trajectory is in the  $x-y$  plane.

But from [Equations 8.3](#) and 8.4, we can write

$$\frac{y}{x} = \tan \phi_0 - \frac{gt}{2V_0 \cos \phi_0} \quad (8.5)$$

By solving Equation 8.3 for time,  $t$ , we get

$$t = \frac{x}{V_0 \cos \phi_0} \quad (8.6)$$

We can then write Equation 8.5 putting  $y$  in terms of  $x$  only. Therefore,

$$y = x \tan \phi_0 - \frac{gx^2}{2V_0^2 \cos^2 \phi_0} \quad (8.7)$$

This equation is in the form of a parabola in  $x$  and  $y$  coordinates, the path the projectile will follow in a vacuum. Solving for the range,  $x$ , when  $y=0$  gives

$$x(2V_0^2 \cos \phi_0 \sin \phi_0 - gx) = 0 \quad (8.8)$$

This says that either  $x=0$  (the trivial solution) or

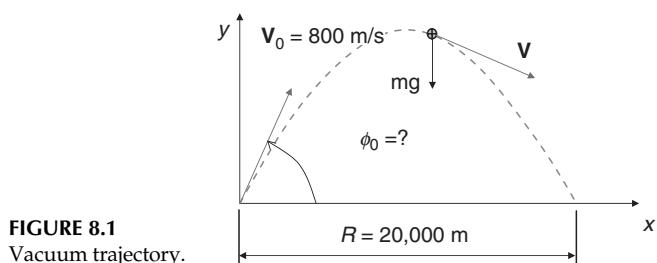
$$x = \frac{2V_0^2}{g} \cos \phi_0 \sin \phi_0 = \frac{V_0^2}{g} \sin 2\phi_0 \quad (8.9)$$

Because the trajectory is a parabola, maximum range is attained at

$$\phi_0 = \frac{\pi}{4} \quad (8.10)$$

since

$$\sin \frac{\pi}{2} = 1 \quad (8.11)$$



When we substitute this into [Equation 8.9](#), the maximum range can be found to be

$$x_{\max} = \frac{V_0^2}{g} \quad (8.12)$$

The maximum ordinate of the trajectory is at half the maximum range and is

$$y_{\max} = \frac{V_0^2}{4g} \quad (8.13)$$

If we differentiate Equation 8.9 with respect to  $\phi_0$  and set this equal to 0, we can prove [Equation 8.10](#) as shown below:

$$\frac{dx_{\max}}{d\phi_0} = \frac{2V_0^2}{g} \cos 2\phi_0 = 0 \quad (8.14)$$

This gives the launch angle for maximum range in a vacuum as  $\frac{\pi}{4}$ .

Except for the maximum range, there are two quadrant elevations (QE) that will allow a projectile to impact at a given distance. We will designate the second QE with a carat, “^.” Its existence is due to the identity

$$\sin \phi = \sin (180^\circ - \phi) \quad (8.15)$$

Then

$$\sin 2\phi_0 = \sin 2(90^\circ - \phi_0) \quad (8.16)$$

Thus

$$\hat{\phi}_0 = 90^\circ - \phi_0 = 90^\circ - \frac{1}{2} \sin^{-1} \left( \frac{gR}{V_0^2} \right) \quad (8.17)$$

where  $x$  has been replaced by the range,  $R$ . The maximum ordinate is achieved when the  $y$ -component of the velocity is 0. By differentiating [Equation 8.4](#) with respect to time and setting the result equal to 0, we get

$$V_0 \sin \phi_0 - gt_s = 0 \quad (8.18)$$

or

$$t_s = \frac{V_0 \sin \phi}{g} \quad (8.19)$$

Substituting this into [Equation 8.4](#) gives

$$y_s = V_0 \sin \phi_0 \left( \frac{V_0 \sin \phi_0}{g} \right) - \frac{1}{2} g \left( \frac{V_0 \sin \phi_0}{g} \right)^2 = \frac{1}{2} \frac{V_0 \sin^2 \phi_0}{g} \quad (8.20)$$

If we note that at impact the  $y$ -coordinate is zero, we can find the time of flight to impact with [Equation 8.4](#)

$$0 = V_0 t_I \sin \phi_0 - \frac{1}{2} g t_I^2 \quad (8.21)$$

or

$$t_I = \frac{2V_0 \sin \phi_0}{g} \quad (8.22)$$

This is double the time to the maximum ordinate and the trajectory in a vacuum is symmetrical about this ordinate. Further evidence of the symmetry may be seen by examining the angle of fall. If we differentiate [Equation 8.7](#) with respect to  $x$  and substitute the value of  $x$  we found at impact in [Equation 8.9](#) in the differentiated result, we see that

$$\left. \frac{dy}{dx} \right|_I = \tan \phi_0 - \frac{\sin 2\phi_0}{\cos^2 \phi_0} \quad (8.23)$$

But

$$\sin 2\phi_0 = 2 \sin \phi_0 \cos \phi_0 \quad (8.24)$$

Therefore,

$$\tan \phi_I = \tan \phi_0 - 2 \frac{\sin \phi_0}{\cos \phi_0} = -\tan \phi_0 \quad (8.25)$$

Thus, in a vacuum trajectory, the projectile impacts at the mirror image of the angle it had when it was launched.

For any given launch velocity,  $V_0$ , maximum range in a vacuum is achieved with an initial launch angle of  $45^\circ$ . To reach any range shorter than the maximum, there are two launch elevations, one greater than  $45^\circ$  and the other less, a high angle and a low angle of fire. The trajectory envelope is a curve that bounds all possible trajectories that attempt to reach all ranges from zero to the maximum range possible for the given launch velocity [1]. We shall now mathematically describe this curve.

We know from [Equation 8.7](#) that

$$y = x \tan \phi_0 - \frac{gx^2}{2V_0^2 \cos^2 \phi_0} \quad (8.7)$$

This can also be written as

$$y = x \tan \phi_0 - \frac{gx^2}{2V_0^2} \sec^2 \phi_0 \quad (8.26)$$

If we make use of the trigonometric identity  $\sec^2 \phi = 1 + \tan^2 \phi$  we can, with substitution and manipulation, write

$$\tan^2 \phi_0 - \frac{2V_0^2}{gx} \tan \phi_0 + \frac{2V_0^2}{gx} y + 1 = 0 \quad (8.27)$$

[Equation 8.27](#) is quadratic in  $\phi_0$  and as such, when solved, yields two roots which correspond to the two elevations that achieve the same range as discussed earlier.

The exceptions to this are when the range is zero or the range is maximum. These conditions yield a repeated root. The other instances a repeated root occurs are whenever the trajectory touches the trajectory envelope. This occurs only once at any given elevation. If the roots of this equation are complex conjugates, the range in question cannot be achieved with the given muzzle velocity. We can solve for all of the double roots to obtain the equation of the trajectory envelope.

We proceed by first completing the square in [Equation 8.27](#) noting that

$$\tan^2 \phi_0 - \frac{2V_0^2}{gx} \tan \phi_0 + \left( \frac{V_0^2}{gx} \right)^2 = \left( \tan \phi_0 - \frac{V_0^2}{gx} \right)^2 \quad (8.28)$$

By adding and subtracting a term,  $\left( \frac{V_0^2}{gx} \right)^2$ , to Equation 8.27, we complete the square of a part of the equation and can operate on the remainder of it.

$$\tan^2 \phi_0 - \frac{2V_0^2}{gx} \tan \phi_0 + \left( \frac{V_0^2}{gx} \right)^2 - \left( \frac{V_0^2}{gx} \right)^2 + \frac{2V_0^2}{gx} y + 1 = 0 \quad (8.29)$$

Breaking apart Equation 8.29 into two terms and setting each equal to zero gives us from Equation 8.28

$$\tan^2 \phi_0 - \frac{2V_0^2}{gx} \tan \phi_0 + \left( \frac{V_0^2}{gx} \right)^2 = \left( \tan \phi_0 - \frac{V_0^2}{gx} \right)^2 = 0 \quad (8.30)$$

and

$$\frac{2V_0^2}{gx} y + 1 = 0 \quad (8.31)$$

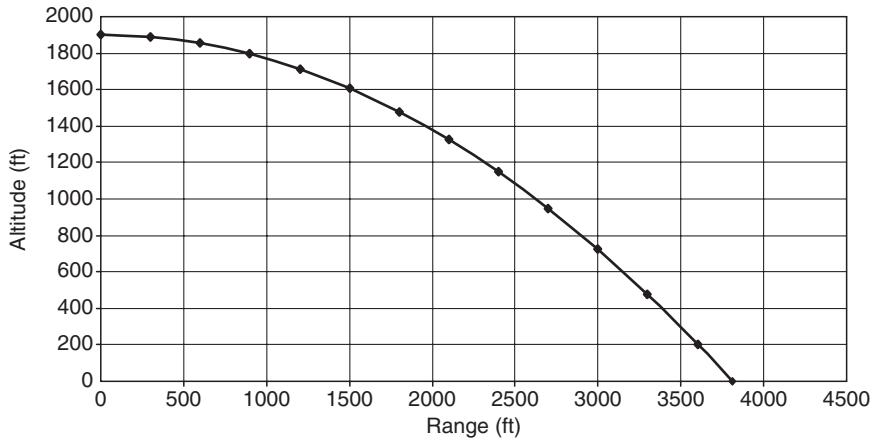
The double root in Equation 8.30 occurs when

$$\tan \phi_0 = \frac{V_0^2}{gx_e} \quad (8.32)$$

where  $x_e = x$  on the envelope curve. We can also pursue the equation for the envelope curve more directly

$$\frac{2V_0^2}{gx_e} y_e - \left( \frac{V_0^2}{gx_e} \right)^2 + 1 = 0 \quad (8.33)$$

where  $y_e$  is the  $y$ -coordinate on the envelope curve. Equation 8.33 can be further rearranged to yield the final equation of the trajectory envelope.



**FIGURE 8.2**  
Trajectory envelope.

$$y_e = \frac{1}{2} \frac{V_0^2}{g} - \frac{gx_e^2}{2V_0^2} \quad (8.34)$$

A typical trajectory envelope is illustrated in Figure 8.2.

To move to a different subject in the study of the vacuum trajectory, when the trajectory of the projectile is relatively flat, certain simplifying assumptions may be made which allow the equations of motion to be solved with greater ease. In particular, if we rewrite [Equation 8.7](#) as

$$y = x \tan \phi_0 - \frac{gx^2}{2V_0^2} \sec^2 \phi_0 \quad (8.35)$$

we now take its derivative with respect to  $\phi_0$ , we get

$$\frac{dy}{d\phi_0} = x \sec^2 \phi_0 - \frac{gx^2}{V_0^2} \tan \phi_0 \sec^2 \phi_0 = x \left( 1 - \frac{gx}{V_0^2} \tan \phi_0 \right) \sec^2 \phi_0 \quad (8.36)$$

Now because

$$\sec^2 \phi_0 = 1 + \tan^2 \phi_0$$

and if  $\tan^2 \phi_0 \ll 1$  then  $\sec^2 \phi_0 \approx 1$ . This occurs when  $\phi_0 < 5^\circ$ . This is the requirement for what is commonly called the flat fire approximation to be valid. We can then translate Equations 8.35 and 8.36 into

$$y \approx x \tan \phi_0 - \frac{gx^2}{2V_0^2} \quad (8.37)$$

and

$$\frac{dy}{d\phi_0} \approx x \left( 1 - \frac{gx}{V_0^2} \tan \phi_0 \right) \quad (8.38)$$

Equation 8.38 can be even further simplified for short ranges if  $-\frac{gx}{V_0^2} \tan \phi_0 \ll 1$  then

$$\frac{dy}{d\phi_0} \approx x \quad (8.39)$$

This is sometimes known as the rigid trajectory because the trajectory appears to rotate rigidly with the elevation angle. The vertical error that arises from use of the flat fire approximation in a vacuum trajectory is

$$\varepsilon_y = \frac{gx^2}{2V_0^2} \tan^2 \phi_0 \quad (8.40)$$

which, as is readily seen, states that as the range or launch angle increases, the error increases. Flat fire is characteristic of the engagements experienced with high-powered, high-velocity tank cannons where initial launch angles for direct-fire ranges of several kilometers are less than 5°. Elevation changes to correct fire are measured in fractions of a degree (known as mils) as well. One mil is equal to 1/6400 of a circle.

### Problem 1

A target is located at 20 km. A projectile muzzle velocity is 800 m/s, assuming a vacuum trajectory, at what QE should one set the weapon to hit the target?

*Answer:*  $\phi_0 = 158.7$  [mil]

### Problem 2

The enemy in the above problem is very smart and has located his unit on the reverse slope of a hill that is 3,000 m in height with its peak located 18,000 m from your firing position. Assuming that the target is at the same level as you (just behind the hill), determine a firing solution (QE, if there is one) to hit him assuming a vacuum trajectory.

*Answer:* It can be hit.—you find the initial QE

### Problem 3

The U.S. pattern 1917 (M1917) “Enfield” rifle was the most numerous rifle used by our troops in the First World War. It was an easier rifle to manufacture than the M1903 “Springfield” (even though the Springfield was officially the U.S. Army’s service rifle) and the troops liked its accuracy better. In fact, the famous Sergeant Alvin York was actually armed with an Enfield, not a Springfield as is commonly believed, when he single handedly captured over 100 German soldiers in the Argonne Forest in 1918. The pattern 1917 used the standard M1 30-'06 cartridge in U.S. service. The bullet had a mass of 174 grains (a grain is a common unit of measure in small arms ammunition and is defined as 1/7000 of a lbm) and a diameter of 0.308 in. This cartridge–rifle combination has a muzzle velocity of 2800 ft/s. Assuming a vacuum trajectory:

- Determine the angle in degrees to set the sights on the rifle (i.e., the QE) if the target is level with the firer and at 200 yards range.

*Answer:*  $\phi_0 = 0.0705^\circ$

- If the target is at the same horizontal range but 20 yards higher, and the firer does not adjust the sights, how much higher or lower will the bullet strike?

*Answer:*  $y_{\text{miss}} = -0.0125$  [in.]

#### Problem 4

You are asked to create a rough safety fan for a maximum range test at Yuma Proving Ground. The test consists of a U.S. M198 155-mm howitzer firing an M549 projectile at maximum charge with rocket off. The projectile weighs 96 lbm. The muzzle velocity is 880 m/s.

1. Using a vacuum trajectory, calculate and plot the trajectory envelope for the test.  
*Answer:  $R = 78,940$  [m]*

2. Determine the longest time of flight of the projectile.  
*Answer:  $t_I = 179.4$  [s]*

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## 8.2 Simple Air Trajectory (Flat Fire)

As we progress in our study of exterior ballistics, we now introduce the concept of drag by substituting air for the medium through which our point mass projectile flies. We do this so that projectile dynamics do not enter yet into the equations of motion. We are essentially still dealing with a spherical, nonrotating cannon ball. Furthermore, to simplify the mathematics, we will insist on a flat fire trajectory, with launch angles below  $5^\circ$ . A flat fire trajectory is depicted in Figure 8.3. The methods and equations we will develop were used in the 1950s for direct-fire calculations over relatively short ranges [1].

We begin with Newton's second law in an inertial reference frame and use vector representations (bold face) where appropriate.

$$\mathbf{F} = m\mathbf{a} \quad (8.41)$$

$$m \frac{d\mathbf{V}}{dt} = \Sigma \mathbf{F} + m\mathbf{g} \quad (8.42)$$

where

$m$  = Projectile mass

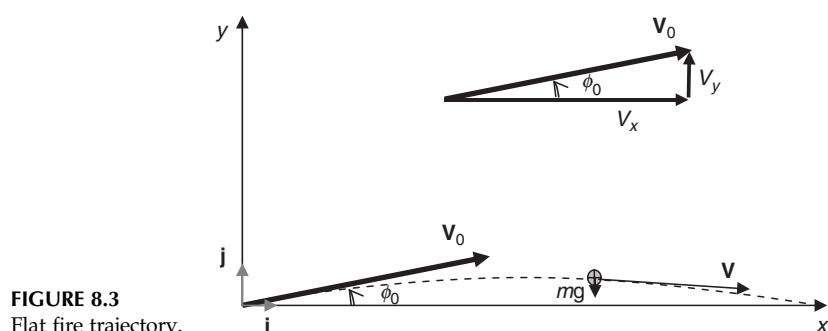
$\mathbf{V}$  = Projectile velocity vector

$t$  = Time

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$\Sigma \mathbf{F}$  = Vector acceleration

$\mathbf{g}$  = Vector acceleration due to gravity



**FIGURE 8.3**  
Flat fire trajectory.

The inertial reference frame allows us to neglect the Coriolis acceleration which is the result of the earth's rotation. Since there is no angle of yaw, the lift and drag forces due to yaw and the Magnus force due to spin are also negligibly small. These will be discussed in detail later. Thus, only the projectile drag forces (base, wave, and skin-friction) are working to slow the projectile down and gravity is pulling it toward the earth. The aerodynamic drag force acting on the projectile is then given by

$$F_D = \frac{1}{2} \rho S C_D VV = \frac{1}{2} \rho V^2 S C_D \quad (8.43)$$

Here  $C_D$  is the drag coefficient, introduced earlier. Another coefficient in common use in ballistics is the ballistic coefficient,  $C$ , which is defined as

$$C = \frac{m}{d^2} \quad (8.44)$$

where  $m$  and  $d$  are the mass and diameter of the projectile. As a matter of convenience, we also define

$$\hat{C}_D^* = \frac{\rho S C_D}{2m} = \frac{\rho \pi}{8} \frac{C_D}{C} \quad (8.45)$$

This allows us to save a little energy in typing since this combination of parameters appears so often. It is known as a starred coefficient [2]. Equation 8.45 stems from the fact that  $S$ , the frontal area of the projectile, is

$$S = \frac{\pi d^2}{4} \quad (8.46)$$

We can combine Equations 8.42 and 8.45 and divide by the mass to get an expression for the time rate of change of velocity (acceleration)

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{2m} \rho S C_D VV + \mathbf{g} \quad (8.47)$$

The negative sign was placed in front of the force above because the drag always opposes the velocity vector (otherwise, it is called thrust). We can separate the velocity, acceleration, and gravitational vectors into components along the coordinate axes, so that they will be convenient to work with.

$$\frac{d\mathbf{V}}{dt} = \dot{V}_x \mathbf{i} + \dot{V}_y \mathbf{j} + \dot{V}_z \mathbf{k} + (V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k}) \quad (8.48)$$

and

$$\mathbf{g} = -g \mathbf{j} \quad (8.49)$$

But because we are in an inertial frame,  $\mathbf{i} = \mathbf{j} = \mathbf{k} = 0$  and therefore

$$\frac{d\mathbf{V}}{dt} = \dot{V}_x \mathbf{i} + \dot{V}_y \mathbf{j} + \dot{V}_z \mathbf{k} \quad (8.50)$$

If we break [Equation 8.47](#) into its components, we get three coupled, ordinary, nonlinear differential equations

$$\dot{V}_x = \hat{C}_D^* V V_x \quad (8.51)$$

$$\dot{V}_y = \hat{C}_D^* V V_y - g \quad (8.52)$$

$$\dot{V}_z = \hat{C}_D^* V V_z \quad (8.53)$$

The equation that couples Equations 8.51 through 8.53 is

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2} \quad (8.54)$$

We can linearize these equations by making a few assumptions. First, let us assume that there is no crosswind, so  $V_z = 0$  and if we further constrain the ratio of the vertical velocity to the horizontal velocity to  $\left| \frac{V_y}{V_x} \right| = \tan \phi < 0.1$ , then  $V$  and  $V_x$  are within 0.5% of each other, and we have constrained the launch and fall angles to be less than  $5.7^\circ$ , the angles introduced in the preceding section for the flat fire approximation.

So with the assumptions that  $V = V_x$  and  $V_z = 0$ , we can develop Equations 8.51 through 8.53 into

$$\dot{V}_x = \hat{C}_D^* V_x^2 \quad (8.55)$$

$$\dot{V}_y = \hat{C}_D^* V_x V_y - g \quad (8.56)$$

$$\dot{V}_z = 0 \quad (8.57)$$

These differential equations use time as the independent variable. It is often convenient to use distance as the independent variable. By making a common transformation of variables to allow distance along the trajectory to be the independent variable instead of time, we can improve our ability to work with these expressions. Performing the transformation results in equations of the form

$$V_x V'_x = -\hat{C}_D^* V_x^2 \quad (8.58)$$

$$V_x V'_y = -\hat{C}_D^* V_x V_y - g \quad (8.59)$$

where the prime denotes differentiation with respect to distance.

By dividing both equations by  $V_x$ , we obtain Equations 8.60 and 8.61 that use the downrange distance,  $x$ , as the independent variable. For these equations, an analytic solution does exist.

$$V'_x = -\hat{C}_D^* V_x \quad (8.60)$$

$$V'_y = -\hat{C}_D^* V_y - \frac{g}{V_x} \quad (8.61)$$

Equation 8.60 can be integrated by separation of variables as

$$V_x = V_{x_0} \exp \left( -\hat{C}_D^* \int_0^x dx_1 \right) \quad (8.62)$$

In this equation and future equations, we use a variable  $x_i$  or  $t_i$  as a dummy variable of integration. [Equation 8.61](#) can also be solved by quadrature methods since it is of the form

$$\frac{dV_y}{dx} + \hat{C}_D^* V_y = -\frac{g}{V_x} \quad (8.63)$$

[Equation 8.61](#) can be solved for  $V_y$  for initial conditions at  $x=0$ ,  $t=0$ , and  $V_y=V_{y_0}$  as

$$V_y = \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \left[ V_{y_0} - \int_0^x \left(\frac{g}{V_x}\right) \exp\left(\int_0^{x_2} \hat{C}_D^* dx_1\right) dx_2 \right] \quad (8.64)$$

If we take the ratio of the  $x$  and  $y$  velocity components, we can obtain the relation for the angle the velocity vector makes with the horizontal. This can be shown to be

$$\tan \phi = \left[ \tan \phi_0 - \frac{1}{V_x} \int_0^x \left(\frac{g}{V_x}\right) \exp\left(\int_0^{x_2} \hat{C}_D^* dx_1\right) dx_2 \right] \quad (8.65)$$

where  $\phi_0$  is the initial launch angle.

To complete our study of the flat fire trajectory, we need to find the elements of it, i.e., the  $x$  and  $y$  values along it, from launch to termination. To do this, we must integrate over time the velocities we have found in [Equations 8.62](#) and 8.64 which we had earlier transformed into distance variables. We know by definition

$$y = \int_0^t V_y dt \quad \text{and} \quad x = \int_0^t V_x dt \quad (8.66)$$

Substituting Equations 8.62 and 8.64 into each of the Equations of 8.66 in turn and performing the integrations, we can show that with the initial conditions of  $x=0$ ,  $t=0$ , and  $y=y_0$

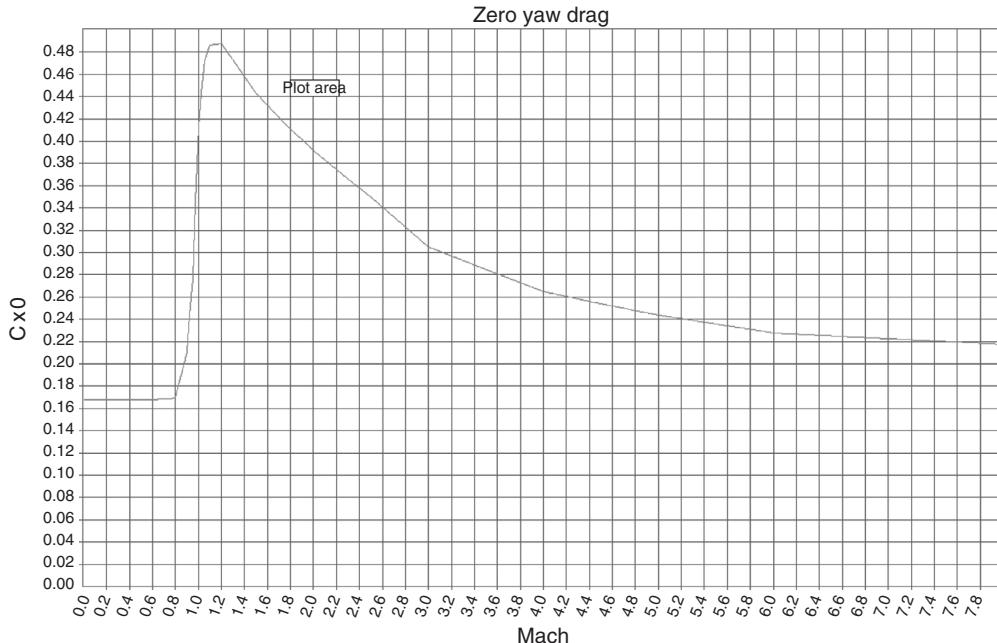
$$y = t \exp\left(\int_0^x \hat{C}_D^* dx_1\right) \left[ V_{y_0} - \int_0^x \left(\frac{g}{V_x}\right) \exp\left(\int_0^{x_2} \hat{C}_D^* dx_1\right) dx_2 \right] + y_0 \quad (8.67)$$

Now we can find  $t$  from  $V_x = \frac{dx}{dt}$ . Separating the variables and substituting Equation 8.62 for  $V_x$ , we get

$$t = \frac{1}{V_{x_0}} \int_0^x \exp\left(\int_0^{x_2} \hat{C}_D^* dx_1\right) dx_2 \quad (8.68)$$

Through a somewhat tedious set of algebraic substitutions and manipulations which are contained in Ref. [1], we can arrive at our desired equation in  $x$  and  $y$ ; the launch angle,  $\phi_0$ ; the dummy range variables  $x_1$ ,  $x_2$ , and  $x_3$ ; the initial launch velocity,  $V_{x_0}$ ; the initial ordinate,  $y_0$ ; and the drag coefficient,  $\hat{C}_D^*$ .

$$y = y_0 + x \tan \phi_0 - \frac{gx^2}{2V_{x_0}^2} \left[ \frac{2}{x^2} \int_0^x \int_0^{x_3} \exp\left(2 \int_0^{x_2} \hat{C}_D^* dx_1\right) dx_2 dx_3 \right] \quad (8.69)$$



**FIGURE 8.4**  
Drag coefficient versus Mach number for a typical projectile.

The disadvantage of these equations is that the variation of drag coefficient has to be simple to evaluate the integrals. Since the drag coefficient does not vary in a simple manner with Mach number, this makes the analytic solutions inaccurate and difficult to accomplish. Figure 8.4 depicts a typical drag curve that varies with Mach number. One can see from this figure that there is no simple analytic solution to this variation. With computer power nowadays, we usually solve or approximate the exact solutions numerically, doing the quadratures by breaking the area under the curve into quadrilaterals and summing the areas.

To integrate these equations analytically, we will examine three forms of the drag coefficient:

1. Constant  $C_D$  that is useful for the subsonic flight regime,  $M < 1$
2.  $C_D$  inversely proportional to the Mach number that is characteristic of the high-supersonic flight regime,  $M \gg 1$
3.  $C_D$  inversely proportional to the square root of the Mach number that is useful in the low-supersonic flight regime,  $M \geq 1$

First, we will examine case of a constant drag coefficient. If we examine Figure 8.4, we can see that this would be a useful approximation for our projectiles behavior if the launch velocity was, say, between Mach 0.8 and 0. We assume that the drag force varies with the square of the velocity (the drag coefficient was the drag force divided by the dynamic pressure,  $\frac{1}{2} \rho V^2$ ) and we set the drag coefficient equal to a constant,  $K_1$ . We shall use terminology consistent with Ref. [1], so that

$$\hat{C}_D^* = \frac{\rho S}{2m} C_D = \frac{\rho S}{2m} K_1 = k_1 \quad (8.70)$$

which we then substitute in [Equation 8.62](#).

$$V_x = V_{x_0} \exp\left(-k_1 \int_0^x dx_1\right) = V_{x_0} e^{-k_1 x} \quad (8.71)$$

We can find  $t$  by substituting [Equation 8.70](#) into [Equation 8.68](#) to give

$$t = \frac{1}{V_{x_0}} \int_0^x \exp\left(\int_0^{x_2} k_1 dx_1\right) dx_2 = \frac{1}{V_{x_0}} \int_0^x e^{k_1 x_2} dx_2 \quad (8.72)$$

or

$$t = \frac{1}{V_{x_0} k_1} (e^{k_1 x} - e^0) = \frac{1}{V_{x_0} k_1} (e^{k_1 x} - 1) \quad (8.73)$$

Noting that from [Equation 8.62](#)

$$\exp\left(- \int_0^x \hat{C}_D^* dx_1\right) = \frac{V_x}{V_{x_0}} \text{ and } \exp\left(\int_0^x \hat{C}_D^* dx_1\right) = \frac{V_{x_0}}{V_x} \quad (8.74)$$

and also noting that  $\frac{V_{y_0}}{V_{x_0}} = \tan \phi_0$ , we can show through manipulation [1] that

$$V_y = V_x \left[ \tan \phi_0 - \frac{gt}{V_{x_0}} \left( 1 + \frac{V_{x_0} k_1 t}{2} \right) \right] \quad (8.75)$$

To find the angle of fall,  $\phi$ , as a function of range,  $x$ , and the instantaneous velocity at  $x$ ,  $V_x$ , we solve [Equation 8.71](#) for  $k_1$

$$k_1 = \frac{1}{x} \ln\left(\frac{V_{x_0}}{V_x}\right) \quad (8.76)$$

We now substitute [Equation 8.76](#) into [Equation 8.73](#), for  $t$ . Taking the result and recalling that  $\tan \phi = V_y/V_x$  for any  $x$ , we use this new equation for  $t$  and transform [Equation 8.65](#) into

$$\tan \phi = \tan \phi_0 - \frac{gt}{V_{x_0}} \left[ 1 + \frac{V_{x_0} t}{2} \frac{1}{x} \ln\left(\frac{V_{x_0}}{V_x}\right) \right] \quad (8.77)$$

Finally, to find the altitude,  $y$ , at any point along the trajectory as a function of the range and the velocity at that range, we transform [Equation 8.65](#) with the constant drag coefficient,  $k_1$ , use the new equation for  $t$  that we derived above and after manipulation arrive at

$$y = y_0 + \tan \phi_0 - \frac{g}{2} \left[ \frac{x}{V_{x_0}} \frac{1}{\ln\left(\frac{V_{x_0}}{V_x}\right)} \right]^2 \left[ \frac{1}{2} \left( \frac{V_{x_0}}{V_x} - 1 \right)^2 + \left( \frac{V_{x_0}}{V_x} - 1 \right) - \ln\left(\frac{V_{x_0}}{V_x}\right) \right] \quad (8.78)$$

A constant drag coefficient is useful when analyzing low-subsonic projectiles, since most of them have nearly constant drag coefficients. Also projectiles at hypersonic speeds (usually described as a Mach number greater than five) can be analyzed with this assumption (look again at [Figure 8.4](#)). Essentially, we are linearizing the problem when we do this.

Our next effort will be to examine a nonconstant drag coefficient, one varying as the inverse of the Mach number. In this case, we assume that the drag force varies linearly with the velocity (because the drag coefficient is the drag force divided by the dynamic pressure,  $\frac{1}{2} \rho V^2$ , and when we divide by the Mach number, we essentially divide by the velocity times a constant). Now we set the drag coefficient equal to  $K_2/M$ , then

$$C_D = \frac{K_2}{M} \quad (8.79)$$

and

$$\hat{C}_D^* = \frac{\rho S}{2m} C_D = \frac{\rho S}{2m} \frac{K_2}{M} \quad (8.80)$$

Recall that the Mach number is  $V/a$  where  $a$  is the speed of sound in air. Then for our flat fire approximation, we can define a constant  $k_2$  such that

$$k_2 = \frac{\rho S}{2m} K_2 a \quad (8.81)$$

then

$$\hat{C}_D^* = \frac{\rho S}{2m} \frac{K_2 a}{V_x} = \frac{k_2}{V_x} \quad (8.82)$$

From [Equations 8.55](#) and [8.56](#), we see that

$$\dot{V}_x = -k_2 V_x \quad (8.83)$$

and

$$\dot{V}_y = -k_2 V_y - g \quad (8.84)$$

Also from [Equation 8.60](#), we see that

$$V'_x = -\hat{C}_D^* V_x \quad (8.85)$$

Using these three equations and proceeding in the same fashion as we did with the constant  $C_D$ , we can derive equations for the  $x$ - and  $y$ -velocities; the time of flight to any range,  $x$ ; the angle of fall,  $\phi$ ; and the trajectory ordinate at any range. These equations are (details in Ref. [1])

$$V_x = V_{x_0} e^{-k_2 t} \quad (\text{in terms of } t) \quad (8.86)$$

$$V_y = \left( V_{y_0} + \frac{g}{k_2} \right) e^{-k_2 t} - \frac{g}{k_2} \quad (\text{in terms of } t) \quad (8.87)$$

$$t = \frac{\frac{x}{V_{x_0}} \ln\left(\frac{V_{x_0}}{V_x}\right)}{\left(1 - \frac{V_x}{V_{x_0}}\right)} \quad (8.88)$$

$$\tan \phi = \tan \phi_0 + \frac{gx}{V_{x_0}^2} \left( \frac{1 - \frac{V_{x_0}}{V_x}}{1 - \frac{V_x}{V_{x_0}}} \right) \quad (8.89)$$

$$y = y_0 + x \tan \phi_0 - \left( \frac{\frac{gt^2}{2} \ln \frac{V_{x_0}}{V_x}}{2 \ln \frac{V_{x_0}}{V_x}} \right) \quad (8.90)$$

These relations, for  $C_D$  proportional to  $1/M$ , are useful in the analysis of high-supersonic projectiles such as kinetic energy armor penetrators where  $2.5 < M < \sim 5$ .

For the case where the drag coefficient varies as the  $\sqrt{M}$ , we assume that the drag varies with velocity to the  $3/2$  power and we set the drag coefficient equal to  $K_3/\sqrt{M}$ , then

$$\hat{C}_D^* = \frac{\rho S}{2m} \frac{K_3}{\sqrt{M}} \quad (8.91)$$

Since

$$\sqrt{M} = \sqrt{\frac{V_x}{a}} \quad (8.92)$$

we can define a new constant as

$$k_3 = \frac{\rho S}{2m} K_3 \sqrt{a} \quad (8.93)$$

We can then write

$$\hat{C}_D^* = \frac{\rho S}{2m} K_3 \sqrt{\frac{a}{V_x}} = \frac{k_3}{\sqrt{V_x}} \quad (8.94)$$

Proceeding as we did in the earlier two cases, we can derive  $V_x$ ,  $V_y$ ,  $t$ ,  $\phi$ , and the ordinate,  $y$ . The details of the derivations are again available in Ref. [1].

$$\sqrt{V_x} = \frac{4V_{x_0}}{(k_3 \sqrt{V_{x_0}} t + 2)} \quad (\text{in terms of } t) \quad (8.95)$$

$$V_y = -\frac{g}{(k_3 \sqrt{V_{x_0}} t + 2)} \left( \frac{k_3^2 V_{x_0} t^3}{3} + 2k_3 \sqrt{V_{x_0}} t^2 + 4t \right) + \frac{4V_{y_0}}{(k_3 \sqrt{V_{x_0}} t + 2)^2} \quad (\text{in terms of } t) \quad (8.96)$$

$$t = \frac{x}{V_{x_0}} \sqrt{\frac{V_{x_0}}{V_x}} \quad (8.97)$$

$$\tan \phi = \tan \phi_0 - \frac{gt}{V_{x_0}} \left[ \frac{1}{3} \left( \frac{V_{x_0}}{V_x} + \sqrt{\frac{V_{x_0}}{V_x} + 1} \right) \right] \quad (8.98)$$

$$y = y_0 + x \tan \phi_0 - \frac{1}{2} g t^2 \left[ \frac{1}{3} \left( 1 + 2 \sqrt{\frac{V_x}{V_{x_0}}} \right) \right] \quad (8.99)$$

These last Equations 8.95 through 8.99 are useful for flight in the low- to moderate-supersonic regime,  $1 < M < 2.5$ .

In summary, we have derived the equations of motion assuming a flat fire trajectory. We use them when the angle of departure and angle of fall are both below  $5.7^\circ$ . We have solved them with three drag assumptions:

1. A constant drag coefficient that is useful in the subsonic and hypersonic regimes and can be used over short distances in all Mach regimes.
2. A drag coefficient inversely proportional to the Mach number that is useful in the high-supersonic regime.
3. A drag coefficient inversely proportional to the square root of the Mach number that is useful in the low-supersonic flight regime.

### Problem 5

The French infantry rifle model 1886 called the Lebel was their standard weapon from 1886 into First World War and even saw limited use in the Second World War. You can see this 51-in. long monster in any movie involving the French Foreign Legion. It used an 8-mm cartridge called the balle D with a bullet mass of 198 grains and a diameter of 0.319 in. This cartridge–rifle combination has a muzzle velocity of 2296 ft/s. Assuming flat fire with  $K_3=0.5$  and using standard sea level met data ( $\rho=0.0751$  lbm/ft<sup>3</sup>,  $a=1120$  ft/s)

1. Create a table containing range (yards), impact velocity (ft/s), time of flight (s), initial QE angle (min), and angle at impact (min) in 200 yard increments out to 1000 yards.
2. If an infantryman is looking at a target at 2000 yards, what angle will the sight have relative to the tube assuming they used standard met in the design?  
*Answer:* About  $10.3^\circ$
3. Comment on the validity of this method with respect to (2) above.

### Problem 6

British 0.303-in. ball ammunition is to be fired in an Mk.1 Maxim machine gun. The bullets mass is 175 grains. When used in this weapon, it has a muzzle velocity of 1820 ft/s. Assuming flat fire with  $K_3=0.5$  and using standard sea level met data ( $\rho=0.0751$  lbm/ft<sup>3</sup>,  $a=1120$  ft/s)

1. Create a table containing range (yards), impact velocity (ft/s), time of flight (s), initial QE angle (min), and angle at impact (min) in 200 yard increments out to 1000 yards.

2. The weapon was used by British units assigned to bolster the Italians in the Alps during the First World War (Italy came in on the Allied side because they wanted the Tyrol region from Austria more than they wanted the Nice region from France). At an altitude of 3000 ft, how much higher or lower will a bullet fired from this weapon impact a level target if the sights are set using the sea level conditions above and the target is at 600 yards? At this altitude assume the density and temperature of the atmosphere are  $\rho = 0.0551 \text{ lbm/ft}^3$  and  $T = 20^\circ\text{F}$ .

*Answer:*  $y = 3.078 \text{ [ft]}$  (too high)

### Problem 7

The main armament in the Italian M13-40 during the Second World War was a 47-mm/32-caliber weapon designed and built by the Ansaldo Arms company. The most effective antitank projectile it carried was an APBC (Armor-Piercing, Ballistic Capped) round which had a muzzle velocity of 2060 ft/s. With this particular projectile–weapon combination, the assumption of constant drag coefficient seems to yield reasonable results. The  $k_1$  value for this case is 0.00025 [1/m]. Using the flat fire, point mass trajectory create a table of range (yards), velocity (ft/s), initial QE (min), and impact QE (min) out to 1000 yards in 200 yard increments.

### Problem 8

A U.S. 37-mm projectile is fired with a muzzle velocity of 2600 [ft/s]. The projectile weighs 1.61 lbm. Assuming  $K_2 = 0.841$  [unitless] and using standard sea level met data ( $\rho = 0.0751 \text{ lbm/ft}^3$ ,  $a = 1120 \text{ ft/s}$ ,  $R = 1716 \left[ \frac{\text{ft-lbf}}{\text{slug}} - R \right]$ )

- Determine the drag coefficient  $C_D$  and drag force on the projectile if the projectile is fired in still air.

*Answer:*  $F_D = 33.04 \text{ [lbf]}$

- Create a table containing range (yards), impact velocity (ft/s), time of flight (s), initial QE angle (min), and angle at impact (min) in 100 yard increments out to 800 yards.
- If this weapon is used at an increased altitude and assuming the density and temperature of the atmosphere are  $\rho = 0.060 \text{ lbm/ft}^3$  and  $T = 30^\circ\text{F}$ , how much higher or lower will the weapon have to be aimed to hit a target at 800 yards.

*Answer:* The weapon must be aimed 0.28 mil or 0.98 min lower

## 8.3 Wind Effects on a Simple Air Trajectory

We continue with our study of a point mass projectile model by adding a further complication to its flat fire trajectory—a crosswind or a range wind, as dynamic atmospheric phenomena. In the basic equations, we have neglected any change in air density with a change in altitude since the effect is small. We also have assumed the equations could be solved in closed form. We want to be able to solve them with winds that are both constant and variable along the flight path.

We begin with a modified version of [Equation 8.47](#) using vector notation

$$\frac{d\mathbf{V}}{dt} = -\frac{\rho S C_D}{2m} \tilde{V}(\mathbf{V} - \mathbf{W}) + \mathbf{g} \quad (8.100)$$

or

$$\frac{d\mathbf{V}}{dt} = -\hat{C}_D^* \tilde{V}(\mathbf{V} - \mathbf{W}) + \mathbf{g} \quad (8.101)$$

where

$m$  = Projectile mass

$\mathbf{W}$  = Wind velocity vector

$\mathbf{V}$  = Projectile velocity vector

$\mathbf{g}$  = Vector acceleration due to gravity

$t$  = Time

$\rho$  = Air density

$\mathbf{a} = \frac{d\mathbf{V}}{dt}$  = Vector acceleration

$S$  = Projectile reference area

$$\hat{C}_D^* = \frac{\rho S C_D}{2m}$$

$C_D$  = Dimensionless drag coefficient

In the above equations, we have replaced the velocity vector,  $\mathbf{V}$ , by the vector  $(\mathbf{V} - \mathbf{W})$  because drag measurements are made relative to the air stream not relative to the ground. We have also replaced the scalar velocity (the speed) with

$$\tilde{V} = |\mathbf{V} - \mathbf{W}| \quad (8.102)$$

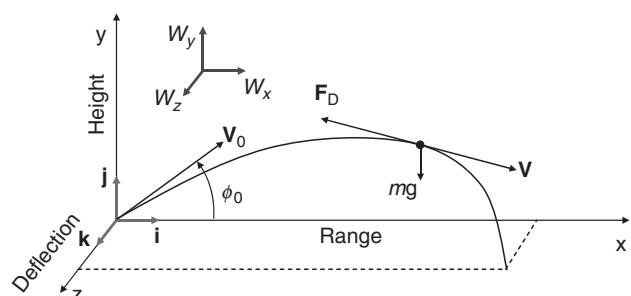
This is the scalar difference of the projectile and wind velocities. A diagram of the problem is shown in Figure 8.5.

We can resolve  $\mathbf{V}$ ,  $\mathbf{W}$ , and  $\mathbf{g}$  into components along the coordinate axes as follows:

$$\mathbf{V} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k} \quad (8.103)$$

$$\mathbf{W} = W_x \mathbf{i} + W_y \mathbf{j} + w_z \mathbf{k} \quad (8.104)$$

$$\mathbf{g} = -g \mathbf{j} \quad (8.105)$$



**FIGURE 8.5**

Coordinate system for projectile launch including wind effects.

Note that

$$\tilde{V}^2 = |\mathbf{V} - \mathbf{W}| \cdot |\mathbf{V} - \mathbf{W}|$$

and

$$|\mathbf{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

This leads us to

$$|\mathbf{V} - \mathbf{W}| \cdot |\mathbf{V} - \mathbf{W}| = (V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2$$

Then

$$\tilde{V} = \sqrt{(V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2} \quad (8.106)$$

If we insert Equations 8.103 and 8.104 into Equation 8.101, we get

$$\frac{d\mathbf{V}}{dt} = [-\hat{C}_D^* \tilde{V}(V_x - W_x)]\mathbf{i} + [-\hat{C}_D^* \tilde{V}(V_y - W_y) - g]\mathbf{j} + [-\hat{C}_D^* \tilde{V}(V_z - W_z)]\mathbf{k}$$

We can separate this vector equation into its three scalar components:

$$\dot{V}_x = \frac{dV_x}{dt} = -\hat{C}_D^* \tilde{V}(V_x - W_x) \quad (8.107)$$

$$\dot{V}_y = \frac{dV_y}{dt} = -\hat{C}_D^* \tilde{V}(V_y - W_y) - g \quad (8.108)$$

$$\dot{V}_z = \frac{dV_z}{dt} = -\hat{C}_D^* \tilde{V}(V_z - W_z) \quad (8.109)$$

Equations 8.107 through 8.109 are the exact equations for a point mass trajectory of a projectile acted upon by gravity, wind, and aerodynamic drag. They are first order, nonlinear, coupled, ordinary differential equations that are coupled through Equation 8.106. The nonlinearity, as previously discussed, creates difficulties when we attempt to solve these expressions analytically. We can only solve the exact equations using numerical methods. This will necessitate making the simplifying assumption of flat fire which will allow us to solve them in closed form. We can alter Equation 8.106 by multiplying by the fraction  $\frac{(V_x - W_x)}{(V_x - W_x)} = 1$  and then simplifying to get

$$\tilde{V} = (V_x - W_x)\sqrt{1 + \varepsilon_y^2 + \varepsilon_z^2} \quad (8.110)$$

where

$$\varepsilon_y = \frac{(V_y - W_y)}{(V_x - W_x)} \quad (8.111)$$

and

$$\varepsilon_z = \frac{(V_z - W_z)}{(V_x - W_x)} \quad (8.112)$$

Using the binomial expansion of the form

$$\sqrt{1 + \zeta} = \left[ 1 + \frac{1}{2}\zeta - \frac{1}{8}\zeta^2 + \dots \right]$$

we can operate on the radical of [Equation 8.110](#) arriving at

$$\tilde{V} = (V_x - W_x) \left[ 1 + \frac{1}{2}(\varepsilon_y^2 + \varepsilon_z^2) - \frac{1}{8}(\varepsilon_y^2 + \varepsilon_z^2)^2 + \dots \right] \quad (8.113)$$

Because a projectile's velocity is usually much greater than winds of even hurricane force, we can assume that

$$|W_x|, |W_y|, \text{ and } |W_z| \ll V_x \quad (8.114)$$

and

$$\varepsilon_y^2 \text{ and } \varepsilon_z^2 \ll 1 \quad (8.115)$$

If we look at the first inequality of [Equation 8.115](#) and consider the assumptions of [Equation 8.114](#), we find that

$$\varepsilon_y = \frac{V_y}{V_x} - \frac{W_y}{W_x} \approx \frac{V_y}{V_x} = \tan \phi \ll 1$$

This was the approximation developed around a similar binomial expansion in [Section 8.2](#). If we recall that this relation restricted us to  $V_y/V_x < 0.1$ , which, by squaring, results in the requirement that winds be at least two orders of magnitude smaller than the velocity,  $V_x$ , and this is easily the case. The second inequality of [Equation 8.115](#) is also satisfied if  $W_z$  and  $V_z$  are comparable in size from

$$\varepsilon_z = \frac{(V_z - W_z)}{(V_x - W_x)} = \frac{(V_z - W_z)}{V_x} \ll 1$$

All this results in  $\tilde{V}$  and  $(V_x - W_x)$  being within about 1% of each other. So if  $\tilde{V} \approx (V_x - W_x)$ , we can rewrite [Equations 8.107](#) through [8.109](#) as

$$\dot{V}_x = \frac{dV_x}{dt} = -\hat{C}_D^*(V_x - W_x)^2 \quad (8.116)$$

$$\dot{V}_y = \frac{dV_y}{dt} = -\hat{C}_D^*(V_x - W_x)(V_y - W_y) - g \quad (8.117)$$

$$\dot{V}_z = \frac{dV_z}{dt} = -\hat{C}_D^*(V_x - W_x)(V_z - W_z) \quad (8.118)$$

Updrafts and downdrafts are usually so small (and usually have the same effect as a crosswind for reasons we shall later describe) that we neglect them completely. Thus, we shall set  $W_y$  equal to zero from now on. We will now look first at the effect where only a crosswind is present (i.e., where  $W_x = W_y = 0$ ) and then examine the effect of a headwind or tailwind. If we make this substitution into [Equations 8.116](#) through [8.118](#), we obtain

$$\dot{V}_x = \frac{dV_x}{dt} = -\hat{C}_D^* V_x^2 \quad (8.119)$$

$$\dot{V}_y = \frac{dV_y}{dt} = -\hat{C}_D^* V_x V_y - g \quad (8.120)$$

$$\dot{V}_z = \frac{dV_z}{dt} = -\hat{C}_D^* V_x (V_z - W_z) \quad (8.121)$$

Equations 8.119 and 8.120 are identical to Equations 8.51 and 8.52 from our earlier work in the zero wind case. If we now change from time to space variables, as we did in the zero wind case, and recall that  $\frac{dt}{dx} = \frac{1}{\bar{d}t}$ , then we arrive at the equations as follows:

$$V'_x = -\hat{C}_D^* V_x \quad (8.122)$$

$$V'_y = -\hat{C}_D^* V_y - \frac{g}{V_x} \quad (8.123)$$

$$V'_z = -\hat{C}_D^* (V_z - W_z) \quad (8.124)$$

Once again, the prime symbol represents differentiation with respect to  $x$ , and Equations 8.122 and 8.123 are identical to those developed for the zero wind case. Now we have already solved differential Equations 8.122 and 8.123 under their previous guise with the result of

$$V_x = V_{x_0} \exp \left( - \int_0^x \hat{C}_D^* dx_1 \right) \quad (8.125)$$

$$V_y = \exp \left( - \int_0^x \hat{C}_D^* dx_1 \right) \left[ V_{y_0} - \int_0^x \left( \frac{g}{V_x} \right) \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_2 \right] \quad (8.126)$$

Equation 8.124 is somewhat more difficult to solve. It is a first order, linear differential equation of the form  $y' + P(x)y = Q$ , whose solution, after the necessary integrations and substitution of initial conditions that at  $x=0$ ,  $V_z=0$ , is

$$V_z = \exp \left( - \int_0^x \hat{C}_D^* dx_1 \right) \int_0^x \hat{C}_D^* W_z \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_2 \quad (8.127)$$

From Equation 8.125, we see that the exponential is  $V_x/V_{x_0}$ , and this can be inserted directly into Equation 8.127. Also if we assume that  $W_z$  is a constant, it can be removed from the integral to give

$$V_z = \frac{V_x}{V_{x_0}} W_z \int_0^x \hat{C}_D^* \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_2 \quad (8.128)$$

We can integrate  $\int_0^x \hat{C}_D^* \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_2$  by parts in Equation 8.128 to yield

$$\int_0^x \hat{C}_D^* \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_2 = \exp \left( \int_0^x \hat{C}_D^* dx_1 \right) \int_0^x \hat{C}_D^* - \int_0^x \left( \int_0^{x_2} \hat{C}_D^* dx_2 \right) \hat{C}_D^* \exp \left( \int_0^{x_2} \hat{C}_D^* dx_1 \right) dx_1 \quad (8.129)$$

The integral of the last term can be solved through a series of substitutions and evaluations at the limits to yield

$$\int_0^x \hat{C}_D^* \exp\left(\int_0^x \hat{C}_D^* dx_1\right) = \exp\left(\int_0^x \hat{C}_D^* dx_1\right) - 1 \quad (8.130)$$

If this is inserted into [Equation 8.128](#), the result is

$$V_z = \frac{V_x}{V_{x_0}} W_z \left[ \exp\left(\int_0^x \hat{C}_D^* dx_1\right) - 1 \right] \quad (8.131)$$

We can further manipulate Equation 8.131 by inserting the value of the exponential from [Equation 8.125](#). In doing so, we get

$$V_z = \frac{V_x}{V_{x_0}} W_z \left( \frac{V_{x_0}}{V_x} - 1 \right) = W_z \left( 1 - \frac{V_x}{V_{x_0}} \right) \quad (8.132)$$

Since  $0 < V_x < V_{x_0}$ ,  $V_z$  always has to be less than the wind speed  $W_z$ . Thus,  $W_z$  is an upper bound on  $V_z$ .

If we examine at the deflection due to a constant crosswind, we can write

$$\begin{aligned} z &= \int_0^t V_z dt = W_z \int_0^t \left( 1 - \frac{V_x}{V_{x_0}} \right) dt = W_z \left( t|_0^t - \frac{1}{V_{x_0}} \int_0^t V_x dt \right) \\ z &= W_z \left( t - \frac{x}{V_{x_0}} \right) \end{aligned} \quad (8.133)$$

[Equation 8.133](#) is known as the lag rule for predicting crosswind effects. It is an exact solution for a constant crosswind. The quantity in the brackets is known as the lag time because a projectile in a real atmosphere would take longer to reach the same range than one fired in a vacuum.

Another interesting point is seen from examination of [Equation 8.132](#). If  $V_x$  is always equal to the initial  $x$  velocity, no matter how hard the wind blows, the projectile will not be affected. Thus, a rocket motor that maintains the initial  $x$  velocity could make the projectile insensitive to wind, a concept called automet. Note also that if the thrust is greater than the initial velocity, the projectile will actually move into the wind.

We consider next the effect of a variable crosswind. A simple way to model this effect on a projectile is to superimpose solutions for constant crosswinds over incremental distances and piece the resultant trajectory together. This technique of superposition works only with linear phenomena. However, since [Equation 8.133](#) is linear in  $x$  and  $t$ , we can apply this method. An alternative approach would be to apply [Equation 8.133](#) in a piecewise fashion using the information from the previous calculation in the subsequent one. To do this, we shall rewrite [Equation 8.133](#) as a difference equation

$$\Delta z_i = w_{z_i} \left[ \Delta t_i - \frac{\Delta x_i}{V_{x_0}} \right] \quad (8.134)$$

where

$\Delta z_i$  is the distance traveled in the  $z$ -direction from time  $i - 1$  to the time  $i$

$\Delta x_i$  is the distance traveled in the  $x$ -direction from time  $i - 1$  to the time  $i$

$\Delta t_i$  is the time between time  $i - 1$  to the time  $i$

$w_{z_i}$  is the constant crosswind acting on the projectile between time  $i - 1$  and time  $i$

$V_{x_{i0}}$  is the  $x$ -velocity at time  $i - 1$

We can rewrite [Equation 8.134](#) as

$$z_i - z_{i-1} = w_{z_i} \left[ (t_i - t_{i-1}) - \frac{(x_i - x_{i-1})}{V_{x_{i0}}} \right] \quad (8.135)$$

To use this method, one must first tabulate  $t$ ,  $V_x$ , and  $x$  as described earlier and then perform the calculation for  $z$  at each interval. With some modifications, a forward difference technique can also be used. These tedious calculations are best done with a computer program for small intervals of time.

We will now examine the effects of a constant range wind, both head-on and a tailwind. We do this by comparing the effects to a flat fire, no-wind flight and will determine the effects on time of flight, impact, and velocity at impact.

We make the initial assumption that there is no crosswind, i.e.,  $W_y = W_z = 0$ , and insert this into [Equations 8.116](#) through [8.118](#), the component differential equations for a point-mass, flat fire trajectory.

$$\dot{V}_x = \frac{dV_x}{dt} = -\hat{C}_D^*(V_x - W_x)^2 \quad (8.136)$$

$$\dot{V}_y = \frac{dV_y}{dt} = -\hat{C}_D^*(V_y - W_y)V_y - g \quad (8.137)$$

$$\dot{V}_z = \frac{dV_z}{dt} = -\hat{C}_D^*(V_z - W_z)V_z \quad (8.138)$$

Because there is no crosswind, [Equation 8.138](#) reduces to

$$V_z = 0$$

By a change of time to space variables and various algebraic manipulations, we can change [Equation 8.136](#) to

$$V'_x + \hat{C}_D^* V_x = \hat{C}_D^* W_x \left( 2 - \frac{W_x}{V_x} \right) \quad (8.139)$$

Similarly, we do the same to [Equation 8.137](#) and arrive at a distance equation in a  $y$ -variable only

$$V'_y + \hat{C}_D^* \left( 1 - \frac{W_y}{V_x} \right) V_y = \frac{-g}{V_x} \quad (8.140)$$

Recall from our earlier discussion that the wind speed is about two orders of magnitude smaller than the projectile velocity, so mathematically, we can express this condition as

$$\frac{W_x}{V_x} \leq 0.01$$

We can then rewrite Equations 8.139 and 8.140 allowing them to be equalities as follows:

$$V'_x + \hat{C}_D^* V_x = 2\hat{C}_D^* W_x \quad (8.141)$$

and

$$V'_y + \hat{C}_D^* V_y = -\frac{g}{V_x} \quad (8.142)$$

We can then solve these equations for  $V_x$  and  $V_y$ .

As we saw in the earlier solutions for the constant crosswind, with appropriate integrations, algebraic manipulation and the insertion of the initial condition that at  $x=0$ ,  $V_x = V_{x0}$ , we see that

$$V_x = \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \int_0^x 2\hat{C}_D^* W_x \exp\left(\int_0^{x_1} \hat{C}_D^* dx_1\right) dx_2 + V_{x0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \quad (8.143)$$

From Equation 8.130, recall that

$\int_0^x \hat{C}_D^* \exp\left(\int_0^{x_1} \hat{C}_D^* dx_1\right) dx_1 = \exp\left(\int_0^x \hat{C}_D^* dx_1\right) - 1$ . Using this fact and by substituting it into Equation 8.143, factoring the result, and considering that  $W_x$  is constant, we arrive at

$$V_x = V_{x0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) + 2W_x \left[1 - \exp\left(-\int_0^x \hat{C}_D^* dx_1\right)\right] \quad (8.144)$$

The first term on the RHS of Equation 8.144 is simply the velocity decay caused by drag of the projectile. The second term is the effect of the range wind on it. If we examine Equation 8.125 which was an analysis for a firing in the absence of range wind, the first term of Equation 8.144 represents  $V_x$ , the  $x$ -velocity with no wind. The second term, when we substitute for the exponential, then represents the effect of the range wind on the flight. Thus, we can see the range wind effects shown as the variable of interest with a tilde ( $\sim$ ) in the following:

$$\tilde{V}_x = V_x + 2W_x \left[1 - \frac{V_x}{V_{x0}}\right] \quad (8.145)$$

This equation shows that at any time,  $t$ , a tailwind (i.e., one blowing in the positive  $x$ -direction) has the effect of increasing the velocity (relative to the ground), while the opposite is true of a headwind. This is important because if we had a table of velocities versus range for the no-wind case, we could then tabulate the effect of range wind.

If we now look at the  $y$ -velocity, we can operate on Equation 8.142 with the initial condition that at  $x=0$ ,  $V_y = V_{y0}$ . This provides us with the solution of the space variable equation

$$V_y = -\exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \int_0^x \frac{g}{V_x} \exp\left(\int_0^{x_1} \hat{C}_D^* dx_1\right) dx_2 + V_{y0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \quad (8.146)$$

At this point, we can introduce the range wind by inserting [Equation 8.144](#) for  $V_x$  arriving at

$$V_y = -g \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \int_0^x \frac{\exp\left(\int_0^x \hat{C}_D^* dx_1\right)}{V_{x0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) + 2W_x \left[1 - \exp\left(-\int_0^x \hat{C}_D^* dx_1\right)\right]} dx_2 \\ + V_{y0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \quad (8.147)$$

This rather complicated integral can be simplified somewhat; however, another approach [1] to the problem that makes use of the no-wind method used previously in [Equation 8.145](#) simplifies things even further. This is seen below:

$$\tilde{V}_y = -\exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \int_0^x \frac{g}{\tilde{V}_x} \exp\left(\int_0^x \hat{C}_D^* dx_1\right) dx_2 + \frac{V_{y0}}{V_{x0}} V_{x0} \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \quad (8.148)$$

or since  $\frac{V_{y0}}{V_{x0}} = \tan \phi_0$

$$\tilde{V}_y = -g \exp\left(-\int_0^x \hat{C}_D^* dx_1\right) \int_0^x \frac{1}{\tilde{V}_x} \exp\left(\int_0^x \hat{C}_D^* dx_1\right) dx_2 + V_x \tan \phi_0 \quad (8.149)$$

Further use of  $\tilde{V}_x$  and some algebraic manipulation gives

$$\tilde{V}_y = V_x \tan \phi_0 - g V_x \int_0^x \frac{1}{V_x \tilde{V}_x} dx_2 \quad (8.150)$$

Recalling [Equation 8.145](#), we can rewrite the denominator of the integral as

$$V_x \tilde{V}_x = V_x^2 + 2W_x \left(V_x - \frac{V_x^2}{V_{x0}}\right) = V_x^2 \left(1 + \frac{2W_x}{V_x} - \frac{2W_x}{V_{x0}}\right) \quad (8.151)$$

If we again use the fact that the wind velocity is at least two orders of magnitude smaller than the projectile velocity, the last two terms in the product on the RHS vanish, leaving

$$V_x \tilde{V}_x \approx V_x^2 \quad (8.152)$$

Then we can rewrite [Equation 8.150](#) as

$$\tilde{V}_y = V_x \tan \phi_0 - g V_x \int_0^x \frac{1}{V_x^2} dx_2 \quad (8.153)$$

This equation has exactly the same form as the flat fire equation for  $V_y$ . Hence, we can say that for a flat fire trajectory, with a small range wind compared to the projectile velocity, the vertical component of velocity is not appreciably affected.

We can now turn our attention to the time of flight of a projectile with a constant range wind by first defining an average downrange velocity following the procedure by McCoy [1] as

$$\tilde{V}_{x_{avg}} = \frac{1}{x} \int_0^x \tilde{V}_x dx_1 \quad (8.154)$$

For  $\tilde{V}_x$ , we substitute [Equation 8.145](#) giving

$$\tilde{V}_{x_{avg}} = \frac{1}{x} \int_0^x \left( V_x + 2W_x - 2W_x \frac{V_x}{V_{x_0}} \right) dx_1$$

Performing the integration on the second term of the integrand and rewriting, we get

$$\tilde{V}_{x_{avg}} = \frac{1}{x} \left( 2W_x x + \int_0^x V_x dx_1 - 2 \frac{W_x}{V_{x_0}} \int_0^x V_x dx_1 \right) \quad (8.155)$$

Rearranging Equation 8.155 and knowing that the velocity averaging also applies to the no-wind case, i.e.,

$$V_{x_{avg}} = \frac{1}{x} \int_0^x V_x dx_1 \quad (8.156)$$

we get

$$\tilde{V}_{x_{avg}} = V_{x_{avg}} + 2W_x \left( 1 - \frac{V_{x_{avg}}}{V_x} \right) \quad (8.157)$$

The time of flight can be expressed as the range divided by the average velocity for either the case of no range wind or with range wind included. Thus, we can write

$$t = \frac{R}{V_{x_{avg}}} \quad (8.158)$$

or

$$\tilde{t} = \frac{R}{\tilde{V}_{x_{avg}}} \quad (8.159)$$

By taking the reciprocal of Equation 8.159, performing judicious substitutions, gathering terms, and finally taking the reciprocal of the result, we can write

$$\tilde{t} = \left[ \frac{t}{1 + 2W_x \left( \frac{t}{R} - \frac{1}{V_x} \right)} \right] \quad (8.160)$$

This shows, as we would expect, that a tailwind ( $W_x$  positive) reduces the time of flight while a headwind ( $W_x$  negative) increases it.

Let us summarize what we have done for crosswinds and range winds. We modified the flat fire equations to account for crosswind and range wind. We use them when the angle of departure and angle of fall are both below  $5.7^\circ$ . We solved the crosswind equations assuming constant and variable crosswinds and introduced the classic lag rule. With variable crosswinds, we saw it is fairly accurate to piece the trajectory together using locally constant values for the crosswind. We have solved the range wind equations assuming only constant range wind. We could treat variable range wind in a manner similar to variable crosswinds, but the difference in results is usually not worth the added effort. For range wind, we used a solution technique that compared the velocities, positions, and time to the no-wind case.

### **Problem 9**

A U.S. 37-mm AP projectile is fired with a muzzle velocity of 2600 [ft/s]. The projectile weighs 1.61 lbm. Assuming flat fire with  $K_2 = 0.841$  [unitless] and using standard sea level met data ( $\rho = 0.0751$  lbm/ft<sup>3</sup>,  $a = 1120$  ft/s)

1. Create a table containing range (yards), impact velocity (ft/s), time of flight (TOF) (s), initial QE angle (min), and angle at impact (min) in 200 yard increments out to 1000 yards assuming no-wind effects.

*Answer:* At 1000 yards,  $V = 1837$  [ft/s]

2. Determine the deflection of the projectile with a 20-min/h crosswind blowing from left to right as viewed from behind the weapon.

*Answer:* At 1000 yards,  $z = 6.217$  [ft]

3. Determine the impact velocity, change in TOF, and how high the projectile will hit if fired at the same QE's with a 20-min/h tailwind and no crosswind.

*Answer:* The projectile will hit 1.402 in. higher than expected.

### **Problem 10**

A British 12-in. projectile has a  $K_3$  of 0.8 and a weight of 850 lbm. If it is fired at an initial QE of 130 mil with a muzzle velocity of 2800 ft/s:

1. Create a table of range (yards), altitude (yards), velocity (ft/s), time of flight (s), inclination angle (degrees), and drift (yards) if the projectile is fired with no wind.
2. Repeat part (1) if the projectile is fired with a headwind of 25 ft/s for the first 3000 yards of flight and a crosswind (left or right—your choice) of 35 ft/s for the remainder of the flight. Tabulate every 1000 yards with the impact location as the last entry in the table.

## **8.4 Generalized Point Mass Trajectory**

In keeping with our practice of introducing ever-increasing complexity into our theory, we will now remove most of the restrictions of the earlier work. We will examine the effects of an unrestricted launch angle and make the high-angle fire of mortars and howitzers amenable to trajectory analysis. We still reserve for later study the effects on flight of a

three-dimensional body whose shape, physical properties, and motions add a significant level of complexity to trajectory analysis.

The aerodynamic behavior of a projectile can be examined from three separate viewpoints: motion affected only by the acceleration of gravity and the initial velocity (vacuum trajectory); motion affected by gravity, initial velocity, and aerodynamic drag (point mass trajectory); motion affected by the projectile's shape, physical properties, and dynamics (which actually manifests itself as changing drag) as well as gravity and launch conditions. We will concentrate on the second viewpoint in this section.

In the equations that follow, we are assuming that the projectile is still a cannon ball with all of its mass concentrated at one point. This allows us to continue to neglect the rigid body kinematics that would be present in a distributed mass. However, we shall include wind effects and earth-rotational effects, and therefore three-dimensional motion. As stated, flat fire restrictions are removed so that the analysis is applicable to all launch angles.

We begin with the same set of equations of motion, except for the addition of a term for the Coriolis force,  $m\Lambda$ .

$$\mathbf{F} = m\mathbf{a} \quad (8.161)$$

$$m \frac{d\mathbf{V}}{dt} = \sum \mathbf{F} + m\mathbf{g} + m\Lambda \quad (8.162)$$

where

$m$  = Projectile mass

$\mathbf{V}$  = Velocity vector

$t$  = Time

$\mathbf{a} = \frac{d\mathbf{V}}{dt}$  = Vector acceleration

$\Sigma \mathbf{F}$  = Vector sum of all aerodynamic forces

$\mathbf{g}$  = Vector acceleration due to gravity

$\Lambda$  = Vector Coriolis acceleration due to rotation of the earth

We also recall from our earlier work with wind effects

$$\frac{d\mathbf{V}}{dt} = -\hat{C}_D^* \tilde{V}(\mathbf{V} - \mathbf{W}) + \mathbf{g} \quad (8.163)$$

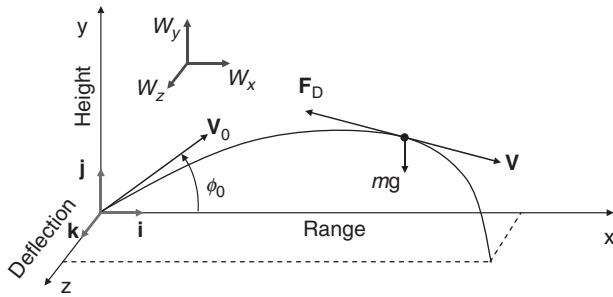
Here  $\mathbf{W}$  is the wind velocity vector and  $\hat{C}_D^* = \rho S C_D / 2m$ . In the above equations, we have replaced the velocity vector  $\mathbf{V}$  by the vector  $(\mathbf{V} - \mathbf{W})$  because drag measurements are made relative to the air stream, not relative to the ground. We have also again replaced the scalar velocity (the speed) with  $\tilde{V} = |\mathbf{V} - \mathbf{W}|$  which is the scalar difference of the projectile and wind velocities. A diagram of this is shown in [Figure 8.6](#).

Without repeating the entire procedure, it can be shown that we may separate Equation 8.163 into individual components to obtain the differential equations for a point mass.

$$\dot{V}_x = \frac{dV_x}{dt} = -\hat{C}_D^* \tilde{V}(V_x - W_x) \quad (8.164)$$

$$\dot{V}_y = \frac{dV_y}{dt} = -\hat{C}_D^* \tilde{V}(V_y - W_y) - g \quad (8.165)$$

$$\dot{V}_z = \frac{dV_z}{dt} = -\hat{C}_D^* \tilde{V}(V_z - W_z) \quad (8.166)$$



**FIGURE 8.6**  
Generalized point mass trajectory.

The scalar velocity,  $\tilde{V}$ , is again

$$\tilde{V} = \sqrt{(V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2} \quad (8.167)$$

In all of the above equations, the wind velocity is variable and is considered positive when it blows in the positive direction of one of the coordinate axes. Equations 8.164 through 8.166 are nonlinear, coupled differential equations which are the exact solution to Newton's laws governing the motion of a projectile affected by wind, gravity, and aerodynamic drag. These equations are coupled through Equation 8.167. Now, as we did in our discussion of flat fire, we would like to evaluate Equations 8.164 through 8.166 by using the downrange distance,  $x$ , as the independent variable. To do this, we simply note that for each of the time derivatives, we can write

$$\dot{V}_x = \frac{dV_x}{dt} = \frac{dV_x}{dt} \frac{dt}{dx} \frac{dx}{dt} = V_x \frac{dV_x}{dx} = V_x V'_x \quad (8.168)$$

And similarly

$$\dot{V}_y = V_x V'_y \quad (8.169)$$

$$\dot{V}_z = V_x V'_z \quad (8.170)$$

We can now write the three equations of motion with  $x$  as the independent variable as follows:

$$V'_x = \frac{1}{V_x} \dot{V}_x = -\hat{C}_D^* \left( \frac{\tilde{V}}{V_x} \right) (V_x - W_x) \quad (8.171)$$

$$V'_y = \frac{1}{V_x} \dot{V}_y = -\hat{C}_D^* \left( \frac{\tilde{V}}{V_x} \right) (V_y - W_y) - \left( \frac{g}{V_x} \right) \quad (8.172)$$

$$V'_z = \frac{1}{V_x} \dot{V}_z = -\hat{C}_D^* \left( \frac{\tilde{V}}{V_x} \right) (V_z - W_z) \quad (8.173)$$

As we noted earlier, the vertical component of the wind,  $W_y$ , is usually extremely small and will be neglected in further treatment. Further, as we mentioned, these equations are impossible to solve in closed form and we must resort to numerical methods for their solution.

Without the restrictions of flat fire, projectiles fired at high angles of departure may traverse the atmosphere to great altitudes. In their flight, they encounter air temperatures and pressures that constantly change. These changes must be accounted for in the numerical computations to adequately solve the trajectory. Hence, knowledge of the standard atmosphere must serve as input to the calculations. There are two standards in common use: Army Standard Metrology and the International Civil Aviation Organization (ICAO) atmosphere. ICAO atmosphere is the most used of the two. Temperature and pressure versus altitude are shown for the ICAO model in Figure 8.7. These atmospheric models are usually incorporated into ballistics codes.

Now, to become familiar with the physics of the Coriolis acceleration which was brought to its final form by Gaspard de Coriolis in 1835, we will study the effects of the earth's rotation on a flat fire, vacuum trajectory example. We do this because we will be able to see these effects without resorting to a computer for calculation. The effect is really due to the fact that the firing point and target are located on the rotating earth, thus when the projectile lands, the earth has rotated through an angle and has thus moved the target. Figure 8.8 shows the geometry of the earth, the latitude of the firing site, and the orientation of the axes.

The Coriolis acceleration is defined as

$$2\omega \times (\mathbf{v}_{B/A})_{xyz} = 2\Omega \times (\mathbf{v})_{xyz} \quad (8.174)$$

We have written Equation 8.174 in this way because the angular velocity we are considering is that of the earth and our projectile velocity is relative to our firing position (and therefore the earth) which moves with the  $x$ - $y$ - $z$  coordinate system. For this equation to be useful to us, we have to write the earth's angular velocity,  $\Omega$ , in terms of our  $x$ - $y$ - $z$  coordinate system. We will see that this acceleration is independent of the projectile weight but dependent upon its velocity. From Figure 8.8, we see that we can readily define  $\Omega$  in terms of our moving coordinate system as

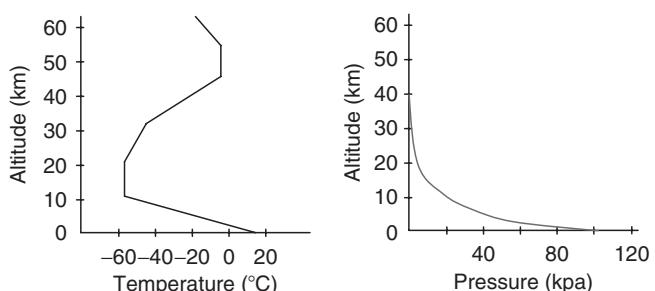
$$\Omega = \Omega \cos L \cos AZ \mathbf{i} + \Omega \sin L \mathbf{j} - \Omega \cos L \sin AZ \mathbf{k} \quad (8.175)$$

If we also note that  $\mathbf{v}$  is defined as

$$(\mathbf{v})_{xyz} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k} \quad (8.176)$$

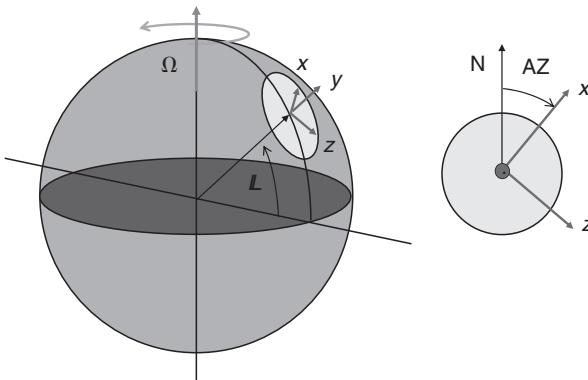
Then inserting Equations 8.175 and 8.176 into Equation 8.174 gives us

$$2\Omega \times (\mathbf{v})_{xyz} = 2\Omega \begin{bmatrix} (V_z \sin L + V_y \cos L \sin AZ) \mathbf{i} \\ (-V_z \cos L \cos AZ - V_x \cos L \sin AZ) \mathbf{j} \\ (V_y \cos L \cos AZ - V_x \sin L) \mathbf{k} \end{bmatrix} \quad (8.177)$$



**FIGURE 8.7**

International Civil Aviation Organization (ICAO) models for atmospheric temperature and pressure.



**FIGURE 8.8**

Angles used for Coriolis acceleration calculations. Picture on the right represents a map of the corresponding area on the globe.

We will write the Coriolis acceleration in terms of a D'Alembert force (i.e., the negative of what we have in [Equation 8.177](#)), so we shall define the Coriolis term in our equation of motion ([Equation 8.162](#)) as

$$\Lambda = -2\Omega \times (\mathbf{v})_{xyz} = 2\Omega \begin{bmatrix} (-V_y \cos L \sin AZ - V_z \sin L) \mathbf{i} \\ (V_x \cos L \sin AZ + V_z \cos L \cos AZ) \mathbf{j} \\ (V_x \sin L - V_y \cos L \cos AZ) \mathbf{k} \end{bmatrix} \quad (8.178)$$

Here we need to define the following variables:

$\Lambda$  = Vector Coriolis acceleration

$\Omega$  = Angular velocity of the earth about its polar axis = 0.00007292(rad/s)

$L$  = Latitude of the firing site, positive in the northern hemisphere, negative in the southern.

$AZ$  = Azimuth angle of fire, measured clockwise from north

$V_x, V_y, V_z$  = Velocity in the  $x, y, z$  directions, respectively, positive along the positive coordinate axes

Now that we have defined some terminology, we shall examine the effect that the Coriolis acceleration has on a vacuum trajectory. While this is stretching the vacuum trajectory much beyond its usefulness in ballistics, we remind the reader that the purpose is to demonstrate the physics that result from Coriolis effects. We begin by recalling [Equation 8.162](#)

$$m \frac{d\mathbf{V}}{dt} = \Sigma \mathbf{F} + m\mathbf{g} + m\Lambda \quad (8.162)$$

Now, since this is a vacuum trajectory the force term on the RHS is zero, and we can divide by the mass,  $m$ , to obtain the vector equation for a vacuum trajectory

$$\frac{d\mathbf{V}}{dt} = \mathbf{g} + \Lambda \quad (8.179)$$

Rewriting [Equation 8.179](#) in terms of its vector components gives (note that the "g" term appears only in [Equation 8.181](#))

$$\frac{dV_x}{dt} = 2\Omega(-V_y \cos L \sin AZ - V_z \sin L) \quad (8.180)$$

$$\frac{dV_y}{dt} = 2\Omega(V_x \cos L \sin AZ + V_z \cos L \cos AZ) - g \quad (8.181)$$

$$\frac{dV_z}{dt} = 2\Omega(V_x \sin L - V_y \cos L \cos AZ) \quad (8.182)$$

We shall now provide examples of the effect. These examples are based on the work of McCoy and can also be found in his work [1]. Let us consider first a purely vertical firing (i.e.,  $V_x = V_z = 0$ ). One may, initially, consider this a trivial example, but for test purposes we occasionally do fire vertically. And, by the way, as we will see, what goes up does not come straight down. Let us also choose due east as positive  $x$ , so  $AZ = 90^\circ$ . With these assumptions, Equations 8.180 through 8.182 become

$$\frac{dV_x}{dt} = -2\Omega V_y \cos L \quad (8.183)$$

$$\frac{dV_y}{dt} = -g \quad (8.184)$$

$$\frac{dV_z}{dt} = 0 \quad (8.185)$$

These equations are well behaved and no longer coupled, so we can solve them independently. We shall integrate Equation 8.183 by first rewriting it, then integrating it.

$$\frac{dV_x}{dt} = -2\Omega \frac{dy}{dt} \cos L \quad (8.186)$$

$$V_x = -2\Omega y \cos L + C \quad (8.187)$$

To determine  $C$ , we know that at  $y = y_0$ ,  $V_x = 0$  so we can write

$$V_x = -2\Omega y \cos L + 2\Omega y_0 \cos L = -2\Omega \cos L(y - y_0) \quad (8.188)$$

If you recall our coordinate system, this means a projectile fired straight up will drift to the west and one fired (or dropped) straight down will drift to the east. Now we will integrate Equation 8.184 to get

$$V_y = -gt + C \quad (8.189)$$

Again, solving for the constant by inserting the initial conditions that at  $t = 0$ ,  $V_y = V_{y_0}$ , we get

$$V_y = V_{y_0} - gt \quad (8.190)$$

Now we shall rewrite and integrate Equation 8.190 a second time making use of the fact that at  $t = 0$ ,  $y = y_0$ , to obtain

$$y = V_{y_0}t - \frac{1}{2}gt^2 + y_0 \quad (8.191)$$

We can now insert Equation 8.191 into Equation 8.188 and rewrite it as

$$\frac{dx}{dt} = -2\Omega \cos L \left( y_0 + V_{y_0}t - \frac{1}{2}gt^2 - y_0 \right) = -2\Omega \cos L \left( V_{y_0}t - \frac{1}{2}gt^2 \right) \quad (8.192)$$

This can be integrated using the initial conditions that at  $t = 0$ ,  $x = 0$  to give

$$x = -\Omega \cos L \left( V_{y_0} t^2 - \frac{1}{3} g t^3 \right) \quad (8.193)$$

Let us now look at the special case of a bomb dropped from a given height with  $V_{y_0} = 0$  and let  $y = 0$ . If we know the altitude from which we are dropping the bomb, we can determine its time of flight from [Equation 8.191](#), thus

$$y_0 = \frac{1}{2} g t^2 \quad (8.194)$$

or

$$t = \sqrt{\frac{2y_0}{g}} \quad (8.195)$$

If we insert this into Equation 8.193, we get

$$x = \frac{1}{3} g \Omega \cos L \left( \frac{2y_0}{g} \right)^{\frac{3}{2}} \quad (8.196)$$

This says that since we are on the positive  $x$ -axis, the bomb will drift to the east. This drift would be greatest at the equator and zero at the poles.

Another example that uses the vacuum trajectory analysis is a projectile that is fired vertically upward with velocity,  $V_{y_0}$ . We can find the time to apogee from [Equation 8.190](#) knowing that at apogee,  $V_y = 0$ .

$$t = \frac{V_{y_0}}{g} \quad (8.197)$$

If we insert this value of  $t$  into Equation 8.193, we get

$$x = -\frac{2}{3} \Omega g t^3 \cos L \quad (8.198)$$

The time to apogee can be put in terms of the height at apogee,  $y_s$ , through Equation 8.191

$$y_s = \frac{1}{2} g t^2 \quad (8.199)$$

Therefore, the time to apogee is

$$t_s = \sqrt{\frac{2y_s}{g}} \quad (8.200)$$

And therefore the Coriolis-caused displacement at apogee along the  $x$ -axis is found by inserting Equation 8.200 into Equation 8.198 giving

$$x_s = -\frac{4}{3} \Omega \sqrt{\frac{2y_s^3}{g}} \cos L \quad (8.201)$$

Lastly, we can apply the Coriolis analysis to the vacuum trajectory, flat fire situation and determine a correction for the acceleration in that case. We begin by making the usual assumptions for the flat fire trajectory of

$$V_y \ll V_x \quad \text{and} \quad V_z \ll V_x \quad (8.202)$$

We substitute these into Equations 8.180 through 8.182 yielding

$$\frac{dV_x}{dt} \approx 0 \quad (8.203)$$

$$\frac{dV_y}{dt} \approx 2\Omega V_x \cos L \sin AZ - g \quad (8.204)$$

$$\frac{dV_z}{dt} \approx 2\Omega V_x \sin L \quad (8.205)$$

Solution of Equation 8.203 with the initial conditions of  $V_x = V_{x_0}$  at  $t = 0$  yields

$$V_x \approx V_{x_0} \quad (8.206)$$

Solution of Equation 8.204 after insertion of Equation 8.206 with the initial conditions of  $V_y = V_{y_0}$  at  $t = 0$  and integrating yields

$$V_y \approx V_{y_0} - gt \left[ 1 - \left( \frac{2\Omega V_{x_0}}{g} \right) \cos L \sin AZ \right] \quad (8.207)$$

Solution of Equation 8.205 with the initial conditions of  $V_z = 0$  at  $t = 0$  yields after insertion of Equation 8.206 and integrating

$$V_z \approx 2\Omega V_{x_0} t \sin L \quad (8.208)$$

If we now integrate Equations 8.206 through 8.208 subject to  $x = 0$ ,  $y = y_0$ , and  $z = 0$  at  $t = 0$  to get the displacements in the  $x$ ,  $y$ , and  $z$  directions, we get

$$x \approx V_{x_0} t \quad (8.209)$$

$$y \approx y_0 + V_{y_0} t - \frac{gt^2}{2} \left[ 1 - \left( \frac{2\Omega V_{x_0}}{g} \right) \cos L \sin AZ \right] \quad (8.210)$$

$$z = \Omega V_{x_0} t^2 \sin L \quad (8.211)$$

If we want to parameterize Equations 8.210 and 8.211 in terms of the downrange distance,  $x$ , we can rewrite Equation 8.209 as

$$t \approx \frac{x}{V_{x_0}} \quad (8.212)$$

We can then insert this value of time into Equations 8.210 and 8.211 to obtain

$$y \approx y_0 + \frac{V_{y_0}}{V_{x_0}} x - \frac{gx^2}{2V_{x_0}^2} \left[ 1 - \left( \frac{2\Omega V_{x_0}}{g} \right) \cos L \sin AZ \right] \quad (8.213)$$

and

$$z \approx \frac{\Omega x^2}{V_{x_0}} \sin L \quad (8.214)$$

[Equation 8.213](#) was rearranged in this form (including the substitution of  $\tan \phi_0$ ) for comparison with [Equation 8.37](#) also modified to include a  $y_0$ .

$$y \approx y_0 + x \tan \phi_0 - \frac{gx^2}{2V_0^2} \quad (8.37)$$

$$y \approx y_0 + x \tan \phi_0 - \frac{gx^2}{2V_{x_0}^2} \left[ 1 - \left( \frac{2\Omega V_{x_0}}{g} \right) \cos L \sin AZ \right] \quad (8.215)$$

From this comparison, we see that the incorporation of the Coriolis acceleration in the flat fire vacuum trajectory manifests itself in a modification to the gravitational term. Thus, as defined in Ref. [1], we can define a Coriolis factor,  $f_C$ , as

$$f_C = \left[ 1 - \left( \frac{2\Omega V_{x_0}}{g} \right) \cos L \sin AZ \right] \quad (8.216)$$

and we could rewrite Equation 8.215 as

$$y \approx y_0 + x \tan \phi_0 - f_C \frac{gx^2}{2V_{x_0}^2} \quad (8.217)$$

If we look closely at Equation 8.215, we note several things: the value of  $\cos L$  is everywhere between 0 and 1 for all possible latitudes; thus, if we were firing due north or due south, there would be no effect on the vertical component of impact; if we fired due east ( $AZ = 90^\circ$ ), the Coriolis effect essentially weakens the gravity term and the bullet would hit high; a due west firing would strike low; and the maximum effect on gravity is to alter it by 1.8%. Since  $\sin L$  fluctuates between +1 and -1, the drift, the  $z$ -component, will vary right or left depending on the hemisphere where the firing occurs.

Now that the physics of the Coriolis effect are understood, the only difference when applied to the non-vacuum point mass trajectory is the fact that the velocity is changing with time due to drag. This is best handled numerically and will not be covered here.

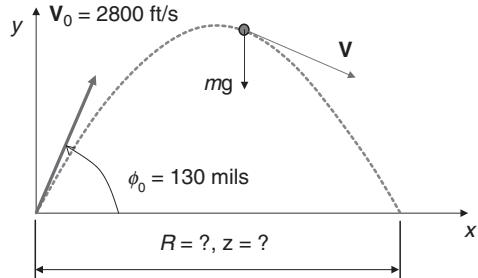
In summary, for the generalized point mass trajectory, we included drag, but ignored the projectile's dynamic effects on drag. We described the origins of the Coriolis acceleration acting on a projectile. The physics was demonstrated through the vacuum trajectory and further examined with the flat fire assumptions. Incorporation of the Coriolis acceleration into the generalized point mass assumption is only affected by the variation of velocity over the trajectory and best handled numerically.

### Problem 11

A projectile fired from a British 12-in. Mark IX naval gun had a muzzle velocity of 2800 ft/s and was fired at a QE of 130 mil ([Figure 8.9](#)). Assuming a vacuum trajectory, at what deflection would the shot hit the ground?

Assume the firing is taking place at  $50^\circ$  south latitude and the round is being fired due north.

*Answer:*  $z = -75.8$  [ft]



**FIGURE 8.9**  
Graphical representation of long range fire for Problem 11.

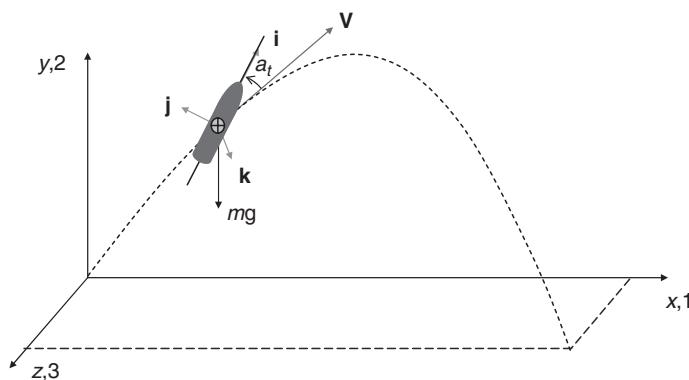
## 8.5 Six Degree-of-Freedom (6-DOF) Trajectory

In keeping with our plan of increasing the complexity of our analyses to approach more closely the physical realities of projectile flight, we will now consider the projectile as a distributed mass. Since projectiles are relatively stiff structures, a six degree-of-freedom model can adequately represent its position and attitude at any time. Each degree of freedom is tied to a coordinate necessary to completely describe the position of a body.

While this model is necessarily more complex than anything we have studied so far, the underlying physical principles remain the same. In the following work, we will use vectors (bold faced, non-italicized letters) in many of the derivations. We continue to do this because of the brevity and elegance of the notation.

In the equations that follow, we assume that the projectile is a rigid body of finite length with its mass distributed based on its geometry. This allows us to account for the effect of projectile attitude on drag and also allows the full dynamics to come into play. We shall use direction cosines with respect to the projectile axis of symmetry (and thus a coordinate system with unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , that translates with the CG but does not rotate and remains aligned with the projectile axis) as opposed to Eulerian angles (angles that are measured relative to the inertial coordinate system). This is illustrated in Figure 8.10.

Once again we restate the equations of motion, which for generality includes a term for rocket propulsion of the projectile. However, because this force is usually assumed to be aligned with the projectile's longitudinal axis, its effect on the motions we will study is



**FIGURE 8.10**  
Coordinate system for 6-DOF model.

uncoupled from the other motions and may be added in afterwards. Consequently, we will ignore it in our further work.

$$\mathbf{F} = m\mathbf{a} \quad (8.218)$$

$$m \frac{d\mathbf{V}}{dt} = \sum \mathbf{F} + m\mathbf{g} + m\boldsymbol{\Lambda} + \sum \mathbf{R}_T \quad (8.219)$$

where

$m$  = Projectile mass

$\mathbf{V}$  = Projectile velocity vector

$t$  = Time

$\mathbf{a} = \frac{d\mathbf{V}}{dt}$  = Vector acceleration

$\sum \mathbf{F}$  = Vector sum of all aerodynamic forces

$\mathbf{g}$  = Vector acceleration due to gravity

$\boldsymbol{\Lambda}$  = Vector Coriolis acceleration due to rotation of the earth

$\sum \mathbf{R}_T$  = Vector sum of all rocket thrust forces (to be ignored)

We can also write the equation for the conservation of angular momentum as

$$\frac{d\mathbf{H}}{dt} = \sum \mathbf{M} + \sum \mathbf{R}_M \quad (8.220)$$

where

$\mathbf{H}$  = Vector angular momentum of the projectile

$\sum \mathbf{M}$  = Vector sum of all aerodynamic moments

$\sum \mathbf{R}_M$  = Vector sum of all rocket thrust moments (to be ignored)

Because the projectile is assumed to be symmetric, every axis transverse to the longitudinal axis through the CG is a principal axis of inertia. The longitudinal axis itself is also, of course, a principal axis of inertia. The definition of the inertia tensor, which we will use, is

$$\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad (8.221)$$

Here the diagonal terms are called the moments of inertia and the off-diagonal terms are called the products of inertia. We know that there is a rotation that can be applied to this tensor such that the off-diagonal elements go to zero. In this orientation, the axes are said to be principal axes of inertia and the tensor is written

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad (8.222)$$

In our coordinate system, we shall define the unit vectors,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  so that they all lie along the projectile's principal axes. Because of this unique situation, the total angular momentum of the projectile can be expressed as the sum of two vectors: the angular momentum about  $\mathbf{i}$  and the angular momentum about any axis perpendicular to  $\mathbf{i}$  through the CG. Since the

$\mathbf{i}$ -axis is what we usually call the polar axis, we will denote the polar moment of inertia as  $I_P$ . With the symmetry of the projectile, the other moments of inertia about axes perpendicular to  $\mathbf{i}$  are known as the transverse moments of inertia,  $I_y = I_z = I_T$ . We can then rewrite the inertia tensor as

$$\mathbf{I} = \begin{bmatrix} I_P & 0 & 0 \\ 0 & I_T & 0 \\ 0 & 0 & I_T \end{bmatrix} \quad (8.223)$$

If a projectile is spinning at spin rate,  $p$ , the angular momentum about the polar axis is defined as

$$\mathbf{H}_P = I_P p \mathbf{i} \quad (8.224)$$

The angular momentum about any transverse axis is defined as

$$\mathbf{H}_T = I_T \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \quad (8.225)$$

With this, we can write the total momentum vector as

$$\mathbf{H} = I_P p \mathbf{i} + I_T \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \quad (8.226)$$

By defining a specific angular momentum,  $\mathbf{h} = \mathbf{H}/I_T$ , we can write

$$\mathbf{h} = \frac{I_P p}{I_T} \mathbf{i} + \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \quad (8.227)$$

If we take the derivative of Equation 8.227 with respect to time, we get

$$\frac{d\mathbf{h}}{dt} = \frac{I_P}{I_T} \dot{p} \mathbf{i} + \frac{I_P p}{I_T} \frac{d\mathbf{i}}{dt} + \left( \frac{d\mathbf{i}}{dt} \times \frac{d\mathbf{i}}{dt} \right) + \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) \quad (8.228)$$

Since the cross product of a vector with itself is zero, we get

$$\frac{d\mathbf{h}}{dt} = \frac{I_P}{I_T} \dot{p} \mathbf{i} + \frac{I_P p}{I_T} \frac{d\mathbf{i}}{dt} + \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) \quad (8.229)$$

In anticipation of a later need, we shall take the dot product and cross product of the vector  $\mathbf{h}$  with the unit vector  $\mathbf{i}$  to get

$$\mathbf{h} \cdot \mathbf{i} = \left[ \frac{I_P p}{I_T} \mathbf{i} + \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \right] \cdot \mathbf{i} = \frac{I_P p}{I_T} \quad (8.230)$$

$$\mathbf{h} \times \mathbf{i} = \left[ \frac{I_P p}{I_T} \mathbf{i} + \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \right] \times \mathbf{i} = \frac{d\mathbf{i}}{dt} \quad (8.231)$$

In Equations 8.230 and 8.231, we have used the orthogonality properties of vectors as follows:

$$\begin{aligned}\mathbf{i} \cdot \mathbf{i} &= 1 \\ \mathbf{i} \cdot \mathbf{j} &= \mathbf{i} \cdot \mathbf{k} = 0 \\ \mathbf{i} \times \mathbf{i} &= 0 \\ \mathbf{i} \times \mathbf{j} &= \mathbf{k} \rightarrow (\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = (\mathbf{k}) \times \mathbf{i} = \mathbf{j} \therefore (\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{j}\end{aligned}$$

We will now examine all of the forces and then the moments acting on the projectile and combine them into Equations 8.219 and 8.220. We have discussed all of these items in Chapter 6, so we shall simply refresh their meanings briefly and move on. The first force acting on the projectile is the drag force, which acts opposite to the velocity vector so we have

$$\text{Drag Force} = \mathbf{F}_D = -\frac{1}{2}\rho S C_D V V \quad (8.232)$$

The second force is the lift force, which we modified for our coordinate system as follows:

$$\text{Lift Force} = \mathbf{F}_L = \frac{1}{2}\rho S C_{L_\alpha} [\mathbf{V} \times (\mathbf{i} \times \mathbf{V})] \quad (8.233)$$

This equation contains a vector triple product in it that we replace with the relationship from vector algebra

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad (8.234)$$

which, for the product in Equation 8.233 can be written as

$$\mathbf{V} \times (\mathbf{i} \times \mathbf{V}) = V^2 \mathbf{i} - (\mathbf{V} \cdot \mathbf{i}) \mathbf{V}$$

When inserted into Equation 8.233, we have

$$\text{Lift force} = \mathbf{F}_L = \frac{1}{2}\rho S C_{L_\alpha} [V^2 \mathbf{i} - (\mathbf{V} \cdot \mathbf{i}) \mathbf{V}] \quad (8.235)$$

The next force is the Magnus force, brought on by the spin or roll of the projectile and taken from Equation 6.13

$$\text{Magnus force} = \mathbf{F}_M = \frac{1}{2}\rho S V \left( \frac{p d}{V} \right) C_{N_{pa}} (\mathbf{V} \times \mathbf{i}) \quad (8.236)$$

However, from Equation 8.230, we know that  $p = \frac{I_T}{I_P}(\mathbf{h} \cdot \mathbf{i})$  and, from vector algebra,  $\mathbf{V} \times \mathbf{i} = -\mathbf{i} \times \mathbf{V}$ . Then, we can manipulate Equation 8.236 to the form

$$\text{Magnus force} = \mathbf{F}_M = -\frac{1}{2}\rho S d C_{N_{pa}} \left( \frac{I_T}{I_P} \right) (\mathbf{h} \cdot \mathbf{i})(\mathbf{i} \times \mathbf{V}) \quad (8.237)$$

Next we need to include the pitch damping force from Equation 6.15 where we will write  $\mathbf{v}'$  as the unit vector along the velocity vector.

$$\text{Pitch damping force} = \frac{1}{2}\rho V S d \left( \frac{d\mathbf{i}}{dt} \right) C_{N_q} + \frac{1}{2}\rho V S d C_{N_{\dot{\alpha}}} \left( \frac{d\mathbf{i}}{dt} - \frac{d\mathbf{v}'}{dt} \right) \quad (8.238)$$

If we assume  $d\mathbf{v}/dt \ll d\mathbf{i}/dt$  (this means that the rate at which the velocity vector is rotating to follow the curve of the trajectory is much smaller than the rate at which the axis of the projectile is moving) and we include [Equation 8.231](#), we get a relation similar to [Equation 6.18](#)

$$\text{Pitch damping force} = \frac{1}{2}\rho VSdC_{N_q}\left(\frac{d\mathbf{i}}{dt}\right) + \frac{1}{2}\rho VSdC_{N_{\dot{\alpha}}}\left(\frac{d\mathbf{i}}{dt}\right) \quad (8.239)$$

or by [Equation 8.231](#)

$$\text{Pitch damping force} = \frac{1}{2}\rho VSd(C_{N_q} + C_{N_{\dot{\alpha}}})(\mathbf{h} \times \mathbf{i}) \quad (8.240)$$

With all our forces now expressed in terms of our defined coefficients, we can divide [Equation 8.219](#), omitting the rocket motor, by the projectile mass and inserting the coefficients to give

$$\begin{aligned} \frac{d\mathbf{V}}{dt} = & \frac{\rho VSC_D}{2m}\mathbf{V} + \frac{\rho SC_{L_a}}{2m}[V^2\mathbf{i} - (\mathbf{V} \cdot \mathbf{i})\mathbf{V}] - \frac{\rho SdC_{N_{pa}}}{2m}\left(\frac{I_T}{I_P}\right)(\mathbf{h} \cdot \mathbf{i})(\mathbf{i} \times \mathbf{V}) \\ & + \frac{\rho VSd(C_{N_q} + C_{N_{\dot{\alpha}}})}{2m}(\mathbf{h} \times \mathbf{i}) + \mathbf{g} + \mathbf{\Lambda} \end{aligned} \quad (8.241)$$

We will now examine the moments involved in [Equation 8.220](#), the first of which is the spin damping moment written as

$$\text{Spin damping moment} = \mathbf{M}_S = \frac{1}{2}\rho V^2 Sd\left(\frac{pd}{V}\right)C_{l_p}\mathbf{i} \quad (8.242)$$

However, if we again insert [Equation 8.230](#) into the above, we get

$$\text{Spin damping moment} = \mathbf{M}_S = \frac{1}{2}\rho VSd^2C_{l_p}\frac{I_T}{I_P}(\mathbf{h} \cdot \mathbf{i})\mathbf{i} \quad (8.243)$$

The rolling moment comes from [Equation 6.7](#) and is

$$\text{Rolling moment} = \mathbf{M}_R = \frac{1}{2}\rho V^2 Sd\delta_F C_{l_b}\mathbf{i} \quad (8.244)$$

The overturning moment can be written from [Equation 6.10](#) as

$$\text{Overturning moment} = \mathbf{M}_{\alpha} = \frac{1}{2}\rho SdVC_{M_{\alpha}}(\mathbf{V} \times \mathbf{i}) \quad (8.245)$$

The Magnus moment can be written from [Equation 6.14](#) and, by using the relations of [Equations 8.234](#) and [8.237](#), we get

$$\text{Magnus moment} = \mathbf{M}_{p\alpha} = \frac{1}{2}\rho Sd^2C_{M_{pa}}\frac{I_T}{I_P}(\mathbf{h} \cdot \mathbf{i})[\mathbf{V} - (\mathbf{V} \cdot \mathbf{i})\mathbf{i}] \quad (8.246)$$

We can obtain the pitch damping moment by rewriting [Equation 6.19](#) as well as using the relation of [Equation 8.230](#) to get

$$\text{Pitch damping moment} = \mathbf{M}_q = \frac{1}{2}\rho VSd^2(C_{M_q} + C_{M_{\dot{\alpha}}})[\mathbf{h} - (\mathbf{h} \cdot \mathbf{i})\mathbf{i}] \quad (8.247)$$

We can now place all of these relations into [Equation 8.220](#), again omitting the rocket term, to yield

$$\frac{d\mathbf{H}}{dt} = \mathbf{M}_S + \mathbf{M}_R + \mathbf{M}_\alpha + \mathbf{M}_{p\alpha} + \mathbf{M}_q \quad (8.248)$$

[Equation 8.248](#) can be changed to a more desirable form by dividing by  $I_T$ , which yields

$$\frac{d\mathbf{h}}{dt} = \frac{\mathbf{M}_S}{I_T} + \frac{\mathbf{M}_R}{I_T} + \frac{\mathbf{M}_\alpha}{I_T} + \frac{\mathbf{M}_{p\alpha}}{I_T} + \frac{\mathbf{M}_q}{I_T} \quad (8.249)$$

This, in turn, may be rewritten by inserting the various moment equations derived above as

$$\begin{aligned} \frac{d\mathbf{h}}{dt} = & \frac{\rho VSd^2C_{l_p}}{2I_p}(\mathbf{h} \cdot \mathbf{i})\mathbf{i} + \frac{\rho V^2Sd\delta_F C_{l_\delta}}{2I_T}\mathbf{i} + \frac{\rho VSdC_{M_\alpha}}{2I_T}(\mathbf{V} \times \mathbf{i}) \\ & + \frac{\rho Sd^2C_{M_{p\alpha}}}{2I_p}(\mathbf{h} \cdot \mathbf{i})[\mathbf{V} - (\mathbf{V} \cdot \mathbf{i})\mathbf{i}] + \frac{\rho VSd^2(C_{M_q} + C_{M_\delta})}{2I_T}[\mathbf{h} - (\mathbf{h} \cdot \mathbf{i})\mathbf{i}] \end{aligned} \quad (8.250)$$

Note that the equations of motion are highly coupled to one another and the reason we call the model a 6 DOF is readily apparent. When we break the equations up into their individual components we have six equations and six unknowns ( $x, y, z, p, \alpha$ , and  $\beta$ ). Let us recall that the  $x, y, z$  axes are axes fixed to the earth, independent of the projectile, while  $\mathbf{i}$  is the unit vector along the axis of symmetry of the projectile and has components along the  $x, y, z$  earth axes. For convenience, clarity, and to facilitate analysis, we will relabel the  $x, y, z$  unit vectors (normally  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ) as  $\mathbf{e}_1, \mathbf{e}_2$ , and  $\mathbf{e}_3$ , respectively, letting the subscripts denote the  $x, y, z$  axes in that order ([see Figure 8.10](#)). Then, in terms of components in the earth-fixed system

$$\mathbf{h} = h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3 \quad (8.251)$$

$$\mathbf{i} = i_1\mathbf{e}_1 + i_2\mathbf{e}_2 + i_3\mathbf{e}_3 \quad (8.252)$$

$$\mathbf{V} = V_1\mathbf{e}_1 + V_2\mathbf{e}_2 + V_3\mathbf{e}_3 \quad (8.253)$$

$$\mathbf{W} = W_1\mathbf{e}_1 + W_2\mathbf{e}_2 + W_3\mathbf{e}_3 \quad (8.254)$$

We shall also define

$$\mathbf{v} = \mathbf{V} - \mathbf{W} = (V_1 - W_1)\mathbf{e}_1 + (V_2 - W_2)\mathbf{e}_2 + (V_3 - W_3)\mathbf{e}_3 \quad (8.255)$$

and further defining

$$v_1 = (V_1 - W_1) \quad v_2 = (V_2 - W_2) \quad v_3 = (V_3 - W_3) \quad (8.256)$$

and

$$v = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad (8.257)$$

We can insert the definition for  $\mathbf{v}$  in place of  $\mathbf{V}$  in [Equations 8.246](#) and 8.250 to yield

$$\begin{aligned} \frac{d\mathbf{V}}{dt} = & -\frac{\rho v S C_D}{2m} \mathbf{v} + \frac{\rho S C_{L_\alpha}}{2m} [v^2 \mathbf{i} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{v}] - \frac{\rho S d C_{N_{p\alpha}}}{2m} \left( \frac{I_T}{I_p} \right) (\mathbf{h} \cdot \mathbf{i}) (\mathbf{i} \times \mathbf{v}) \\ & + \frac{\rho v S d (C_{N_q} + C_{N_\delta})}{2m} (\mathbf{h} \times \mathbf{i}) + \mathbf{g} + \mathbf{\Lambda} \end{aligned} \quad (8.258)$$

and

$$\begin{aligned}\frac{d\mathbf{h}}{dt} = & \frac{\rho v S d^2 C_{lp}}{2I_p} (\mathbf{h} \cdot \mathbf{i}) \mathbf{i} + \frac{\rho v^2 S d \delta_F C_{ls}}{2I_T} \mathbf{i} + \frac{\rho v S d C_{Ma}}{2I_T} (\mathbf{v} \times \mathbf{i}) \\ & + \frac{\rho S d^2 C_{Mpa}}{2I_p} (\mathbf{h} \cdot \mathbf{i}) [\mathbf{v} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{i}] + \frac{\rho v S d^2 (C_{Mq} + C_{Ma})}{2I_T} [\mathbf{h} - (\mathbf{h} \cdot \mathbf{i}) \mathbf{i}]\end{aligned}\quad (8.259)$$

Our goal is to examine Equations 8.258 and 8.259 and break each into three equations, one for each coordinate direction (actually for the acceleration in each coordinate direction). But before we attempt to break Equations 8.258 and 8.259 into their components, it will be best to solve for some of the vector quantities that occur in them. Beginning with the second term of Equation 8.258 we can, with appropriate vector multiplication, obtain

$$\begin{aligned}(\mathbf{v} \cdot \mathbf{i}) \mathbf{v} = & (v_1^2 i_1 + v_1 v_2 i_2 + v_1 v_3 i_3) \mathbf{e}_1 + (v_1 v_2 i_1 + v_2^2 i_2 + v_2 v_3 i_3) \mathbf{e}_2 \\ & + (v_1 v_3 i_1 + v_2 v_3 i_2 + v_3^2 i_3) \mathbf{e}_3\end{aligned}\quad (8.260)$$

Another useful relation is that

$$(\mathbf{v} \cdot \mathbf{i}) = (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3) \cdot (i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3)\quad (8.261)$$

or

$$(\mathbf{v} \cdot \mathbf{i}) = v_1 i_1 + v_2 i_2 + v_3 i_3\quad (8.262)$$

However, we can show that

$$\cos \alpha_t = \frac{(\mathbf{v} \cdot \mathbf{i})}{v} = \frac{v_1 i_1 + v_2 i_2 + v_3 i_3}{v}\quad (8.263)$$

The next relations in Equation 8.258 are

$$(\mathbf{h} \cdot \mathbf{i}) = h_1 i_1 + h_2 i_2 + h_3 i_3\quad (8.264)$$

and

$$(\mathbf{i} \times \mathbf{v}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ i_1 & i_2 & i_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (i_2 v_3 - i_3 v_2) \mathbf{e}_1 + (i_3 v_1 - i_1 v_3) \mathbf{e}_2 + (i_1 v_2 - i_2 v_1) \mathbf{e}_3\quad (8.265)$$

Now, also in Equation 8.258 is the term

$$(\mathbf{h} \cdot \mathbf{i})(\mathbf{i} \times \mathbf{v}) = (h_1 i_1 + h_2 i_2 + h_3 i_3)[(i_2 v_3 - i_3 v_2) \mathbf{e}_1 + (i_3 v_1 - i_1 v_3) \mathbf{e}_2 + (i_1 v_2 - i_2 v_1) \mathbf{e}_3]\quad (8.266)$$

But with the fact that  $(\mathbf{h} \cdot \mathbf{i}) = \frac{I_p p}{I_T}$  as shown earlier, then we can write

$$(\mathbf{h} \cdot \mathbf{i})(\mathbf{i} \times \mathbf{v}) = \frac{I_p p}{I_T} (i_2 v_3 - i_3 v_2) \mathbf{e}_1 + \frac{I_p p}{I_T} (i_3 v_1 - i_1 v_3) \mathbf{e}_2 + \frac{I_p p}{I_T} (i_1 v_2 - i_2 v_1) \mathbf{e}_3\quad (8.267)$$

Another relation we have to deal with is

$$(\mathbf{h} \times \mathbf{i}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ h_1 & h_2 & h_3 \\ i_1 & i_2 & i_3 \end{vmatrix} = (h_2 i_3 - h_3 i_2) \mathbf{e}_1 + (h_3 i_1 - h_1 i_3) \mathbf{e}_2 + (h_1 i_2 - h_2 i_1) \mathbf{e}_3\quad (8.268)$$

The next relation we generate with the help of [Equation 8.230](#)

$$(\mathbf{h} \cdot \mathbf{i})\mathbf{i} = \frac{I_{pp}}{I_T} i_1 \mathbf{e}_1 + \frac{I_{pp}}{I_T} i_2 \mathbf{e}_2 + \frac{I_{pp}}{I_T} i_3 \mathbf{e}_3 \quad (8.269)$$

In [Equation 8.259](#), we need

$$(\mathbf{v} \times \mathbf{i}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ i_1 & i_2 & i_3 \end{vmatrix} = (i_3 v_2 - i_2 v_3) \mathbf{e}_1 + (i_1 v_3 - i_3 v_1) \mathbf{e}_2 + (i_2 v_1 - i_1 v_2) \mathbf{e}_3 \quad (8.270)$$

And finally in [Equation 8.259](#)

$$(\mathbf{v} \cdot \mathbf{i})\mathbf{i} = (v_1 i_1 + v_2 i_2 + v_3 i_3)(i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3) \quad (8.271)$$

or

$$\begin{aligned} (\mathbf{v} \cdot \mathbf{i})\mathbf{i} = & (v_1 i_1^2 + v_2 i_1 i_2 + v_3 i_1 i_3) \mathbf{e}_1 + (v_1 i_1 i_2 + v_2 i_2^2 + v_3 i_2 i_3) \mathbf{e}_2 \\ & + (v_1 i_1 i_3 + v_2 i_2 i_3 + v_3 i_3^2) \mathbf{e}_3 \end{aligned} \quad (8.272)$$

Let us now look at [Equation 8.258](#) with all of the vector quantities broken into their components

$$\begin{aligned} \frac{dV_1}{dt} \mathbf{e}_1 + \frac{dV_2}{dt} \mathbf{e}_2 + \frac{dV_3}{dt} \mathbf{e}_3 = & -\frac{\rho v S C_D}{2m} (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3) \\ & + \frac{\rho S C_{L_\alpha}}{2m} [v^2 i_1 \mathbf{e}_1 + v^2 i_2 \mathbf{e}_2 + v^2 i_3 \mathbf{e}_3 - v \cos \alpha_t (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3)] \\ & - \frac{\rho S d C_{N_{pa}}}{2m} \left( \frac{I_{pp}}{I_p} \right) \left( \frac{I_{pp}}{I_T} \right) [(v_3 i_2 - v_2 i_3) \mathbf{e}_1 + (v_1 i_3 - v_3 i_1) \mathbf{e}_2 + (v_2 i_1 - v_1 i_2) \mathbf{e}_3] \\ & + \frac{\rho v S d (C_{N_q} + C_{N_{\dot{\alpha}}})}{2m} [(h_2 i_3 - h_3 i_2) \mathbf{e}_1 + (h_3 i_1 - h_1 i_3) \mathbf{e}_2 + (h_1 i_2 - h_2 i_1) \mathbf{e}_3] \\ & + g_1 \mathbf{e}_1 + g_2 \mathbf{e}_2 + g_3 \mathbf{e}_3 + \Lambda_1 \mathbf{e}_1 + \Lambda_2 \mathbf{e}_2 + \Lambda_3 \mathbf{e}_3 \end{aligned} \quad (8.273)$$

Similarly, let us perform the same operation on [Equation 8.259](#)

$$\begin{aligned} \frac{dh_1}{dt} \mathbf{e}_1 + \frac{dh_2}{dt} \mathbf{e}_2 + \frac{dh_3}{dt} \mathbf{e}_3 = & \frac{\rho v S d^2 C_{l_p}}{2 I_p} \left( \frac{I_{pp}}{I_T} \right) (i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3) \\ & + \frac{\rho v^2 S d \delta_F C_{l_b}}{2 I_T} (i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3) + \frac{\rho v S d C_{M_\alpha}}{2 I_T} [(v_2 i_3 - v_3 i_2) \mathbf{e}_1 + (v_3 i_1 - v_1 i_3) \mathbf{e}_2 + (v_1 i_2 - v_2 i_1) \mathbf{e}_3] \\ & + \frac{\rho S d^2 C_{M_{pa}}}{2 I_p} \left( \frac{I_{pp}}{I_T} \right) [(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3) - v \cos \alpha_t (i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3)] \\ & + \frac{\rho v S d^2 (C_{M_q} + C_{M_{\dot{\alpha}}})}{2 I_T} \left[ (h_1 \mathbf{e}_1 + h_2 \mathbf{e}_2 + h_3 \mathbf{e}_3) - \left( \frac{I_{pp}}{I_T} \right) (i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + i_3 \mathbf{e}_3) \right] \end{aligned} \quad (8.274)$$

We will first operate on [Equation 8.273](#) by collecting all of the terms with the unit vectors  $\mathbf{e}_1$ , then  $\mathbf{e}_2$  and  $\mathbf{e}_3$ , and by putting them into the equations for linear and angular momentum.

$$\begin{aligned}\frac{dV_1}{dt} = & -\frac{\rho v S C_D}{2m} v_1 + \frac{\rho S C_{L_\alpha}}{2m} [v^2 i_1 - v v_1 \cos \alpha_t] - \frac{\rho S d C_{N_{pa}} p}{2m} (v_3 i_2 - v_2 i_3) \\ & + \frac{\rho v S d (C_{N_q} + C_{N_\alpha})}{2m} (h_2 i_3 - h_3 i_2) + g_1 + \Lambda_1\end{aligned}\quad (8.275)$$

$$\begin{aligned}\frac{dV_2}{dt} = & -\frac{\rho v S C_D}{2m} v_2 + \frac{\rho S C_{L_\alpha}}{2m} [v^2 i_2 - v v_2 \cos \alpha_t] - \frac{\rho S d C_{N_{pa}} p}{2m} (v_1 i_3 - v_3 i_1) \\ & + \frac{\rho v S d (C_{N_q} + C_{N_\alpha})}{2m} (h_3 i_1 - h_1 i_3) + g_2 + \Lambda_2\end{aligned}\quad (8.276)$$

$$\begin{aligned}\frac{dV_3}{dt} = & -\frac{\rho v S C_D}{2m} v_3 + \frac{\rho S C_{L_\alpha}}{2m} [v^2 i_3 - v v_3 \cos \alpha_t] - \frac{\rho S d C_{N_{pa}} p}{2m} (v_2 i_1 - v_1 i_2) \\ & + \frac{\rho v S d (C_{N_q} + C_{N_\alpha})}{2m} (h_1 i_2 - h_2 i_1) + g_3 + \Lambda_3\end{aligned}\quad (8.277)$$

Next is [Equation 8.274](#) where the same procedure will be followed.

$$\begin{aligned}\frac{dh_1}{dt} = & \frac{\rho v S d^2 C_{l_p} p}{2I_T} i_1 + \frac{\rho v^2 S d \delta_F C_{l_s}}{2I_T} i_1 + \frac{\rho v S d C_{M_\alpha}}{2I_T} (v_2 i_3 - v_3 i_2) \\ & + \frac{\rho S d^2 C_{M_{pa}} p}{2I_T} [v_1 - v i_1 \cos \alpha_t] + \frac{\rho v S d^2 (C_{M_q} + C_{M_\alpha})}{2I_T} \left[ h_1 - \left( \frac{I_p p}{I_T} \right) i_1 \right]\end{aligned}\quad (8.278)$$

$$\begin{aligned}\frac{dh_2}{dt} = & \frac{\rho v S d^2 C_{l_p} p}{2I_T} i_2 + \frac{\rho v^2 S d \delta_F C_{l_s}}{2I_T} i_2 + \frac{\rho v S d C_{M_\alpha}}{2I_T} (v_3 i_1 - v_1 i_3) \\ & + \frac{\rho S d^2 C_{M_{pa}} p}{2I_T} [v_2 - v i_2 \cos \alpha_t] + \frac{\rho v S d^2 (C_{M_q} + C_{M_\alpha})}{2I_T} \left[ h_2 - \left( \frac{I_p p}{I_T} \right) i_2 \right]\end{aligned}\quad (8.279)$$

$$\begin{aligned}\frac{dh_3}{dt} = & \frac{\rho v S d^2 C_{l_p} p}{2I_T} i_3 + \frac{\rho v^2 S d \delta_F C_{l_s}}{2I_T} i_3 + \frac{\rho v S d C_{M_\alpha}}{2I_T} (v_1 i_2 - v_2 i_1) \\ & + \frac{\rho S d^2 C_{M_{pa}} p}{2I_T} [v_3 - v i_3 \cos \alpha_t] + \frac{\rho v S d^2 (C_{M_q} + C_{M_\alpha})}{2I_T} \left[ h_3 - \left( \frac{I_p p}{I_T} \right) i_3 \right]\end{aligned}\quad (8.280)$$

We can simplify Equations 8.275 through 8.280 considerably by defining the following coefficients:

$$\begin{aligned}\tilde{C}_D &= \frac{\rho v S C_D}{2m} & \tilde{C}_{l_p} &= \frac{\rho v S d^2 C_{l_p} p}{2I_T} \\ \tilde{C}_{L_\alpha} &= \frac{\rho S C_{L_\alpha}}{2m} & \tilde{C}_{l_s} &= \frac{\rho v^2 S d \delta_F C_{l_s}}{2I_T} \\ \tilde{C}_{N_{pa}} &= \frac{\rho S d C_{N_{pa}} p}{2m} & \tilde{C}_{M_\alpha} &= \frac{\rho v S d C_{M_\alpha}}{2I_T} \\ \tilde{C}_{N_q} &= \frac{\rho v S d (C_{N_q} + C_{N_\alpha})}{2m} & \tilde{C}_{M_{pa}} &= \frac{\rho S d^2 C_{M_{pa}} p}{2I_T} \\ \tilde{C}_{M_q} &= \frac{\rho v S d^2 (C_{M_q} + C_{M_\alpha})}{2I_T}\end{aligned}$$

With these coefficients, we can write Equations 8.275 through 8.277 in a more compact form:

$$\begin{aligned}\frac{dV_1}{dt} = & -\tilde{C}_D v_1 + \tilde{C}_{L_a}(v^2 i_1 - vv_1 \cos \alpha_t) - \tilde{C}_{N_{pa}}(v_3 i_2 - v_2 i_3) \\ & + \tilde{C}_{N_q}(h_2 i_3 - h_3 i_2) + g_1 + \Lambda_1\end{aligned}\quad (8.281)$$

$$\begin{aligned}\frac{dV_2}{dt} = & -\tilde{C}_D v_2 + \tilde{C}_{L_a}(v^2 i_2 - vv_2 \cos \alpha_t) - \tilde{C}_{N_{pa}}(v_1 i_3 - v_3 i_1) \\ & + \tilde{C}_{N_q}(h_3 i_1 - h_1 i_3) + g_2 + \Lambda_2\end{aligned}\quad (8.282)$$

$$\begin{aligned}\frac{dV_3}{dt} = & -\tilde{C}_D v_3 + \tilde{C}_{L_a}(v^2 i_3 - vv_3 \cos \alpha_t) - \tilde{C}_{N_{pa}}(v_2 i_1 - v_1 i_2) \\ & + \tilde{C}_{N_q}(h_1 i_2 - h_2 i_1) + g_3 + \Lambda_3\end{aligned}\quad (8.283)$$

We can do the same with Equations 8.278 through 8.280.

$$\frac{dh_1}{dt} = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta}) i_1 + \tilde{C}_{M_a}(v_2 i_3 - v_3 i_2) + \tilde{C}_{M_{pa}}(v_1 - vi_1 \cos \alpha_t) + \tilde{C}_{M_q}\left[h_1 - \left(\frac{I_{pp}}{I_T}\right)i_1\right] \quad (8.284)$$

$$\frac{dh_2}{dt} = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta}) i_2 + \tilde{C}_{M_a}(v_3 i_1 - v_1 i_3) + \tilde{C}_{M_{pa}}(v_2 - vi_2 \cos \alpha_t) + \tilde{C}_{M_q}\left[h_2 - \left(\frac{I_{pp}}{I_T}\right)i_2\right] \quad (8.285)$$

$$\frac{dh_3}{dt} = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta}) i_3 + \tilde{C}_{M_a}(v_1 i_2 - v_2 i_1) + \tilde{C}_{M_{pa}}(v_3 - vi_3 \cos \alpha_t) + \tilde{C}_{M_q}\left[h_3 - \left(\frac{I_{pp}}{I_T}\right)i_3\right] \quad (8.286)$$

Now that we have the six, coupled, equations for our six accelerations, we would like to determine the position of the projectile in space and time. We do this by creating a vector,  $\mathbf{X}$ , to the center of mass of the projectile. If we note that  $\mathbf{X} = [x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3]$  in the earth-fixed coordinate system, then we can break the individual components into

$$x = x_0 + \int_0^t V_1 dt \quad (8.287)$$

$$y = y_0 + \int_0^t V_2 dt \quad (8.288)$$

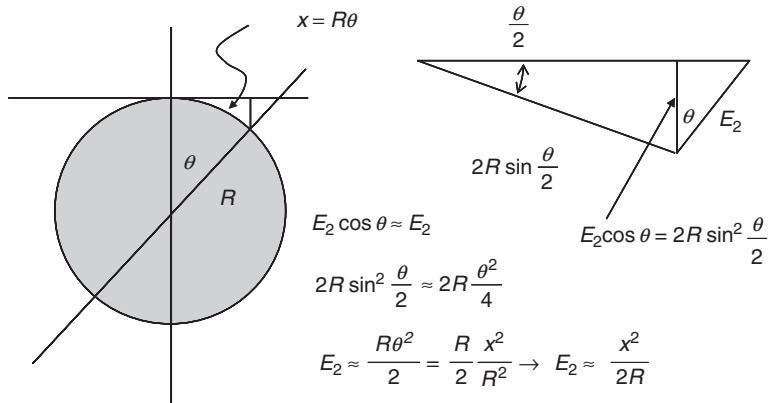
$$z = z_0 + \int_0^t V_3 dt \quad (8.289)$$

Recognize that when firing a long range weapon, we usually do so with grid coordinates on a map of the earth. A map is, in theory, created by peeling the geometry off a globe. Thus, the coordinates and distances are correct in the downrange and cross range directions ( $x$  and  $z$ ). However, the altitude,  $y$ , has to be corrected for the curvature of the earth. This is depicted with the applicable equations in Figure 8.11. A similar rotation occurs with the gravity vector as depicted in Figure 8.12.

With this relationship, we can write the projectile position vector in earth coordinates as

$$\mathbf{E} \approx [E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 + E_3 \mathbf{e}_3] = \left[ x\mathbf{e}_1 + \left(y + \frac{x^2}{2R}\right)\mathbf{e}_2 + z\mathbf{e}_3 \right] = \begin{bmatrix} x \\ y + \frac{x^2}{2R} \\ z \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (8.290)$$

Here  $R$  is the average radius of the earth, taken to be 6,951,844 yards or 6,356,766 m. The use of earth coordinates is recommended at ranges beyond about 2000 yards (at 2000 yards



**FIGURE 8.11**  
Altitude error over long trajectories.

there is a 10.36-in. difference in height [1]). Furthermore, the acceleration of gravity varies with altitude (and, in fact, latitude and longitude as well) and we need to consider this.

To complete the equations of motion, we must consider the form of the Coriolis acceleration vector. We have discussed this extensively previously so we shall simply write the components of this vector as

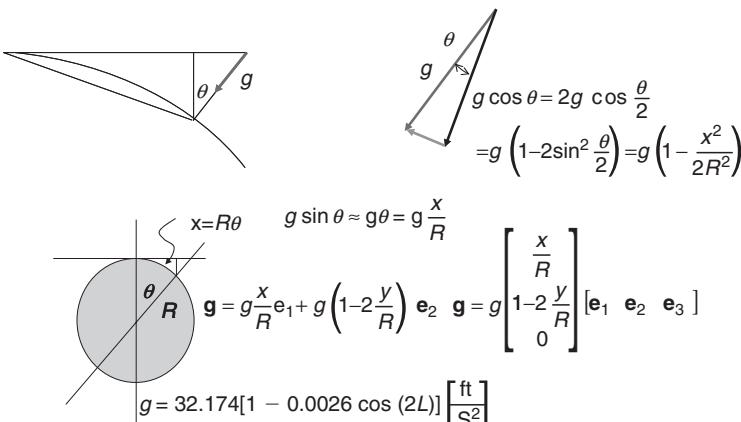
$$\Lambda_1 = 2\Omega(-V_2 \cos L \sin AZ - V_3 \sin L) \quad (8.291)$$

$$\Lambda_2 = 2\Omega(V_1 \cos L \sin AZ + V_3 \cos L \cos AZ) \quad (8.292)$$

$$\Lambda_3 = 2\Omega(V_1 \sin L - V_2 \cos L \cos AZ) \quad (8.293)$$

or as a vector

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = 2\Omega \begin{bmatrix} -V_2 \cos L \sin AZ - V_3 \sin L \\ V_1 \cos L \sin AZ + V_3 \cos L \cos AZ \\ V_1 \sin L - V_2 \cos L \cos AZ \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (8.294)$$



**FIGURE 8.12**  
Rotation of the gravity vector due to earth curvature and associated equations.

We now have the differential equations of motion but need initial conditions to solve them. Let us examine the projectile at the instant of muzzle exit without worrying about how it attained its state of motion there (this is the job of the interior ballistian). We shall define the initial tube angle in azimuth and elevation as  $\theta_0$  and  $\phi_0$ , respectively. Then our initial velocity vector can be defined as

$$\mathbf{V}_0 = \begin{bmatrix} V_{1_0} \\ V_{2_0} \\ V_{3_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = V_0 \begin{bmatrix} \cos \phi_0 \cos \theta_0 \\ \sin \phi_0 \cos \theta_0 \\ \sin \theta_0 \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (8.295)$$

And, if we also take the wind into account, we have

$$\mathbf{v}_0 = \mathbf{V}_0 - \mathbf{W}_0 = \begin{bmatrix} v_{1_0} \\ v_{2_0} \\ v_{3_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = \begin{bmatrix} V_{1_0} - W_{1_0} \\ V_{2_0} - W_{2_0} \\ V_{3_0} - W_{3_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (8.296)$$

Here the usual relationships for these vectors apply. These are

$$V_0 = \sqrt{V_{1_0}^2 + V_{2_0}^2 + V_{3_0}^2} \quad (8.297)$$

$$v_0 = \sqrt{v_{1_0}^2 + v_{2_0}^2 + v_{3_0}^2} \quad (8.298)$$

The initial orientations of the body-fixed unit vectors in the earth-fixed system are

$$\begin{aligned} \mathbf{i}_0 &= i_{1_0} \mathbf{e}_1 + i_{2_0} \mathbf{e}_2 + i_{3_0} \mathbf{e}_3 = \begin{bmatrix} i_{1_0} \\ i_{2_0} \\ i_{3_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \\ &= \begin{bmatrix} \cos(\phi_0 + \alpha_0) \cos(\theta_0 + \beta_0) \\ \sin(\phi_0 + \alpha_0) \cos(\theta_0 + \beta_0) \\ \sin(\theta_0 + \beta_0) \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \end{aligned} \quad (8.299)$$

$$\begin{aligned} \mathbf{j}_0 &= j_{1_0} \mathbf{e}_1 + j_{2_0} \mathbf{e}_2 + j_{3_0} \mathbf{e}_3 = \begin{bmatrix} j_{1_0} \\ j_{2_0} \\ j_{3_0} \end{bmatrix} [\mathbf{e}] \\ &= \frac{1}{\sqrt{Q}} \begin{bmatrix} -\cos^2(\theta_0 + \beta_0) \sin(\phi_0 + \alpha_0) \cos(\phi_0 + \alpha_0) \\ \cos^2(\theta_0 + \beta_0) \cos(\phi_0 + \alpha_0) + \sin^2(\theta_0 + \beta_0) \\ -\sin(\theta_0 + \beta_0) \cos(\theta_0 + \beta_0) \sin(\phi_0 + \alpha_0) \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \end{aligned} \quad (8.300)$$

$$\begin{aligned} \mathbf{k}_0 &= k_{1_0} \mathbf{e}_1 + k_{2_0} \mathbf{e}_2 + k_{3_0} \mathbf{e}_3 = \begin{bmatrix} k_{1_0} \\ k_{2_0} \\ k_{3_0} \end{bmatrix} [\mathbf{e}] \\ &= \frac{1}{\sqrt{Q}} \begin{bmatrix} -\sin(\theta_0 + \beta_0) \\ 0 \\ \cos(\theta_0 + \beta_0) \cos(\phi_0 + \alpha_0) \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \end{aligned} \quad (8.301)$$

In the above equations,  $\alpha_0$  and  $\beta_0$  are the initial pitch and yaw angles, respectively, of the projectile. Thus, they add directly to the weapon azimuth and elevation angles. The quantity  $Q$  we define following Ref. [1] as

$$Q = \sin^2(\theta_0 + \beta_0) + \cos^2(\theta_0 + \beta_0)\cos^2(\phi_0 + \alpha_0) \quad (8.302)$$

If we now consider the rotation,  $(\boldsymbol{\omega})_{ijk}$  of the projectile about its axis of symmetry (thus relative to the **i-j-k** triad) and we define an arbitrary initial projectile rotation as

$$(\boldsymbol{\omega}_0)_{ijk} = \omega_{i_0} \mathbf{i}_0 + \omega_{j_0} \mathbf{j}_0 + \omega_{k_0} \mathbf{k}_0 \quad (8.303)$$

Here this initial angular velocity is dependent upon the initial orientation of the unit vector,  $\mathbf{i}_0$ . Then the initial velocity of the unit vector can be written as follows:

$$\begin{aligned} \frac{d\mathbf{i}_0}{dt} &= (\boldsymbol{\omega}_0)_{ijk} \times \mathbf{i}_0 = \begin{vmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 \\ \omega_{i_0} & \omega_{j_0} & \omega_{k_0} \\ 1 & 0 & 0 \end{vmatrix} = [\omega_{k_0} \mathbf{j}_0 - \omega_{j_0} \mathbf{k}_0] = \begin{bmatrix} \dot{i}_{1_0} \\ \dot{i}_{2_0} \\ \dot{i}_{3_0} \end{bmatrix} [\mathbf{e}] \\ &= \begin{bmatrix} \omega_{k_0} j_{1_0} - \omega_{j_0} k_{1_0} \\ \omega_{k_0} j_{2_0} - \omega_{j_0} k_{2_0} \\ \omega_{k_0} j_{3_0} - \omega_{j_0} k_{3_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \end{aligned} \quad (8.304)$$

Note that Equation 8.304 is a tensor equation. Tensors are higher order vectors but can be treated the same. If we insert the results of Equations 8.300 and 8.301 into the above, we get

$$\dot{i}_{1_0} = \frac{1}{\sqrt{Q}} [\omega_{j_0} \sin(\theta_0 + \beta_0) - \omega_{k_0} \cos^2(\theta_0 + \beta_0) \sin(\phi_0 + \alpha_0) \cos(\phi_0 + \alpha_0)] \quad (8.305)$$

$$\dot{i}_{2_0} = \frac{1}{\sqrt{Q}} [\omega_{k_0} \cos^2(\theta_0 + \beta_0) \cos(\phi_0 + \alpha_0) + \omega_{k_0} \sin^2(\theta_0 + \beta_0)] \quad (8.306)$$

$$\begin{aligned} \dot{i}_{3_0} &= \frac{1}{\sqrt{Q}} [-\omega_{j_0} \cos(\theta_0 + \beta_0) \cos(\phi_0 + \alpha_0) \\ &\quad - \omega_{k_0} \sin(\theta_0 + \beta_0) \cos(\theta_0 + \beta_0) \sin(\phi_0 + \alpha_0)] \end{aligned} \quad (8.307)$$

Continuing with our statement of the initial conditions, a positive pitch rotates the nose of the projectile upward and a positive yaw rotates the nose to the left as viewed from the rear. The initial value of the modified angular momentum vector is given by

$$\mathbf{h}_0 = \frac{I_p p_0}{I_T} \mathbf{i}_0 + \left( \mathbf{i}_0 \times \frac{d\mathbf{i}_0}{dt} \right) \quad (8.308)$$

We can rewrite  $d\mathbf{i}_0/dt$  as

$$\frac{d\mathbf{i}_0}{dt} = \dot{i}_{1_0} \mathbf{e}_1 + \dot{i}_{2_0} \mathbf{e}_2 + \dot{i}_{3_0} \mathbf{e}_3 \quad (8.309)$$

which then allows us to write

$$\mathbf{i}_0 \times \frac{d\mathbf{i}_0}{dt} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ i_{1_0} & i_{2_0} & i_{3_0} \\ \dot{i}_{1_0} & \dot{i}_{2_0} & \dot{i}_{3_0} \end{vmatrix} = (i_{2_0} \dot{i}_{3_0} - i_{3_0} \dot{i}_{2_0}) \mathbf{e}_1 + (i_{3_0} \dot{i}_{1_0} - i_{1_0} \dot{i}_{3_0}) \mathbf{e}_2 + (i_{1_0} \dot{i}_{2_0} - i_{2_0} \dot{i}_{1_0}) \mathbf{e}_3 \quad (8.310)$$

We can then incorporate Equation 8.310 into Equation 8.308 to yield

$$\mathbf{h}_0 = \begin{bmatrix} h_{1_0} \\ h_{2_0} \\ h_{3_0} \end{bmatrix} [\mathbf{e}] = \begin{bmatrix} \frac{I_p p_0}{I_T} i_{1_0} + i_{2_0} \dot{i}_{3_0} - i_{3_0} \dot{i}_{2_0} \\ \frac{I_p p_0}{I_T} i_{2_0} + i_{3_0} \dot{i}_{1_0} - i_{1_0} \dot{i}_{3_0} \\ \frac{I_p p_0}{I_T} i_{3_0} + i_{1_0} \dot{i}_{2_0} - i_{2_0} \dot{i}_{1_0} \end{bmatrix} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (8.311)$$

Here the initial value of the spin rate  $p_0$  is determined by the axial velocity and the twist rate,  $n$  (in calibers per revolution) of the weapon through  $p_0 = 2\pi V_0/nd$ . We have thus completed all of the initial conditions necessary to perform the calculation.

As we will discuss in a later section, the projectile's motion can be characterized as epicyclic. The tip of a vector drawn from the CG of the projectile to the nose will trace out a curve that contains two cyclic modes: a fast mode, known as nutation and a slow mode, known as precession. If the round is stable, these modes will eventually damp down to near zero leaving only some movement because of nonlinear forces and moments. We shall explore this more later.

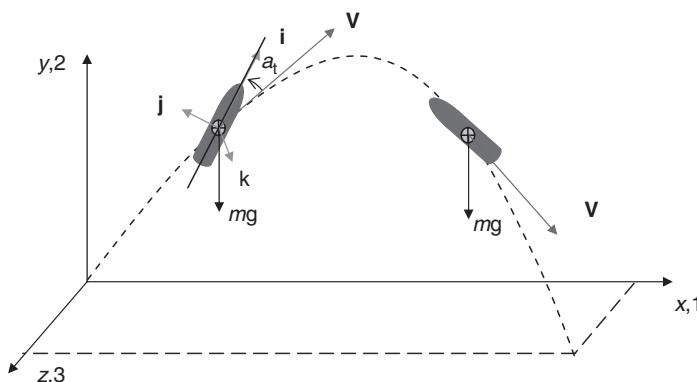
Some other terms come up in succeeding sections that require definitions. Since they are essential to the understanding of trajectories, we will define them now.

A projectile's yaw of repose is the yaw created by the action of gravity on the projectile as it attempts to follow its trajectory curve. As stated earlier, the nose of the projectile is usually above the trajectory. There is then a net aerodynamic force through the CP which wants to rotate the nose up. With a right-hand spinning projectile, this results in a yaw of the nose to the right. This is called the yaw of repose.

Failure to trail is a situation that arises when the base of the projectile does not follow the nose (it flies base first after apogee). This is depicted in Figure 8.13.

The trail angle is the quadrant elevation angle (particular to a gun, projectile, and charge combination) above which the projectile will not turn over and will fail to trail.

We can summarize this section by saying that for a rigid projectile, the 6-DOF model is as accurate as one can get to the trajectories. If the model yields an inaccurate answer, the problem is usually a wrong assumption in the metrology, initial conditions, or projectile mass properties. Lastly, the only practical method of solving these equations is by numerical methods and with the speed of computers today, the codes run very efficiently



**FIGURE 8.13**

A projectile that has failed to trail.

and quickly. This last statement makes it difficult to generate meaningful problems for the interested reader. We have endeavored to create useful exercises by stipulating a large number of conditions and requiring the reader to examine the accelerations of the projectile at a point in space.

### Problem 12

A British bomber is flying at a speed of 200 mph in still air. If the 0.303-in. machine guns are fired sideways, calculate the axial acceleration vector and the angular acceleration vector acting on the projectile through use of the 6-DOF equations if the projectile is

1. Fired to the right

$$\text{Answer: } \mathbf{a} = [-1376\mathbf{e}_1 + 5.89\mathbf{e}_2 - 1184\mathbf{e}_3] \left[ \frac{\text{ft}}{\text{s}^2} \right]$$

$$\frac{d\mathbf{h}}{dt} = [39.85\mathbf{e}_1 - 35,922\mathbf{e}_2 - 2.45\mathbf{e}_3] \left[ \frac{\text{rad}}{\text{s}^2} \right]$$

2. Fired to the left

$$\text{Answer: } \mathbf{a} = [-1376\mathbf{e}_1 - 70.29\mathbf{e}_2 + 1184\mathbf{e}_3] \left[ \frac{\text{ft}}{\text{s}^2} \right]$$

$$\frac{d\mathbf{h}}{dt} = [39.85\mathbf{e}_1 + 35,922\mathbf{e}_2 + 2.45\mathbf{e}_3] \left[ \frac{\text{rad}}{\text{s}^2} \right]$$

3. Discuss the effect of the angular momentum on the projectile nose (which way does it tip?)

Please ignore the Coriolis acceleration, assume there is no yaw at muzzle exit and assume a muzzle velocity of 2440 ft/s, the weapon has a right-hand twist.

Projectile information:

$C_{D_0} = 0.35$	$(C_{M_q} + C_{M_\alpha}) = -16.2$	$I_P = 0.00026 \text{ [lbm-in.}^2\text{]}$
$C_{D_\delta^2} = 3.46$	$(C_{N_q} + C_{N_\alpha}) = 0.003$	$I_T = 0.00258 \text{ [lbm-in.}^2\text{]}$
$C_{M_\alpha} = 2.36$	$C_{M_p\alpha} = 0.02$	$m = 0.025 \text{ [lbm]}$
$C_{L_\alpha} = 2.81$	$\rho = 0.060 \left[ \frac{\text{lbm}}{\text{ft}^3} \right]$	$p = 2033 \left[ \frac{\text{rev}}{\text{s}} \right]$
$C_{N_{p\alpha}} = -0.67$		

Please supply all answers in an inertial coordinate system labeled 1, 2, and 3 with 1 being along the aircraft flight path and 3 being off the right side of the plane. Treat all missing coefficients as equal to zero.

### Problem 13

One of the interesting aspects of the forces acting on a projectile occurs as the projectile leaves an aircraft sideways. This problem is encountered all the time in the AC-130 gunship. Let us examine a 105-mm HE projectile being fired into a city from both the top of a building and from the AC-130 in flight. The velocity of the projectile is 1510 ft/s. With the information provided

1. Calculate the total acceleration vector for both cases.
2. Comment on the differences.

Positional information:

33.5° north latitude

Azimuth of velocity vector: 80° True

Angle of velocity vector to horizontal : -10°

Wind is calm

$\alpha = +2^\circ$ (nose up),  $\beta = -1.5^\circ$ (nose to the left looking downrange)

The projectile nose is rotating to the right of the velocity vector at 0.5 rad/s

The aircraft is flying at 300 mph to the north

Projectile information:

$$\begin{array}{lll} C_{D_0} = 0.39 & (C_{M_q} + C_{M_\alpha}) = -6.5 & I_P = 0.547 \text{ [lbm - ft}^2\text{]} \\ C_{D_{\delta^2}} = 8.0 & (C_{N_q} + C_{N_\alpha}) = 0.005 & I_T = 5.377 \text{ [lbm - ft}^2\text{]} \\ C_{M_\alpha} = 3.80 & C_{M_{p\alpha}} = 0.05 & m = 32.1 \text{ [lbm]} \\ C_{L_\alpha} = 1.9 & \rho = 0.060 \left[ \frac{\text{lbm}}{\text{ft}^3} \right] & p = 220 \left[ \frac{\text{rev}}{\text{s}} \right] \\ C_{N_{p\alpha}} = -0.01 & & \end{array}$$

Please supply all answers in an inertial coordinate system labeled 1, 2, and 3 with 1 being due north and 3 being due east.

### Problem 14

The Paris gun was built by Germany in the First World War to shell Paris from 75 miles away. The weapon was a 210-mm diameter bore with the shells pre-engraved to compensate for wear of the tube. During firing of this weapon, all things such as wind effects, Coriolis, etc. had to be accounted for (they really could have used a good 6-DOF model and a computer). Write the acceleration vector for this projectile at an instant in its trajectory when the velocity (relative to the ground) is 2500 ft/s and the following conditions apply (please note that there is "no" rocket motor):

Positional information:

48.75° north latitude

Azimuth of velocity vector: 300° True

Angle of velocity vector to horizontal: +10°

Wind is blowing at 20 mph due south and horizontal

$\alpha = 1^\circ$ ,  $\beta = 1.5^\circ$

The projectile nose is rotating up at 2 rad/s

Projectile information:

$$\begin{array}{lll} C_D = 0.28 & (C_{M_q} + C_{M_\alpha}) = -16.5 & I_P = 19.13 \text{ [lbm - ft}^2\text{]} \\ C_{M_\alpha} = 3.50 & (C_{N_q} + C_{N_\alpha}) = 0.005 & I_T = 66.40 \text{ [lbm - ft}^2\text{]} \\ C_{L_\alpha} = 2.50 & C_{M_{p\alpha}} = 0.55 & m = 220 \text{ [lbm]} \\ C_{N_{p\alpha}} = -0.02 & \rho = 0.060 \left[ \frac{\text{lbm}}{\text{ft}^3} \right] & p = 150 \left[ \frac{\text{rev}}{\text{s}} \right] \end{array}$$

Note that the above numbers are guesses at the projectiles characteristics, they do not represent the real projectile's performance as no data is available from any source researched.

Please supply all answers in an inertial coordinate system labeled 1, 2, and 3 with 1 being due west and 3 being due north.

*Answer:*

$$\text{Linear acceleration vector is } \mathbf{a} = [-86\mathbf{e}_1 - 32\mathbf{e}_2 - 35\mathbf{e}_3] \left[ \frac{\text{ft}}{\text{s}^2} \right]$$

$$\text{Angular acceleration vector is } \frac{d\mathbf{h}}{dt} = [-17\mathbf{e}_1 - 35\mathbf{e}_2 + 43\mathbf{e}_3] \left[ \frac{\text{rad}}{\text{s}^2} \right]$$


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## 8.6 Modified Point Mass Trajectory

The 6-DOF model's equations, when fully developed, described an epicyclical motion with fast (nutational) and slow (precessional) modes that (hopefully) would damp out early on, allowing the projectile to assume a yaw of repose for the remainder of the flight. This yaw of repose, which remains nearly constant, we assume will account for most of the drag induced by the yawing of the projectile. If we can simplify the 6-DOF model, which is computationally expensive to run, by accounting only for the yaw of repose, we could get a model that will allow the projectile to drift the proper amount and still be quite accurate.

Once again, following McCoy [1], we will make the point mass assumption in the equations that follow. Recall that because of this assumption, the projectile is essentially represented as a cannon ball with all of its mass concentrated at one point. We shall then add some details, which will account for the yawing of the projectile, by assuming the projectile yaw is relatively constant or varies little with time compared to the steady state yaw angle. This assumption is usually valid except in high-angle fire situations.

We begin with the usual equations of motion and Newton's second law

$$\mathbf{F} = m\mathbf{a} \quad (8.312)$$

particularized as

$$m \frac{d\mathbf{V}}{dt} = \sum \mathbf{F} + m\mathbf{g} + m\boldsymbol{\Lambda} \quad (8.313)$$

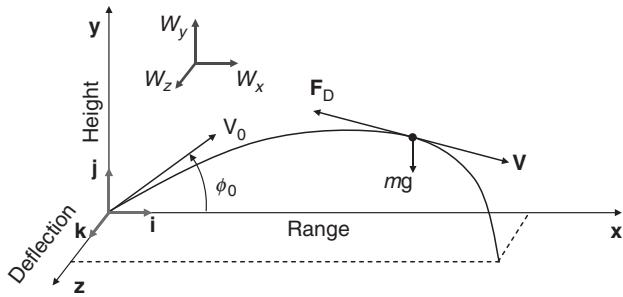
Here the variables are the same as we described in the 6-DOF section. In the above equations, we replace the velocity vector  $\mathbf{V}$  by the vector ( $\mathbf{V}-\mathbf{W}$ ) because drag measurements are made relative to the air stream not relative to the ground. We will also again replace the scalar velocity (the speed) with the difference between the projectile and wind velocities.

$$\mathbf{v} = \mathbf{V} - \mathbf{W} \rightarrow \tilde{V} = |\mathbf{V} - \mathbf{W}| \quad (8.314)$$

The diagram of the problem is shown in [Figure 8.14](#).

From our work on 6-DOF model recall [Equations 8.258](#) and [8.259](#) which we rewrite here neglecting the pitch damping and rocket forces.

$$\frac{d\mathbf{V}}{dt} = -\frac{\rho v S C_D}{2m} \mathbf{v} + \frac{\rho S C_{L_a}}{2m} [v^2 \mathbf{i} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{v}] - \frac{\rho S d C_{N_{pa}}}{2m} \left( \frac{I_T}{I_P} \right) (\mathbf{h} \cdot \mathbf{i}) (\mathbf{i} \times \mathbf{v}) + \mathbf{g} + \boldsymbol{\Lambda} \quad (8.315)$$



**FIGURE 8.14**  
Modified point mass trajectory.

$$\begin{aligned}\frac{d\mathbf{h}}{dt} = & \frac{\rho v S d^2 C_{lp}}{2I_p} (\mathbf{h} \cdot \mathbf{i}) \mathbf{i} + \frac{\rho v^2 S d \delta_F C_{ls}}{2I_T} \mathbf{i} + \frac{\rho v S d C_{M_\alpha}}{2I_T} (\mathbf{v} \times \mathbf{i}) \\ & + \frac{\rho S d^2 C_{M_{pa}}}{2I_p} (\mathbf{h} \cdot \mathbf{i}) [\mathbf{v} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{i}] + \frac{\rho v S d^2 (C_{M_q} + C_{M_{\dot{\alpha}}})}{2I_T} [\mathbf{h} - (\mathbf{h} \cdot \mathbf{i}) \mathbf{i}]\end{aligned}\quad (8.316)$$

Recall also from 6-DOF model (Equation 8.229) that introduces the polar and transverse moments of inertia

$$\frac{d\mathbf{h}}{dt} = \frac{I_p}{I_T} \dot{p} \mathbf{i} + \frac{I_p p}{I_T} \frac{d\mathbf{i}}{dt} + \left( \mathbf{i} \times \frac{d^2 \mathbf{i}}{dt^2} \right) \quad (8.317)$$

Equations 8.315 and 8.316 may be simplified by introducing the tilde ( $\sim$ ) coefficients.

$$\begin{array}{ll}\tilde{C}_D = \frac{\rho v S C_D}{2m} & \tilde{C}_{lp} = \frac{\rho v S d^2 C_{lp} p}{2I_T} \\ \tilde{C}_{L_\alpha} = \frac{\rho S C_{L_\alpha}}{2m} & \tilde{C}_{ls} = \frac{\rho v^2 S d \delta_F C_{ls}}{2I_T} \\ \tilde{C}_{N_{pa}} = \frac{\rho S d C_{N_{pa}} p}{2m} & \tilde{C}_{M_\alpha} = \frac{\rho v S d C_{M_\alpha}}{2I_T} \\ \tilde{C}_{N_q} = \frac{\rho v S d (C_{N_q} + C_{N_{\dot{\alpha}}})}{2m} & \tilde{C}_{M_{pa}} = \frac{\rho S d^2 C_{M_{pa}} p}{2I_T} \\ \tilde{C}_{M_q} = \frac{\rho v S d^2 (C_{M_q} + C_{M_{\dot{\alpha}}})}{2I_T}\end{array}$$

Using this notation, the modified equations are written as

$$\frac{d\mathbf{V}}{dt} = -\tilde{C}_D \mathbf{v} + \tilde{C}_{L_\alpha} [v^2 \mathbf{i} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{v}] - \tilde{C}_{N_{pa}} \left( \frac{I_T}{I_p} \right) (\mathbf{h} \cdot \mathbf{i}) (\mathbf{i} \times \mathbf{v}) + \mathbf{g} + \boldsymbol{\Lambda} \quad (8.318)$$

$$\frac{d\mathbf{h}}{dt} = (\tilde{C}_{lp} + \tilde{C}_{ls}) \mathbf{i} + \tilde{C}_{M_\alpha} (\mathbf{v} \times \mathbf{i}) + \tilde{C}_{M_{pa}} (\mathbf{h} \cdot \mathbf{i}) [\mathbf{v} - (\mathbf{v} \cdot \mathbf{i}) \mathbf{i}] + \tilde{C}_{M_q} [\mathbf{h} - (\mathbf{h} \cdot \mathbf{i}) \mathbf{i}] \quad (8.319)$$

or, alternatively

$$\frac{d\mathbf{V}}{dt} = -\tilde{C}_D \mathbf{v} + \tilde{C}_{L_\alpha} [\mathbf{v} \times (\mathbf{i} \times \mathbf{v})] - \tilde{C}_{N_{pa}} (\mathbf{v} \times \mathbf{i}) + \mathbf{g} + \boldsymbol{\Lambda} \quad (8.320)$$

$$\frac{d\mathbf{h}}{dt} = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta})\mathbf{i} + \tilde{C}_{M_\alpha}(\mathbf{v} \times \mathbf{i}) + \tilde{C}_{M_{pa}}(\mathbf{h} \cdot \mathbf{i})[\mathbf{v} - (\mathbf{v} \cdot \mathbf{i})\mathbf{i}] + \tilde{C}_{M_q}[\mathbf{h} - (\mathbf{h} \cdot \mathbf{i})\mathbf{i}] \quad (8.321)$$

We have shown earlier that since the unit vector,  $\mathbf{i}$ , is always perpendicular to its derivative  $d\mathbf{i}/dt$ , the dot product of  $\mathbf{i}$  and  $d\mathbf{i}/dt$  is identically zero. We shall combine Equations 8.317 and 8.321 to yield

$$\frac{I_p}{I_T}\dot{p}\mathbf{i} + \frac{I_p}{I_T}p\frac{d\mathbf{i}}{dt} + \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta})\mathbf{i} + \tilde{C}_{M_\alpha}(\mathbf{v} \times \mathbf{i}) + \tilde{C}_{M_{pa}}[\mathbf{i} \times (\mathbf{v} \times \mathbf{i})] + \tilde{C}_{M_q} \left[ \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right] \quad (8.322)$$

We will now take the dot product of  $\mathbf{i}$  with Equation 8.322 to yield

$$\frac{I_p}{I_T}\dot{p} = (\tilde{C}_{l_p} + \tilde{C}_{l_\delta}) \rightarrow \frac{dp}{dt} = \frac{I_T}{I_p}(\tilde{C}_{l_p} + \tilde{C}_{l_\delta}) \quad (8.323)$$

Here use has been made of the facts that a cross product results in a vector that is orthogonal to both of the original vectors and that the dot product of orthogonal vectors is identically zero. These relationships are written in mathematical terms here

$$\begin{aligned} \mathbf{i} \cdot \frac{d\mathbf{i}}{dt} &= \mathbf{i} \cdot (\boldsymbol{\omega} \times \mathbf{i}) = 0 & \mathbf{i} \cdot \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) &= 0 \\ \mathbf{i} \cdot [\mathbf{i} \times (\mathbf{v} \times \mathbf{i})] &= 0 & \mathbf{i} \cdot (\mathbf{i} \times \mathbf{v}) &= \mathbf{i} \cdot (\mathbf{v} \times \mathbf{i}) = 0 \end{aligned}$$

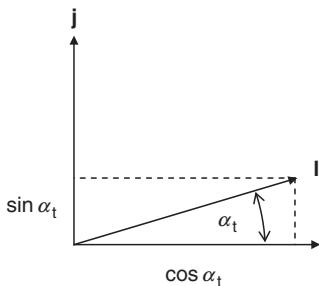
Equation 8.323 has an important consequence; for a rotationally symmetric projectile, the spin is decoupled from the yawing motion. Now, if we substitute Equation 8.323 into Equation 8.322, we get

$$\begin{aligned} (\tilde{C}_{l_p} + \tilde{C}_{l_\delta})\mathbf{i} + \frac{I_p}{I_T}p\frac{d\mathbf{i}}{dt} + \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) &= (\tilde{C}_{l_p} + \tilde{C}_{l_\delta})\mathbf{i} + \tilde{C}_{M_\alpha}(\mathbf{v} \times \mathbf{i}) + \tilde{C}_{M_{pa}}[\mathbf{i} \times (\mathbf{v} \times \mathbf{i})] \\ &\quad + \tilde{C}_{M_q} \left[ \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right] \end{aligned} \quad (8.324)$$

or

$$\frac{I_p}{I_T}p\frac{d\mathbf{i}}{dt} + \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) = \tilde{C}_{M_\alpha}(\mathbf{v} \times \mathbf{i}) + \tilde{C}_{M_{pa}}[\mathbf{i} \times (\mathbf{v} \times \mathbf{i})] + \tilde{C}_{M_q} \left[ \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right] \quad (8.325)$$

With Equations 8.320, 8.323, and 8.325, we have merely restated our 6-DOF model. Murphy [2] formulated the differential equation of motion as a second order equation in terms of complex variables and solved it. The particular solution was the (relatively) constant yaw of repose, and the complimentary solution was the transient epicyclic motion. In the modified point mass approach, we extract the particular solution and ignore the transient motion, instead concentrating on the yaw of repose, the drift, and the effect of the yaw drag. We assume that the epicyclic pitching and yawing motion are negligible everywhere along the trajectory, i.e., in many instances, reasonable, since it should damp early in the trajectory and thus contributes little to the drift. We proceed by defining another unit vector triad in the same sense as our  $\mathbf{i}$ - $\mathbf{j}$ - $\mathbf{k}$  triad. Instead of it being aligned



**FIGURE 8.15**  
Projectile axial unit vector,  $\mathbf{l}$ , illustrated.

with the geometric axis of the projectile, we align it with the velocity vector and utilize  $\mathbf{l-m-n}$  as the principal directions. We can then define  $\mathbf{l}$  (Figure 8.15) as

$$\mathbf{l} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad (8.326)$$

and formally define our vector yaw of repose as

$$\boldsymbol{\alpha}_R = \mathbf{l} \times (\mathbf{i} \times \mathbf{l}) \quad (8.327)$$

But we know that, where  $\alpha_t$  is the total angle of attack,

$$\mathbf{l} \times (\mathbf{i} \times \mathbf{l}) = (1)^2 \mathbf{i} - (\mathbf{l} \cdot \mathbf{i}) \mathbf{l} = \mathbf{i} - (1)(1) \cos \alpha_t \mathbf{l} \quad (8.328)$$

so that in terms of  $\alpha_t$

$$\boldsymbol{\alpha}_R = \mathbf{l} \times (\mathbf{i} \times \mathbf{l}) = \mathbf{i} - (\cos \alpha_t) \mathbf{l} \quad (8.329)$$

For simplicity [1,2], if we choose the plane that  $\mathbf{l}$  lies in the plane that  $\mathbf{j}$  lies in as well, we can write Equation 8.329 as

$$\boldsymbol{\alpha}_R = \mathbf{i} - \cos \alpha_t (\cos \alpha_t \mathbf{i} + \sin \alpha_t \mathbf{j}) = (1 - \cos^2 \alpha_t) \mathbf{i} + \sin \alpha_t \cos \alpha_t \mathbf{j}$$

then

$$\alpha_R = \sqrt{(1 - \cos^2 \alpha_t)^2 + \cos^2 \alpha_t \sin^2 \alpha_t} = \sqrt{\sin^4 \alpha_t + \cos^2 \alpha_t \sin^2 \alpha_t}$$

and

$$\alpha_R = \sin \alpha_t \sqrt{\sin^2 \alpha_t + \cos^2 \alpha_t} = \sin \alpha_t$$

We shall now differentiate Equation 8.329 with respect to time.

$$\frac{d\boldsymbol{\alpha}_R}{dt} = \frac{d\mathbf{i}}{dt} - (\cos \alpha_t) \frac{d\mathbf{l}}{dt} + \sin \alpha_t \mathbf{l} \quad (8.330)$$

We have made the assumption early in this analysis that the yaw of repose is relatively constant, thus  $\frac{d\boldsymbol{\alpha}_R}{dt} \approx 0$ . We also note that for a small yaw angle  $\sin \alpha_R \approx 0 \ll \cos \alpha_R$ . If we incorporate these approximations into Equation 8.330, we get

$$\frac{d\mathbf{i}}{dt} = (\cos \alpha_t) \frac{d\mathbf{l}}{dt} \quad (8.331)$$

Taking the time derivative of [Equation 8.331](#) yields

$$\frac{d^2\mathbf{i}}{dt^2} = (\cos \alpha_t) \frac{d^2\mathbf{l}}{dt^2} - (\sin \alpha_t) \frac{d\mathbf{l}}{dt} \quad (8.332)$$

But the small angle approximation still applies so that

$$\frac{d^2\mathbf{i}}{dt^2} = (\cos \alpha_t) \frac{d^2\mathbf{l}}{dt^2} \quad (8.333)$$

We can also solve [Equation 8.329](#) for  $\mathbf{i}$  to get

$$\mathbf{i} = \boldsymbol{\alpha}_R + (\cos \alpha_t) \mathbf{l} \quad (8.334)$$

But, since  $\mathbf{v}$  and  $\mathbf{l}$  are parallel and cross products of parallel vectors are zero, we can write

$$\mathbf{i} \times \mathbf{v} = [\boldsymbol{\alpha}_R + (\cos \alpha_t) \mathbf{l}] \times \mathbf{v} = \boldsymbol{\alpha}_R \times \mathbf{v} + (\cos \alpha_t) \mathbf{l} \times \mathbf{v} = \boldsymbol{\alpha}_R \times \mathbf{v} \quad (8.335)$$

and

$$\mathbf{v} \times (\mathbf{i} \times \mathbf{v}) = \mathbf{v} \times (\boldsymbol{\alpha}_R \times \mathbf{v}) = v^2 \boldsymbol{\alpha}_R - (\mathbf{v} \cdot \boldsymbol{\alpha}_R) \mathbf{v} = v^2 \boldsymbol{\alpha}_R \quad (8.336)$$

Also in a similar fashion operating on Equation 8.335, we can write

$$\mathbf{v} \times \mathbf{i} = \mathbf{v} \times [\boldsymbol{\alpha}_R + (\cos \alpha_t) \mathbf{l}] = \mathbf{v} \times \boldsymbol{\alpha}_R + \mathbf{v} \times (\cos \alpha_t) \mathbf{l} = \mathbf{v} \times \boldsymbol{\alpha}_R \quad (8.337)$$

We can also show that

$$\mathbf{i} \times (\mathbf{v} \times \mathbf{i}) = v \cos \alpha_t \boldsymbol{\alpha}_R + v(\sin^2 \alpha_t) \mathbf{l} \quad (8.338)$$

We now have relations in Equations 8.331, 8.333, and 8.334 for  $\mathbf{i}$ ,  $d\mathbf{i}/dt$ , and  $d^2\mathbf{i}/dt^2$ , respectively and can substitute them into [Equations 8.320](#) and [8.325](#) to eliminate  $\mathbf{i}$ . We shall start with Equation 8.320, and also noting that

$$\mathbf{v} \times \boldsymbol{\alpha}_R = (v\mathbf{l}) \times \boldsymbol{\alpha}_R = v(\mathbf{l} \times \boldsymbol{\alpha}_R) \quad (8.339)$$

we get

$$\frac{d\mathbf{V}}{dt} = -\tilde{C}_D \mathbf{v} + \tilde{C}_{L_\alpha} v^2 \boldsymbol{\alpha}_R - \tilde{C}_{N_{pa}} v(\mathbf{l} \times \boldsymbol{\alpha}_R) + \mathbf{g} + \boldsymbol{\Lambda} \quad (8.340)$$

It is also worth noting that

$$\frac{d\mathbf{V}}{dt} = \frac{d}{dt} V\mathbf{l} = \frac{dV}{dt} \mathbf{l} + V \frac{d\mathbf{l}}{dt} = \dot{V}\mathbf{l} + V \frac{d\mathbf{l}}{dt} \quad (8.341)$$

We will now attack each term of Equation 8.325, but first we define

$$\gamma = (\mathbf{l} \cdot \mathbf{i}) = \cos \alpha_t \quad (8.342)$$

Then this and the succeeding relations follow as

$$\frac{I_P}{I_T} p \frac{d\mathbf{i}}{dt} \approx \frac{I_P}{I_T} p \cos \alpha_t \frac{d\mathbf{l}}{dt} = \gamma \frac{I_P}{I_T} p \frac{d\mathbf{l}}{dt} \quad (8.343)$$

$$\left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) = \gamma \left( \boldsymbol{\alpha}_R \times \frac{d^2\mathbf{l}}{dt^2} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d^2\mathbf{l}}{dt^2} \right) \quad (8.344)$$

$$\left[ \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right] = \gamma \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \quad (8.345)$$

Combining the terms of [Equation 8.325](#), we get

$$\begin{aligned} \gamma \frac{I_P}{I_T} p \frac{d\mathbf{l}}{dt} + \gamma \left( \boldsymbol{\alpha}_R \times \frac{d^2\mathbf{l}}{dt^2} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d^2\mathbf{l}}{dt^2} \right) &= \tilde{C}_{M_\alpha} v (\mathbf{l} \times \boldsymbol{\alpha}_R) + \tilde{C}_{M_{pa}} [v \gamma \boldsymbol{\alpha}_R + v \sin^2 \alpha_t \mathbf{l}] \\ &\quad + \tilde{C}_{M_q} \left[ \gamma \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \right] \end{aligned} \quad (8.346)$$

At this point we continue with our simplifying assumptions and neglect the Coriolis term in comparison with the gravitational acceleration term and also neglect the  $\sin^2 \alpha_t$  in comparison to  $\gamma$ . Thus, we can rewrite [Equations 8.340](#) (including the relation of [Equation 8.341](#)) and 8.346 as

$$\dot{V}\mathbf{l} + V \frac{d\mathbf{l}}{dt} = -\tilde{C}_D \mathbf{v} + \tilde{C}_{L_\alpha} v^2 \boldsymbol{\alpha}_R - \tilde{C}_{N_p \alpha} v (\mathbf{l} \times \boldsymbol{\alpha}_R) + \mathbf{g} \quad (8.347)$$

and

$$\begin{aligned} \gamma \frac{I_P}{I_T} p \frac{d\mathbf{l}}{dt} + \gamma \left( \boldsymbol{\alpha}_R \times \frac{d^2\mathbf{l}}{dt^2} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d^2\mathbf{l}}{dt^2} \right) &= \tilde{C}_{M_\alpha} v (\mathbf{l} \times \boldsymbol{\alpha}_R) + \tilde{C}_{M_{F\alpha}} v \gamma \boldsymbol{\alpha}_R \\ &\quad + \tilde{C}_{M_q} \left[ \gamma \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) + \gamma^2 \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \right] \end{aligned} \quad (8.348)$$

We shall now take the vector cross product of  $\mathbf{l}$  with Equations 8.347 and 8.348 and observe at how each term behaves. First the LHS

$$1 \times \left( \dot{V}\mathbf{l} + V \frac{d\mathbf{l}}{dt} \right) = 1 \times \dot{V}\mathbf{l} + \mathbf{l} \times V \frac{d\mathbf{l}}{dt} = 0 + V \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \quad (8.349)$$

then each term on the RHS of Equation 8.347

$$\mathbf{l} \times (-\tilde{C}_D \mathbf{v}) = \mathbf{l} \times (-\tilde{C}_D v \mathbf{l}) = -\tilde{C}_D v (\mathbf{l} \times \mathbf{l}) = 0 \quad (8.350)$$

$$\mathbf{l} \times \tilde{C}_{L_\alpha} v^2 \boldsymbol{\alpha}_R = \tilde{C}_{L_\alpha} v^2 (\mathbf{l} \times \boldsymbol{\alpha}_R) \quad (8.351)$$

$$\begin{aligned} \mathbf{l} \times (-\tilde{C}_{N_p \alpha} v) (\mathbf{l} \times \boldsymbol{\alpha}_R) &= -\tilde{C}_{N_p \alpha} v [\mathbf{l} \times (\mathbf{l} \times \boldsymbol{\alpha}_R)] = \tilde{C}_{N_p \alpha} v [\mathbf{l} \times (\boldsymbol{\alpha}_R \times \mathbf{l})] \\ &= \tilde{C}_{N_p \alpha} v [\boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \mathbf{l}] \end{aligned} \quad (8.352)$$

and

$$\mathbf{l} \times \mathbf{g} = \mathbf{l} \times \mathbf{g} \quad (8.353)$$

Continuing on [Equation 8.348](#), first the LHS

$$\mathbf{l} \times \gamma \frac{I_p}{I_t} p \frac{d\mathbf{l}}{dt} = \gamma \frac{I_p}{I_t} p \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \quad (8.354)$$

$$\mathbf{l} \times \gamma \left( \boldsymbol{\alpha}_R \times \frac{d^2\mathbf{l}}{dt^2} \right) = \gamma \mathbf{l} \times \left( \boldsymbol{\alpha}_R \times \frac{d^2\mathbf{l}}{dt^2} \right) = \gamma \left[ \left( \mathbf{l} \cdot \frac{d^2\mathbf{l}}{dt^2} \right) \boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \frac{d^2\mathbf{l}}{dt^2} \right] \quad (8.355)$$

$$\mathbf{l} \times \gamma^2 \left( \mathbf{l} \times \frac{d^2\mathbf{l}}{dt^2} \right) = \gamma^2 \left[ \mathbf{l} \times \left( \mathbf{l} \times \frac{d^2\mathbf{l}}{dt^2} \right) \right] = \gamma^2 \left[ \left( \mathbf{l} \cdot \frac{d^2\mathbf{l}}{dt^2} \right) \mathbf{l} - \frac{d^2\mathbf{l}}{dt^2} \right] \quad (8.356)$$

Then each term on the RHS of Equation 8.348

$$\mathbf{l} \times \tilde{C}_{M_\alpha} v (\mathbf{l} \times \boldsymbol{\alpha}_R) = \tilde{C}_{M_\alpha} v [\mathbf{l} \times (\mathbf{l} \times \boldsymbol{\alpha}_R)] = -\tilde{C}_{M_\alpha} v [\boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \mathbf{l}] \quad (8.357)$$

$$\mathbf{l} \times \tilde{C}_{M_{p\alpha}} v \gamma \boldsymbol{\alpha}_R = \tilde{C}_{M_{p\alpha}} v \gamma (\mathbf{l} \times \boldsymbol{\alpha}_R) \quad (8.358)$$

and continuing

$$\mathbf{l} \times \tilde{C}_{M_q} \gamma \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) = \tilde{C}_{M_q} \gamma \left[ \mathbf{l} \times \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) \right] \quad (8.359)$$

$$\mathbf{l} \times \tilde{C}_{M_q} \gamma \left( \boldsymbol{\alpha}_R \times \frac{d\mathbf{l}}{dt} \right) = \tilde{C}_{M_q} \gamma \left[ \left( \mathbf{l} \cdot \frac{d\mathbf{l}}{dt} \right) \boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \frac{d\mathbf{l}}{dt} \right] = -\tilde{C}_{M_q} \gamma (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \frac{d\mathbf{l}}{dt} \quad (8.360)$$

and finally

$$\mathbf{l} \times \tilde{C}_{M_q} \gamma^2 \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) = \tilde{C}_{M_q} \gamma^2 \left[ \mathbf{l} \times \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) \right] = \tilde{C}_{M_q} \gamma^2 \left[ 0 - (\mathbf{l} \cdot \mathbf{l}) \frac{d\mathbf{l}}{dt} \right] = -\tilde{C}_{M_q} \gamma^2 \frac{d\mathbf{l}}{dt} \quad (8.361)$$

We will now insert [Equations 8.349](#) through 8.353 into [Equation 8.347](#) to get

$$V \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) = \tilde{C}_{L_\alpha} v^2 (\mathbf{l} \times \boldsymbol{\alpha}_R) + \tilde{C}_{N_{p\alpha}} v [\boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \mathbf{l}] + (\mathbf{l} \times \mathbf{g}) \quad (8.362)$$

We then do the same with Equations 8.354 through 8.361, inserting them into Equation 8.348 yielding

$$\begin{aligned} & \gamma \frac{I_p}{I_t} p \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) + \gamma \left[ \left( \mathbf{l} \cdot \frac{d^2\mathbf{l}}{dt^2} \right) \boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \frac{d^2\mathbf{l}}{dt^2} \right] + \gamma^2 \left[ \left( \mathbf{l} \cdot \frac{d^2\mathbf{l}}{dt^2} \right) \mathbf{l} - \frac{d^2\mathbf{l}}{dt^2} \right] \\ &= -\tilde{C}_{M_\alpha} v [\boldsymbol{\alpha}_R - (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \mathbf{l}] + \tilde{C}_{M_{p\alpha}} v \gamma (\mathbf{l} \times \boldsymbol{\alpha}_R) - \tilde{C}_{M_q} \left[ \gamma (\mathbf{l} \cdot \boldsymbol{\alpha}_R) \frac{d\mathbf{l}}{dt} + \gamma^2 \frac{d\mathbf{l}}{dt} \right] \end{aligned} \quad (8.363)$$

We now have a pair of equations that essentially are comprised of two vector variables: the yaw of repose,  $\boldsymbol{\alpha}_R$  and the vector,  $\mathbf{l} \times \boldsymbol{\alpha}_R$ . Cumbersome as it may seem, this is a linear

system that can be solved readily through the use of matrices and their determinants to yield  $\alpha_R$  in terms of  $\mathbf{l}$  and the other scalars and coefficients. The solution for  $\alpha_R$  after much manipulation is

$$\alpha_R = \frac{-\tilde{C}_{M_{pa}} v \gamma V \left[ \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) - (\mathbf{l} \times \mathbf{g}) \right] + \tilde{C}_{L_\alpha} v^2 \gamma^2 \left[ \left( \mathbf{l} \cdot \frac{d^2 \mathbf{l}}{dt^2} \right) \mathbf{l} - \frac{d^2 \mathbf{l}}{dt^2} \right] + \tilde{C}_{L_\alpha} v^2 \gamma \frac{I_p}{I_T} p \left( \mathbf{l} \times \frac{d\mathbf{l}}{dt} \right) + \tilde{C}_{L_\alpha} \tilde{C}_{M_q} v^2 \gamma^2 \frac{d\mathbf{l}}{dt}}{\tilde{C}_{N_{pa}} \tilde{C}_{M_{pa}} v^2 \gamma - \tilde{C}_{L_\alpha} v^2 \gamma \left( \mathbf{l} \cdot \frac{d^2 \mathbf{l}}{dt^2} \right) + \tilde{C}_{L_\alpha} \tilde{C}_{M_\alpha} v^3} \quad (8.364)$$

We can further simplify this expression by returning to [Equation 8.347](#) and taking the dot product of it with  $\mathbf{l}$ . Through vector algebra and the use of the fact that  $(\mathbf{l} \cdot \alpha_R) = 0$ , we see that

$$\dot{V} = -\tilde{C}_D v + (\mathbf{l} \cdot \mathbf{g}) = -\frac{\rho S C_D v^2}{2m} + (\mathbf{l} \cdot \mathbf{g}) \quad (8.365)$$

When this is substituted back into [Equation 8.347](#), we get

$$-\tilde{C}_D \mathbf{v} + (\mathbf{l} \cdot \mathbf{g}) \mathbf{l} + V \frac{d\mathbf{l}}{dt} = -\tilde{C}_D \mathbf{v} + \tilde{C}_{L_\alpha} v^2 \alpha_R - \tilde{C}_{N_{pa}} v (\mathbf{l} \times \alpha_R) + \mathbf{g} \quad (8.366)$$

which can be rewritten as

$$V \frac{d\mathbf{l}}{dt} = \tilde{C}_{L_\alpha} v^2 \alpha_R - \tilde{C}_{N_{pa}} v (\mathbf{l} \times \alpha_R) + \mathbf{g} - (\mathbf{l} \cdot \mathbf{g}) \mathbf{l} \quad (8.367)$$

and by neglecting the Magnus force term as it is small and noting that  $\mathbf{l} \times (\mathbf{g} \times \mathbf{l}) = \mathbf{g} - (\mathbf{l} \cdot \mathbf{g}) \mathbf{l}$ , we get

$$V \frac{d\mathbf{l}}{dt} = \tilde{C}_{L_\alpha} v^2 \alpha_R + [\mathbf{l} \times (\mathbf{g} \times \mathbf{l})] \quad (8.368)$$

Remember that we wish to find a useful form with which we can calculate the quasi-steady state yaw of repose as shown in [Equation 8.364](#). That equation encompasses different vector functions and time derivatives of  $\mathbf{l}$ , but our ultimate goal is to find expressions that only involve the measurable quantities of the aeroballistic coefficients, spin, gravity, and velocity and not the unit vector,  $\mathbf{l}$ . To do this, though it may seem a devious process, we begin by taking the time derivative of [Equation 8.368](#), getting

$$\dot{V} \frac{d\mathbf{l}}{dt} + V \frac{d^2 \mathbf{l}}{dt^2} = 0 + \frac{d}{dt} [\mathbf{l} \times (\mathbf{g} \times \mathbf{l})] \quad (8.369)$$

We use [Equation 8.365](#) and substitute it in [Equation 8.369](#), arriving at

$$-\tilde{C}_D v \frac{d\mathbf{l}}{dt} + (\mathbf{l} \cdot \mathbf{g}) \frac{d\mathbf{l}}{dt} + V \frac{d^2 \mathbf{l}}{dt^2} = \frac{d\mathbf{g}}{dt} - (\mathbf{l} \cdot \mathbf{g}) \frac{d\mathbf{l}}{dt} - \left( \frac{d\mathbf{l}}{dt} \cdot \mathbf{g} \right) \mathbf{l} - \left( \mathbf{l} \cdot \frac{d\mathbf{g}}{dt} \right) \mathbf{l} \quad (8.370)$$

Noticing that  $\frac{d\mathbf{g}}{dt} = 0$  (if not, we really will have problems), we can rewrite this as

$$V \frac{d^2 \mathbf{l}}{dt^2} = -2(\mathbf{l} \cdot \mathbf{g}) \frac{d\mathbf{l}}{dt} - \left( \frac{d\mathbf{l}}{dt} \cdot \mathbf{g} \right) \mathbf{l} + \tilde{C}_D v \frac{d\mathbf{l}}{dt} \quad (8.371)$$

If we examine the RHS of Equation 8.371 term by term and realize from [Equation 8.368](#) that  $d\mathbf{l}/dt$  can be solved for, then

$$-2(\mathbf{l} \cdot \mathbf{g}) \frac{d\mathbf{l}}{dt} = -\frac{2}{V} [(\mathbf{l} \cdot \mathbf{g}) \tilde{C}_{L_a} v^2 \boldsymbol{\alpha}_R + (\mathbf{l} \cdot \mathbf{g}) \mathbf{g} - (\mathbf{l} \cdot \mathbf{g})^2 \mathbf{l}] \quad (8.372)$$

also

$$\left( \frac{d\mathbf{l}}{dt} \cdot \mathbf{g} \right) \mathbf{l} = \frac{1}{V} [\tilde{C}_{L_a} v^2 (\boldsymbol{\alpha}_R \cdot \mathbf{g}) + g^2 - (\mathbf{l} \cdot \mathbf{g})^2] \mathbf{l} \quad (8.373)$$

and

$$\tilde{C}_D v \frac{d\mathbf{l}}{dt} = \frac{\tilde{C}_D v}{V} [\tilde{C}_{L_a} v^2 \boldsymbol{\alpha}_R + \mathbf{g} - (\mathbf{l} \cdot \mathbf{g}) \mathbf{l}] \quad (8.374)$$

Putting these expressions back into Equation 8.371 complicates things considerably, viz.

$$\begin{aligned} V \frac{d^2 \mathbf{l}}{dt^2} = & -\frac{2}{V} [\tilde{C}_{L_a} v^2 (\mathbf{l} \cdot \mathbf{g}) \boldsymbol{\alpha}_R + (\mathbf{l} \cdot \mathbf{g}) \mathbf{g} - (\mathbf{l} \cdot \mathbf{g})^2 \mathbf{l}] - \frac{1}{V} [\tilde{C}_{L_a} v^2 (\boldsymbol{\alpha}_R \cdot \mathbf{g}) \mathbf{l} \\ & + g^2 \mathbf{l} - (\mathbf{l} \cdot \mathbf{g})^2 \mathbf{l}] + \tilde{C}_D v \frac{d\mathbf{l}}{dt} \end{aligned} \quad (8.375)$$

If we assume that the terms containing our modified lift,  $\tilde{C}_{L_a}$ , and drag,  $\tilde{C}_D$ , coefficients are either small with respect to the other terms or cancel one another, the third term in Equation 8.375 disappears and the others simplify to give eventually

$$V^2 \frac{d^2 \mathbf{l}}{dt^2} = [3(\mathbf{l} \cdot \mathbf{g})^2 - g^2] \mathbf{l} - 2(\mathbf{l} \cdot \mathbf{g}) \mathbf{g} \quad (8.376)$$

Recalling that the triad  $(\mathbf{l}, \mathbf{m}, \mathbf{n})$  are unit vectors, with  $\mathbf{l}$  in the direction of the velocity vector,  $\mathbf{v}$ , we can write  $\mathbf{l} = \frac{\mathbf{v}}{v}$ , and can transform Equation 8.368 into

$$\frac{d\mathbf{l}}{dt} = \frac{1}{V} \left\{ \tilde{C}_{L_a} v^2 \boldsymbol{\alpha}_R + \frac{1}{v^2} [\mathbf{v} \times (\mathbf{g} \times \mathbf{v})] \right\} \quad (8.377)$$

Likewise our Equation 8.375, by dividing both sides by  $V$ , can be transformed to

$$\begin{aligned} \frac{d^2 \mathbf{l}}{dt^2} = & -\frac{2}{V^2} \left[ \tilde{C}_{L_a} v (\mathbf{v} \cdot \mathbf{g}) \boldsymbol{\alpha}_R + \frac{1}{v} (\mathbf{v} \cdot \mathbf{g}) \mathbf{g} - \frac{1}{v^3} (\mathbf{v} \cdot \mathbf{g})^2 \mathbf{v} \right] \\ & - \frac{1}{V^2} \left[ \tilde{C}_{L_a} v (\boldsymbol{\alpha}_R \cdot \mathbf{g}) \mathbf{v} + \frac{g^2}{v} \mathbf{v} - \frac{1}{v^3} (\mathbf{v} \cdot \mathbf{g})^2 \mathbf{v} \right] + \frac{\tilde{C}_D v}{V} \frac{d\mathbf{l}}{dt} \end{aligned} \quad (8.378)$$

Now, if we examine each term of [Equation 8.364](#), we will have to deal with such terms as  $\mathbf{l} \times \frac{d\mathbf{l}}{dt}$ ,  $\mathbf{l} \cdot \frac{d^2 \mathbf{l}}{dt^2}$ , and  $\left( \mathbf{l} \cdot \frac{d^2 \mathbf{l}}{dt^2} \right) \mathbf{l} - \frac{d^2 \mathbf{l}}{dt^2}$ . We can perform all the substitutions of these terms with

what we have shown in Equations 8.377 and 8.378, and then examine the resulting Equation 8.364 for terms that can be neglected for comparative magnitudes. When all the algebra is completed, the resulting equation, applicable to spinning projectiles [1], is

$$\alpha_R = \frac{-2I_{pp} \left( \mathbf{v} \times \frac{d\mathbf{V}}{dt} \right)}{\rho S d v^4 C_{M_\alpha}} \quad (8.379)$$

For non-spinning projectiles, we can simplify Equation 8.364 by removing the spin terms and neglecting terms of small magnitude. The resulting expression is

$$\alpha_R = \frac{\tilde{C}_{M_q} \gamma^2 [\mathbf{v} \times (\mathbf{v} \times \mathbf{g})]}{\tilde{C}_{M_\alpha} V v^3} \quad (8.380)$$

The vector mechanics work out so that when there is a positive overturning moment (statically unstable projectile), the yaw of repose vector points to the right for a right-hand spin. The yaw of repose for a statically stable non-spinning projectile is such that the nose points slightly above the trajectory. Either Equation 8.379 or 8.380 can be inserted into Equation 8.362 and numerically integrated simultaneously with Equation 8.363 to yield the velocity and position at any time. This forms the basis of the modified point mass method.

### Problem 15

The Paris gun was built by Germany in the First World War to shell Paris from 75 miles away. The weapon was a 210-mm diameter bore with the shells pre-engraved to account for wear of the tube. During firing of this weapon, all things such as wind effects, Coriolis, etc. had to be accounted for. When the United States entered the war, the doughboys (the nickname for American troops) were to take the St. Mihiel salient where the gun was located. We shall assume that the Germans have turned the gun to fire on the Americans. The projectile is at some point in space defined below. To demonstrate your knowledge of the modified point mass equations

1. Draw the situation.
2. Calculate the vector yaw of repose for this projectile using Equation 8.327.  
*Answer:*  $\alpha_R = [0.008\mathbf{e}_2 + 0.002\mathbf{e}_3][\text{rad}]$
3. Write the acceleration vector for this projectile using Equation 8.341 at the instant in its trajectory when the velocity (relative to the ground) is 2100 ft/s and the conditions below apply.

*Note:* You do not need all of the information below. It is provided to you so you can compare the differences in formulations with the 6-DOF model.

$$\text{Answer: } \frac{d\mathbf{V}}{dt} = \left[ -61\mathbf{e}_1 - 29\mathbf{e}_2 - 10\mathbf{e}_3 \right] \left[ \frac{\text{ft}}{\text{s}^2} \right]$$

4. Why we do not need to obtain the angular acceleration vector  $d\mathbf{h}/dt$ ?

Positional information:

48° north latitude

Azimuth of velocity vector: 190° True

Angle of velocity vector to horizontal : +1°

Wind is blowing at 15 mph due south and horizontal  
 $\alpha = 0.5^\circ$ ,  $\beta = 0.25^\circ$

The projectile nose is rotating down at 1 rad/s

Projectile information:

$$\begin{array}{lll} C_D = 0.28 & (C_{M_q} + C_{M_\alpha}) = -16.5 & I_P = 19.13 [\text{lbfm} \cdot \text{ft}^2] \\ C_{M_\alpha} = 3.50 & (C_{N_q} + C_{N_\alpha}) = 0.005 & I_T = 66.40 [\text{lbfm} \cdot \text{ft}^2] \\ C_{L_\alpha} = 2.50 & C_{M_{p\alpha}} = 0.55 & m = 220 [\text{lbfm}] \\ C_{N_{p\alpha}} = -0.02 & \rho = 0.060 \left[ \frac{\text{lbfm}}{\text{ft}^3} \right] & p = 130 \left[ \frac{\text{rev}}{\text{s}} \right] \\ C_{I_p} = -0.01 & & \end{array}$$

Supply all answers in an inertial coordinate system labeled 1, 2, and 3 with 1 being due south and 3 being due west.

### Problem 16

If we were to use a modified point mass assumption for both of the cases sited in [Problem 13](#)

1. Calculate the vector yaw of repose for both cases.

*Answer:*  $\alpha_R = [0.0003\mathbf{e}_1 + 0.0060\mathbf{e}_2 - 0.0009\mathbf{e}_3]$  for the building  
 $\alpha_R = [-0.0009\mathbf{e}_1 - 0.0007\mathbf{e}_2 - 0.0000\mathbf{e}_3]$  for the gunship

2. Draw and explain what this vector represents.
3. Comment on whether this model is applicable for each case and why.

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## References

1. McCoy, R.L., *Modern Exterior Ballistics*, Schiffer Military History, Atglen, PA, 1999.
2. Murphy, C.H., *Free Flight Motion of Symmetric Missiles*, Ballistics Research Laboratory Report No. 1216, Aberdeen Proving Ground, MD, 1963.

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## Further Reading

- Capecelatro, A. and Salzarulo, L., *Quantitative Physics for Scientists and Engineers—Mechanics*, Auric Associates, Newark, NJ, 1980.
- Sears, F.W., Zemanski, M.W., and Young, H.D., *University Physics*, 6th ed., Addison-Wesley, Reading, MA, 1982.

# 9

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## Linearized Aeroballistics

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The aeroballistics topics discussed so far have built up to where the reader has an appreciation for the techniques required to analyze projectile motion to a great degree of accuracy. The culmination of this study was the development of the equations for a six degree-of-freedom (6-DOF) model which accurately describes the motion of a rigid body through air. With a 6-DOF model in hand, the aeroballistician can examine the effects of a *given* configuration. The word *given* was italicized for emphasis because the aeroballistician must know the configuration properties before he or she analyzes the projectile. The implications of this are that without other tools to determine what needs to be changed in a design to alter the projectile behavior, one must simply guess at a new configuration, determine the aerodynamic coefficients, and reanalyze. This process can be very inefficient. The solution to this problem is to develop a theory that can be used to quickly determine what must be changed in a projectile to alter its flight behavior, make the changes, and reassess. This will be the topic for the remainder of this section.

Linearized theory was (at least in the opinion of the authors) refined to an exceptional degree by Murphy [1] in 1963. Other authors before and since [2–4] have developed similar theories and a good description of these can be found in McCoy [5]. The reason the theory is called linearized is the fact that the aerodynamic coefficients are assumed to be functions of the angle of attack in a linear sense. In other words,

$$F_j \propto C_{j0} (\sin \alpha_t) \text{ or } M_j \propto C_{j0} d (\sin \alpha_t) \quad (9.1)$$

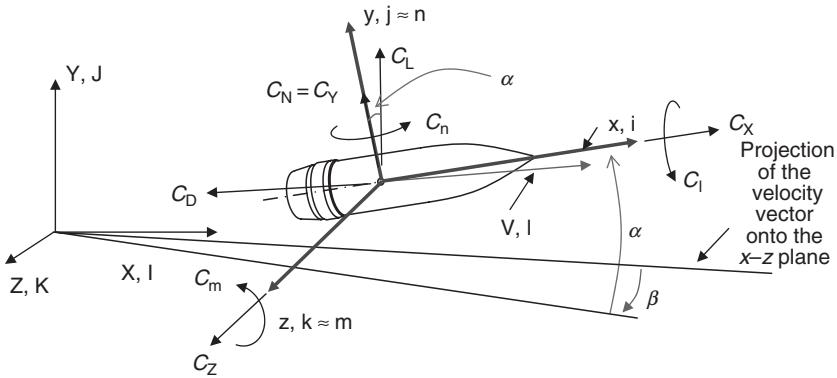
Here the subscript  $j$  indicates any parameter of interest as introduced earlier. There are good points and bad points (as always) with this technique. The good news is that the mathematics become simple enough to determine quantities of interest extremely quickly and find means of changing a projectile's flight characteristics quickly. The bad news is that the use of linear coefficients prevents us from duplicating some motions that occur frequently enough in projectile flight to warrant the inclusion of their nonlinear brethren—and the math becomes complicated to boot.

We will continue the practice of using the definitions of the appropriate vectors and scalars based on Ref. [5]. The choice is somewhat arbitrary, but for several years now the authors have used this lucid work as a supplementary textbook and it is a matter of convenience. Our coordinate system is defined as in [Figure 9.1](#).

The aerodynamic coefficients introduced in the beginning of this chapter were written for both forces and moments as

$$F_j = \frac{1}{2} \rho V^2 S C_j \quad (9.2)$$

$$M_j = \frac{1}{2} \rho V^2 S d C_j \quad (9.3)$$



**FIGURE 9.1**  
Coordinate system for projectile aerodynamic coefficients.

We have also defined the angular rates of the projectile as

$$p = \text{Roll (spin) rate} \quad (9.4)$$

$$q = \text{Pitch rate} \quad (9.5)$$

$$r = \text{Yaw rate} \quad (9.6)$$

The projectile angular position with respect to the velocity vector was given by

$$\alpha = \text{Angle of attack} \quad (9.7)$$

$$\beta = \text{Angle of sideslip} \quad (9.8)$$

The aerodynamic coefficients are functions of the rates expressed in Equations 9.4 through 9.6 as well as angular positions expressed in Equations 9.7 and 9.8. Additionally, these coefficients are also functions of the time rate of change of  $\alpha$  and  $\beta$  which do not normally coincide with  $q$  and  $r$ . Thus, we can write

$$C_j = C_j(\alpha, \beta, \dot{\alpha}, \dot{\beta}, p, q, r) \quad (9.9)$$

With this nomenclature, any coefficient can normally be expressed as a series expansion in the seven variables

$$C_j = C_{j0} + C_{j_\alpha} \alpha + C_{j_\beta} \beta + C_{j_{\dot{\alpha}}} \dot{\alpha} \left( \frac{\dot{\alpha} d}{V} \right) + C_{j_{\dot{\beta}}} \dot{\beta} \left( \frac{\dot{\beta} d}{V} \right) + C_{j_p} \left( \frac{pd}{V} \right) + C_{j_q} \left( \frac{qd}{V} \right) + C_{j_r} \left( \frac{rd}{V} \right) + \dots \quad (9.10)$$

In Equation 9.10, we have included the terms in parentheses to maintain the nondimensional characteristics of the coefficient. We can see that this expansion results in a large number of terms that must be carried. Seldom in aeroballistics do we require terms in this expression beyond second order, but they can be included if data is available. When we discuss linear aeroballistics, we are limiting ourselves to the eight terms displayed in Equation 9.10.

The linearization implies that

$$C_{jk} = \frac{\partial C_j}{\partial k} \Big|_{k=0} \quad (9.11)$$

Further simplifications will be made as we progress which will assist us in tackling the mathematics. We shall make use in this section of starred coefficients. These coefficients are defined in terms of their un-starred counterparts as

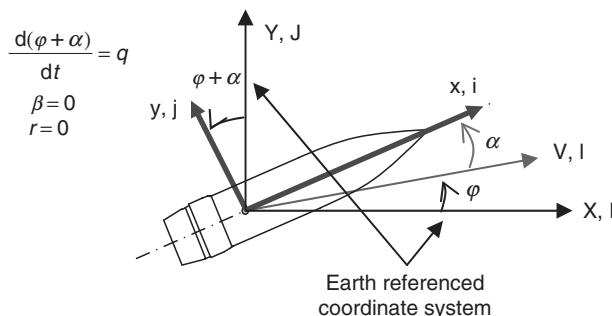
$$C_{jk}^* = \frac{\rho S d}{2m} C_{jk} \quad (9.12)$$

## 9.1 Linearized Pitching and Yawing Motions

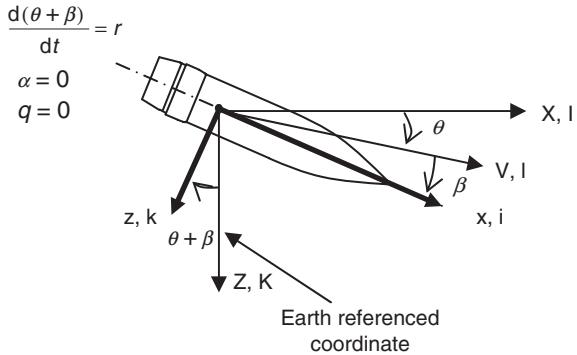
In the beginning of this chapter, we discussed terminology that allowed us to describe the pitching and yawing motion of a projectile. Because of the symmetry of typical projectiles, we combined pitch and yaw into a total yaw without much ado. In this section, we will discuss the two motions separately and then formally make the assumptions that allowed us to combine them. This approach was formulated by Murphy [1] and what follows is basically that development with the coordinate system altered to fit our needs.

If we have a projectile as depicted in [Figure 9.1](#) and allow it only to move in a truly pitching motion, we can look down the z-axis and we would see what is depicted in [Figure 9.2](#). Some interesting observations can be made from this figure. First, we see that the velocity vector,  $\mathbf{V}$ , and the associated unit vector,  $\mathbf{l}$ , are pitched up at angle  $\phi$  to the earth-fixed coordinate system. The projectile is actually pointed above this angle by the pitch angle  $\alpha$ . The vector along which the projectile is pointed is the geometric axis unit vector,  $\mathbf{x}$ , and the spin (principal) axis unit vector,  $\mathbf{i}$ . In an axially symmetric projectile, these are identical. We can see that through a rigid body rotation this forces the unit vectors of the transverse geometric axis,  $\mathbf{y}$ , and transverse principal axis,  $\mathbf{j}$ , to be rotated from the earth-fixed Y-axis through an angle of  $\phi + \alpha$ . If we assume that the projectile is constrained to pitch only, then the time rate of change of this total angle is  $q$  and the yaw angle and yaw rate are equal to zero as depicted in the figure.

In a similar manner, we can constrain our projectile to motion in the yaw plane only which is depicted in [Figure 9.3](#). In this case, the velocity and its associated unit vector are yawed with respect to the earth-fixed coordinate system by angle  $\theta$ . The projectile geometric axis as well as the principal axis are yawed at angle  $\beta$  with respect to the velocity



**FIGURE 9.2**  
Projectile in a pure pitching motion.



**FIGURE 9.3**  
Projectile in a pure yawing motion.

vector. This results in a rotation of the transverse principle axis from the earth-fixed coordinate system of  $\theta + \beta$  as depicted in the illustration. The rate of change of this total angle is the yaw rate,  $r$ , and because of our constraints there is no pitching motion as identified in the figure.

We shall now develop the equations of motion for each of these two specialized cases with the purpose of combining them in the end. For the purpose of this development, we shall define the force in the Y- and Z-directions using force coefficients  $C_Y$  and  $C_Z$ , respectively.

If we examine our projectile constrained to a pitching motion only, we can define the force coefficient as

$$C_Y = C_{Y_0} + C_{Y_\alpha} \alpha + C_{Y_\dot{\alpha}} \left( \frac{\dot{\alpha}d}{V} \right) + C_{Y_q} \left( \frac{qd}{V} \right) \quad (9.13)$$

Here we have restricted ourselves to the linear coefficients. We can see that this pitching motion causes a force in the Y-direction that is affected by angle of attack, rate of change of angle of attack, and pitching rate. An item worthy of note is that for a perfectly symmetrical projectile,  $C_{Y_0}$  would be zero. It is included here for completeness and can be present if an asymmetry exists.

The corresponding moment for pitching motion only is given by

$$C_m = C_{m_0} + C_{m_\alpha} \alpha + C_{m_{\dot{\alpha}}} \left( \frac{\dot{\alpha}d}{V} \right) + C_{m_q} \left( \frac{qd}{V} \right) \quad (9.14)$$

Here the same comments about the nondimensionalization and  $C_{m_0}$  apply as well.

Now we will examine the equations of motion. The force and moment equations are given by

$$\mathbf{F} = m\mathbf{a} \quad (9.15)$$

$$\mathbf{M} = I\dot{\boldsymbol{\alpha}} \quad (9.16)$$

If we define angles  $\hat{\phi}$  and  $\hat{\theta}$  as

$$\hat{\phi} = \phi + \alpha \quad (9.17)$$

$$\hat{\theta} = \theta + \beta \quad (9.18)$$

Then our scalar equations of a projectile in flight exhibiting pure pitching motion are

$$m \frac{dV_x}{dt} \approx m \frac{dV}{dt} = F_x \quad (9.19)$$

$$m \frac{d^2Y}{dt^2} = F_Y \cos \hat{\phi} + F_x \sin \phi - mg \quad (9.20)$$

$$\frac{d^2\hat{\phi}}{dt^2} = \frac{M_Z}{I_Z} \quad (9.21)$$

If we examine a small time of the projectile flight, we can assume constant velocity. If we further limit the pitching motion to small angles, we can assume

$$F_x = -F_D \approx 0 \quad (9.22)$$

$$\cos \hat{\phi} \approx 1 \quad (9.23)$$

$$\sin \hat{\phi} \approx \hat{\phi} \quad (9.24)$$

These assumptions can be used in Equations 9.19 and 9.20 to yield

$$m \frac{dV}{dt} = 0 \quad (9.25)$$

$$m \frac{d^2Y}{dt^2} = F_Y - mg \quad (9.26)$$

We know that

$$F_Y = \frac{1}{2} \rho V^2 S C_Y \quad (9.27)$$

If we then substitute [Equations 9.13](#) and 9.27 into Equation 9.26, we obtain

$$m \frac{d^2Y}{dt^2} = \frac{1}{2} \rho V^2 S \left[ C_{Y_0} + C_{Y_\alpha} \alpha + C_{Y_{\dot{\alpha}}} \left( \frac{\dot{\alpha} d}{V} \right) + C_{Y_q} \left( \frac{qd}{V} \right) \right] - mg \quad (9.28)$$

or, using our definition of starred coefficients, we have

$$\frac{d^2Y}{dt^2} = \frac{V^2}{d} \left[ C_{Y_0}^* + C_{Y_\alpha}^* \alpha + C_{Y_{\dot{\alpha}}}^* \left( \frac{\dot{\alpha} d}{V} \right) + C_{Y_q}^* \left( \frac{qd}{V} \right) \right] - g \quad (9.29)$$

Equation 9.29 can be combined with Equation 9.21 to develop a single equation for projectile motion. With this, the dynamic equation for the pure pitching motion of a projectile can then be described as

$$\ddot{\alpha} + \hat{H}_{1d} \dot{\alpha} - \hat{M}_1 \alpha = \hat{A}_1 + \hat{G}_d \quad (9.30)$$

This linear, second order, differential equation with constant coefficients was established by Murphy [1] and modified here (the terms with the “d” subscript) to account for the assumption of zero drag. In this expression, we identify the coefficients as follows:

$$\hat{H}_{1d} = - \left[ C_{Y_\alpha}^* + \frac{1}{k_Z^2} (C_{m_q}^* + C_{m_\alpha}^*) \right] \left( \frac{V}{d} \right) \quad (9.31)$$

$$\hat{M}_1 = \left( \frac{1}{k_Z^2} C_{m_\alpha}^* \right) \left( \frac{V}{d} \right)^2 \quad (9.32)$$

$$\hat{A}_1 = \left( \frac{1}{k_Z^2} C_{m_0}^* \right) \left( \frac{V}{d} \right)^2 \quad (9.33)$$

$$\hat{G}_d = - \left( \frac{1}{k_Z^2} C_{m_q}^* \right) \left( \frac{g}{d} \right) \quad (9.34)$$

$$k_Z^2 = \frac{I_Z}{md^2} \quad (9.35)$$

If we include drag (and thus ignore Equation 9.22) yet leave all of the other assumptions in place, we obtain a result identical to Murphy [1]. This results in Equations 9.30, 9.31, and 9.34 being modified to

$$\ddot{\alpha} + \hat{H}_1 \dot{\alpha} - \hat{M}_1 \alpha = \hat{A}_1 + \hat{G} \quad (9.36)$$

$$\hat{H}_1 = - \left[ C_{Y_\alpha}^* + C_D^* + \frac{1}{k_Z^2} (C_{m_q}^* + C_{m_\alpha}^*) \right] \left( \frac{V}{d} \right) \quad (9.37)$$

$$\hat{G} = - \left( \frac{1}{k_Z^2} C_{m_q}^* - C_D^* \right) \left( \frac{g}{d} \right) \quad (9.38)$$

It is more convenient to examine the differential Equations 9.30 and 9.36 with dimensionless distance (defined as  $s/d$ ) instead of time as the independent variable. The time derivatives of dimensionless distance can then be written as

$$\frac{ds}{dt} = \left( \frac{V}{d} \right) \quad (9.39)$$

and

$$\frac{d^2 s}{dt^2} = \left( \frac{\dot{V}}{d} \right) \quad (9.40)$$

With this, we can use the relations

$$\frac{d}{dt}(\cdot) = \frac{d}{ds}(\cdot) \frac{ds}{dt} = \left( \frac{V}{d} \right) \frac{d}{ds}(\cdot) \quad (9.41)$$

$$\frac{d^2}{dt^2}(\cdot) = \left( \frac{V}{d} \right)^2 \frac{d^2}{ds^2}(\cdot) + \left( \frac{\dot{V}}{d} \right) \frac{d}{ds}(\cdot) \quad (9.42)$$

to rewrite Equations 9.30 and 9.36, respectively as

$$\alpha'' + H_{1d}\alpha' - M_1\alpha = A_1 + G_d \quad (9.43)$$

$$\alpha'' + H_1\alpha' - M_1\alpha = A_1 + G \quad (9.44)$$

The coefficients in these equations are given by

$$H_{1d} = -\left[ C_{Y_\alpha}^* + \frac{1}{k_Y^2} (C_{m_q}^* + C_{m_{\dot{\alpha}}}^*) \right] \quad (9.45)$$

$$H_1 = -\left[ C_{Y_\alpha}^* + 2C_D^* + \frac{1}{k_Z^2} (C_{m_q}^* + C_{m_{\dot{\alpha}}}^*) \right] \quad (9.46)$$

$$M_1 = \frac{1}{k_Z^2} C_{m_\alpha}^* \quad (9.47)$$

$$A_1 = \frac{1}{k_Z^2} C_{m_0}^* \quad (9.48)$$

$$G_d = -\left( \frac{1}{k_Z^2} C_{m_q}^* \right) \left( \frac{gd}{V_0^2} \right) \quad (9.49)$$

$$G = -\left( \frac{1}{k_Z^2} C_{m_q}^* - C_D^* \right) \left( \frac{gd}{V_0^2} \right) \quad (9.50)$$

where  $V_0$  is the muzzle (or a reference) velocity of the projectile.

A similar procedure can be followed to define motion constrained to the yaw plane only. This gives the result (details covered in Ref. [1]) of

$$\beta'' + H_{2d}\beta' - M_2\beta = A_2 \quad (9.51)$$

$$\beta'' + H_2\beta' - M_2\beta = A_2 \quad (9.52)$$

The coefficients in these equations are given by

$$H_{2d} = -\left[ C_{Z_\beta}^* + \frac{1}{k_Z^2} (C_{n_r}^* + C_{n_{\dot{\beta}}}^*) \right] \quad (9.53)$$

$$H_2 = -\left[ C_{Z_\beta}^* + 2C_D^* + \frac{1}{k_Y^2} (C_{n_r}^* + C_{n_{\dot{\beta}}}^*) \right] \quad (9.54)$$

$$M_2 = \frac{1}{k_Y^2} C_{n_\beta}^* \quad (9.55)$$

$$A_2 = \frac{1}{k_Y^2} C_{n_0}^* \quad (9.56)$$

$$k_Y^2 = \frac{I_Y}{md^2} \quad (9.57)$$

Assuming a projectile is axially symmetric implies that any plane orthogonal to the polar axis is a principal axis. This forces the two transverse moments of inertia to be equal and, with an assumption of small yaw, allows us to write

$$I_T = I_y = I_z \approx I_Y = I_Z \quad (9.58)$$

This symmetry also allows us to equate the pitch and yaw coefficients. Thus, we define

$$C_{N_\alpha} \equiv C_{Y_\alpha} = C_{Z_\beta} \quad (9.59)$$

$$C_{M_q} \equiv C_{m_q} = C_{n_r} \quad (9.60)$$

$$C_{M_\alpha} \equiv C_{m_\alpha} = C_{n_\beta} \quad (9.61)$$

$$C_{M_\dot{\alpha}} \equiv C_{m_\dot{\alpha}} = C_{n_\dot{\beta}} \quad (9.62)$$

Complex numbers are commonly used to define pitch and yaw angles. This is extremely convenient because it allows us to collapse two differential equations into one. We shall define the complex yaw angle as  $\xi$  which shall be defined thusly

$$\xi \equiv \alpha + i\beta \quad (9.63)$$

This definition allows one to look downrange as a projectile flies along a trajectory and visualize the imaginary part of the equation affecting the yaw of the projectile and the real part of the equation as affecting pitch. This is illustrated in Figure 9.4. In this figure, the origin is the trajectory of the projectile looking downrange.

The two differential equations of motion Equations 9.44 and 9.52 can then be combined by first multiplying Equation 9.52 by the imaginary number,  $i$ , and adding them together. This results in

$$\xi'' + H\xi' - M\xi = A + G\left(\frac{V_0}{V}\right)^2 \quad (9.64)$$

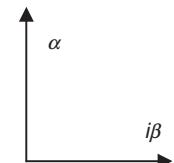
The coefficients in this equation are given by

$$H = -\left[C_{N_\alpha}^* + 2C_D^* + \frac{1}{k_T^2} (C_{M_q}^* + C_{M_\dot{\alpha}}^*)\right] \quad (9.65)$$

$$M = \frac{1}{k_T^2} C_{M_\alpha}^* \quad (9.66)$$

$$A = \frac{1}{k_T^2} (C_{m_0}^* + iC_{n_0}^*) \quad (9.67)$$

$$G = -\left(\frac{1}{k_Z^2} C_{M_q}^* - C_D^*\right) \left(\frac{gd}{V_0^2}\right) \quad (9.68)$$



**FIGURE 9.4**  
Complex yaw plane.

The solution to [Equation 9.64](#) can be found for a non-spinning projectile to be [1]

$$\xi = K_1 e^{i\psi_1} + K_2 e^{i\psi_2} + K_3 e^{i\psi_{30}} + \xi_g \quad (9.69)$$

In this equation, each term  $K_j$  is known as an arm to be described subsequently. Mathematically, we can express these terms as

$$K_j = K_{j0} e^{\lambda_j s} \quad (9.70)$$

Here we see that each arm is a function of its initial value (that occurring at the muzzle of the gun) and an exponential damping term. The exponential damping term decides whether the amplitude of the motion will decay, grow, or remain constant. The damping terms are given by [1,5].

$$\lambda_1 = \lambda_2 = -\frac{1}{2} H \quad (9.71)$$

The exponential terms in [Equation 9.69](#) contain phase angles,  $\psi_j$ . These phase angles represent the instantaneous angle that each arm makes with the imaginary axis. These can be written in terms of their initial value and a turning frequency as

$$\psi_j = \psi_{j0} + \psi'_j s \quad (9.72)$$

The turning frequencies are given for a non-spinning projectile by [1,5].

$$\psi'_1 = -\psi'_2 = \sqrt{-M} \quad (9.73)$$

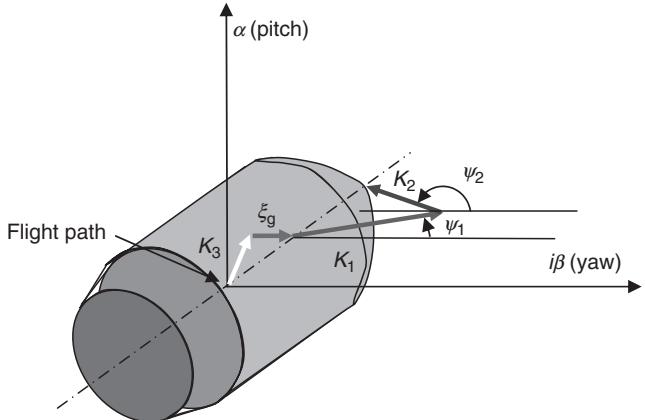
The third term on the RHS of [Equation 9.69](#) is the so-called trim arm. This is a measure of the amount that a fin-stabilized projectile will trim (i.e., fly with constant pitch or yaw) during flight. It is given by

$$K_3 e^{i\psi_{30}} = -\frac{i(C_{m_0} + iC_{n_0})}{C_{M_\alpha}} \quad (9.74)$$

The fourth term on the RHS of [Equation 9.69](#) is the yaw caused by interaction of the projectile with the gravity vector, sometimes called the yaw of repose. It is defined as

$$\xi_g = \frac{i(C_{M_q} - k_T^2 C_D) \left( \frac{gd}{V^2} \right)}{C_{M_\alpha}} \quad (9.75)$$

To visualize the physical meaning of [Equation 9.69](#), we shall imagine we have a projectile and we are looking downrange along the trajectory such that the complex plane lies perpendicular to the trajectory curve. Our projectile will be at some arbitrary yaw angle. This is depicted in [Figure 9.5](#). We need to note that the arms usually do not point to the nose of the projectile, they point to the symmetry axis; however, it is easiest to visualize the situation by scaling them to point to the nose. Imagine that we follow the projectile depicted in [Figure 9.5](#) as it traverses the trajectory. We would see the nose motion swirling around. Throughout this time, we would also see the length of each of the arms changing (growing, decaying, or remaining the same) as dictated by [Equation 9.69](#).



**FIGURE 9.5**  
Example of tricyclic arms.

Additionally, we would see the arms rotating around their respective origins at rates described by [Equation 9.72](#). All through this time, our viewpoint would be changing because we have our gaze fixed on the complex plane and it is rotating into the paper because of the curvature of the trajectory.

In the development of [Equation 9.64](#) and its solution [Equation 9.69](#), the spin of the projectile was neglected. Because of this, these equations are specific to fin- or drag-stabilized projectiles that have relatively small spin rates. References [1], [2], and [5] develop the equation of motion for spinning projectiles in exactly the same manner. The results essentially incorporate the third angular component known as the roll or spin. The differential equation for a spinning projectile is given by

$$\xi'' + (H - iP)\xi' - (M + iP\Gamma)\xi = -iPG \quad (9.76)$$

In this formulation, we can utilize axial symmetry and thus, define our coefficients as follows:

$$H = C_{L_\alpha}^* - C_D^* - \frac{1}{k_T^2} (C_{M_q}^* + C_{M_{\dot{\alpha}}}^*) \quad (9.77)$$

$$M = \frac{1}{k_T^2} C_{M_\alpha}^* \quad (9.78)$$

$$T = C_{L_\alpha}^* + \frac{1}{k_P^2} C_{M_p\alpha}^* \quad (9.79)$$

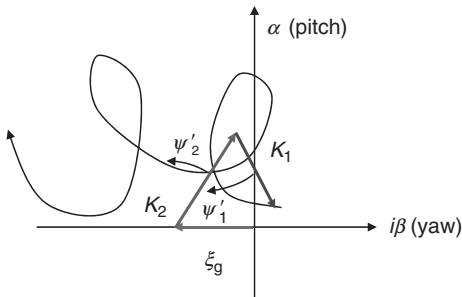
$$G = \frac{gd}{V_0^2} \quad (9.80)$$

$$P = \left( \frac{I_p}{I_T} \right) \left( \frac{pd}{V} \right) \quad (9.81)$$

The solution to [Equation 9.76](#) is

$$\xi = K_{10} e^{\lambda_1 s} \exp [i(\psi_{10} + \psi'_1 s)] + K_{20} e^{\lambda_2 s} \exp [i(\psi_{20} + \psi'_2 s)] + \xi_g \quad (9.82)$$

This equation is essentially the same form as [Equation 9.69](#) except for the deletion of the trim arm. It is also noteworthy that we have expanded the slow and fast arm terms



**FIGURE 9.6**  
Example of tricyclic arms for fin-stabilized projectile.

and exponents to display their exponential behavior. The expression is also commonly written as

$$\xi = K_1 e^{i\psi_1} + K_2 e^{i\psi_2} + \xi_g \quad (9.83)$$

where the definitions of Equations 9.70 and 9.71 apply. The  $\lambda_j$  terms are known as the exponential damping coefficients and the  $\psi_j$  terms are the precessional and nutation frequencies of the projectile. These are commonly defined as a complex pair where

$$\lambda_{1,2} + i\psi_{1,2} = \frac{1}{2} \left[ -H + iP \pm \sqrt{4M + H^2 - P^2 + 2iP(2T - H)} \right] \quad (9.84)$$

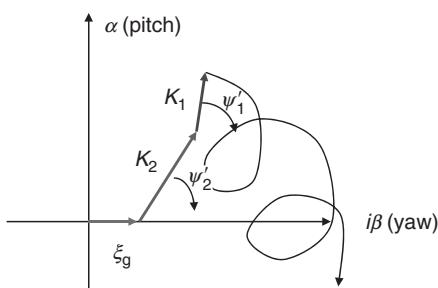
As a parting note, we need to discuss the behavior of the fast and slow arms and the associated motion that they undergo. For a non-spinning projectile, we shall examine Equation 9.73. In this expression, the sign of  $M$  is important. For a non-spinning projectile,  $M$  is negative that tells us that the arms turn in opposite directions with  $K_1$  being positive (clockwise) and  $K_2$  negative (counter-clockwise). This is depicted in Figure 9.6.

Likewise for a spinning projectile, we need to examine the derivative with respect to  $s$  of Equation 9.84. In this case, we would find that

$$\psi'_{1,2} = \frac{1}{2} \left( P \pm \sqrt{P^2 - 4M} \right) \quad (9.85)$$

Here we shall see in the following section that for stability, this must result in a solution that has no imaginary part. So both values of the root will have the same sign thus the two arms turn in the same direction as shown in Figure 9.7.

Initial conditions that are present when the projectile leaves the muzzle of the weapon are important as our starting point for the values of the fast and slow arms. These can even



**FIGURE 9.7**  
Example of tricyclic arms for spin-stabilized projectiles.

cause drastically different flight behavior when nonlinear coefficients are introduced later. The initial sizes of the fast and slow arms can be expressed as functions of the precession and nutation rates, the damping exponents, and the initial yaw and yaw rates [5] as

$$K_{10}e^{i\psi_{10}} = \frac{\xi'_0 - (\lambda_2 + i\psi'_2)\xi_0}{\lambda_1 - \lambda_2 + i(\psi'_1 - \psi'_2)} \quad (9.86)$$

$$K_{20}e^{i\psi_{20}} = \frac{\xi'_0 - (\lambda_1 + i\psi'_1)\xi_0}{\lambda_2 - \lambda_1 + i(\psi'_2 - \psi'_1)} \quad (9.87)$$

Because the damping exponents are usually an order of magnitude or more smaller than the precession and nutation rates, these equations can be simplified to

$$K_{10}e^{i\psi_{10}} = \frac{i\xi'_0 + \psi'_2\xi_0}{\psi'_2 - \psi'_1} \quad (9.88)$$

$$K_{20}e^{i\psi_{20}} = \frac{i\xi'_0 + \psi'_1\xi_0}{\psi'_1 - \psi'_2} \quad (9.89)$$

These equations are important because they allow one to determine the initial amplitudes of the arms given an assumed or measured initial yaw, yaw rate, and muzzle exit conditions for a known projectile geometry.

The expressions introduced in this section are the basis for stability criterion to be established next. In the next section, we shall discuss the behavior of these equations and use them to define stability criteria for a projectile.

### Problem 1

A 155-mm M549A1 Projectile has the following properties and initial conditions:

$C_D = 0.3$		
$C_{L_\alpha} = 0.13$	$\rho = 0.0751 \left[ \frac{\text{lbm}}{\text{ft}^3} \right]$	$I_P = 505.5 \left[ \text{lbm-in.}^2 \right]$
$C_{M_\alpha} = 4.28$	$d = 155 \text{ [mm]}$	$I_T = 6610 \left[ \text{lbm-in.}^2 \right]$
$C_{l_p} = 0.024$		$m = 96 \text{ [lbm]}$
$C_{M_q} + C_{M_\alpha} = -26$	$V_{\text{muzzle}} = 3000 \left[ \frac{\text{ft}}{\text{s}} \right]$	
$C_{M_{P_\alpha}} = 0.876$		

At an instant in time after launch when

$$\begin{aligned} p &= 220 \text{ [Hz]} \\ \phi &= \delta = 4^\circ \\ V &= 1764 \left[ \frac{\text{ft}}{\text{s}} \right] \end{aligned}$$

Determine

1. The yaw of repose

*Answer:*  $\beta_R = 0.00172 \text{ [rad]}$

2. The precessional frequency in Hz

*Answer:*  $\frac{d\psi_2}{dt} = 1.9 \text{ [Hz]}$

3. The nutational frequency in Hz

$$\text{Answer: } \frac{d\psi_1}{dt} = 14.9 \text{ [Hz]}$$


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## 9.2 Gyroscopic and Dynamic Stabilities

In the previous section, we developed a pair of equations and their solutions using linear aeroballistic coefficients that allow us to examine the motion of a projectile in pitch, yaw, and roll. These equations will now be examined in detail so that we can establish criteria for a stable projectile. In so doing, we will examine some interesting characteristics of motion which will be displayed as curves in the complex plane.

We shall repeat the equations and their solutions here for ease of reference but leave the coefficient definitions [in Section 9.1](#) to preserve space. The governing equations are as follows:

For a non-spinning or slowly spinning projectile,

$$\xi'' + H\xi' - M\xi = A + G\left(\frac{V_0}{V}\right)^2 \quad (9.90)$$

with the solution

$$\xi = K_{10}e^{\lambda_1 s} \exp[i(\psi_{10} + \psi'_1 s)] + K_{20}e^{\lambda_2 s} \exp[i(\psi_{20} + \psi'_2 s)] + K_3 e^{i\psi_{30}} + \xi_g \quad (9.91)$$

For a spinning projectile,

$$\xi'' + (H - iP)\xi' - (M + iPT)\xi = -iPG \quad (9.92)$$

with the solution

$$\xi = K_{10}e^{\lambda_1 s} \exp[i(\psi_{10} + \psi'_1 s)] + K_{20}e^{\lambda_2 s} \exp[i(\psi_{20} + \psi'_2 s)] + \xi_g \quad (9.93)$$

For our general development of stability, we shall focus on Equation 9.92 and its solution, Equation 9.93, since the trim term in Equation 9.91 can be easily dealt with separately.

If we examine Equation 9.93, we can readily see that nasty things can happen to us mathematically because of the exponential terms. Since  $K_{10}$  and  $K_{20}$  are constants (they are the initial magnitudes of the fast and slow arms, respectively), we can focus on the exponential terms that they are multiplied by as a means of determining whether they will grow, shrink, or remain the same.

We shall consider the exponential functions of  $\psi$  and  $\psi'$  first using the fast arm terms as examples. The term  $\psi_{10}$  is a constant and will be ignored. This leaves the term  $\psi'_1 s$  which is multiplied by  $i$  in the exponent. If  $\psi'_1$  is purely real, then, when multiplied by  $i$ , it becomes purely imaginary in the exponent (because  $s$  must be real), the solution is oscillatory and this will cause the fast arm to increase and decrease in amplitude (i.e., oscillate), neither increasing nor decreasing beyond the established limits of oscillation. This would be a gyroscopically stable projectile. If it has an imaginary component then, when multiplied by  $i$  in the exponent, the solution has a real part. This real part will be multiplied by  $s$  and continue to grow throughout the flight as  $s$  continually increases. This would result in a gyroscopically unstable projectile.

The question to answer at this point is “What governs whether the exponents have real or imaginary parts?” This can be answered by examination of a version of [Equation 9.84](#), whereby all aerodynamic forces and moments are ignored except for the largest (pitching) moment. This has been shown [1,5] to result in a governing equation of

$$\xi'' - iP\xi' - M\xi = -iPG \quad (9.94)$$

With the solution

$$\xi = K_1 \exp [i(\psi_{10} + \psi'_1 s)] + K_2 \exp [i(\psi_{20} + \psi'_2 s)] + \xi_g \quad (9.95)$$

Resulting in

$$\psi'_{1,2} = \frac{1}{2}(P \pm \sqrt{P^2 - 4M}) \quad (9.96)$$

Here the subscripts 1 and 2 represent the fast and slow arms, respectively.

Using [Equation 9.96](#), we recall that for a gyroscopically stable projectile,  $\psi'$  must be real, therefore for gyroscopic stability, we require that

$$(P^2 - 4M) > 0 \quad (9.97)$$

This expression has some interesting implications. If we look back at the definition of our parameter,  $M$  in [Equation 9.66](#), we see that it is dependent upon the pitching moment coefficient. This happens to always be negative for a fin-stabilized projectile since the fins impart a restoring moment. Unless there is some unique drag device, this moment is positive in a non-fin-stabilized projectile. Because of this, a fin-stabilized projectile is always gyroscopically stable because  $P^2$  must be positive. However, a non-fin-stabilized projectile must have a spin sufficient to make  $P^2 > 4M$ . We therefore define a statically stable projectile as one in which  $M < 0$ . With this definition, a statically stable projectile is always gyroscopically stable.

Gyroscopic stability is a necessary but not sufficient condition for a stable projectile. The second condition required is that of dynamic stability. Let us once again examine [Equation 9.93](#), but this time we shall assume that we have a gyroscopically stable projectile. This means that the exponential terms containing  $\psi'$  decay or remain constant, leaving the terms containing  $\lambda$  as potentially destabilizing. We can readily see that, since these are multiplied by the downrange distance,  $s$ , they must be negative to assure that the fast and slow arms decay in magnitude. With this, we shall define a dynamically stable projectile as one in which both  $\lambda$ 's are negative throughout the flight. Recall that we calculate  $\lambda$  as the real part of [Equation 9.84](#). For convenience, we shall express them directly as

$$\lambda_{1,2} = -\frac{1}{2} \left[ H \mp \frac{P(2T - H)}{\sqrt{P^2 - 4M}} \right] \quad (9.98)$$

It should be noted here that, as is common in ballistics, there are always exceptions to any rule. Some successful projectiles have been fielded where instability occurs for a very short time in a flight or in a range where a certain projectile will never be fired. Of course, it is always best to avoid these situations but sometimes lack of design space makes it unavoidable. In these instances, rational examination of the instability is necessary and should be well documented.

We have shown mathematically how we define stability and the parameters that affect stability. Sometimes, it is desirable to quantify how stable a projectile is. We do this through use of a gyroscopic and dynamic stability factors. We define the gyroscopic stability factor as

$$S_g = \frac{P^2}{4M} \quad (9.99)$$

Here, with our earlier discussion,  $S_g > 1$  to assure gyroscopic stability. In a similar fashion, we can define a dynamic stability factor as

$$S_d = \frac{2T}{H} \quad (9.100)$$

Where for a symmetric projectile to be deemed stable, whether spinning or non-spinning, we require

$$\frac{1}{S_g} < S_d(2 - S_d) \quad (9.101)$$

For a statically stable projectile, we require that  $0 < S_d < 2$  for dynamic stability. This leads to an interesting condition where one can spin a statically stable projectile too fast, resulting in instability. This condition translated to dimensionless spin rate is given by

$$P < \sqrt{\frac{4M}{S_d(2 - S_d)}} \quad (9.102)$$

for a statically stable projectile.

It is interesting to combine Equations 9.96 and 9.98 in various ways writing them in terms of the dimensionless parameters  $P$ ,  $M$ ,  $H$ , and  $T$ . The details of this can be found in Refs. [1] and [5] with the result

$$P = \psi'_1 + \psi'_2 \quad (9.103)$$

$$M = \psi'_1 \psi'_2 - \lambda_1 \lambda_2 \quad (9.104)$$

$$H = -(\lambda_1 + \lambda_2) \quad (9.105)$$

$$PT = -(\psi'_1 \lambda_1 + \psi'_2 \lambda_2) \quad (9.106)$$

If we again examine Equation 9.91 or 9.93, we see that the magnitude of the precessional and nutational arms is highly dependent upon initial conditions. Without going into details (which are described quite well in Ref. [5]), we can express these initial conditions in terms of the complex angle of attack and damping parameters as

$$K_{10} e^{i\psi_{10}} = \frac{\xi'_0 - (\lambda_2 + i\psi'_2)\xi_0}{\lambda_1 - \lambda_2 + i(\psi'_1 - \psi'_2)} \quad (9.107)$$

$$K_{20} e^{i\psi_{20}} = \frac{\xi'_0 - (\lambda_1 + i\psi'_1)\xi_0}{\lambda_2 - \lambda_1 + i(\psi'_2 - \psi'_1)} \quad (9.108)$$

In these equations,  $\xi_0$  and  $\xi'_0$  are the initial complex yaw and yaw rates, respectively. These parameters are determined by measurements as the projectile leaves the gun tube or are assumed values.

We now have solid criterion by which we can determine whether a projectile will be stable or not. These developments have been made assuming that the projectile aerodynamic coefficients behave in a linear fashion. As such, a projectile is either stable or it is not. This stability, even with our linear model, will change during the flight based on Mach number and angle of attack. We will discuss in a later section how a nonlinearity can help or hurt matters. The true power of these equations is that they can tell us which coefficients need to be altered to affect stability. This can be used in instances where we want to change a physical configuration to make a projectile “drop out of the sky” or design a round such that it damps more quickly and thus can fly with lower drag. Other uses for these equations allow for tweaking the flight characteristics for better flight behavior in general.

### Problem 2

Up until the late 1960s, many U.S. and foreign ships carried the Bofors 40-mm gun as a general light support weapon. Originally designed as an antiaircraft weapon, this gun served as an antitank weapon if the situation required it and its high rate of fire made it quite successful as an antipersonnel weapon. Assume the properties of the system are given below:

1. Calculate the gyroscopic stability factor at the beginning and at the end of the flight assuming a terminal velocity of 2450 ft/s and the spin rate is 10% lower than the initial value.

*Answer:* At the beginning of flight  $S_g = 5.814$

2. Is the projectile stable throughout the flight?

*Answer:* Yes

3. Assuming that this is the longest time of flight for the projectile, at what spin rate will the projectile become unstable?

*Answer:*  $p_{\text{unstable}} < 1887 \left[ \frac{\text{rad}}{\text{s}} \right]$

4. Where will the instability occur?

*Answer:* At the muzzle of the weapon

Projectile and weapon information

$C_{M_\alpha} = 3.10$	$\rho = 0.067 \left[ \frac{\text{lbfm}}{\text{ft}^3} \right]$	$I_P = 1.231 \left[ \text{lbfm-in.}^2 \right]$
$C_{l_p} = -0.011$	$d = 40 \left[ \text{mm} \right]$	$I_T = 6.263 \left[ \text{lbfm-in.}^2 \right]$
	$V_{\text{muzzle}} = 2850 \left[ \frac{\text{ft}}{\text{s}} \right]$	$m = 1.985 \left[ \text{lbfm} \right]$
		$n = \frac{1}{30} \left[ \frac{\text{rev}}{\text{cal}} \right]$

Please note that this weapon actually has a progressive twist but when faced with this situation you only need the muzzle velocity and the twist at the muzzle to calculate initial spin.

### Problem 3

For the projectile described in Problem 13 of [Chapter 8](#),

1. Determine the precessional damping exponent.

*Answer:*  $\lambda_2 = -0.0000673$

2. Determine the nutational damping exponent.

*Answer:*  $\lambda_1 = -0.0001151$

3. With (1) and (2) above which of these modes will damp out?

*Answer:* Both

4. Determine the dynamic stability factor,  $S_d$ .

*Answer:*  $S_d = 0.74$

5. Consider a cargo projectile with identical properties to our projectile in [Problem 2](#).

The designer did not secure the cargo well enough so that the cargo fails to spin up completely during gun launch in a worn tube. When this happens, immediately after muzzle exit, the round spins down (and the cargo spins up a little more) so that the projectile finally reaches a spin rate of 100 Hz. The velocity is unaffected.

- a. Determine the gyroscopic stability factor for each of the two situations.

*Answer:*  $S_g = 3.192$  and  $S_g = 0.659$

- b. Will both projectiles fly properly? Why or why not?

*Answer:* No, the second projectile will tumble.

#### Problem 4

A 155-mm HE projectile is fired from a cannon. The muzzle velocity of the projectile is 800 m/s and the twist of the rifling is 1:20. The projectile and filler properties are given below. Assuming the aerodynamic forces and moments are negligible and that the projectile is dynamically stable:

1. The initial spin rate of the complete projectile.

$$\text{Answer: } p_{\text{muzzle}} = 1621.5 \left[ \frac{\text{rad}}{\text{s}} \right]$$

2. The spin rate of the projectile in flight assuming the fill does not spin up in the bore and both shell and fill come into dynamic equilibrium.

$$\text{Answer: } p_{\text{total}} = 1259.2 \left[ \frac{\text{rad}}{\text{s}} \right]$$

3. Determine the gyroscopic stability factors for (1) and (2).

*Answer:*  $S_g = 16.54$  and  $S_g = 9.97$

4. Is the projectile stable in (1) and (2)?

*Answer:* Yes to both.

Projectile and weapon information

$$C_{M_a} = 1.07$$

$$C_{l_p} = -0.012$$

$$I_{P_{\text{shell}}} = 431 \text{ [lbm-in.}^2\text{]}$$

$$I_{P_{\text{fill}}} = 124 \text{ [lbm-in.}^2\text{]}$$

$$\rho = 0.067 \left[ \frac{\text{lbm}}{\text{ft}^3} \right]$$

$$d = 155 \text{ [mm]}$$

$$V_{\text{muzzle}} = 800 \left[ \frac{\text{m}}{\text{s}} \right]$$

$$I_{P_{\text{total}}} = 555 \text{ [lbm-in.}^2\text{]}$$

$$I_{T_{\text{total}}} = 3335 \text{ [lbm-in.}^2\text{]}$$

$$m = 106 \text{ [lbm]}$$

$$n = \frac{1}{20} \left[ \frac{\text{rev}}{\text{cal}} \right]$$

#### Problem 5

For the projectile given in Problem 4, determine the precession and nutation frequencies in Hertz.

$$\text{Answer: } \frac{d\psi_1}{dt} = 42.05 \text{ [Hz]} \text{ and } \frac{d\psi_2}{dt} = 0.67 \text{ [Hz]}$$

### Problem 6

Assume the projectile in [Problem 1](#) has slipped its rotating band and the spin at the same instant in time is 130 Hz. Is the projectile stable?

*Answer:* No

### Problem 7

What is the minimum spin (Hz) required to stabilize the projectile in Problem 1?

*Answer:*  $p_{\min} = 140 \text{ [Hz]}$

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## 9.3 Yaw of Repose

In [Section 9.1](#), we introduced the yaw of repose for a projectile and defined it in [Equation 9.75](#) using the symbol  $\xi_g$ . The subscript “g” was used to denote that this quantity comes about through the action of gravity on the projectile. In terms of our dimensionless parameters, we can rewrite [Equation 9.75](#) as

$$\xi_g = \frac{PG}{M + iP\bar{T}} \quad (9.109)$$

A qualitative look at this expression leads to some extremely interesting results. First and foremost is that the spin rate directly affects the yaw. The greater the spin (and therefore the larger the value of  $P$ ), the greater the yaw of repose is.

The second useful item to note is that the flatter the trajectory, the greater the yaw of repose is. In fact, if we look at the term  $G$ , it is linear in the cosine of angle of attack,  $\phi$ . Thus, when the projectile approaches maximum ordinate, the yaw of repose should be a maximum given all of the other parameters remain constant. Because of the decay of the other terms, the result is that the yaw of repose is usually a maximum shortly before or after reaching maximum ordinate.

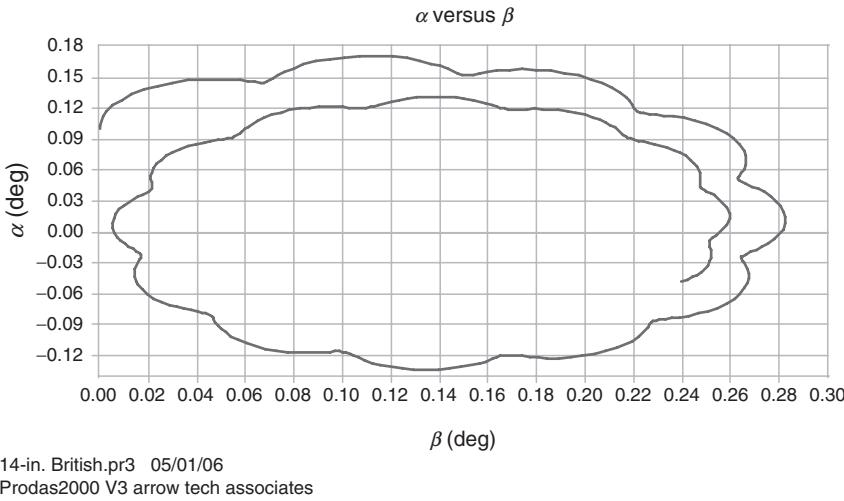
The sign of the yaw of repose is important. In our convention, the term  $P$  is positive for a right-hand twist. Thus, a positive value of  $\xi_g$  causes the projectile to nose over to the right. Note that there can also be a significant pitch component to this quantity, this is easily seen as the real part of [Equation 9.109](#).

If we examine a plot of pitch ( $\alpha$ ) versus yaw ( $\beta$ ) for a British 14-in. projectile in [Figure 9.8](#), we can imagine the yaw of repose as the vector pointing to the right (viewed from the rear) to the center of the precessional path similar to [Figure 9.7](#). We can see that the magnitude as well as the direction of this vector change as the projectile moves downrange. In [Figure 9.8](#), the projectile was analyzed using the PRODAS software and was fired with a muzzle velocity of 2483 ft/s, spin rate of 71 Hz corresponding to a 1:30 twist with an initial pitch angle of 0.1°. There was no initial yaw or pitch/yaw rate. This projectile has progressed through only one and one half yaw cycles (about 1.7 s) when the analysis was stopped to yield a nice clear illustration.

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## 9.4 Roll Resonance

Until this point, we have assumed that the projectiles under study has been axially symmetric. This rarely happens in practice because of manufacturing tolerances in a given projectile design. In [Chapter 10](#), we shall discuss the means of handling a slight mass asymmetry. In this section, we shall discuss the implications of a geometric (including slight mass) asymmetry as applied to a fin-stabilized projectile.



**FIGURE 9.8**

Pitching and yawing motion for a British 14-in. Mk.I projectile fired at 2483 ft/s with a 0.1° initial pitch angle.

Fin asymmetries commonly occur when a finned projectile is manufactured or can be the result of damage owing to rough handling. In the field of explosively formed penetrators which are normally drag- or fin-stabilized, inconsistencies can (and usually do) arise due to the explosive formation process. In either case, this effect may be coupled with some mass asymmetry as well.

In [Equation 9.74](#), the trim arm was introduced which would force a statically stable projectile to fly with an angle of attack. It is for this reason that all fin- and drag-stabilized projectiles are designed to roll slightly to increase accuracy. One can see from the way that this equation was written there is no change in the orientation of  $K_{30}$ . It was fixed, oriented at the initial angle  $\psi_{30}$ .

To begin our assessment of this specific type of asymmetry, we shall start with the governing equation for a spin-stabilized projectile, [Equation 9.76](#), because the roll is going to play a part. We shall alter the RHS to incorporate a forcing term representing the lifting force and moment that is caused by the asymmetry (say, for example, a bent fin). We shall write this in such a way that the direction of the applied force and moment rotates with the projectile.

$$\xi'' + (H - iP)\xi' - (M + iPT)\xi = -iA_3 e^{i\psi} \quad (9.110)$$

where

$$A_3 = \left( \frac{\rho S d}{2m} \right) \left( \frac{1}{k_T^2} \right) (C_{m_0} + iC_{n_0}) + (\psi' - 1)(C_{Y_0} + iC_{Z_0}) \quad (9.111)$$

$$\psi' = \frac{pd}{V}, \text{ dimensionless turning rate} \quad (9.112)$$

$$\psi = \int_0^s \psi' ds, \text{ dimensionless distance} \quad (9.113)$$

This development was put forth in Refs. [1], [4], and [5]. If we look closely at these equations, we see that the forcing function,  $A_3$ , rotates with the projectile.

If we solve [Equation 9.100](#), assuming a solution for the particular part of

$$\xi_p = K_3 \exp [i(\psi + \psi_0)] \quad (9.114)$$

where  $\psi_0$  is some arbitrary angle that contains the plane of the asymmetry, we obtain a general solution for a constant roll rate of

$$\xi = K_1 e^{i\psi_1} + K_2 e^{i\psi_2} + K_3 \exp [i(\psi + \psi_0)] \quad (9.115)$$

and, after inserting the initial conditions, say, of  $\psi_0 = 0$  we obtain

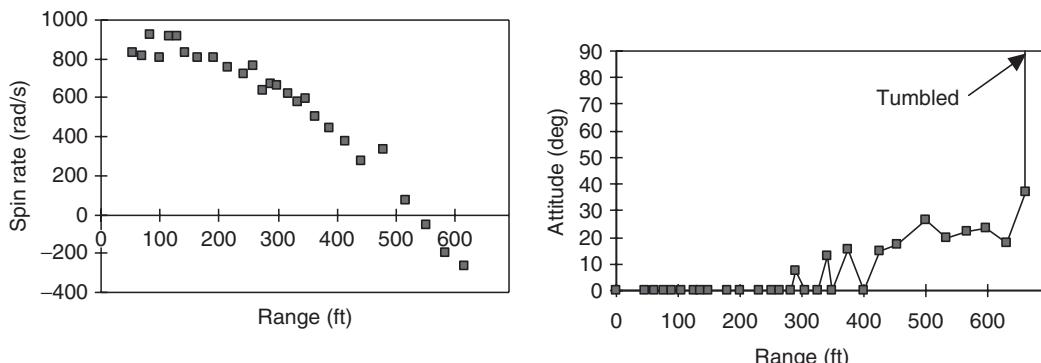
$$K_3 = \frac{-iA_3}{\psi'^2 - P\psi' + M - i(\psi'H - PT)} \quad (9.116)$$

This is the expression for the yaw component caused by a lift force and corresponding moment constrained to rotate at the projectile spin rate. If the spin rate is zero, the orientation of this lift force will be fixed and the projectile will drift more and more in that direction. This is not desirable from an accuracy standpoint so we must have some spin.

The denominator in [Equation 9.106](#) is normally dominated by its real part because  $H$  and the product  $PT$  are small by comparison. However, much like a resonance in a spring-mass system, if the roll frequency ever approaches either one of the precession or nutation frequencies (and remains there for some time), the denominator in [Equation 9.106](#) approaches zero and the yaw becomes very large [1,5]. This usually occurs when the nutational frequency is approached and is called roll resonance or spin-pitch resonance [1]. Since projectiles are usually changing spin rate throughout their flight, this is only a problem if there is a slow change of spin rate when the frequencies are close.

Another way of looking at this is to imagine a projectile where this asymmetry is present. Since the asymmetry is at the same frequency as the nutation rate, every time the projectile is at the outer limit of its motion it gets kicked a little further, similar to pushing a child on a swing. This disturbance grows as long as the two motions stay coupled (i.e., at the same frequency); however, if they became out of phase, the problem would correct itself.

An example of roll resonance is depicted in Figure 9.9. In this case, an explosively formed penetrator (EFP) was the device under test. Keep in mind that only total angle of



**FIGURE 9.9**

Explosively formed penetrator experiencing roll resonance (courtesy of Eric Volkmann, Alliant Techsystems).

attack is measured here so the yawing motion is not constrained to a single plane. We see that as the EFP approached a spin rate of  $\sim$ 300 rad/s it locked in and flew very far off of the target.

---

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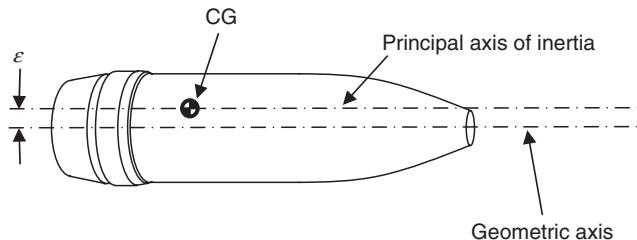
# 10

## *Mass Asymmetries*

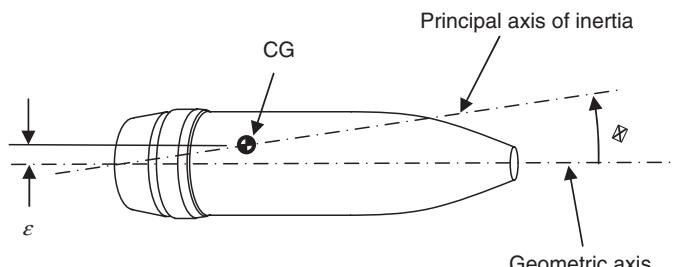
Until this point we have assumed that the projectile has been an axially symmetric body. This allowed us to simplify the equations of motion considerably. Projectiles are rarely axially symmetric. The asymmetry usually comes about through manufacturing tolerances, damage due to rough handling, cargo slippage or, more recently, they are simply designed that way. The purpose of this section is simply to introduce the geometry of mass asymmetries, which will be introduced into the equations of motion for the projectile in later sections.

Mass asymmetries come in two categories: static imbalance and dynamic imbalance. In a static imbalance, the center of gravity (CG) of the projectile is not located on the geometric axis of symmetry. The geometric axis of symmetry can be defined by imagining a projectile with the same exterior dimensions as the unbalanced projectile but of uniform density. The symmetry axis would then be centrally located in the body of revolution (i.e., a perfectly axially symmetric body). In a statically imbalanced projectile, this axis would be shifted to pass through the CG but remain parallel to the geometric axis. This is illustrated in Figure 10.1.

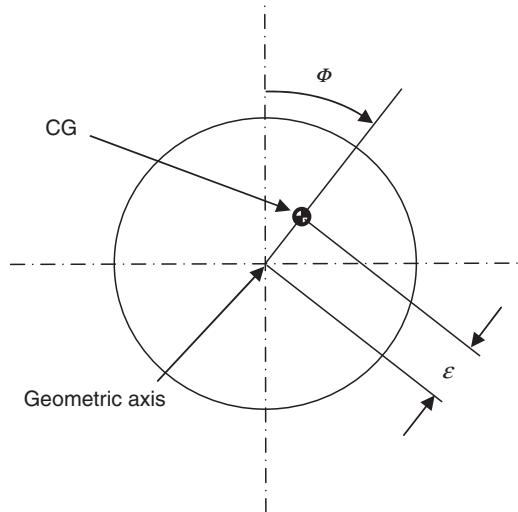
A dynamically imbalanced projectile also has a CG that is offset from the geometric axis of symmetry. In this case, however, the mass distribution is such that the principal axis of inertia resides at some angle to the geometric axis as well. This is illustrated in Figure 10.2.



**FIGURE 10.1**  
Statically imbalanced projectile.



**FIGURE 10.2**  
Dynamically imbalanced projectile.



**FIGURE 10.3**

Center of gravity (CG) offset viewed from rear of projectile.

Whether a projectile is statically or dynamically imbalanced, we shall define the plane in which the CG offset is located relative to some reference plane (we shall arbitrarily use the  $x-y$  plane as the reference, which we have defined in earlier sections) using the symbol  $\Phi$ . This is illustrated in Figure 10.3 as viewed from the rear of the projectile.

The effect of these mass asymmetries on projectile flight can dramatically affect accuracy, especially in direct fire systems. Consider a projectile with an imbalance in the gun tube. While in the tube, the projectile is constrained to rotate about the tube geometric axis. If we idealize this situation to say that the tube is perfectly straight, inflexible, and fits the projectile snugly, we can further state that the projectile is constrained to rotate about its own geometric axis. Note that there is a wealth of literature dedicated to the real situation, e.g., [1–10].

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# 11

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## *Lateral Throwoff*

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Earlier in the text, we stated that projectiles rarely leave the tube with their velocity vectors aligned with the geometric axis of the gun tube. Chapters 11 and 12 describe this behavior. The result of this behavior is weapon inaccuracy and it must be well understood by the practicing ballistian because, although it is not practical to completely eliminate the behavior, we would like to reduce it to acceptable levels. The first component of this behavior is known as lateral throwoff. It is a dynamic response of the projectile to either a static or a dynamic imbalance and will now be described in detail.

If we imagine a projectile with a mass asymmetry as depicted in [Figure 10.3](#), we can imagine the spinning motion as viewed from the rear. If we ignore the axial velocity by simply spinning the projectile at a high rate, say, between two flexible supports on a test stand, we would see a wobble develop as a result of the centrifugal action on the center of mass. All the time the projectile is being spun up in the gun, the tube walls and stiffness of the supporting members prevent this wobble (to the extent the clearances allow) from developing. At the instant, the projectile is free from the constraints of the tube we expect it to become affected by this centrifugal loading. This is lateral throwoff because the effect is to fling the projectile in a direction off the tube centerline.

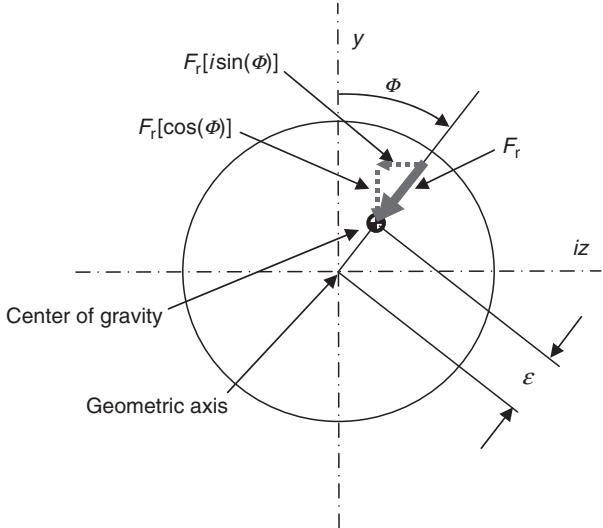
We can use the analogy of a vacuum trajectory to examine the lateral throwoff effect generated by either a static or a dynamic imbalance. Consider the projectile asymmetry from [Figure 10.3](#). If we examine the projectile over a short period of flight, ignoring gravity as well as assuming no drag because of the vacuum assumption, we would see the dynamic forces acting on the projectile as depicted in [Figure 11.1](#). In this figure, the only force acting is the centrifugal force due to spin. This dynamic action will result in the force vector changing direction, though since there is no angular acceleration or deceleration it maintains a constant magnitude. It is worth noting that we have resorted to our complex plane in this example as it is convenient to use in our development. At the instant, in time depicted here, we can break the force into a component in the  $y$ -direction and one in the  $iz$ -direction.

We are not necessarily concerned with the force acting on the CG per se. We want to see where the projectile moves because of this force. To accomplish this, we need to use Newton's second law. We know that

$$F_r = ma_r \quad (11.1)$$

This is the centripetal force. The centrifugal force would be equal but opposite in sign. From dynamics [1], we recall that

$$a_r = -rp^2 \quad (11.2)$$



**FIGURE 11.1**

Dynamic force acting on a statically or dynamically imbalanced projectile.

In the case we are considering here, we see that

$$r = \varepsilon \text{ and } \Phi = pt \quad (11.3)$$

With this, we can write the magnitude of the force as

$$F_r = m\varepsilon p^2 \quad (11.4)$$

and the centripetal acceleration in the complex plane as

$$a = -\frac{F_r}{m} [\cos(pt) + i \sin(pt)] = -\varepsilon p^2 [\cos(pt) + i \sin(pt)] \quad (11.5)$$

The complex velocity can therefore be expressed as

$$V = -\varepsilon p^2 \int_0^t [\cos(pt) + i \sin(pt)] dt \quad (11.6)$$

Evaluating the integral and assuming that as the projectile leaves the muzzle we have an initial orientation of the mass asymmetry of  $\Phi = \Phi_0$  yields

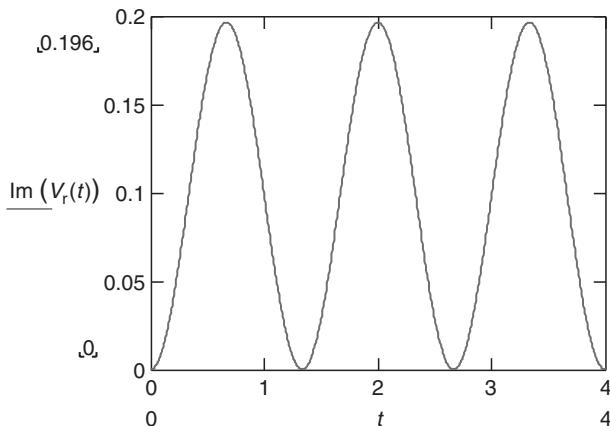
$$V = -\varepsilon p [\sin(pt + \Phi_0) - i \cos(pt + \Phi_0)] = \varepsilon p [-\sin(pt + \Phi_0) + i \cos(pt + \Phi_0)] \quad (11.7)$$

To see how much lateral movement has developed, we can integrate again

$$r = \varepsilon p \int_0^t [-\sin(pt + \Phi_0) + i \cos(pt + \Phi_0)] dt \quad (11.8)$$

The evaluation of which yields

$$r = \varepsilon [\cos(pt + \Phi_0) + i \sin(pt + \Phi_0)] \quad (11.9)$$



**FIGURE 11.2**

Velocity in the  $z$ -direction of a 100-lbm projectile spinning at 270 Hz with a 0.25-in. CG offset.

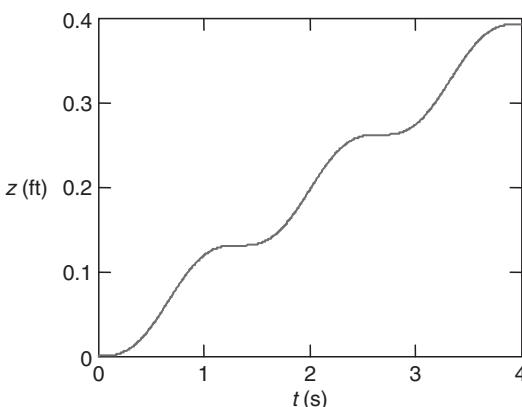
As an example, if we were only concerned with motion in the crossrange direction, we could state

$$z = \text{Im}\{\varepsilon[\cos(pt + \Phi_0) + i\sin(pt + \Phi_0)]\} = \varepsilon \sin(pt + \Phi_0) \quad (11.10)$$

To apply numbers to this example, let us consider a projectile that weighs 100 lbm and is spinning at a rate of 270 Hz. We shall assume the projectile has a CG offset of 0.25 in. If this were the case, the velocity in the  $z$ -direction as well as the motion for the first 4 s of flight can be seen in Figures 11.2 and 11.3. Here we have assumed that the CG offset has emerged from the weapon at the twelve o'clock position.

The most interesting observation between the figures is that for this arbitrary emergence of the CG offset, we see that the projectile would like to move laterally to the right for a right-hand spin. This is commonly known as drift. Just to put things into perspective, the muzzle velocity consistent with the 270-Hz spin rate is about 2,750 ft/s so the projectile would only have gone about 0.4 ft to the right after it traversed 11,000 ft downrange.

We must always bear in mind that this example was an idealized situation. In the case of a real projectile, there are other forces acting which complicate the motion; however, it is instructive to look at simplifications such as this to see the phenomenon at work. We will now move on to examine the dynamic behavior in terms of the equations of motion of a projectile from statically imbalanced and dynamically imbalanced projectiles. We shall see how this affects lateral throwoff.



**FIGURE 11.3**

Displacement in the  $z$ -direction of a 100-lbm projectile spinning at 270 Hz with a 0.25-in. CG offset.

## 11.1 Static Imbalance

In [Figure 10.1](#), we saw the effect on the principal axis of a static imbalance. Although this rarely happens in production (imbalances are usually of the dynamic type) it can happen and presents an interesting case. We shall follow the analysis procedure documented by McCoy [2] in the development, correcting terms to fit our coordinate system.

If we examine the velocity of the center of mass of the projectile as it leaves the gun tube, we see a scene as depicted in [Figure 11.4](#). If the projectile is constrained as it continues down the gun tube, the motion of the CG would resemble a spiral or helix similar to a thread on a bolt, except that the pitch of the helix would continue to increase as the axial velocity increases. We could express this mathematically using a cylindrical set of coordinates with  $x$  indicating the axial distance,  $r$  indicating the radius of the CG from the centerline, and  $\Phi$  indicating the angular position from the vertical plane. If we assume the tube is straight, then the axial component of the velocity vector will be constrained along the tube and our unit vector,  $l$ , will describe the direction adequately. We shall use the unit vectors  $e_r$  and  $e_\Phi$  to represent the radial and angular positions, respectively. If we use our instantaneous spin rate,  $p$ , as defined in [Equation 11.3](#), we can write the tangential component of velocity as

$$V_\Phi = rpe_\Phi = \varepsilon pe_\Phi \quad (11.11)$$

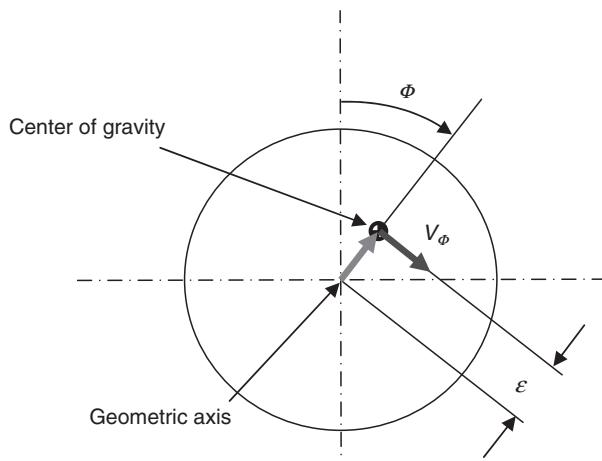
The axial velocity is simply

$$V_x = Vl \quad (11.12)$$

Then the velocity vector could be written in cylindrical coordinates as

$$\mathbf{V} = Vl + \varepsilon pe_\Phi \quad (11.13)$$

Or, if we like to remain in Cartesian coordinates, we can combine [Equation 11.13](#) with [Equation 11.7](#) to yield



**FIGURE 11.4**  
Velocity of a statically imbalanced projectile's CG.

$$\mathbf{V} = V\mathbf{i} + \varepsilon p(-\sin(\phi_0)\mathbf{n} + i \cos(\phi_0)\mathbf{m}) \quad (11.14)$$

These Cartesian coordinates are useful when we want to write the velocity vector at the muzzle of the weapon. The lateral throwoff caused by a static imbalance can be described as the tangent of the angle of the projectile CG as it exits. For small angles (usually the case), this is approximately the angle itself in radians. With this, we can define the lateral throwoff at the muzzle owing to a static imbalance as

$$T_L = \frac{\varepsilon p_0}{V_0} [-\sin(\phi_0) + i \cos(\phi_0)] \quad (11.15)$$

where we have used  $t=0$  at the muzzle and specified the spin rate and muzzle velocity. We must keep in mind that this is an angular measure for small angles or, more precisely, a tangent of an angle. We can use the relationship

$$ie^{i\theta} = -\sin\theta + i \cos\theta \quad (11.16)$$

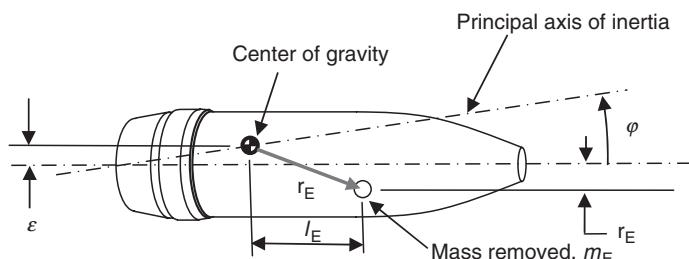
to write

$$T_L = i \frac{\varepsilon p_0}{V_0} e^{i\phi_0} \quad (11.17)$$

If the projectile has a rotating band that forces it to spin based on the rifling twist, this expression can be written in terms of the projectile diameter and twist rate as well. This is extremely straightforward and left as an exercise for the reader.

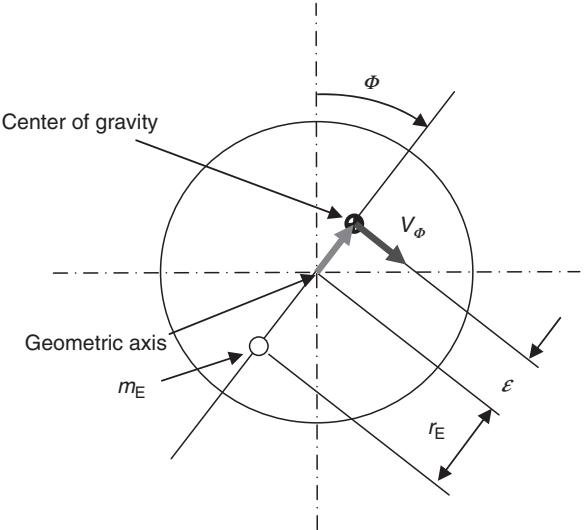
## 11.2 Dynamic Imbalance

The diagram of Figure 10.2 represents the most common case of a projectile asymmetry, a dynamic imbalance. In this case, a lateral throwoff effect as described in Section 11.1 will result as the projectile leaves the muzzle of the gun and there will also be significant flight dynamic effects as the projectile moves downrange. Usually, this mass asymmetry is small and can be treated as a small amount of mass removed from or added to a projectile at a point defined by a radial set of coordinates from the CG. We shall use the former approach following the development of Ref. [2]. This is depicted in Figure 11.5. Figure 11.6 depicts how this removed mass is oriented relative to the CG offset in the radial direction.



**FIGURE 11.5**

Dynamically imbalanced projectile with mass removed.



**FIGURE 11.6**  
Velocity of a dynamically imbalanced projectile's CG.

The development put forward in Ref. [2] assumes that the removed mass is much smaller than the overall mass of the projectile. As established earlier, we will use two orthogonal coordinate systems. The first is our  $i-j-k$  triad which is oriented along the projectile axis as depicted in Figure 9.1. This coordinate system does not roll with the projectile. We shall also make use of a second non-rolling coordinate system using the  $l-n-m$  system depicted in the same figure. In this case, the coordinate system is oriented along the velocity vector. The coordinate systems are related to one another, assuming small yaw angles, through the relationships

$$\mathbf{i} = \gamma \mathbf{l} + \alpha \mathbf{n} + \beta \mathbf{m} \quad (11.18)$$

$$\mathbf{j} = -\alpha \mathbf{l} + \mathbf{n} \quad (11.19)$$

$$\mathbf{k} = -\beta \mathbf{l} + \mathbf{m} \quad (11.20)$$

where  $\alpha$  is the pitch angle,  $\beta$  is the yaw angle, and  $\gamma$  is defined as

$$\gamma = \cos \alpha \cos \beta \approx 1 \quad (11.21)$$

The angular momentum of the projectile is the vector sum of all of the angular momenta and is closely approximated by

$$\mathbf{H} = I_{Pp}\mathbf{i} + I_T \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) - m_E(\mathbf{r}_E \times \mathbf{v}_E) \quad (11.22)$$

Here  $\mathbf{H}$  is the total angular momentum and  $\mathbf{v}_E$  is the velocity of the removed mass. This velocity can be broken into two components, one owing to the rotation about the spin axis and the other owing to the yawing motion of the projectile as follows:

$$\mathbf{v}_E = p(\mathbf{i} \times \mathbf{r}_E) + \left[ \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) \times \mathbf{r}_E \right] \quad (11.23)$$

Then inserting this relationship into [Equation 11.22](#) and combining terms gives us, after utilization of the vector triple product

$$\begin{aligned}\mathbf{H} &= (I_P - m_E r_E^2)p\mathbf{i} + I_T \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) - m_E \\ &\quad \times \left[ -p(\mathbf{r}_E \cdot \mathbf{i})\mathbf{r}_E - \left( \mathbf{r}_E \cdot \frac{d\mathbf{i}}{dt} \right)(\mathbf{r}_E \times \mathbf{i}) + (\mathbf{r}_E \cdot \mathbf{i}) \left( \mathbf{r}_E \times \frac{d\mathbf{i}}{dt} \right) \right]\end{aligned}\quad (11.24)$$

If we examine [Figure 11.5](#), we see that

$$(\mathbf{r}_E \cdot \mathbf{i}) = l_E \quad (11.25)$$

And we note that for a spin-stabilized projectile, the yaw rate,  $di/dt$ , is much smaller than the spin rate,  $p$ , we can eliminate terms in Equation 11.24 to yield

$$\mathbf{H} \approx I_P p \mathbf{i} + I_T \left( \mathbf{i} \times \frac{d\mathbf{i}}{dt} \right) + m_E p l_E \mathbf{r}_E \quad (11.26)$$

We can express the mass asymmetry vector  $\mathbf{r}_E$  in terms of the projectile geometric axes as

$$\mathbf{r}_E = l_E \mathbf{i} + r_E \cos \Phi \mathbf{j} + r_E \sin \Phi \mathbf{k} \quad (11.27)$$

This can be expressed in our coordinate system attached to the velocity vector through the relationships in [Equations 11.18](#) through [11.20](#) as

$$\mathbf{r}_E = (l_E \gamma - r_E \alpha \cos \Phi - r_E \beta \sin \Phi) \mathbf{l} + (l_E \alpha + r_E \cos \Phi) \mathbf{n} + (l_E \beta + r_E \sin \Phi) \mathbf{m} \quad (11.28)$$

We can simplify this expression somewhat if we use the fact that both  $\alpha$  and  $\beta$  are much smaller than  $\gamma$ . In this case, the expression would simplify to

$$\mathbf{r}_E = (l_E \gamma) \mathbf{l} + (l_E \alpha + r_E \cos \Phi) \mathbf{n} + (l_E \beta + r_E \sin \Phi) \mathbf{m} \quad (11.29)$$

We can take the derivative of Equation 11.29 using the fact that the coordinate system is effectively not rotating to write

$$\frac{d\mathbf{r}_E}{dt} = (l_E \dot{\gamma}) \mathbf{l} + (l_E \dot{\alpha} - r_E p \sin \Phi) \mathbf{n} + (l_E \dot{\beta} + r_E p \cos \Phi) \mathbf{m} \quad (11.30)$$

Here we have used the fact that

$$\frac{d\Phi}{dt} = p \quad (11.31)$$

As in our previous analyses, we shall consider a short period of flight. By doing this, we can neglect all forces and moments except the pitching (overturning) moment. This allows us to equate the rate of change of angular momentum to the applied pitching moment

$$\frac{d\mathbf{H}}{dt} \approx I_P p \frac{d\mathbf{i}}{dt} + I_T \left( \mathbf{i} \times \frac{d^2\mathbf{i}}{dt^2} \right) + m_E p l_E \frac{d\mathbf{r}_E}{dt} = m C_{M_\alpha}^* V^2 (\mathbf{l} \times \mathbf{i}) \quad (11.32)$$

This expression can be written as a set of three equations in terms of each component as follows:

$$I_P p \dot{\gamma} + I_T \left( \alpha \frac{d^2 \beta}{dt^2} - \beta \frac{d^2 \alpha}{dt^2} \right) + m_E p l_E^2 \dot{\gamma} = 0 \quad (11.33)$$

$$I_P p \dot{\alpha} + I_T \left( \beta \frac{d^2 \gamma}{dt^2} - \gamma \frac{d^2 \beta}{dt^2} \right) + m_E p l_E (l_E \dot{\alpha} - r_E p \sin \Phi) = -m C_{M_\alpha}^* V^2 \beta \quad (11.34)$$

$$I_P p \dot{\beta} + I_T \left( \gamma \frac{d^2 \alpha}{dt^2} - \alpha \frac{d^2 \gamma}{dt^2} \right) + m_E p l_E (l_E \dot{\beta} + r_E p \cos \Phi) = m C_{M_\alpha}^* V^2 \alpha \quad (11.35)$$

The details of this are provided in Ref. [2]. If we change the temporal derivatives into spatial derivatives along a dimensionless downrange distance,  $s$ , and define the following:

$$P = \left( \frac{I_P}{I_T} \right) \left( \frac{pd}{V} \right) \quad (11.36)$$

$$M = \frac{md^2}{I_T} C_{M_\alpha}^* \quad (11.37)$$

$$I_E = m_E r_E l_E \quad (11.38)$$

We can rewrite Equations 11.33 through 11.35 as

$$P \left( 1 + \frac{m_E l_E^2}{I_T} \right) \gamma' + \alpha \beta'' - \beta \alpha'' = 0 \quad (11.39)$$

$$P \left( 1 + \frac{m_E l_E^2}{I_T} \right) \alpha' + \beta \gamma'' - \gamma \beta'' + M \beta - \left( \frac{I_E I_T}{I_P^2} \right) P^2 \sin \Phi = 0 \quad (11.40)$$

$$P \left( 1 + \frac{m_E l_E^2}{I_T} \right) \beta' + \gamma \alpha'' - \alpha \gamma'' - M \alpha + \left( \frac{I_E I_T}{I_P^2} \right) P^2 \cos \Phi = 0 \quad (11.41)$$

Here the primed quantities are differentiated with respect to  $s$ . With small yaw as well as classical size assumptions (see Refs. [2,3]), we can neglect several of these terms because they are either products of small numbers or summed with a much larger number. This results in Equation 11.39 vanishing altogether and the other two transforming into

$$P \alpha' - \beta'' + M \beta = \left( \frac{I_E I_T}{I_P^2} \right) P^2 \sin \Phi \quad (11.42)$$

$$P \beta' + \alpha'' - M \alpha = - \left( \frac{I_E I_T}{I_P^2} \right) P^2 \cos \Phi \quad (11.43)$$

If we now multiply Equation 11.42 by  $-i$  and add it to Equation 11.43, we obtain

$$(\alpha'' + i \beta'') + P(\beta' - i \alpha') - M(\alpha + i \beta) = - \left( \frac{I_E I_T}{I_P^2} \right) P^2 (\cos \Phi + i \sin \Phi) \quad (11.44)$$

If we invoke our definition of complex yaw angle, we can write this as

$$\xi'' - iP\xi' - M\xi = \left( \frac{I_E I_T}{I_P^2} \right) P^2 e^{i\Phi} \quad (11.45)$$

The solution to this differential equation was discussed in Section 9.1. The difference here is that the forcing term on the RHS is somewhat different. If we use a solution written as

$$\xi = K_1 e^{i\theta_1} + K_2 e^{i\theta_2} + K_4 e^{i\Phi} \quad (11.46)$$

where  $K_1$  and  $K_2$  are the solutions to the homogeneous part of the equation and  $K_4$  is our new term which depends on the spin rate and the mass asymmetry we can solve for the magnitude of the trim arm caused by the asymmetry. If we solve Equation 11.46 for the particular solution, we find that this new trim arm caused by the mass asymmetry is given by

$$K_4 = \frac{I_E}{I_T - I_P + \frac{I_P^2 M}{I_T P}} \quad (11.47)$$

The third term in the denominator is usually very small so this term has been approximated (see Refs. [2,4]) as

$$K_4 \approx \frac{I_E}{I_T - I_P} \quad (11.48)$$

This trim arm due to a mass asymmetry is usually small.

Throughout this development,  $I_P$  and  $I_T$  have been used as the moments of inertia even though, in the purest sense, the mass asymmetry removes the axially symmetric properties of the projectile. For most cases, it is sufficient to use these quantities based on an axially symmetric projectile.

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# 12

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## Swerve Motion

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Following our procedure of slowly introducing complexity into the description of projectile behavior we shall now develop equations to characterize the remainder of what is known in general as swerve motion. We saw in Chapter 11 that a mass asymmetry can cause projectile motion transverse to the original line of fire even in a vacuum. We stated in that section that a dynamic projectile imbalance was more common than a static imbalance but either can actually occur.

Chapter 6 explained many aspects of projectile behavior that arise due to the presence of the air stream. All of the coefficients were functions of the angle of the attack observed by the projectile relative to that air stream. If we examine how a statically or dynamically imbalanced projectile would behave as viewed from above the trajectory curve based on its spin, we would see motion as depicted in Figures 12.1 and 12.2. We must keep in mind that the motion in these figures is greatly exaggerated for ease of viewing.

We can imagine, by looking at these figures that the aerodynamic forces would be considerable because even in the case of the statically imbalanced projectile, motion laterally across the trajectory will manifest itself in an angle of attack and therefore affect the flight characteristics.

In this section, we shall describe and evaluate the aerodynamic forces that arise from this behavior and include them in our equations of motion for projectile flight. We shall also include the effect of configurational asymmetries such as bent fins or damaged form because these will result in similar behavior even without the mass asymmetry present. In fact, to a varying degree, every projectile has a combination of both form and mass asymmetries present.

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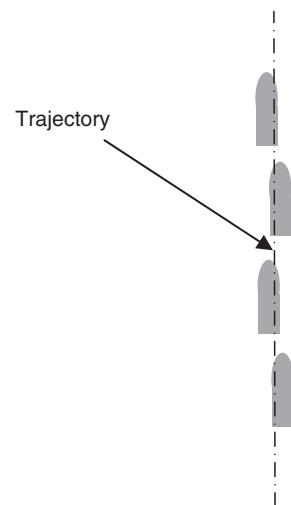
### 12.1 Aerodynamic Jump

McCoy [1] has shown that the equation of motion for the point mass solution plus swerving motion is given by

$$\frac{d^2y}{ds^2} + i \frac{d^2z}{ds^2} = C_L \alpha^* \xi - \frac{gd}{V_0^2} \exp(2C_D^* s) \quad (12.1)$$

with a solution of

$$y + iz = (y_0 + iz_0) + \left( \frac{dy}{ds} \Big|_0 + i \frac{dz}{ds} \Big|_0 \right) s + C_{L_a}^* I_L - \frac{s^2 gd}{2V_0^2} \left[ \frac{\exp(2C_D^* s) - 2C_D^* s - 1}{2(C_D^* s)^2} \right] \quad (12.2)$$



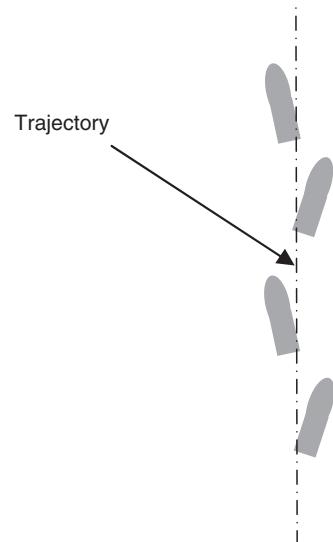
**FIGURE 12.1**  
Motion of a statically imbalanced projectile.

where

$$I_L = \int_0^s \int_0^{s_1} \xi ds_1 ds_2 \quad (12.3)$$

Here we have used  $s_1$  and  $s_2$  as dummy variables representing integrations with respect to  $s$ . [Equation 12.2](#) describes the position of the projectile in a direction perpendicular to the trajectory based on flat fire point mass assumptions.

[Equation 9.72](#) was developed as a solution for  $\xi$ . Reference [1] has shown that the solution to the double integral of Equation 12.3 can be obtained by substitution of Equation 9.72 into Equation 12.3 resulting in



**FIGURE 12.2**  
Motion of a dynamically imbalanced projectile.

$$I_L = - \left[ \left( \frac{\lambda_1 - i\psi'_1}{\lambda_1^2 + \psi_1'^2} \right) K_{10} e^{i\psi_{10}} + \left( \frac{\lambda_2 - i\psi'_2}{\lambda_2^2 + \psi_2'^2} \right) K_{20} e^{i\psi_{20}} \right] s + (R_{11} - iR_{12}) K_{10} e^{i\psi_{10}} \left\{ \exp \left[ (\lambda_1 + i\psi'_1)s \right] - 1 \right\} + (R_{21} - iR_{22}) K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2)s \right] - 1 \right\} + i \frac{PG_0}{M} s^2 \frac{\left[ \exp(2C_D^* s) - 2C_D^* s - 1 \right]}{(2C_D^* s)^2} \quad (12.4)$$

Here we have used

$$G_0 = \frac{gd}{V_0^2} \quad (12.5)$$

$$R_{11} = \frac{\lambda_1^2 - \psi_1'^2}{(\lambda_1^2 + \psi_1'^2)^2} \quad (12.6)$$

$$R_{12} = \frac{2\lambda_1 \psi'_1}{(\lambda_1^2 + \psi_1'^2)^2} \quad (12.7)$$

$$R_{21} = \frac{\lambda_2^2 - \psi_2'^2}{(\lambda_2^2 + \psi_2'^2)^2} \quad (12.8)$$

$$R_{22} = \frac{2\lambda_2 \psi'_2}{(\lambda_2^2 + \psi_2'^2)^2} \quad (12.9)$$

If we make the assumption that  $\lambda_{1,2}^2 \ll \psi_{1,2}'^2$ , the above parameters become

$$R_{11} \approx -\frac{1}{\psi_1'^2}, \quad R_{12} \approx 0, \quad R_{21} \approx -\frac{1}{\psi_2'^2}, \quad \text{and} \quad R_{22} \approx 0 \quad (12.10)$$

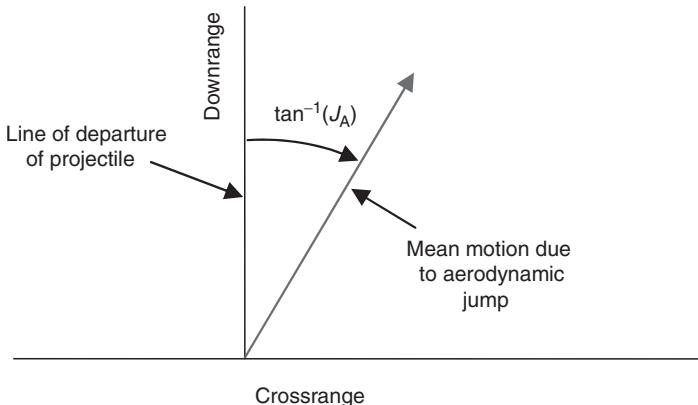
Inserting these assumptions into Equation 12.4 yields the following result:

$$I_L = i \left( \frac{1}{\psi_1'} K_{10} e^{i\psi_{10}} + \frac{1}{\psi_2'} K_{20} e^{i\psi_{20}} \right) s - \frac{1}{\psi_1'^2} K_{10} e^{i\psi_{10}} \left\{ \exp \left[ (\lambda_1 + i\psi'_1)s \right] - 1 \right\} - \frac{1}{\psi_2'^2} K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2)s \right] - 1 \right\} + i \frac{PG_0}{M} s^2 \frac{\left[ \exp(2C_D^* s) - 2C_D^* s - 1 \right]}{(2C_D^* s)^2} \quad (12.11)$$

This result is important because it depicts the three components of swerve motion. The first term on the RHS is called the aerodynamic jump,  $J_A$ , and it is what we will examine for the remainder of this section. The second two terms are the epicyclic swerve,  $S_E$ , and will be discussed in Section 12.2. The third term is called drift,  $D_R$ , and will be discussed in Section 12.3. To keep things simple, we will restate the aerodynamic jump as

$$J_A = i C_{L_\alpha}^* \left( \frac{1}{\psi_1'} K_{10} e^{i\psi_{10}} + \frac{1}{\psi_2'} K_{20} e^{i\psi_{20}} \right) \quad (12.12)$$

We should note a few things about Equation 12.12. First, we must keep in mind that in Equation 12.11 this aerodynamic jump term is multiplied by a downrange distance,  $s$ ,



**FIGURE 12.3**  
Graphical representation of aerodynamic jump.

implying that it is actually an angular measure (for small angles). A second observation is that the aerodynamic jump is completely dependent upon the initial conditions of the projectile and how these couple in with the fast and slow arm turning rates.

If we insert our approximated initial fast and slow arm amplitudes from Equations 9.78 and 9.79 into Equation 12.12, we obtain

$$J_A = iC_{L_\alpha}^* \left[ \frac{-i\xi'_0 - \psi'_2 \xi'_0}{\psi'_1 (\psi'_1 - \psi'_2)} + \frac{i\xi'_0 + \psi'_1 \xi'_0}{\psi'_2 (\psi'_1 - \psi'_2)} \right] \quad (12.13)$$

This can be rewritten [1] as

$$J_A = k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) (iP\xi_0 - \xi'_0) \quad (12.14)$$

This result shows that by knowing the projectile mass properties and launch conditions, we can determine to what angle a projectile will “jump.” We can envision this jump effect as shown in Figure 12.3.

While we have said a great deal mathematically about aerodynamic jump, we have not really described the physics behind it. Because of the presence of aerodynamic lift on the projectile, there is a strong influence of angle of attack on the resultant motion. We saw earlier that a projectile, through purely dynamic means, can yaw because of either spin or some geometric asymmetry. When this happens, the aerodynamic forces change, either improving or worsening the situation. This interaction of the aerodynamic forces with the projectile manifests itself in the jump angle as depicted in Figure 12.3.

## 12.2 Epicyclic Swerve

The second two terms in Equation 12.11 describe the epicyclic swerve of a projectile. We can define this parameter specifically [1] as

$$S_E = -C_{L_\alpha}^* \left( \frac{1}{\psi_1'^2} K_{10} e^{i\psi_{10}} \left\{ \exp \left[ (\lambda_1 + i\psi'_1) s \right] - 1 \right\} + \frac{1}{\psi_2'^2} K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2) s \right] - 1 \right\} \right) \quad (12.15)$$

McCoy [1] has shown that this equation can be put into a more useful form through use of the relation

$$C_{L_\alpha}^* = k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) \psi'_1 \psi'_2 \quad (12.16)$$

If we insert Equation 12.16 into Equation 12.15, we obtain

$$S_E = -k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) \left( \frac{\psi'_1}{\psi'_2} K_{10} e^{i\psi_{10}} \left\{ \exp \left[ (\lambda_1 + i\psi'_1) s \right] - 1 \right\} + \frac{\psi'_1}{\psi'_2} K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2) s \right] - 1 \right\} \right) \quad (12.17)$$

Following McCoy, we shall examine two special cases of this equation. The first is where we have a projectile that is non-spinning (statically stable) and the second is a spin-stabilized projectile with a good gyroscopic stability (measured at muzzle exit) of at least 1.5.

For the non-spinning projectile, the following conditions apply:

$$M < 0, P = 0, \text{ and } \psi'_2 = -\psi'_1 \quad (12.18)$$

If we insert these conditions into Equation 12.17, we get

$$S_{E_{\text{non-spin}}} = k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) \left( K_{10} e^{i\psi_{10}} \left\{ \exp \left[ (\lambda_1 + i\psi'_1) s \right] - 1 \right\} + K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2) s \right] - 1 \right\} \right) \quad (12.19)$$

Now we can invoke the fact that the spin is equal to zero and insert [Equation 9.59](#), in which we shall neglect the trim and yaw of repose, into Equation 12.19 to yield

$$S_{E_{\text{non-spin}}} = k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) (\xi - \xi_0) \quad (12.20)$$

This relationship will produce a motion in exactly the same manner as the aerodynamic jump developed in [Section 12.1](#). It essentially couples the yawing motion of the projectile to the swerving motion. Both will thus damp together and the more yaw, the greater the epicyclic swerve.

If we examine the spin-stabilized projectile, we can write

$$M > 0 \text{ and } \psi'_1^2 \gg \psi'_2^2 \quad (12.21)$$

With the above mathematical statements, McCoy [1] has stated that an excellent approximation of Equation 12.15 for a spinning projectile is

$$S_{E_{\text{spin}}} = -k_T^2 \left( \frac{C_{L_\alpha}}{C_{M_\alpha}} \right) \left( \frac{\psi'_1}{\psi'_2} \right) \left( K_{20} e^{i\psi_{20}} \left\{ \exp \left[ (\lambda_2 + i\psi'_2) s \right] - 1 \right\} \right) \quad (12.22)$$

An interesting comparison may be drawn between the epicyclic swerving behavior of a spinning projectile and a non-spinning projectile. If we compare Equations 12.22 and 12.20,

we see that in the latter the yawing motion and swerve are locked together and operate in the same fixed plane. This is because the lift generated by the motion never rotates. In a spin-stabilized projectile, the lift vector is always rotating, thus the center of mass of the projectile will move in a helical manner around the flight path. Furthermore, the motion will be locked to the rate of turning of the slow arm and will damp or increase as the slow arm does.

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### 12.3 Drift

The last term in [Equation 12.11](#) describes the drift of a projectile. We can define this parameter specifically [1] as

$$D_R = i \frac{PG_0}{M} s^2 \left[ \frac{\exp(2C_D^* s) - 2C_D^* s - 1}{(2C_D^* s)^2} \right] \quad (12.23)$$

If we expand the term in brackets in a power series, we can rewrite this equation as

$$D_R = i \left( \frac{PG_0}{M} \right) s^2 \left[ 1 + \frac{2}{3} (C_D^* s) + \frac{1}{3} (C_D^* s)^2 + \dots \right] \quad (12.24)$$

Examination of the drift equation in this form has some advantages. First, we can see that if a projectile has no spin,  $P=0$  and there is no drift. If we look at a fin- or drag-stabilized projectile where  $M < 0$ , we see that the projectile will drift in the direction opposite to the spin. That is, a left-hand spin will produce a right-hand drift and vice versa. In a statically unstable (spin-stabilized) projectile where  $M > 0$ , we see that the projectile will drift in the same direction as the spin. It must be noted that this drift is very small compared to the other swerve components as well as Coriolis drift. In fact, to even measure it, some researchers [1] have fired two projectiles simultaneously out of side-by-side gun barrels with both left- and right-hand twist to remove Coriolis and wind drift components which would affect each equally.

The interested reader should consult Ref. [1] for further information on this topic.

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# 13

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## *Nonlinear Aeroballistics*

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Until this point we have concerned ourselves with linear behavior of the aerodynamic coefficients only. This is very convenient for direct fire projectiles and projectiles which fly with very little yaw. It had the benefit of allowing us to make a black or white decision with regard to projectile stability as well—the projectile was either stable or not. In real systems, several of the coefficients are only linear over a small range of angles of attack. This can be either helpful or hurtful to a particular design.

Limit-cycle motion is motion that develops over time in a projectile, whereby the projectiles angle of attack grows until a certain (sometimes rather large) angle is achieved. As the angle of attack increases (or some other parameter such as the air density changes), the coefficients change so that the projectile will actually become stable at some large angle of attack. At first, this may seem like it is a desirable quality in a projectile; however, range is sacrificed due to the larger drag generally associated with this large yaw. Some systems have been fielded unwittingly in this condition and it was only after a large number of firings in the field that this was determined to be an issue.

This nonlinear behavior arises out of the interaction between the air and the surfaces of the projectile. It is a rather complicated mechanism that can arise (many times in a discontinuous manner) from boundary layer separation, fin masking, vortex shedding, etc. All of which are fluid dynamic phenomenon. This is and continues to be a challenging area of aeroballistic research, where experimental, theoretical, and computational techniques are pushed to the limit of their usefulness.

The next two sections will look at this behavior to some degree of detail; however, because of space constraints, the reader is encouraged to consult the literature for more detailed mathematical and theoretical treatment.

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### **13.1 Nonlinear Forces and Moments**

In general, we can divide nonlinear forces and moments into two categories: geometric and aerodynamic nonlinearities. The geometric nonlinearities arise from the cosine terms in the equations of motion that were eliminated when we assumed a small yaw angle. This small angle assumption is generally valid for most projectiles in flight. If a projectile is flying with large yaw, the cosine terms must be retained and the resulting equations are more difficult to solve. Since this behavior is usually designed out of projectiles, we shall focus on the second type of nonlinearity, the aerodynamic nonlinearity.

The aerodynamic nonlinearity can exist even at angles of attack that are consistent with the small yaw assumption. They arise due to the fluid-mechanic interaction of the air with the solid projectile body. This interaction can consist of phenomena such as vortex shedding, separation, shock interactions, etc.

The most dominant force acting on the body is the drag force. In all of our previous discussions, we have stated that the forces that arise due to other sources are small and that is still true for the case of nonlinearities; however, the moments caused by these other forces cannot be neglected. We can define a nonlinear drag coefficient as

$$C_D = C_{D_0} + C_{D_{\delta^2}} \delta^2 + \dots \quad (13.1)$$

In this equation, the first term on the RHS is the zero-yaw-drag coefficient and the second term is the cubic-drag coefficient. More coefficients can be added but typically the expression is truncated at the first term.

There are essentially two common ways of determining the cubic coefficient: experimentally or computationally. Experimental evaluation is more common although recent advances in computational fluid dynamics (CFD) [1] have shown that it is possible to extract coefficients directly from analyses. In either case, the overall drag coefficient at multiple angles of attack is determined from either a direct force measurement (in the case of a wind tunnel or CFD model) or the velocity decay (in a free flight test), and the results are plotted as  $C_D$  versus angle of attack. The slope of the resulting line (hopefully it is a line) is then the cubic-drag coefficient and the  $y$ -intercept is the zero-yaw-drag coefficient. In the case of a free flight firing where the projectile is dragging down continuously, Murphy [2] and McCoy [3] suggest an averaging scheme that has been successfully demonstrated based on a great deal of experience.

The above technique is known as a quasi-linear approach because it defines a linear function that is a solution to a nonlinear equation. The same approach is used to determine the nonlinear moments, which are generally assumed to have the same form as Equation 13.1.

In general, both the zero-yaw-drag coefficient and the cubic-drag coefficient are positive values. In the case of the pitching or overturning moment of a spin-stabilized projectile, the zero-yaw overturning moment coefficient is positive while the cubic overturning moment coefficient is negative [3]. This condition can have some interesting effects on stability as summarized by McCoy [3].

The overall equation of motion that includes all of the nonlinear terms that is equivalent to our linear equation ([Equation 9.76](#)) with the gravitational term neglected is

$$\xi'' + (H_0 + H_2 \delta^2 - iP) \xi' - [M_0 + M_2 \delta^2 + iP(T_0 + T_2 \delta^2)] \xi = 0 \quad (13.2)$$

We can define our coefficients as follows:

$$H_0 = \frac{\rho S d}{2M} \left[ C_{L_{\alpha 0}} - C_{D_0} - \frac{1}{k_T^2} (C_{M_q} + C_{M_{\dot{\alpha}}})_0 \right] \quad (13.3)$$

$$H_2 = \frac{\rho S d}{2M} \left[ C_{L_{\alpha 2}} - C_{D_2} - \frac{1}{k_T^2} (C_{M_q} + C_{M_{\dot{\alpha}}})_2 \right] \quad (13.4)$$

$$M_0 = \frac{\rho S d}{2m} \frac{1}{k_T^2} C_{M_{\alpha 0}} \quad (13.5)$$

$$M_2 = \frac{\rho S d}{2m} \frac{1}{k_T^2} C_{M_{\alpha 2}} \quad (13.6)$$

$$T_0 = \frac{\rho S d}{2m} \left( C_{L_{\alpha_0}} + \frac{1}{k_p^2} C_{M_{p\alpha_0}} \right) \quad (13.7)$$

$$T_2 = \frac{\rho S d}{2m} \left( C_{L_{\alpha_2}} + \frac{1}{k_p^2} C_{M_{p\alpha_2}} \right) \quad (13.8)$$

$$P = \left( \frac{I_p}{I_T} \right) \left( \frac{pd}{V} \right) \quad (13.9)$$

The solution to [Equation 13.2](#) is

$$\xi = K_{10} e^{\lambda_{1s}} \exp \left[ i(\psi_{10} + \psi'_1 s) \right] + K_{20} e^{\lambda_{2s}} \exp \left[ i(\psi_{20} + \psi'_2 s) \right] \quad (13.10)$$

Where again, we are reminded that the gravitational term has been neglected. McCoy [3] has written expressions for the coefficients in terms of the damping exponents and turning rates as follows:

$$\delta_{e1}^2 = K_1^2 + 2K_2^2 \quad (13.11)$$

$$\delta_{e2}^2 = K_2^2 + 2K_1^2 \quad (13.12)$$

$$\psi'_1 + \psi'_2 = P + M_2 \left( \frac{K_1^2 + K_2^2}{\psi'_1 - \psi'_2} \right) \approx P \quad (13.13)$$

$$\delta_e^2 = \frac{\psi'_1 \delta_{e2}^2 - \psi'_2 \delta_{e1}^2}{\psi'_1 - \psi'_2} \quad (13.14)$$

$$\psi'_1 \psi'_2 = M_0 + M_2 \delta_e^2 \quad (13.15)$$

$$\lambda_1 = \frac{-H_0 \psi'_1 + P(T_0 + T_2 \delta_{e1}^2) - H_2 [\psi'_1 (K_1^2 + K_2^2) + \psi'_2 K_2^2]}{\psi'_1 - \psi'_2} \quad (13.16)$$

$$\lambda_2 = \frac{-H_0 \psi'_2 - P(T_0 + T_2 \delta_{e2}^2) + H_2 [\psi'_2 (K_1^2 + K_2^2) + \psi'_1 K_1^2]}{\psi'_1 - \psi'_2} \quad (13.17)$$

In terms of some of these parameters, McCoy [3] has derived a form for the nonlinear lift coefficient as

$$C_{L_a} = C_{L_{\alpha_0}} + C_{L_{\alpha_2}} \left( \frac{\psi'^2_2 \delta_{e1}^2 K_1 e^{i\psi_1} + \psi'^2_1 \delta_{e2}^2 K_2 e^{i\psi_2}}{\psi'^2_2 K_1 e^{i\psi_1} + \psi'^2_1 K_2 e^{i\psi_2}} \right) \quad (13.18)$$

As is readily apparent, these expressions are significantly more complex than their linear cousins. Because of this, they are generally solved using numerical schemes. The interested reader is referred to Refs. [4–12] for a more detailed treatment as well as examples of this behavior.

### Problem 1

If the projectile in [Problem 1](#) of [Chapter 9](#) happens to be flying at a limit-cycle yaw of 4° with a spin rate of 130 Hz and velocity 1764 ft/s. What would the nonlinear pitching moment have to be for the projectile to be marginally stable?

*Hints:*

1. Assume all of the other coefficients are linear.
2. Recall the definition of the nonlinear pitching moment (you have the linear part in [Problem 1 of Chapter 9](#)).

*Answer:*  $C_{M_{\alpha_2}} = -119.079$

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## 13.2 Bilinear and Trilinear Moments

We have discussed nonlinear forces and moments and their implications in the previous section. At this point, we shall turn our attention to nonlinear moments in which the cubic behavior itself can be described by a bilinear or trilinear curve. This is evident when the cubic coefficient is plotted versus yaw angle. A bilinear coefficient would have two different linear slopes, while a trilinear moment would have three. This is quite useful since many experimental data can be fitted using these curves. In particular, we shall examine the Magnus moment and its implications because this is the dominant moment in spin-stabilized projectile flight behavior [3].

If we are examining projectile flight data, it is often tempting to fit a higher order polynomial curve to deal with the nonlinearity. This is usually not advisable since the abrupt changes in behavior at certain angles of attack are caused by fluid–solid interactions such as boundary layer separation, vortex shedding, etc.

To describe the behavior of projectiles with nonlinear Magnus moment coefficients, we shall use two examples: one with a linear cubic Magnus moment and one with a bilinear Magnus moment. We are interested in two things: first, the effect of initial conditions on projectile stability and second, limit-cycle motion.

In the excellent treatment by McCoy [3], for illustrative purposes, the author suggested assuming a linear pitch damping moment with a cubic Magnus moment coefficient. This will force  $H_2$  to be zero and allow [Equations 13.16](#) and [13.17](#) to be written as

$$\lambda_1 = \frac{-H_0\psi'_1 + P(T_0 + T_2\delta_{e1}^2)}{\psi'_1 - \psi'_2} \quad (13.19)$$

$$\lambda_2 = \frac{-H_0\psi'_2 - P(T_0 + T_2\delta_{e2}^2)}{\psi'_1 - \psi'_2} \quad (13.20)$$

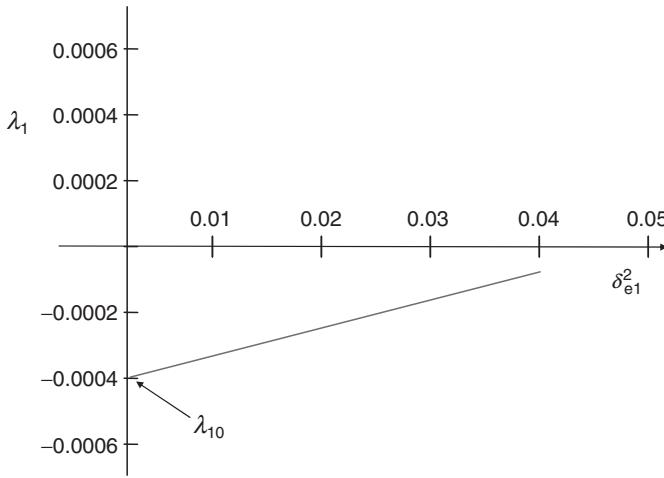
We can put these equations into the form

$$\lambda_1 = \lambda_{1_0} + \lambda_{1_2}\delta_{e1}^2 \quad (13.21)$$

$$\lambda_2 = \lambda_{2_0} + \lambda_{2_2}\delta_{e2}^2 \quad (13.22)$$

where, we can define

$$\lambda_{1_0} = \frac{-H_0\psi'_1 + PT_0}{\psi'_1 - \psi'_2} \quad (13.23)$$



**FIGURE 13.1**

Plot of fast mode damping coefficient versus yaw for linear cubic fast mode.

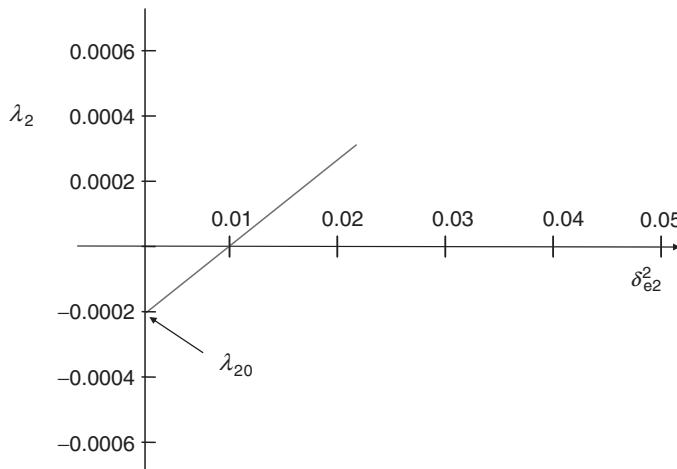
$$\lambda_{2_0} = \frac{-H_0\psi'_2 - PT_0}{\psi'_1 - \psi'_2} \quad (13.24)$$

$$\lambda_{1_2} = -\lambda_{2_2} = \frac{PT_2}{\psi'_1 - \psi'_2} \quad (13.25)$$

With these expressions, we can draw plots of damping coefficients versus yaw angle in a manner similar to the coefficients.

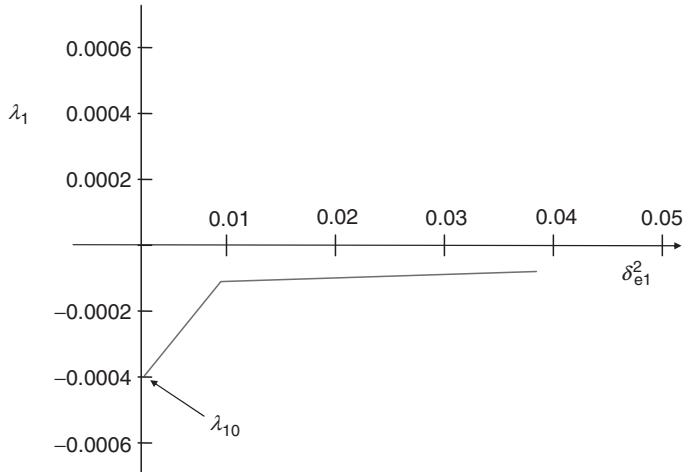
At this juncture, we need to recall that these damping exponents will decrease the yaw of their particular mode if they are negative, and increase the yaw if they are positive. Thus, negative values are stabilizing and positive values are destabilizing. As a simple example, let us look at a projectile that has a linear cubic Magnus moment. In analyzing this projectile, we create two plots of damping coefficient versus yaw. These are depicted as in Figures 13.1 and 13.2.

In Figure 13.1, we can see the fast mode damping coefficient is negative for all yaw angles of interest (if the projectile is flying at an angle above 11°, we probably have a



**FIGURE 13.2**

Plot of slow mode damping coefficient versus yaw for linear cubic slow mode.



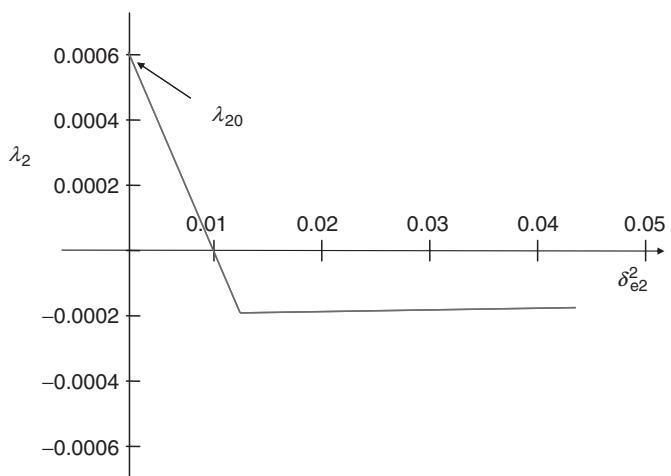
**FIGURE 13.3**

Plot of fast mode damping coefficient versus yaw for bilinear cubic fast mode.

problem). Thus, the fast mode will always damp for this projectile. Examination of Figure 13.1 reveals that as long as the projectiles yaw angle is below  $5.73^\circ$ , the slow arm will damp to zero (recall that the yaw angle is equal to  $\sin(\delta)$ ); above this angle, it will grow without bound. Although this angle is fairly large for a projectile, there have been instances documented where a slowly launched missile was stable when fired from one side of a fast warship, but unstable when launched from the other [2,8]. The instability was caused by the vector addition of the ships own speed with the launch velocity.

Figures 13.3 and 13.4 show the fast and slow damping exponents for a projectile with bilinear cubic Magnus moment behavior. This is an interesting example because it illustrates how a projectile can enter into limit-cycle motion. Limit-cycle motion is motion in which the projectile cones in a predictable manner about the velocity vector.

If we examine Figure 13.3, we see that, similar to our earlier case, the fast arm damping coefficient is everywhere negative. Because of this, the fast mode will always damp to zero. The interesting part of the story is shown in Figure 13.4. Here we see that for small angles, the projectiles slow arm will continue to grow because the damping exponent is positive. Once the amplitude of the motion grows beyond  $5.74^\circ$ , the sign of the coefficient changes driving the motion back to zero. However, the motion cannot be driven all the



**FIGURE 13.4**

Plot of slow mode damping coefficient versus yaw for bilinear cubic slow mode.

way back to zero because as soon as the angle decreases below  $5.74^\circ$ , the now positive damping coefficient will again cause it to increase. The end result will be a projectile that cones about the velocity vector at a  $5.74^\circ$  angle.

These examples assumed that the velocity of the projectile has had no affect on the exponents. We must always keep in mind that there are many interrelated phenomena that affect these coefficients—the real world is a complicated place. This discussion should provide you with a feel for the physics of the projectile behavior.

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## **Part III**

# **Terminal Ballistics**



# 14

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## *Introductory Concepts*

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Terminal ballistics is the regime that the projectile enters at the conclusion of its flight. It has been delivered into its flight by the interior ballistian, pursued and guided through its flight by the exterior ballistian, and now at its target becomes the responsibility of the terminal ballistian. The basic objective of firing the projectile is to defeat some type of target and we will study the widely varying phenomena of terminal effects that are the tools of the terminal ballistian. These end effects are dependent on the design and mission of the projectile. The most common of the missions are as follows: fragmentation of the projectile body by its cargo of high explosives; penetration or perforation of the target by the application of kinetic or chemical energy; blast at the target area delivered by the chemical energy of the explosive cargo; and the dispersal of the cargo for lethal or other missions, e.g., smoke, illumination, propaganda dispersal, etc.

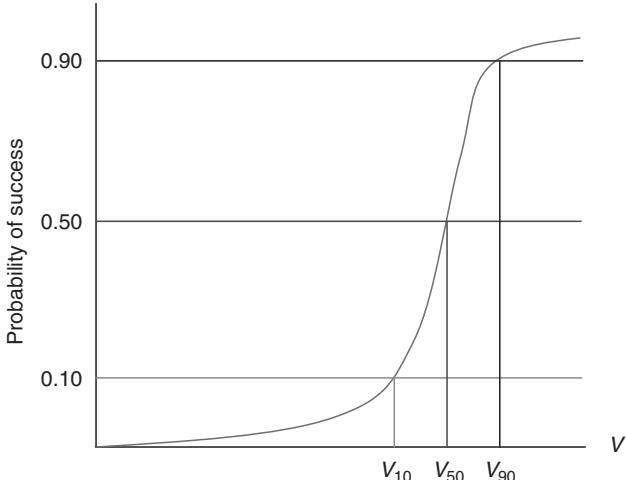
Since most terminal ballistic phenomena involve the generation and effects of stress waves in solids, we will spend some time examining the details of this field. We must gain some knowledge of terminal ballistic terminology to be able to study the theories of kinetic energy penetration of solid targets; detonation, deflagration, and burning of energetic materials; the fundamentals of shaped charges; fragmentation theories; blast effects; and lethality with the study of wound ballistics.

We shall begin by introducing some concepts that we shall use throughout our study of this field.

In examination of penetration theories, we need to consider the following items: What constitutes defeat of the target? What is the source of the data for which we have to create a theory? Does the theory track with respect to momentum balance or energy balance? How many empirically derived constants are there in the model (this tells us how universal the theory will be)? What simplifications and assumptions were made?

Penetration is defined as an event during which a projectile creates a discontinuity in the original surface of the target. Perforation requires that, after projectile or its remnants are removed, light may be seen through the target. Since penetration is a somewhat stochastic event, we need to define some statistical parameters.  $V_{10}$  is the velocity at which a given projectile will defeat a given target 10% of the time.  $V_{50}$  is the velocity at which a given projectile will defeat a given target 50% of the time, and  $V_{90}$  is the velocity at which a given projectile will defeat a given target 90% of the time. These quantities are depicted in [Figure 14.1](#).

The 50% penetration velocity is commonly used as both experimental measurement as well as a production check. The following procedure illustrates its usage in an experiment. The reader should [refer to Figure 14.2](#) to illustrate the meaning. First, we should estimate  $V_{50}$  through a calculation. Once this is accomplished, we fire a projectile with a  $V_s$  as close to  $V_{50}$  as we can achieve. Let us say, the velocity of this experimental firing is a bit over our estimate (at 1 in Figure 14.2). Assuming shot 1 only partially penetrated, we increase the velocity considerably, and let us say that we achieve complete penetration at 2 in the



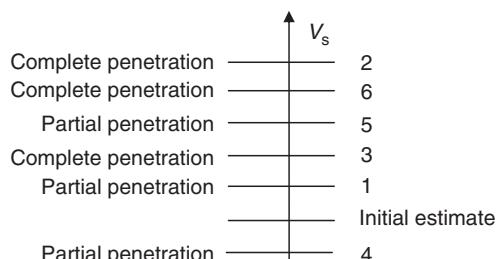
**FIGURE 14.1**  
Statistical velocities defined.

figure. We now assume that  $V_{50}$  is midway between 1 and 2. We now would attempt to fire at the velocity halfway between 1 and 2 (at 3) and, say, we get complete penetration. We would next lower the velocity to get a partial penetration, say at 4, then we would increase it to get a complete penetration (but let us say, we get only a partial penetration at 5). We would then have to increase the next shot velocity to 6.

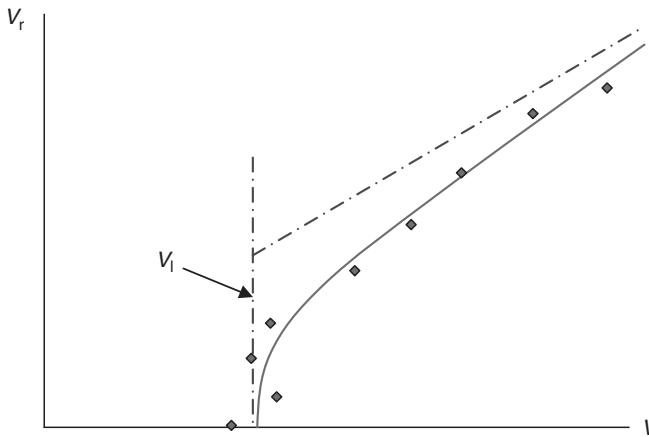
We would continue the above procedure, commonly known as an up and down test, until we obtained three complete penetrations and three partial penetrations with the difference between the highest and the lowest velocities in the set less than 200 ft/s. At that point, we would calculate the experimental  $V_{50}$  from

$$V_{50} = \frac{\sum_{i=1}^6 V_i}{6} \quad (14.1)$$

The limit velocity,  $V_i$  (sometimes called the ballistic limit when referring to the armor), is the velocity below which a given projectile will not defeat a given target. The technique for determining it was invented by the U.S. Army Ballistics Research Laboratory (BRL), Aberdeen, Maryland. The object is to fire a few projectiles that achieve complete penetration, measuring the residual velocity through the use of flash x-rays, and then generate a curve as shown in Figure 14.3. Now we plot the residual velocity after penetration versus the striking velocity. Usually, there will be a lower limit that develops below which the armor is not penetrated or the projectile gets stuck in the armor.



**FIGURE 14.2**  
Illustration of the  $V_{50}$  experimental procedure.



**FIGURE 14.3**  
Limit velocity illustrated.

From experimental evidence, we know that the following factors affect the limit velocity: material hardness, yaw at impact, projectile density, projectile nose shape, and length to diameter ratio of the projectile. For the material hardness, in general, the harder the target, the higher  $V_{50}$  becomes; while the harder the penetrator, the lower  $V_{50}$  becomes and there is more residual penetrator. With respect to yaw at impact, the more yaw, the greater chance for breakup or ricochet and the higher  $V_{50}$  becomes. With projectile density, we find that the more dense the projectile is, the lower  $V_{50}$  becomes. A blunter nose translates, in general, to a higher  $V_{50}$ . If the target is overmatched significantly, however, the nose shape has negligible effect. The length to diameter ratio can go either way and a great deal depends on the obliquity of impact.

We will now introduce some concepts which we shall use in our examination of penetration events.



# 15

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## *Penetration Theories*

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Now that we have a firm grounding in some penetration concepts such as limit velocity, we can proceed to discuss various penetration theories. We shall discuss, in some detail, penetration mechanisms in a variety of materials, all of which, to different degrees, serve to protect some vital target. Because these materials behave very differently from one another, they must be treated separately. It is this large difference in behavior, as well as mechanical properties, which makes the selection of a material for ballistic protection an important one.

We shall move successively through metals, concrete, soil, ceramic, and composite armors so that the reader gets a feel for how they behave. In all instances, the day-to-day analysis techniques of these materials are progressing, especially in the areas of numerical methods.

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### **15.1 Penetration and Perforation of Metals**

Metals are by and large the most common target of medium to large caliber projectiles. Although small caliber ammunition is generally used against soft targets, there are times when even they are called upon to penetrate metal objects. This section will discuss several models of penetration into two of the most common metals: steel and aluminum. While these formulas are not exactly perfect for other materials, usually a material will behave like one or the other.

Projectiles may impact metallic targets under a wide range of velocities. The nature of the target material is such that different velocities must be handled using somewhat different techniques. At very low velocities ( $<250$  m/s), the penetration is usually coupled to the overall structural dynamics of the target. Responses are on the order of 1 ms. As the impact velocity increases (500–2000 m/s), the local behavior of the target (and sometimes penetrator) material dominates the problem. This local zone is approximately 2–3 projectile diameters from the center of impact. With further increases in velocity (2000–3000 m/s), the high pressures involved allow the materials to be modeled as fluids in the early stages of impact. At impact speeds greater than 12,000 m/s, energy exchange occurs at such a high rate that some of the colliding material will vaporize. This energy exchange must be accounted for. We will not treat this last case as it is beyond the normal scope of military applications.

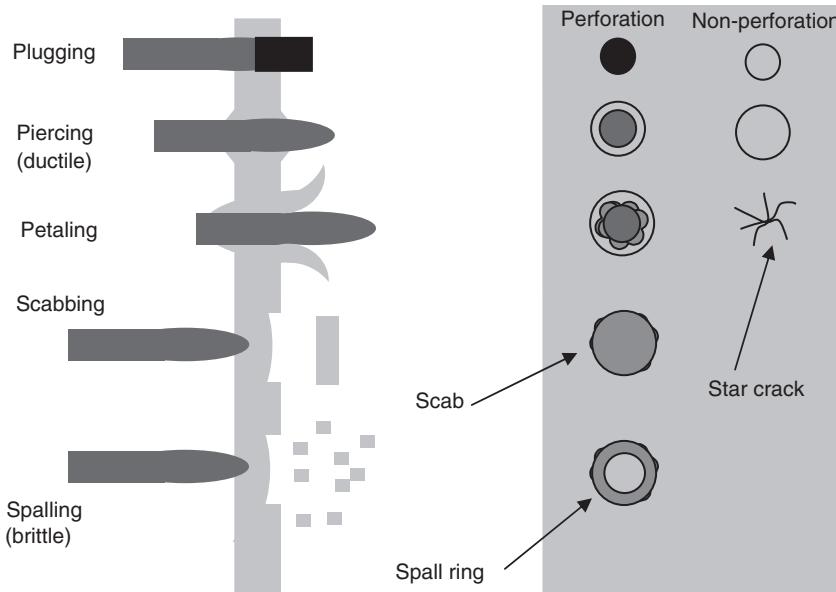
A typical sequence of events that occur during a projectile impact is developed here [1]. Given that a projectile strikes a target, compressive waves propagate into both the projectile and the target. Relief waves propagate inward from the lateral free surfaces of the penetrator, cross at the centerline, and generate a high tensile stress. If the impact were normal, we would have a two-dimensional stress state. If the impact were oblique, bending stresses will

be generated in the penetrator. When the compressive wave reached the free surface of the target, it would rebound as a tensile wave. The target may fracture at this point as will be seen in Section 16.3. The projectile may change direction if it perforates (usually toward the normal of the target surface).

Because of the differences in target behavior based on the proximity of the distal surface, we must categorize targets into four broad groups. A semi-infinite target is one where there is no influence of distal boundary on penetration. A thick target is one in which the boundary influences penetration after the projectile is some distance into the target. An intermediate thickness target is a target where the boundaries exert influence throughout the impact. Finally, a thin target is one in which stress or deformation gradients are negligible throughout the thickness.

There are several methods by which a target will fail when subjected to an impact. The major variables are the target and penetrator material properties, the impact velocity, the projectile shape (especially the ogive), the geometry of the target supporting structure, and the dimensions of the projectile and target.

The failure modes of the target are depicted in Figure 15.1. They will now be described. Spalling is very common and is the result of wave reflection from the rear face of the plate. It is common for materials stronger in compression than in tension. Scabbing is similar to spalling, but the fracture results predominantly from large plate deformation which begins a crack at a local inhomogeneity. These failure mechanisms will be expounded upon in Section 16.3. Brittle fracture occurs usually in weak and lower density targets. Radial cracking occurs in materials that are fairly ductile and usually when the projectile impact velocity is very close to the ballistic limit. Plugging occurs when the radial and circumferential stresses are high and the projectile impact velocity is close to the ballistic limit.



**FIGURE 15.1**  
Target failure modes.

Because of the very high loading rates and correspondingly high temperatures, we need to describe some phenomena that occur during penetration events. Terms such as these occur throughout the literature, so it is good to understand what they mean.

The concept of adiabatic shearing is encountered in impacts where a plug has been formed. On initial impact, a local ring of intense shear is generated. Since this occurs very quickly ( $\sim\mu s$ ), the target does not have sufficient time to build up any motion. Locally intense heat is generated. Because of the time scale and a large deformation rate, the heat cannot be conducted away. Since the material properties are weaker at this high temperature, the material tends to yield readily and flow plastically. The process then feeds on itself. Finally, a plug is formed and breaks free. If the minimum perforation velocity is exceeded by more than about 5%–10%, the plug will usually break up. Blunt noses on projectiles tend to increase the propensity to fail a target by adiabatic shear.

Hydrodynamic erosion is an important concept in terminal ballistics. Metal cutting tools such as water jets or soft metal penetrators and shaped charge jets can defeat a target by hydrodynamic erosion. During hydrodynamic erosion, the penetrator material forces the target material aside in a manner similar to a punch being pushed into the target material except that the hole will be larger. This phenomenon usually occurs at impact velocities over 1000 m/s. Deposition of the penetrator material on the walls of the hole is an indication that this failure mechanism played a part in the penetration.

The hydrodynamic transition velocity is the velocity below which the projectile and target act as essentially elastic bodies and above which both target and projectile can be treated as fluids. This concept is illustrated by the penetration sequence of Brooks [1]. For all penetration velocities, the target material is accelerated radially away from the axis of penetration. At low velocities, elastic strain keeps the target material in contact with the penetrator. At high velocities, the material is thrown away from the projectile, so that the hole becomes bigger than the projectile diameter. The radial acceleration of the material is greatest at the tip of the projectile. At the hydrodynamic transition velocity, the tip of the penetrator deforms laterally. The projectile tip becomes spherically blunted and forms a stable shape which penetrates the target for the remainder of the event. The transition velocity varies inversely with the tip radius. Hydrodynamic transition velocity is possibly related to the rate of rod erosion and plastic wave propagation.

Shear banding is a form of adiabatic shearing in which layers of material in a like state of shear tend to form. There are discontinuities in stress and strain instead of a gradual increase in shear strain near the disturbed region. Uranium and tungsten tend to display this phenomenon. Normal material models used in finite element codes do not show this effect. A model that includes thermal softening is required.

The analytical models in use today to solve these types of problems can be organized into three broad categories: empirical or quasi-analytical, approximate analytical, and numerical. In empirical or quasi-analytical models, algebraic equations are developed from large amounts of experimental data. These models are generally curve fits (results based). They usually do not incorporate physics and tend to be configuration dependent. An approximate analytical model attempts to examine the physics of a particular aspect of the penetration process or failure mechanism such as petaling, plugging, etc. The mathematics becomes tractable because we must make simplifying assumptions. They are usually limited to particular situations. Numerical models usually attempt to solve the full equations of continuum mechanics using finite difference or finite element techniques. This is the most general method. The problem with numerical models is that good material models are required and this can be expensive.

Most analytical models can only consider one damage mechanism (like plugging or fracture) or conservation law before they become mathematically intractable. Some allow as many as two mechanisms. The approach is to make simplifying assumptions.

Typical assumptions are to assume localized influence where the projectile is only influenced by a small region of the target, to ignore rigid body motions, and to ignore thermal, friction, shock heating, and any material behavioral changes due to these mechanisms, that the target is initially stress free, etc. One important thing to recognize is that a complicated model does not necessarily yield a more accurate answer.

Perforation of finite thickness plates in which plugging is the predominant penetration mode is divided into three stages. In the first stage, locally, the material ahead of the projectile is compressed and the mass is added to the projectile (i.e., the projectile decelerates somewhat and the added mass accelerates). In the second stage, more material is accelerated but shearing is occurring on the surface area of the plug. In the third stage, the plug has completely sheared out and both the plug as well as the projectile move with the same velocity. If this model is used for an oblique impact, one must use the line-of-sight thickness. At velocities from 1200 to 5000 m/s, the model used usually involves hydrodynamic erosion of the projectile tip as the first stage as well. This can be followed by both plugging and further tip erosion. In the third stage, we usually consider the projectile to be completely eroded and the plug is ejected from the armor [1].

Some models account for the flexibility of the target. This is usually required as the impact velocities approach the limit velocity. In this case, a significant amount of energy is consumed in both elastically and plastically bending the target plate.

We shall examine the underlying assumptions in a few penetration theories before moving on to detailed examination of the theories themselves. Theories which are derived from a momentum balance are typically used for thin plates. These theories can be used with minor modifications when the target petals. They usually require that the projectile remains intact.

Theories which are derived from an energy balance are typically used for thick and moderately thick plates. With moderately thick targets, plugging can occur. Thick plates are usually defeated by a piercing phenomenon which also has distinct phases. The first phase is a radial displacement of the target material. Sometimes, there is plugging at this stage. This stage is followed by plastic flow and yielding of the target. The target material may well be able to be treated like a fluid during this phase.

Many empirically based predictive relationships are based on energy approaches. A particularly popular model takes the form of

$$E = kd^m t^n \quad (15.1)$$

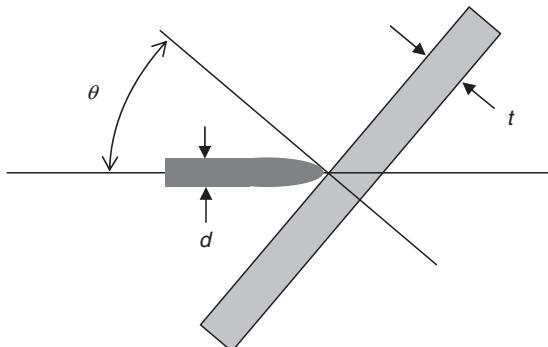
where we have

$$m + n \approx 3 \quad (15.2)$$

In these equations,  $E$  is the perforation energy,  $d$  is the projectile diameter,  $t$  is the plate thickness, and  $k$  is an empirically derived constant (see Figure 15.2). If we let  $m = 1.5$  and  $n = 1.4$ , we get the famous DeMarre formula for normal impact. If we would like to include an angle of obliquity in the above formula, it is common practice to use

$$E = kd^m t^n \sec^p \theta \quad (15.3)$$

Here  $p$  is an experimental parameter based on the projectile–armor combination and  $\theta$  is the angle of obliquity measured from the normal to the plate. Sometimes, the armor fabrication process will affect the penetration. In this case, there is a function called the figure of merit (FOM) where the perforation velocity of the armor is compared to that of mild steel.



**FIGURE 15.2**

Projectile impact problem illustrated.

$$\text{FOM} = \frac{V_1}{V_{\text{l}_{\text{mild steel}}}} \quad (15.4)$$

Note that in Equation 15.4, the velocity used does not necessarily have to be the limit velocity. Another useful relationship commonly employed by the projectile designer is

$$E_{\text{perf}} = \frac{1}{2} m V_{\text{perf}}^2 \quad (15.5)$$

Inserting [Equation 15.3](#) into Equation 15.5 yields

$$V_{\text{perf}}^2 = 2k \frac{d^m t^n}{m} \sec^p \theta \quad (15.6)$$

Now taking the square root and assimilating terms, we get

$$V_{\text{perf}} = k \sqrt{\frac{d^m t^n}{m}} \sec^{j(\theta)} \theta \quad (15.7)$$

In 1886, DeMarre developed a famous formula for the penetration of a plate given a normal impact.

$$\frac{mV^2}{d^3} = \alpha \frac{t^{1.4}}{d^{1.5}} \quad (15.8)$$

Here  $m$  is the penetrator mass,  $V$  is the impact velocity,  $d$  is the diameter of the projectile, and  $t$  is the plate thickness with  $\alpha$  being an empirically derived constant. As a word of caution, many of these formulas are dangerous because of the units in the empirically derived constant, it is commonplace to see CGS units in these formulas as well. Over time, many have modified the DeMarre formula and used it in this form

$$\frac{mV^2}{d^3} = \alpha \left( \frac{t}{d} \right)^\beta \quad (15.9)$$

Here  $\beta$  is an empirically derived constant as well. In the form above, the DeMarre formula is used when considering a normal impact. Some researchers have extended its use to include an oblique impact and it would then take the following form:

$$\frac{mV^2}{d^3} = \alpha \left[ \frac{\operatorname{tg}(\theta)}{d} \right]^\beta \quad (15.10)$$

where  $g(\theta)$  is a function of the angle of obliquity and is most often taken as  $\sec \theta$ .

We sometimes define the specific limit energy (SLE) as

$$\frac{mV_l^2}{d^3} \equiv \text{SLE} \quad (15.11)$$

H. Berthe who worked at Frankford arsenal in 1941 determined that for piercing type problems (i.e., thin plate perforation where a hole is laterally or radially widened by the penetrator), the constant,  $\beta$ , should be equal to 1, thus yielding

$$mV_l^2 \sim td^2 \quad (15.12)$$

Around the same time (1942), Zener and Holloman from Watertown arsenal came up with a formula for use when plugging or petaling is the predominant penetration mode. They stated that in this case,  $\beta$  should equal to 2, thus yielding

$$mV_l^2 \sim t^2 d \quad (15.13)$$

In 1943, Curtis and Taub attempted to modify the DeMarre formula to account for a mode change during the penetration event. In a thick plate, the mode changes at some point from a piercing to a plugging at the rear surface. This results in a decrease in energy consumed per unit path length, so the DeMarre formula had to be further modified to

$$\frac{mV_l^2}{d^3} = \alpha \left( \frac{t}{d} + \gamma \right) \quad (15.14)$$

Here  $\alpha$  and  $\gamma$  are constants and  $\gamma < 0$ . If we define  $t'$  as depicted in Figure 15.3, then  $\gamma$  is a quadratic function of  $t'$ . Also  $t' \sim d$  and is the distance after the mode changes.

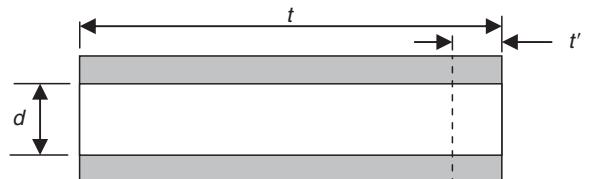
S. Jacobson, working at the Picatinny Arsenal in New Jersey further refined the concept that there is a different energy relationship for each of the two modes. For plugging, this is

$$E_{\text{plug}} = \text{force} \cdot \text{distance} \approx \pi dt Y_s \cdot t \quad (15.15)$$

where  $Y_s$  is the shear yield strength of the material. For the piercing mode, we have

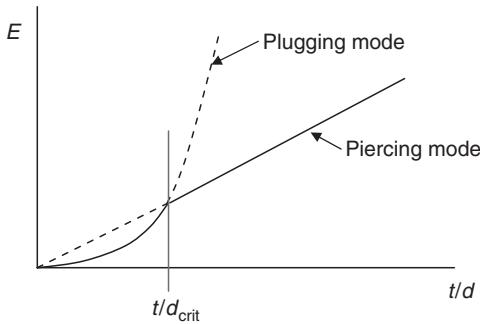
$$E_{\text{piercing}} = Y_{\text{flow}} \cdot V \approx \frac{\pi}{4} d^2 t Y_{\text{flow}} \quad (15.16)$$

where  $V$  is the volume of the plug and  $Y_{\text{flow}}$  is the flow or plastic yield stress of the target material.



**FIGURE 15.3**

Section of a target plate that defines  $t$  and  $t'$ .



**FIGURE 15.4**

Energy in penetration modes based on the model of Jacobson.

We can rewrite Equations 15.15 and 15.16 as

$$E_{\text{plug}} = k_{\text{plug}} d^3 \left( \frac{t}{d} \right)^2 Y_s \quad (15.17)$$

$$E_{\text{piercing}} = k_{\text{piercing}} d^3 \left( \frac{t}{d} \right) Y_{\text{flow}} \quad (15.18)$$

If we graph both expressions, we obtain a plot as illustrated in Figure 15.4. To obtain  $t/d_{\text{crit}}$ , we solve Equations 15.17 and 15.18 where

$$E_{\text{plug}} = E_{\text{piercing}} \quad (15.19)$$

using the relations that

$$Y_s \approx 0.6 Y_{\text{flow}} \quad (15.20)$$

$$k_{\text{piercing}} = \frac{\pi}{4} \text{ and } k_{\text{plug}} = \pi \quad (15.21)$$

Then combining Equations 15.17 and 15.18, we get

$$\frac{\pi}{4} d^3 \left( \frac{t}{d} \right) Y_{\text{flow}} = \pi d^3 \left( \frac{t}{d} \right)^2 0.6 Y_{\text{flow}} \rightarrow \left( \frac{t}{d} \right)_{\text{crit}} = 0.42 \quad (15.22)$$

This value of  $t/d$  is the point where the mode of penetration changes from plugging to piercing. Thus, against targets whose thickness is such that an attack by a penetrator whose  $t/d$  ratio is greater than 0.42, we can expect that the penetration mode will be piercing, otherwise plugging is to be expected.

Lambert and Zukas proposed a model in 1982 while working for BRL to cover more general cases of penetration. If we examine Equation 15.14, we can see that as the plate thickness goes to zero, the residual velocity should approach the striking velocity and the limit velocity should approach zero. Expressed mathematically, we require that

$$\lim_{t \rightarrow 0} V_1 \rightarrow 0 \quad (15.23)$$

However, if we look at Equation 15.14, we note that if  $V_1=0$  and  $t=0$  it requires the product  $\gamma\alpha$  to equal zero, which is not physically possible. Therefore, the Lambert model replaces  $\gamma$  by  $[\exp(-t/d) - 1]$  as below.

$$\frac{mV_1^2}{d^3} = \alpha \left[ \frac{t}{d} + \exp\left(-\frac{t}{d}\right) - 1 \right] \quad (15.24)$$

This forces

$$V_1 = 0 \text{ at } t = 0 \quad (15.25)$$

and

$$V_1 = \infty \text{ at } t = \infty \quad (15.26)$$

Since the penetrator volume is proportional to  $d^2l$  and since there should be a dependence on this volume in the specific limit energy, we want to keep the dimension of diameter cubed in Equation 15.24, thus we shall write

$$d^3 \rightarrow d^{3-c}l^c = d^2l \left(\frac{l}{d}\right)^{c-1} = d^3 \left(\frac{l}{d}\right)^c \quad (15.27)$$

where  $c$  is a constant. We can then incorporate this into Equation 15.24 as

$$\frac{mV_1^2}{d^3} = \left(\frac{l}{d}\right)^c \alpha \left[ \frac{t}{d} + \exp\left(-\frac{t}{d}\right) - 1 \right] \quad (15.28)$$

Next we will include obliquity effects by adding in the angle of obliquity,  $\theta$  through replacement of  $t$  by  $t \sec^k \theta$ . In this case, if  $k=1$ , we have the true path length through the armor plate (line-of-sight thickness). We shall define

$$z = \frac{t}{d} \sec^k \theta \quad (15.29)$$

We can now rewrite Equation 15.29 as

$$\frac{mV_1^2}{d^3} = \alpha \left(\frac{l}{d}\right)^c \left[ \frac{t}{d} \sec^k \theta + \exp\left(-\frac{t}{d} \sec^k \theta\right) - 1 \right] \quad (15.30)$$

If we solve Equation 15.30 for the limit velocity, we obtain

$$V_1 = \sqrt{\alpha \left(\frac{l}{d}\right)^c \left[ \frac{t}{d} \sec^k \theta + \exp\left(-\frac{t}{d} \sec^k \theta\right) - 1 \right] \frac{d^3}{m}} \quad (15.31)$$

The Lambert model was used to examine the firing of 200 long-rods into rolled homogeneous armor (RHA). The test conditions were as follows:

$$\begin{aligned} 0.5 &\leq m[\text{g}] \leq 3630 & 0.6 &\leq t[\text{cm}] \leq 15 \\ 0.2 &\leq d[\text{cm}] \leq 0.5 & 0^\circ &\leq \theta \leq 60^\circ \\ 4 &\leq \frac{l}{d} \leq 30 & 7.8 &\leq \rho \left[ \frac{\text{g}}{\text{cm}^3} \right] \leq 19.0 \end{aligned}$$

A least squares fit of the results yielded the following:  $\alpha = (4000)^2$ ,  $c = 0.3$ , and  $k = 0.75$ . If we insert these into [Equation 15.31](#), we get

$$V_1 = \left(\frac{l}{d}\right)^{0.15} (4000) \sqrt{\frac{d^3}{m} \left[ \frac{t}{d} \sec^{0.75} \theta + \exp\left(-\frac{t}{d} \sec^{0.75} \theta\right) - 1 \right] \left[\frac{m}{s}\right]} \quad (15.32)$$

Please note the CGS units. The authors suggest that the model is applicable where  $t/d > 1.5$ . Also we must note that nose geometry has a significant influence for  $t/d < 1.0$ . RHA or good quality steel is the target (the specific properties are unimportant).

One measure of lethal effects once a projectile has perforated the target material is the residual velocity.  $V_r$  is the symbol for the residual velocity of the penetrator. That is the velocity that the penetrator moves with once it perforates the target. Mathematically, it is defined in the Lambert model as

$$V_r = \begin{cases} 0, & 0 \leq V_s \leq V_1 \\ a(V_s^p - V_1^p)^{\frac{1}{p}}, & V_s > V_1 \end{cases} \quad (15.33)$$

If we assume that  $V_s$  is large so that the absorption of momentum by the target is negligible, then the momentum balance can be written in terms of identifiable penetrator mass and velocity ( $m_r$  and  $V_r$ ), and the large quantity of unidentifiable target and penetrator ejecta with each particle  $m_i$  having a particular velocity,  $V_i$ . Thus, the momentum balance is

$$m_r V_r + \sum_{i=1}^n m_i V_i \rightarrow m_s V_s \text{ as } V_s \rightarrow \infty \quad (15.34)$$

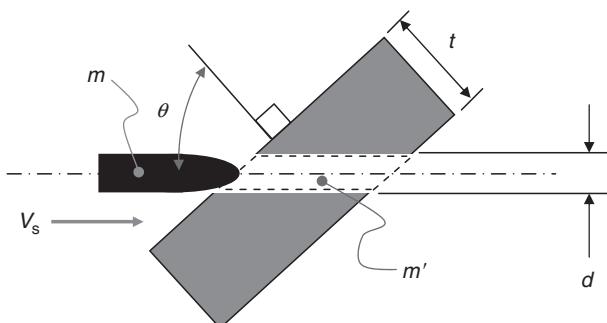
Even though [Equation 15.34](#) is mathematically satisfying, in practice, it is usually difficult to measure the mass and velocity of all of the fragments, so most of the  $m_i V_i$  will remain unknown.

We shall now consider a general case of impact as illustrated in [Figure 15.5](#). Here we shall let  $m'$  be the mass of the ejecta. We can then write

$$m' = \rho \frac{\pi}{4} d^3 z \quad (15.35)$$

where

$$z = \frac{t}{d} \sec^{0.75} \theta \quad (15.36)$$



**FIGURE 15.5**  
General case of projectile impact.

therefore

$$m' = \rho \frac{\pi}{4} d^3 \left( \frac{t}{d} \sec^{0.75} \theta \right) = \rho \frac{\pi}{4} d^2 t \sec^{0.75} \theta \quad (15.37)$$

If we now assume that

$$\sum_{i=1}^n m_i V_i = h m' V_r \quad (15.38)$$

This is equivalent to stating that  $m'$  is the mass of material pushed ahead of the penetrator,  $m'$  is ejected with speed  $V_r$  (plugging theory), and the total momentum of the ejecta jumble is proportional to  $m' V_r$ . We can also write, in the limiting case, that the residual momentum approaches the initial momentum or, mathematically

$$\frac{M_r}{M} \rightarrow 1 \quad (15.39)$$

If we substitute Equation 15.38 into [Equation 15.34](#), we get

$$m V_r + h m' V_r \rightarrow m_s V_s \text{ as } V_s \rightarrow \infty \quad (15.40)$$

which can be rearranged to yield

$$\frac{V_r}{V_s} \rightarrow \left( \frac{m_s}{m_r + h m'} \right) \text{ as } V_s \rightarrow \infty \quad (15.41)$$

We know that if penetration occurred, [Equation 15.33](#) applies, so we have

$$V_r = a(V_s^p - V_1^p)^{\frac{1}{p}} \quad (15.42)$$

We can divide Equation 15.42 by  $V_s$  to get

$$\frac{V_r}{V_s} = a \left[ 1 - \left( \frac{V_1}{V_s} \right)^p \right]^{\frac{1}{p}} \quad (15.43)$$

which means that as  $V_s$  approaches infinity, the second term in the parentheses approaches zero or

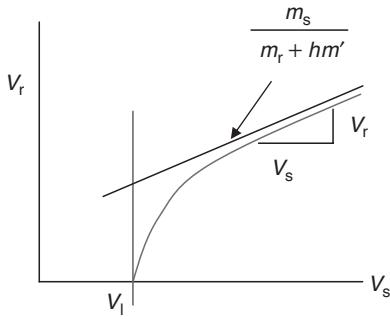
$$\frac{V_r}{V_s} \rightarrow a \text{ as } V_s \rightarrow \infty \quad (15.44)$$

This is illustrated in [Figure 15.6](#).

If we look at Equations 15.44 and 15.41, we see that

$$a = \left( \frac{m_s}{m_r + h m'} \right) \quad (15.45)$$

Furthermore, we can assume in the plugging mode that the penetrators mass does not change significantly during penetration, so we get  $m_s = m_r = m$ . We can then write Equation 15.45 as



**FIGURE 15.6**  
Asymptote on limit velocity.

$$a = \left( \frac{m}{m + hm'} \right) \quad (15.46)$$

There is empirical evidence that suggests that  $h \approx 1/3$ , so we can write

$$a = \left( \frac{m}{m + \frac{1}{3}m'} \right) \quad (15.47)$$

If we assume that the penetrator remains intact throughout the perforation event, we can write

$$\text{KE}_{\text{impact}} = \text{KE}_{\text{limit}} + \text{KE}_{\text{residual}} \quad (15.48)$$

This can also be expressed as

$$V_r^2 \sim V_s^2 - V_l^2 \rightarrow V_r \sim (V_s^2 - V_l^2)^{\frac{1}{2}} \quad (15.49)$$

which, if written as

$$V_r = a(V_s^2 - V_l^2)^{\frac{1}{2}} \quad (15.50)$$

would say that  $p = 2$ . If we looked at momentum, we would get

$$V_s \sim V_l + V_r \quad (15.51)$$

which could be written as

$$V_s = a(V_l + V_r) \quad (15.52)$$

Equation 15.52 implies that for a momentum balance,  $p = 1$ . Thus, it is clear that the value for  $p$  should fall between 1 and 2. Lambert accounted for this by choosing

$$p = 2 + z = 2 + \frac{t}{d} \sec^{0.75} \theta \quad (15.53)$$

where both  $p$  and  $z$  grow monotonically as

$$\frac{t}{d} \rightarrow \infty \text{ and/or } \theta \rightarrow \frac{\pi}{2} \quad (15.54)$$

also

$$p \rightarrow 2 \text{ as } t \rightarrow 0 \quad (15.55)$$

Lambert also found that a better empirical fit was obtained if he let

$$p = 2 + \frac{z}{3} = 2 + \frac{t}{3d} \sec^{0.75} \theta \quad (15.56)$$

A numerical model for penetration was proposed by A. Tate has to determine penetration of metals [2]. The base equation for this model is

$$\frac{1}{2} \rho_p (V_i - u)^2 + Y_p = \frac{1}{2} \rho_t u^2 + R_t \quad (15.57)$$

Here  $\rho_p$  is the density of the projectile material,  $\rho_t$  is the density of the target material,  $V_i$  is the impact velocity,  $u$  is the instantaneous projectile velocity, and  $Y_p$  and  $R_t$  are the ballistic resistances of the projectile and target, respectively, defined as

$$Y_p = 1.7 \sigma_p \quad (15.58)$$

$$R_t = \sigma_t \left[ \frac{2}{3} + \ln \left( 0.57 \frac{E_t}{\sigma_t} \right) \right] \quad (15.59)$$

where  $\sigma_p$  is the yield strength of the projectile material,  $\sigma_t$  is the yield strength of the target material, and  $E_t$  is the modulus of elasticity of the target material. The way the Tate model is used is to integrate Equation 15.57 numerically until the velocity goes to zero or perforation occurs. When the projectile stops, a second integration determined the depth of penetration. When perforation occurs, the value of  $u$  is the residual velocity. Tate states that the accuracy of this method is within 20%. One of the models downsides is that it does not handle oblique impacts but it can at least be altered by the line-of-sight thickness.

If a penetrator hits a target at a great enough angle, it may ricochet. The ricochet process can be described as follows. During impact, both the projectile and the target are compressed elastically. When this energy is released, it will change the projectiles motion. Deformations because of resisting force of the target will change the direction of the penetrator. Rotating moments are generated by internal forces acting in the projectile.

In general, thin plates do not allow ricochet except at extreme angles of attack. Tate has produced a ricochet formula for the critical ricochet angle (oblique impacts at angles greater than this will ricochet).

$$\tan^3 \beta > \frac{2}{3} \frac{\rho_p V^2}{Y_p} \left( \frac{L^2 + D^2}{LD} \right) \left[ 1 + \left( \frac{\rho_p}{\rho_t} \right)^{\frac{1}{2}} \right] \quad (15.60)$$

Here  $Y_p$  is a characteristic strength usually taken as the Hugoniot elastic limit (described in [Section 16.3](#)), the subscripts "p" and "t" are projectile and target, respectively, and  $L$ ,  $D$ , and  $V$  are the length, diameter, and velocity of the penetrator, respectively.

As vehicles become lighter weight, aluminum is being used more and more as armor. It is therefore necessary to determine the penetration capabilities of projectiles into aluminum.

Aluminum behaves a little differently than steel during penetration by ogival projectiles in its tendency to be pierced rather than to develop plugs. A penetrator is usually of significantly greater density than the target in most cases. One significant difference is the evidence of a layer of aluminum with an altered microstructure on the penetrated surface. This indicates a melt layer which is believed to assist in penetration.

A simple model of projectile penetration into aluminum was put forward by Forrestal et al. in 1992 [3]. A distinct advantage of this model is its simplicity. A possible disadvantage is that the empirical nature is not universal. Even though the study was performed specifically with 7075-T651 targets, it yields a fairly good representation of aluminum penetration. The model assumes normal impact of the projectile and that the projectile is rigid. This may, at first, seem to be a restrictive assumption, but the method provides reasonable estimates for slightly yawed projectiles if the angle is below about 5° and possibly further.

We first define the caliber-radius-head as

$$\psi = \frac{s}{d} \quad (15.61)$$

Here  $d$  is the diameter of the projectile,  $\psi$  is the caliber-radius-head, and  $s$  is the ogive radius. We can also define a nose length as

$$l = \frac{d}{2} \sqrt{4\psi - 1} \quad (15.62)$$

This geometry is illustrated in Figure 15.7.

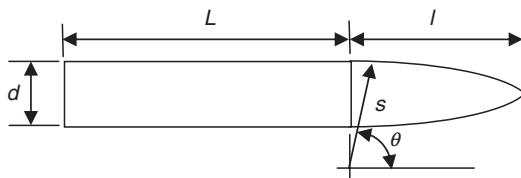
We shall say that the resistance force of the aluminum target on the penetrator in this case will have two components: one normal to the surface (normal stresses) and one tangential to the surface (shear stresses and friction). If we lump the shear stress in with the stress owing to friction and furthermore assume that the tangential stress is proportional to normal stress, we can write

$$\sigma_t = \mu \sigma_n \quad (15.63)$$

Here  $\sigma_t$  is the tangential stress,  $\sigma_n$  is the normal stress, and  $\mu$  is the proportionality constant (coefficient of sliding friction).

Forrestal et al. [4] developed a formula for the axial force on an ogival nose.

$$F_z = 2\pi s \int_{\theta_0}^{\pi/2} \left\{ \left[ \sin \theta - \left( \frac{s - \frac{d}{2}}{s} \right) \right] (\cos \theta + \mu \sin \theta) \right\} \sigma_n(V_z, \theta) d\theta \quad (15.64)$$



**FIGURE 15.7**  
Ogival penetrator for the model of Forrestal et al.

where

$$\theta_0 = \sin^{-1} \left( \frac{s - \frac{d}{2}}{s} \right) \quad (15.65)$$

Here  $V_z$  is the instantaneous velocity during penetration. The stress function  $\sigma_n(V_z, \theta)$  is assumed to be similar to that of a spherically symmetric expanding cavity (defined below). If we let  $V$  be the constant velocity at which the tip of the projectile radially expands the hole, then we can write the radial stress at the cavity surface as

$$\frac{\sigma_r}{Y} = A + B \left( \sqrt{\frac{\rho_t}{Y}} V \right)^2 \quad (15.66)$$

Here  $\sigma_r$  is the radial stress,  $Y$  is the material yield stress,  $\rho_t$  is the target density, and  $A$  and  $B$  are constants defined as

$$A = \frac{2}{3} \left[ 1 + \left( \frac{2E}{3Y} \right)^n I \right] \quad (15.67)$$

$$B = \frac{3}{2} \quad (15.68)$$

where

$$I = \int_0^{1-\left(\frac{3Y}{2E}\right)} \frac{(-\ln x)^n}{1-x} dx \quad (15.69)$$

In these expressions,  $E$  is Young's modulus and  $n$  is the strain hardening exponent (assumes power-law strain hardening). For an assumed incompressible 7075-T651 aluminum, Forrestal et al. [3] provide  $I = 3.896$  and  $A = 4.609$ .

Empirically, curve-fitting the stress-strain curves (thus including compressibility) for 7075-T651 yielded slightly different results with  $A = 4.418$  and  $B = 1.068$ .

To approximate the normal stress on the ogive, we can replace the spherically symmetric velocity,  $V$  in Equation 15.66 with  $V_z \cos \theta$ , then we have

$$\frac{\sigma_n(V_z, \theta)}{Y} = A + B \left( \sqrt{\frac{\rho_t}{Y}} V_z \cos \theta \right)^2 \quad (15.70)$$

If we insert Equation 15.70 into [Equation 15.64](#), we obtain

$$F_z = 2\pi s Y \int_{\theta_0}^{\pi/2} \left\{ \left[ \sin \theta - \left( \frac{s - \frac{d}{2}}{s} \right) \right] (\cos \theta + \mu \sin \theta) \right\} \left( A + B \frac{\rho_t}{Y} V_z^2 \cos^2 \theta \right) d\theta \quad (15.71)$$

Now we integrate to obtain

$$F_z = \frac{\pi d^2}{4} Y \left( \alpha + \beta \frac{\rho_t V_z^2}{Y} \right) \quad (15.72)$$

where

$$\alpha = A \left[ 1 + 4\mu\psi^2 \left( \frac{\pi}{2} - \theta_0 \right) - \mu(2\psi - 1) \sqrt{4\psi - 1} \right] \quad (15.73)$$

$$\beta = B \left[ \frac{\frac{8\psi-1}{24\psi^2} + \mu\psi^2 \left( \frac{\pi}{2} - \theta_0 \right) - \mu(2\psi - 1)(6\psi^2 + 4\psi - 1) \sqrt{4\psi - 1}}{24\psi^2} \right] \quad (15.74)$$

Now that we have an expression for force as a function of velocity, we need to come up with how this varies during penetration.

We can write Newton's second law as

$$-F_z = m \frac{dV_z}{dt} \quad (15.75)$$

We can convert this time integral to a distance integral and rewrite it as follows:

$$-F_z = m V_z \frac{dV_z}{dz} \quad (15.76)$$

One can write the mass of our projectile in terms of the parameters we have already described. The mass of the cylindrical section of the projectile is

$$m_{\text{cylinder}} = \rho_p \frac{\pi d^2}{4} L \quad (15.77)$$

We can write the mass of the ogive as

$$m_{\text{ogive}} = \rho_p \frac{\pi d^3}{8} k \quad (15.78)$$

where

$$k = \left( 4\psi^2 - \frac{4\psi}{3} + \frac{1}{3} \right) \sqrt{4\psi - 1} - 4\psi^2(2\psi - 1) \sin^{-1} \left( \sqrt{\frac{4\psi - 1}{2\psi}} \right) \quad (15.79)$$

Now the total mass of the projectile is

$$m = m_{\text{cylinder}} + m_{\text{ogive}} = \rho_p \frac{\pi d^2}{4} \left( L + \frac{kd}{2} \right) \quad (15.80)$$

If we insert Equations 15.80 and 15.72 into Equation 15.76, we get, after some rearrangement

$$-dz = \rho_p \left( L + \frac{kd}{2} \right) \frac{V_z}{\alpha Y + \beta \rho_t V_z^2} dV_z \quad (15.81)$$

This can be integrated as

$$-\int_0^P dz = \rho_p \left( L + \frac{kd}{2} \right) \int_{V_0}^0 \frac{V_z}{\alpha Y + \beta \rho_t V_z^2} dV_z \quad (15.82)$$

The result of this integration is

$$P = \frac{1}{2\beta} \left( \frac{\rho_p}{\rho_t} \right) \left( L + \frac{kd}{2} \right) \ln \left[ 1 + \left( \frac{\beta}{\alpha} \right) \left( \frac{\rho_t V_0^2}{Y} \right) \right] \quad (15.83)$$

Here  $P$  is the final penetration depth and  $V_0$  is the impact velocity.

If the penetration depth,  $P$ , is greater than the target thickness, perforation will occur.

When this is the case, it is useful to be able to calculate the residual velocity of the penetrator which we do by integrating [Equation 15.82](#) with different limits of integration.

$$-\int_0^T dz = \rho_p \left( L + \frac{kd}{2} \right) \int_{V_0}^{V_r} \frac{V_z}{\alpha Y + \beta \rho_t V_z^2} dV_z \quad (15.84)$$

Here  $T$  is the target thickness. Performing the integration yields

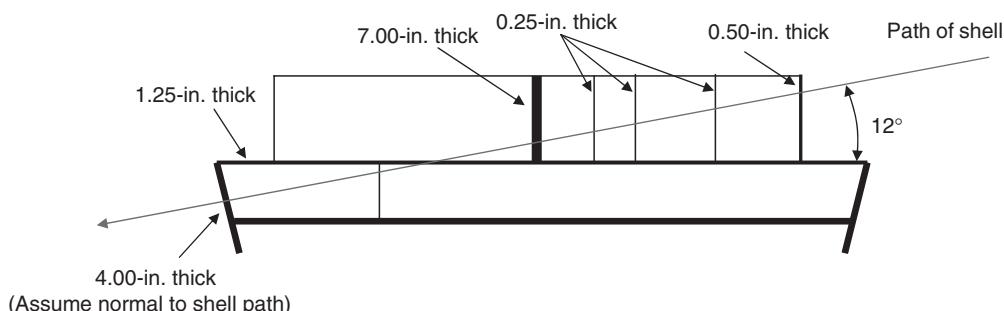
$$V_r = \sqrt{\left( \frac{\alpha Y}{\beta \rho_t} + V_0^2 \right) \exp \left[ -\frac{2\beta \rho_t T}{\rho_p (L + \frac{kd}{2})} \right] - \frac{\alpha Y}{\beta \rho_t}} \quad (15.85)$$

This model has proven to be fairly accurate (within 15%) once the coefficients have been tuned. It is fairly sensitive to the friction coefficient,  $\mu$ , incorporated in both  $\alpha$  and  $\beta$ , which Forrestal et al. [4] suggest should be between 0 and 0.06.

### Problem 1

A German 280-mm armor-piercing projectile weighs 666 lbm and is about 34 in. in length. It strikes a British warship in the 1/2-in. thick vertical side plating at an angle of  $12^\circ$  from horizontal along the path depicted below. The initial impact velocity is 2000 ft/s. Determine the residual velocity of the shell after passing through each compartment and how far through the ship it will go (i.e., in which compartment will it stop).

Assume the density of the armor plate to be  $\rho = 0.283$  lbm/in.<sup>3</sup>



*Answer:* The projectile is arrested by the 1.25-in. deck.

**Problem 2**

An explosively formed penetrator impacts a 4-in. thick RHA plate at a velocity of 1500 m/s. The penetrator parameters are given below. Determine if the penetrator will perforate the target using the Lambert/Zukas model given

1. A normal impact  
*Answer:*  $V_1 = 1299 \left[ \frac{\text{m}}{\text{s}} \right]$  yes
2. An impact at  $30^\circ$  obliquity  
*Answer:*  $V_1 = 1389 \left[ \frac{\text{m}}{\text{s}} \right]$  yes

Penetrator information

$$l = 95[\text{mm}] \quad m = 1.25[\text{lbfm}]$$
$$d = 22[\text{mm}] \quad V_s = 1500 \left[ \frac{\text{m}}{\text{s}} \right]$$

**Problem 3**

A German 7.5-cm Gr 34A1 projectile is fired at a 2-in. thick armor plate at a  $30^\circ$  obliquity. The impact velocity is 400 m/s. The penetrator parameters are given below.

1. Determine whether the penetration mode will be plugging or piercing through use of the Jacobson model for a normal impact.  
*Answer:* Piercing
2. Determine if the penetrator will perforate the armor though use of the Lambert model.  
*Answer:* No
3. Comment on the validity of the model.

Penetrator information

$$l = 39[\text{cm}] \quad m = 5.75[\text{kg}]$$
$$d = 7.5[\text{cm}] \quad V_s = 400 \left[ \frac{\text{m}}{\text{s}} \right]$$

**Problem 4**

A Japanese 20-mm projectile with the properties below impacts the 1/2-in. thick aluminum armor plate on a U.S. plane's rear gun mount at  $30^\circ$  obliquity. If the projectile and the armor have the following properties:

1. Determine how deep the projectile will penetrate into the armor (assume  $\mu = 0.03$ ).  
*Answer:*  $P = 53.1[\text{mm}] = 2.09[\text{in.}]$
2. If the projectile perforates the armor, determine its residual velocity.  
*Answer:*  $V_r = 423 \left[ \frac{\text{m}}{\text{s}} \right]$

Estimated penetrator information

$$s = 40[\text{mm}] \quad m = 128[\text{g}]$$

$$d = 20[\text{mm}] \quad V_s = 500 \left[ \frac{\text{m}}{\text{s}} \right] \quad \rho_p = 0.283 \left[ \frac{\text{lbm}}{\text{in.}^3} \right]$$

$$L = 60[\text{mm}]$$

Estimated armor information

$$A = 4.418Y = 39,000[\text{psi}] \quad \rho_t = 0.098 \left[ \frac{\text{lbm}}{\text{in.}^3} \right]$$

$$B = 1.068$$


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## 15.2 Penetration and Perforation of Concrete

Concrete penetrating munitions have always been important in the military arsenal. Bunkers, buildings, and walls are used as cover by an enemy and it is required to perforate the structure and deliver some type of lethal or nonlethal effect behind the obstruction.

Concrete comes in a variety of forms which have variable strengths, reinforcement geometry, and material properties owing to curing. Each of these forms behaves somewhat differently when impacted by a projectile. There is some evidence that once the impact velocity of a projectile is great enough, one can ignore reinforcement and only the concrete strength becomes important. As a consequence of the high compressive strength of concrete relative to its tensile strength, it tends to spall readily.

A relatively simple model of projectile penetration into concrete was put forward by Forrestal et al. in 1994 [5]. This model has an advantage in its simplicity. But a slight disadvantage is that its empirical nature makes its global applicability somewhat limited. We shall use this model as a fairly good representation of concrete penetration physics. The model assumes normal impact of the projectile and that the projectile is rigid. This may seem to be restrictive assumptions, however, the method provides reasonable estimates for slightly yawed projectiles if the angle is below about 5° based on this author's own work.

The point of departure is the determination of the force on the nose of the projectile which is defined in a manner similar to a fluid mechanics analysis as

$$F = \frac{\pi d^2}{4} (\tau_0 A + NB\rho V^2) \quad (15.86)$$

With  $N$  defined as

$$N = \frac{8\psi - 1}{24\psi^2} \quad (15.87)$$

In these equations, the projectile properties are as follows (see Figure 15.7):  $d$  is the diameter of the projectile;  $\psi$  is the caliber-radius-head, defined in Equation 15.88;  $V$  is the projectile velocity (assuming rigid body motion); and  $s$  (used in Equation 15.88) is the ogive radius.

The caliber-radius-head is defined as

$$\psi = \frac{s}{d} \quad (15.88)$$

The target properties used in [Equation 15.86](#) are as follows:  $\rho$  is the density of the target, the product  $\tau_0 A$  is a shear strength parameter obtained from a triaxial strength test, and  $B$  is a compressive strength parameter. In this model, the parameters are set as

$$B = 1 \quad (15.89)$$

$$\tau_0 A = Sf'_c \quad (15.90)$$

Here,  $S$  is a dimensionless empirical constant that depends upon the unconfined compressive strength  $f'_c$ .

If we define the instantaneous depth of penetration as  $z$ , we find that for  $z > 2d$ , we can write

$$F = \frac{\pi d^2}{4} (Sf'_c + N\rho V^2), \quad z > 2d \quad (15.91)$$

This equation is valid for deep penetration depths. For depths less than two projectile diameters, the penetration process is affected by surface cratering. Beyond two projectile diameters, the hole caused by the projectile will be approximately equal to the projectile diameter. This is known as the tunnel region. We shall define the penetration depth as  $P$ .

Below two projectile diameters the damage to the concrete will, in general, be a conical taper called the crater. This is illustrated in Figure 15.8.

In the surface crater region, the force on the projectile nose is proportional to the penetration depth or, mathematically

$$F = cz, \quad 0 < z < 2d \quad (15.92)$$

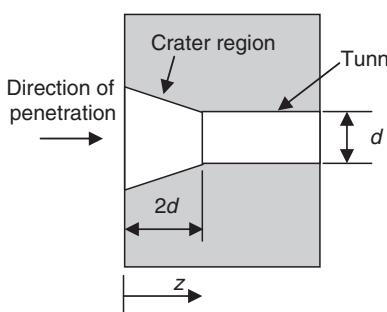
Here  $c$  is a constant which we will soon define.

If we begin with Newton's second law, we see that

$$F = ma = m \frac{d^2z}{dt^2} \quad (15.93)$$

Here  $m$  is the mass of the projectile. Since we know the force acting on the projectile will tend to slow it down, we can equate Equations 15.92 and 15.93.

$$m \frac{d^2z}{dt^2} = -cz \quad (15.94)$$



**FIGURE 15.8**  
Illustration of a concrete penetration.

We can rewrite Equation 15.94 as

$$\frac{d^2z}{dt^2} = -\omega^2 z \quad (15.95)$$

where we have defined

$$\omega^2 = \frac{c}{m} \quad (15.96)$$

If we assume a solution of the form

$$z = A_1 \sin \omega t \quad (15.97)$$

We can write

$$\frac{dz}{dt} = A_1 \omega \cos \omega t \quad (15.98)$$

$$\frac{d^2z}{dt^2} = -A_1 \omega^2 \sin \omega t \quad (15.99)$$

Our initial conditions are such that at  $t = 0$ ,  $dz/dt = V_s$ , where  $V_s$  is our striking velocity, so

$$V_s = A_1 \omega \rightarrow A_1 = \frac{V_s}{\omega} \quad (15.100)$$

Then we have for  $z < 2d$

$$z = \frac{V_s}{\omega} \sin \omega t \quad (15.101)$$

$$\frac{dz}{dt} = V_s \cos \omega t \quad (15.102)$$

$$\frac{d^2z}{dt^2} = -\omega V_s \sin \omega t \quad (15.103)$$

We now use a compatibility condition that at  $z = 2d$ , both Equations 15.103 and 15.91 must yield the same answer. We shall call the time it takes the projectile to reach  $2d$ ,  $t_1$  and the velocity at that point will be  $V_1$ , thus at  $z = 2d$  we have

$$F|_{t=t_1} = \frac{\pi d^2}{4} \left( Sf'_c + N \rho V_1^2 \right), \quad z = 2d \quad (15.104)$$

$$\frac{d^2z}{dt^2} = -\omega V_s \sin \omega t_1, \quad z = 2d \quad (15.105)$$

Since  $F = ma$ , we can combine the above equations to write

$$m\omega V_s \sin \omega t_1 = \frac{\pi d^2}{4} \left( Sf'_c + N \rho V_1^2 \right), \quad z = 2d \quad (15.106)$$

Also at  $t = t_1$ , Equations 15.101 and 15.102 can be written as

$$2d = \frac{V_s}{\omega} \sin \omega t_1 \quad (15.107)$$

$$V_1 = V_s \cos \omega t_1 \quad (15.108)$$

We now rearrange Equation 15.108 to

$$V_s = \frac{\omega 2d}{\sin \omega t_1} \quad (15.109)$$

Now insert Equation 15.109 into [Equation 15.106](#) giving us

$$m\omega^2 2d = \frac{\pi d^2}{4} \left( Sf_c' + N \rho V_1^2 \right) \quad (15.110)$$

And if we make use of [Equation 15.96](#), we can obtain  $c$  as

$$c = \frac{\pi d}{2} \left( Sf_c' + N \rho V_1^2 \right) \quad (15.111)$$

We now need to find  $V_1$  which we do by squaring Equations 15.107 and 15.108 and adding them, resulting in

$$V_1^2 + \frac{4cd^2}{m} = V_s^2 \sin^2 \omega t_1 + V_s^2 \cos^2 \omega t_1 \quad (15.112)$$

Making use of a trigonometric identity and rearranging brings us to

$$c = \frac{m}{4d^2} (V_s^2 - V_1^2) \quad (15.113)$$

If we now equate Equations 15.113 and 15.111, we get

$$V_1^2 = \frac{mV_s^2 - 2\pi d^3 Sf_c'}{m + 2\pi d^3 N\rho} \quad (15.114)$$

Once we have  $V_1$  and  $c$ , the determination of the time  $t_1$  is found simply through use of Equation 15.108.

$$t_1 = \frac{1}{\omega} \cos^{-1} \left( \frac{V_1}{V_s} \right) = \sqrt{\frac{m}{c}} \cos^{-1} \left( \frac{V_1}{V_s} \right) \quad (15.115)$$

To summarize the analysis procedure for the crater region, we must first find  $V_1$  through use of Equation 15.114, then we find  $c$  through use of Equation 15.113, and finally, we find  $t_1$  through use of Equation 15.115.

If  $V$  goes to zero before time,  $t_1$  is reached, the projectile never penetrates deeper than the crater region and our analysis would be complete. The depth of penetration in this case would be found from [Equation 15.102](#).

$$V = 0 = V_s \cos \omega t \quad (15.116)$$

This would occur when

$$\omega t = \frac{\pi}{2} \rightarrow t = \frac{\pi}{2} \sqrt{\frac{m}{c}} \rightarrow \sin \omega t = 1 \quad (15.117)$$

If we insert this result into [Equation 15.101](#), we obtain the achieved depth of penetration,  $P$ .

$$P = V_s \sqrt{\frac{m}{c}} \quad (15.118)$$

The striking velocity that would make this true would be determined from [Equation 15.114](#) with  $V_1$  set equal to zero. So for a projectile to stop before creating a tunnel, the velocity is given by

$$V_{s_{\text{No tunnel}}} \leq \sqrt{\frac{2\pi d^3 Sf'_c}{m}} \quad (15.119)$$

If the projectile penetrates beyond 2 diameters into the concrete, it will enter the so-called tunnel region. When the projectile continues into the tunnel region, there is a change in the governing equation as discussed earlier. To determine the depth of penetration, we begin by combining [Equations 15.91](#) and [15.93](#) to obtain

$$m \frac{d^2 z}{dt^2} = \frac{\pi d^2}{4} \left( Sf'_c + N\rho V^2 \right), \quad 2d < z < P \quad (15.120)$$

We can transform our independent variable from time to distance and we can write

$$mV \frac{dV}{dz} = \frac{\pi d^2}{4} \left( Sf'_c + N\rho V^2 \right), \quad 2d < z < P \quad (15.121)$$

If we rewrite [Equation 15.121](#) as follows:

$$\frac{dV}{dz} = \frac{\pi d^2}{4m} \left( \frac{Sf'_c}{V} + N\rho V \right) \quad (15.122)$$

Now we integrate it from  $V_1$  to zero and  $2d$  to  $P$ , so we can write

$$\int_{V_1}^0 \left( \int_{2d}^P \frac{dV}{dz} dz \right) = \int_{V_1}^0 \left[ \int_{2d}^P \frac{\pi d^2}{4m} \left( \frac{Sf'_c}{V} + N\rho V \right) dz \right] \quad (15.123)$$

which results in

$$P = \frac{2m}{\pi d^2 N\rho} \ln \left( 1 + \frac{N\rho V_1^2}{Sf'_c} \right) - 2d, \quad 2d < P \quad (15.124)$$

If we have determined through use of [Equations 15.113](#) through [15.115](#) that a projectile will penetrate beyond the tunnel region, we can write a procedure to determine the depth of penetration as follows. First, calculate  $V_1$ ,  $c$ , and  $t_1$  as described earlier for the crater region. Then calculate  $P$  from [Equation 15.124](#). If this is greater than the concrete thickness, the projectile will perforate. If not, the projectile will penetrate to depth  $P$ . It would be good to see if a spall thickness is created (as will be described in [Section 16.3](#)) by the impact and if this is the case, we could add the spall thickness to  $P$  and perforation may still result (though with low residual velocity).

Since the dimensionless parameter  $S$  is obtained or verified experimentally, it would be nice to know how close we came to our estimate by direct calculation. If one had an experiment where a given projectile penetrated to depth  $P$ , we can back calculate  $S$  as follows. We start with [Equation 15.124](#) and rearrange thusly

$$S = \frac{N\rho V_1^2}{f'_c} \frac{1}{\left\{ \exp \left[ (P - 2d) \frac{\pi d^2 N \rho}{2m} \right] - 1 \right\}} \quad (15.125)$$

In an experiment, we usually are given the striking velocity, so we want to replace  $V_1$  in this equation with  $V_s$ , so we use [Equation 15.114](#).

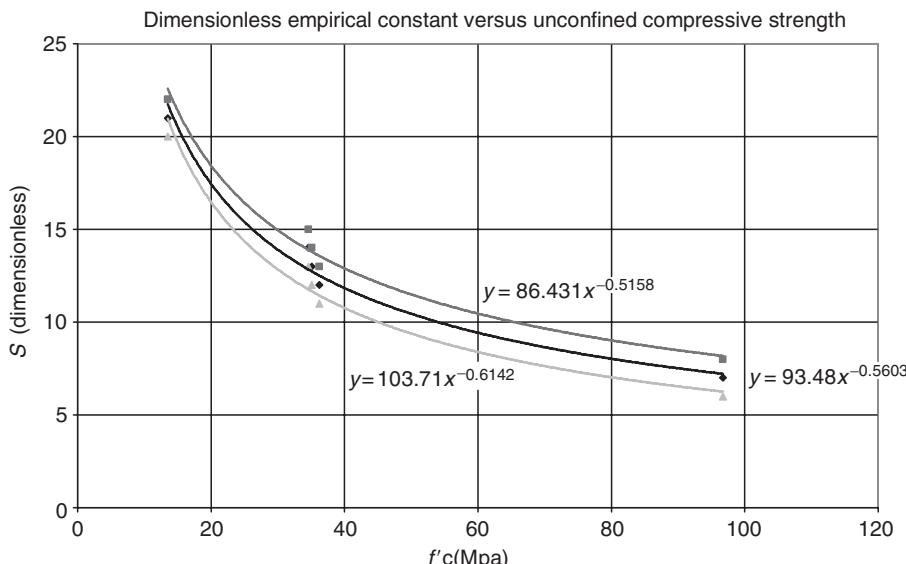
$$S = \frac{N\rho(mV_s^2 - 2\pi d^3 S f'_c)}{f'_c(m + 2\pi d^3 N \rho)} \frac{1}{\left\{ \exp \left[ (P - 2d) \frac{\pi d^2 N \rho}{2m} \right] - 1 \right\}} \quad (15.126)$$

which can be simplified to

$$S = \frac{N\rho V_s^2}{f'_c} \frac{1}{\left( 1 + \frac{2\pi d^3 N \rho}{m} \right) \left\{ \exp \left[ (P - 2d) \frac{\pi d^2 N \rho}{2m} \right] - 1 \right\}} \quad (15.127)$$

With this equation, one can find  $S$  if you know the striking velocity and the concrete strength. Forrestal et al. [5] have calibrated this equation with several experiments. A reproduction of their chart is shown in Figure 15.9 with the addition of upper and lower bounds based on their data. The equation used to determine  $S$  given the unconfined compressive strength  $f'_c$  is

$$S = 93.48 f'_c^{-0.5603} \quad (15.128)$$



**FIGURE 15.9**

Determination of dimensionless parameter  $S$  for Forrestal et al. [5] concrete penetration model.

Here recall that  $f'_c$  is in MPa and  $S$  is dimensionless. Bounding equations are shown in Figure 15.9. These equations were obtained through use of a curve fit routine.

### Problem 5

A .50-caliber projectile is fired at an extremely thick concrete wall of 2100 psi unconfined compressive strength and density of 0.084 lbm/in.<sup>3</sup> It strikes with no obliquity and a 2000 ft/s velocity. How far does it penetrate?

*Answer:*  $P = 16.7[\text{cm}]$

#### Projectile Information

$$s = 63.50[\text{mm}] \quad m = 662[\text{grains}]$$

$$d = 12.70[\text{mm}] \quad V_s = 2000 \left[ \frac{\text{ft}}{\text{s}} \right]$$

### 15.3 Penetration and Perforation of Soils

In recent times, the penetration of soils has gained importance in the terminal ballistic field. Enemy strong points have been encountered below a soil layer. Land mines need to be defeated below various types of soils as well. It is therefore necessary to determine the penetration capabilities of projectiles into soils with the intention of defeating a buried target.

As a reasonable approach to determine soil penetration, we shall use the method of Forrestal and Luk [6]. While other approaches exist, this rather simple procedure is excellent for introducing the physics of the problem.

Soils vary widely in their behavior under penetration loadings. Because the behavior is somewhat complicated, more parameters are needed to describe a soil than a material such as a metal. The first thing we have to realize is that soil can be in a state where the density is less than its locked density. The locked density is where the soil behaves like a solid or fluid in compression (i.e., its states are defined by a hydrostat). We therefore need to introduce two densities:  $\rho_0$ , its initial density and  $\rho^*$ , its locked density. We also need to define  $\eta^*$ , its locked volumetric strain. Here we define  $\eta^*$  as

$$\eta^* = 1 - \frac{\rho_0}{\rho^*} \quad (15.129)$$

Two typical models used for soils come directly from our failure theories of structures. They are the Tresca (maximum shear stress) theory and the Mohr–Coulomb theory of failure. Both of these were introduced in Section 4.2. Here we shall use a combination of the two. A Mohr–Coulomb yield criteria with a Tresca flow rule. For the Tresca criterion, once a shear stress failure level is achieved, the material strength is not increased with increasing load. With the Mohr–Coulomb criterion, the yield stress in the material increases with compressive load. The combination of the two allows the material to resist more load as compression is applied up to a point, then further increase in the compressive loading will not affect the material strength.

Similar to the aluminum penetration model, we again define the caliber-radius-head as

$$\psi = \frac{s}{d} \quad (15.130)$$

Here  $d$  is the diameter of the projectile,  $\psi$  is the caliber-radius-head, and  $s$  is the ogive radius. We again define nose length as

$$l = \frac{d}{2} \sqrt{4\psi - 1} \quad (15.131)$$

The method considers the resistance force of the soil on the penetrator to have two components: a normal force (normal stresses) and a tangential force (shear stresses and friction). If we lump the shear stress in with the stress owing to friction and furthermore assume that the tangential stress is proportional to normal stress, we can again write

$$\sigma_t = \mu \sigma_n \quad (15.132)$$

Here  $\sigma_t$  is the tangential stress,  $\sigma_n$  is the normal stress, and  $\mu$  is the proportionality constant (a coefficient of sliding friction).

Forrestal et al. [4] developed a formula for the axial force on an ogival nose which we introduced in Section 15.1 and we again use here

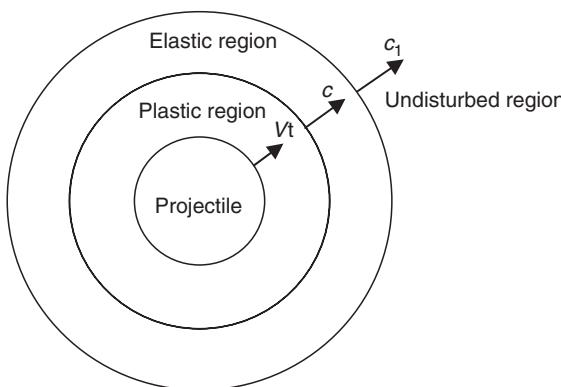
$$F_z = 2\pi s \int_{\theta_0}^{\pi/2} \left\{ \left[ \sin \theta - \left( \frac{s - \frac{d}{2}}{s} \right) \right] (\cos \theta + \mu \sin \theta) \right\} \sigma_n(V_z, \theta) d\theta \quad (15.133)$$

where

$$\theta_0 = \sin^{-1} \left( \frac{s - \frac{d}{2}}{s} \right) = \sin^{-1} \left( \frac{2\psi - 1}{2\psi} \right) \quad (15.134)$$

Here  $V_z$  is the instantaneous velocity during penetration. The stress function  $\sigma_n(V_z, \theta)$  is assumed to be similar to that of a spherically symmetric expanding cavity.

At this point, we are going to depart from the mathematics to look at the penetration event in a qualitative manner. Let us assume that we are at some axial location in the ogive of the projectile and we are looking in the direction of penetration at time,  $t$ . What we would see is illustrated in Figure 15.10. The projectile would be opening a cavity at a rate which we shall call  $V_t$ . The plastic zone would be expanding at some rate  $c$ . Here  $c$  is the speed of the plastic wave (dependent upon the Hugoniot jump conditions to be discussed in Section 16.1). The elastic zone would be expanding at a rate  $c_1 t$ . Here  $c_1$  is



**FIGURE 15.10**

Elastic and plastic compression zones at a section of an ogive penetrating into soil looking in the direction of penetration.

the speed of the dilatational wave in the material. It can be shown that  $V$  and  $c$  are related (as per the shock theory that will follow in Chapter 16) and we can define a parameter,  $\gamma$  as

$$\gamma = \frac{V}{c} = \left[ \left( 1 + \frac{\tau_c}{2E} \right)^3 - (1 - \eta^*) \right]^{\frac{1}{3}} \quad (15.135)$$

In Equation 15.135,  $E$  is Young's modulus. With the above physics, Forrestal and Luk [6] derived material response models for each of the three failure models we have discussed earlier. For the detailed derivation, the interested reader is referred to that paper. The basic idea was to have a general function for the force acting on the projectile nose that we can integrate using Newton's second law to obtain the velocity and penetration distance as a function of time.

If we insert expressions that relate the radial expansion velocity of the cavity,  $V$ , to the projectile penetration velocity,  $V_z$ , we can put the expression for the retarding force in this form

$$F_z = \alpha_s + \beta_s V_z^2 \quad (15.136)$$

where

$$\alpha_s = \frac{\pi d^2}{4} \tau_c A \left[ 1 + 4\mu\psi^2 \left( \frac{\pi}{2} - \theta_0 \right) - \mu(2\psi - 1) \sqrt{4\psi - 1} \right] \quad (15.137)$$

$$\beta_s = \frac{\pi d^2}{4} \rho_0 B \left[ \frac{8\psi - 1}{24\psi^2} + \mu\psi^2 \left( \frac{\pi}{2} - \theta_0 \right) - \frac{\mu(2\psi - 1)(6\psi^2 + 4\psi - 1)\sqrt{4\psi - 1}}{24\psi^2} \right] \quad (15.138)$$

Here the coefficients  $A$  and  $B$  are dependent upon the material model used for the soil.

Recall that the definition for the Tresca criterion implies that once a material reaches its state of maximum shear stress, it begins to deform plastically and cannot support any more load. For a soil that behaves in a Tresca type manner, we have

$$A = \frac{2}{3} \left\{ 1 - \ln \left[ \frac{\left( 1 + \frac{\tau_c}{2E} \right)^3 - (1 - \eta^*)}{\left( 1 + \frac{\tau_c}{2E} \right)^3} \right] \right\} \quad (15.139)$$

$$B = \frac{3}{2(1 - \eta^*)} + \frac{\frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2}{\left[ \left( 1 + \frac{\tau_c}{2E} \right)^3 - (1 - \eta^*) \right]^{\frac{2}{3}}} - \frac{\left[ \left( 1 + \frac{\tau_c}{2E} \right)^3 - (1 - \eta^*) \right]^{\frac{1}{3}}}{2 \left( 1 + \frac{\tau_c}{2E} \right)^4} \left[ 1 + \frac{3 \left( 1 + \frac{\tau_c}{2E} \right)^3}{(1 - \eta^*)} \right] \quad (15.140)$$

Recall that the definition for the Mohr–Coulomb criterion implies that as the compressive forces increase, it becomes harder to have the material fail in shear. For a soil that behaves in a Mohr–Coulomb type manner, we have

$$A = \frac{1}{\alpha} \left( \frac{1 + \frac{\tau_c}{2E}}{\gamma} \right)^{2\alpha} - \frac{1}{\lambda} \quad (15.141)$$

$$B = \frac{3}{(1 - \eta^*)(1 - 2\alpha)(2 - \alpha)} + \frac{1}{\gamma^2} \left( \frac{1 + \frac{\tau_c}{2E}}{\gamma} \right)^{2\alpha} \left\{ \frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2 - \frac{\gamma^3 [2(1 - \eta^*)(2 - \alpha) + 3\gamma^3]}{(1 - \eta^*)(1 - 2\alpha)(2 - \alpha) \left( 1 + \frac{\tau_c}{2E} \right)^4} \right\} \quad (15.142)$$

Also note that the Tresca criterion behaves the same as the Mohr–Coulomb criteria with  $\lambda = 0$ . We define

$$\alpha = \frac{3\lambda}{3 + 2\lambda} \quad (15.143)$$

Because of a singularity in the governing equations, there is a special set of equations for the Mohr–Coulomb criterion when we have  $\lambda = 3/4$ . In this case,

$$A = 2 \left( \frac{1 + \frac{\tau_c}{2E}}{\gamma} \right) - \frac{4}{3} \quad (15.144)$$

$$B = \frac{-2 \ln \gamma}{(1 - \eta^*)} + \frac{\left( 1 + \frac{\tau_c}{2E} \right) \left[ \frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2 \right]}{\gamma^3} - \frac{2}{3} \left[ \frac{1}{\left( 1 + \frac{\tau_c}{2E} \right)^3} - \frac{3 \ln \left( 1 + \frac{\tau_c}{2E} \right)}{(1 - \eta^*)} \right] \quad (15.145)$$

When a material behaves according to the model that combines both Mohr–Coulomb and Tresca behaviors, things become slightly more complicated. The parameters  $A$  and  $B$  will be dependent upon the rate of loading. One must keep in mind that this failure criterion implies that up to some stress level, the material will have improved resistance to compressive loading because of the internal friction of the grains and after a limit load is reached ( $\tau_m$ ), the material simply yields regardless of load. Thus, we can consider three velocity regimes:  $V < V_{\min}$ , where the yielding is completely Mohr–Coulomb behavior;  $V_{\min} < V < V_{\max}$ , where the yielding closest to the projectile is by Tresca criterion and the yielding near the elastic–plastic interface is according to the Mohr–Coulomb criterion; and  $V > V_{\max}$ , where the entire yield region is according to the Tresca model. We shall consider each of these cases.

If  $V < V_{\min}$ , we stated that the yielding is completely according to the Mohr–Coulomb model. Thus, Equations 15.141 through 15.145 apply. The equation required to determine  $V_{\min}$  is

$$V_{\min} = \sqrt{\frac{\tau_c}{\alpha \rho_0 B} \left[ \frac{\tau_m}{\tau_c} - \left( \frac{1 + \frac{\tau_c}{2E}}{\gamma} \right)^{2\alpha} \right]} \quad (15.146)$$

Recall that  $\tau_m$  is the stress level at which the material behaves according to the Tresca model. We shall discuss how we determine  $V$  shortly.

If  $V_{\min} < V < V_{\max}$ , the zone of yielding material has two subzones: a zone next to the projectile that behaves according to the Tresca model and a zone next to the elastic region that behaves according to the Mohr–Coulomb model. We shall first write the equation for  $V_{\max}$ .

$$V_{\max} = \sqrt{\frac{\tau_c \gamma^2}{\rho_0 \left[ \frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2 \right]}} \left[ \frac{1}{\alpha} \left( \frac{\tau_m}{\tau_c} \right) - \frac{1}{\lambda} - \frac{2}{3} \right] \quad (15.147)$$

If we define a coordinate,  $\xi$ , that varies from 0 at the projectile surface to 1 at the elastic-plastic interface, we can determine a coordinate,  $\xi_m$ , where the yield behavior changes from Tresca to Mohr–Coulomb. Unfortunately, this crossover point has to be solved numerically with the equation that follows:

$$\begin{aligned} & \frac{\left( 1 + \frac{\tau_c}{2E} \right)^{2\alpha}}{\left[ (1 - \eta^*)\xi_m^3 + \gamma^3 \right]^{\frac{2\alpha}{3}}} + \frac{\alpha \rho_0 V^2}{\tau_c \gamma^2} \left\{ \frac{\gamma^3 [2(1 - \eta^*)(2 - \alpha)\xi_m^3 + 3\gamma^3]}{(1 - \eta^*)(1 - 2\alpha)(2 - \alpha)[(1 - \eta^*)\xi_m^3 + \gamma^3]^{\frac{4}{3}}} \right\} \\ & + \frac{\alpha \rho_0 V^2 \left( 1 + \frac{\tau_c}{2E} \right)^{2\alpha}}{\tau_c \gamma^2 \left[ (1 - \eta^*)\xi_m^3 + \gamma^3 \right]^{\frac{2\alpha}{3}}} \left\{ \frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2 - \frac{\gamma^3 [2(1 - \eta^*)(2 - \alpha) + 3\gamma^3]}{(1 - \eta^*)(1 - 2\alpha)(2 - \alpha) \left( 1 + \frac{\tau_c}{2E} \right)^4} \right\} - \frac{\tau_m}{\tau_c} = 0 \end{aligned} \quad (15.148)$$

Keep in mind here that we know all of the information (including  $V$ ) and we are solving for  $\xi_m$ . A good math code will generally solve this equation quickly.

Once we have  $\xi_m$ , then  $A$  and  $B$  are given at the projectile surface (Tresca) by

$$A = \frac{1}{\alpha} \left( \frac{\tau_m}{\tau_c} \right) - \frac{1}{\lambda} + \frac{2}{3} \left( \frac{\tau_m}{\tau_c} \right) \ln \left[ 1 + (1 - \eta^*) \left( \frac{\xi_m}{\gamma} \right)^3 \right] \quad (15.149)$$

$$B = \frac{1}{2(1 - \eta^*)} \left\{ 3 - \frac{3 + 4(1 - \eta^*) \left( \frac{\xi_m}{\gamma} \right)^3}{\left[ 1 + (1 - \eta^*) \left( \frac{\xi_m}{\gamma} \right)^3 \right]^{\frac{4}{3}}} \right\} \quad (15.150)$$

These equations account for the fact that the yielding is Mohr–Coulomb outside of  $\xi = \xi_m$ .

If  $V > V_{\max}$ , the yielding is completely according to the Tresca model. Thus,  $A$  and  $B$  are given by

$$A = \frac{2}{3} - 2 \left( \frac{\tau_m}{\tau_c} \right) \ln \left[ \frac{\gamma}{\left( 1 + \frac{\tau_c}{2E} \right)} \right] \quad (15.151)$$

$$B = \frac{3}{2(1 - \eta^*)} + \frac{\frac{3\tau_c}{E} + \eta^* \left( 1 - \frac{3\tau_c}{2E} \right)^2}{\gamma^2} - \frac{\gamma}{2 \left( 1 + \frac{\tau_c}{2E} \right)^4} \left[ 1 + \frac{3 \left( 1 + \frac{\tau_c}{2E} \right)^3}{(1 - \eta^*)} \right] \quad (15.152)$$

To approximate the normal stress on the ogive, we can replace the spherically symmetric velocity,  $V$ , in our previous equations with  $V_z \cos \theta$ . We can write an equation for the normal stress function on the ogive [6] as

$$\sigma_n(V_z, \theta) = \tau_c A + \rho_0 B [V_z \cos \theta]^2 \quad (15.153)$$

We can write Newton's second law as

$$-F_z = m \frac{dV_z}{dt} \quad (15.154)$$

We can then convert this time integral to a distance integral as before to yield

$$-F_z = m V_z \frac{dV_z}{dz} \quad (15.155)$$

If we substitute [Equation 15.136](#) into the above and integrate, we get an equation for the acceleration, velocity, and depth of penetration, respectively, as a function of time.

$$a = -\frac{\frac{\alpha_s}{m}}{\cos^2 \left[ \tan^{-1} \left( \sqrt{\frac{\beta_s}{\alpha_s}} V_0 \right) - \frac{t \sqrt{\alpha_s \beta_s}}{m} \right]} \quad (15.156)$$

$$V_z = \sqrt{\frac{\alpha_s}{\beta_s}} \tan \left[ \tan^{-1} \left( \sqrt{\frac{\beta_s}{\alpha_s}} V_0 \right) - \frac{t \sqrt{\alpha_s \beta_s}}{m} \right] \quad (15.157)$$

$$z = \frac{m}{\beta_s} \ln \left\{ \frac{\cos \left[ \tan^{-1} \left( \sqrt{\frac{\beta_s}{\alpha_s}} V_0 \right) - \frac{t \sqrt{\alpha_s \beta_s}}{m} \right]}{\cos \left[ \tan^{-1} \left( \sqrt{\frac{\beta_s}{\alpha_s}} V_0 \right) \right]} \right\} \quad (15.158)$$

If we determine the distance where the velocity of the projectile slows to zero, we obtain the depth of penetration as

$$P = \frac{m}{2\beta_s} \ln \left( 1 + \frac{\beta_s V_0^2}{\alpha_s} \right) \quad (15.159)$$

Here  $P$  is the final penetration depth and  $V_0$  is the impact velocity.

So now that we have developed penetration formulas for soils, what do we do with them? The use of models such as this one, as nice as it is, usually carries with it some practical issues. A detailed model like this requires detailed material properties which, in practice, one rarely has. It usually will require a test or two to calibrate it. Forrestal and Luk [6] suggest using a value of 0.13 for  $\eta^*$ . The authors claim the model is relatively insensitive to it. The model was derived for normal penetration, but the authors claim good results up to impact yaw angles of 30°. In this case, they used the line-of-sight penetration depth. As one might expect, the accuracy of this particular model varies significantly with the properties of the soil. Rocks, roots, and soil layers further complicate everything. Nevertheless, the model is an excellent tool and describes the physics of soil penetrations well. This is a highly active area of current research.

### Problem 6

A .50-caliber projectile is fired at a soil berm with properties established below. How far does it penetrate?

Answer:  $P = 84.9[\text{cm}]$

Projectile information

$$s = 63.50[\text{mm}] \quad m = 662[\text{grains}]$$

$$d = 12.70[\text{mm}] \quad V_s = 2000 \left[ \frac{\text{ft}}{\text{s}} \right]$$

Soil information (assume Mohr–Coulomb behavior)

$$\text{Initial density} \quad \rho_0 = 1860 \left[ \frac{\text{kg}}{\text{m}^3} \right] \quad \mu = 0.1$$

$$\text{Locked density} \quad \rho^* = 2125 \left[ \frac{\text{kg}}{\text{m}^3} \right] \quad \lambda = 0.33$$

$$\tau_c = 1500 \left[ \frac{\text{lbf}}{\text{in.}^2} \right] \quad E = 2 \times 10^7 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

$$\tau_m = 2500 \left[ \frac{\text{lbf}}{\text{in.}^2} \right]$$

---

## 15.4 Penetration and Perforation of Ceramics

The desire to decrease the weight of vehicles coupled with constant improvements in manufacture has increased interest in the use of ceramics as armor. The design of an armored vehicle using ceramics requires an understanding of their behavior under impact loads. The advantages of ceramic armor are its relatively low density, high hardness, and high compressive strength. The disadvantages are that ceramics are usually brittle, have low tensile strength which when coupled with high compressive strength can be a problem from a spallation standpoint, they allow the protection to be degraded in a multi-hit situation, and they are somewhat expensive. Their complex structural behavior makes them difficult to model although this is only a disadvantage to the designers.

The response of a ceramic to penetration is unique amongst all of the other materials discussed in this text. The material behaves differently depending on the radial confinement and whether it is backed or not. For reasons such as spallation, they are usually backed by a fiber reinforced composite, plastic, elastomer, or metal plate. If a ceramic is not backed, it will most likely spall when subjected to a high-shock load. This spallation can be analyzed by the techniques we will discuss in Chapter 16 on shock theory.

Although not exhaustive, this is a list of common ceramics currently either in use or being studied for armor applications:

Boron carbide ( $\text{B}_4\text{C}$ )

Silicon carbide ( $\text{SiC}$ )

Titanium di-boride ( $\text{TiB}_2$ )

Aluminum nitride ( $\text{AlN}$ )

Alumina ( $\text{Al}_2\text{O}_3$ )

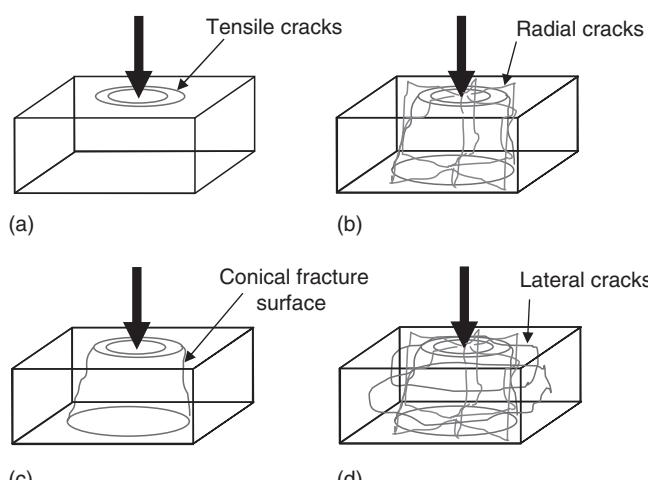
Historically, terra cotta (ceramic) armor has been found in Chinese tombs dating from 400 BC. Before First World War, the practice of placing coal bunkers around magazines to take advantage of comminution, a phenomena that we shall discuss shortly.

If a ceramic is backed, one can take advantage of its high compressive strength to resist penetration. This will cause the tip of the penetrator to deform. Large stresses then build up in the penetrator. If the striking velocity is low enough, the penetrator will break up or ricochet. This process is called interface defeat or infinite dwell. If the penetrator survives the initial impact, the ceramic begins to fail. This process is complicated which is why it is difficult to model, but it is key to understanding the behavior and utilizing the ceramic to the maximum extent possible.

The ceramic penetration process has been documented by Cheeseman [7]. After an initial dwell and several reflections of the shocks and rarefactions, the following events occur and will either continue to perforation or stop when the penetration is arrested. Initially, tensile cracks appear near the penetrator forming circular rings. These cracks propagate along the principal stress planes which are usually  $25^{\circ}$ – $75^{\circ}$  from the surface normal.

Once the cracks reach the distal boundary, they coalesce into conical form. At this point, if the ceramic was not backed, a plug would be ejected and the material would be perforated. If the plate is backed, then at the time when the conoid is formed, the stress is redistributed circumferentially and radial cracks appear. After the appearance of radial cracks, lateral cracking in the plane of the impact surface forms. This process is illustrated in Figure 15.11. With backing material present that holds the ceramic plug in place, the material has nowhere to go so micro-cracking begins. This pulverizes the ceramic material. This is known as the comminuted zone. The process of comminution and the sand-like character of the comminuted material erode the penetrator at a rapid rate. The powdered material continually gets in the way of the penetrator. This material flows radially outward and rearward. A similar effect occurs during shaped charge jet penetration into sand bags.

The penetration of a ceramic armor is highly dependent upon the boundary conditions. It is known that confinement increases the penetration resistance (increasing  $V_{50}$ ). This effect is not because of the strength of the confinement material. A stiffer backing also increases  $V_{50}$  to a point. There does appear to be an upper limit though. The key to good design appears to be the movement of the neutral axis out of the ceramic material and into the backing material [7].



**FIGURE 15.11**  
Ceramic fracture process illustrated.

One of the parameters that must be considered when designing ceramic armor is the fact that there can be large dynamic deflections during an impact. This can be more than twice the static deflection left after a penetration event. Care must be taken in mounting sensitive components in the sway space of the armor. Impact to the component may impede fighting efficiency of the vehicle. This effect is still being investigated.

One can see from the process that modeling this event (either numerically or analytically) is nontrivial. The numerical approach is the subject of intense research. A cursory look at the problem shows that we need a model for the ceramic before fracture, a crack propagation model, a micro-cracking model, a model that handles the comminution and, after all that we have to model the behavior of the backing material.

Florence [8] developed a simplified model to determine the limit velocity for an aluminum backed ceramic armor plate. This model assumes that the projectile was a short cylindrical rod, and the conoid is idealized and the loading on the backing plate was assumed to occur across the base of this conoid. The backing material is assumed to fail when the maximum strain in it exceeds its failure strain.

$$\varepsilon_r = 1.82f(a)\frac{K}{S} \quad (15.160)$$

Here  $\varepsilon_r$  is the maximum strain in the aluminum and the other parameters are defined below. The parameter  $K$  is the kinetic energy of the penetrator given by

$$K = m_p \frac{V_s^2}{2} \quad (15.161)$$

Here  $m_p$  is the penetrator mass and  $V_s$  is the striking velocity. The strength parameter,  $S$ , is given by

$$S = \sigma_Y h_m \quad (15.162)$$

where  $\sigma_Y$  is the aluminum yield strength and  $h_m$  is the thickness of the aluminum plate. The momentum parameter,  $f(a)$ , is given by

$$f(a) = \frac{m_p}{\pi a^2 [m_p + (m_c + m_m)\pi a^2]} \quad (15.163)$$

Here the mass subscripts “p,” “c,” and “m” refer to the mass of the projectile, ceramic, and backing plate material, respectively. We can rearrange these formulas to obtain the limit velocity of the projectile–armor combination [8] as

$$V_l = \sqrt{\frac{\varepsilon_r S}{(0.91)m_p f(a)}} \quad (15.164)$$

This can be used exactly like the limit velocity in the Lambert model of [Section 15.1](#).

More complicated models exist for ceramic penetrations. Walker and Anderson [9] proposed a penetration model for ball ammunition penetrating ceramic backed by a metal plate. The model assumes axisymmetric behavior, that a velocity profile in both the target and the penetrator can be specified analytically, that the rear of the projectile only experiences elastic waves (i.e., the plastic waves are arrested before reaching the rear surface), and that the shear behavior of the target can be specified as a pressure-dependent flow stress

(Mohr–Coulomb) for the ceramic with a constant flow shear stress (Von-Mises) for the metal. The model is quite detailed and limitations on space prevent the inclusion of the model here; however, the interested reader is directed to the paper for a full description of the model. The model still has to be solved by computer but the nice thing is that one can program it into MathCAD or MATLAB and make many calculations quickly. Since the model uses readily available parameters, it can be run for any materials consistent with the velocity and material behavior assumptions. The authors claim 15% accuracy which is good.

Zaera and Sanchez-Galvez [10] proposed an interesting penetration model based on Tate's penetration equation. The model is elegant for its simplicity and seems to correlate well with medium-caliber ammunition. The model neglects mushrooming of the projectile and only includes deformation because of erosion. It assumes rigid, perfectly plastic behavior in a zone confined to be near the projectile tip.

The three basic equations are as follows. For the penetration velocity,  $u$  we have

$$\frac{1}{2}\rho_p(v-u)^2 + Y_p = \frac{1}{2}\rho_t u^2 + R_t \quad (15.165)$$

Here  $\rho_p$  is the density of the projectile material,  $\rho_t$  is the density of the target material,  $v$  is the projectile velocity,  $u$  is the penetration velocity,  $Y_p$  is the dynamic yield strength in the projectile, and  $R_t$  is the ballistic resistance of the target. The time rate of change of the projectile length owing to erosion is

$$\frac{dL}{dt} = -(v-u) \quad (15.166)$$

Here  $L$  is the length of the projectile. Finally, the deceleration of the projectile is given by

$$\frac{dv}{dt} = -\frac{Y_p}{\rho_p L} \quad (15.167)$$

At some point in time, the pressure on the projectile nose will be unable to erode it further, thus Equation 15.167 will switch to

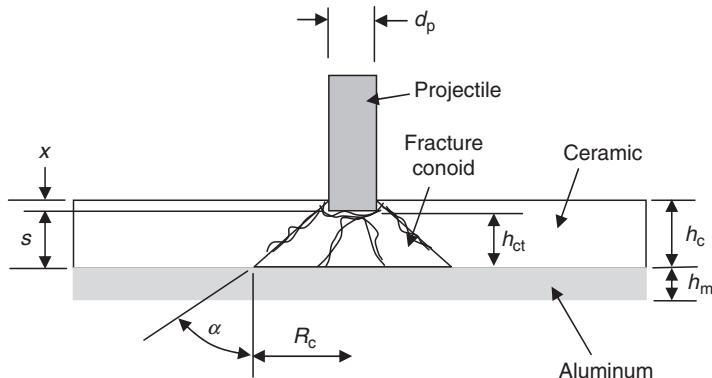
$$\frac{dv}{dt} = -\frac{R_t + \frac{1}{2}\rho_t v^2}{\rho_p L} \quad (15.168)$$

To simplify the geometry in the model, the concept of equivalent length is invoked. In this case, the length is adjusted based on the amount of material present in the projectile. The equivalent diameter is given by

$$d_{eq} = \frac{\int_0^{L_p} d^3(z) dz}{\int_0^{L_p} d^2(z) dz} \quad (15.169)$$

The equivalent length is then

$$L_{eq} = \frac{4m_p}{\pi \rho_p d_{eq}^2} \quad (15.170)$$



**FIGURE 15.12**

Model of Zaera and Sanchez-Galvez illustrated.

As described earlier, when a projectile impacts ceramic armor, a fracture conoid develops after interaction of the stress waves with the boundaries. We shall assume the time for this event to be

$$t_{\text{conoid}} = \frac{h_c}{c_L} + \frac{h_c}{v_{\text{rad.crack}}} \quad (15.171)$$

Here  $h_c$  is the thickness of the ceramic (shown in Figure 15.12),  $c_L$  is the longitudinal wave speed in the material, and  $v_{\text{rad.crack}}$  is the speed of radial crack growth. We shall also assume, based on observations [10], that

$$v_{\text{rad.crack}} = \frac{1}{5} c_L \quad (15.172)$$

During the penetration event, assuming the limit velocity is exceeded, the projectile tip will meet the crack front at some time. This will effectively change the mode of penetration. The equation for this is given by

$$z + s_{\text{crack}} = h_c \quad (15.173)$$

The linear momentum equation assumes a constant velocity in the projectile ( $v$ ), a jump discontinuity in velocity based on our flow rule at the ceramic-projectile interface ( $u$ ), and also assumes a uniform velocity ( $w$ ) in the metal backing plate. If we call  $p_c$  the momentum, we can write

$$\frac{dp_c}{dt} = Y_c \pi \frac{d_{\text{eq}}^2}{4} - f_m \pi R_c \quad (15.174)$$

Here  $f_m$  is the force exerted by the backing plate,  $R_c$  is the base radius of the fracture conoid, and  $Y_c$  is the penetration strength of the ceramic.

If we define  $h_{\text{ct}}$  as the instantaneous thickness of the ceramic and  $\alpha$  as the conoid semi-apex angle, we can define  $R_c$  based on geometry as

$$R_c = \frac{d_{\text{eq}}}{2} + h_{\text{ct}} \tan \alpha \quad (15.175)$$

We can now integrate [Equation 15.174](#) to yield

$$p_c = \pi \rho_c h_{ct} \left[ u \left( \frac{d_{eq}^2}{16} + \frac{R_c^2}{12} + \frac{d_{eq} R_c}{12} \right) + w \left( \frac{d_{eq}^2}{48} + \frac{R_c^2}{4} + \frac{d_{eq} R_c}{12} \right) \right] \quad (15.176)$$

Since  $w$  was introduced, we have to alter [Equation 15.165](#) to

$$\frac{1}{2} \rho_p (v - u)^2 + Y_p = \frac{1}{2} \rho_t (u - w)^2 + R_t \quad (15.177)$$

Keep in mind here that when the projectile is in the ceramic,  $R_t = Y_c$  and when in the backing plate,  $R_t = Y_m$ .

Once the ceramic fractures and is comminuted, then its strength is significantly reduced. This is accounted for by using

$$Y_c = \begin{cases} Y_{co} & t \leq t_{conoid} \\ Y_{co} \left( \frac{u-w}{u_{phase1}} \right)^2 & t > t_{conoid} \end{cases} \quad (15.178)$$

In this expression,  $u_{phase1}$  is the value of  $u$  at  $t = t_{conoid}$ .

Zaera and Sanchez-Galvez chose an energy approach to the penetration of the metal backing plate. The work dissipated by plastic deformation is given by

$$E_p = \pi h_m Y_m \delta \left( \frac{2}{3} h_m + \frac{1}{2} \delta \right) \quad (15.179)$$

Here  $\delta$  is the deflection at the center of the plate and the subscript "m" refers to the backing plate itself. The time rate of change of plastic work is then

$$\frac{dE_p}{dt} = \pi h_m Y_m \frac{d\delta}{dt} \left( \frac{2}{3} h_m + \delta \right) = \pi h_m Y_m w \left( \frac{2}{3} h_m + \delta \right) \quad (15.180)$$

The work to deform the interface is given by

$$T = \pi R_c^2 f_m \delta \quad (15.181)$$

Therefore, the time rate of change of this work is

$$\frac{dT}{dt} = \pi R_c^2 f_m \frac{d\delta}{dt} = \pi R_c^2 f_m w \quad (15.182)$$

The kinetic energy of the plate material is

$$E_k = \frac{1}{2} \pi R^2 h_m \rho_m w^2 \quad (15.183)$$

It then follows that the time rate of change of kinetic energy is

$$\frac{dE_k}{dt} = \pi R^2 h_m \rho_m w \frac{dw}{dt} \quad (15.184)$$

Equating Equations 15.180, 15.182, and 15.184 gives us

$$R_c^2 f_m = h_m Y_m \left( \frac{2}{3} h_m + \delta \right) + R^2 h_m \rho_m \frac{dw}{dt} \quad (15.185)$$

When the projectile reaches the backing plate, the equation for the deceleration can be written as

$$\frac{dv}{dt} = - \frac{Y_m + \frac{1}{2} \rho_m (v - w)^2}{\rho_p L} \quad (15.186)$$

We can use Equation 15.180 once more for the time rate of change of plastic energy and modify Equation 15.182 for the time rate of change of work as

$$T = \pi \frac{d_{eq}^2}{4} Y_m \delta \quad (15.187)$$

and differentiating with respect to time

$$\frac{dT}{dt} = \pi \frac{d_{eq}^2}{4} Y_m \frac{d\delta}{dt} = \pi \frac{d_{eq}^2}{4} Y_m w \quad (15.188)$$

The kinetic energy for the backing plate is

$$E_k = \frac{1}{2} m_m w^2 \quad (15.189)$$

Its time rate of change is

$$\frac{dE_k}{dt} = m_m w \frac{dw}{dt} \quad (15.190)$$

This leads us to the equation for the deceleration in the plate as

$$\frac{dw}{dt} = \frac{\pi \frac{d_{eq}^2}{4} Y_m - \pi h_m Y_m \left( \frac{2}{3} h_m + \delta \right)}{m_m} \quad (15.191)$$

We need to define  $m_m$  as the effective mass of the plate given by

$$m_m = \pi \rho_m \left[ R^2 h_m - \frac{d_{eq}^2}{4} (h_m - h_{mt}) \right] \quad (15.192)$$

In this case,  $h_{mt}$  is the distance left to the free surface of the plate (i.e., distance remaining to be penetrated).

The armor is said to be perforated when

$$h_{mt} = 0 \quad (15.193)$$

This would be a piercing/petaling type perforation. Additionally, the armor can be defeated by plugging if at any time

$$v = w \quad (15.194)$$

Thus, the plug and projectile would be moving at the same rate.

The line-of-sight thickness can be used to handle obliquity. Thus, we would set

$$h'_t = \frac{h_t}{\cos \theta} \quad (15.195)$$

where  $\theta$  is our obliquity angle measured from the plate normal. We also have to be careful that all our measurements are transformed to these lengths. Physical data shows that after about a  $20^\circ$  obliquity, the fracture of the ceramic starts to deviate from this model. The authors show fair agreement up to  $50^\circ$  [10].

This model is much simpler than others and provides reasonable results. Unfortunately, it still has to be coded into a computer to solve the equations simultaneously (and as the penetrator moves into the backing plate, sequentially). It is nice because it can account for obliquity. If one generally has to get more detailed than this, direct numerical simulation is probably the best approach.

We have presented some analytic equations for the penetration of ceramic armor by projectiles. Ceramics are nearly always used with some type of backing plate. These models, though fairly complicated, allow rapid analysis of designs. They do, however, need to be coded to be used. If more detailed results are required, one must resort to direct numerical simulation.

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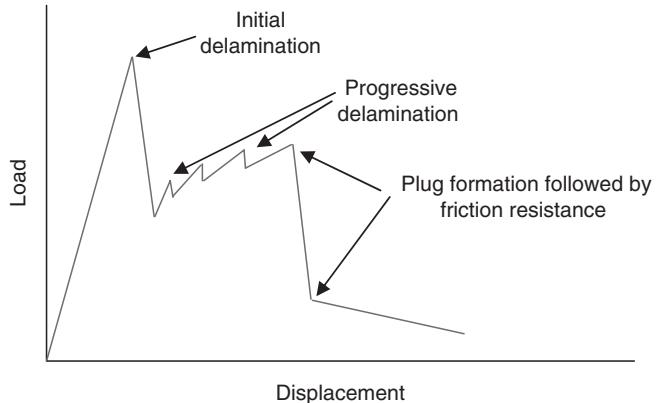
## 15.5 Penetration and Perforation of Composites

Composites have arguably been used as armor materials since the middle ages. Advantages of using composite materials are their relatively low density, their tailorable properties, and fair to high strength. The disadvantages of composites are the inconsistency of hand lay up, the dependency of strength on manufacturing process, and the somewhat expensive nature of their manufacture. Additionally, composites pose a problem to the designers because they are difficult to model.

Composites resist penetration primarily by dissipating energy. Because of the complex structure of the material, this energy dissipation manifests itself in the failure of portions of the laminate, fiber breakage, matrix cracking, and delamination. Since composite properties can vary from isotropic to a complicated anisotropic, the behavior will depend upon the configuration.

In chopped fiber composites the material properties are usually isotropic. A notable exception to this is in injection moldings where the fibers tend to align with the flow directions near mold gate areas or areas of higher velocity flow. An isotropic composite is usually treated as we do a metal and those formulas should work well. We recommend that one try the Lambert model first or the Tate model.

Continuous fiber composites behave differently from metallic plates. A typical load-displacement curve is shown in [Figure 15.13](#). In this figure, after an initial delamination point, where the load carrying capability is degraded, we see increases and decreases in load carrying ability based on successive delaminations of material followed by a final



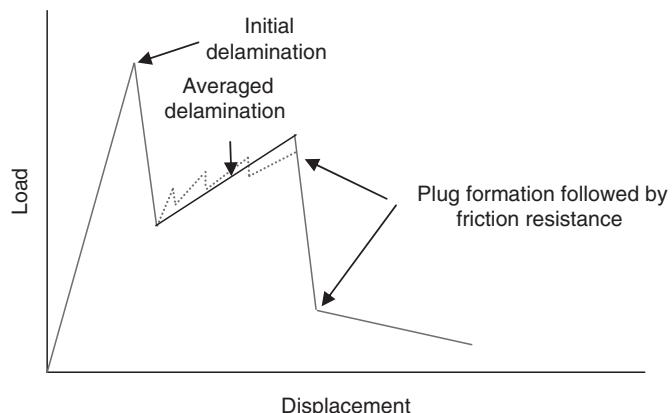
**FIGURE 15.13**  
Load–displacement curve for a typical continuous fiber reinforced composite.

plug shear out. This delamination actually promotes energy dissipation by forcing the fibers to elongate. In many composites, shear failure of the fibers as well as tensile failures dominate during an impact [1].

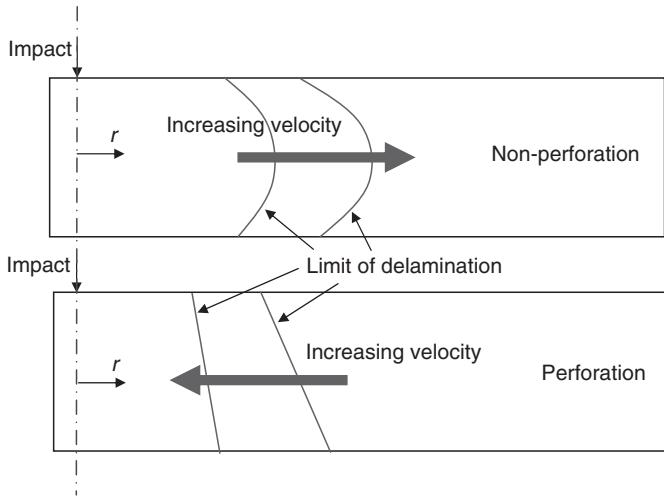
It is extremely difficult to obtain an analytical model for the penetration of continuous fiber composites. This is due to the change of energy dissipation as the composite is damaged. Finite element methods have been utilized to determine limit velocities [1], but there are nuances to each analysis that must be explained.

The first issue that must be dealt with is how to handle the damage and its effect on the remaining strength of the composite. Some researchers have actually modeled each lamina with its correct directional properties and assumed a failure criteria based on interlaminar shear strength [1]. When the interlaminar shear strength is exceeded, the layer no longer supports shear and the overall bending stiffness is reduced. This can be accounted for explicitly having the model change internal constraints between layers or implicitly by tracking the overall smeared bending stiffness of the composite and reducing it based on the lamina that failed. Another means of handling the behavior of the composite is to average the stiffness change because of the progressive failure of lamina as shown in Figure 15.14 [1]. The issue with this approach is that test data from some sort of penetration event is required.

A second issue with analyzing fiber reinforced composites is the actual failure of the fibers themselves. The fibers can themselves delaminate from the matrix. They can also fail



**FIGURE 15.14**  
Load–displacement curve for a typical fiber reinforced composite modeled with averaged properties after initial delamination.



**FIGURE 15.15**

Extent of delamination in a composite with respect to increasing velocity during both perforating and non-perforating impacts.

in tension and are usually very sensitive to fracture. These issues of necessity complicate the analysis.

Cheeseman [7] has performed extensive work in the area of composite materials under impact loads and has made the following observations regarding their behavior. First, delaminations tend to prefer moving along the fiber direction. Additionally, compression of the composite material (e.g., at clamped locations) tends to suppress delamination, as one would expect. The extent of delamination increases linearly as the distal surface is approached if perforation occurs. However, the delamination increases then decreases if no penetration occurred. As the impact velocity increases, the delamination decreases indicating that the bending of the target becomes less significant. This is illustrated in Figure 15.15.

With the information presented here, we have seen that the penetration of composite armor is by no means simple. We have discussed some issues with modeling these types of materials and their general behavior during a penetration event. This is an area of intense active research.

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# 16

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## *Shock Physics*

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Shock physics is the study of how high intensity, highly transient events affect materials. There are essentially two areas where this applies. The first area is shocks in nonreacting materials. This field of study is important because it allows one to determine whether materials will survive a dynamic event or not. It tells us information about the material that would not be predicted by static equilibrium solid mechanics. The second area is that of reacting material behavior. This area is important because it allows us to see whether a shock is sufficient to begin and foster a chemical reaction such as a detonation. Both of these areas are the subject of whole textbooks, however, we shall only devote sufficient space to introduce them to the reader.

An important subset of nonreacting shocks is how stresses developed by these input loads propagate and reflect off free surfaces potentially leading to spallation. Spallation is an important process in ballistics whereby the target of a projectile may be compromised without perforation leading to damaging behind-armor effects.

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### 16.1 Shock Hugoniots

A most lucid treatment of the Rankine–Hugoniot jump equations is found in the book *Explosives Engineering* by Cooper [1]. In the shocking of a solid, it is critical that we understand these equations completely. The purpose of this section will be to gain an understanding of the equations required to characterize the shock front in a solid (or fluid).

First we shall describe a Hugoniot. Simply put, a Hugoniot (Hyoo' gon nee oh) is a curve that contains all possible equilibrium states at which a material can exist. It is an empirically derived curve that relates any two of the following variables to one another: Pressure,  $p$ ; shock velocity,  $U$ ; particle velocity,  $u$ ; specific volume,  $v$  (or density,  $\rho$ ). It is not an equation of state although it can be used in a similar manner. It is sometimes used as if it was an isentrope even though it is not the same. It is not the same because entropy increases across a shock. It is derived experimentally and therefore the experiment will have all the irreversibilities present.

A velocity Hugoniot is an empirical relationship that relates particle velocity in a material to the velocity of a shock front moving through that material. For most materials, it is a simple linear relationship expressed in the form

$$U = c_0 + su \quad (16.1)$$

Here  $U$  is the speed of propagation of the shock front,  $c_0$  is the bulk speed of sound in the medium (not really a sound speed per se but the  $y$ -intercept of the Hugoniot curve),  $u$  is the particle velocity, and  $s$  is an empirically obtained velocity coefficient. In some materials,

the curve is bilinear or trilinear usually indicating a phase change, though some authors have fitted quadratics or cubics to the curves.

The real power of this simple relationship is seen when we use it in conjunction with our equations of mass conservation, conservation of momentum, and conservation of energy as repeated below.

$$\frac{\rho_1}{\rho_0} = \frac{v_0}{v_1} = \frac{U - u_0}{U - u_1} \quad (16.2)$$

$$p_1 - p_0 = \rho_0(u_1 - u_0)(U - u_0) \quad (16.3)$$

$$e_1 - e_0 = \frac{p_1 u_1 - p_0 u_0}{\rho_0(U - u_0)} - \frac{1}{2}(u_1^2 - u_0^2) \quad (16.4)$$

Here the subscript “0” represents conditions ahead of the shock wave and “1” represents conditions after the passage of the wave. It is useful at this stage to examine an example problem.

### Example Problem 1

A slab of polystyrene has the following properties:

$$\rho_0 = 1.044 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_0 = 2.746 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s = 1.319$$

The particle velocity in an experiment is known to be  $u_1 = 1.37 \text{ km/s}$ . Calculate the shock velocity and shock pressure.

The shock velocity follows from [Equation 16.1](#). Where, plugging in numbers we have

$$U = (2.746) \left[ \frac{\text{km}}{\text{s}} \right] + (1.319)(1.37) \left[ \frac{\text{km}}{\text{s}} \right] = 4.553 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.5)$$

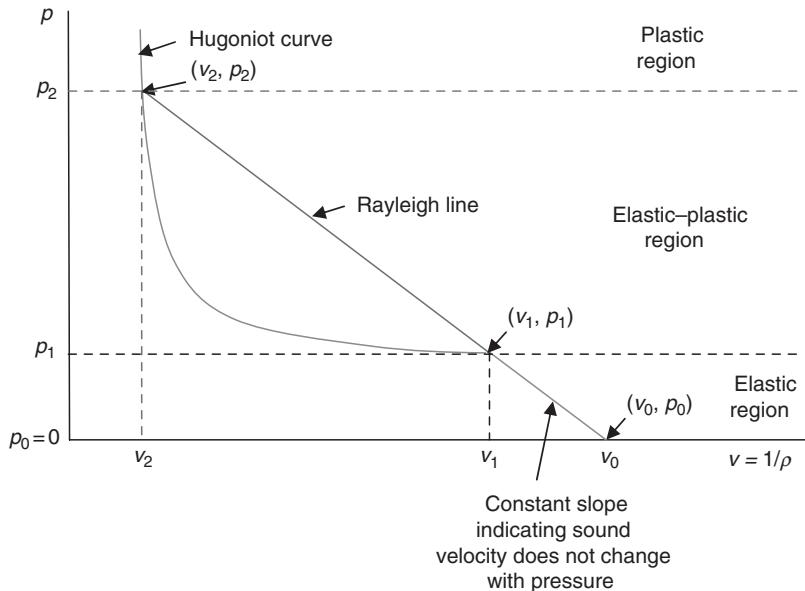
The pressure is obtained from conservation of momentum (with  $p_0$  and  $u_0 = 0$ ) using [Equation 16.3](#)

$$p_1 = \rho_0 u_1 U = (1.044) \left[ \frac{\text{g}}{\text{cm}^3} \right] (1.37) \left[ \frac{\text{km}}{\text{s}} \right] (4.553) \left[ \frac{\text{km}}{\text{s}} \right] = 6.512 \text{ [GPa]} \quad (16.6)$$

Now wait a minute. How did those units work out? It is good to remember that with density in  $\text{g/cm}^3$  and velocities in  $\text{km/s}$  we obtain answers in GPa. This is done so that we do not have a lot of zeros or  $10^x$  powers around. Here is the breakout

$$\begin{aligned} & (1.044) \left[ \frac{\text{g}}{\text{cm}^3} \right] (100)^3 \left[ \frac{\text{cm}^3}{\text{m}^3} \right] \left( \frac{1}{1000} \right) \left[ \frac{\text{kg}}{\text{g}} \right] (1.37) \left[ \frac{\text{km}}{\text{s}} \right] (4.553) \left[ \frac{\text{km}}{\text{s}} \right] (1000)^2 \left[ \frac{\text{m}^2}{\text{km}^2} \right] \\ & = 6.512 \times 10^9 \left[ \frac{\text{kg}}{\text{m-s}^2} \right] 6.512 \times 10^9 \left[ \frac{\text{kg}}{\text{m-s}^2} \right] = 6.512 \times 10^9 \left[ \frac{\text{kg-m}}{\text{s}^2} \right] \\ & = 6.512 \times 10^9 \left[ \frac{\text{N}}{\text{m}^2} \right] = 6.512 \times 10^9 \text{ [Pa]} = 6.512 \text{ [GPa]} \end{aligned} \quad (16.7)$$

You can see why we will not carry the units around in these examples any longer.



**FIGURE 16.1**  
A  $p$ - $v$  diagram showing elastic, elastic–plastic, and plastic region of a material.

If we combine [Equation 16.1](#) with our continuity and momentum equations [1], we obtain the  $p$ - $v$  Hugoniot in the following form:

$$p_1 = \frac{c_0^2(v_0 - v_1)}{[v_0 - s(v_0 - v_1)]^2} \quad (16.8)$$

For simplicity, we assumed  $p_0$  and  $u_0$  were equal to zero in Equation 16.8. We need to recall that the specific volume,  $v$ , is equal to  $1/\rho$ . This Hugoniot then tells us how pressure varies with density. Equation 16.8 is very powerful in the sense that it can tell us to what pressure a material will jump if we know the change in density or specific volume. This “jump” will occur through the formation of a shock wave. This can be seen on a  $p$ - $v$  diagram such as Figure 16.1. In this figure, we have noted the elastic, elastic–plastic, and plastic regions to be discussed later.

We shall now look at another example.

### Example Problem 2

A slab of aluminum has the following properties:

$$\begin{aligned}\rho_0 &= 2.785 \left[ \frac{\text{g}}{\text{cm}^3} \right] \\ c_0 &= 5.328 \left[ \frac{\text{km}}{\text{s}} \right] \\ s &= 1.338\end{aligned}$$

If we shock this material with a pressure of 40.2 GPa, what will the density of the material be behind the shock front? If the material is initially at rest, how fast will the particles move behind the shock wave and what will the velocity of the shock wave be?

To determine the density of the material behind the shock front, we need [Equation 16.8](#). We note here, however, that this equation is in terms of the specific volume. We need to convert our initial data as follows:

$$\rho_0 = 2.785 \left[ \frac{\text{g}}{\text{cm}^3} \right] = \frac{1}{v_0} \rightarrow v_0 = 0.359 \left[ \frac{\text{cm}^3}{\text{g}} \right] \quad (16.9)$$

Now let us rewrite [Equation 16.8](#). We need to rearrange our equation into a quadratic so that we can solve it easily

$$[v_0 - s(v_0 - v_1)]^2 p_1 - c_0^2(v_0 - v_1) = 0 \quad (16.10)$$

Now if we put in our values noting that km/s, cm<sup>3</sup>/g, and GPa are consistent units, we can write

$$v_1^2 + 0.213v_1 - 0.133 = 0 \quad (16.11)$$

If we solve this using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (16.12)$$

we get

$$v_1 = \frac{-0.213 \pm \sqrt{(0.213)^2 - (4)(1)(-0.133)}}{2(1)} = -0.107 \pm 0.380 \left[ \frac{\text{cm}^3}{\text{g}} \right] \quad (16.13)$$

$$v_1 = 0.273 \left[ \frac{\text{cm}^3}{\text{g}} \right] \quad (16.14)$$

We chose this root because it is impossible to have a negative density. Thus, our density behind the shock wave is

$$\rho_1 = \frac{1}{v_1} = \frac{1}{0.273 \left[ \frac{\text{cm}^3}{\text{g}} \right]} = 3.664 \left[ \frac{\text{g}}{\text{cm}^3} \right] \quad (16.15)$$

Nearly double the density. To find the speed at which the shock wave will propagate, we need to do a little algebra. We know from our Hugoniot relation that

$$U = c_0 + su \quad (16.16)$$

We also know that from [Equation 16.2](#) we can write, assuming that  $u_0 = 0$

$$\frac{v_0}{v_1} = \frac{U}{U - u_1} \quad (16.17)$$

If we put some numbers in here, we have

$$U = (5.328) \left[ \frac{\text{km}}{\text{s}} \right] + (1.338)u_1 \left[ \frac{\text{km}}{\text{s}} \right] = 5.328 + 1.338u_1 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.18)$$

and

$$\frac{v_0}{v_1} = \frac{0.359}{0.273} = 1.315 = \frac{U}{U - u_1} \quad (16.19)$$

Substitution of [Equation 16.18](#) into [Equation 16.19](#) yields

$$1.315 = \frac{5.328 + 1.338u_1 \left[ \frac{\text{km}}{\text{s}} \right]}{5.328 + 0.338u_1 \left[ \frac{\text{km}}{\text{s}} \right]} \quad (16.20)$$

Solving for  $u_1$  gives us

$$u_1 = 1.88 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.21)$$

Our shock velocity then follows directly from [Equation 16.18](#)

$$U = (5.328) \left[ \frac{\text{km}}{\text{s}} \right] + (1.338)(1.88) \left[ \frac{\text{km}}{\text{s}} \right] = 7.84 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.22)$$

We could also have solved this using [Equation 16.25](#).

A jump as described in the previous paragraph will take place through the formation of a shock wave and proceed along what is called a Rayleigh line. The equation of the Rayleigh line is derived by a combination of the mass and momentum equations and, for convenience, setting  $u_0 = 0$ . This results in

$$p_1 - p_0 = \frac{U^2}{v_0} - \frac{U^2}{v_0^2} v_1 \quad (16.23)$$

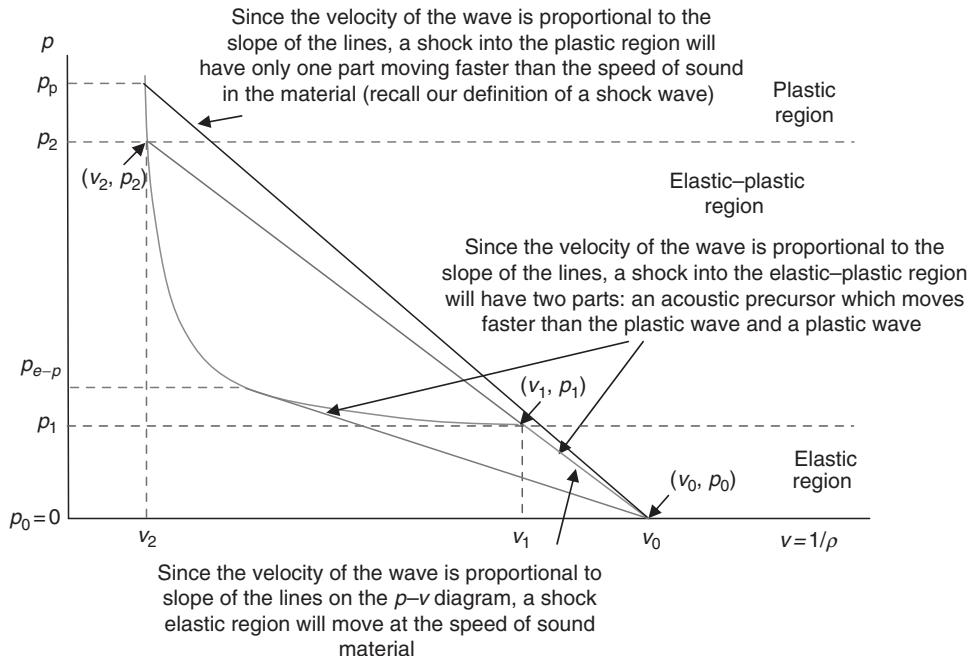
The slope of the Rayleigh line is then

$$\text{slope} = \frac{U^2}{v_0^2} = \rho_0^2 U^2 \quad (16.24)$$

Recall from thermodynamics that the area under a  $p$ - $v$  diagram represents the work done on or by the system. Then if we shock a system up a Rayleigh line and allow it to relax along the Hugoniot, the network we have done on the system is determined from the area between the curves. [Figure 16.2](#) shows how, depending on the pressure to which we shock a material, the wave speeds will vary. In fact, if we shock a material into the elastic-plastic regimes there will be two shocks, an elastic wave (precursor) that will move at the longitudinal wave speed (speed of sound) in the solid and a plastic wave which will move at a slower speed. We shall discuss this further later.

If we assume  $p_0$  and  $u_0$  are equal to zero and combine the momentum equation ([Equation 16.3](#)) with our  $U$ - $u$  Hugoniot equation ([Equation 16.1](#)), we obtain the  $p$ - $u$  Hugoniot in the following form:

$$p_1 = \rho_0 u_1 (c_0 + s u_1) \rightarrow p_1 = \rho_0 c_0 u_1 + \rho_0 s u_1^2 \quad (16.25)$$



**FIGURE 16.2**  
A  $p$ - $v$  diagram describing the wave behavior in the elastic, elastic-plastic, and plastic regimes.

This relationship gives the pressure as a function of material velocity,  $u$ , when the material is initially at rest. If the material was not initially at rest, our equation would be a little more complicated

$$p_1 = \rho_0 c_0 (u_1 - u_0) + \rho_0 s (u_1 - u_0)^2 \quad (16.26)$$

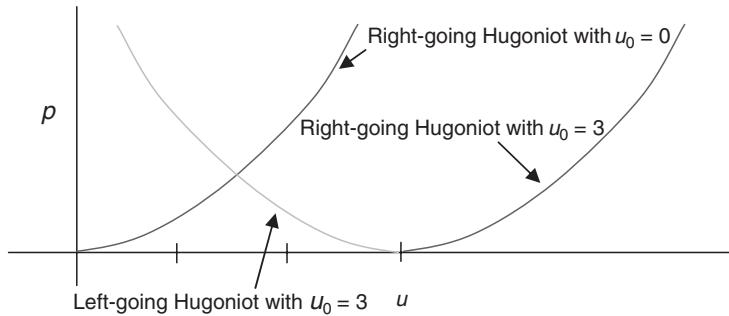
This equation was obtained by taking [Equation 16.25](#) and subtracting the same equation with  $u = u_0$ . This would be appropriate if the wave was moving to the right ( $u_1 > u_0$ ); thus, it is aptly called a “right-going Hugoniot” in following with the derivation set forth in Ref. [1]. If the wave was moving to the left ( $u_1 < u_0$ ), we would have a left-going Hugoniot and the equation would be

$$p_1 = \rho_0 c_0 (u_0 - u_1) + \rho_0 s (u_0 - u_1)^2 \quad (16.27)$$

The effect of having a nonzero  $u_0$  is to shift the  $x$ -intercept of the curve as depicted in [Figure 16.3](#).

We have these wonderful equations for a left- and right-going wave (the Hugoniot) so what do we do with them? By using Equations 16.26 and 16.27, we can calculate how a wave will propagate (transmit) and reflect when two dissimilar materials impact one another or when a shock crosses an interface where they are initially in contact. First, we shall define the impedance,  $Z$ . The impedance of a material is the product of its density and the velocity that a shock wave travels in that material.

$$Z = \rho U \quad (16.28)$$



**FIGURE 16.3**

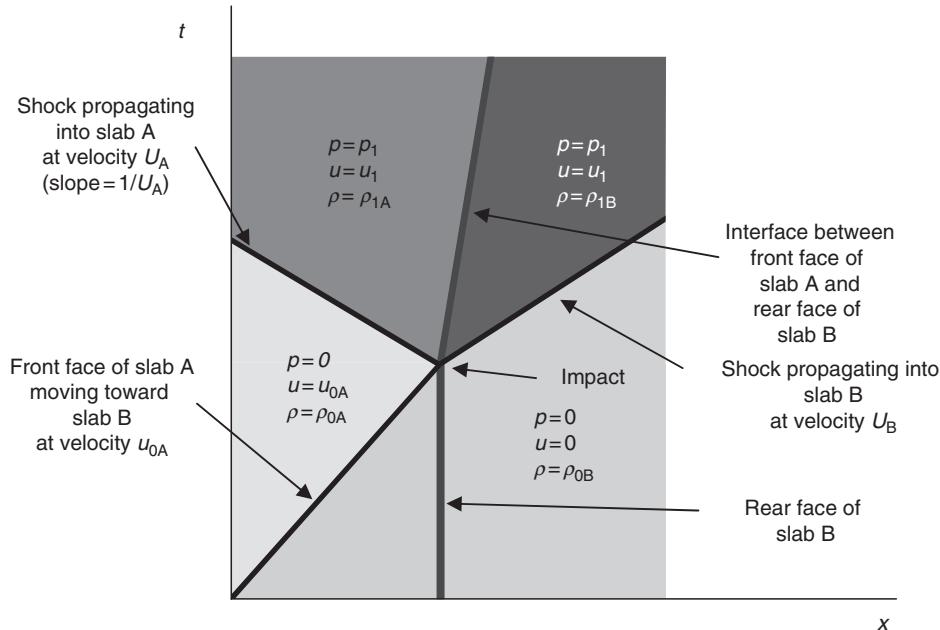
Effect of initial material velocity on a Hugoniot curve.

A material's acoustic impedance is the product of the material density times the speed of sound (an infinitesimally small disturbance) in that material.

$$Z_{\text{Acoustic}} = \rho c \quad (16.29)$$

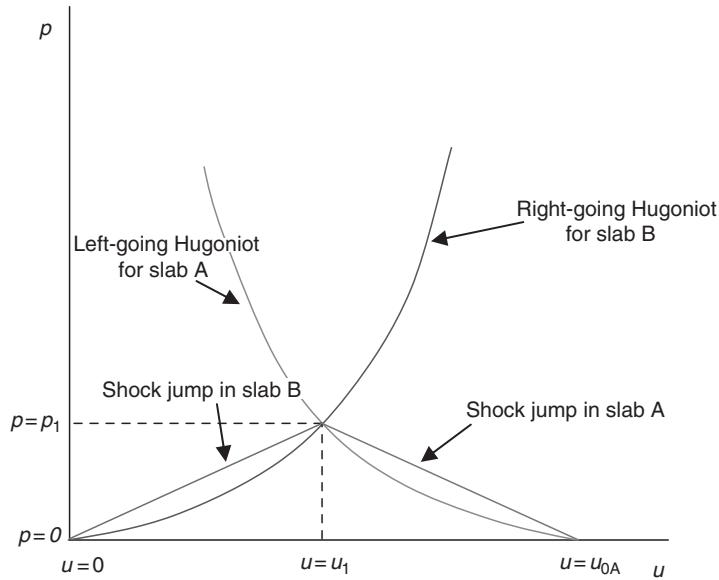
When a shock wave crosses a boundary between materials of the same impedance, there will be no reflection and all of the wave will be transmitted into the new material—the wave acts as though the interface is not there. If the materials are not in intimate contact, this will not be the case.

We shall now introduce a means of looking at shocks known as a  $t$ - $x$  plot. A  $t$ - $x$  (time-displacement) plot is used as a method of keeping track of material motion in a wave propagation problem. An example of this type of plot is in Figure 16.4 for two slabs which



**FIGURE 16.4**

Time-displacement plot of a slab impact problem. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)



**FIGURE 16.5**

A  $p$ - $u$  Hugoniot plot for an impact event. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

will impact one another. Because time is the ordinate, the slopes of the lines are the reciprocal of the velocity.

When two slabs impact one another, the following conditions must apply: The pressure at the interface must be consistent across the interface and the velocity of the particles at the interface must be the same in both materials. Consider that we have slab "B" sitting at rest and slab "A" impacts it with the initial conditions that slabs A and B are both stress free, but slab A is moving (i.e., all of the particles of slab A have the same particle velocity). Once impact occurs, a shock wave of equal strength will pass into each material. A right-going wave in B and a left-going wave in A. We can see this on a  $p$ - $u$  plot in Figure 16.5. Let us consider another example problem.

### Example Problem 3

An experiment is set up in which a magnesium slab is launched at a slab of brass. The velocity at impact is measured to be 2.0 km/s. Determine

1. The particle velocity in the two materials at the interface
2. The shock pressure at the interface
3. The speed at which the shock wave travels in the brass
4. The speed at which the shock wave travels in the magnesium

The slabs have the following properties:

Magnesium	Brass
$\rho_{0\text{Mg}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0\text{Brass}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0\text{Mg}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0\text{Brass}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$
$s_{\text{Mg}} = 1.256$	$s_{\text{Brass}} = 1.434$

*Solution:* The first thing we do is write the  $p-u$  Hugoniot equations for both materials, by convention assume that the magnesium plate is flying from left to right, then we need a right-going Hugoniot in the target (brass) and a left-going Hugoniot for the flyer (magnesium). We shall examine the brass first. A right-going Hugoniot is described by [Equation 16.26](#), but since the brass was not initially moving we can use [Equation 16.25](#). Inserting values for the brass we have

$$p_1[\text{GPa}] = (8.450) \left[ \frac{\text{g}}{\text{cm}^3} \right] (3.726) \left[ \frac{\text{km}}{\text{s}} \right] u_1 \left[ \frac{\text{km}}{\text{s}} \right] + (8.450) \left[ \frac{\text{g}}{\text{cm}^3} \right] (1.434) u_1^2 \left[ \frac{\text{km}}{\text{s}} \right]^2 \quad (16.30)$$

$$p_1[\text{GPa}] = 31.485u_1 + 12.117u_1^2 \quad (16.31)$$

Since we know that the compatibility relation requires pressure to be identical in both materials at the interface, we can write the left-going Hugoniot for the magnesium, equate the two expressions, and solve for the particle velocity (which must also be the same in both materials at the interface). The left-going Hugoniot in the magnesium is given by [Equation 16.27](#). Inserting our values yields

$$p_1[\text{GPa}] = 2.229u_1^2 - 16.932u_1 + 24.948 \quad (16.32)$$

If we equate Equations 16.31 and 16.32, we obtain

$$u_1^2 + 4.897u_1 - 2.523 = 0 \quad (16.33)$$

Now if we solve this using the quadratic formula, we get

$$u_1 = 0.471 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.34)$$

Here we used the positive velocity since the other root is meaningless. To determine the pressure at the interface we can put this value back into either Equation 16.25 or 16.27 to yield

$$p_1 = 17.52[\text{GPa}] \quad (16.35)$$

To find the speed that the shock wave moves in each material, we call upon the  $U-u$  Hugoniots for each ([Equation 16.1](#)). For the brass, we have

$$U_{\text{Brass}} = (3.726) \left[ \frac{\text{km}}{\text{s}} \right] + (1.434)u_1 \left[ \frac{\text{km}}{\text{s}} \right] = (3.726) \left[ \frac{\text{km}}{\text{s}} \right] + (1.434)(0.471) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.36)$$

$$U_{\text{Brass}} = 4.401 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.37)$$

Note that this velocity is to the right because we used a right-going Hugoniot. For the magnesium, we have

$$U_{\text{Mg}} = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.256)(u_0 - u_1) \left[ \frac{\text{km}}{\text{s}} \right] = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.256)(2.0 - 0.471) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.38)$$

$$U_{\text{Mg}} = 6.436 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.39)$$

This velocity is to the left because we used a left-going Hugoniot. Notice that we used  $u_0 - u_1$  in place of  $u_1$  because the shock velocity is relative to the wave.

When a shock wave propagates from a lower impedance material into a higher impedance material, as always, the compatibility condition is such that the pressure must also be continuous at the interface and the particle velocities must be equal. The higher impedance material will cause the pressure to increase and this higher pressure wave will propagate back into the lower impedance material (but at a lower velocity) and into the higher impedance material at a lower velocity than the original wave. The particle velocity will be the same (and lower) in both materials. We shall illustrate this with an example.

#### Example Problem 4

An experiment is set up in which a magnesium slab is shocked while in contact with a slab of brass. The particle velocity at the interface is measured to be 2.0 km/s. Determine

1. The pressure generated at the interface
2. The speed at which the transmitted shock wave travels in the brass
3. The particle velocity in the magnesium before the impact
4. The speed at which the original shock pulse traveled in the magnesium
5. The pressure of the original shock pulse in the magnesium

The slabs have the following properties:

Magnesium	Brass
$\rho_{0\text{Mg}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0\text{Brass}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0\text{Mg}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0\text{Brass}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$
$s_{\text{Mg}} = 1.256$	$s_{\text{Brass}} = 1.434$

*Solution:* If we examine [Figure 16.6](#), we see that we should be able to determine the answer to part (1) from the right-going Hugoniot in the brass.

A right-going Hugoniot is described by [Equation 16.26](#), but since the brass was not initially moving we can use [Equation 16.25](#) but to stay consistent with our diagram we will say the particle velocity is  $u_2$  for this case

$$p_2 = \rho_0 c_0 u_2 + \rho_0 s u_2^2 \quad (16.40)$$

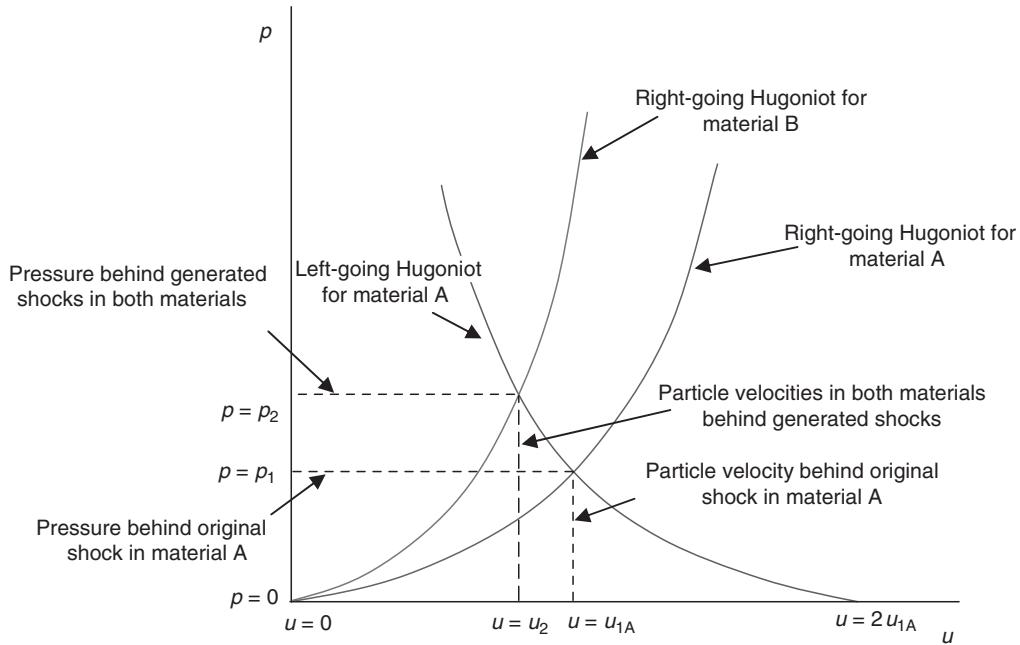
Inserting values for the brass we have

$$p_2[\text{GPa}] = 31.485u_2 + 12.117u_2^2 \quad (16.41)$$

We were provided with  $u_2$  so we can write

$$p_2[\text{GPa}] = 31.485(2) + 12.117(2)^2 \quad (16.42)$$

$$p_2 = 111.438[\text{GPa}] \quad (16.43)$$



**FIGURE 16.6**  
A  $p$ - $u$  diagram for low to high impedance shock propagation.

The speed at which the transmitted shock wave travels in the brass can be found directly from our  $U$ - $u$  Hugoniot [Equation 16.1](#). For the brass, we have

$$U_{\text{Brass}} = (3.726) \left[ \frac{\text{km}}{\text{s}} \right] + (1.434)u_2 \left[ \frac{\text{km}}{\text{s}} \right] = (3.726) \left[ \frac{\text{km}}{\text{s}} \right] + (1.434)(2.0) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.44)$$

$$U_{\text{Brass}} = 6.594 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.45)$$

The particle velocity in the magnesium before impact is found by noting that we have the point  $(u_2, p_2)$  on the left-going Hugoniot which, by definition, has to pass through point  $(2u_{1A}, 0)$  as well. Our equation for the left-going Hugoniot is [Equation 16.27](#). Putting this in terms of our diagram, we can write

$$p_2 = \rho_{0A}c_{0A}(2u_{1A} - u_2) + \rho_{0A}s_A(2u_{1A} - u_2)^2 \quad (16.46)$$

Inserting our values for magnesium, we can write

$$u_{1A}^2 - 0.202u_{1A} - 13.297 = 0 \quad (16.47)$$

From which we obtain the solution

$$u_{1A} = 3.749 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.48)$$

The speed at which the original shock pulse travels in the magnesium falls out directly from our  $U-u$  Hugoniot again.

$$U_{\text{Mg}} = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.256) u_{1A} \left[ \frac{\text{km}}{\text{s}} \right] = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.256)(3.749) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.49)$$

$$U_{\text{Mg}} = 9.225 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.50)$$

The pressure of the original shock pulse in the magnesium then follows from the momentum equation

$$p_{1A} = \rho_{0A} u_{1A} U_{\text{Mg}} \quad (16.51)$$

$$p_{1A} = (1.775) \left[ \frac{\text{g}}{\text{cm}^3} \right] (3.749) \left[ \frac{\text{km}}{\text{s}} \right] (9.225) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.52)$$

$$p_{1A} = 61.388[\text{GPa}] \quad (16.53)$$

When a shock wave propagates from a higher impedance material into a lower impedance material, the compatibility condition still requires the pressure be continuous at the interface and the particle velocities be equal. The lower impedance material will cause the pressure to decrease and this lower pressure (relief) wave will propagate back into the higher impedance material (at a higher velocity), and also into the lower impedance material at a higher velocity than the original wave. The particle velocity will be the same (and higher) in both materials. Another example will illustrate the point.

### Example Problem 5

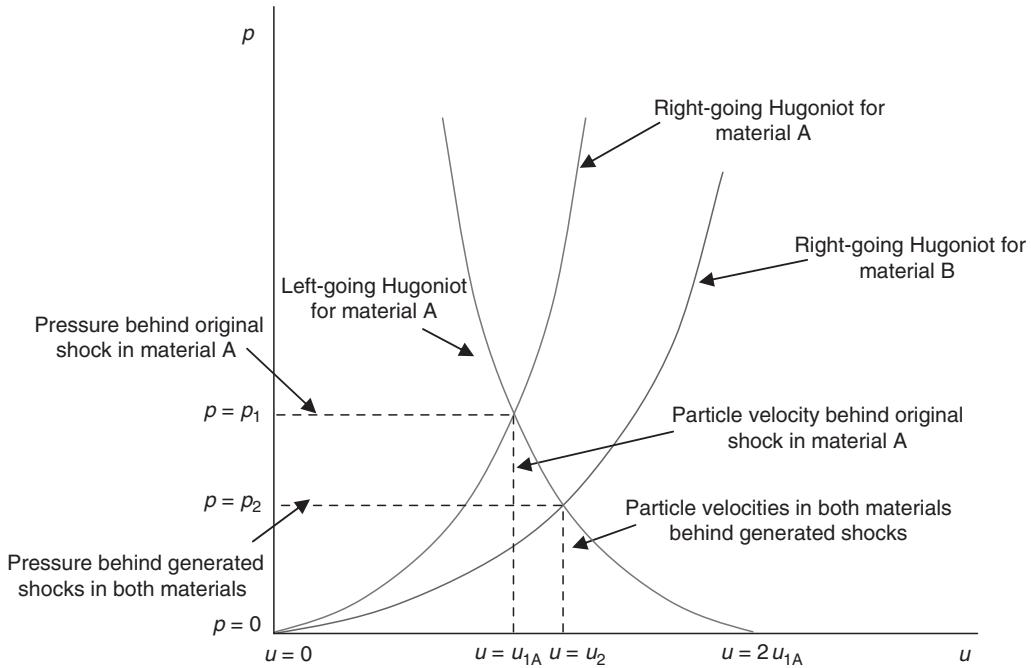
An experiment is set up in which a brass slab is shocked while in contact with a slab of magnesium. The particle velocity at the interface is measured to be 2.0 km/s. Determine

1. The pressure generated at the interface
2. The speed at which the transmitted shock wave travels in the magnesium
3. The particle velocity in the brass before the impact
4. The speed at which the original shock pulse traveled in the brass
5. The pressure of the original shock pulse in the brass

The slabs have the following properties:

Magnesium	Brass
$\rho_{0\text{Mg}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0\text{Brass}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0\text{Mg}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0\text{Brass}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$
$s_{\text{Mg}} = 1.256$	$s_{\text{Brass}} = 1.434$

*Solution:* If we examine Figure 16.7, we see that we should be able to determine the answer to part (1) from the right-going Hugoniot in the magnesium.



**FIGURE 16.7**

A  $p$ - $u$  diagram for high to low impedance shock propagation. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

A right-going Hugoniot is described by [Equation 16.26](#), but since the magnesium was not initially moving we can use [Equation 16.25](#) but to stay consistent with our diagram we will say the particle velocity is  $u_2$  for this case

$$p_2 = \rho_0 c_0 u_2 + \rho_0 s u_2^2 \quad (16.54)$$

Inserting values for the magnesium we have

$$p_2[\text{GPa}] = 8.016u_2 + 2.212u_2^2 \quad (16.55)$$

$$p_2[\text{GPa}] = 8.016(2) + 2.212(2)^2 \quad (16.56)$$

$$p_2 = 24.879[\text{GPa}] \quad (16.57)$$

The speed at which the transmitted shock wave travels in the magnesium can be found directly from [Equation 16.1](#).

$$U_{\text{Mg}} = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.246)u_2 \left[ \frac{\text{km}}{\text{s}} \right] = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (1.246)(2.0) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.58)$$

$$U_{\text{Mg}} = 7.008 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.59)$$

The particle velocity in the brass before impact is found by noting that we have the point  $(u_2, p_2)$  on the left-going Hugoniot which, by definition, has to pass through point  $(2u_{1A}, 0)$

as well. Our equation for the left-going Hugoniot is [Equation 16.27](#). Putting this in terms of [Figure 16.7](#), we can write

$$p_2 = \rho_{0A}c_{0A}(2u_{1A} - u_2) + \rho_{0A}s_A(2u_{1A} - u_2)^2 \quad (16.60)$$

Inserting our values for brass we have

$$u_{1A}^2 - 0.701u_{1A} - 0.812 = 0 \quad (16.61)$$

From this, we see that

$$u_{1A} = 1.317 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.62)$$

The speed at which the original shock pulse travels in the brass falls out directly from our  $U-u$  Hugoniot again [Equation 16.1](#).

$$U_{\text{Brass}} = 5.615 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.63)$$

The pressure of the original shock pulse in the brass then follows from the momentum equation.

$$p_{1A} = \rho_{0A}u_{1A}U_{\text{Brass}} \quad (16.64)$$

$$p_{1A} = 62.487 \text{ [GPa]} \quad (16.65)$$

When two shock waves collide in the same material, the pressure will jump to a new value that is greater than the sum of the two individual pressure pulses. Let us assume that we have a wave originally traveling to the right at pressure  $p_1$  and a stronger wave originally traveling to the left at pressure  $p_2$  in a material. We need to reflect the Hugoniots of these waves as shown in [Figure 16.8](#) to solve for the resulting pressure  $p_3$ . We shall examine this again by example.

### Example Problem 6

An experiment is set up in which a magnesium slab is shocked from both ends. The pressure generated in the left-going shock is 20 GPa. The pressure generated in the right-going shock is 10 GPa. Determine

1. The particle velocity in the right-going shock
2. The particle velocity in the left-going shock
3. The resultant particle velocity in the material
4. The resultant pressure generated

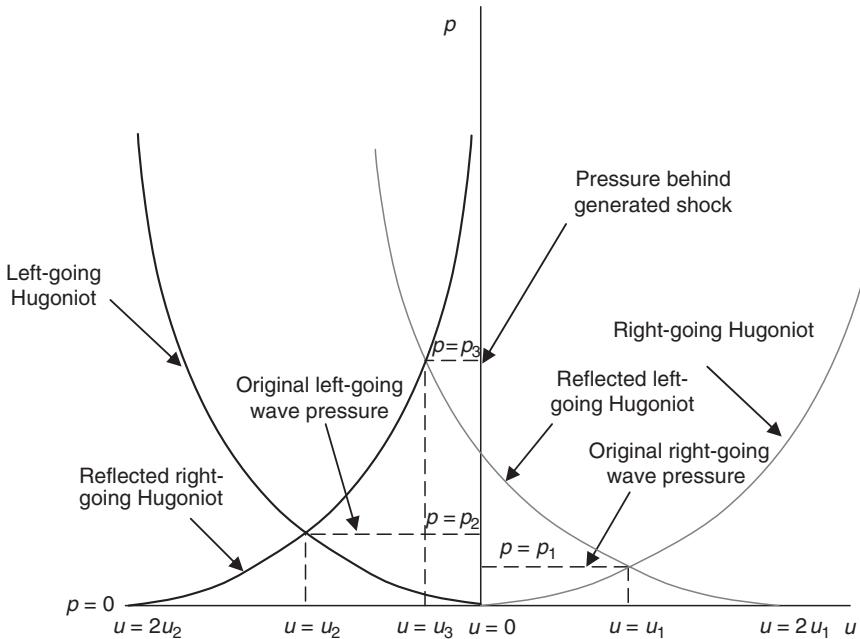
The slab has the following properties:

Magnesium

$$\rho_{0Mg} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0Mg} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{Mg} = 1.256$$



**FIGURE 16.8**

Collision of two shock waves. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

*Solution:* If we examine Figure 16.8, we see that we should be able to determine the answer to part (1) from the right-going Hugoniot in the magnesium.

A right-going Hugoniot is described by [Equation 16.26](#), but since the magnesium was not initially moving we can use [Equation 16.25](#) but to stay consistent with our diagram we will say the particle velocity is  $u_1$  for this case.

$$p_1 = \rho_0 c_0 u_1 + \rho_0 s u_1^2 \quad (16.66)$$

Inserting values for the magnesium we have

$$u_1^2 + 3.624u_1 - 4.521 = 0 \quad (16.67)$$

Solving this we obtain

$$u_1 = 0.982 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.68)$$

The particle velocity in the left-going shock is found again by noting that we have the left-going Hugoniot passing through the origin. Our equation for the left-going Hugoniot is

$$p_2 = \rho_0 c_0(u_2 - 0) + \rho_0 s(u_2 - 0)^2 = \rho_0 c_0 u_2 + \rho_0 s u_2^2 \quad (16.69)$$

Inserting our values for magnesium we have

$$u_2^2 + 3.624u_2 - 9.042 = 0$$

and therefore

$$u_2 = -1.699 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.70)$$

The resultant particle velocity is found by taking this data, reflecting the Hugoniots around  $u_1$  and  $u_2$  and eliminating the pressure (since it is equal to  $p_3$ ) from the equation. We shall reflect the right-going Hugoniot first. This will result in a left-going Hugoniot where we know points  $(u_1, p_1)$  and  $(2u_1, 0)$ .

$$p_3 = \rho_0 c_0(u_3 - 2u_1) + \rho_0 s(u_3 - 2u_1)^2 \quad (16.71)$$

or

$$p_3 = 2.212u_3^2 - 16.705u_3 + 24.275 \quad (16.72)$$

Now we need to examine the right-going Hugoniot where we know points  $(u_2, p_2)$  and  $(2u_2, 0)$ .

$$p_3 = \rho_0 c_0(u_3 - 2u_2) + \rho_0 s(u_3 - 2u_2)^2 \quad (16.73)$$

or

$$p_3 = 2.212u_3^2 + 23.049u_3 + 52.779 \quad (16.74)$$

If we now subtract Equation 16.72 from Equation 16.74, we can solve for  $u_3$  so we have

$$39.754u_3 + 28.524 = 0 \quad (16.75)$$

Therefore,

$$u_3 = -0.717 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.76)$$

The pressure then can be found from either Equation 16.72 or 16.74.

$$p_3 = 37.390[\text{GPa}] \quad (16.77)$$

We have now completed our introduction of the Hugoniot curve and examined the use of Hugoniots for an impact problem. We have demonstrated the behavior of shocks across an interface and have examined infinite shock behavior in a single material (incipient shock and collision of two shocks). These shocks were considered infinite because the driving pressure was always present behind them, generating continued motion. Further reading is provided in the references. We shall now move on to discuss rarefaction waves.

**Problem 1**

An experiment is set up in which a steel slab is shocked from both ends. The pressure generated in the left-going shock is 20 GPa. The pressure generated in the right-going shock is 10 GPa. Draw the  $p$ - $u$  diagram and determine

1. The particle velocity in the right-going shock.

$$\text{Answer: } u_1 = 0.256 \left[ \frac{\text{km}}{\text{s}} \right]$$

2. The particle velocity in the left-going shock.

$$\text{Answer: } u_2 = -0.479 \left[ \frac{\text{km}}{\text{s}} \right]$$

3. The resultant particle velocity in the material.

$$\text{Answer: } u_3 = -0.223 \left[ \frac{\text{km}}{\text{s}} \right]$$

4. The resultant pressure generated.

$$\text{Answer: } p_3 = 32.872[\text{GPa}]$$

The slab has the following properties:

Steel

$$\rho_{0\text{Steel}} = 7.896 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0\text{Steel}} = 4.569 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Steel}} = 1.490$$

**Problem 2**

A strange jeweler wants to make an earring by launching a quartz slab at a slab of gold. His high-tech instrumentation measures the induced velocity in the gold as 0.5 km/s. Determine

1. The impact velocity.

$$\text{Answer: } u_0 = 3.420 \left[ \frac{\text{km}}{\text{s}} \right]$$

2. The shock pressure at the interface.

$$\text{Answer: } p_1 = 36.960[\text{GPa}]$$

3. The speed at which the shock wave travels in the gold.

$$\text{Answer: } U_{\text{Au}} = 3.842 \left[ \frac{\text{km}}{\text{s}} \right]$$

4. The speed at which the shock wave travels in the quartz.

$$\text{Answer: } U_Q = 5.743 \left[ \frac{\text{km}}{\text{s}} \right]$$

The slabs have the following properties:

Quartz	Gold
$\rho_{0Q} = 2.204 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0Au} = 19.24 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0Q} = 0.794 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0Au} = 3.056 \left[ \frac{\text{km}}{\text{s}} \right]$
$S_Q = 1.695$	$S_{Au} = 1.572$

## 16.2 Rarefaction Waves

We have examined infinite waves in the previous section (i.e., waves in which the pressure does not abate). In the shocking of a real material, the pressure pulse only lasts for a finite time and then the material must expand back to a relaxed state. Nature accomplishes this expansion through a rarefaction wave.

A rarefaction wave is the manner in which nature restores a material to its unshocked state after the passage of a shock wave. Unlike a shock front (which is a nearly discontinuous jump in pressure), a rarefaction or relief wave will occur over some finite distance which will gradually increase with time. We typically assume that rarefaction waves occur rapidly enough that the process may be considered adiabatic.

Recall that passing a shock wave through a material increases its internal energy as shown through the Rankine–Hugoniot equation

$$e_1 - e_0 = \frac{1}{2} \left( \frac{1}{\rho_0} - \frac{1}{\rho_1} \right) (p_0 + p_1) = \frac{1}{2} (p_0 + p_1)(v_0 - v_1) \quad (16.78)$$

As a consequence of the second law of thermodynamics, we can write

$$dE = TdS - pdV \quad (16.79)$$

Since we assumed that the rarefaction process is adiabatic, we know that

$$dQ = TdS = 0 \quad (16.80)$$

Since, on the Rankine or Kelvin scales,  $T$  must be positive and, except in a special case nonzero, then  $dS$  must equal zero for this equation to be true. Thus, the rarefaction or relief process must be isentropic. This presents us with a bit of a dilemma. Except for an ideal gas, we do not have an isentropic relation to allow us to quantify the expansion process. If we had such a relationship, it would, in theory, allow us to eliminate one of the variables in our Equation 16.79 which, through Equation 16.80, can be rewritten as

$$dE = -pdV \rightarrow E = E(p, V) \quad (16.81)$$

We have stated before that a Hugoniot curve is neither an equation of state nor an isentrope. Here we will use it as if it was one and accept any errors that result.

We now know that we can handle a rarefaction wave through use of the Hugoniot. The simplest way to illustrate how to obtain the rarefaction wave velocity is to consider the case where the initial material velocity is equal to zero, we can then write

$$p|_{u_0=0} = \rho_0 u U_R \quad (16.82)$$

Taking the first derivative, we obtain

$$\frac{dp|_{u_0=0}}{du} = \rho_0 U_R \quad (16.83)$$

We also saw that we can write the  $p-u$  Hugoniot as

$$p = \rho_0 c_0 u + \rho_0 s u^2 \quad (16.84)$$

Taking the derivative of Equation 16.84, we obtain

$$\frac{dp}{du} = \rho_0 c_0 + 2\rho_0 s u \quad (16.85)$$

Eliminating  $dp/du$  between Equations 16.83 and 16.85 yields

$$\rho_0 U_R = \rho_0 c_0 + 2\rho_0 s u \quad (16.86)$$

or

$$U_R = c_0 + 2s u \quad (16.87)$$

which is our final relation for the speed of the head of the rarefaction wave. This is depicted in [Figure 16.9](#). If we recall the speed of our shock wave ( $U-u$  Hugoniot), we would see

$$U = c_0 + s u \quad (16.88)$$

If we were to shock a material with a certain pulse length,  $\lambda_1$ , over a particular time,  $t_1$ , the shock would have moved a distance

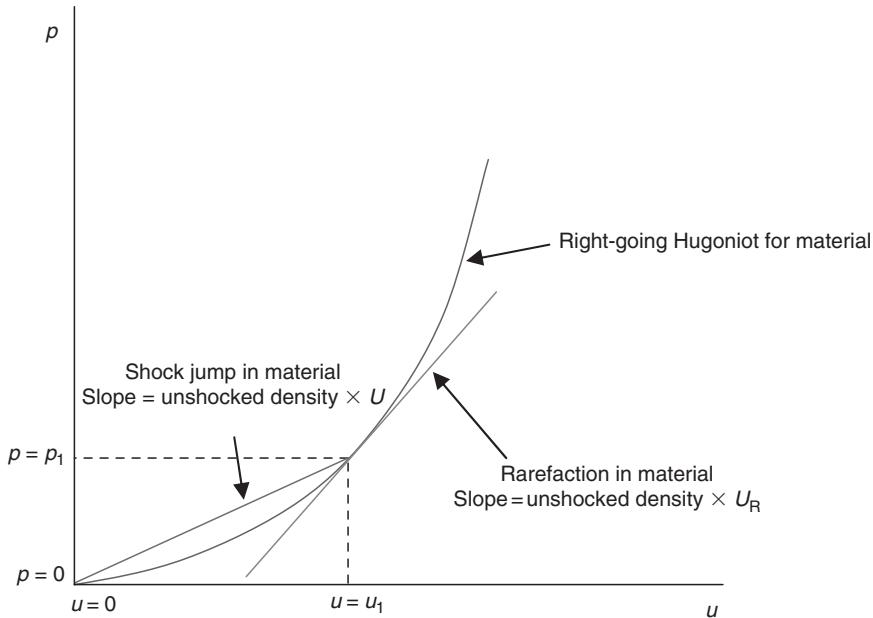
$$\lambda_1 = U t_1 \quad (16.89)$$

The instant the applied load has ceased, a relief wave would begin moving into the material and at a time  $t > t_1$  would be located at a distance from the point of shock initiation of

$$d_2 = U_R(t - t_1) \quad (16.90)$$

Since we saw from our examination of Equations 16.87 and 16.88 that  $U_R > U$ , we can determine the distance at which the relief wave will catch up to the shock wave through

$$U t = U_R(t - t_1) \quad (16.91)$$



**FIGURE 16.9**  
Speed of the rarefaction wave head.

If we insert Equations 16.87 and 16.88 into this expression, we obtain

$$(c_0 + su)t = (c_0 + 2su)(t - t_1) \quad (16.92)$$

which simplifies to

$$sut = c_0t_1 + 2sut_1 \quad (16.93)$$

Thus, the time required for the rarefaction wave to catch up with the initial shock is determined from

$$t = \frac{c_0t_1 + 2sut_1}{su} \quad (16.94)$$

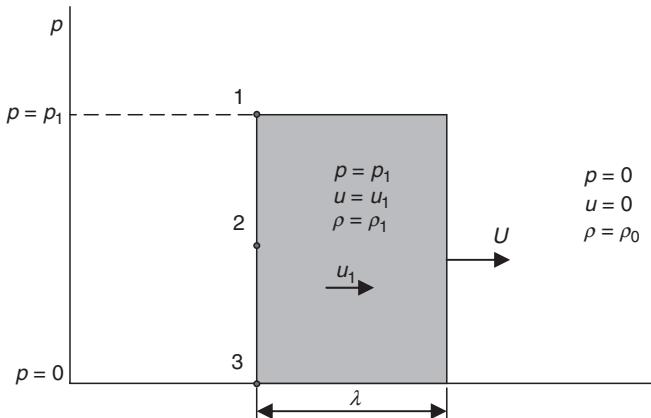
where we should know everything on the RHS from the material and the strength of the initial pulse.

We could then use Equation 16.95 to determine the catch-up distance.

$$\lambda_c = Ut \quad (16.95)$$

The text by Paul Cooper [1] offers the clearest treatment of rarefaction wave physics that these authors have ever encountered. We shall endeavor to follow that method of explanation here. Consider a finite square shock pulse of wavelength,  $\lambda$ , as shown in Figure 16.10.

Recall that the shock velocity is dependent upon the pressure ratio across the disturbance. Unlike the compression shock, where the increasing pressure caused the part of the wave initially behind the leading edge of the shock to catch up and form a front, at the rear



**FIGURE 16.10**

Simple model of a rarefaction wave.  
(From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

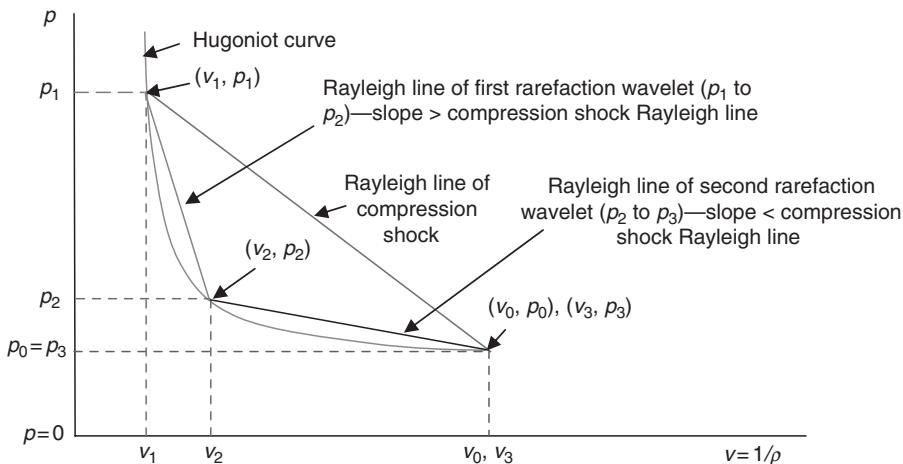
end of this disturbance, the pressure is decreasing. This causes the rearmost portions of the rarefaction wave to fall further and further behind the incident shock. Additionally, since the rarefaction wave is passing into an effectively denser material, the head of the wave will be moving faster than the compression shock.

On a  $p$ - $v$  diagram, we would see what appears in Figure 16.11 if we considered only points 1, 2, and 3 in our square pulse shown in Figure 16.10.

From the  $p$ - $v$  diagram in Figure 16.11, we can see that our wavelet from  $p_1$  to  $p_2$  will move faster than the compression shock and our wavelet from  $p_2$  to  $p_3$  will move slower. Over time, the shape of the pulse will change as depicted in Figure 16.12.

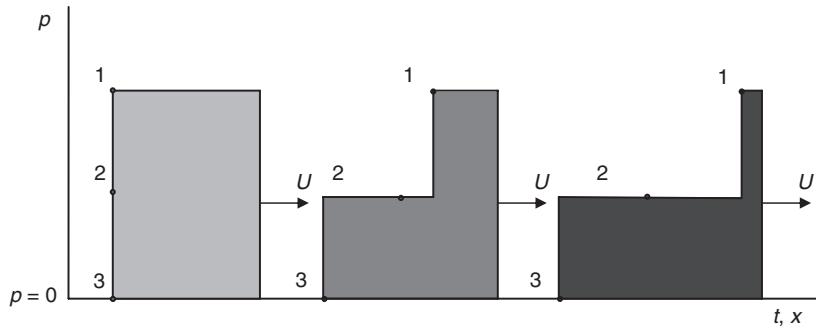
Figure 16.12 is a very crude discretization to facilitate understanding. The more elements we break the wave into, the closer the rarefaction wave gets as we approach the continuous (actual) situation. This is illustrated in Figure 16.13. If we wanted to draw a  $t$ - $x$  plot of the rarefaction wave illustrated in Figure 16.13, the result would appear as in Figure 16.14.

We shall now examine some classic rarefaction problems in detail. The first is quite important for use in terminal ballistics—the reflection of a square wave at a free surface.



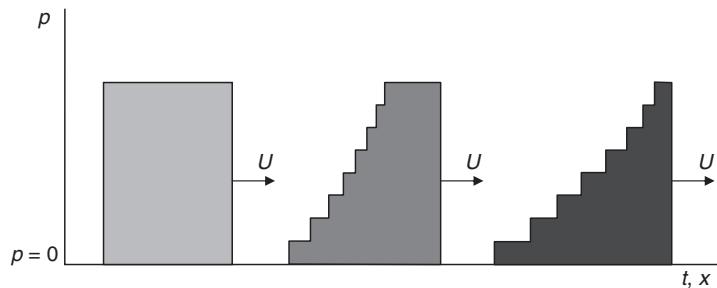
**FIGURE 16.11**

A  $p$ - $v$  diagram of a simple rarefaction wave. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)



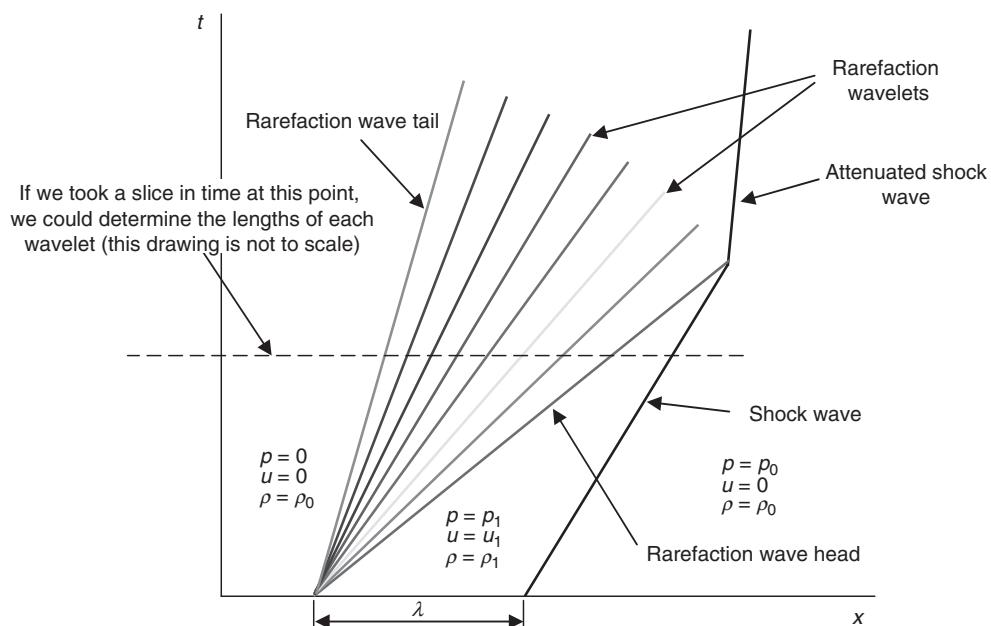
**FIGURE 16.12**

Rarefaction wave modeled as two wavelets catching up to incident shock. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)



**FIGURE 16.13**

Rarefaction wave modeled as eight wavelets catching up to incident shock.



**FIGURE 16.14**

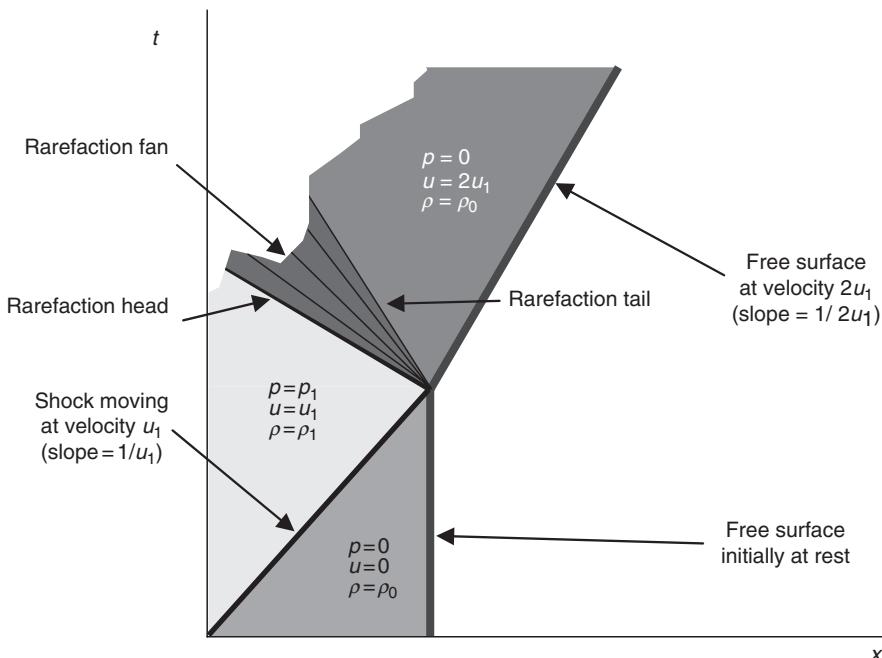
Rarefaction wave modeled as eight wavelets on a  $t$ - $x$  plot. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

When a compressive pulse reaches a free surface in a material, recall that the condition of zero stress on the surface must be maintained. Nature accomplishes this through the generation of a relief (rarefaction) wave at the surface such that the total stress is zero. The relief wave will exactly cancel the compressive wave. This has implications in stress behavior which we shall see later and we shall also see how we can treat that scenario a little differently. The interaction with the free surface also results in a material velocity that is double the material velocity behind the original compressive pulse. Let us consider the  $t$ - $x$  plot of a shock wave that encounters a free surface as depicted in Figure 16.15. The plot of this interaction on a  $p$ - $u$  diagram is shown in Figure 16.16. In these figures, we see that after the compression shock encounters the free surface a rarefaction wave propagates back into the material dropping the pressure down to zero and doubling the material velocity.

The rarefaction wave will have to travel back into material that is still approaching it at an induced velocity created by the incident shock. This requires us to understand the difference between Lagrangian and Eulerian coordinate systems. This is shown in Figure 16.17.

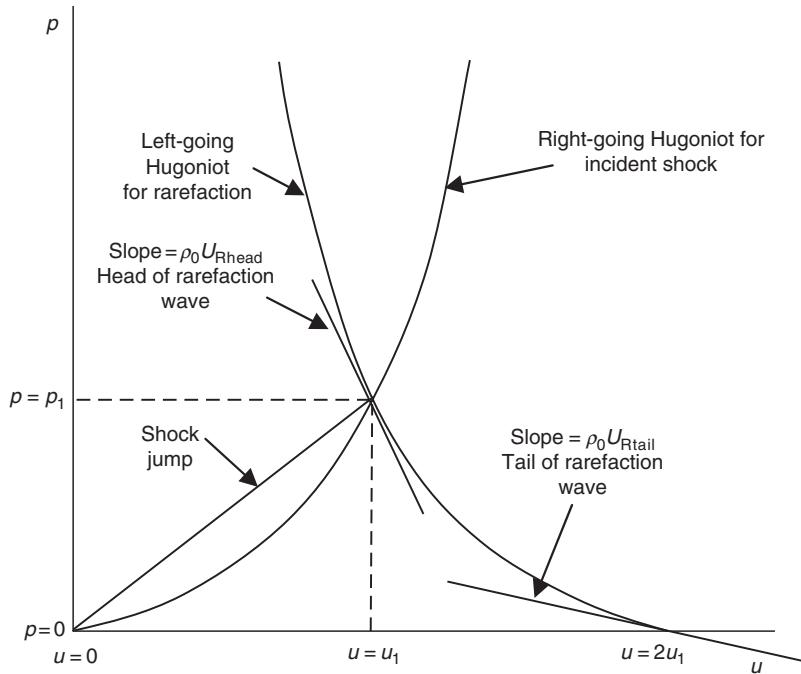
We have previously described Lagrangian coordinates as a coordinate system that is moving with the shock. Eulerian coordinates are stationary relative to the laboratory. All the velocities we examined thus far were Lagrangian (this made our equations simple). When we want velocities in Eulerian coordinates, we need to account for the motion of the material the shock is moving into. For instance, as previously mentioned, in our reflected shock,  $U_R$  is the Lagrangian velocity of the reflected wave. The Eulerian velocity of this same wave would be  $U_R - u_1$ . Or  $U_R + u_1$  if you consider  $U_R$  as negative in our lab and  $u_1$  as positive.

The interaction with a free surface will now be illustrated with an example.



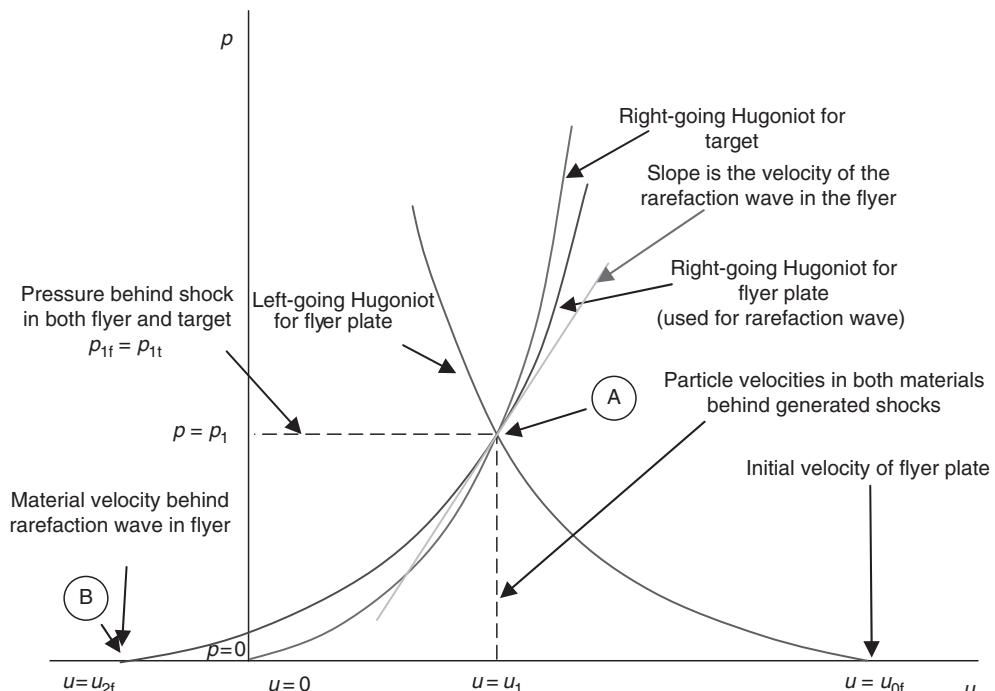
**FIGURE 16.15**

A  $t$ - $x$  plot of a shock wave interacting with a free surface. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)



**FIGURE 16.16**

A  $p$ - $u$  Hugoniot plot of a shock wave interacting with a free surface. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)



**FIGURE 16.17**

Rarefaction wave speed determination.

### Example Problem 7

An experiment is set up in which a magnesium slab is shocked with a constant pressure of 5.0 GPa. Determine

1. The particle velocity in the magnesium behind the incident shock before an encounter with a free surface
2. The velocity of the free surface after the interaction
3. The particle velocity behind the surface after the interaction
4. The Lagrangian velocity of the leading edge of the rarefaction
5. The Eulerian velocity of the leading edge of the rarefaction

The material has the following properties:

Magnesium

$$\rho_{0\text{Mg}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$
$$c_{0\text{Mg}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$$
$$s_{\text{Mg}} = 1.256$$

*Solution:* We can determine the answer to part (1) from the right-going Hugoniot in the material. A right-going Hugoniot in a nonmoving material is described by [Equation 16.25](#). Inserting values for the magnesium we have

$$5.0[\text{GPa}] = (1.775) \left[ \frac{\text{g}}{\text{cm}^3} \right] (4.516) \left[ \frac{\text{km}}{\text{s}} \right] u_1 \left[ \frac{\text{km}}{\text{s}} \right] + (1.775) \left[ \frac{\text{g}}{\text{cm}^3} \right] (1.256) u_1^2 \left[ \frac{\text{km}}{\text{s}} \right]^2 \quad (16.96)$$

Following through we have

$$5.0[\text{GPa}] = 8.016u_1 + 2.229u_1^2 \quad (16.97)$$

or

$$u_1^2 + 3.596u_1 - 2.243 = 0 \quad (16.98)$$

which results in

$$u_1 = -1.798 \pm 2.340 \rightarrow u_1 = 0.542 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.99)$$

The velocity of the free surface is simply

$$u_2 = 2u_1 = (2)(0.542) \left[ \frac{\text{km}}{\text{s}} \right] = 1.084 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.100)$$

The particle velocity behind the reflected wave is the same as the free surface velocity. The Lagrangian velocity of the leading edge of the rarefaction is given by [Equation 16.87](#)

$$U_R = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (2)(1.256)u_1 \left[ \frac{\text{km}}{\text{s}} \right] = (4.516) \left[ \frac{\text{km}}{\text{s}} \right] + (2)(1.256)(0.542) \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.101)$$

resulting in

$$U_R = 5.877 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.102)$$

The Eulerian velocity is found by noting that the reflected wave is moving in the negative direction and the material behind it is moving in the positive direction, so we can write

$$U_{R_{\text{Lab}}} = U_R + u_1 = -5.877 \left[ \frac{\text{km}}{\text{s}} \right] + 0.542 \left[ \frac{\text{km}}{\text{s}} \right] = -5.335 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.103)$$

We will now examine two cases where a flyer plate (a thin plate) impacts a thick target. The flyer plate assumption allows us to ignore reflections of shocks from the free boundaries transverse to our impact direction. Case 1 is that of a flyer plate with an impedance less than or equal to that of the target (Ref. [1] treats these individually but the case where they are equal is really the limiting case for a lower impedance flyer). Case 2 is that of a flyer plate with a greater impedance than the target. An important item to note is that Hugoniots are derived from compressive data; thus, if we have a tensile wave, we usually use a linear model on the  $p-u$  Hugoniot diagrams when negative values in pressure (tension) occur. The slope of these lines is  $\rho_0 c_L$ . Here  $c_L$  is the longitudinal speed of sound in the material.

If the flyer plate has an impedance less than or equal to that of the target on impact, a compressive shock will propagate into both objects. The shock in the flyer will reflect from the free surface of it and return as a rarefaction wave to the interface. When the rarefaction wave reaches the interface, two things happen: the flyer will rebound off the target and a new rarefaction wave will propagate into the flyer. Recall that waves reflect as like waves when the boundary condition stipulates a higher impedance. A rarefaction wave will also propagate into the target. This new rarefaction wave in the target will eventually catch up to the shock front in the target and reduce its strength. In the flyer, since it has free surfaces now, the waves will reflect in opposite sense until they equilibrate. The  $t-x$  plot and the  $p-u$  Hugoniots follow in Example Problem 8.

### Example Problem 8

An experiment is set up in which a brass slab is shocked by impact from with a magnesium flyer plate that is 1 mm in thickness. The impact velocity was measured to be 2.0 km/s. Determine

1. The material velocity behind the generated shock
2. The pressure generated at the interface
3. The time duration of the shock pulse in the target
4. The velocity with which the magnesium plate will rebound

The materials have the following properties:

Magnesium

$$\rho_{0\text{Mg}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0\text{Mg}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Mg}} = 1.256$$

$$c_{L\text{Mg}} = 5.770 \left[ \frac{\text{km}}{\text{s}} \right]$$

Brass

$$\rho_{0\text{Brass}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0\text{Brass}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Brass}} = 1.434$$

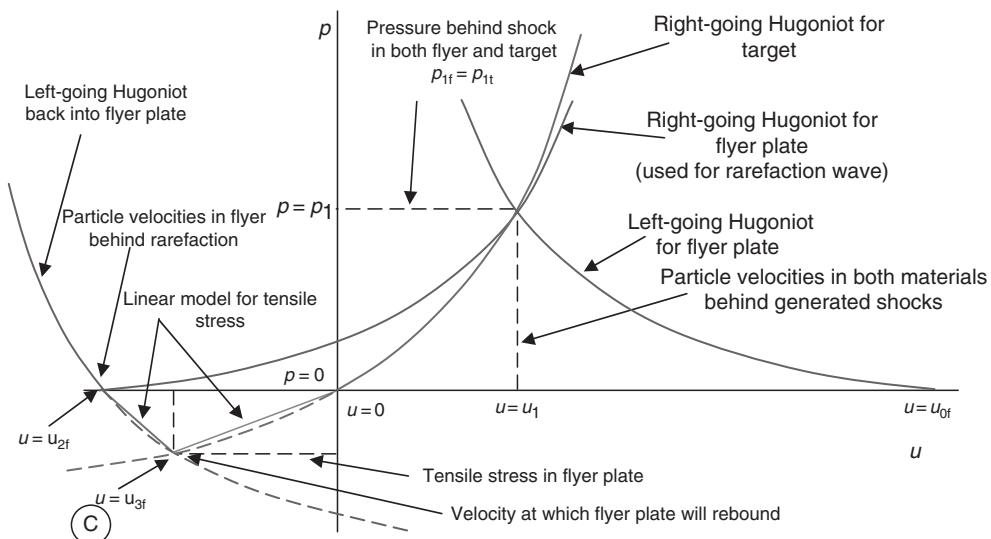
$$c_{L\text{Brass}} = 4.700 \left[ \frac{\text{km}}{\text{s}} \right]$$

*Solution:* Figure 16.18 tells us that to get the pressure generated at the interface, we need to calculate the left-going Hugoniot in the flyer and solve it for the pressure since we have the impact velocity and we know the target was initially at rest. The particle velocity in the magnesium before impact is given and we have located it on our diagram on the left-going Hugoniot which, by definition, has to pass through point  $(u_0, 0)$  as well. Our equation for the left-going Hugoniot is [Equation 16.87](#), which after insertion of the given values yields

$$p_1 = 2.229u_1^2 - 9.309u_1 + 9.702 \quad (16.104)$$

Here we assume the units are correct and we know the answer will be in GPa. Also for our right-going Hugoniot in the brass, we can use [Equation 16.85](#) to write

$$p_1 = 12.117u_1^2 + 31.485u_1 \quad (16.105)$$



**FIGURE 16.18**

A  $p$ - $u$  plot of a flyer plate interaction with target of higher impedance.

Equating Equations 16.86 and 16.87 yields

$$u_1^2 + 4.126u_1 - 0.981 = 0 \quad (16.106)$$

Then

$$u_1 = 0.225 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.107)$$

With Figure 16.18, it had to be positive. The pressure now comes from inserting this value in either Equation 16.104 or 16.105.

$$p_1 = (2.229)(0.225)^2 - (9.309)(0.225) + 9.702 = 7.720[\text{GPa}] \quad (16.108)$$

We have stated previously that the flyer plate will remain in contact with the target until the shock wave propagates to the rear face of the flyer, reflects as a rarefaction wave, and then reaches the front face. After this occurs, waves will continue moving back and forth in the flyer until the material velocity equilibrates. To determine the time of impact, we break the problem into two parts: the time it takes for the shock to reach the rear face and the time it takes for the first rarefaction to reach the impact surface.

The time it takes the shock to reach the rear face is determined by noting that the speed of wave propagation is the slope of the jump on the  $p-u$  Hugoniot divided by the initial density. Thus, we can write

$$U = \frac{p_1 - p_0}{\rho_0(u_1 - u_0)} \quad (16.109)$$

Inserting our values we obtain

$$U = -2.450 \left[ \frac{\text{km}}{\text{s}} \right] = -2.450 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.110)$$

Why did not we use Equation 16.80 or 16.78? The reason is that if we used Equation 16.80, we would actually obtain the Eulerian velocity which would be

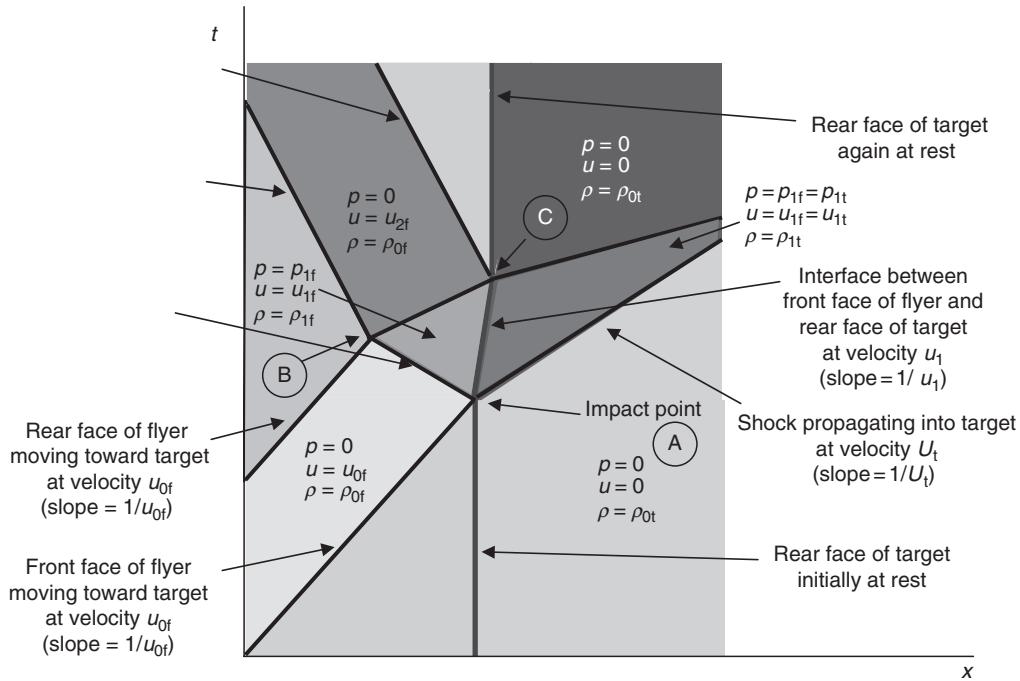
$$p_1 - p_0 = \rho_0(u_1 - u_0)(U - u_0) \quad (16.111)$$

$$U = -0.450 \left[ \frac{\text{km}}{\text{s}} \right] = -0.450 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.112)$$

If we were interested in the velocity alone, relative to the lab, this would be the correct answer. However, the material in the flyer is moving toward the interface during the shock event so it would appear to an observer on the shock that the face will move to meet the wave.

We shall return to the problem. If the shock was moving toward the rear surface of our flyer plate at 2.450 mm/ $\mu$ s, then it would reach the rear of the plate in

$$\Delta t = \frac{l}{U} = \frac{1[\text{mm}]}{2.450 \left[ \frac{\text{mm}}{\mu\text{s}} \right]} = 0.408[\mu\text{s}] \quad (16.113)$$



**FIGURE 16.19**

A  $t$ - $x$  plot of flyer plate interaction with a target of higher impedance. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

To determine the speed of the leading edge of the rarefaction wave, we need to examine Figure 16.19. Here we see that the speed of the head of the rarefaction wave is the slope of the  $p$ - $u$  Hugoniot curve at the material pressure. [Equation 16.83](#) was written for a left-going rarefaction wave. In our case, the slope is the negative of this value which we know. Here we need to use [Equation 16.104](#) since this is the Hugoniot for the flyer. Taking the derivative we have

$$\frac{dp}{du} \Big|_{u=u_1} = 4.458u_1 - 9.309 = (4.458)(0.225) - 9.309 = -8.306 \quad (16.114)$$

The rarefaction velocity is the negative of this value divided by the density of the material, so we have

$$U_{R\text{head}} = \frac{8.306}{1.775} \left[ \frac{\text{km}}{\text{s}} \right] = 4.679 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.115)$$

So the time it takes the rarefaction to reach the front face is

$$\Delta t = \frac{l}{U} = \frac{1[\text{mm}]}{4.679 \left[ \frac{\text{mm}}{\mu\text{s}} \right]} = 0.214[\mu\text{s}] \quad (16.116)$$

Then the total time for the shock pulse is the time between impact and the rarefaction wave reaching the interface or

$$t_{\text{shock}} = 0.408[\mu\text{s}] + 0.214[\mu\text{s}] = 0.622[\mu\text{s}] \quad (16.117)$$

To determine the velocity at which the magnesium plate will rebound, let us look at [Figure 16.18](#).

We have the densities of both materials, we have the longitudinal sound speeds, we have  $u_{2f}$  so we can find  $u_{3f}$  by solving the following equations simultaneously:

$$\rho_{0_{\text{Mg}}} c_{L_{\text{Mg}}} = \frac{p_3 - 0}{u_{3f} - u_{2f}} \rightarrow p_3 = \rho_{0_{\text{Mg}}} c_{L_{\text{Mg}}} (u_{3f} - u_{2f}) \quad (16.118)$$

$$\rho_{0_{\text{Brass}}} c_{L_{\text{Brass}}} = \frac{p_3 - 0}{u_{3f} - 0} \rightarrow p_3 = \rho_{0_{\text{Brass}}} c_{L_{\text{Brass}}} u_{3f} \quad (16.119)$$

Combining Equations 16.118 and 16.119 gives us

$$\rho_{0_{\text{Brass}}} c_{L_{\text{Brass}}} u_{3f} = \rho_{0_{\text{Mg}}} c_{L_{\text{Mg}}} (u_{3f} - u_{2f}) \quad (16.120)$$

$$(8.450)(4.700)u_{3f} = (1.775)(5.770)(u_{3f} - u_{2f}) \quad (16.121)$$

A neat way to find  $u_{2f}$  is to note that the two Hugoniots for the magnesium are reflected about the velocity  $u_1$ . So we can write

$$u_{2f} - u_1 = u_1 - u_{0f} \quad (16.122)$$

$$u_{2f} = (2)(0.225) \left[ \frac{\text{km}}{\text{s}} \right] - 2.0 \left[ \frac{\text{km}}{\text{s}} \right] = -1.550 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.123)$$

Then we can rewrite Equation 16.120 as

$$(39.715)u_{3f} = (10.242)(u_{3f} - 1.550) \quad (16.124)$$

$$u_{3f} = -0.539 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.125)$$

A  $t-x$  plot of this event is shown in [Figure 16.19](#).

If the flyer plate has an impedance greater than that of the target on impact, a compressive shock will again propagate into both objects. This shock will again reflect from the free surface of the flyer and return as a rarefaction wave to the interface. When the rarefaction wave reaches the interface, several things will happen: The rarefaction will again reflect in the opposite sense (as a shock) because the material into which it is propagating is of lower impedance, a new shock wave will propagate into the flyer as it digs into the target, and the rarefaction wave will propagate into the target. This new rarefaction wave in the target will again eventually catch up to the shock front in the target and reduce its strength. In the flyer, the process will repeat until equilibrium is reached. The physics of this event is again best described by an example problem.

### Example Problem 9

An experiment is set up in which a magnesium slab is shocked by impact from with a brass flyer plate that is 1 mm in thickness. The impact velocity was measured to be 2.0 km/s. Determine

1. The material velocity behind the generated shock
2. The pressure generated at the interface
3. The time duration of the initial shock pulse in the target
4. The material velocity behind the first rarefaction
5. The pressure behind the first rarefaction
6. The speed of the head of the first rarefaction wave in the target

The materials have the following properties:

Magnesium	Brass
$\rho_{0_{\text{Mg}}} = 1.775 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0_{\text{Brass}}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0_{\text{Mg}}} = 4.516 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0_{\text{Brass}}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$
$s_{\text{Mg}} = 1.256$	$s_{\text{Brass}} = 1.434$
$c_{L_{\text{Mg}}} = 5.770 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{L_{\text{Brass}}} = 4.700 \left[ \frac{\text{km}}{\text{s}} \right]$

*Solution:* Figure 16.20 tells us that to obtain the pressure generated at the interface, we need to calculate the left-going Hugoniot in the flyer and solve it for the pressure since we have the impact velocity and we know the target was initially at rest. We again do this by simultaneously solving the left-going Hugoniot in the flyer and the right-going Hugoniot in the target.

The particle velocity in the brass after impact is located at point A in Figure 16.20 on the left-going Hugoniot which, by definition, has to pass through point  $(u_{0f}, 0)$  as well. Our equation for the left-going Hugoniot (Equation 16.87) with the appropriate numbers inserted is

$$p_1 = (31.485)(2.0 - u_1) + (12.117)(2.0 - u_1)^2 \quad (16.126)$$

or

$$p_1 = 12.117u_1^2 - 80.165u_1 + 111.650 \quad (16.127)$$

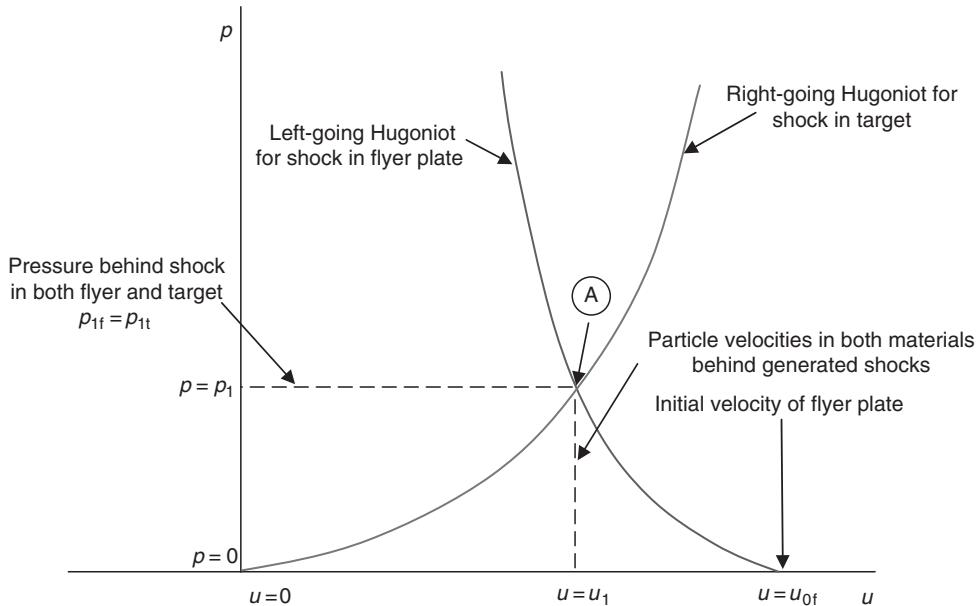
Here we again know the answer will be in GPa. Also for our right-going Hugoniot in the magnesium, we can write using Equation 16.85

$$p_1 = (1.775)(4.516)u_1 + (1.775)(1.256)u_1^2 \quad (16.128)$$

$$p_1 = 2.229u_1^2 + 8.016u_1 \quad (16.129)$$

Equating Equations 16.127 and 16.129 yields

$$u_1^2 - 8.918u_1 + 11.291 = 0 \quad (16.130)$$



**FIGURE 16.20**

A  $p$ - $u$  plot of a flyer plates initial interaction with target of lower impedance. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

Then

$$u_1 = 1.528 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.131)$$

Here we used the least positive value because the velocity  $u_1$  has to be less than our initial velocity. The pressure now comes from inserting this value in either [Equation 16.127](#) or [16.129](#).

$$p_1 = (2.229)(1.528)^2 + (8.016)(1.528) = 17.453[\text{GPa}] \quad (16.132)$$

We have stated previously that the flyer plate will remain in contact and dig into the target in this case. Even though this is the case, the shock wave will still propagate to the rear face of the flyer, reflect as a rarefaction wave, and reach the front face. It is at this time that the initial pulse into the target will end. To determine the time of this event, we again break the problem into two parts: the time it takes for the shock to reach the rear face and the time it takes for the first rarefaction to reach the impact surface.

The time it takes the shock to reach the rear face is determined by noting that the speed of wave propagation is the slope of the jump on the  $p$ - $u$  Hugoniot divided by the initial density. Thus, we can write

$$U = \frac{p_1 - p_0}{\rho_0(u_1 - u_0)} \quad (16.133)$$

$$U = \frac{(17.453 - 0)}{(8.450)(1.528 - 2.0)} \quad (16.134)$$

$$U = -4.376 \left[ \frac{\text{km}}{\text{s}} \right] = -4.376 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.135)$$

The shock will thus reach the rear of the plate in

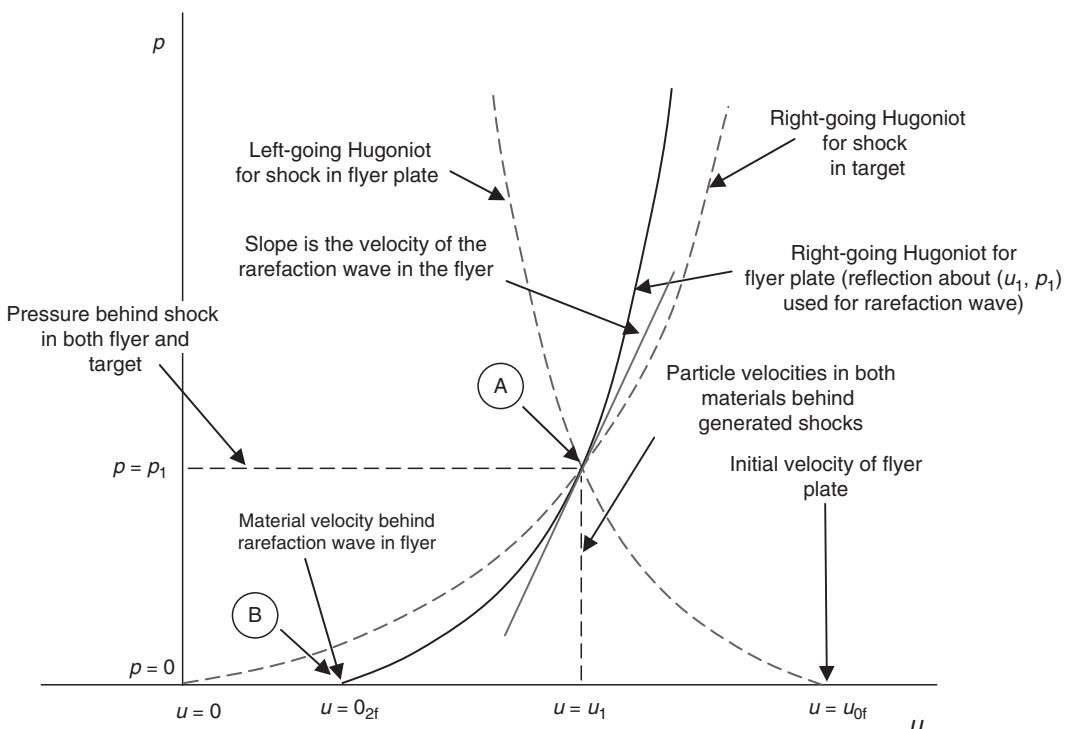
$$\Delta t = \frac{l}{U} = \frac{1[\text{mm}]}{4.376 \left[ \frac{\text{mm}}{\mu\text{s}} \right]} = 0.228[\mu\text{s}] \quad (16.136)$$

To determine the speed of the leading edge of the rarefaction wave, we need to examine Figure 16.21. Here we recall that the speed of the head of the rarefaction wave times the initial density is the slope of the  $p-u$  Hugoniot curve at the material pressure. The slope is the negative of this value which we know. We need to use [Equation 16.127](#) since this is the Hugoniot for the flyer. Then

$$\frac{dp}{du} \Big|_{u=u_1} = 24.234u_1 - 80.165 = (23.234)(1.528) - 80.165 = -43.135 \quad (16.137)$$

The rarefaction velocity is the negative of this value, so we have

$$U_{\text{Rhead}} = \frac{(43.135)}{(8.450) \left[ \frac{\text{g}}{\text{cm}^3} \right]} = 5.105 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.138)$$



**FIGURE 16.21**

A  $p-u$  plot of a flyer plates rarefaction behavior during an interaction with target of lower impedance.

So the time it takes the rarefaction to reach the front face is

$$\Delta t = \frac{l}{U} = \frac{1[\text{mm}]}{5.105 \left[ \frac{\text{mm}}{\mu\text{s}} \right]} = 0.196[\mu\text{s}] \quad (16.139)$$

Then the total time for the shock pulse is the time between impact and the rarefaction wave reaching the interface or

$$t_{\text{shock}} = 0.228[\mu\text{s}] + 0.196[\mu\text{s}] = 0.424[\mu\text{s}] \quad (16.140)$$

The material velocity behind the first rarefaction in the flyer plate is found by solving the right-going Hugoniot in the flyer plate for  $p_2 = 17.453 \text{ GPa}$ . So we have

$$p_2 = 17.453 = \rho_0 c_0 (u_1 - u_{2f}) + \rho_0 s (u_1 - u_{2f})^2 \quad (16.141)$$

Inserting some numbers in here we have

$$u_{2f}^2 - 5.654u_{2f} + 4.865 = 0 \quad (16.142)$$

Then

$$u_{2f} = 1.059 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.143)$$

Again  $u_{2f}$  had to be less than  $u_1$ .

Our next task is to find the pressure behind the first rarefaction wave in the flyer plate. We shall refer to Figure 16.22 throughout this part of the discussion.

The rarefaction will drop our pressure along the Hugoniot from point A to point C as shown in Figure 16.22. We need to reflect our right-going Hugoniot in the flyer plate about material velocity  $u_{2f}$  and solve simultaneously with our right-going Hugoniot in the target. To reflect our flyer plate Hugoniot, we shall write the equation for a left-going Hugoniot centered at  $u_{2f}$ .

$$p_3 = \rho_0 c_0 (u_{2f} - u_{3f}) + \rho_0 s (u_{2f} - u_{3f})^2 \quad (16.144)$$

or

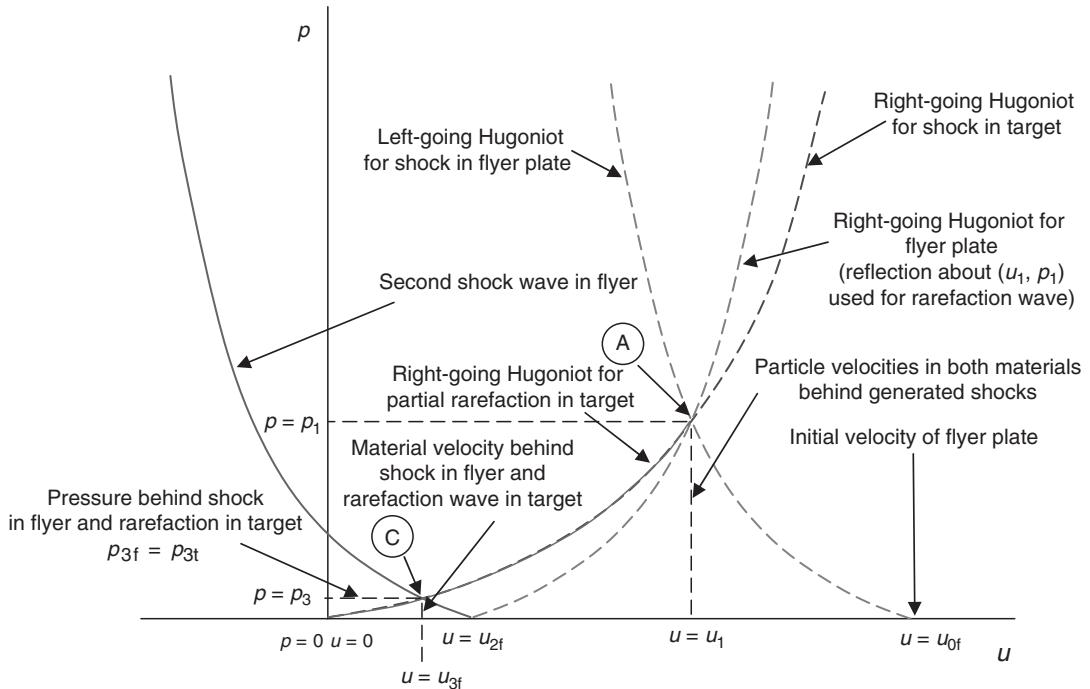
$$p_3 = 12.117u_{3f}^2 - 57.149u_{3f} + 46.932 \quad (16.145)$$

You know the drill by now. We have to simultaneously solve this equation with Equation 16.129 from before since we are looking for the intersection of the two Hugoniots

$$p_3 = 2.229u_{3f}^2 + 8.016u_{3f} \quad (16.146)$$

This leaves us with

$$u_{3f}^2 - 6.590u_{3f} + 4.746 = 0 \quad (16.147)$$



**FIGURE 16.22**

A  $p-u$  plot of a flyer plates behavior during the second shock interaction with target of lower impedance. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

Then

$$u_{3f} = 0.822 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.148)$$

The speed of the head of the rarefaction wave in the target will be different from the speed of the rarefaction wave in the flyer. Recall that the speed of the head of the rarefaction wave is the slope of the Hugoniot at the shock pressure. An examination of Figure 16.23 shows this clearly.

We can find this slope by differentiating the Hugoniot for the target, Equation 16.129, at  $u = u_1$ .

$$\frac{dp}{du} \Big|_{u=u_1} = 4.458u_1 + 8.016 = (4.458)(1.528) + 8.016 = 14.828 \quad (16.149)$$

$$U_R \text{ head target} = \frac{(14.828)}{(1.775)} \left[ \frac{\text{km}}{\text{s}} \right] = 8.353 \left[ \frac{\text{mm}}{\mu\text{s}} \right] \quad (16.150)$$

A  $t-x$  plot of this event is shown in Figure 16.24.

To close out the subject of rarefaction waves, we will discuss how to use our previous techniques to determine if spalling or scabbing of a material will occur. We shall discuss a different method in the following section but this is a good way to introduce the physics involved.

Recall that in our earlier discussions we stated that in a compressive wave, the material velocity follows the wave and in a rarefaction the opposite is true. This behavior implies

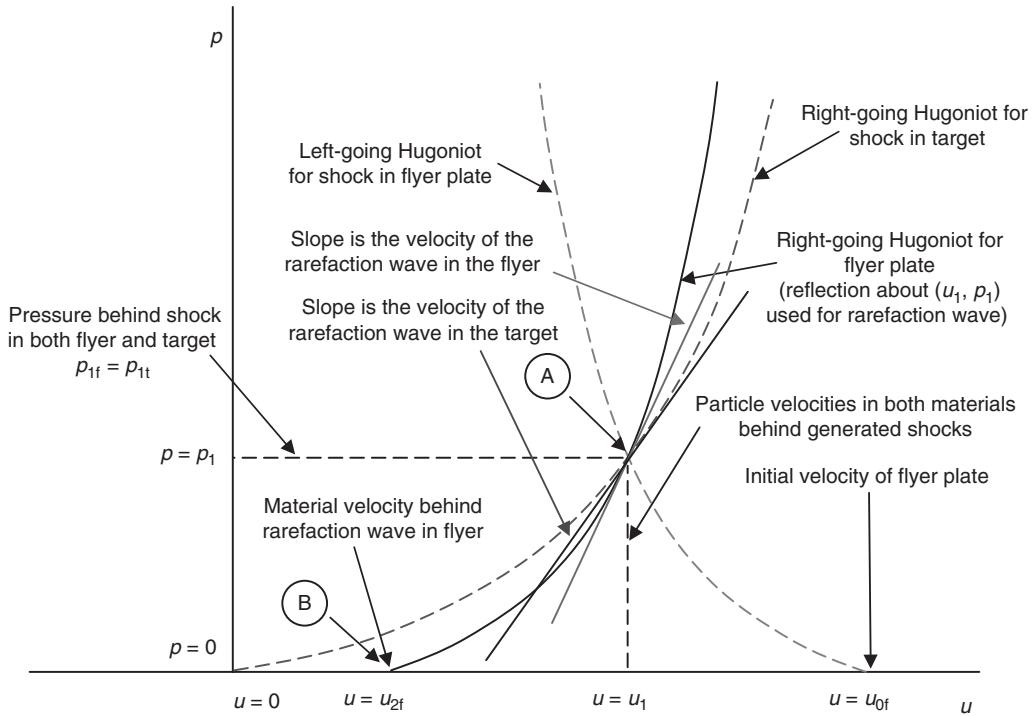


FIGURE 16.23

A  $p$ - $u$  plot of the rarefaction behavior into the target during a flyer plate impact into a lower impedance target.

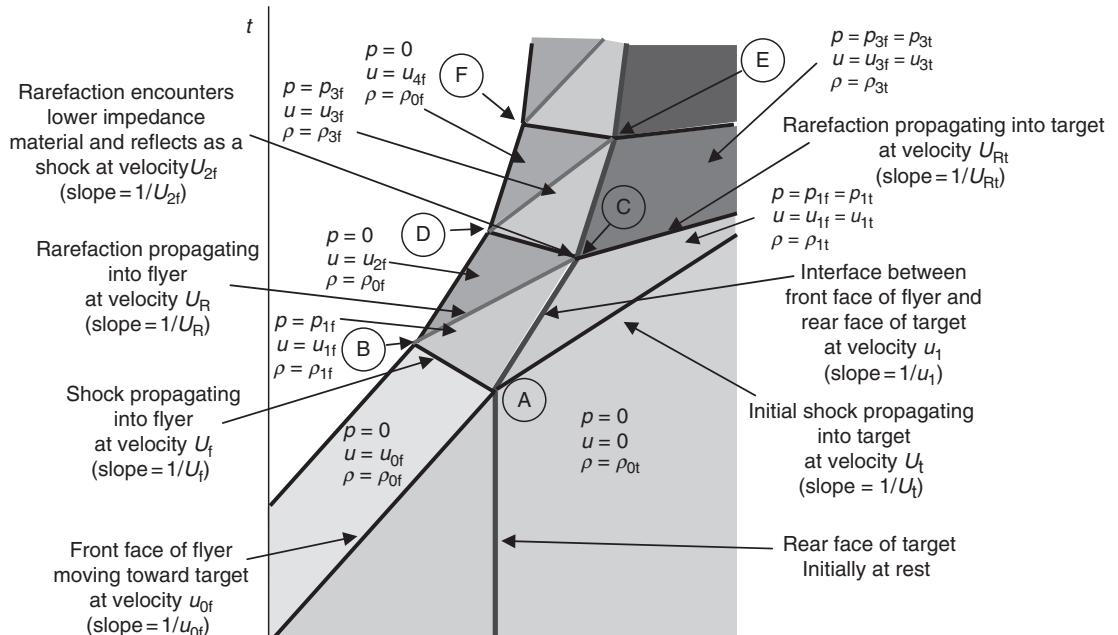


FIGURE 16.24

A  $t$ - $x$  plot of flyer plate interaction with a target of lower impedance. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

that if two rarefaction waves collide, tension of the material will result (tensile waves will propagate away from the plane of collision). If this tensile stress exceeds the material's (dynamic) ultimate tensile stress, the material will scab or spall. Also recall that we generally assume linear behavior of the material in tension (so the Hugoniots of the generated tensile waves will be straight lines). Once more we shall illustrate the theory through an example problem.

### Example Problem 10

An experiment is set up in which a brass plate is shocked by an explosive from both sides. The shock pressure was measured to be 4.0 GPa. Determine if the brass will spall

The material has the following properties:

Brass

$$\rho_{0\text{Brass}} = 8.450 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0\text{Brass}} = 3.726 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Brass}} = 1.434$$

$$c_{L\text{Brass}} = 4.700 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$\sigma_{UTS\text{Dynamic}} = 2.1 \text{[GPa]}$$

*Solution:* The only piece of information we have is the shock pressure ( $p_1$ ), but we do know the equations for the two Hugoniot curves and the approximate tensile isentropes. The situation is illustrated in [Figure 16.25](#).

We can locate  $u_1$  on the left-going Hugoniot which, by definition, has to pass through point  $(u_1, 0)$  as well. Our equation for the left-going Hugoniot is

$$p_1 = \rho_0 c_0 (u_1 - 0) + \rho_0 s (u_1 - 0)^2 \quad (16.151)$$

Inserting values

$$4.0 = (8.450)(3.726)(u_1 - 0) + (8.450)(1.434)(u_1 - 0)^2 \quad (16.152)$$

$$u_1^2 + 2.598u_1 - 0.330 = 0 \quad (16.153)$$

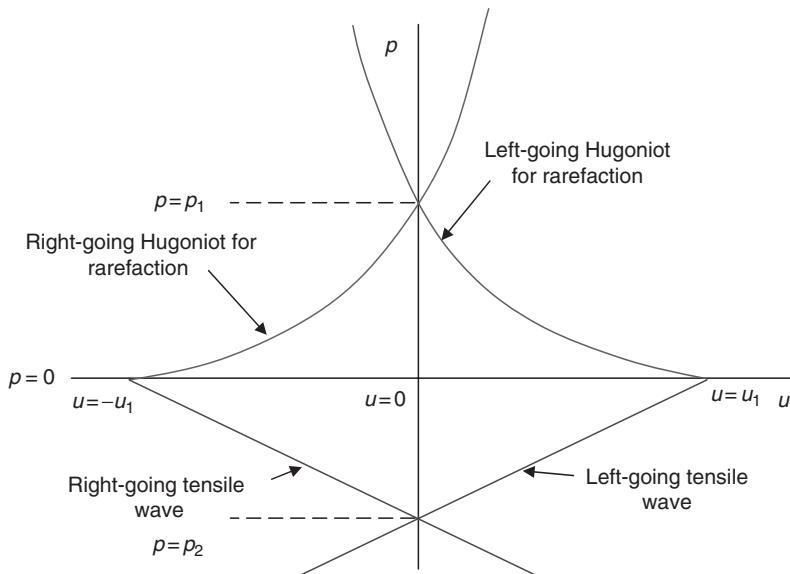
$$u_1 = 0.121 \left[ \frac{\text{km}}{\text{s}} \right] \quad (16.154)$$

Now we have located our  $x$ -axis intercept on the above diagram. All that is left to do is determine the equation for the tensile isentrope and solve for the pressure. Recall that the slope of this isentrope is defined as

$$\rho_{0\text{Brass}} c_{L\text{Brass}} = \frac{0 - p_2}{0 - u_1} \rightarrow p_2 = \rho_{0\text{Brass}} c_{L\text{Brass}} u_1 \quad (16.155)$$

$$p_2 = (8.450)(4.700)(0.121) = 4.805 \text{[GPa]} \quad (16.156)$$

Since this value is greater than the dynamic tensile strength of the material, the part will spall.



**FIGURE 16.25**

A  $p$ - $u$  plot of the collision of two rarefaction waves. (From Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996. With permission.)

### Problem 3

An experiment is set up in which a tungsten penetrator is fired against a rigid target. The impact velocity is 500 m/s. Determine the shock pressure, tensile stress, and also if the penetrator will break up.

*Answer:*  $p_1 = 44.672[\text{GPa}]$ ,  $p_2 = 53.26[\text{GPa}]$ , and it will spall

The material has the following properties:

Tungsten

$$\rho_{0W} = 19.224 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0W} = 4.029 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_W = 1.237$$

$$c_{LW} = 5.541 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$\sigma_{UTS_{\text{Dynamic}}} = 2.0[\text{GPa}]$$

### Problem 4

A 4-in. long steel bar impacts a 12-in. thick slab of 4340 steel at 1000 m/s and bounces off. Assuming the impact is normal and using one-dimensional equations, determine

1. The duration of the impact event.

*Answer:*  $\Delta t = 33.00[\mu\text{s}]$

2. The pressure developed at the interface.

*Answer:*  $p_1 = 20.980[\text{GPa}]$

#### 4340 Steel

$$\rho_{0_{\text{Steel}}} = 7.896 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0_{\text{Steel}}} = 4.569 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Steel}} = 1.490$$

$$c_{L_{\text{Steel}}} = 5.941 \left[ \frac{\text{km}}{\text{s}} \right]$$

---

### 16.3 Stress Waves in Solids

A stress wave is generated in a solid whenever an impact occurs—it is the way nature reacts to this violent event. The stress wave affects both the penetrator and the target. It is a major consideration in the breakup of the penetrator and is the primary cause of scabbing and spalling of the target.

Stress waves in solids are either elastic or elastic-plastic in nature. By this we mean that in the elastic regime the material returns to its original shape, while in the plastic regime the material is distorted permanently. How we treat the materials involved depends on the rate and intensity of loading. If these loads and rates are high enough, we can treat the materials as fluids. We will often refer to a target as being semi-infinite with the effect that geometrically only the impact surface is present and there is no reflection of the stress wave once it enters the target. This further implies that material can only compress or move backward from the free surface.

We also classify materials for the purpose of modeling as follows: isotropic (material properties are independent of direction), anisotropic (material properties are dependent upon direction), or orthotropic (material properties vary in three-orthogonal directions). Inertial effects are said to be important when the motion of the mass of the material is a major consideration in the behavior. We further stipulate that a dilatational wave is one that only involves normal stresses and a distortional wave is one where shear stresses are involved [2].

When an impact occurs in a material, several things happen simultaneously [3]: longitudinal (dilatational) waves propagate into the material; transverse (distortional) waves propagate at right angles to the longitudinal waves; Rayleigh surface waves propagate along the surface and into the material a small distance; in a material that has layers with different properties (such as a laminate or a composite), a Love shear wave may occur; and depending on the geometry of loading torsional or flexural waves may be generated. We shall only examine the first two in detail and we will call the velocity of a longitudinal and a shear waves as  $c_L$  and  $c_S$ , respectively.

The acoustic velocity (velocity of sound) in a solid medium is greatly influenced by the boundary conditions. Using a cylindrical steel bar as an example, the material is considered “bounded” if the wave encounters a boundary in the radial direction. Otherwise, the material is “unbounded” [2].

we say the following about the acoustic velocities:

Extended (unbounded)	Bounded
----------------------	---------

$$c_L^2 = \frac{\lambda + 2\mu}{\rho} = \frac{E(1 - \nu)}{\rho(1 + \nu)(1 - 2\nu)} \quad \frac{E}{\rho} \quad (16.157)$$

$$c_S^2 = \frac{\mu}{\rho} = \frac{G}{\rho} = \frac{E}{2\rho(1 + \nu)} \quad \frac{G}{\rho} \quad (16.158)$$

where

$E$  is the modulus of elasticity

$\lambda$  and  $\mu$  are the Lamé parameters

$\nu$  is the Poisson's ratio

$G$  is the shear modulus

$\rho$  is the density

We must note that since shear waves are, by definition, perpendicular to the main wave front, the form of the equation does not change between the bounded and the unbounded conditions. In a real wave, some mechanical energy is converted to heat. This is not considered in the models that we have just introduced.

In our discussions of compressible fluids, a wave simply rebounded off a solid boundary. However, in a solid medium, a compression wave will reflect off a free surface as a tensile wave. If this tensile wave's intensity is greater than the material's ultimate tensile strength, the material will fracture. If the intensity of the loading is such that the yield strength is exceeded, there will be two waves: an elastic wave (precursor in a rate independent [RI] material) and a plastic wave (very intense but rapidly attenuated in most materials). At high loading rates, with a material that has a concave-up strain rate dependency, a shock can form with the plastic wave overtaking the elastic wave. We have seen this in our earlier work.

A material's stress-strain behavior is characterized as either rate independent or rate dependent. A rate independent material has stress-strain curves which are unaffected by a change in loading rate. Examples of rate independent materials are aluminum and some steels. Examples of rate dependent materials are titanium and most steels. If the intensity of the load is about two orders of magnitude above the materials' strength, we can consider both target and penetrator as viscous fluids. In computer solutions, to impact phenomena, this is where the term "hydro-code" comes from.

Proceeding into the analysis, we need to introduce indicial notation because this is a compact way of writing the equations. For any vector,  $\mathbf{F}$ , in an  $x$ ,  $y$ , and  $z$  space, we can write it based on its components as

$$\mathbf{F} = F_x \mathbf{i}_x + F_y \mathbf{i}_y + F_z \mathbf{i}_z \quad (16.159)$$

In indicial notation, this vector is written as  $F_i$  where  $i = 1, 2, 3$  which is equivalent to our  $x$ ,  $y$ , and  $z$  space. We then have

$$F_i = F_1 \mathbf{i}_x + F_2 \mathbf{i}_y + F_3 \mathbf{i}_z \quad (16.160)$$

In this notation, a pair of distinct indices indicate a tensor.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (16.161)$$

Repeated indices indicate a sum, for instance, the trace of our previous tensor is

$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (16.162)$$

A derivative with respect to a coordinate is indicated by a comma, thus

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j} = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} & \frac{\partial \sigma_{xy}}{\partial y} & \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} & \frac{\partial \sigma_{yy}}{\partial y} & \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zx}}{\partial x} & \frac{\partial \sigma_{zy}}{\partial y} & \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} \quad (16.163)$$

Two repeated subscripts after the comma indicate a second derivative as follows:

$$u_{i,jj} = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \begin{bmatrix} \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \end{bmatrix} \quad (16.164)$$

Two other terms are frequently seen: the tensor called the Kronecker delta,  $\delta_{ij}$ , and the alternating tensor,  $\epsilon_{ijk}$ . The Kronecker delta takes on the values as

$$\delta_{ij} = 1 \text{ if } i = j \text{ or } \delta_{ij} = 0 \text{ otherwise} \quad (16.165)$$

The alternating tensor takes on the values as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \\ 0 & \text{if any two indices are alike} \\ -1 & \text{if } ijk = 321, 213, \text{ or } 132 \end{cases} \quad (16.166)$$

Let us return to the physics of stress waves in a solid. In an elastic solid, we require three relations to describe the material behavior: an equation of motion that requires the force to be converted into stress (force/unit area), an equation relating stress to strain for which we will use Hooke's law, and an equation relating strain to displacement.

If we begin with an equation of motion (Newton's second law), we have, using indicial notation to change from the vector form

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{u}} \rightarrow F_i = ma_i = m\ddot{u}_i \quad (16.167)$$

Note that  $u$  here is the material displacement/position. If we divide Equation 16.167 by a unit volume, we get

$$\frac{F_i}{V} = \frac{m}{V} \ddot{u}_i \quad (16.168)$$

We know that the mass per unit volume is defined as the density, and if we call the body force per unit mass  $f_i$ , we get

$$\frac{F_i}{V} = \frac{mf_i}{V} \left( \rho \frac{V}{m} \right) = \left( \frac{m}{V} \right) \ddot{u}_i \rightarrow \rho f_i = \rho \ddot{u}_i \quad (16.169)$$

We need to consider the internal forces in terms of stresses in our equation, so we shall add another term to the LHS to account for this with a derivation to follow. Thus, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = \rho \ddot{u}_i \quad (16.170)$$

This is the equation of motion for a differential element of a continuum.

Although Equation 16.170 is a three-dimensional equation, we shall illustrate its derivation in two dimensions. Assume we have a cube of material with volume  $dxdydz$ . The mass of the cube is the density times this volume and the body forces on the cube are  $f_x$  and  $f_y$  for simplicity. We can then draw the situation (with  $dz$  into the paper) in two dimensions as shown in Figure 16.26.

If we write the force balance in the  $x$ -direction, we obtain

$$\begin{aligned} \rho f_x (dxdydz) + \sigma_{xx}(dydz) + \frac{\partial \sigma_{xx}}{\partial x} dx(dydz) - \sigma_{xx}(dydz) \\ + \tau_{yx}(dxdz) + \frac{\partial \tau_{yx}}{\partial y} dy(dxdz) - \tau_{yx}(dxdz) = \rho (dxdydz) \ddot{u}_x \end{aligned} \quad (16.171)$$

After we cancel terms and divide by the volume  $dxdydz$ , we obtain

$$\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = \rho \ddot{u}_x \quad (16.172)$$

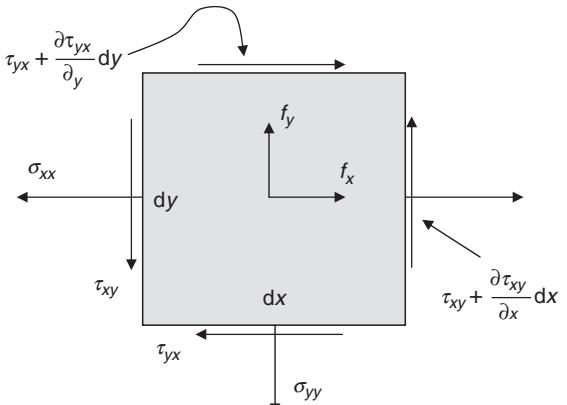
Examined in three dimensions, the equation would be

$$\rho f_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \ddot{u}_x \rightarrow \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \ddot{u}_i \quad (16.173)$$

If we recall Hooke's law in its one-dimensional form, we get

$$\sigma = E \epsilon \quad (16.174)$$

In three dimensions, it is written for a homogeneous material using two material constants, called the Lamé constants as



**FIGURE 16.26**

Differential element for calculation of stresses.

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad (16.175)$$

Here we define the constants as

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (16.176)$$

$$\mu = G = \frac{E}{2(1+\nu)} \quad (16.177)$$

Here  $G$  is the shear modulus and  $\nu$  is Poisson's ratio.

A strain-displacement relationship is the final equation necessary for our description of wave motion. For a homogeneous continuum, it is usually written as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (16.178)$$

To obtain the material displacement as a function of forces and accelerations, we shall first combine Equations 16.175 and 16.178

$$\sigma_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \lambda \frac{\partial u_j}{\partial x_j} \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (16.179)$$

If we take the derivative of Equation 16.179 with respect to  $x_j$ , we get

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \lambda \frac{\partial^2 u_j}{\partial x_j \partial x_j} \delta_{ij} + \mu \left( \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \quad (16.180)$$

Since  $\delta_{ij}$  is not equal to zero only when  $i=j$ , we can interchange  $i$  and  $j$  freely in the first term on the RHS of Equation 16.180 to yield

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \lambda \frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \left( \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_i} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_j \partial x_i} \quad (16.181)$$

If we insert Equation 16.181 into [Equation 16.173](#), we get

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + \rho f_i = \rho \ddot{u}_i \quad (16.182)$$

or

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu) \frac{\partial^2 u_i}{\partial x_j \partial x_i} + \rho f_i = \rho \ddot{u}_i \quad (16.183)$$

[Equations 16.172, 16.174](#), 16.178, and 16.183 are the equations necessary to describe wave motion in a material.

We want to simplify these equations to look like the wave equation. To do so, first we define

$$\Delta = \varepsilon_{jj} = \frac{\partial u_i}{\partial x_j} \quad (16.184)$$

If we ignore the body forces, we can rewrite Equation 16.183 as

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} = \rho \ddot{u}_i \quad (16.185)$$

If we differentiate the above equation, we get

$$\rho \frac{\partial \ddot{u}_i}{\partial x_i} = \mu \frac{\partial^3 u_i}{\partial x_j \partial x_j \partial x_i} + (\lambda + \mu) \frac{\partial^2 \Delta}{\partial x_i \partial x_i} \quad (16.186)$$

From our earlier definition, we can see that

$$\frac{\partial \ddot{u}_i}{\partial x_i} = \frac{\partial^2 \Delta}{\partial t^2} \quad (16.187)$$

We can also see that

$$\frac{\partial^3 u_i}{\partial x_j \partial x_j \partial x_i} = \frac{\partial^2 \Delta}{\partial x_j \partial x_j} = \frac{\partial^2 \Delta}{\partial x_i \partial x_i} \quad (16.188)$$

So we now rewrite Equation 16.186 as

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 \Delta}{\partial x_i \partial x_i} \rightarrow \frac{\partial^2 \Delta}{\partial t^2} = \left( \frac{\lambda + 2\mu}{\rho} \right) \frac{\partial^2 \Delta}{\partial x_i \partial x_i} \quad (16.189)$$

which is the classical wave equation of the form

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x_i \partial x_i} \quad (16.190)$$

The solution to this equation is

$$\psi = f(x - ct) + g(x + ct) \quad (16.191)$$

We previously stated that boundaries have a significant effect on wave propagation. If the medium were infinite, waves would propagate spherically at the speed of sound (wave velocity) in the material. The wave velocity in a material is defined for one-dimensional wave motion as

$$c = \sqrt{\frac{E}{\rho}} \quad (16.192)$$

For a bar impact, if the ratio of the radius of the bar to the wavelength is much less than 1, we can use these simplified equations. If we limit our study to longitudinal waves, our wave equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (16.193)$$

Much like the discussion we had about the fluid in a shock tube after the bursting of a diaphragm, when a bar is stressed by a suddenly applied load, not all parts of the bar immediately feel the impact. The waves created traverse the material and distribute the stresses and strains accordingly. We will examine first the longitudinal wave (also called dilatational, irrotational, or primary ( $P$ ) wave). This wave moves in the same direction as the pulse was applied. Next we will examine a transverse wave (also called a distortional, rotational, shear, or secondary ( $S$ ) wave). This wave moves normal to the applied pulse.

As in compressible flow there are several ways we can describe the motion of the material: stress versus time; particle velocity versus time; stress versus distance; or particle velocity versus distance. The two velocities we will use quite frequently are the speed of sound in the material,  $c$ , and the particle velocity at a point,  $v$ . The symbol  $u$  represents axial displacement. We shall make some simplifying assumptions in this treatment. We assume that the bar has a length to diameter ratio of at least 10:1. We shall neglect transverse strain. We shall neglect lateral inertia. We shall neglect body forces and internal dissipation (i.e., friction and damping).

If we look at Newton's second law for a longitudinal impact of force,  $F_L$ , and bar mass,  $m$ , we have

$$F_L dt = d(mv_L) \quad (16.194)$$

If we note that the stress,  $\sigma = F_L/A$  and the mass,  $m = \rho Adl$ , we can rewrite the above equation as

$$\sigma Adt = \rho Adl dv_L \quad (16.195)$$

where  $dl$  is the distance the pulse has moved in time  $dt$ . We can simplify the above to

$$\sigma = \rho \frac{dl}{dt} dv_L \quad (16.196)$$

But the speed of the pulse is  $dl/dt$ , so we can write for either a longitudinal or a shear wave (changing the differential to a finite difference) as

$$\sigma = \rho c_L \Delta v_L \quad (16.197)$$

$$\tau = \rho c_S \Delta v_S \quad (16.198)$$

As in the case of a wave in a fluid, when a wave in a solid reaches a boundary, it is reflected. The normal stress on a free surface must be equal to zero so a compression wave reflects as a tensile wave and vice versa. It can be shown that the shape of the reflected pulse is the same as that of the incident pulse but opposite in sign. The position (displacement) of the incident and reflected pulses (right and left running characteristics) is

$$u_I = f(x - ct) \quad (16.199)$$

$$u_R = g(x + ct) \quad (16.200)$$

In these and all subsequent equations, displacements, velocities, stresses, and strains with the subscript "I" denote those occurring due to the incident pulse, whereas the subscript "R" denotes the reflected pulse effects. At the boundary ( $x = l$ ), we have

$$u_I|_{x=l} = f(l - ct) \quad (16.201)$$

$$u_R|_{x=l} = g(l + ct) \quad (16.202)$$

Also we need to note that the strain at  $x = l$  is

$$\varepsilon_I|_{x=l} = \frac{\partial u_I}{\partial x}\Big|_{x=l} = \frac{\partial}{\partial(x-ct)}f(x-ct)\frac{\partial(x-ct)}{\partial x}\Big|_{x=l} = f'(l-ct) \quad (16.203)$$

$$\varepsilon_R|_{x=l} = \frac{\partial u_R}{\partial x}\Big|_{x=l} = \frac{\partial}{\partial(x+ct)}g(x+ct)\frac{\partial(x+ct)}{\partial x}\Big|_{x=l} = g'(l+ct) \quad (16.204)$$

At the free boundary, the stress must be zero so we have

$$\sigma_{\text{net}}|_{x=l} = \sigma_I + \sigma_R = 0 \quad (16.205)$$

But since  $\sigma = E\varepsilon$ , we can write

$$\sigma_{\text{net}}|_{x=l} = 0 = E[f'(l-ct) + g'(l+ct)] \quad (16.206)$$

$$f'(l-ct) = -g'(l+ct) \quad (16.207)$$

We can define the net velocity at a point as

$$v_{\text{net}} = v_I + v_R = \frac{\partial u_I}{\partial t} + \frac{\partial u_R}{\partial t} \quad (16.208)$$

The terms on the RHS are

$$v_I|_{x=l} = \frac{\partial u_I}{\partial t}\Big|_{x=l} = \frac{\partial}{\partial(x-ct)}f(x-ct)\frac{\partial(x-ct)}{\partial t}\Big|_{x=l} = -cf'(l-ct) \quad (16.209)$$

$$v_R|_{x=l} = \frac{\partial u_R}{\partial t}\Big|_{x=l} = \frac{\partial}{\partial(x+ct)}g(x+ct)\frac{\partial(x+ct)}{\partial t}\Big|_{x=l} = cg'(l+ct) \quad (16.210)$$

But at  $x = l$ , we can insert Equation 16.207 giving us

$$v_{\text{net}} = 2cg(l+ct) \quad (16.211)$$

Thus with a free boundary, the particle velocity and displacement are both double the incident value when the waves overlap.

If the boundary was rigid, Equations 16.205 through 16.207 are no longer true, but we know that the velocity must be zero, so we can write

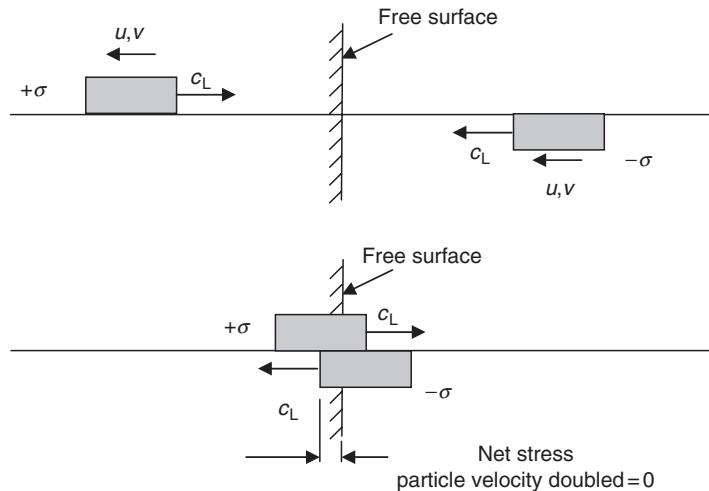
$$v_{\text{net}} = 0 = -cf'(l-ct) + cg'(l+ct) \quad (16.212)$$

$$cf'(l-ct) = cg'(l+ct) \quad (16.213)$$

We can then write Equation 16.206 as

$$\sigma_{\text{net}}|_{x=l} = E[f'(l-ct) + g'(l+ct)] = 2Ef'(l-ct) \quad (16.214)$$

Thus at a rigid boundary, the stress is doubled while the displacement and particle velocities are zero.



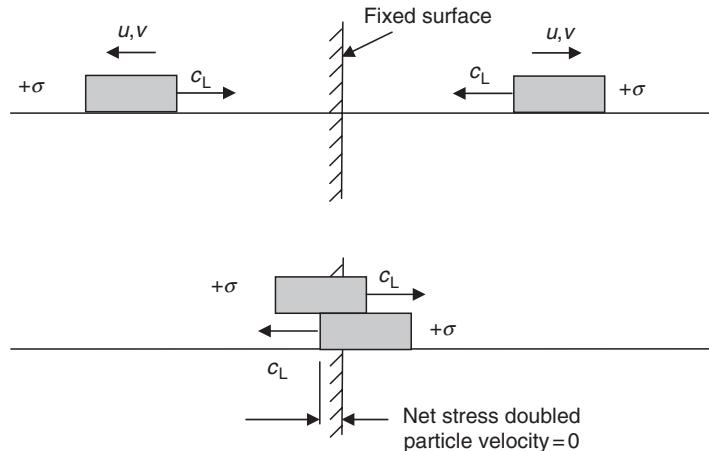
**FIGURE 16.27**

Wave interaction at a free boundary. (From Zukas, J.A., et al., *Impact Dynamics*, Krieger Publishing Co., Malabar, FL, 1992. With permission.)

These equations allow us to visualize wave interactions with fixed or free ends as follows. When a tensile wave encounters a free boundary it is reflected as a compressive wave. If we have a free surface, we can imagine a phantom pulse coming in from outside the bar as depicted in Figure 16.27. With a fixed boundary, the imagined pulse is in the same sense as the incident pulse as depicted in Figure 16.28.

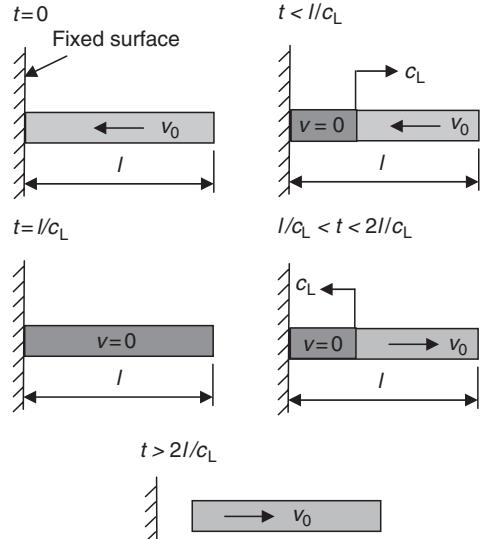
When a bar elastically impacts a surface, a stress wave of strength  $\rho v_0 c_L$  moves into the bar, stopping the motion behind it. At time  $t = l/c_L$ , the bar is stationary and in compression and all of the kinetic energy has been converted to strain energy which can be written as

$$\frac{1}{2} A_0 l \rho v_0^2 = \frac{A_0 l}{2E} (\rho c_L v_0^2) \quad (16.215)$$



**FIGURE 16.28**

Wave interaction at a fixed boundary. (From Zukas, J.A., et al., *Impact Dynamics*, Krieger Publishing Co., Malabar, FL, 1992. With permission.)



**FIGURE 16.29**

Elastic bar impact. (From Zukas, J.A., et al., *Impact Dynamics*, Krieger Publishing Co., Malabar, FL, 1992. With permission.)

When this wave encounters the free end, it reflects as a tensile wave with all of the particles behind it moving at velocity  $v_0$  away from the impact surface. This is depicted in Figure 16.29.

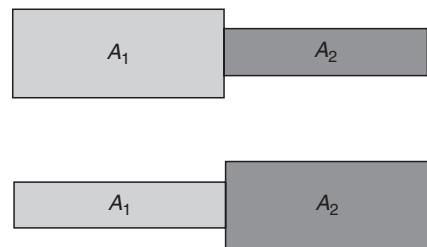
When a wave encounters a change in cross section (as illustrated in Figure 16.30) or in a new material, part of it is transmitted and part is reflected. The conditions which must be satisfied at the interface are that the forces must be equal and the velocities must be equal. The general equations for this interaction are

$$\sigma_T = \frac{2A_1\rho_2c_2}{A_1\rho_1c_1 + A_2\rho_2c_2} \sigma_I \quad (16.216)$$

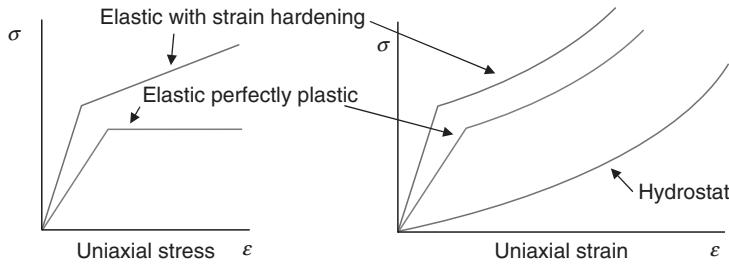
$$\sigma_R = \frac{A_2\rho_2c_2 - A_1\rho_1c_1}{A_1\rho_1c_1 + A_2\rho_2c_2} \sigma_I \quad (16.217)$$

Here  $\sigma_T$  is the transmitted stress,  $\sigma_R$  is the reflected stress, and  $\sigma_I$  is the incident stress. The implications of these equations are that if  $A_2/A_1 \rightarrow 0$ , the bar is effectively free and  $\sigma_R$  approaches  $-\sigma_I$ . If  $A_2/A_1 \rightarrow \infty$ , the bar is effectively fixed and  $\sigma_R$  approaches  $\sigma_I$ . Also  $\sigma_R$  equals 0 if  $A_2\rho_2c_2 = A_1\rho_1c_1$  and if  $\rho_2c_2 \gg \rho_1c_1$ , the stress in the transmitted pulse is approximately twice the incident stress.

When we look at shock waves in solids, we usually use plates to simplify the problem. In plates, we assume uniaxial strain (three-dimensional stress). In bars, we assume uniaxial stress (three-dimensional strain). Stress-strain diagrams of these two behaviors are illustrated in Figure 16.31. The following analysis was originally developed in Ref. [2] and



**FIGURE 16.30**  
Bars of varying cross section.



**FIGURE 16.31**

Comparison of uniaxial stress and uniaxial strain models in stress–strain diagrams.

neglects thermo-mechanical coupling as well as assuming one-dimensional deformation (i.e., the constraints are set up such that lateral strains are zero).

If we break the strain up into an elastic part (superscript “e”) and a plastic part (superscript “p”), we can write the strain in three-orthogonal directions as

$$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p, \quad \varepsilon_2 = \varepsilon_2^e + \varepsilon_2^p, \quad \varepsilon_3 = \varepsilon_3^e + \varepsilon_3^p \quad (16.218)$$

In uniaxial strain, we have

$$\varepsilon_2 = \varepsilon_3 = 0 \rightarrow \varepsilon_2^e = -\varepsilon_2^p \text{ and } \varepsilon_3^e = -\varepsilon_3^p \quad (16.219)$$

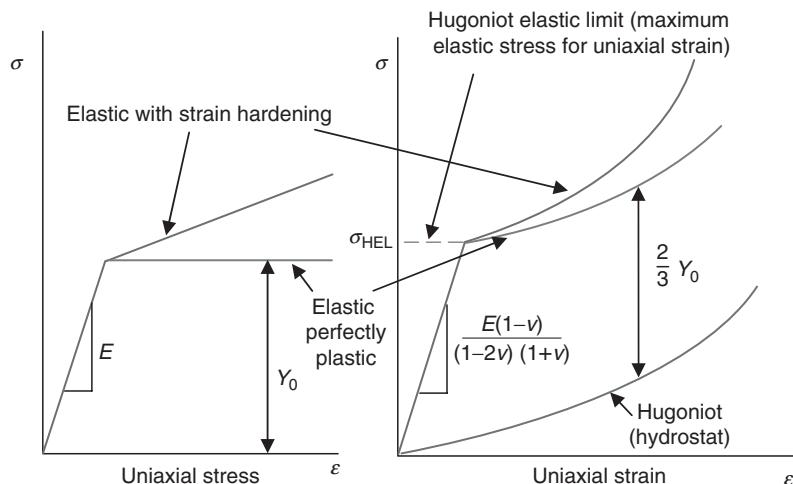
Because of symmetry, we can write

$$\varepsilon_2^p = \varepsilon_3^p \quad (16.220)$$

The material is incompressible so

$$\varepsilon_1^p + \varepsilon_2^p + \varepsilon_3^p = 0 \rightarrow \varepsilon_1^p = 2\varepsilon_2^e \quad (16.221)$$

This behavior is illustrated in Figure 16.32.



**FIGURE 16.32**

Comparison of uniaxial stress and uniaxial strain models in stress–strain diagrams with parameters established.

Thus, the total strain is

$$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p = \varepsilon_1^e + 2\varepsilon_2^e \quad (16.222)$$

If we note that  $\sigma_3 = \sigma_2$ , we can write

$$\varepsilon_1 = \frac{\sigma_1(1 - 2\nu)}{E} + \frac{2\sigma_2(1 - 2\nu)}{E} \quad (16.223)$$

If we use a yield criterion such as von Mises, we can write

$$\sigma_1 - \sigma_2 = Y_0 \quad (16.224)$$

$$\sigma_1 = \frac{E}{3(1 - 2\nu)}\varepsilon_1 + \frac{2}{3}Y_0 = K\varepsilon_1 + \frac{2}{3}Y_0 \quad (16.225)$$

The bulk compressibility term,  $K$ , causes the stress to increase regardless of yield strength or strain hardening. This is important as we shall later see and is depicted in [Figure 16.31](#). The reason that uniaxial strain is applicable in our work is that in the initial phases of impact the material does not have time to expand laterally. Later on in the impact, a condition closer to uniaxial stress may occur as the lateral deformation progresses. At extremely high pressures ( $\sim 100$  GPa,  $\sim 14.5 \times 10^6$  psi), the material will behave like a compressible fluid and will follow the Hugoniot curve (hydrostat). At lower pressures, deviation from the Hugoniot curve will occur.

If the applied stress is above the Hugoniot elastic limit (HEL), two stress waves will propagate through the material as was discussed in the previous sections. The first is an elastic wave with speed

$$c_E^2 = \frac{E(1 - \nu)}{\rho_0(1 - 2\nu)(1 + \nu)} \quad (16.226)$$

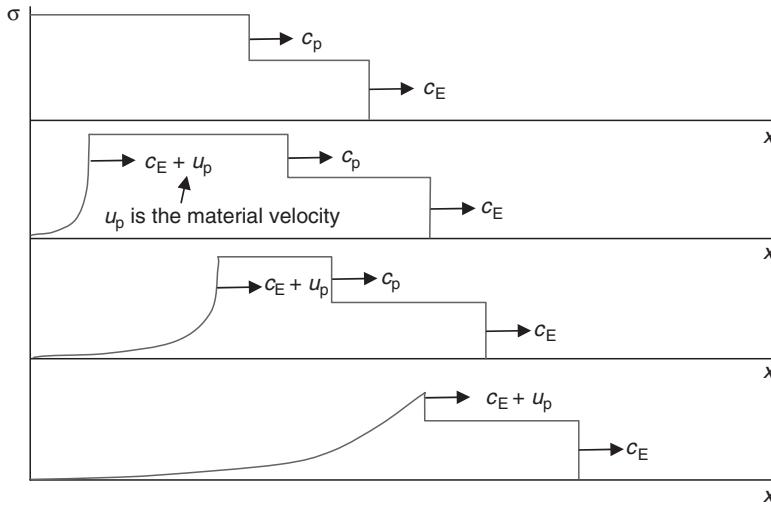
The second is a plastic wave with speed

$$c_p^2 = \frac{\sigma_B - \sigma_{HEL}}{\rho_{HEL}(\varepsilon_B - \varepsilon_A)} \quad (16.227)$$

In the above expression,  $\sigma_B$  and  $\varepsilon_B$  are the stress and strain caused by the pulse,  $\varepsilon_A$  is the strain at the Hugoniot elastic limit, and  $\rho_{HEL}$  is the material density at the HEL. After the applied pulse is over, an elastic unloading wave is generated. This unloading wave usually travels faster than the compressive wave and, if the material region is long enough, we will eventually catch up and unload the initial pulse. The point at which this occurs is called the catch-up distance. This behavior is illustrated in [Figure 16.33](#).

The spalling of armor from a non-penetrating or partially penetrating hit can be significant. Some projectiles are even designed so that they simply create spall.

When a finite thickness material is impacted on one side by an object that either does or does not penetrate, a stress wave will be generated which can cause spalling or scabbing. This is to be expected in materials that are strong in compression but weak in tension. We are going to examine the impact event as a saw-tooth pulse in one dimension and assume that the pulse propagates without change in stress or intensity. We define the failure strength of a material as the point where the tensile stress reaches some critical value  $\sigma_F$ . The length of the incident compressive pulse is defined as  $\lambda$  and its magnitude is



**FIGURE 16.33**

Diagram depicting plastic wave attenuation. (From Zukas, J.A., et al., *Impact Dynamics*, Krieger Publishing Co., Malabar, FL, 1992. With permission.)

specified as  $\sigma_m$ . The wave is reflected from the free surface with a net maximum tensile stress  $\sigma_T$  which will always occur at the leading edge of the wave (Figure 16.34).

At any time, we can write

$$\sigma_T = \sigma_m - \sigma_I \quad (16.228)$$

Here  $\sigma_I$  is the part of the compression wave remaining at an instant in time. If  $\sigma_T$  ever exceeds  $\sigma_F$ , a fracture will occur. Thus at fracture, we can write

$$\sigma_F = \sigma_m - \sigma_I \quad (16.229)$$

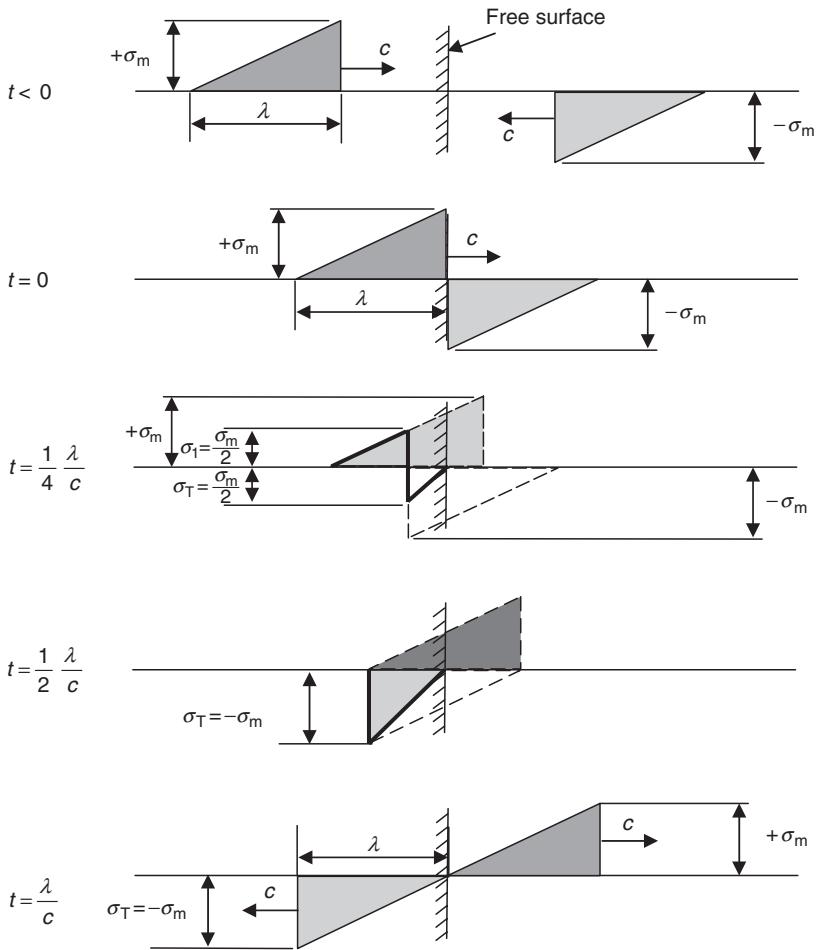
If we assume that this occurs at some instant, we will generate a spall thickness  $t_1$  and we can write this spall thickness as

$$\frac{\sigma_I}{\lambda - 2t_1} = \frac{\sigma_m}{\lambda} \quad (16.230)$$

It can be shown that by eliminating  $\sigma_I$  between Equations 16.228 and 16.229, and using Equation 16.230, we can write the spall thickness as

$$t_1 = \frac{\sigma_F \lambda}{\sigma_m 2} \quad (16.231)$$

Thus, if the initial pulse amplitude into the material is equal to its tensile strength, the material will fail at a distance one half of the pulse wavelength from the rear face. We also need to note that if  $\sigma_m < \sigma_F$ , there will be no fracture and if  $\sigma_m \gg \sigma_F$ , there will be multiple fractures.



**FIGURE 16.34**

Triangular pulse encounter with a free surface.

If multiple fractures occur, the portion of the pulse trapped in a fractured piece will leave with that piece (actually forcing it away) and the part of the pulse which remains in the original target plate is defined through

$$\lambda_2 = \lambda - 2t_1 \quad (16.232)$$

$$\sigma_{m_2} = \sigma_I \quad (16.233)$$

If this occurs, we would enter these values back into our original equations to obtain

$$t_2 = \frac{\sigma_F}{\sigma_{m_2}} \frac{\lambda_2}{2} \quad (16.234)$$

This process is repeated until conditions no longer permit spalling (i.e.,  $\sigma_{m_n} < \sigma_F$ ).

We shall use the principle of impulse and momentum to determine the velocity of the spalled piece. The momentum of the spall is

$$mv_{t_1} = (\rho t_1 A)v_{t_1} \quad (16.235)$$

The impulse imparted to the spall is

$$\int F dt = \left[ \frac{(\sigma_m + \sigma_f)}{2} A \right] \frac{2t_1}{c} \quad (16.236)$$

Here the average stress acting over the time the wave is trapped in the spalled piece has been used. If we make the substitution for  $\sigma_f$  and combine Equations 16.235 and 16.236, we get

$$v_{t_1} = \frac{2\sigma_m - \sigma_f}{\rho c} \quad (16.237)$$

If there is a second spall layer, the velocity of that will be

$$v_{t_2} = \frac{2\sigma_m - 3\sigma_f}{\rho c} \quad (16.238)$$

If there are more spall layers, their velocities will be

$$v_{t_n} = \frac{2\sigma_m - (2n-1)\sigma_f}{\rho c} \quad (16.239)$$

The number of spall layers a wave will produce is given by

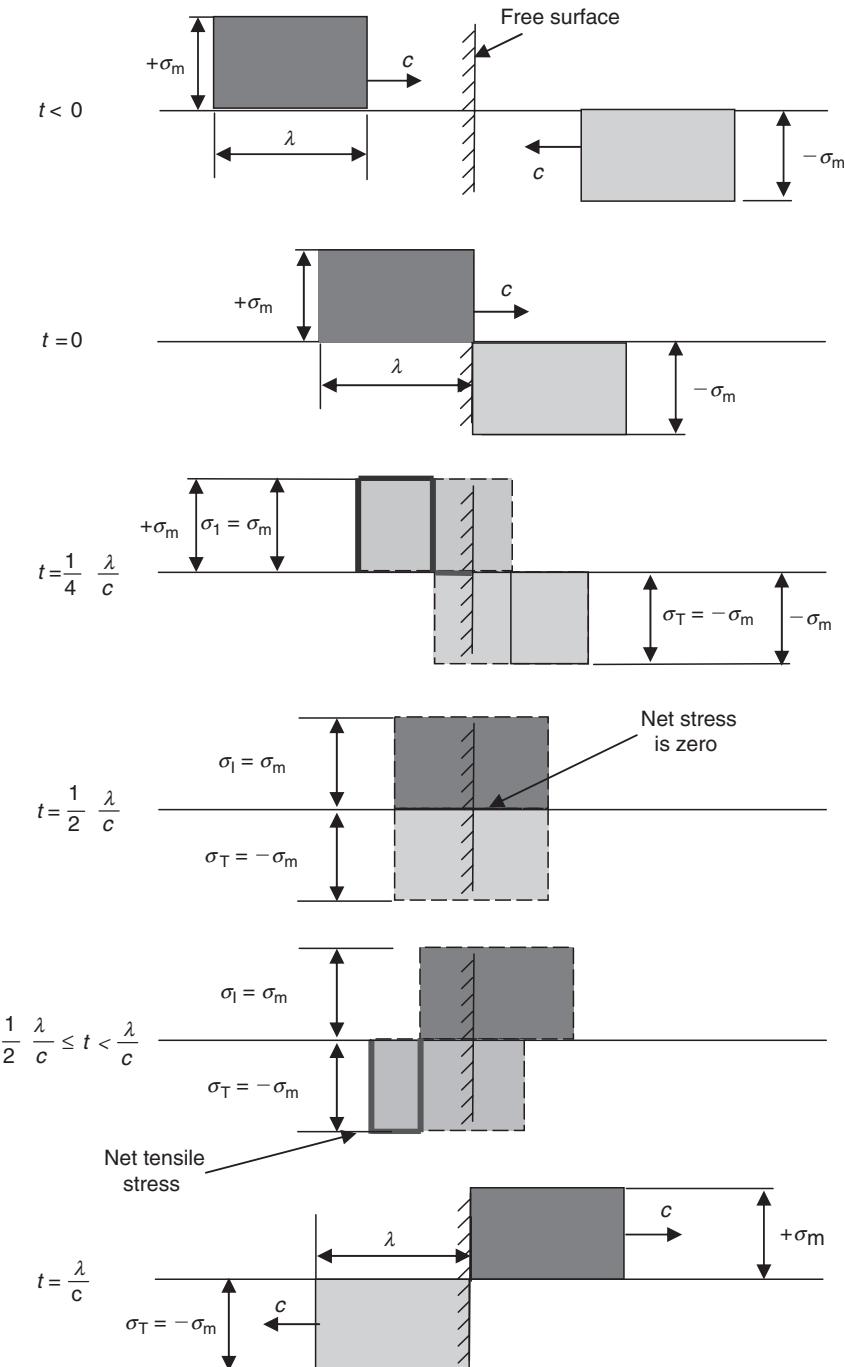
$$n = \frac{\sigma_m}{\sigma_f} \quad (16.240)$$

Unlike a triangular pulse, a theoretically square pulse (shown in [Figure 16.35](#)) can only spall one piece of material because of its discontinuous nature. The thickness will be either zero (if  $\sigma_m < \sigma_f$ ) or  $\lambda/2$  (if  $\sigma_m \geq \sigma_f$ ). The velocity imparted to the spalled piece will be given by

$$v_t = \frac{\sigma_m}{\rho c} \quad (16.241)$$

The previous formulas only yield qualitative results. Dynamic fracture can be divided into four phases: nucleation of micro-cracks at many locations in the material, symmetric growth of the fracture nuclei, coalescence of the fractures, and spallation owing to formation of a large fracture surface.

Spallation is such a common occurrence in armor that some terms have been established to describe it. The incipient spall threshold is that combination of stress amplitude and pulse duration below which no damage is detected in a specimen at 100X magnification. The complete spall threshold is the combination of stress amplitude and pulse duration at which a large piece of material will spall. Because of the complicated nature of the



**FIGURE 16.35**

Square pulse encounter with a free surface.

phenomena, it is difficult to predict exactly when and how a material will spall. There are various models all of which attempt to describe the spallation process by some physical means, one of which was introduced in [Section 16.2](#).

### Problem 5

A 4-in. long steel bar impacts a 12-in. thick slab of 4340 steel at 1000 m/s and bounces off. Assuming the impact is normal and using one-dimensional equations, determine

1. The duration of the impact event (use Hugoniots).

*Answer:*  $t_{\text{shock}} = 35.87[\mu\text{s}]$

2. The pressure developed at the interface (use Hugoniots).

*Answer:*  $p_1 = 20.980[\text{GPa}]$

3. The thickness of the first spalled piece (if any) assuming the input pulse is a constant square wave pulse throughout the impact event.

*Answer:*  $t_1 = 3.75[\text{in.}]$

Illustrate your answer to (2) above

Illustrate your answer to (3) above

4340 Steel  
 Modulus of elasticity =  $30.0[\times 10^6 \text{ psi}]$   
 Modulus of rigidity (shear) =  $11.5[\times 10^6 \text{ psi}]$   
 Poisson's ratio = 0.29  
 Ultimate tensile stress =  $250,000[\text{lbf/in.}^2]$

$$\rho_{0_{\text{Steel}}} = 7.896 \left[ \frac{\text{g}}{\text{cm}^3} \right]$$

$$c_{0_{\text{Steel}}} = 4.569 \left[ \frac{\text{km}}{\text{s}} \right]$$

$$s_{\text{Steel}} = 1.490$$

$$c_{L_{\text{Steel}}} = 5.941 \left[ \frac{\text{km}}{\text{s}} \right]$$

### Problem 6

A Japanese 20-mm projectile with the properties below impacts a 7-in. thick concrete wall at  $0^\circ$  obliquity. The concrete has a 1500-psi unconfined compressive strength and density of  $0.080 \text{ lbm/in.}^3$ . The concrete dynamic tensile strength is 1000 psi. If the projectile has the following properties:

1. Determine the duration of the impact event using the assumption of non-penetration (use Hugoniots).  
*Answer:*  $t_{\text{shock}} = 24.85[\mu\text{s}]$
2. Determine whether the concrete will spall and if so determine the extent (in inches of thickness) of the total spallation—list all assumptions.  
*Answer:*  $t_1 = 2.32[\text{in.}]$
3. Determine if the projectile perforates the concrete accounting for the spallation.  
*Answer:* The projectile will perforate.
4. Using your ability to determine the timing of the penetration events explain why or why not the above model is valid, i.e., prove it using the numbers.

## Estimated penetrator information

$$s = 40[\text{mm}] \quad m = 128[\text{g}]$$

$$d = 20[\text{mm}] \quad V_s = 550 \left[ \frac{\text{m}}{\text{s}} \right] \quad \rho_p = 0.283 \left[ \frac{\text{lbm}}{\text{in.}^3} \right]$$

$$L = 60[\text{mm}]$$

Steel	Concrete
$\rho_{0_{\text{Steel}}} = 7.896 \left[ \frac{\text{g}}{\text{cm}^3} \right]$	$\rho_{0_{\text{Concrete}}} = 2.232 \left[ \frac{\text{g}}{\text{cm}^3} \right]$
$c_{0_{\text{Steel}}} = 4.569 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{0_{\text{Concrete}}} = 4.0 \left[ \frac{\text{km}}{\text{s}} \right]$ (estimate)
$s_{\text{Steel}} = 1.490$	$s_{\text{Concrete}} = 1.4$ (estimate)
$c_{\text{L}_{\text{Steel}}} = 5.941 \left[ \frac{\text{km}}{\text{s}} \right]$	$c_{\text{L}_{\text{Concrete}}} = 4.0 \left[ \frac{\text{km}}{\text{s}} \right]$ (estimate)

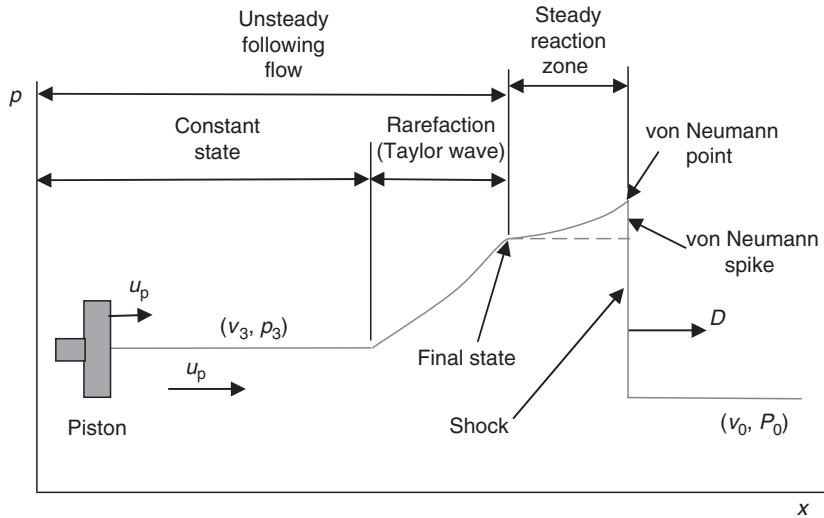
## 16.4 Detonation Physics

Now that we have talked about shock in nonreacting solids, it is appropriate to discuss how these shocks behave in detonating materials. While the interested reader is again referred to the references to find more detailed treatment, we shall endeavor to introduce the concept of detonation by building on what we have discussed previously.

In 1950, Zel'dovich, von Neumann, and Doering developed the so-called ZND model for detonation [4]. This model is sometimes known as “the simple model” for a reaction. The model is a one-dimensional model that neglects transport properties. In this model, the leading part of the detonation wave is a nonreacting shock, a jump discontinuity called the von Neumann spike. In the model, shocks of sufficient strength raise the density (and the temperature) above the ignition point beginning the reaction. In the gas behind the reaction zone’s final state is the following flow which was denoted as moving with velocity  $u_p$  in our previous work.

In the ZND model, there are essentially two conditions that can exist: the unsupported case and the overdriven case. In the unsupported case, an initial shock starts the reaction and it can continue if the conditions are right or it can die out. In the overdriven case, there is a force that continually drives the wave forward similar to the infinite shock pulses that we have examined earlier. Our approach here will be to physically describe the types of waves on a  $p$ - $x$  diagram and then to relate these descriptions to the Hugoniot curves.

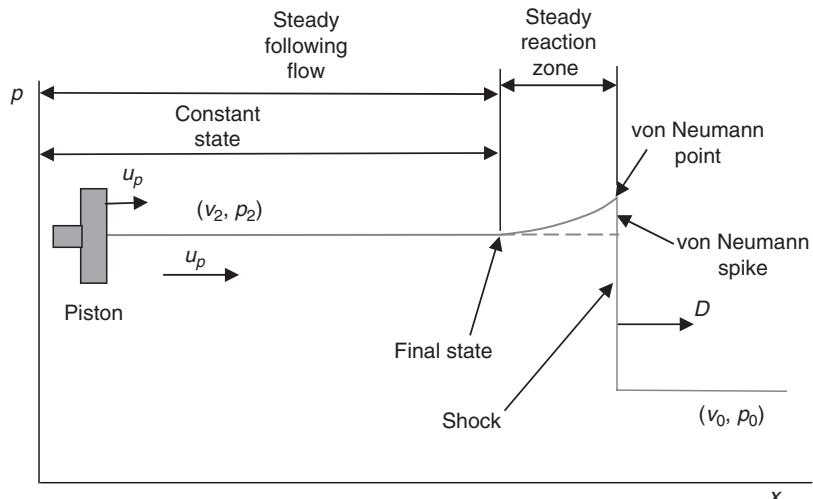
The unsupported case is depicted in Figure 16.36 as a  $p$ - $x$  diagram. In this figure, there is an initial shock that begins the reaction. The detonation wave velocity is  $D$ . This is commonly known as the von Neumann spike. This spike begins the chemical reaction which takes place in the reaction zone immediately behind the shock. The reacted products are said to be in their final state when they leave the reaction zone. Once the reaction is completed, there is a rarefaction wave that follows the reaction zone. This is followed by the constant state where the chemically altered gases follow the rarefaction. Sometimes,



**FIGURE 16.36**  
Unsupported detonation wave.

we like to imagine that there is a piston that causes the induced velocity,  $u_p$ , and this is also depicted in the figure. Later on, we shall introduce restrictions on this piston velocity that is consistent with our unsupported definition.

The overdriven case is depicted in a  $p$ - $x$  diagram as in Figure 16.37. Again there is an initial shock that begins the reaction. This spike begins the chemical reaction which takes place in the reaction zone immediately behind the shock. The reacted products are said to



**FIGURE 16.37**  
Overdriven detonation wave.

be in their final state when they leave the reaction zone. In this case, however there is no rarefaction wave. Our imaginary piston is pushing the reacted gas at such a velocity that the rarefaction cannot form. We shall soon see that this piston velocity, in either the unsupported or overdriven case determines completely the geometry and the velocity of the detonation wave.

The ZND model has two main parts. First, we must determine all possible steady solutions for the detonation wave velocity,  $D$ . This will determine what the final state is. Then we must find a following flow (piston velocity,  $u_p$ ) that is a function of the detonation velocity. If this is greater than the minimum value of  $D$ , the wave is overdriven. If it is less than the minimum  $D$ , the wave is unsupported. If it is equal to the minimum  $D$ , the wave is a steady detonation wave. For now, we shall assume that the reaction takes place instantaneously. Thus, the steady reaction zone is a jump discontinuity.

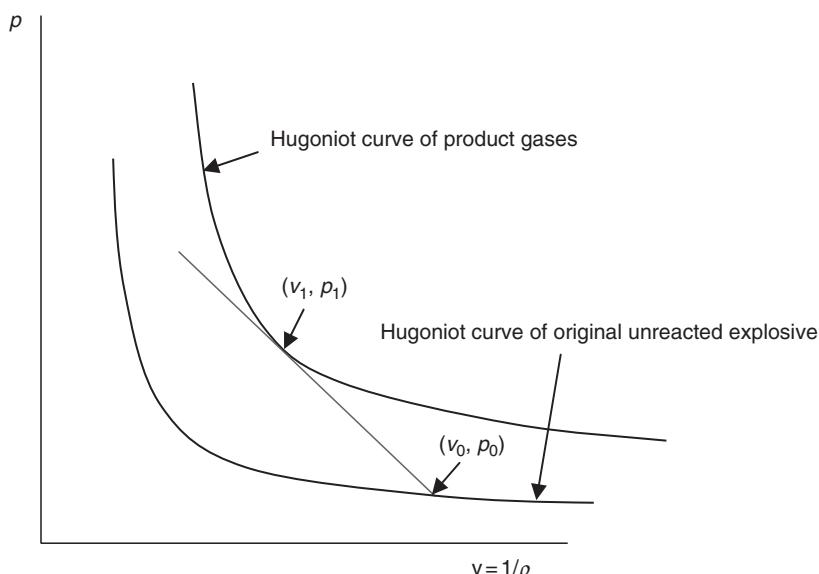
With a reactive flow, there are some nuances associated with the Hugoniot curves. The first we must recognize is that once the reaction has taken place, we have a different material than the solid unreacted material we started with. Because of this material change, we have a different Hugoniot. It will be shifted toward the concave side as depicted in Figure 16.38. Thus, any further shocks or rarefactions take place using this new curve.

If we assume that the products of the reaction are instantaneously produced by the shock (i.e., the reaction zone is infinitesimally small in thickness), we obtain the simplest theory. If we rewrite the conservation of mass equation using the detonation velocity, we obtain

$$\rho_0 D = \rho_1 (D - u_p) \quad (16.242)$$

Similarly, we can write the conservation of momentum as

$$p_1 - p_0 = \rho_0 D u_p \quad (16.243)$$



**FIGURE 16.38**

Hugoniot curve for reacted and unreacted material—overdriven detonation waveHH.

If we eliminate  $u_p$  from these two equations, we obtain the equation for the Rayleigh line.

$$\rho_0^2 D^2 - \frac{(p_1 - p_0)}{(v_0 - v_1)} = 0 \quad (16.244)$$

Here we have used the specific volume because we like to deal with  $p-v$  diagrams.

From our Rayleigh line Equation 16.244, we can see that it passes through the point  $(v_0, p_0)$  and has a slope of  $-\rho_0 D^2$ . Some interesting things can be gleaned from this. First, we know that  $\rho_0$  is positive and finite. If the Rayleigh line was horizontal, it would represent a detonation velocity of zero; hence, the detonation would not go anywhere. This is known as a constant pressure detonation. If the line was vertical, this would represent an infinite detonation velocity; so the detonation would happen everywhere at once. This is known as a constant volume detonation. This is illustrated in Figure 16.39.

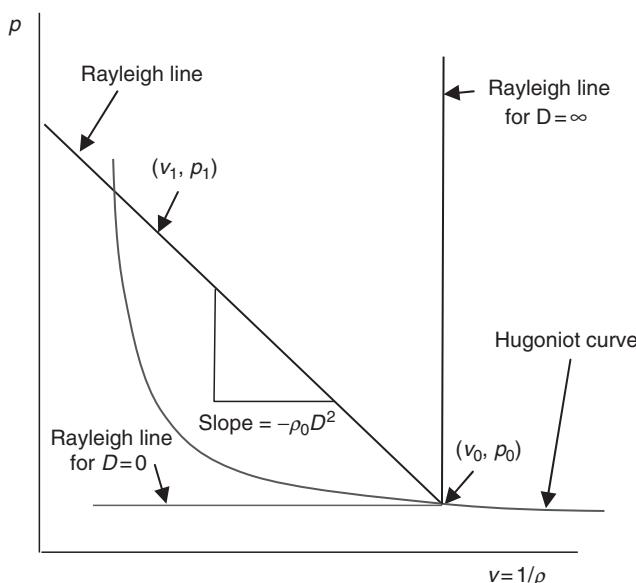
If we eliminate  $D$  between Equations 16.242 and 16.244, we obtain the equation for the Hugoniot curve

$$u_p^2 = (p_1 - p_0)(v_0 - v_1) \quad (16.245)$$

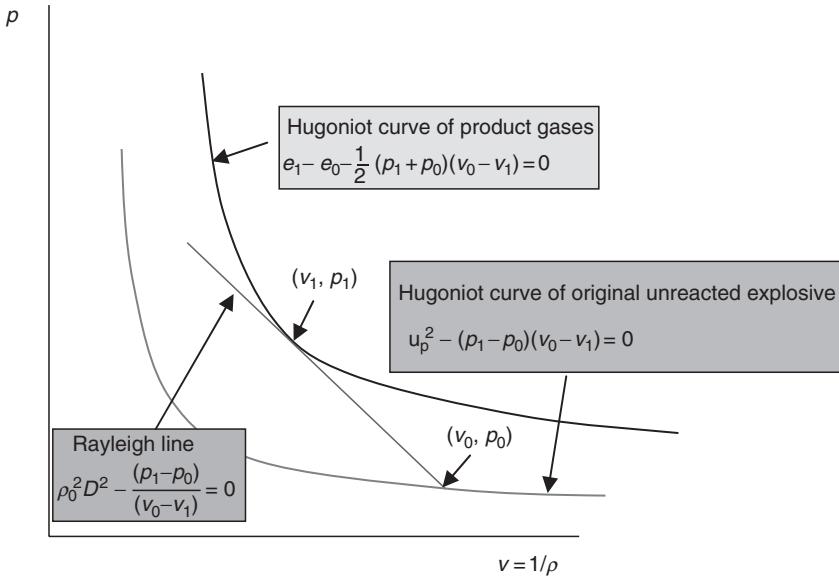
Thus, if we are given  $u_p$  and  $D$ , everything else is known because we can find the intersection of the Rayleigh line and the Hugoniot curve. If we write the energy equation using specific volume, we obtain

$$e_1 - e_0 - \frac{1}{2}(p_1 + p_0)(v_0 - v_1) = 0 \quad (16.246)$$

In this case, remember that the reaction is complete at state "1" and we have the energy of the unreacted explosive at state "0." We can then intersect this with the Rayleigh line (Equation 16.244) to determine the state of the explosive products. This is illustrated in Figure 16.40.



**FIGURE 16.39**  
Constant pressure and constant volume detonation.



**FIGURE 16.40**

Hugoniots of unreacted and reacted explosive.

If we assume a polytropic gas (an ideal gas with constant specific heats), we can write the equation of state as

$$pv = RT \quad (16.247)$$

The energy equation then would be

$$e = C_v T - \lambda q \quad (16.248)$$

with

$$q = \Delta h_r^0 \quad (16.249)$$

Here  $C_v$  is the (constant) specific heat at constant volume,  $T$  is the absolute temperature,  $q$  is the heat released from the reaction, and  $\Delta h_r^0$  is the heat of reaction of the complete reaction. In this equation,  $\lambda$  is a parameter which varies from 0 to 1 indicating the degree of reaction:

$\lambda = 0$  means the reaction has not even begun.

$\lambda = 1$  means the reaction is complete.

In this simplest model, there are only two states, 0 and 1.

We can rearrange Equation 16.247 as follows:

$$T = \frac{pv}{R} \quad (16.250)$$

If we recall the relationship between specific heat at constant volume and the gas constant as

$$R = C_v(\gamma - 1) \quad (16.251)$$

where  $\gamma$  is the ratio of specific heats, we can then say that

$$T = \frac{pv}{C_v(\gamma - 1)} \quad (16.252)$$

Inserting Equation 16.252 into [Equation 16.248](#) yields

$$e = \frac{pv}{(\gamma - 1)} - \lambda q \quad (16.253)$$

Putting this result directly into our Hugoniot equation gives us

$$\frac{p_1 v_1}{(\gamma - 1)} - \frac{p_0 v_0}{(\gamma - 1)} - \lambda q - \frac{1}{2}(p_1 + p_0)(v_0 - v_1) = 0 \quad (16.254)$$

By defining

$$\mu^2 = \frac{(\gamma - 1)}{(\gamma + 1)} \quad (16.255)$$

We can express Equation 16.254 as

$$\left( \frac{p_1}{p_0} + \mu^2 \right) \left( \frac{v_1}{v_0} - \mu^2 \right) - 1 + \mu^4 - \mu^2 \frac{2\lambda q}{p_0 v_0} = 0 \quad (16.256)$$

This is the equation of a hyperbola in the  $(v/v_0, p/p_0)$  plane centered at  $v/v_0 = \mu^2$  and  $p/p_0 = -\mu^2$ . This is a Hugoniot curve that defines all possible end states of the detonation reaction. If this is solved simultaneously with a Rayleigh line ([Equation 16.244](#)), their intersection defines the state of the gas emerging from the reaction. The issue now is that the slope of the Rayleigh line is dependent upon the detonation velocity so one of three families of solutions exists:

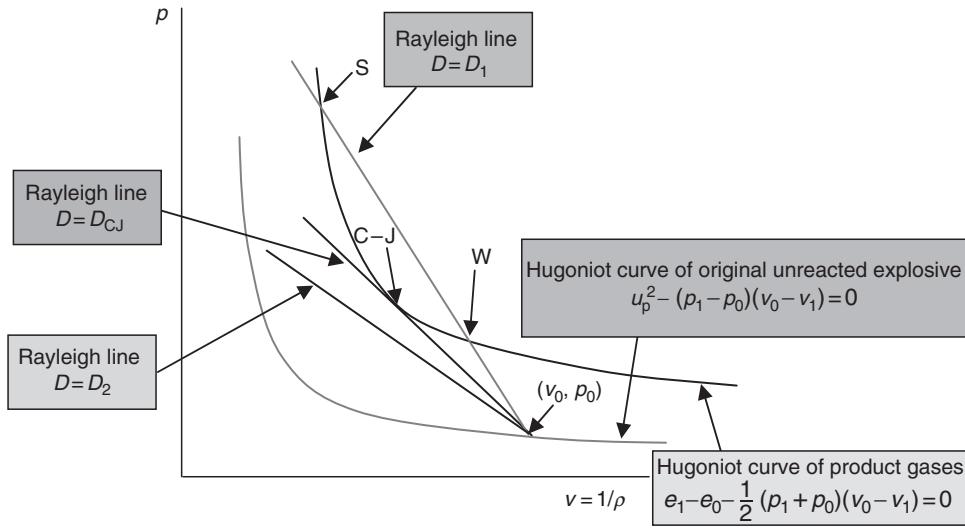
Two intersections of the Hugoniot by the Rayleigh line.

One intersection of the Hugoniot by the Rayleigh line.

No intersections of the Hugoniot by the Rayleigh line.

This is depicted in [Figure 16.41](#).

If the detonation wave speed,  $D$ , is sufficiently high, say  $D = D_1$ , then there will be two intersections of the Rayleigh line with the Hugoniot. If the detonation wave speed,  $D$ , is sufficiently high, say  $D = D_{CJ}$ , then there will be one intersection of the Rayleigh line with the Hugoniot. If the detonation wave speed,  $D$ , is sufficiently low, say  $D = D_2$ , then there will be no intersection of the Rayleigh line with the Hugoniot. If there are no solutions, then the detonation will (under the assumptions of the model) die out. If there are two solutions, we generally call the upper solution the strong solution and the lower one the weak solution (S and W in Figure 16.41). If there is only one solution, we call this the Chapman-Jouguet solution.



**FIGURE 16.41**  
Possible intersections of Rayleigh line and Hugoniot curves.

For the strong solution, any disturbance created behind the wave will overtake the wave. Examine the Hugoniot in Figure 16.41. The slope of a line tangent at S is greater than the detonation wave Rayleigh line; therefore, any disturbance will move faster than the detonation wave and will eventually catch up with it. Induced flow is subsonic relative to the wave (i.e.,  $c > D_1 - u$ ). In the weak solution, the induced velocity is supersonic with respect to the detonation wave. The slope of a line tangent at W is smaller than the detonation wave Rayleigh line; therefore, any disturbance will move slower than the detonation wave and will fall farther and farther behind. Induced flow is supersonic relative to the wave (i.e.,  $c < D_1 - u_p$ ).

For the Chapman–Jouguet (or C–J) solution, any disturbance created behind the wave will maintain its distance from the wave. Once more look at the Hugoniot of Figure 16.41. Since the line tangent at the CJ point is the Rayleigh line, any disturbance will propagate at the same speed as the detonation wave and will keep pace with it. Induced flow is sonic relative to the wave (i.e.,  $c = D_1 - u_p$ ). If we recall the slope of the Rayleigh line as

$$\left(\frac{dp}{dv}\right)_{\text{Rayleigh}} = -\frac{p_1 - p_0}{v_0 - v_1} \quad (16.257)$$

We shall divide our Hugoniot [Equation 16.246](#) by  $(v_0 - v_1)^2$  to obtain

$$\frac{(e_1 - e_0)}{(v_0 - v_1)^2} - \frac{1}{2} \frac{(p_1 + p_0)}{(v_0 - v_1)} = 0 \quad (16.258)$$

Now we multiply by 2 and separate the first term into

$$\frac{2\left(\frac{de}{dv}\right)_{\text{Hugoniot}}}{(v_0 - v_1)} - \frac{(p_1 + p_0)}{(v_0 - v_1)} = 0 \quad (16.259)$$

Let us distribute the negative sign on the second term to write

$$\frac{2\left(\frac{de}{dv}\right)_{\text{Hugoniot}}}{(v_0 - v_1)} + \frac{(-p_1 - p_0)}{(v_0 - v_1)} = 0 \quad (16.260)$$

We can add and subtract  $p_1/(v_0 - v_1)$  to obtain

$$\frac{2\left(\frac{de}{dv}\right)_{\text{Hugoniot}}}{(v_0 - v_1)} + \frac{(p_1 - p_0)}{(v_0 - v_1)} + \frac{2p_1}{(v_0 - v_1)} = \left(\frac{dp}{dv}\right)_{\text{Hugoniot}} \quad (16.261)$$

The only way for Equation 16.261 to equal [Equation 16.257](#) is for

$$\left(\frac{de}{dv}\right)_{\text{Hugoniot-CJ}} = -p_1 \quad (16.262)$$

If we recall from thermodynamics that on an isentrope

$$\left(\frac{de}{dv}\right)_s = -p \quad (16.263)$$

Therefore, the Rayleigh line and Hugoniot curve lie on the isentrope at the C-J point. The implications of this are

$$\gamma = \frac{C_p}{C_v} = \frac{\left(1 - \frac{p_0}{p_1}\right)}{\left(\frac{v_0}{v_1} - 1\right)} \quad (16.264)$$

We can use this fact and assuming  $p_0 \approx 0$  by substituting back into our Rayleigh and Hugoniot equations to state that at the C-J point the following are true:

$$p_{\text{CJ}} = \frac{\rho_0 D^2}{(\gamma + 1)} \quad (16.265)$$

$$v_{\text{CJ}} = \frac{v_0 \gamma}{(\gamma + 1)} = \frac{1}{\rho_{\text{CJ}}} \quad (16.266)$$

$$u_{\text{P CJ}} = \frac{D}{(\gamma + 1)} \quad (16.267)$$

$$c_{\text{CJ}} = \frac{D \gamma}{(\gamma + 1)} \quad (16.268)$$

We have stated that in this simplest theory the reaction occurs instantaneously. Thus, as soon as unreacted material passes through the detonation wave, it is instantaneously converted to a new material. We can determine this final state by the intersection of the Rayleigh line with the reacted material Hugoniot curve. In this theory, there are three cases we must consider:  $D < D_{\text{CJ}}$ ,  $D = D_{\text{CJ}}$ , and  $D > D_{\text{CJ}}$ .

If  $D < D_{\text{CJ}}$ , the Rayleigh line does not intersect the Hugoniot curve of the reaction products, we will not have a steady reaction—the reaction will die out. If  $D = D_{\text{CJ}}$ , the Rayleigh line intersects the Hugoniot curve of the reaction products at one point, the detonation wave will continue to move into the unreacted material and the detonation products will move away from the wave, relative to the wave, at the sonic velocity. There is only one solution—the reaction will be steady. If  $D > D_{\text{CJ}}$ , the Rayleigh line intersects the Hugoniot curve of the reaction products at two points (strong and weak). We will ignore the weak solution as inadmissible because the pressure will have to drop. For the strong solution, the detonation wave will continue to move into the unreacted material. In this case, the detonation products will move away from the wave, relative to the wave, at a subsonic velocity.

The speed of the reaction products,  $u_p$ , is also a parameter we must consider. Sometimes, this problem is known as the piston problem since we can imagine a piston pushing the reaction products at a speed  $u_p$ . Once we have determined the detonation velocity we can then find  $u_p$ .

First, we shall examine a strong solution where

$$u_p > u_{\text{PCJ}} \quad (16.269)$$

In this case, any decrease in piston velocity will generate a rarefaction wave which will catch up to the detonation wave and the flow will equilibrate to the new velocity. If we have a situation where

$$u_p = u_{\text{PCJ}} \quad (16.270)$$

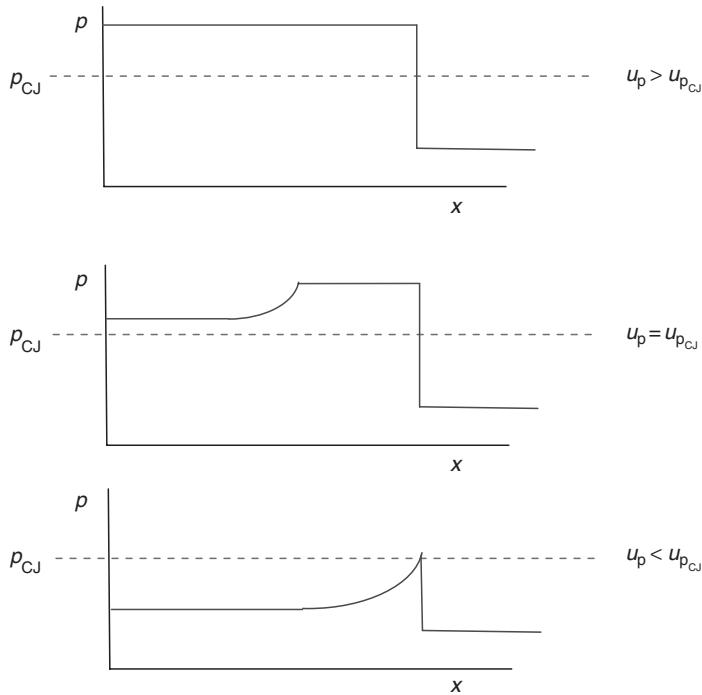
and there is a rarefaction generated, it cannot catch up to the front because it will move at the sonic velocity. If we have a situation where

$$u_p < u_{\text{PCJ}} \quad (16.271)$$

Then we need a rarefaction wave to reduce the flow velocity from the detonation wave speed at the front (which, recall, must move at a speed of at least  $D_{\text{CJ}}$ ) to the speed of the piston. This rarefaction wave will be time-dependent. If the piston was moving at zero velocity, then the tail of the rarefaction would stay attached to the detonation wave while the head of the rarefaction would remain about halfway between the detonation wave and the piston. This would be exactly halfway for a polytropic gas with  $p_0 = 0$ . In common problems, it will be typical to have the piston velocity less than or equal to zero. All of these conditions are illustrated in [Figure 16.42](#).

If we initiate a detonation at a point  $x = 0$  and  $t = 0$  and we have  $u_p < u_{\text{PCJ}}$ , then a  $t$ - $x$  plot of this situation would look like [Figure 16.43](#). The detonation front would move at velocity  $D_{\text{CJ}}$  and after a time  $t = t_1$  it would be at position  $x = D_{\text{CJ}}t_1$ . There would also be a centered rarefaction wave that, in the same time, would move to position  $x = u_p t_1$ . This centered rarefaction wave is sometimes called a Taylor wave. A particle path is also depicted in the figure.

An equation of state is required to close the set of equations and solve a reacting flow problem. There are some equations of state that do not treat the chemical reaction explicitly. When we have such a case, empirical values are obtained for the relationships. Thus, each new reaction must be calibrated through an experiment. We shall look at an equation of state that does treat the reaction. In this case, all that is needed is the composition of the reactants, the initial density, and the heats of formation.



**FIGURE 16.42**

Varying behavior of explosive reaction products—overdriven detonation waveHH. (From Fickett, W. and Davis, W.C., *Detonation: Theory and Experiment*, Dover Publications Inc., Mineola, NY, 1979. With permission.)

The Kistiakowsky–Wilson (K–W) equation of state is given by

$$\frac{pv}{RT} = 1 + xe^{bx} \quad (16.272)$$

Here

$$x = \frac{k}{v(T + \theta)^a} \quad (16.273)$$

where  $k$  is the effective mixture co-volume determined through

$$k = \kappa \sum_{i=1}^m \chi_i k_i \quad (16.274)$$

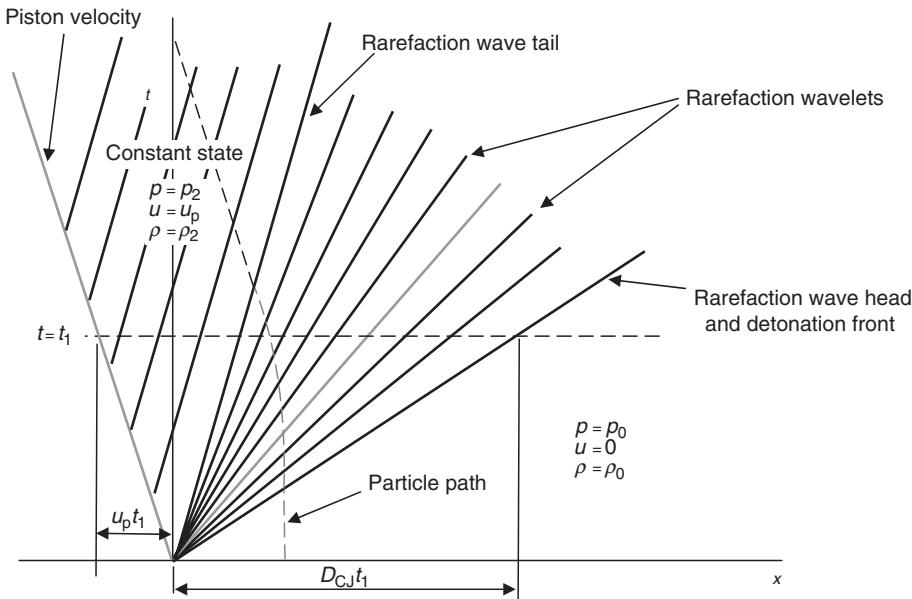
In these equations:

$a$ ,  $b$ ,  $\kappa$ ,  $\theta$ , and  $k_i$  are empirical constants.

$k_i$  is the co-volume of each species,  $i$ .

$\chi_i$  is the mole fraction of each species,  $i$ .

Unless better data is available, it is common to use  $a = 0.25$ ,  $b = 0.30$ ,  $\kappa = 1$ , and  $\theta = 0$ .



**FIGURE 16.43**

A  $t$ - $x$  diagram of reaction products—Overdriven detonation waveHH. (From Fickett, W. and Davis, W.C., *Detonation: Theory and Experiment*, Dover Publications Inc., Mineola, NY, 1979. With permission.)

Kamlet and Jacobs empirically fit data to come up with the following definitions at the CJ state:

$$p_{\text{CJ}} = \zeta \rho_0^2 \phi \quad (16.275)$$

$$D_{\text{CJ}} = A \sqrt{\phi} (1 + B \rho_0) \quad (16.276)$$

$$\phi = N \sqrt{MW_{\text{avg}} \Delta h_r^0} \quad (16.277)$$

In these equations:

$\zeta$ ,  $A$ , and  $B$  are empirical constants in SI units (m, kg, s).

$\zeta = 0.762$ .

$A = 22.3$ .

$B = 0.0013$ .

$N$  is the number of moles per unit mass in kg-mol/kg.

$MW_{\text{avg}}$  is the average molecular weight of the gaseous products in kg/kg-mol.

$\Delta h_r^0$  is the specific heat of reaction of the gaseous products in J/kg.

$D$  will be in m/s.

$p$  will be in Pa if  $\rho_0$  is in kg/m<sup>3</sup>.

If there are no solids in the reaction products then [4]

$$N = \frac{1}{MW_{\text{avg}}} \quad (16.278)$$

Equation 16.277 then reduces to

$$\phi = \sqrt{N\Delta h_r^0} \quad (16.279)$$

Equation 16.276 would then be

$$D_{\text{CJ}} = A(1 + B\rho_0) \left( \sqrt[4]{N\Delta h_r^0} \right) \quad (16.280)$$

Equation 16.275 would correspondingly be

$$p_{\text{CJ}} = \zeta \rho_0^2 \sqrt{N\Delta h_r^0} \quad (16.281)$$

With this in mind, we shall now discuss a procedure for the simplest theory that allows us to calculate the behavior of the reaction.

To estimate reaction product behavior, we must first develop the balanced chemical reaction. With this, we need to estimate the heat of detonation. Usually, we know the heat of formation of the unreacted explosive. We then calculate the heat of formation of the gas mixture through

$$\bar{\Delta h}_{\text{product gas}}^0 = \sum_i N_i \bar{\Delta h}_f^0 \quad (16.282)$$

At this point, we must guess at the ideal temperature of the explosive products. This guess is  $T_2^*$ . We next calculate the ideal ratio of specific heats through

$$\gamma = 1 + \frac{R}{C_v} \quad (16.283)$$

We also know that

$$R = \frac{R_u}{MW} \quad (16.284)$$

The universal gas constant is

$$R_u = 1.99 \left[ \frac{\text{cal}}{\text{g-mol} \cdot \text{K}} \right] \quad (16.285)$$

We can obtain the specific heat at constant volume through

$$C_v = A + BT \quad (16.286)$$

where the constants  $A$  and  $B$  are provided in Table 16.1. We can calculate the average specific heat of the products at our assumed temperature, then use this value in Equation 16.283.

If we use our notation for averages and estimated values, Equation 16.283 becomes

$$\gamma_2^* = 1 + \frac{NR_u}{\overline{C_v^*}} \quad (16.287)$$

**TABLE 16.1**

Coefficients for Specific Heat at Constant Volume Calculation

Molecule	Heat of Formation $\Delta h_f^0$ (cal/mol)	A	B	Co-volume ( $k$ ) (cm <sup>3</sup> /g-mol)
H <sub>2</sub>	0	5.02	0.28	153
CO <sub>2</sub>	94,450	10.30	0.42	687
CO	26,840	5.82	0.33	386
H <sub>2</sub> O(g)	57,801	7.13	0.67	108
N <sub>2</sub>	0	5.68	0.37	353
OH	5,930	5.20	0.26	108
O <sub>2</sub>	0	5.86	0.28	333
NO	-21,600	6.00	0.15	233
C(s)	0	4.52	0.20	0

$$C_v \text{ (cal/g-mol·K)} = A + B [T \text{ (K)}]$$

If we recall the energy equation which we will rewrite as

$$\Delta e = \overline{C}_v(T_2 - T_1) - q \quad (16.288)$$

where

$$q = \Delta h_r^0 \quad (16.289)$$

We can rearrange this to

$$T_2 = \frac{\Delta e}{\overline{C}_v} + \frac{q}{\overline{C}_v} + T_1 \quad (16.290)$$

The first term on the RHS is the kinetic energy, second is heat released. We know from the energy equation that

$$\Delta e = \frac{1}{2}(p_2 + p_1)(v_1 - v_2) \quad (16.291)$$

If we factor  $p_2$  and  $v_2$  out of Equation 16.291, we get

$$\Delta e = \frac{1}{2}p_2v_2 \left( 1 + \frac{p_1}{p_2} \right) \left( \frac{v_1}{v_2} - 1 \right) \quad (16.292)$$

If we state here that  $p_1 \ll p_2$ , we can write

$$\Delta e = \frac{1}{2}p_2v_2 \left( \frac{v_1}{v_2} - 1 \right) \quad (16.293)$$

Recall our definition of the specific heat ratio

$$\gamma \equiv \frac{C_p}{C_v} = \frac{\left( 1 - \frac{p_1}{p_2} \right)}{\left( \frac{v_1}{v_2} - 1 \right)} \quad (16.294)$$

Again if  $p_1 \ll p_2$ , we can write

$$\gamma \equiv \frac{C_p}{C_v} = \frac{1}{\left(\frac{v_1}{v_2} - 1\right)} \quad (16.295)$$

Substitution of Equation 16.295 into 16.293 yields

$$\Delta e = \frac{1}{2} \frac{p_2 v_2}{\gamma} \quad (16.296)$$

If we now use the ideal gas relation, we obtain

$$\Delta e = \frac{1}{2} \frac{N R_u T_2}{\gamma} \quad (16.297)$$

We can now write [Equation 16.290](#) as

$$T_2 = \frac{\frac{1}{2} N R_u T_2}{\gamma C_v} + \frac{q}{C_v} + T_1 \quad (16.298)$$

Now we can use Equation 16.297 to estimate  $T_2^*$

$$T_2^* = \frac{\frac{1}{2} N R_u T_2^*}{\gamma_2^* C_v^*} + \frac{q}{C_v^*} + T_1 \quad (16.299)$$

To use this equation, we substitute our guessed temperature into the RHS with our calculated  $\gamma_2^*$  and  $C_v^*$ . If the LHS comes out reasonably close to the RHS, we are done and our guess was correct. If it does not agree, we use the new value to calculate a new  $\gamma_2^*$  and  $C_v^*$ , and repeat the process until the solution converges.

To determine the detonation velocity, recall [Equation 16.244](#) which we can rearrange as

$$D = v_0 \sqrt{\frac{(p_1 - p_0)}{(v_0 - v_1)}} \quad (16.300)$$

We can factor this equation and use our definition of  $\gamma$  to make it look as follows:

$$D = v_0 \sqrt{\frac{p_1 \left(1 - \frac{p_0}{p_1}\right)}{v_1 \left(\frac{v_0}{v_1} - 1\right)}} = v_0 \sqrt{\frac{p_1 \gamma}{v_1}} \quad (16.301)$$

If we multiply and divide the inside by  $v_1^2$ , we obtain

$$D = \frac{v_0}{v_1} \sqrt{p_1 v_1 \gamma} \quad (16.302)$$

We can use [Equation 16.266](#) to alter  $v_0/v_1$  to yield

$$D = \frac{(\gamma + 1)}{\gamma} \sqrt{p_1 v_1 \gamma} \quad (16.303)$$

This can be rearranged as

$$D = (\gamma + 1) \sqrt{\frac{p_1 v_1}{\gamma}} \quad (16.304)$$

And inserting the ideal gas equation of state we obtain

$$D = (\gamma + 1) \sqrt{\frac{NRT}{\gamma}} \quad (16.305)$$

We can now calculate the ideal detonation velocity  $D^*$  through

$$D^* = (\gamma_2^* + 1) \sqrt{\frac{NR_u T_2^*}{\gamma_2^* (MW_{\text{explosive}})}} \quad (16.306)$$

Once we have these ideal values  $T_2^*$ ,  $\gamma_2^*$ , and  $D^*$ , we need to calculate the real values based upon the co-volume correction of [Equation 16.272](#). Using [Table 16.1](#), we determine a co-volume for the product gas mixture through

$$k = \sum_i N_i k_i \quad (16.307)$$

Now we find our correction factor  $x_1$  from [Equation 16.273](#) modified below

$$x_1 = \frac{k}{v_2(T_2^*)^a} \quad (16.308)$$

We can now use [Tables 16.2](#) through [16.5](#) with interpolation to obtain  $\frac{D}{D^*}$ ,  $\frac{T_2}{T_2^*}$ ,  $x_2$ , and  $\frac{\gamma_2}{\gamma_2^*}$ . These are the actual (nonideal) detonation wave velocity, temperature, and specific heat ratio. To determine the pressure, we now can use

$$p_2 = \rho_0 D^2 \left( 1 - \frac{x_1}{x_2} \right) \quad (16.309)$$

and to find the induced or material velocity we use

$$u_p = D \left( 1 - \frac{x_1}{x_2} \right) \quad (16.310)$$

While more realistic models exist for examining detonation, we will refer the interested reader to the references for further study.

**TABLE 16.2**

Specific Heat Ratio Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	$\gamma_2/\gamma_2^*$	$\Delta$	
0.1	0.991	-0.004	0.989	-0.006	0.987	-0.007	0.985	-0.008	0.984	-0.010	0.982	-0.011					
0.2	0.987	-0.003	0.983	-0.004	0.980	-0.004	0.977	-0.005	0.974	-0.006	0.971	-0.007					
0.3	0.984	-0.001	0.979	-0.001	0.976	-0.002	0.972	-0.003	0.968	-0.004	0.964	-0.004					
0.4	0.983	-0.001	0.978	-0.001	0.974	-0.002	0.969	-0.002	0.964	-0.002	0.960	-0.003					
0.5	0.982	0.000	0.977	0.000	0.972	-0.001	0.967	-0.001	0.962	-0.001	0.957	-0.001					
0.6	0.982	0.000	0.977	0.000	0.971	0.000	0.966	0.000	0.961	-0.001	0.956	-0.001					
0.7	0.982	0.000	0.977	0.000	0.971	0.000	0.966	-0.001	0.960	-0.001	0.955	-0.001					
0.8	0.982	0.001	0.977	0.000	0.971	0.000	0.965	0.000	0.959	0.000	0.954	-0.001					
0.9	0.983	0.000	0.977	0.001	0.971	0.001	0.965	0.001	0.959	0.000	0.953	0.000					
1.0	0.983	0.001	0.978	0.000	0.972	0.000	0.966	0.000	0.959	0.000	0.953	-0.001					
1.1	0.984	0.001	0.978	0.001	0.972	0.001	0.966	0.000	0.959	0.000	0.952	0.000					
1.2	0.985	0.001	0.979	0.001	0.973	0.000	0.966	0.000	0.959	0.000	0.952	0.000					
1.3	0.986	0.000	0.980	0.000	0.973	0.001	0.966	0.001	0.959	0.001	0.952	0.000					
1.4	0.986	0.001	0.980	0.001	0.974	0.000	0.967	0.000	0.960	0.000	0.952	0.000					
1.5	0.987	0.001	0.981	0.000	0.974	0.001	0.967	0.000	0.960	0.000	0.952	-0.001					
1.6	0.988	0.000	0.981	0.001	0.975	0.000	0.967	0.000	0.960	-0.001	0.951	0.000					
1.7	0.988	0.001	0.982	0.000	0.975	0.000	0.967	0.000	0.959	0.000	0.951	-0.001					
1.8	0.989	0.000	0.982	0.001	0.975	0.000	0.967	0.000	0.959	0.000	0.950	0.000					
1.9	0.989	0.001	0.983	0.000	0.975	0.000	0.967	0.000	0.959	-0.001	0.950	-0.001					
2.0	0.990	0.000	0.983	0.000	0.975	0.000	0.967	0.000	0.958	0.000	0.949	0.000					
2.1	0.990	0.000	0.983	0.000	0.975	0.000	0.967	0.000	0.958	-0.001	0.949	-0.001					
2.2	0.990	0.000	0.983	0.000	0.975	0.000	0.967	-0.001	0.957	-0.001	0.948	-0.001					
2.3	0.990	0.000	0.983	0.000	0.975	0.000	0.966	0.000	0.956	0.000	0.947	-0.001					
2.4	0.990	0.000	0.983	0.000	0.975	-0.001	0.966	-0.001	0.956	-0.001	0.946	-0.002					
2.5	0.990	0.000	0.983	0.000	0.974	0.000	0.965	-0.001	0.955	-0.002	0.944	-0.002					
2.6	0.990	0.000	0.983	-0.001	0.974	-0.001	0.964	-0.001	0.953	-0.001	0.942	-0.001					
2.7	0.990	0.000	0.982	0.000	0.973	-0.001	0.963	-0.001	0.952	-0.001	0.941	-0.002					
2.8	0.990	0.000	0.982	-0.001	0.972	-0.001	0.962	-0.001	0.951	-0.002	0.939	-0.002					
2.9	0.990	-0.001	0.981	0.000	0.971	-0.001	0.961	-0.002	0.949	-0.002	0.937	-0.002					
3.0	0.989	0.000	0.981	-0.001	0.970	-0.001	0.959	-0.001	0.947	-0.001	0.935	-0.002					
3.1	0.989	-0.001	0.980	-0.001	0.969	-0.001	0.958	-0.002	0.946	-0.002	0.933	-0.003					
3.2	0.988	0.000	0.979	-0.001	0.968	-0.002	0.956	-0.002	0.944	-0.002	0.930	-0.002					
3.3	0.988	-0.001	0.978	-0.001	0.966	-0.001	0.954	-0.002	0.942	-0.003	0.928	-0.002					
3.4	0.987	-0.001	0.977	-0.002	0.965	-0.002	0.952	-0.002	0.939	-0.002	0.926	-0.003					

**TABLE 16.3**

Temperature Ratio Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	
0.1	0.994	-0.005	0.992	-0.006	0.991	-0.008	0.989	-0.009	0.988	-0.011	0.986	-0.012					
0.2	0.989	-0.005	0.986	-0.006	0.983	-0.007	0.980	-0.008	0.977	-0.010	0.974	-0.011					
0.3	0.984	-0.004	0.980	-0.006	0.976	-0.007	0.972	-0.009	0.967	-0.009	0.963	-0.011					
0.4	0.980	-0.004	0.974	-0.005	0.969	-0.007	0.963	-0.008	0.958	-0.010	0.952	-0.011					
0.5	0.976	-0.005	0.969	-0.006	0.962	-0.007	0.955	-0.008	0.948	-0.009	0.941	-0.011					
0.6	0.971	-0.004	0.963	-0.006	0.955	-0.007	0.947	-0.009	0.939	-0.010	0.930	-0.011					
0.7	0.967	-0.005	0.957	-0.006	0.948	-0.008	0.938	-0.009	0.929	-0.011	0.919	-0.012					
0.8	0.962	-0.005	0.951	-0.006	0.940	-0.008	0.929	-0.009	0.918	-0.010	0.907	-0.012					
0.9	0.957	-0.005	0.945	-0.007	0.932	-0.008	0.920	-0.010	0.908	-0.011	0.895	-0.012					

(continued)

**TABLE 16.3 (continued)**

Temperature Ratio Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	$T_2/T_2^*$	$\Delta$	
1.0	0.952	-0.006	0.938	-0.007	0.924	-0.008	0.910	-0.010	0.897	-0.012	0.883	-0.013					
1.1	0.946	-0.006	0.931	-0.007	0.916	-0.009	0.900	-0.010	0.885	-0.012	0.870	-0.013					
1.2	0.940	-0.006	0.924	-0.008	0.907	-0.010	0.890	-0.011	0.873	-0.012	0.857	-0.014					
1.3	0.934	-0.006	0.916	-0.008	0.897	-0.010	0.879	-0.011	0.861	-0.013	0.843	-0.014					
1.4	0.928	-0.007	0.908	-0.009	0.887	-0.010	0.868	-0.012	0.848	-0.013	0.829	-0.015					
1.5	0.921	-0.007	0.899	-0.009	0.877	-0.010	0.856	-0.012	0.835	-0.014	0.814	-0.015					
1.6	0.914	-0.008	0.890	-0.009	0.867	-0.011	0.844	-0.013	0.821	-0.014	0.799	-0.015					
1.7	0.906	-0.008	0.881	-0.010	0.856	-0.012	0.831	-0.013	0.807	-0.014	0.784	-0.015					
1.8	0.898	-0.008	0.871	-0.010	0.844	-0.012	0.818	-0.013	0.793	-0.015	0.769	-0.016					
1.9	0.890	-0.009	0.861	-0.011	0.832	-0.012	0.805	-0.014	0.778	-0.015	0.753	-0.016					
2.0	0.881	-0.009	0.850	-0.011	0.820	-0.013	0.791	-0.014	0.763	-0.015	0.737	-0.017					
2.1	0.872	-0.009	0.839	-0.011	0.807	-0.013	0.777	-0.014	0.748	-0.016	0.720	-0.016					
2.2	0.863	-0.010	0.828	-0.012	0.794	-0.013	0.763	-0.015	0.732	-0.015	0.704	-0.017					
2.3	0.853	-0.010	0.816	-0.012	0.781	-0.014	0.748	-0.015	0.717	-0.016	0.687	-0.017					
2.4	0.843	-0.011	0.804	-0.012	0.767	-0.014	0.733	-0.015	0.701	-0.017	0.670	-0.017					
2.5	0.832	-0.011	0.792	-0.013	0.753	-0.014	0.718	-0.016	0.684	-0.016	0.653	-0.017					
2.6	0.821	-0.011	0.779	-0.013	0.739	-0.014	0.702	-0.015	0.668	-0.016	0.636	-0.017					
2.7	0.810	-0.011	0.766	-0.013	0.725	-0.015	0.687	-0.016	0.652	-0.017	0.619	-0.017					
2.8	0.799	-0.012	0.753	-0.014	0.710	-0.015	0.671	-0.016	0.635	-0.016	0.602	-0.017					
2.9	0.787	-0.012	0.739	-0.014	0.695	-0.015	0.655	-0.016	0.619	-0.017	0.585	-0.017					
3.0	0.775	-0.013	0.725	-0.014	0.680	-0.015	0.639	-0.016	0.602	-0.017	0.568	-0.017					
3.1	0.762	-0.012	0.711	-0.014	0.665	-0.015	0.623	-0.016	0.585	-0.016	0.551	-0.017					
3.2	0.750	-0.013	0.697	-0.015	0.650	-0.016	0.607	-0.016	0.569	-0.017	0.534	-0.016					
3.3	0.737	-0.014	0.682	-0.015	0.634	-0.016	0.591	-0.016	0.552	-0.016	0.518	-0.017					
3.4	0.723	-0.013	0.667	-0.015	0.618	-0.016	0.575	-0.016	0.536	-0.016	0.501	-0.016					

**TABLE 16.4**

Detonation Velocity Ratio Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$	$\Delta$	
0.1	1.094	0.090	1.093	0.089	1.090	0.089	1.090	0.087	1.089	0.086	1.088	0.085					
0.2	1.184	0.088	1.182	0.087	1.179	0.087	1.177	0.086	1.175	0.085	1.173	0.083					
0.3	1.272	0.088	1.269	0.087	1.266	0.086	1.263	0.084	1.260	0.083	1.256	0.082					
0.4	1.360	0.088	1.356	0.087	1.352	0.085	1.347	0.084	1.343	0.083	1.338	0.082					
0.5	1.448	0.088	1.443	0.087	1.437	0.085	1.431	0.084	1.426	0.082	1.420	0.081					
0.6	1.536	0.089	1.530	0.087	1.522	0.086	1.515	0.084	1.508	0.082	1.501	0.080					
0.7	1.625	0.089	1.617	0.087	1.608	0.086	1.599	0.084	1.590	0.082	1.581	0.081					
0.8	1.714	0.090	1.704	0.088	1.694	0.086	1.683	0.084	1.672	0.082	1.662	0.080					
0.9	1.804	0.090	1.792	0.088	1.780	0.086	1.767	0.084	1.754	0.082	1.742	0.079					
1.0	1.894	0.091	1.880	0.089	1.866	0.086	1.851	0.084	1.836	0.082	1.821	0.080					
1.1	1.985	0.092	1.969	0.089	1.952	0.087	1.935	0.085	1.918	0.082	1.901	0.079					
1.2	2.077	0.093	2.058	0.090	2.039	0.087	2.020	0.084	2.000	0.081	1.980	0.078					
1.3	2.170	0.094	2.148	0.091	2.126	0.088	2.104	0.084	2.081	0.081	2.058	0.079					
1.4	2.264	0.095	2.239	0.091	2.214	0.088	2.188	0.085	2.162	0.081	2.137	0.078					
1.5	2.359	0.095	2.330	0.092	2.302	0.087	2.273	0.084	2.243	0.081	2.215	0.077					
1.6	2.454	0.096	2.422	0.092	2.389	0.088	2.357	0.084	2.324	0.081	2.292	0.077					
1.7	2.550	0.097	2.514	0.092	2.477	0.088	2.441	0.084	2.405	0.080	2.369	0.076					
1.8	2.647	0.097	2.606	0.093	2.565	0.088	2.525	0.084	2.485	0.079	2.445	0.075					
1.9	2.744	0.098	2.699	0.093	2.653	0.088	2.609	0.083	2.564	0.079	2.520	0.074					

**TABLE 16.4 (continued)**

Detonation Velocity Ratio Table for Simple Formula Calculation

$\gamma_2^* = 1.15$		$\gamma_2^* = 1.19$		$\gamma_2^* = 1.23$		$\gamma_2^* = 1.27$		$\gamma_2^* = 1.31$		$\gamma_2^* = 1.35$		
$x_1$	$D_2/D_2^*$	$\Delta$	$D_2/D_2^*$									
2.0	2.842	0.099	2.792	0.093	2.741	0.088	2.692	0.082	2.643	0.078	2.594	0.074
2.1	2.941	0.099	2.885	0.093	2.829	0.088	2.774	0.082	2.721	0.077	2.668	0.073
2.2	3.040	0.100	2.978	0.093	2.917	0.087	2.856	0.082	2.798	0.076	2.741	0.071
2.3	3.140	0.100	3.071	0.093	3.004	0.087	2.938	0.081	2.874	0.076	2.812	0.071
2.4	3.240	0.100	3.164	0.093	3.091	0.086	3.019	0.080	2.950	0.074	2.883	0.069
2.5	3.340	0.100	3.257	0.093	3.177	0.086	3.099	0.080	3.024	0.074	2.952	0.069
2.6	3.440	0.101	3.350	0.092	3.263	0.085	3.179	0.078	3.098	0.072	3.021	0.067
2.7	3.541	0.100	3.442	0.092	3.348	0.084	3.257	0.077	3.170	0.072	3.088	0.066
2.8	3.641	0.100	3.534	0.092	3.432	0.083	3.334	0.076	3.242	0.070	3.154	0.064
2.9	3.741	0.100	3.626	0.091	3.515	0.082	3.410	0.076	3.312	0.069	3.218	0.063
3.0	3.841	0.100	3.717	0.090	3.597	0.082	3.486	0.075	3.381	0.067	3.281	0.061
3.1	3.941	0.100	3.807	0.089	3.679	0.081	3.561	0.071	3.448	0.066	3.342	0.060
3.2	4.041	0.099	3.896	0.089	3.760	0.080	3.632	0.072	3.514	0.065	3.402	0.059
3.3	4.140	0.099	3.985	0.088	3.840	0.078	3.704	0.071	3.579	0.064	3.461	0.057
3.4	4.239	0.098	4.073	0.086	3.918	0.077	3.775	0.069	3.643	0.062	3.518	0.056

**TABLE 16.5**

Correction Factor Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$
0.1	0.177	0.159	0.173	0.159	0.171	0.157	0.169	0.156	0.167	0.155	0.165	0.154	0.177	0.152	0.166	0.154	0.154
0.2	0.336	0.152	0.332	0.150	0.328	0.150	0.325	0.148	0.322	0.147	0.319	0.146	0.336	0.145	0.322	0.145	0.322
0.3	0.488	0.145	0.482	0.145	0.478	0.144	0.473	0.143	0.469	0.142	0.465	0.141	0.488	0.145	0.473	0.143	0.473
0.4	0.633	0.142	0.627	0.141	0.622	0.139	0.616	0.139	0.611	0.138	0.606	0.138	0.633	0.142	0.622	0.141	0.622
0.5	0.775	0.139	0.768	0.137	0.761	0.137	0.755	0.136	0.749	0.135	0.744	0.134	0.775	0.139	0.761	0.135	0.761
0.6	0.914	0.135	0.905	0.136	0.898	0.134	0.891	0.133	0.884	0.133	0.878	0.132	0.914	0.135	0.905	0.133	0.905
0.7	1.049	0.133	1.041	0.132	1.032	0.132	1.024	0.131	1.017	0.131	1.010	0.130	1.049	0.133	1.032	0.131	1.032
0.8	1.182	0.132	1.173	0.131	1.164	0.130	1.155	0.130	1.148	0.129	1.140	0.129	1.182	0.132	1.164	0.129	1.164
0.9	1.314	0.130	1.304	0.129	1.294	0.129	1.285	0.128	1.277	0.127	1.269	0.127	1.314	0.130	1.304	0.128	1.304
1.0	1.444	0.129	1.433	0.128	1.423	0.127	1.413	0.127	1.404	0.127	1.396	0.126	1.444	0.129	1.433	0.127	1.433
1.1	1.573	0.128	1.561	0.127	1.550	0.126	1.540	0.126	1.531	0.125	1.522	0.125	1.573	0.128	1.561	0.126	1.561
1.2	1.701	0.126	1.688	0.126	1.676	0.126	1.666	0.124	1.656	0.124	1.647	0.124	1.701	0.126	1.688	0.124	1.688
1.3	1.827	0.126	1.814	0.125	1.802	0.124	1.790	0.124	1.780	0.124	1.771	0.123	1.827	0.126	1.814	0.124	1.814
1.4	1.953	0.124	1.939	0.124	1.926	0.124	1.914	0.123	1.904	0.122	1.894	0.122	1.953	0.124	1.939	0.123	1.939
1.5	2.077	0.124	2.063	0.123	2.050	0.122	2.037	0.123	2.026	0.122	2.016	0.122	2.077	0.124	2.063	0.123	2.063
1.6	2.201	0.123	2.186	0.122	2.172	0.122	2.160	0.122	2.148	0.122	2.138	0.121	2.201	0.123	2.186	0.122	2.186
1.7	2.324	0.122	2.308	0.122	2.294	0.122	2.282	0.121	2.270	0.121	2.259	0.121	2.324	0.122	2.308	0.121	2.308
1.8	2.446	0.122	2.430	0.122	2.416	0.121	2.403	0.120	2.391	0.120	2.380	0.120	2.446	0.122	2.430	0.121	2.430
1.9	2.568	0.121	2.552	0.121	2.537	0.120	2.523	0.120	2.511	0.120	2.500	0.119	2.568	0.121	2.552	0.120	2.552
2.0	2.689	0.121	2.673	0.120	2.657	0.120	2.643	0.120	2.631	0.119	2.619	0.119	2.689	0.121	2.673	0.120	2.673
2.1	2.810	0.120	2.793	0.120	2.777	0.120	2.763	0.119	2.750	0.119	2.738	0.119	2.810	0.120	2.793	0.119	2.793
2.2	2.930	0.120	2.913	0.119	2.897	0.119	2.882	0.119	2.869	0.119	2.857	0.119	2.930	0.120	2.913	0.119	2.913
2.3	3.050	0.119	3.032	0.119	3.016	0.119	3.001	0.119	2.988	0.118	2.976	0.118	3.050	0.119	3.032	0.118	3.032
2.4	3.169	0.119	3.151	0.119	3.135	0.118	3.120	0.118	3.106	0.118	3.094	0.118	3.169	0.119	3.151	0.118	3.151
2.5	3.288	0.119	3.270	0.118	3.253	0.118	3.238	0.118	3.224	0.118	3.212	0.118	3.288	0.119	3.270	0.118	3.270
2.6	3.407	0.118	3.388	0.118	3.371	0.118	3.356	0.118	3.342	0.118	3.330	0.118	3.407	0.118	3.388	0.118	3.388
2.7	3.525	0.118	3.506	0.118	3.489	0.118	3.474	0.117	3.460	0.118	3.448	0.117	3.525	0.118	3.506	0.118	3.506
2.8	3.643	0.118	3.624	0.117	3.607	0.117	3.591	0.117	3.578	0.117	3.565	0.117	3.643	0.118	3.624	0.117	3.624

(continued)

**TABLE 16.5 (continued)**

Correction Factor Table for Simple Formula Calculation

$\gamma_2^* = 1.15$			$\gamma_2^* = 1.19$			$\gamma_2^* = 1.23$			$\gamma_2^* = 1.27$			$\gamma_2^* = 1.31$			$\gamma_2^* = 1.35$		
$x_1$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	$x_2$	$\Delta$	
2.9	3.761	0.117	3.741	0.117	3.724	0.117	3.708	0.117	3.695	0.117	3.682	0.117					
3.0	3.878	0.117	3.858	0.117	3.841	0.117	3.825	0.117	3.812	0.117	3.799	0.117					
3.1	3.995	0.117	3.975	0.117	3.958	0.117	3.942	0.117	3.929	0.117	3.916	0.117					
3.2	4.112	0.117	4.092	0.117	4.075	0.117	4.059	0.117	4.046	0.117	4.033	0.117					
3.3	4.229	0.117	4.209	0.117	4.192	0.116	4.176	0.116	4.163	0.116	4.150	0.116					
3.4	4.346	0.116	4.326	0.116	4.308	0.116	4.292	0.116	4.279	0.116	4.266	0.116					

**Problem 7**

Tetryl ( $C_7H_5N_5O_8$ ) is detonated in standard sea level air. Assuming nonideal behavior and  $\rho = 0.86 \left[ \frac{g}{cm^3} \right]$ ,  $MW_{mix} = 213 \left[ \frac{g}{g\text{-mol}} \right]$  and  $\Delta \bar{h}_f^0 = +4.67 \left[ \frac{kcal}{g\text{-mol}} \right]$

1. Determine the reaction equation assuming no dissociation
2. Determine the temperature of the products behind the detonation wave,  $T_2$   
*Answer:  $T_2 = 3308[K]$*
3. Determine the speed of the detonation wave,  $D$   
*Answer:  $D = 4742 \left[ \frac{m}{s} \right]$*
4. Determine the pressure behind the detonation wave,  $p_2$   
*Answer:  $p_2 = 5.29[GPa]$*
5. Determine the induced velocity of the gas behind the wave,  $u_2$   
*Answer:  $u_2 = 1296 \left[ \frac{m}{s} \right]$*

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# 17

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## *Introduction to Explosive Effects*

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Explosive effects are an important consideration when dealing with projectiles that are designed to deliver blast, fragments, or even deep penetrating effects such as a shaped charge jet. The earlier sections on penetration focused on the penetration events that occurred when a relatively solid projectile impacted the target. This impact resulted in either a non-penetration/partial penetration or a perforation. The latter effect was the sole cause of damage considered. Before the advent of the KE long-rod, even armor-piercing projectiles carried some explosive that would burst the projectile (hopefully) after passage through the armor of the target. This further damage mechanism would use fragmentation to destroy the soft targets protected by the armor.

Some projectiles are designed as strictly HE carriers. While these projectiles may have some armor-penetration capability, their primary job is to kill soft targets. A soft target is one that does not require a large amount of KE to kill or one that requires a large number of small perforations to destroy. Classically, soft targets are personnel, trucks, aircraft, radars, etc. While a single, well-placed KE projectile would kill these targets, their vulnerable areas are small; so to increase the probability of kill, a large number of slower moving or lower mass fragments are required.

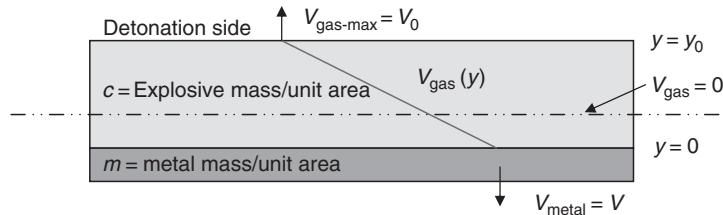
A further adaptation of focused explosive energy is the shaped charge which will be the subject of [Chapter 18](#). These devices can penetrate deep into armor and do not require any delivery KE to be effective. The explosive effects we shall discuss here will be used in Chapter 18 but further adapted for shaped charge jet analysis.

In this chapter, we shall first discuss how an explosive wave propagates to generate velocity in the metal casing that it is adjacent to. This will allow us to calculate the velocity and direction of fragment flight. After this, we shall discuss the penetration mechanisms (very similar to ogival-nosed projectiles and KE long-rods) of fragments.

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### 17.1 Gurney Method

The objective of the Gurney method is to obtain algebraic relationships for metal velocity when an explosive in contact with it is detonated. R.W. Gurney was a researcher who worked at the U.S. Army BRL in the 1940s and studied explosively driven metal plates during that time. The method is valid for both shaped charge analysis and fragmentation problems. The Gurney method assumes that all explosive chemical energy is converted into the KE of the fragments and expansion of the explosive products. We call the Gurney energy,  $E$ , the energy that is converted from chemical energy to KE and thus propels the metal and explosive products. This is in actuality only a portion of the energy generated during an explosion. We further shall assume that the gaseous detonation products expand uniformly with constant density.



**FIGURE 17.1**

Open-faced sandwich configuration with velocity gradient. (From Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989. With permission.)

The method is based on both a conservation of momentum and energy and results in answers which are usually within 10% of experimental results. The governing parameter in the Gurney method is the mass to charge ( $m/c$ ) ratio. The method works in its basic form for  $0.1 \leq m/c \leq 10.0$ . It is believed that the accuracy of this method comes about through offsetting errors [1]. The method ignores rarefaction waves in the explosive which would cause the calculated velocity to be too high, while at the same time the method assumes density is constant rather than being greatest at the surface of the charge. This latter assumption causes the calculated velocity to be too low. With these offsetting errors, the method is surprisingly accurate.

A slapper detonator or open-faced sandwich consists of explosive on one side and a metal plate on the other. This configuration is depicted in Figure 17.1. This configuration is used extensively in explosive characterization tests but has been used in ordnance as well. When the explosive is detonated, a velocity gradient is assumed to be set up as depicted in the figure. In Figure 17.1, the  $y$ -coordinate is associated with a layer of particles (a Lagrangian system) and thus can move. The velocities are interpreted as velocities after all the detonation product gases have expanded to several times their initial volume.

If we assume a constant density throughout the gas products, we can show that

$$\rho_{\text{gas}}y_0 = c \quad (17.1)$$

Here  $y_0$  is typically taken as the initial thickness of the explosive since, based on our assumptions, Equation 17.1 holds true for all time.

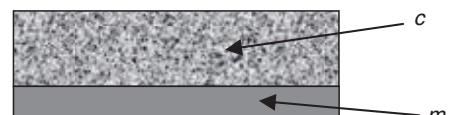
The velocity distribution for this configuration is given as

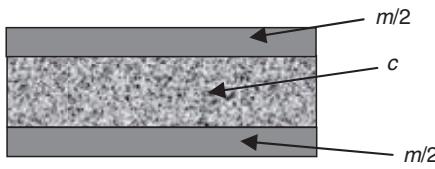
$$V_{\text{gas}} = (V_0 + V) \frac{y}{y_0} - V \quad (17.2)$$

Without going into the detailed derivation (the derivation can be found in Ref. [1], pp. 47–49), we can write the final expression for an open-faced sandwich as depicted in Figure 17.2 as

$$V = \sqrt{2E} \left\{ \frac{1}{3} \left[ \left( \frac{2m}{c} \right)^2 + \frac{5m}{c} + 1 \right] \right\}^{-\frac{1}{2}} \quad (17.3)$$

**FIGURE 17.2**  
Open-faced sandwich.





**FIGURE 17.3**  
Flat sandwich.

The velocities for the metal fragments in the flat sandwich, cylinder, and tamper configurations can also be derived [1] as follows. For the flat sandwich as depicted in Figure 17.3, we have

$$V = \sqrt{2E} \left( \frac{m}{c} + \frac{1}{3} \right)^{-\frac{1}{2}} \quad (17.4)$$

Many configurations in common use for military applications require a cylindrical configuration where a tube of metal is filled with explosive material. This is also a common configuration for use in shaped charge jet analysis. For a cylindrical geometry as depicted in Figure 17.4, we can write

$$V = \sqrt{2E} \left( \frac{m}{c} + \frac{1}{2} \right)^{-\frac{1}{2}} \quad (17.5)$$

In some instances, it is necessary that the metallic plates are not of the same mass. This is commonly referred to as the tamper configuration. The formula which expresses the metal velocities for this configuration is

$$V_n = \sqrt{2E} \left[ \frac{1+A^3}{3(1+A)} + \frac{n}{c} A^2 + \frac{m}{c} \right]^{-\frac{1}{2}} \quad (17.6)$$

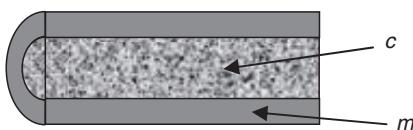
where

$$V_n = A V_m \quad (17.7)$$

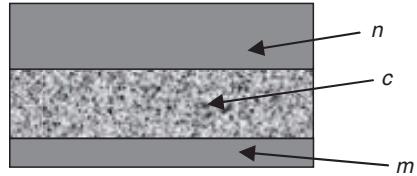
$$A = \frac{1+2\frac{m}{c}}{1+2\frac{n}{c}} \quad (17.8)$$

In these cases, the subscript "n" refers to the thicker tamper plate and the subscript "m" refers to the thinner driven plate. This is illustrated in Figure 17.5.

In some instances, it is informative to examine the behavior of a spherical geometry. The equation that describes the metal velocity for this configuration which is illustrated in Figure 17.6 is given as Equation 17.9. The derivation for this expression is found in Refs. [2,3].



**FIGURE 17.4**  
Cylindrical geometry.



**FIGURE 17.5**  
Tamper configuration.

$$V = \sqrt{2E} \left( \frac{m}{c} + \frac{3}{5} \right)^{-\frac{1}{2}} \quad (17.9)$$

The term  $\sqrt{2E}$  has units of velocity and is sometimes called the Gurney characteristic velocity, Gurney velocity, or the Gurney constant. If analyzing an explosive for which there is no Gurney velocity, an approach recommended by Kennedy (1970) [2] is to use  $E \sim 0.7H_D$ . Here  $H_D$  is the heat of detonation (the negative of the heat of formation of the explosive). For most explosives,  $0.61 < E/H_D < 0.76$ . As the  $m/c$  ratio approaches zero, the velocity of the fragments approaches a constant value. For a flat sandwich, open-faced sandwich, and asymmetric sandwich (tamper) this value is  $\sqrt{6E}$ . For a cylinder, this value is  $\sqrt{4E}$ . And for a sphere, this value is  $\sqrt{(10/3)E}$ .

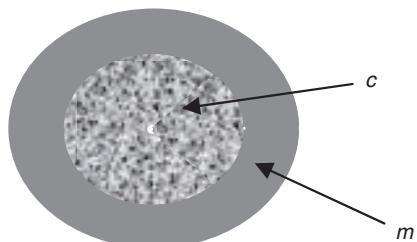
The Gurney method is fairly accurate, but of all the configurations it is least accurate for the open-faced sandwich configuration. In this case, the metal velocity would be predicted too high. Unfortunately, more complex methods are not always worth the increased accuracy.

## 17.2 Taylor Angles

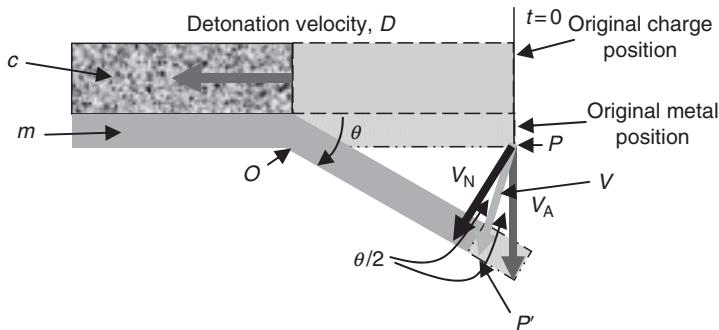
The previous section explained a means of determining the velocity to which a metal, initially in contact with the explosive, will be projected. This section focuses on the Taylor method that predicts the angle at which the metal will be thrown given a detonation event.

In the Gurney method, the equations assumed that the metal moves normal to its surface. If an explosive wave strikes the metal at some angle, this assumption is no longer valid and the metal will be projected at some angle. It is in these instances that we need to invoke the Taylor angle approximation. In this method, we assume that the metal is accelerated to its final velocity instantaneously. We also assume this is a pure rotation so no thickness change or change in length of the metal occurs.

Consider a detonation wave that is propagating from right to left as depicted in Figure 17.7. During this time, the explosive wave moves from the initial position to point  $O$ , the point initially at  $P$  moves to  $P'$ . If the detonation wave passes point  $P$  at time  $t=0$ , then we can show that



**FIGURE 17.6**  
Spherical geometry.



**FIGURE 17.7**

Taylor angle geometry. (From Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989. With permission.)

$$\overline{OP} = Dt \quad (17.10)$$

and

$$\overline{PP'} = Vt \quad (17.11)$$

Then it follows from geometric arguments that

$$\sin \frac{\theta}{2} = \frac{\overline{PP'}}{2\overline{OP}} = \frac{Vt}{2Dt} = \frac{V}{2D} \quad (17.12)$$

If we know  $D$  from the explosive properties and we can estimate  $V$  from the Gurney method, we can get an idea of what  $\theta$  will be. Experiments usually use smear cameras and measure  $V_A$  which relates to  $V$  through

$$V_A = D \tan \theta = \frac{V_N}{\cos \theta} \quad (17.13)$$

Usually  $V$ ,  $V_N$ , and  $V_A$  are within a few percent of one another. This allows us to use them somewhat interchangeably. Also, for most explosives,  $V/2D$  is approximately constant [1].

If we examine a typical HE shell and assume a detonation velocity  $D$  from the fuze, and given that we know the geometry, we can generate a reasonable estimate for the spray pattern of the fragments. We do this by dividing the shell into segments and solving for the Gurney velocities and Taylor angles in each segment. We can curve-fit the data. Spreadsheet programs are great for this task. However, there are specialized codes that perform this task for us as well.

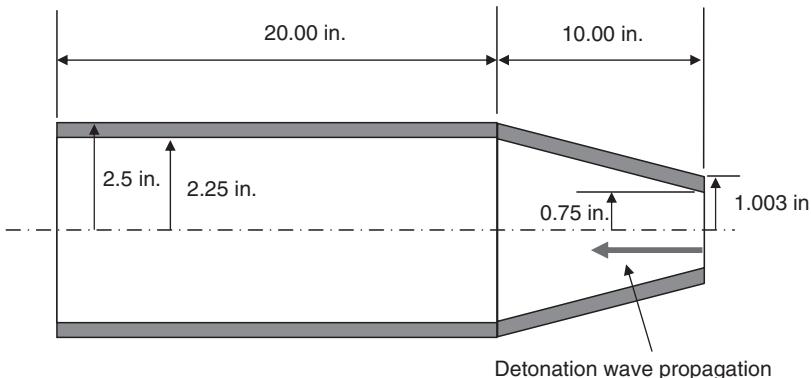
We shall illustrate the procedure with an example.

### Example Problem 1

A projectile is to be fabricated from steel and filled with TNT as depicted in Figure 17.8. For a detonation of the fill, graph the fragment velocities in m/s and Taylor angles in degrees versus distance from the nose of the projectile. The required properties for this calculation are given as follows:

TNT Gurney velocity  $(2E)^{1/2} = 2.039 \text{ km/s}$

TNT detonation velocity ( $D$ ) = 6730 m/s



**FIGURE 17.8**  
Projectile with an HE fill.

$$\begin{aligned} \text{TNT density} &= 1.63 \text{ g/cc} \\ \text{Steel density} &= 0.283 \text{ lbm/in.}^3 \end{aligned}$$

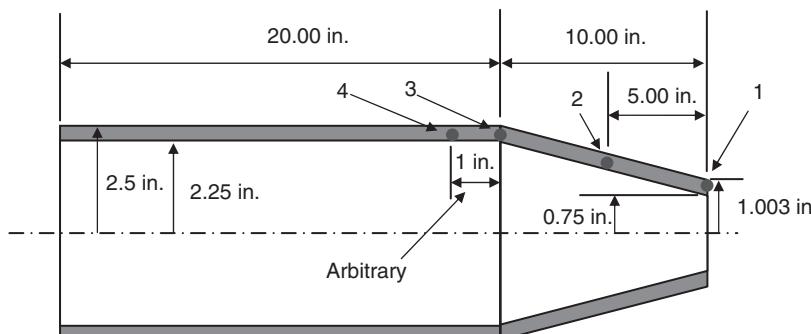
*Solution:* Let us get everything in consistent units. The density of TNT first.

$$\rho_{\text{TNT}} = (1.63) \left[ \frac{\text{g}}{\text{cm}^3} \right] (2.54)^3 \left[ \frac{\text{cm}^3}{\text{in.}^3} \right] \frac{(2.046)}{(1000)} \left[ \frac{\text{lbm}}{\text{g}} \right] = 0.059 \left[ \frac{\text{lbm}}{\text{in.}^3} \right] \quad (17.14)$$

The next step is to get the sectional densities calculated for the fill and the case. We only need to use four stations as depicted in Figure 17.9 because in the areas of constant cross section, we only need one data point but the data will be slightly different at the transition from the cone. We shall only list the calculations for the first location and depict the results in a table using the same procedure.

For cross section 1, we have

$$M_1 = \rho_{\text{steel}}(A_{1_{\text{case}}}) = (0.283) \left[ \frac{\text{lbm}}{\text{in.}^3} \right] \pi(1.003^2 - 0.750^2)[\text{in.}^2] = 0.394 \left[ \frac{\text{lbm}}{\text{in.}} \right] \quad (17.15)$$



**FIGURE 17.9**  
Projectile with an HE fill discretized.

For the fill, we want the dimension normal to the surface, so we need to determine the angle of the surface as

$$\alpha = \tan^{-1} \left( \frac{2.25 - 0.75}{10} \right) \rightarrow \alpha = 8.531^\circ \quad (17.16)$$

$$C_1 = \rho_{\text{TNT}}(A_{1\text{fill}}) = (0.059) \left[ \frac{\text{lbm}}{\text{in.}^3} \right] \pi \left[ \frac{(0.750^2)}{\cos^2(8.531^\circ)} \right] [\text{in.}^2] = 0.107 \left[ \frac{\text{lbm}}{\text{in.}} \right] \quad (17.17)$$

Now the fragment velocity follows directly from

$$V = \sqrt{2E} \left( \frac{M}{C} + \frac{1}{2} \right)^{-\frac{1}{2}} \quad (17.18)$$

$$V_1 = (2.039) \left[ \frac{\text{km}}{\text{s}} \right] (1000) \left[ \frac{\text{m}}{\text{km}} \right] \left( \frac{0.394}{0.107} + \frac{1}{2} \right)^{-\frac{1}{2}} = 997 \left[ \frac{\text{m}}{\text{s}} \right] \quad (17.19)$$

For the Taylor angle, we first need to find the angle  $\theta/2$  from our formula

$$\sin \frac{\theta}{2} = \frac{V}{2D} = \frac{(997) \left[ \frac{\text{m}}{\text{s}} \right]}{2(6730) \left[ \frac{\text{m}}{\text{s}} \right]} = 0.074 \rightarrow \frac{\theta}{2} = 4.25^\circ \quad (17.20)$$

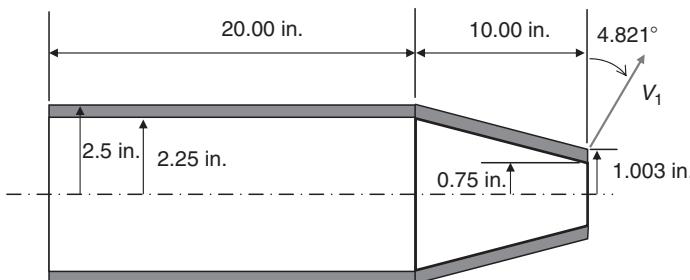
This Taylor angle would tend to tilt the fragment at  $4.25^\circ$  in the direction of the detonation wave (toward the base) but at this point, our nose is canted  $8.531^\circ$  toward the projectile axis; so the actual angle is  $4.25^\circ - 8.531^\circ$  or  $-4.281^\circ$  (see Figure 17.10).

If we take all of our data and put it in a table we get [Table 17.1](#). [Figure 17.11](#) shows the graph of these data.

A similar plot could be drawn for the Taylor angle tabulated in Table 17.1. It must be noted that the slight velocity increase at the ogive/bourrelet transition (10 in. from the nose) is an artifact of the way the projectile was discretized. We would normally assume that there is a smooth tangency point at that location.

### Problem 1

A Bangalore torpedo was a device built by the United States during the Second World War to clear beach (or any other) obstacles. It consisted of a long tube filled with explosive



**FIGURE 17.10**  
Taylor angle at the projectile nose tilted to account for ogive angle.

**TABLE 17.1**

Gurney Velocities and Taylor Angles for Projectile Fragments

Position	Axial Location (in.)	$M = rV/L$	$C = rV/L$	Fragment Velocity (m/s)	$\theta/2$ (deg)	Taylor Angle ( $\alpha$ ) (deg)
1	0.000	0.394	0.107	995	4.240	-4.291
2	5.000	0.732	0.426	1370	5.841	-2.690
3	10.000	1.056	0.959	1612	6.877	-1.653
4	11.000	1.056	0.938	1599	6.825	6.825
5	20.000	1.056	0.938	1599	6.825	6.825

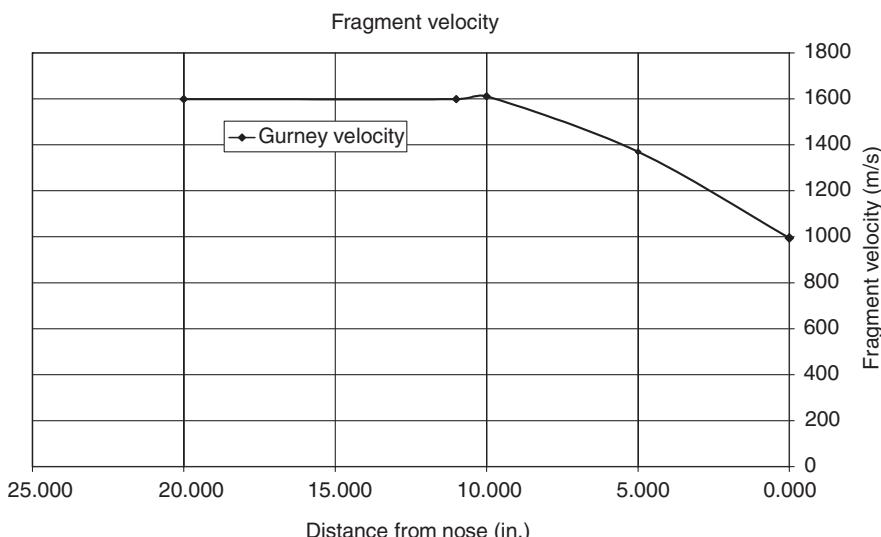
that was detonated on the end. Assume that we have a similar device made of steel and filled with Composition B. The device is 3 ft long. The ID is constant at 2 in. The OD varies with length. The first foot of length is 2-1/4 in. in diameter, the next foot of length is 2-3/4 in. in diameter and the last foot of length is 3 in. in diameter. Assuming that we detonate the device at the 2-1/4 in. end:

1. Draw a graph of the fragment velocities versus length in ft and ft/s.
2. Draw a graph of the Taylor angles in ft and degrees from the device axis.

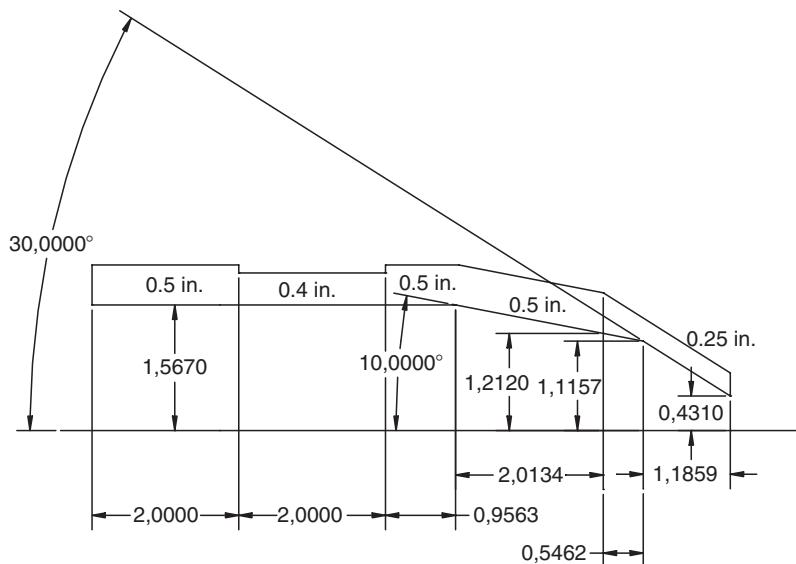
Assume that the tube is steel with a density of 0.283 lbm/in.<sup>3</sup> Assume that the filler density is 1.70 g/cc. Assume that the detonation velocity is 7.89 mm/μs and the Gurney constant is 2.7 mm/μs.

### Problem 2

Assume that we used the Paris gun so often that it finally blew up. We want to determine the velocity of the fragments and their Taylor angles. Assume the section where the explosion took place is centered over a jacket transition. Therefore, the analysis consists

**FIGURE 17.11**

Gurney velocity versus distance from projectile nose.



**FIGURE 17.12**  
Projectile geometry for Problem 3.

of two sections each 4 ft long. The ID of the weapon is 210 mm. The OD of the forward section is constant at 350 mm. The OD of the jacketed section is also constant at 420 mm. Assume the explosion begins at the projectile and propagates rearward. Assume that the Gurney constant for the filler/propellant combination is 1.8 km/s.

1. Draw a graph of the fragment velocities versus length in ft and ft/s.
2. Draw a graph of the Taylor angles in ft and degrees from the bore axis.

Assume that the tube is steel with a density of 0.283 lbm/in.<sup>3</sup> Assume that the filler/propellant density averages to about 0.6 g/cc. Assume that the detonation velocity is 16,500 ft/s.

### Problem 3

A projectile is to be fabricated from steel and filled with TNT as depicted in Figure 17.12. For a detonation of the fill, graph the fragment velocities in m/s and Taylor angles in degrees versus distance from the nose of the projectile. The required properties for this calculation are given as follows:

$$\text{TNT Gurney velocity } (2E)^{1/2} = 2.039 \text{ km/s}$$

$$\text{TNT detonation velocity } (D) = 6730 \text{ m/s}$$

$$\text{TNT density} = 1.63 \text{ g/cc}$$

$$\text{Steel density} = 0.283 \text{ lbm/in.}^3$$

### 17.3 Mott Formula

The preceding sections outlined the procedure to determine the velocity and directions that fragments of an exploded projectile will fly when the fuze is initiated. In this section, we

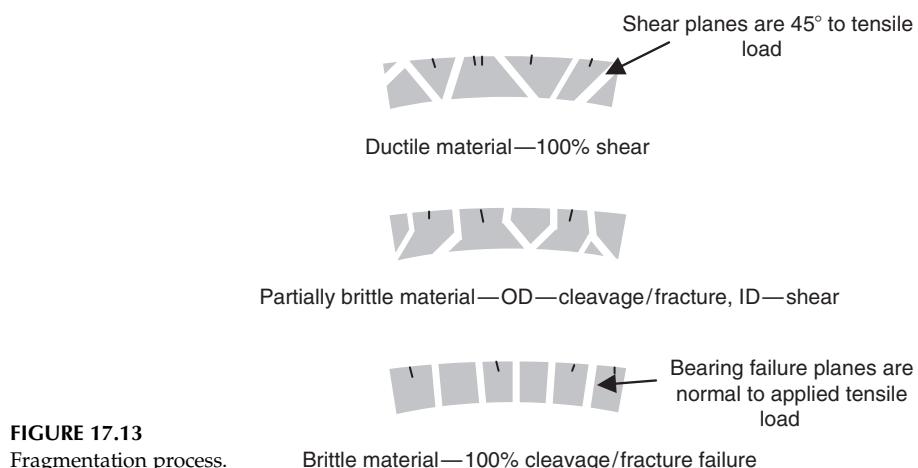
will examine the Mott formula, a method by which we can estimate the mass of the fragments. We begin by describing the fragmentation process itself.

When we detonate an HE fill in a metallic cylinder (projectile), several things occur. First, a detonation wave propagates along the axis of detonation. This results in pressure being generated with the attendant stress being transferred to the metallic casing. At this point, the case expands and ruptures by shear or brittle failure. If the case expands significantly and removes significant energy from the detonation products, we have a condition known as a terminal detonation. If the case expands very little before fragmenting, the result is known as a prompt detonation. Once the case ruptures, fragments fly in directions dependent upon the Taylor angle and their individual geometries. At some point, the fragments may impact a target. The processes of detonation, acceleration, and flight have been dealt with in detail in our prior work (both in the previous sections as well as the exterior ballistics section). Here we shall concentrate on the fragmentation process and penetration of the fragments themselves.

There are several factors that affect the fragmentation process: explosive brisance (see glossary), charge to mass ratio, casing diameter, casing wall thickness, and mechanical properties of the casing. The fragmentation of the casing usually begins at the outside diameter through formation of sharp radial cracks. These cracks then join with shear cracks from the inside of the material (or not, if the material is extremely brittle). The cracks then coalesce into long, longitudinal cracks. If the casing material is ductile enough, as the case expands radially and during this process, the wall will thin out somewhat. Finally, the casing will fragment completely. This is depicted in Figure 17.13.

Some general rules for case fragmentation based on material properties are presented here. In general, a more brittle material such as gray cast iron will produce a very large number of small fragments. This is desirable when lethal effects are to be localized to the projectile area. A precision delivery would be required to use this property most effectively. A more ductile material will generally produce a smaller number of large fragments. These fragments will be more lethal at longer ranges. This has the advantage of being able to account for some inaccuracy in projectile delivery. It is generally accepted that changes in the material microstructure affect this phenomenon.

The fragmentation process directly relates to the effectiveness of the weapon system. More fragments means a greater probability of a fragment hit,  $P_h$ . Larger fragment size translates to a greater probability of a kill, given a hit,  $P_{hk}$ . This trade-off must be made through an effectiveness analysis. In other words, if the target we are looking to kill is



**FIGURE 17.13**

Fragmentation process.

susceptible to even small fragment impacts, then we are better off with smaller fragment sizes as that will maximize our probability of killing more targets. If, however, we can only kill the target of interest with a large fragment, we must take the degradation in the hit probability. Mathematically, we want to maximize the effectiveness through

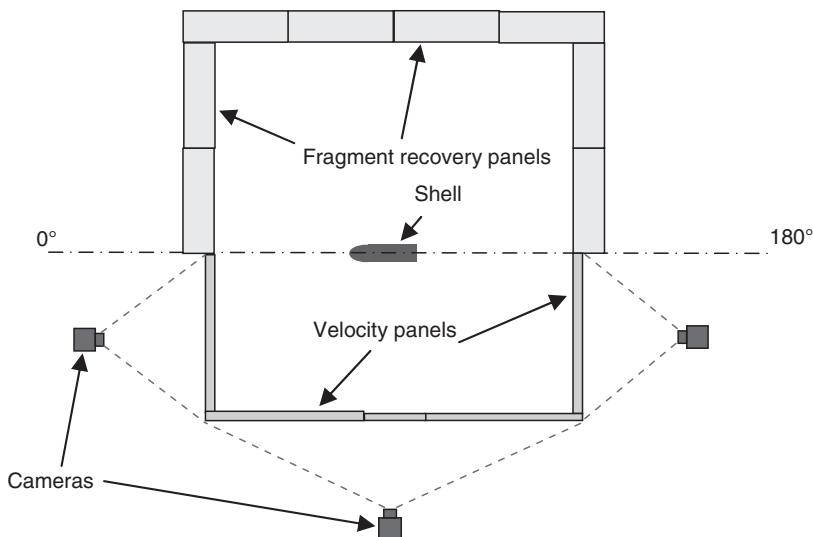
$$E_{hk} = E_h P_{hk} \quad (17.21)$$

Here  $E_{hk}$  is our expected number of impacts that kill a given target and  $E_h$  is the expected number of fragments that impact the target. So what we have learned here is that more, small fragments means greater  $E_h$  and lower  $P_{hk}$ , while fewer, larger fragments means smaller  $E_h$  and larger  $P_{hk}$ . If we would like to quantify the total probability of a kill,  $P_k$ , on a given target, we can write

$$P_k = 1 - e^{-E_{hk}} \quad (17.22)$$

There are several ways the fragmentation process can be controlled: explosive selection, case material selection, heat treatment of the casing, prestressing, preforming, or explosive wave shaping. One of the important things to remember is that the projectile body design has to survive rough handling and gun launch. Sometimes, this is at odds with the desired fragmentation effect and trades must be made. For a given target as well as any collateral damage effects, control of the fragmentation process translates to control of the following: fragment velocity, number of fragments, mass of the fragments, shape of the fragments, and distribution of the fragments (i.e., the fragmentation pattern). We have already mentioned how some of these contradict one another.

We have discussed some simple analytical approaches to determine fragment velocities and patterns in previous sections. However, experimentally, an arena test is the best verification. An arena test is one in which we detonate the projectile of interest and surround it with evaluation panels. A typical arena test setup is depicted in Figure 17.14. Two types of panels are commonly used: velocity panels and fragment recovery panels.



**FIGURE 17.14**  
Typical arena test setup.

Velocity panels are thin aluminum sheets between which there are sometimes placed light sources. High-speed films taken during the fragmentation event reveal bright spots caused by perforation. Since the distance is well known, the average velocity can be calculated from the speed of the camera and time of arrival (appearance of the bright spot).

The recovery panels allow the velocity to be estimated from depths of penetration into the panels. In mild steel panels, the depth of penetration can be estimated through

$$P = cm_p^{\frac{1}{3}} \left( \frac{V_s}{1000} \right)^{\frac{4}{3}} \quad (17.23)$$

Here for mild steel,  $P$  = depth of penetration in inches,  $c = 0.112$ ,  $m_p$  = fragment weight in ounces, and  $V_s$  = striking velocity in ft/s.

For composition board (Celotex) panels, we can write

$$V_s = 1865 \frac{P^{\frac{1}{3}}}{m_p^{0.1}} \quad (17.24)$$

In Equation 17.24, we have  $P$  is the depth of penetration in inches,  $m_p$  is the fragment weight in grams, and  $V_s$  is the striking velocity in ft/s. In all cases, if the projectile which creates the fragment is moving at a high velocity, this must be vectorially added to the fragment velocity in the effectiveness analysis. Mathematically, this is given by

$$V_0^2 = V_{\text{projectile}}^2 + V_{\text{frag}}^2 \quad (17.25)$$

Here  $V_0$  is the resultant initial fragment velocity,  $V_{\text{frag}}$  is the fragment velocity resulting from the detonation, and  $V_{\text{projectile}}$  is the projectile velocity at the time of detonation.

As we have discussed in the section on exterior ballistics, an object that moves through air will lose velocity because of the mechanisms of drag. This effect is usually more pronounced on fragments because of their irregular and sometimes inconsistent shapes which present varying frontal areas to the air stream. To simplify matters somewhat, it is typical to use a drag model that assumes a constant drag coefficient for fragments. This model is given by

$$V_s = V_0 e^{-k_1 x} \quad (17.26)$$

Here we define the constant  $k_1$  as we have in the exterior ballistics section using

$$k_1 = \frac{\rho S}{2m} C_D \quad (17.27)$$

In these equations,  $V_s$  is velocity of the fragment at impact,  $V_0$  is the initial fragment velocity caused by the explosion (Gurney velocity),  $x$  is the distance from the point of detonation to the point of impact,  $S$  is the presented area of the fragment,  $C_D$  is the fragment drag coefficient,  $\rho$  is the density of the ambient air in the vicinity of the detonation, and  $m$  is the mass of the fragment.

Typical drag curves for fragments can be found in Ref. [4].

The mass of fragments is a critical piece of data in any effectiveness analysis. It is a daunting task to determine how a naturally fragmenting warhead breaks up. If a warhead contains preformed fragments, we can assume that the fragment size will be based on the

**TABLE 17.2**  
Mott Formula Coefficients for Typical Projectile Fills

Explosive	B(lbm <sup>1/2</sup> in. <sup>-7/16</sup> )
Composition B	0.0554
Cyclotol (75/25)	0.0493
Pentolite (50/50)	0.0620
TNT	0.0779
Composition A-3	0.0549
RDX/Wax (95/5)	0.0531
Tetrl	0.0681

preformed geometry. Mott [5] proposed the following semi-empirical equation for predicting the number of fragments in a naturally fragmenting warhead

$$N(m) = \frac{M_0}{2M_K^2} \exp\left(-\sqrt{\frac{m}{M_K}}\right) \quad (17.28)$$

Here  $N(m)$  is the number of fragments greater than mass  $m$ ,  $m$  is the mass of the fragment (lbm),  $M_0$  is the mass of the projectile (lbm), and  $M_K$  is a distribution factor defined in Equation 17.29 (lbm<sup>1/2</sup>).

$$M_K = B t^{16} d^8 \left(1 + \frac{t}{d}\right)^{\frac{5}{8}} \quad (17.29)$$

Here  $B$  is a constant specific for the particular explosive/metal combination,  $t$  is the wall thickness in inches, and  $d$  is the inside diameter of the projectile (in.). The Mott coefficient,  $B$ , for mild steel cylinders combined with particular explosives is given in Table 17.2 [2]. We also know that charge to mass ratio has an effect; this is implicit in the combination of  $B$ ,  $t$ , and  $d$ .

When an HE warhead explodes, fragments fly in all directions. As previously mentioned, these fragments seldom penetrate heavily armored targets—they are only effective against light armor or soft targets. Because of this, we usually examine fragment impacts against thin targets. Usually, this means the target is thinner than any characteristic dimension of the fragment. Simple shapes are usually considered for ease of analysis; the shapes are usually cubes and spheres. The penetration behavior of a fragment is typically characterized by its residual mass and velocity once it has perforated the target material.

The fragment momentum equation is given by [3]

$$m_0 V_s = m_{rp} V_{rp} + m_p V_{rm} + I \quad (17.30)$$

Here  $m_0$  and  $V_s$  are the mass and impact velocity of the fragment relative to the target, respectively;  $m_{rp}$  and  $V_{rp}$  are the residual mass and velocity of the mass center of the fragment pieces that perforate the target, respectively;  $m_p$  and  $V_{rm}$  are the residual mass and velocity of the mass center of the target pieces that have broken free of the target, respectively; and  $I$  is the impulse transmitted to the target owing to both the target

stopping pieces of the penetrator and the absorption of the shear energy by the target that is required to set the mass,  $m_p$  free.

The energy equation for a fragment impact is given by [3]

$$\frac{1}{2}m_0V_s^2 = \frac{1}{2}m_{rp}V_{rp}^2 + \frac{1}{2}m_pV_{rm}^2 + \frac{1}{2}(m_0 - m_{rp})V_0^2 + E_f + W_s \quad (17.31)$$

Here  $E_f$  is the energy associated with the plastic deformation of masses  $m_0$  and  $m_p$ . It is calculated as if mass  $m_p$  was not attached to the target.  $W_s$  is the work associated with the shearing mass  $m_p$ , while it is attached to the target. The third term on the RHS represents KE of the initial impact that remains with the target.

The residual velocity of a fragment after it perforates a soft target is important in estimating its lethality. Recht [3] has shown that an equation can be written for residual velocity of a fragment as

$$V_r = \frac{\sqrt{V_s^2 - V_x^2}}{1 + \frac{m_p}{m_{rp}}} \quad (17.32)$$

In this equation,  $V_x$  is a characteristic velocity which is normally replaced by  $V_{50}$ . After one calculates  $V_r$ , the impulse transmitted to the target can be calculated as a function of  $V_x$  through

$$\frac{I}{m_0V_s} = 1 - \left(\frac{m_{rp}}{m_0}\right) \sqrt{1 - \left(\frac{V_x}{V_s}\right)^2} \quad (17.33)$$

This impulse can be normalized to  $V_{50}$  to determine the optimum velocity of a fragment. For a thin plate, if the penetration velocity is close to  $V_{50}$ , the impulse transmitted to the plate is maximized. In most damage theories, more damage occurs to a component with more impulse applied. This means that if one would like to damage a component behind thin armor, for maximum effect, one would like a fragment that gets through the outer armor without a problem yet impacts the component near its  $V_{50}$ .

Much like long-rod penetrators, fragments tend to lose mass as the penetration event progresses. When a blunt fragment impacts a plate, material is eroded from the contact surface. This process occurs continually until the relative velocity between what remains of the fragment and the contact surface drops below the plastic wave velocity in the fragment material. Recht [3] developed the following equation for determination of fragment residual mass:

$$\frac{m_{re}}{m_0} = 1 + \frac{m_p}{2m_0} \ln \left[ \frac{1 + \frac{1}{Q}}{1 + \frac{\left(\frac{V_s}{U_c}\right)^2}{Q}} \right] \quad (17.34)$$

In this expression,

$m_p$  = plate plug mass (same as earlier)

$$Q = \frac{\sigma_e}{\rho_p U_c^2} \text{ (dimensionless parameter)}$$

$\sigma_e$  = dynamic yield strength of fragment material

$\rho_p$  = density of fragment

$U_c$  = plastic wave speed in the fragment material

With this material, we have completed the treatment of fragmentation. These formulas can be used with fair accuracy to predict fragment behavior from HE devices.

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## References

1. Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989.
  2. Cooper, P.W., *Explosives Engineering*, Wiley-VCH, New York, NY, 1996.
  3. Carleone, J. Ed., *Tactical Missile Warheads*, American Institute of Aeronautics and Astronautics, Washington DC, 1993.
  4. Hoerner, S.F., *Fluid Dynamic Drag*, Hoerner Fluid Dynamics Publishing, Vancouver, WA, 1965.
  5. Mott, N.F., "Fragmentation of shell cases," *Proceedings of the Royal Society*, Vol. A189, London, 1947, pp. 300–308.
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## Further Reading

Zukas, J.A. and Walters, W.P., Ed., *Explosive Effects and Applications*, Springer, New York, NY, 1997.



# 18

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## *Shaped Charges*

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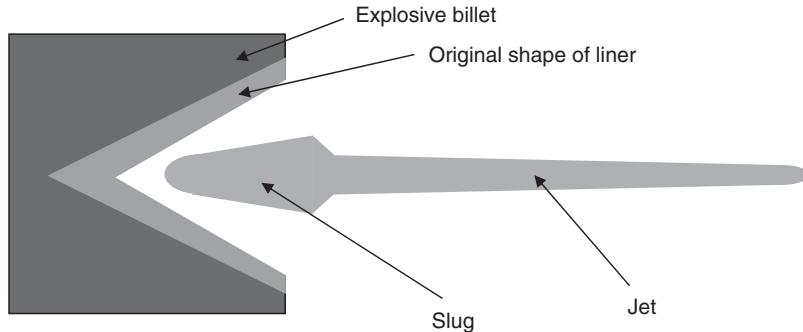
Although shaped charges can trace their origin to the early 1900s (and some authors suggest even further back), it was not until the Second World War that their use proliferated. Monroe in the United States and von Foerster and von Neumann in Europe discovered that a hollow charge, i.e., a block of explosive with a cavity on the target side, caused a deeper penetration than a similar charge that had no cavity. About the time of the Second World War, the combatants determined that if they lined this cavity with a metal and pulled the charge back from the surface, they achieved an even deeper penetration. The penetration depths achieved were on the order of several warhead diameters. These warheads were and still are so effective that they continue to be developed by nearly every nation. It is the goal of this section to describe their behavior and analysis.

Shaped charge warheads fall under the category of chemical energy (CE) warheads because they do not require any KE from the delivery system to be effective. This property makes them ideal for use in items such as shoulder fired weapons, grenades, mines, and even static cutting charges. The oil industry as well as the steel industry use them in large numbers to clear plugs or open up pores in rock to allow oil to flow into well shafts. These devices are also used to cut large masses of steel plate and bars.

The process through which a shaped charge works is as follows:

1. An explosion is generated which passes a detonation wave over the liner.
2. The liner collapses from the rear forward and is squeezed by the pressure of the expanding gases.
3. A jet of material forms, the tip of which moves at high velocity toward the target.
4. The remaining liner material is formed into a slug which follows the jet at a much lower velocity (approximately 1/10 the tip velocity).
5. The tip then penetrates the target material and the overall length of the jet is decreased until either the target is perforated or the entire jet is consumed.

This process generates high temperatures and pressures. As we have previously discussed, pressure much higher than the ultimate stress in the material allows us to model the material as an inviscid fluid. This has led to several common misconceptions. Shaped charges do not burn through the armor plate. This is believed to have been the misconstrual of the acronym HEAT which actually stands for High Explosive Anti-Tank. As we have stated earlier, high temperatures are generated during a penetration event, but it is the KE of the jet that does the work. Shaped charges do not turn the liner into a liquid. When pressures are orders of magnitude above the yield strength of the material (and they



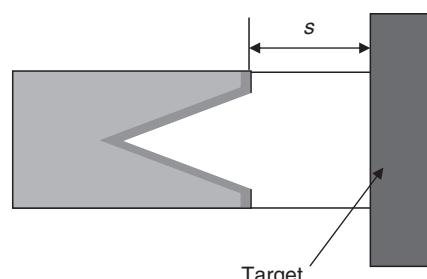
**FIGURE 18.1**  
Shaped charge jet formation.

are during a jet formation), we can treat the problem as a fluid dynamics problem even though the liner material really is not a fluid. If we could somehow magically stop the detonation process, we would have a solid rod of material. The formation of a typical shaped charge jet is shown as Figure 18.1.

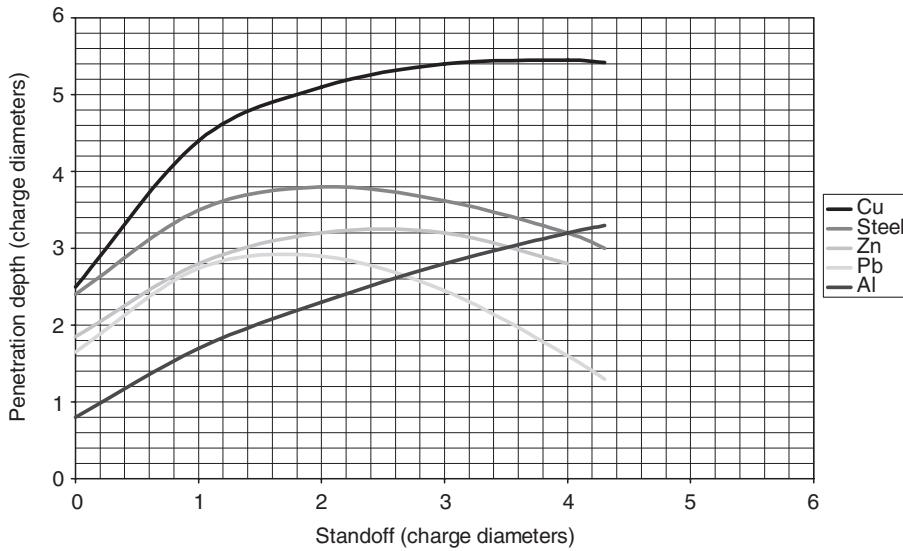
The standoff,  $s$ , of a shaped charge is the distance from the base of the liner or cavity to the target. This is illustrated in Figure 18.2. It is known that the standoff distance in shaped charges has an optimum value for armor penetration. This is depicted in Figure 18.3. The penetration performance is very sensitive to the standoff and performance decays rapidly if it is too large or too small. Explosive reactive armor (ERA) is an effective way to defeat a shaped charge by both breaking the jet up on impact, feeding additional material to erode the jet, and altering the standoff. Standoff plates (you can see these in many Second World War photographs of German vehicles) and sandbags defeat shaped charges by respectively affecting the standoff or forcing the jet to be consumed.

In addition to standoff, detonation symmetry is also very important. A slight asymmetric geometry of the liner or charge ignition will result in inefficient or improper formation. This is why most liners designed for military use are machined to precise tolerances. Charge to liner mass ( $C/M$ ) ratio greatly affects the velocity of the jet. If this ratio is too high, the liner can fragment and fail to penetrate. If this ratio is too low, the jet velocity will not be high enough for efficient penetration. Many authors use the inverse of this parameter as the ( $M/C$ ) ratio. The liner geometry has a pronounced effect on the jet formation because it affects how the explosive wave collapses the liner and forms the jet.

Liner material also has an effect on penetration. This is illustrated in Figure 18.3 for several different materials.



**FIGURE 18.2**  
Standoff,  $s$ , of a shaped charge.



**FIGURE 18.3**  
Effect of standoff on jet penetration using 45° conical liners.

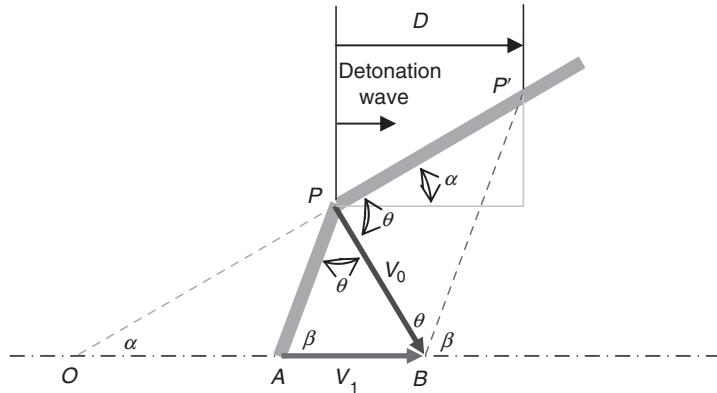
## 18.1 Shaped Charge Jet Formation

The previous section introduced some general terms commonly used in discussing shaped charges. In this section, we shall examine methods of predicting jet formation. Shaped charge jet penetration is critically dependent upon proper formation of the jet. The ability to predict this formation allows the designer to predict performance and even to optimize the design. Although computational techniques now allow great accuracy in predicting jet formation and penetration, it is always good practice to use a simplified analytical technique as a check of the computer models. While the analytic solution, with its associated idealizations, is not as accurate as the computational solution, it will be close enough to gain an appreciation of whether the code is outputting erroneous answers or not.

Birkhoff and others developed a theory in 1948 [1] that assumed the pressures generated by the explosive products are so great that the liner material strength could be neglected. Because of this, liners are typically modeled as inviscid, incompressible fluids. This was important because the modeling was greatly simplified. Birkhoff assumed that the liner particles were instantly accelerated to their final collapse velocity. It was further assumed that this velocity was constant throughout the formation. We know from experience that this is incorrect, as the tip of the jet moves faster than the tail or slug. This analysis method was later modified by Pugh in 1952 to include the velocity gradient. The model only became slightly more complicated but the accuracy improved.

The theory that was developed is now known as the Birkhoff–MacDougal–Pugh–Taylor theory. It is a fairly accurate, simple-to-use theory that allows for rapid estimates of jet and slug velocities. The theory assumes no velocity gradient in the jet and that the particles of the liner are instantly accelerated to their final velocity.

The theory models the liner collapse as follows. We shall use the nomenclature introduced by Walters [1] to describe this process which is illustrated in Figure 18.4. When we initiate an explosive behind a liner, after a time, the detonation wave will pass any point of interest as depicted in Figure 18.4. The liner is assumed to collapse inward at a velocity,  $V_0$ .



**FIGURE 18.4**

Illustration of liner collapse. (From Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989. With permission.)

We assume an instantaneous angle ( $2\beta$ ) between the moving walls of the liner which is greater than the initial angle ( $2\alpha$ ). We assume the detonation wave moves at a constant velocity,  $D$ . If we imagine ourselves in a Lagrangian reference frame attached to point  $P$  in Figure 18.4, the liner material can be assumed to move inward along  $PP'$  and out along  $PA$  with the pressure forces perpendicular to this motion. From the geometry in Figure 18.4, we can show that [1]

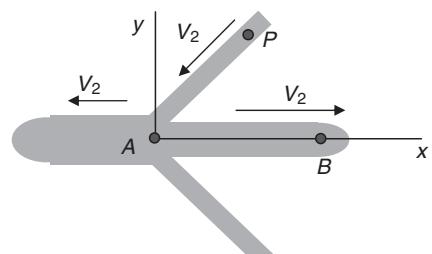
$$V_1 = \frac{V_0 \cos\left(\frac{\beta - \alpha}{2}\right)}{\sin \beta} \quad (18.1)$$

The trigonometry for this is fairly detailed and well developed in Ref. [1]. If an observer was moving with point  $A$  as depicted in Figure 18.5, he would see point  $P$  approaching at a velocity

$$V_2 = V_1 \cos \beta + V_0 \sin\left(\frac{\beta - \alpha}{2}\right) \quad (18.2)$$

We can solve for the detonation velocity,  $D$  through

$$\frac{D}{\cos \alpha} = \frac{V_0 \cos\left(\frac{\beta - \alpha}{2}\right)}{\sin(\beta - \alpha)} \quad (18.3)$$



**FIGURE 18.5**

Jet formation in the Lagrangian frame. (From Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989. With permission.)

If we were riding along in our coordinate system at point *A*, we would see both the slug and the jet moving away from us at velocity  $V_2$  and the liner moving toward us at the same velocity. As a reminder, we are assuming inviscid, incompressible flow in this case. If our coordinate system was stationary (Eulerian), however, we would see the jet velocity as

$$V_j = V_1 + V_2 \quad (18.4)$$

And the slug velocity as

$$V_s = V_1 - V_2 \quad (18.5)$$

The mass of the system must be conserved, therefore at any time,  $t$  we can write

$$m = m_j + m_s \quad (18.6)$$

Here  $m_j$  is the jet mass per unit length into the paper and  $m_s$  is the slug mass per unit length into the paper. Also  $m$  is the liner mass per unit length into the paper. If we now write the conservation of axial momentum, we obtain

$$mV_2 \cos \beta = m_s V_2 - m_j V_2 \quad (18.7)$$

We can solve Equations 18.6 and 18.7 simultaneously to write

$$m_j = \frac{1}{2}m(1 - \cos \beta) \quad (18.8)$$

$$m_s = \frac{1}{2}m(1 + \cos \beta) \quad (18.9)$$

It must be noted that this model assumes that the jet and slug velocities as well as their cross-sectional areas are constant. With all of these assumptions, we can write the velocities of the jet and slug, respectively, in terms of our known detonation velocity as

$$V_j = \frac{D}{\cos \alpha} \sin(\beta - \alpha) \left[ \operatorname{cosec} \beta + \cot \beta + \tan\left(\frac{\beta - \alpha}{2}\right) \right] \quad (18.10)$$

$$V_s = \frac{D}{\cos \alpha} \sin(\beta - \alpha) \left[ \operatorname{cosec} \beta - \cot \beta - \tan\left(\frac{\beta - \alpha}{2}\right) \right] \quad (18.11)$$

We can see from these equations that as  $\alpha \rightarrow 0$ , the jet velocity approaches a theoretical maximum.

$$V_{\max} = D \left[ 1 + \cos \beta - \sin \beta \tan\left(\frac{\beta}{2}\right) \right] \quad (18.12)$$

But  $\beta \rightarrow 0$  as  $\alpha \rightarrow 0$  so

$$V_{\max} = 2D \quad (18.13)$$

Thus, the maximum jet velocity can never exceed twice the detonation velocity of the explosive.

Another noteworthy observation is that as  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$ ,  $V_s \rightarrow 0$ . Also as  $\alpha \rightarrow 0$ , we approach a cylindrical geometry of the liner. Cylindrical liners are well known for their high velocity and low mass jets. If we could somehow generate an explosive wave that moved perpendicular to a conical liner, we would see that  $\beta = \alpha$  and the velocities of the jet and slug, respectively, could be expressed as

$$V_j = \frac{V_0}{\sin \alpha} (1 + \cos \alpha) \quad (18.14)$$

$$V_s = \frac{V_0}{\sin \alpha} (1 - \cos \alpha) \quad (18.15)$$

With this type of detonation wave, the jet velocity could be increased without bound by decreasing  $\alpha$ . However, we must note that as  $\alpha \rightarrow 0$ ,  $V_0 \rightarrow 0$  and  $m_j \rightarrow 0$ . Therefore, the momentum would also approach zero as shown in Equation 18.16.

$$m_j V_j = \frac{m V_0}{2} \sin \alpha \rightarrow 0 \quad (18.16)$$

To perform calculations either by hand or with the help of a spreadsheet, the following steps are provided:

- Determine the steady state jet and slug velocities from [Equations 18.10](#) and [18.11](#).
- Calculate the masses from [Equations 18.8](#) and [18.9](#).
- Determine the momentum or energy or other parameters of interest from the results.

This procedure tends to overpredict jet velocities somewhat. Also since no velocity gradient is present, jet stretching will not be predicted. Let us now look at an example of the procedure.

### **Example Problem 1**

A conical-shaped charge liner is to be fabricated from steel and filled with TNT as the explosive. The thickness of the liner is to be 0.1 in. and the half-angle,  $\alpha$  is to be  $45^\circ$ . The length of the liner is 5 in. and the charge OD is 12 in. Determine the following using the Birkhoff et al. theory:

1. Mass of the jet
2. Mass of the slug
3. Velocity of the jet
4. Velocity of the slug

The required properties for this calculation are given as follows:

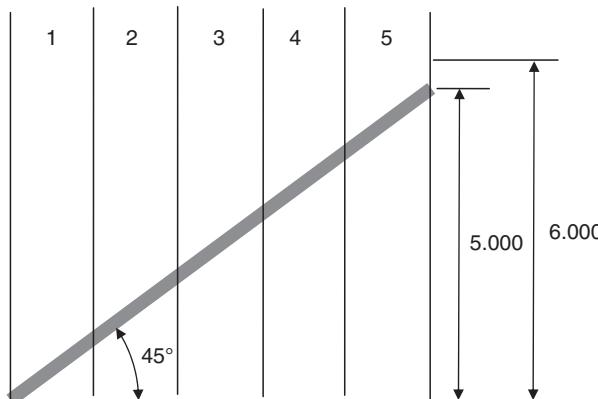
TNT Gurney velocity  $(2E)^{1/2} = 2.039$  km/s

TNT detonation velocity  $(D) = 6730$  m/s

TNT density = 1.63 g/cc

Steel density = 0.283 lbm/in.<sup>3</sup>

*Solution:* The first thing we need to do is get everything in consistent units. The density of TNT first.



**FIGURE 18.6**  
Discretization of a shaped charge liner.

$$\rho_{\text{TNT}} = (1.63) \left[ \frac{\text{g}}{\text{cm}^3} \right] (2.54)^3 \left[ \frac{\text{cm}^3}{\text{in.}^3} \right] \left( \frac{2.046}{1000} \right) \left[ \frac{\text{lbfm}}{\text{g}} \right] = 0.059 \left[ \frac{\text{lbfm}}{\text{in.}^3} \right]$$

Now we need to break the problem into sections and determine the Gurney velocity for each section. (For this case, we shall use five 1-in. long sections as shown in Figure 18.6.)

We need to determine, for each section, the liner mass to charge mass ratio to determine our velocity,  $V_0$ , for our later calculations. With our truncated cones, we will simply assume each section is a cylinder at the average radius of the section. Bill Walters\* suggests that to determine this ratio we use dimensions of the charge perpendicular to the liner. Then we can write the masses of the liner and charge as follows:

For cross-section 1, we have

$$M_1 = \rho_{\text{steel}} 2\pi \left( \frac{r_1 + r_0}{2} \right) t = (0.283) \left[ \frac{\text{lbfm}}{\text{in.}^3} \right] (2)\pi(0.1)[\text{in.}] \left( \frac{1+0}{2} \right) [\text{in.}] = 0.088 \left[ \frac{\text{lbfm}}{\text{in.}} \right] \quad (18.17)$$

$$\begin{aligned} C_1 &= \rho_{\text{TNT}} \pi \left[ \frac{r_c^2}{\cos^2 \alpha} - \left( \frac{r_1 + r_0}{2} \right)^2 \right] = (0.059) \left[ \frac{\text{lbfm}}{\text{in.}^3} \right] \pi \left[ \frac{(6)^2}{\cos^2(45)} - \left( \frac{1+0}{2} \right)^2 \right] [\text{in.}^2] \\ &= 13.299 \left[ \frac{\text{lbfm}}{\text{in.}} \right] \end{aligned} \quad (18.18)$$

Now the liner segment velocity follows directly from

$$V = \sqrt{2E} \left( \frac{M}{C} + \frac{1}{2} \right)^{-\frac{1}{2}} \quad (18.19)$$

$$V_{01} = (2.039) \left[ \frac{\text{km}}{\text{s}} \right] (1000) \left[ \frac{\text{m}}{\text{km}} \right] \left( \frac{0.088}{13.299} + \frac{1}{2} \right)^{-\frac{1}{2}} = 2864 \left[ \frac{\text{m}}{\text{s}} \right] \quad (18.20)$$

We can now use Equation 18.3 to find the angle,  $\beta$ .

$$\frac{D}{\cos \alpha} = \frac{V_0 \cos \left[ \frac{(\beta - \alpha)}{2} \right]}{\sin (\beta - \alpha)} \quad (18.21)$$

\* Personal correspondence with Bill Walters, 20 June 2002.

**TABLE 18.1**

Results of Computations for Jet and Slug Velocities

Position	$M = rV/L$	$C = rV/L$	Segment Velocity, $V_0$ (m/s)	$\beta$ (deg)	$V_j$ (m/s)	$V_s$ (m/s)
1	0.089	13.299	2864	62.310	5115.384	1280.944
2	0.267	12.928	2826	62.074	5063.423	1262.104
3	0.445	12.187	2784	61.818	5006.654	1241.659
4	0.622	11.075	2734	61.516	4939.138	1217.534
5	0.800	9.592	2669	61.122	4850.150	1186.366
Total	2.223	59.082	Average	61.768	4994.950	1237.721

It is convenient to solve this using iteration. Once we have these results, we can determine the jet mass and the slug mass using an average of the angles,  $\beta$ . As you can see from our overall results contained in Table 18.1, when using this method this angle does not vary too much. Our average  $\beta$  is  $61.768^\circ$  so we have

*Answer:*

$$1. m_j = \frac{1}{2} (2.223)[\text{lbfm}] [1 - \cos(61.768^\circ)] = 0.586[\text{lbfm}] \quad (18.22)$$

$$2. m_s = \frac{1}{2} (2.223)[\text{lbfm}] [1 + \cos(61.768^\circ)] = 1.637[\text{lbfm}] \quad (18.23)$$

The overall liner mass is the sum of all our individual masses tabulated in Table 18.1 (or it could be calculated directly from the geometry). It is 2.223 lbm.

The jet and slug velocities are obtained for a conical liner from Equations 18.10 and 18.11.

$$V_j = \frac{(6730) \left[ \frac{\text{m}}{\text{s}} \right]}{\cos(45^\circ)} \sin(\beta - 45^\circ) \left[ \operatorname{cosec} \beta + \cot \beta + \tan\left(\frac{\beta - 45^\circ}{2}\right) \right] \quad (18.24)$$

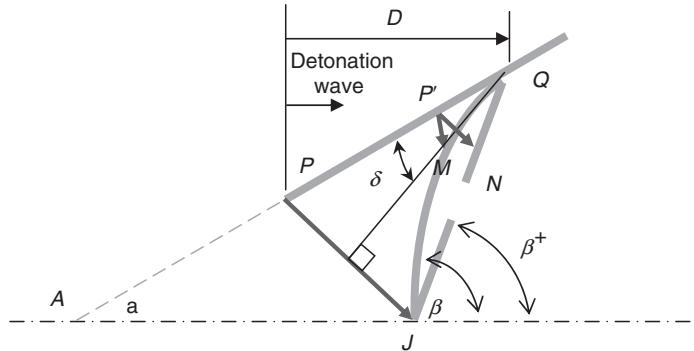
The answers are shown in Table 18.1. We could also have taken an average as well. For the slug velocity, we have

$$V_s = \frac{(6730) \left[ \frac{\text{m}}{\text{s}} \right]}{\cos(45^\circ)} \sin(\beta - 45^\circ) \left[ \operatorname{cosec} \beta - \cot \beta - \tan\left(\frac{\beta - 45^\circ}{2}\right) \right] \quad (18.25)$$

All of our data for this problem is summarized in Table 18.1.

We shall just briefly discuss the PER theory, details of which can be found in Ref. [1]. The PER theory was developed by Pugh, Eichelberger, and Rostoker at the U.S. Army BRL. The theory assumes a variable velocity during liner collapse which improves the correlation with experiment. Typically, as a liner collapses, the collapse velocity decreases as the detonation wave progresses from the apex of the cone to its base. This makes sense based on what we have learned so far since there is usually a smaller explosive mass compared to the liner mass. The end result is that the tip of the formed jet moves faster than the tail or slug, stretching the jet.

When the velocity of collapse decreases with time, the collapse angle,  $\beta$ , actually increases as does the amount of material entering the jet. This is illustrated in Figure 18.7. If we examine this figure, we see that as the detonation wave travels from point



**FIGURE 18.7**

Geometry of the PER theory. (From Walters, W.P. and Zukas, J.A., *Fundamentals of Shaped Charges*, CMC Press, Baltimore, MD, 1989. With permission.)

$P$  to  $Q$ , the element originally at  $P$  collapses to  $J$ . From the figure, we also see that the element at  $P'$  arrives at  $M$  at the same time that  $P$  reaches  $J$ . If the collapse velocity were constant, point  $P'$  would arrive at  $N$  instead and the collapsed shape would be conical as we have seen in [Figure 18.4](#).

Since the derivation of this theory is adequately addressed in Ref. [1], we will not derive the detailed mathematics behind it. The interested reader is referred to that work for the details.

The results based on Figure 18.7 yield an instantaneous velocity for the tip of the jet and the tail of the slug as given below:

$$V_j = V_0 \operatorname{cosec} \frac{\beta}{2} \cos \left[ \alpha - \frac{\beta}{2} + \sin^{-1} \left( \frac{V_0}{2u} \right) \right] \quad (18.26)$$

$$V_s = V_0 \sec \frac{\beta}{2} \sin \left[ \alpha - \frac{\beta}{2} + \sin^{-1} \left( \frac{V_0}{2u} \right) \right] \quad (18.27)$$

Here  $u$  is defined as

$$u = \frac{D}{\cos \alpha} \quad (18.28)$$

At any time, mass must be either in the liner, the slug, or the jet, so we can write

$$dm = dm_j + dm_s \quad (18.29)$$

where

$$\frac{dm_j}{dm} = \sin^2 \frac{\beta}{2} \quad (18.30)$$

$$\frac{dm_s}{dm} = \cos^2 \frac{\beta}{2} \quad (18.31)$$

We can now see that [Equations 18.17](#) through [18.22](#) depend upon the cone angle,  $2\alpha$ ; the detonation velocity,  $D$ ; the collapse angle,  $\beta$ ; and  $V_0$ .

Now we shall let  $t$  be the elapsed time between the instant the detonation wave passes the apex of the cone and define

$$T = \frac{x}{D} = \frac{x}{u \cos \alpha} \quad (18.32)$$

We can then express the position of any particle of the liner, initially at a distance,  $x$  from the apex in cylindrical coordinates as

$$Z = x + V_0(t - T) \sin A \quad (18.33)$$

$$r = x \tan \alpha - V_0(t - T) \cos A \quad (18.34)$$

where we define

$$A = \alpha + \delta \quad (18.35)$$

From this, the angle  $\beta$  can be shown to be

$$\tan \beta = \frac{\sin \alpha + 2 \sin \delta \cos \alpha - x \sin \alpha (1 - \tan A \tan \delta) \frac{V'_0}{V_0}}{\cos \alpha - 2 \sin \delta \sin A + x \sin \alpha (\tan A + \tan \delta) \frac{V'_0}{V_0}} \quad (18.36)$$

where

$$V'_0 = \frac{dV_0}{dx} \quad (18.37)$$

[Equations 18.17](#) through [18.27](#) are typically solved by computer to determine the formation parameters and describe the jet formation. It is beyond our scope to discuss the coding of the equations. Results of this model are shown in Ref. [1].

### Problem 1

A conical-shaped charge liner is to be fabricated from copper and filled with Composition B as the explosive. The thickness of the liner is to be 0.1 in. and the half-angle,  $\alpha$ , is to be  $45^\circ$ . The length of the liner is 3 in. and the charge OD is 7 in. Determine the following using the Birkhoff et al. theory:

1. Mass of the jet

*Answer:*  $m_j = 0.302[\text{lbm}]$

2. Mass of the slug

*Answer:*  $m_s = 0.990[\text{lbm}]$

3. Velocity of the jet

*Answer:*  $V_j = 4752 \left[ \frac{\text{m}}{\text{s}} \right]$

4. Velocity of the slug

*Answer:*  $V_s = 1086 \left[ \frac{\text{m}}{\text{s}} \right]$

Note that depending on how you discretize the problem you may get a somewhat (but not too) different answer.

The required properties for this calculation are given as follows:

Composition B Gurney velocity  $(2E)^{1/2} = 2.35 \text{ km/s}$

Composition B detonation velocity ( $D$ ) =  $7890 \text{ m/s}$

Composition B density =  $1.717 \text{ g/cc}$

Copper density =  $0.323 \text{ lbm/in.}^3$

## Problem 2

A conical-shaped charge liner is to be fabricated from copper and filled with Composition B as the explosive. The thickness of the liner is to be 0.15 in. and the half-angle,  $\alpha$ , is to be  $30^\circ$ . The length of the liner is 5 in. and the charge OD is 8 in. Determine the following using the Birkhoff et al. theory:

1. Mass of the jet

*Answer:*  $m_j = 0.410[\text{lbm}]$

2. Mass of the slug

*Answer:*  $m_s = 1.787[\text{lbm}]$

3. Velocity of the jet

*Answer:*  $V_j = 7500 \left[ \frac{\text{m}}{\text{s}} \right]$

4. Velocity of the slug

*Answer:*  $V_s = 960 \left[ \frac{\text{m}}{\text{s}} \right]$

5. Estimate the jet length assuming constant velocity of the tip and slug if the standoff is 1 m (use the fastest tip velocity and the average slug velocity)

*Answer:*  $L \approx 0.875[\text{m}]$

The required properties for this calculation are given as follows:

Composition B Gurney velocity  $(2E)^{1/2} = 2.79 \text{ km/s}$

Composition B detonation velocity ( $D$ ) =  $7910 \text{ m/s}$

Composition B density =  $1.717 \text{ g/cc}$

Copper density =  $0.323 \text{ lbm/in.}^3$

Steel density =  $0.283 \text{ lbm/in.}^3$

---

## 18.2 Shaped Charge Jet Penetration

Now that we have discussed how shaped charge jets are formed, we will move to how they penetrate their targets. As mentioned previously, shaped charge jets are formed at relatively close standoffs. The jet stretches from the instant it is formed with velocities ranging from 10 (tip) to 2 km/s (tail). Because of this stretching, the jet will eventually break up thereby reducing penetration because of drift/tumbling of the jet segments. This is known as particulation.

The penetration performance of shaped charge jets is dependent upon whether or not they are continuous. The further away from a target that the jet is formed, the more the jet

will stretch. If this standoff distance is large enough, the jet will particulate. This particulation complicates the penetration calculation.

The simplest penetration formula is attributed to Birkhoff [2] who assumed a constant velocity of the jet and thus described jet penetration through a momentum balance

$$\frac{1}{2}\rho_j(V - U)^2 = \frac{1}{2}\rho_t U^2 \quad (18.38)$$

In the above equation,  $\rho_j$  is the jet density,  $\rho_t$  is the target density,  $U$  is the velocity of the bottom of the hole in the target, and  $V$  is the (constant) velocity of the jet. By solving for  $U$  in the above equation and noting that the total penetration can be described as follows, we can obtain an expression for the depth of penetration

$$P(t) = \int_0^t U(t)dt \quad (18.39)$$

Here  $P(t)$  is the total penetration of the jet at time,  $t$ .

From the above integral, we obtain the formula from the penetration of a continuous velocity jet (called the density law)

$$P = l_j \left( \frac{\rho_j}{\rho_t} \right)^{\frac{1}{2}} \quad (18.40)$$

Here  $l_j$  is the length of the jet. Equation 18.40 states that, for a constant velocity jet, the penetration is only dependent upon the jet length and the density ratio. If the jet is segmented, Pack and Evans [2] proposed the following relation:

$$2\rho_j(V - U)^2 = \rho_t U^2 \quad (18.41)$$

which implies

$$P = l_j \left( \frac{2\rho_j}{\rho_t} \right)^{\frac{1}{2}} \quad (18.42)$$

In Equation 18.42,  $l_j$  is the length of the jet including the gaps between segments and  $\rho_j$  is the jet density calculated based on the length (including gaps) so that the overall density will be lower than a continuous jet. In this case,  $P$  ends up usually being lower. We must note that there are cases in which a particulated jet can actually penetrate deeper into the target material than a non-particulated jet [2].

As the jet velocity decreases, there is a point where the constitutive strength of the target material becomes important. There are formulas by Pack and Evans as well as by Eichelberger that account for this [2].

The expressions developed so far assume that the jet velocity is constant. If this assumption does not provide an accurate enough answer, we can use the formulas derived by DiPersio and Simon [2] to account for jet stretching. This technique uses three different formulas dependent upon where particulation occurs.

If the jet is continuous throughout the penetration event, we can write

$$P = s \left[ \left( \frac{V_0}{V_{\min}} \right)^{\frac{1}{\gamma}} - 1 \right] \quad (18.43)$$

If particulation occurs sometime during the penetration event

$$P = \left\{ \frac{\left[ (1 + \gamma)(V_0 t_1)^{\frac{1}{1+\gamma}} s^{\frac{\gamma}{1+\gamma}} - V_{\min} t_1 \right]}{\gamma} \right\} - s \quad (18.44)$$

If particulation occurs before penetration

$$P = \frac{(V_0 - V_{\min})t_1}{\gamma} \quad (18.45)$$

In Equations 18.43 through 18.45,  $V_0$  is the jet tip velocity,  $s$  is the distance from the target surface to the virtual origin of the jet (this is a theoretical origin derived from examination of a velocity-distance curve—to be explained later),  $t_1$  is the time from jet formation to particulation,  $V_{\min}$  is the minimum jet velocity capable of penetrating the target material, and  $\gamma$  is defined as

$$\gamma = \left( \frac{\rho_t}{\rho_j} \right)^{\frac{1}{2}} \quad (18.46)$$

$V_{\min}$  is a value that is usually between 2 and 8 km/s. There are various methods to calculate  $V_{\min}$  but we usually assume 2 km/s for the purposes of rough analysis. Some authors use  $U_{\min}$ —as we shall see later.

$V_{\min}$  (or  $U_{\min}$ ) for a metallic target can be calculated through [3]

$$V_{\min} \left[ \frac{\text{cm}}{\mu\text{s}} \right] = U_{\min} \left[ \frac{\text{cm}}{\mu\text{s}} \right] = 0.044 + (0.000206)(\text{BHN}) \quad (18.47)$$

Note that this expression uses the targets Brinnell hardness number (BHN) as a parameter affecting penetration.

Another form of the nonuniform-velocity jet equations similar to the DiPersio and Simon equations is as follows [3]. For very short standoffs, defined as

$$0 \leq s \leq (1 + \gamma)V_{\min}t_1 \left[ \frac{(1 + \gamma)V_{\min}}{V_0} \right]^{\frac{1}{\gamma}} \quad (18.48)$$

The depth of penetration can be found through

$$P = s \left\{ \left[ \frac{V_0}{(1 + \gamma)V_{\min}} \right]^{\frac{1}{\gamma}} - 1 \right\} \quad (18.49)$$

or

$$P = \left\{ \frac{\left[ (1 + \gamma)(V_0 t_1)^{\frac{1}{1+\gamma}} s^{\frac{\gamma}{1+\gamma}} - V_{\min} t_1 \right]}{\gamma} \right\} - s \quad (18.50)$$

For moderate standoffs, defined as

$$(1 + \gamma)V_{\min}t_1 \left[ \frac{(1 + \gamma)V_{\min}}{V_0} \right]^{\frac{1}{\gamma}} \leq s \leq V_0 t_1 \quad (18.51)$$

The depth of penetration is given by

$$P = \frac{(1 + \gamma)}{\gamma} (V_0 t_1)^{\frac{1}{1+\gamma}} s^{\frac{\gamma}{1+\gamma}} - \frac{1}{\gamma} \sqrt{(1 + \gamma)(V_{\min} t_1)(V_0 t_1)^{\frac{1}{1+\gamma}} s^{\frac{\gamma}{1+\gamma}}} - s \quad (18.52)$$

and for long standoffs where

$$V_0 t_1 \leq s \leq \frac{V_0 t_1}{\gamma} \left( \frac{V_0}{V_{\min}} - 1 \right) \quad (18.53)$$

The penetration can be found through

$$P = \frac{V_0 t_1}{\gamma^2} \sqrt{(V_{\min} t_1)(V_0 t_1 + \gamma s)} \quad (18.54)$$

Mott, Pack, and Hill [3] developed a theory that accounts for the material behavior of the jet. This theory is known as the MPH theory. In this theory, the density of a shaped charge jet is given by

$$\rho_j = \frac{m_j}{V_j} = \frac{m_j}{A_j l_j} \quad (18.55)$$

In Equation 18.55, we have used the density,  $\rho_j$ ; mass,  $m_j$ ; cross-sectional area,  $A_j$ ; length,  $l_j$ ; and volume,  $V_j$ , of the jet. The MPH theory states that the penetration depth is given by

$$P = l_j \sqrt{\frac{\lambda \rho_j}{\rho_t}} \quad (18.56)$$

Here the parameter  $\lambda$  is a factor which accounts for how the material behaves:

- For pure hydrodynamic behavior,  $\lambda = 1$
- For a particulated jet,  $\lambda = 2$
- For a jet which is hydrodynamic but particulates,  $1 < \lambda < 2$

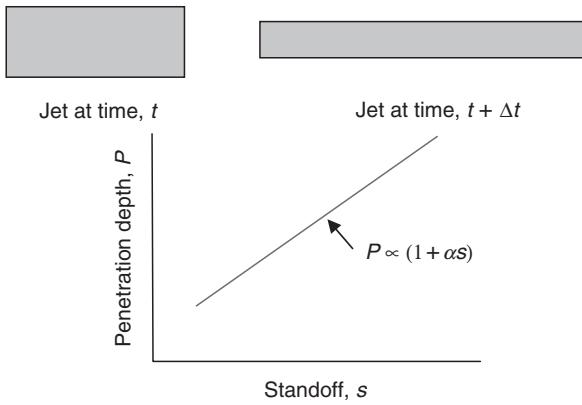
If we insert [Equation 18.46](#) into Equation 18.55, we obtain

$$P = \sqrt{\frac{\lambda m_j l_j}{\rho_t A_j}} \quad (18.57)$$

We can modify this formula for the effect of standoff distance by assuming the distribution of the jet mass is linear with its length or, mathematically

$$l_j = l_0(1 + \alpha s) \quad (18.58)$$

Here  $l_0$  and  $\alpha$  are constants and  $s$  is the standoff distance.



**FIGURE 18.8**  
Hydrodynamic jet behavior.

If we assume pure hydrodynamic behavior, then the volume of the jet is a constant and  $\lambda = 1$ . We know that

$$V_j = A_j l_j \quad (18.59)$$

and

$$P = \sqrt{\frac{m_j l_j}{\rho_t A_j}} = \sqrt{\frac{m_j l_j^2}{\rho_t V_j}} = l_j \sqrt{\frac{m_j}{\rho_t V_j}} \quad (18.60)$$

If we include the effects of standoff, we can write

$$P = l_0(1 + \alpha s) \sqrt{\frac{m_j}{\rho_t V_j}} \quad (18.61)$$

This states that  $P$  varies linearly with  $s$  or, mathematically

$$P \propto (1 + \alpha s) \quad (18.62)$$

This behavior is shown graphically in Figure 18.8.

If we assume the jet particulates, then the cross-sectional area of the jet is a constant and  $\lambda = 2$ . Then [Equation 18.57](#) can be written as

$$P = \sqrt{\frac{2m_j l_j}{\rho_t A_j}} \quad (18.63)$$

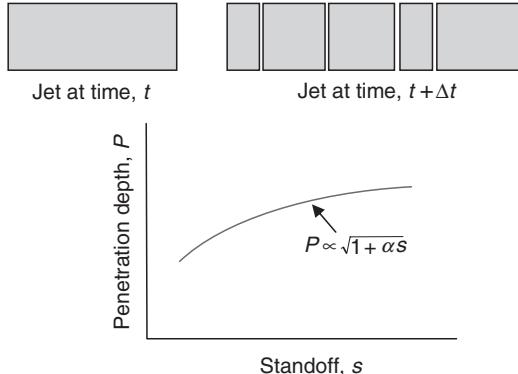
If we include the effects of standoff, we can write

$$P = \sqrt{\frac{2m_j l_0(1 + \alpha s)}{\rho_t A_j}} \quad (18.64)$$

This states that  $P$  varies with the square root of  $(1 + \alpha s)$  or, mathematically

$$P \propto \sqrt{1 + \alpha s} \quad (18.65)$$

This behavior is illustrated in [Figure 18.9](#).



**FIGURE 18.9**  
Particulating jet behavior.

If we assume the jet is somewhat hydrodynamic and also particulates, then both the volume and area of the jet are variables and  $1 < \lambda < 2$ . In this case, [Equation 18.57](#) applies directly. If we modify this expression to include the effects of standoff, we can write

$$P = \sqrt{\frac{\lambda m_j l_0 (1 + \alpha s)}{\rho_t A_j}} \quad (18.66)$$

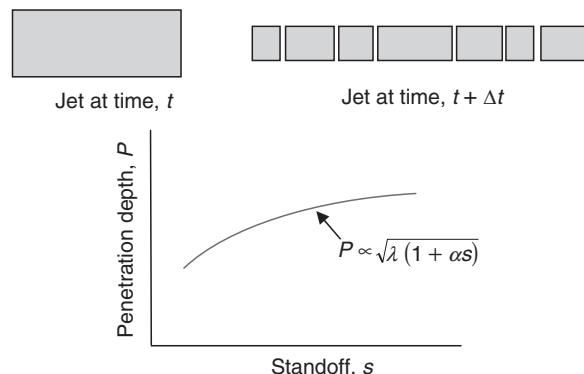
This states that  $P$  varies with the square root of  $\lambda(1+\alpha s)$  or, mathematically

$$P \propto \sqrt{\lambda(1 + \alpha s)} \quad (18.67)$$

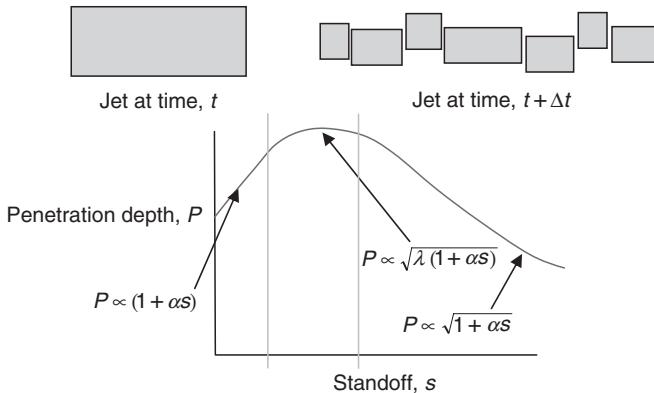
This behavior is illustrated in Figure 18.10.

The MPH theory can be modified to account for jet waver. Jet waver is the phenomenon whereby the particles in the jet move off the flight axis as illustrated in [Figure 18.11](#). This is caused by imperfections in the formation, strain hardening of the jet material, and subsequent breakup that provides for asymmetric particles. These particles begin to rotate with the end result being a jet that does not completely exert its energy in deepening the hole in the target but widens the hole as the particles impact the sides. We can account for this by adjusting the area term through

$$A_j = A_1(1 + Bs^2) \quad (18.68)$$



**FIGURE 18.10**  
Mixed mode jet behavior.



**FIGURE 18.11**  
Wavering jet behavior.

Here  $A_1$  and  $B$  are empirically obtained constants. This expression can be substituted directly into our three penetration equations to yield

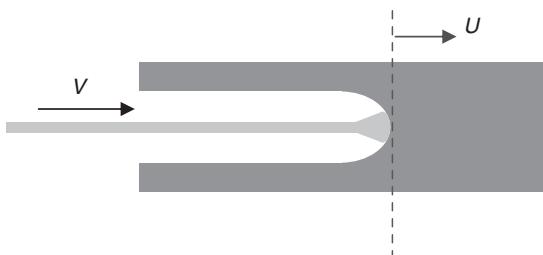
$$P = \sqrt{\frac{m_j l_0 (1 + \alpha s)}{\rho_t A_1 (1 + B s^2)}} \quad (18.69)$$

$$P = \sqrt{\frac{2 m_j l_0 (1 + \alpha s)}{\rho_t A_1 (1 + B s^2)}} \quad (18.70)$$

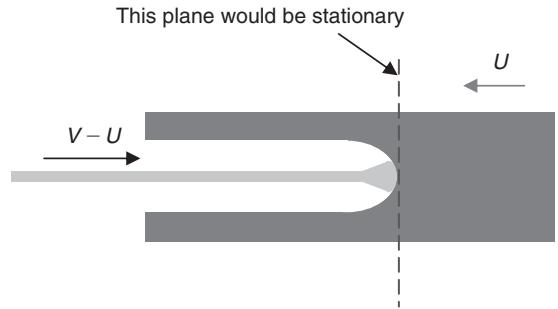
$$P = \sqrt{\frac{\lambda m_j l_0 (1 + \alpha s)}{\rho_t A_1 (1 + B s^2)}} \quad (18.71)$$

Here Equations 18.69 through 18.71 replace [Equations 18.61, 18.64, and 18.66](#), respectively. As we can see from Figure 18.11, we can use each of these equations to determine the depth of penetration dependent upon the standoff distance.

Once we have decided upon the proper particulation model to use for the penetration event, we determine the depth of penetration. To do this, we need to examine the penetration event from a Lagrangian viewpoint. If we were watching the stationary target as shown in Figure 18.12, we would see a hole that is deepening while the jet was shortening. In this figure, the rear of the jet would have a faster velocity,  $V$ , than the speed at which the hole was advancing,  $U$ . It is convenient to analyze this problem from a Lagrangian viewpoint. If we invoke the principle of superposition, we will obtain a situation as depicted in [Figure 18.13](#). In this case, the jet velocity, relative to the hole velocity, would be  $V - U$  and an observer moving with the hole would see target material approaching them at velocity  $U$ .



**FIGURE 18.12**  
Eulerian view of a jet penetration.



**FIGURE 18.13**  
Lagrangian view of a jet penetration.

If we use an analysis technique that lets us imagine a jet of constant length passing through the target material and somehow relate this to a hole depth, we would have the visualization depicted in Figure 18.14.

With this model, we can write the conservation of momentum equations in the variables that we have defined previously as

$$\rho_j(V - U)^2 = \frac{\rho_t U^2}{\lambda} \quad (18.72)$$

With the exception of the coefficient  $\lambda$ , this equation is identical to the Birkhoff equation (Equation 18.38). If we collect the velocity terms and take the square root of both sides, we get

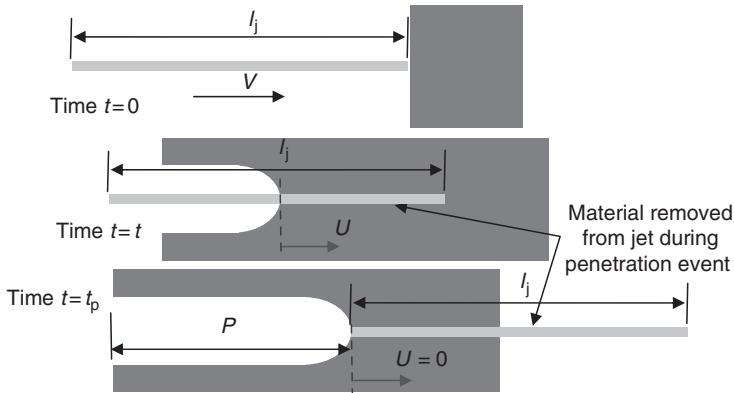
$$\frac{U}{(V - U)} = \sqrt{\frac{\lambda \rho_j}{\rho_t}} \quad (18.73)$$

The depth of penetration is still given by Equation 18.39 and essentially results in the penetration velocity times the penetration time, so we can write

$$P = Ut_p \quad (18.74)$$

But we can state  $t_p$  as

$$t_p = \frac{l_j}{(V - U)} \quad (18.75)$$



**FIGURE 18.14**  
Model that assumes constant jet length during penetration event.

Substitution of [Equation 18.75](#) into [Equation 18.74](#) yields

$$P = l_j \sqrt{\frac{\lambda \rho_j}{\rho_t}} \quad (18.76)$$

Again similar to the Birkhoff et al. theory with the exception of  $\lambda$ , for different configurations, we would assign a different value of  $\lambda$ :

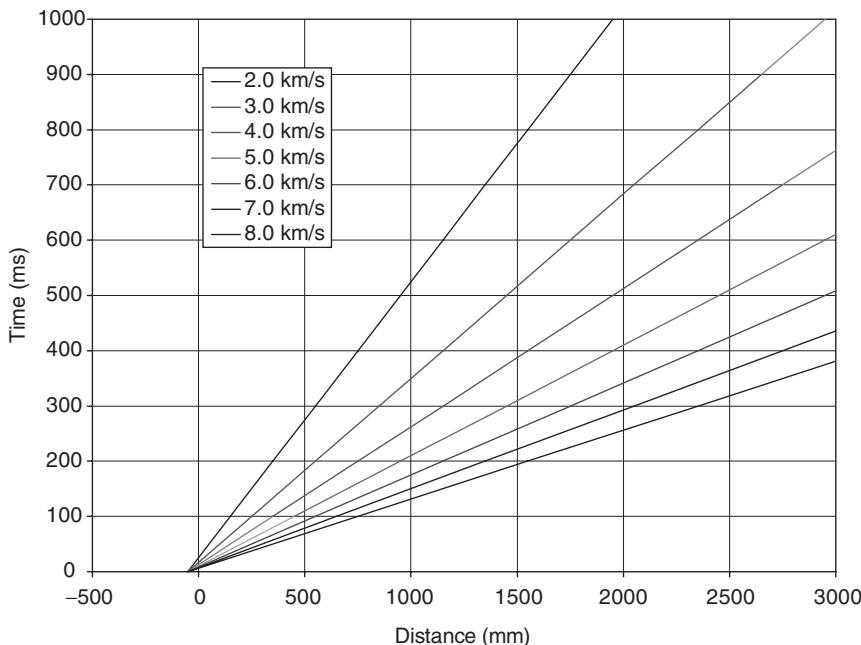
- For fluid (hydrodynamic) jets,  $\lambda = 1$
- For particulating jets.  $\lambda = 2$
- For mixed mode jets,  $1 < \lambda < 2$

We could also account for standoff by adjusting  $l_j$  accordingly.

We shall now discuss the virtual origin concept. Many researchers have determined relationships that use the virtual origin to describe shaped charge jet behavior [1,2,4]. The virtual origin is an empirically derived distance that is obtained from multiple jet tests. We have stated (repeatedly) that a real-shaped charge jet has a gradient in velocity from the tip of the jet to the tail. The velocity is highest at the tip. If we assume that this velocity gradient is linear, then when the jet impacts the target we can say that the distance the tip (or any part) has traveled can be written as

$$x = Vt + s \quad (18.77)$$

Let us consider a situation where we fire three identical jets at different standoffs, say 1, 1.5, and 2.5 m. If the tip velocity in each case is (say) 8 km/s and the tail velocity in each case is variable. Further assume that we get a time–distance curve for each jet as shown in [Figure 18.15](#).



**FIGURE 18.15**  
Virtual origin concept.

With this, the virtual origin will be where all parts of each jet with the same velocity line up and intercept the  $x$ -axis. This is depicted in the figure.

We have examined several models for the penetration behavior of shaped charge jets. This is greatly dependent on their formation and participation. These models are by no means the end of all jet penetration analytical tools. There is still a great deal of work that is ongoing to describe this important behavior.

### Problem 3

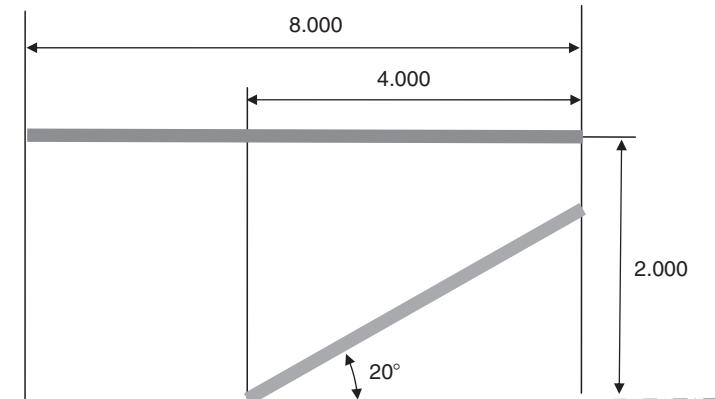
A conical liner as shown in Figure 18.16 is to be fabricated from copper and filled with Composition A-3 as the explosive. The thickness of the liner is to be 0.10 in. The length of the  $20^\circ$  conical liner is 4 in. and the charge OD is 4 in. The case is fabricated from steel and is an 8-in. long cylinder, 0.12-in. thick. Determine the following using the Birkhoff et al. theory ignoring effects of confinement (if the region over the liner is discretized into four segments that should be sufficient):

1. Masses of the jet and slug.  
*Answer:*  $m_j = 0.067[\text{lbm}]$  and  $m_s = 0.488[\text{lbm}]$
2. Velocities of the jet and slug.  
*Answer:*  $V_j = 8800 [\text{ft/s}]$  and  $V_s = 570 [\text{ft/s}]$
3. The velocities of the case material (plot this as fragment velocity versus case length).
4. The direction in which the case material fragments will be projected (plot this as departure angle versus case length).
5. If the standoff is 8 in., determine the maximum penetration into RHA plate at a  $15^\circ$  angle of obliquity using the formula of Dipersio and Simon and assuming no participation.  
*Answer:*  $P = 11.89[\text{in.}]$

The required properties for this calculation are given as follows:

Composition A-3 Gurney velocity  $(2E)^{1/2} = 2.63 \text{ km/s}$

Composition A-3 detonation velocity  $(D) = 8.14 \text{ km/s}$



**FIGURE 18.16**  
Shaped charge of problem 3.

Composition A-3 density = 1.59 g/cc

Copper density = 0.323 lbm/in.<sup>3</sup>

Steel density = 0.283 lbm/in.<sup>3</sup>

#### Problem 4

A conical-shaped charge liner as shown in Figure 18.17 is to be fabricated from copper and filled with Composition B as the explosive. The thickness of the liner is to be 0.15 in. and the half-angle,  $\alpha$ , is to be 30°. The length of the liner is 5 in. and the charge OD is 8 in. Determine the following using the Birkhoff et al. theory:

1. Mass of the jet.

Answer:  $m_j = 0.410[\text{lbm}]$

2. Mass of the slug.

Answer:  $m_s = 1.787[\text{lbm}]$

3. Velocity of the jet.

Answer:  $V_{j\text{Avg}} = 7491 \left[ \frac{\text{m}}{\text{s}} \right]$

4. Velocity of the slug.

Answer:  $V_{s\text{Avg}} = 965 \left[ \frac{\text{m}}{\text{s}} \right]$

5. Estimate the jet length assuming constant velocity of the tip and slug if the standoff is 1 m (use the fastest tip velocity and the average slug velocity).

Answer:  $L \approx 0.875[\text{m}]$

6. Using the above data and assuming that the virtual origin is 10 cm behind the standoff measurement, estimate the penetration ability into mild steel assuming the jet does not particulate using the formula of DiPersio and Simon.

Answer:  $P = 2.98[\text{m}]$

7. Compare the answer in (6) above to that for the density law.

Answer:  $P = 0.935[\text{m}]$

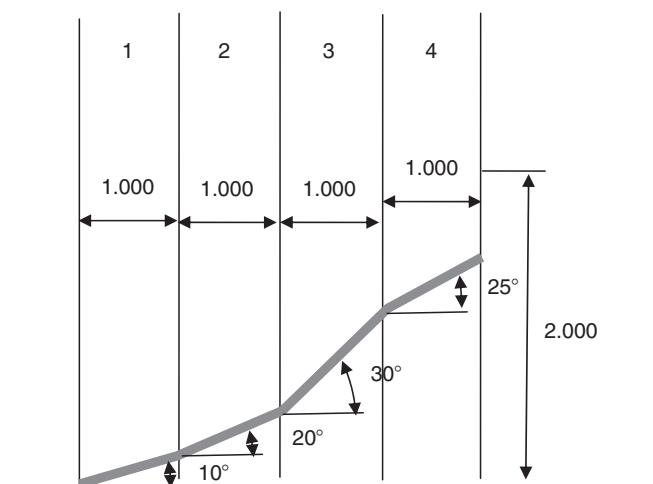


FIGURE 18.17

Shaped charge of problem 4.

The required properties for this calculation are given as follows:

Composition B Gurney velocity  $(2E)^{1/2} = 2.79$  km/s

Composition B detonation velocity ( $D$ ) = 7910 m/s

Composition B density = 1.717 g/cc

Copper density = 0.323 lbm/in.<sup>3</sup>

Steel density = 0.283 lbm/in.<sup>3</sup>

### Problem 5

A trumpet liner is to be fabricated from copper and filled with Composition A-3 as the explosive. The thickness of the liner is to be 0.12 in. We shall approximate the trumpet liner as indicated below where the half-angle,  $\alpha$ , is to be variable. The length of the liner is 4 in. and the charge OD is 4 in. Determine the following using the Birkhoff et al. theory:

1. Masses of the jet and slug.

*Answer:*  $m_s = 0.545$  [lbm] and  $m_j = 0.045$  [lbm]

2. Velocities of the jet and slug.

*Answer:*  $V_j = 5200$  [ft/s] and  $V_s = 200$  [ft/s]

3. If the standoff is 8 in., determine if the jet will perforate 5 in. of rolled homogeneous armor plate at a 70° angle of obliquity using the formula of Dipersio and Simon and assuming no particulation.

*Answer:* No,  $P = 3.3$  [in.]

The required properties for this calculation are given as follows:

Composition A-3 Gurney velocity  $(2E)^{1/2} = 2.63$  km/s

Composition A-3 detonation velocity ( $U_D$ ) = 8.14 km/s

Composition A-3 density = 1.59 g/cc

Copper density = 0.323 lbm/in.<sup>3</sup>

Steel density = 0.283 lbm/in.<sup>3</sup>

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# 19

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## *Wound Ballistics*

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Until this point we have dealt with the penetration of projectiles into inanimate objects. One of the more tragic aspects of ballistics is the fact that they are used against living creatures. This is not meant to imply that hunting is good or bad, but simply that there are instances when people, intentionally or not, fire weapons at other people or animals and the effects of the bullet impact must be understood.

When a projectile is fired at a living creature some amount of incapacitation is desired. If a projectile is of the nonlethal type, trauma to the target must be minimized and either a fluid must be injected, the victim must be rendered physically immobile, or some other effect must be obtained. If the projectile is of the lethal type, ideally one hit should subdue the victim (through any protection) rendering them incapable of harm.

Because the subject of wound ballistics is as complicated as the target's anatomy, we shall only conduct a cursory review here, pointing the interested reader to some excellent references for further detail. We shall only treat subjects which affect humans, though these may affect animals in a similar manner.

An interesting statistic is that over 58% of combat casualties in the British army during the First World War were caused by fragments rather than bullets [1]. This is interesting since we all have seen movies (accurate or not) of wild charges into machine gun fire. This is probably the case with most conflicts.

In the sections on aeroballistics, we have learned to treat projectile flight through a fluid medium (air). While these equations still hold in a human body, the simplifications we made do not always hold and we must take steps to include properties such as the elasticity of tissue. The initial conditions such as entrance angle become largely important when dealing with a wound. Additionally, a bullet is usually unstable in a human body, causing it to yaw greatly or even tumble. Thus, bullet geometry, mass properties, and material strength matter a great deal as far as the extent of damage is concerned.

Before we discuss the details any further, it must be understood that there are many people who have diligently studied the field of wound ballistics during their entire careers. These researchers have drawn on their wide experience, some from the engineering viewpoint and some from the medical viewpoint, to reach conclusions and develop theories about wound physics. They are probably all correct even though their viewpoints may be vastly different. The reality is that "anything" can happen when a bullet interacts with a human. It has been these authors' experience that the experts can be categorized into two broad camps: the medical camp and the engineering camp. The medical camp sees the wounds (even wounds that were caused by an identical bullet at an identical entrance angle into an identical location) as individually different and must be treated through a medical procedure based on the caregivers' experience, observations, and understanding. The medical camp believes that the psychological and physiological effects of a wound will always be different and that no conclusions can be drawn based on weapon type, etc. The engineering camp believes that wounding can be quantified through physics. They believe

that relationships (potentially very complex) can be drawn based on energy, momentum, material properties, etc., which can be used to quantify the effect of projectiles against persons. The truth is probably a combination of both camps, but to date no one has found the Holy Grail that would bring it all together.

The work by Peters [2] states that there are several misconceptions about wounding that must be addressed. One misconception is that the temporary cavity is the major cause of tissue damage. This has probably grown out of extremely interesting videos that have been published showing massive temporary cavities in projectile firings through gelatin blocks. It is difficult to imagine, as a human, that these cavities would not cause huge amounts of damage. In fact, this topic is rather hotly debated by experts. We shall pass no judgment here, simply state that the work of Peters suggests that less than 20% of all tissue damage is caused by the temporary cavity.

Another misconception pointed out by Peters is that tissue damage is proportional to kinetic energy of the projectile. Peters suggests that there is a relationship but it is nonlinear. It was thought (and possibly still is) that the sizes of the maximum temporary cavity and the permanent cavity were somehow proportional to energy deposited in the target by the projectile. Peters suggests that there is a nonlinear relationship but additionally, over some ranges of the data, it can be linearized which is possibly why the conclusion was drawn.

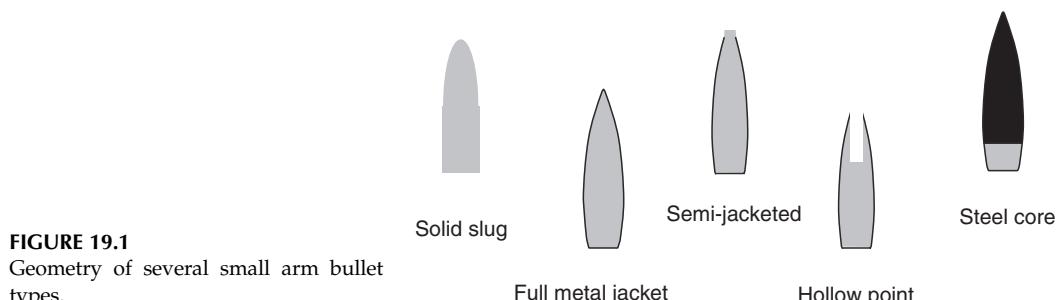
Engineers who look at a person as an engineering structure at some point assume that the volume of the permanent cavity must, in some way, result from material ejected from the wound. That is, that the permanent cavity volume must equal the volume of material ejected. This is not the case since a permanent cavity remains even when the bullet stops in the target. The cause of this permanent cavity is primarily through inelastic deformation of the tissue.

Peters and other researchers have shown that temporary cavities in humans or animals will be of different size than those developed in gelatin blocks. Currently, this is a very active area of research. There are even differences in cavity formation between animals and humans to the extent that no scaling law has been universally established.

One of the most interesting aspects of wound ballistics is the inertial effect on a human body. In many Hollywood action films, we routinely see people being picked up and thrown several feet backward by impacts of small arms projectiles.

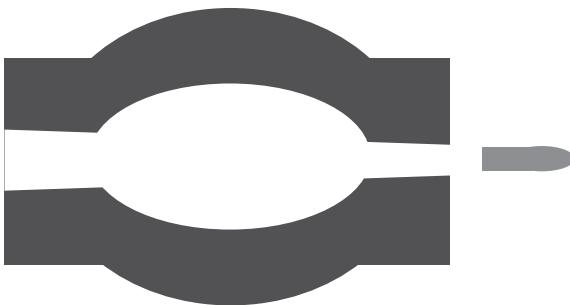
When the numbers are worked out with a 7.62-mm projectile at point blank range, the energy exchange (assuming the bullet remains lodged in the target) is such that the rearward velocity is less than 0.2 mil/h. In fact, most human targets usually fall toward the shooter (unless they were running away when hit).

We shall next discuss some bullet types that are illustrated in Figure 19.1. A solid slug is nothing more than a soft metal (usually a lead alloy) projectile that is engraved along its body length by the rifling to impart spin. A full metal jacket (FMJ) projectile is a solid slug



**FIGURE 19.1**

Geometry of several small arm bullet types.



**FIGURE 19.2**  
Temporary cavity.

that is coated with a material such as copper to better withstand firing stresses and whose residue can easily be removed from the inside of the gun tube. A semi-jacketed projectile or open-tipped projectile is jacketed up to a small region of the nose. This region, being softer than the jacketed region and unable to withstand the radial stresses upon impact, expands as it enters the target theoretically causing a more extensive wound. A hollow point projectile is similar to a semi-jacketed projectile except that the tip is actually concave, which uses fluid mechanics coupled with the lower radial strength upon penetration, to open larger. Finally, the steel-core projectile has a hard core for penetration of metallic structures or textile armor. There are a huge number of other projectile types such as slit jackets, dum-dums, etc., but usually they fall into one of the aforementioned categories.

In the earlier paragraphs, we mentioned some terms such as temporary cavity and permanent cavity. We will now define some of these terms.

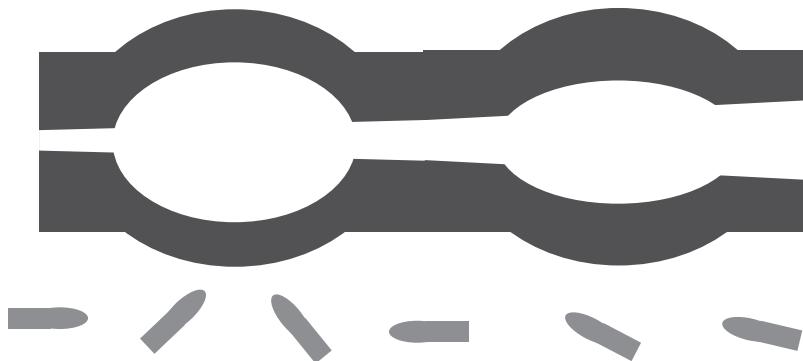
A laceration is a cut through tissue. A projectile's primary means of incapacitation is through laceration. Because of the complicated nature of the human body, a projectile which penetrates can do anything from causing minor bleeding if no major organ or artery is damaged to rapid death if a vital organ is hit. If a projectile impacts bone tissue or even meets a severe gradient in density, it can be deflected considerably.

We learned a great deal about stress waves previously. When a projectile enters a human being, it sends stress waves through the body. These waves and associated rarefactions can cause damage, but it is generally agreed that, primarily, these waves will damage nerves and can, possibly, collapse organs.

The temporary cavity is created through the process of cavitation introduced earlier in the fluid mechanics section (Figure 19.2). It results from the adherence of the fluid molecules to the surface of the projectile, and when the shear stress drops to zero on the surface, the flow separates. This separation bubble can grow to 40 times the projectile diameter as the projectile passes through the body. Once the projectile has passed by, however, the radial energy that it imparted to the tissue is removed and the elasticity of the tissue causes it to immediately collapse to a much smaller size. The largest extent that this bubble reaches is known as the maximum temporary cavity while the small, equilibrium cavity is known as the permanent cavity.

Projectile yaw has a dramatic effect on cavitation. As stated earlier, a projectile is usually unstable in a human body. This causes it to yaw considerably and possibly tumble. As one can imagine, because of the relatively immense presented area of a projectile flying with a large yaw, the separation and associated cavitation can be huge. In fact, if a projectile rotates 180°, it will usually exit the target base first. This is depicted in [Figure 19.3](#).

Analysis of this flight behavior is extremely difficult because projectiles perform differently depending upon what tissue they happen to be passing through. The following is a short list of just a few of the different types of tissues that affect bullet passage through a living creature:



**FIGURE 19.3**  
Cavitation due to projectile yaw.

- Bone
- Skull and brain
- Thorax/ribs
- Lung
- Intestine/stomach/bladder
- Muscle

Each of these tissue types will have a different effect on the projectile. It is even important if an organ is flaccid (empty) or not or whether the target is living or dead. For simplicity, the most general research is carried out on muscle tissue and that is where a great deal of work has been expended to come up with a suitable surrogate material.

Assuming we are discussing muscle tissue penetration, the first thing we must recognize is that tissue has a nonnegligible tearing stress that must be overcome. This additional stress must be incorporated into our drag model. We cannot emphasize the complexity of the problem enough. Even though, in the discussion that follows, we shall assume a penetration into homogeneous muscle tissue we must always keep in mind that a penetration event is much more complicated. We know that as a projectile enters muscle tissue, what was once relatively simple aeroballistics becomes a more complicated problem of continuum mechanics: in air, there was no yield stress to overcome (this is the major difference); the viscosity and density of muscle are different than air. If the impact angle is low enough, the nose of the projectile will enter first. The usual decrease in shear stress as we progress along the projectile will occur and at some point the shear stress will reach zero and the tissue will separate from the projectile forming a cavitation bubble. Throughout this event, the projectile will slow down due to drag. There will also be a larger overturning moment than in air because of the large force on the small area of the nose (higher density in the dynamic pressure term) and in addition the separation will take place ahead of the CG, increasing the moment arm. The drag force will also include the force required to overcome the cohesive stresses in the tissue (tearing stress) which is not usually included in aerodynamic models. What was once a transonic/supersonic flow field becomes a transonic (at best) or subsonic flow field. This is because the speed of sound in muscle tissue is around 1500 m/s (4920 ft/s).

In comparison to the aerodynamic models we have presented earlier, Peters et al. [3] have developed a drag model that accounts for the tearing of the tissue. The equation of motion is given by

$$-m \frac{dV}{dt} = \frac{1}{2} \rho V^2 A C_D + \frac{1}{2} \rho (aU)^2 A C_D \quad (19.1)$$

or it can be written in terms of distance traveled as

$$-mV \frac{dV}{dx} = \frac{1}{2} \rho A C_D [V^2 + (aU)^2] \quad (19.2)$$

In these equations,  $m$  is the mass of the projectile,  $V$  is its velocity,  $\rho$  is the density of the tissue,  $A$  is the presented area\* of the projectile,  $C_D$  is the projectile drag coefficient,  $x$  is the distance the projectile has progressed into the tissue,  $a$  is a modification to  $C_D$ , and  $U$  is a characteristic velocity of the tissue (more on these last two terms will follow).

If we examine Equation 19.2, we see that if we exclude the second term on the RHS, we get our classic equation for aerodynamic drag (assuming, of course that the area is a cross-sectional area,  $S$ , of the projectile). The second term accounts for the energy loss associated with the tearing of the tissue and its movement away from the projectile.

The characteristic velocity is defined as

$$U = U_6 \left( \frac{d}{d_6} \right)^{-\frac{1}{3}} \quad (19.3)$$

These are empirically derived values. In Equation 19.3,  $d$  is the diameter of the actual projectile,  $d_6$  is the diameter of a 6-mm projectile (in case you want the units of  $d$  in a different system), and  $U_6$  is a characteristic velocity for different materials determined through experiments with a 6-mm diameter projectile. The parameter  $a$  in Equations 19.1 and 19.2 is a function of the projectile type and the angle of attack of the projectile.

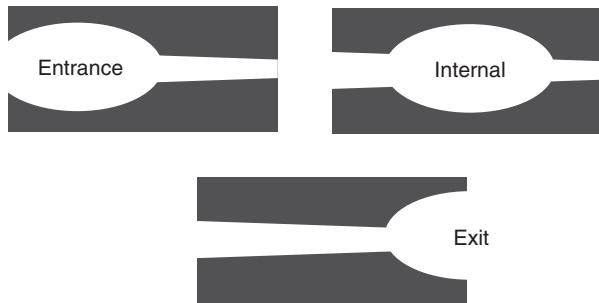
The stability criteria developed in Part II of this book work fairly well for behavior in the human body. As stated earlier, the density terms must be increased as well as the effect of Mach number. It is also recommended to add a force term as was included in Equation 19.2, however, that would require a re-derivation of the stability equations which is beyond our scope.

If a projectile has features which would cause it to expand upon impact with the more dense human tissue, it will cause greater trauma. These were mentioned earlier as hollow point and slit-jacketed bullets. The opening of the hollow point or jacket allows more of the projectiles energy to be transferred to the body and the flatter surface directs the flow of the tissue in a more radial direction. If a bullet is unstable in the body and it tumbles, there is more surface area presented for the body to slow the projectile down and thus more energy would be expended on the body. A greater amount of cavitation will occur as well due to greater radial flow of the tissue. Depending on whether this expansion happens at the entrance to the body, the exit, or somewhere in between, the wound would be affected as depicted in Figure 19.4.

To defend against projectile impacts, body armor has been developed. Body armor for humans has been designed since projectiles were first fired. Metallic armors were good against ball ammunition but armor-piercing rounds can go through them easily. Textile/composite armor has met with better success at stopping penetration, but it can still happen. Even with textile armor some depth of penetration or organ damage is still possible. In addition, the same mechanisms that we discussed about non-penetrating

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\* Here the presented area is different than the cross-sectional area we used in the sections on exterior ballistics which is why  $A$  was used and not  $S$ .



**FIGURE 19.4**

Wounds that are affected by the time when tumbling or bullet head expansion occurs.

damage are applicable here as well such as shock waves and momentum transfer (which is even greater for a non-penetrating hit than a pass-through).

In summary, we have touched upon several aspects of wound ballistics. A more comprehensive treatment is provided in Ref. [4]. It is a complicated and hotly debated subject, yet one that is extremely fascinating.

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## Further Reading

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# *Appendix*

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## A. Glossary

**Active Homing** A method of guidance whereby the device is guided by electronics that contain both a transmitter and receiver, so that the munition can be adjusted onto the target.

**Autofrettage** A process by which the inner layer of material in a gun tube is yielded plastically and held in compression by the outer layer. This increases the fatigue life of the weapon by limiting the cyclic stress amplitude during repeated firings.

**Autonomous Munition** A munition that needs no input from an outside source once fired.

**Azimuth** The rotation of a weapon about the pintle or turret ring (side to side) as opposed to elevation (q.v.) (which is up and down).

**Ballistic Cap** See [Windshield](#).

**Balloting** The lateral motion of the projectile in the gun tube. This can be one of three modes: the whole projectile moving side to side with its centerline remaining parallel to the bore axis, the projectile nose and base rotating about the center of gravity (centerline of projectile at an angle to bore axis), projectile remaining pushed to one side of the bore and rotating in a cyclic motion at the rifling twist rate.

**Band Seat** The annular groove in a projectile into which the rotating band is swaged or welded.

**Base** The rear end of a projectile.

**Base Gap** A gap between the explosive fill and metal base or wall of a projectile that can be very dangerous. If the weapon is fired, setback forces compress the air in the gap with a resultant heating. This process occurs over milliseconds so that the heat cannot be transferred away. The resultant heat can detonate the explosive fill in the bore of the weapon usually resulting in a loss of the weapon and the crew.

**Battery** A group of three to six field artillery pieces.

**Bayonet** A knife or spike that attaches to the muzzle of a rifle used in hand-to-hand combat.

**Bayonet Lug** A boss or protrusion located near the muzzle of a rifle that is the attachment point for the bayonet.

**Bent** A latch which engages the sear, preventing the firing pin from moving forward until released by the sear.

**Berdan Primer** A primer whose anvil is an integral part of the cartridge case.

**Bipod** A pair of supports that are used to steady a mortar or a gun, so that it can be aimed or to increase accuracy by limiting muzzle movement.

**Boat Tail** The angled rear end of a projectile.

**Bolt** The device in a small arm that houses the firing pin. A bolt can be manually operated or automatically operated. The bolt usually obturates the breech of the weapon as well.

**Booster** A section of explosive charge, usually attached to the fuze whose purpose is to accept the initiation from the primary detonator and amplify the detonation to more reliably and completely initiate the main charge of a projectile. It can be made from the same material

as the main charge or different material. Its key characteristic is proximity to the primary initiation train so that reliable and timely ignition is assured.

**Bore Evacuator** A device connected to the bore of a gun by ports, that fills with high-pressure gas upon firing. After the projectile exits the muzzle of the weapon, this high-pressure gas pushes any remaining smoke and burning embers out of the tube before the breech is opened. It is used with vehicles that have closed firing compartments so that the crew is not affected by smoke or any burning debris entering the compartment. On warships there is an external system that blows the hot gases out (a bore scavenger).

**Bourrelet (Pronounced Boor'-rel-lay)** Regions of the projectile where the diameter is full caliber (usually divided into a forward and aft bourrelet and separated by the undercut (qv.)).

**Boxer Primer** A primer whose anvil is enclosed as part of the primer itself.

**Breech Block** Device which allows access to the chamber for the loading of ammunition into the weapon and closes to maintain pressure in the chamber during the ballistic cycle. Normally, a breech block is designed so that gravity drops it into place. Used almost exclusively with cartridge cased ammunition.

**Breech Plug** Device which allows access to the chamber for the loading of ammunition into the weapon and closes to maintain pressure in the chamber during the ballistic cycle. It generally screws into the breech of the weapon with an interrupted thread and can obturate the propellant gases if a cartridge case is not used.

**Brilliant Munition** A precision munition that can classify potential targets and potentially select the one with the highest value.

**Brisance** A property of an explosive that relates its shattering effect. This is related to the rate of energy release in the explosive. A "brisant" explosive will shatter its container rather than expand it to burst like a balloon.

**Burster** A charge of energetic material in a projectile or munition that is intended to burst the outer casing of the device and spread the contents over some defined area.

**Butt** The end of a rifle that rests on the shoulder of the firer.

**Caliber** The largest diameter of the bourrelet (and the projectile excepting the rotating band).

**Caliber** The smallest internal diameter of a gun tube. Also a unit of measure for a tube length. A 155-mm 39 caliber gun tube is  $39 \times 155$  mm or 6045 mm in length.

**Canards** Control surfaces mounted to an airframe or projectile ahead of the center of gravity or center of pressure.

**Candle** The device carried in an illumination projectile that burns upon expulsion from the projectile after parachute deployment. The purpose of the candle is to illuminate the battlefield to allow combat to take place at night. Some candles illuminate in the infrared spectrum so that they only aid soldiers equipped with infrared optical equipment.

**Canister** A projectile resembling a shotgun shell containing a burster charge and a large number of metal balls or flechettes. The purpose of a canister round is to incapacitate personnel in relatively close proximity to the weapon.

**Cannelure** Circumferential groove cut in a rotating band or projectile jacket to reduce engraving pressure and allow a place for material to flow during the engraving process.

**Cap** A relatively thin metal device attached to the nose of an armor-piercing projectile to grease the main penetrator by biting into the armor. It also can help penetration if the projectile strikes at an oblique angle by rotating the projectile on impact normal to the armor plate.

**Carriage** The component on a weapon platform that connects the gun assembly to the trails and wheels. [See also Upper carriage](#) and [Lower carriage](#).

**Cartridge** The assembly that contains case, propellant, and projectile.

**Cartridge Case** The metal or energetic material that is attached to the base of some projectiles. The purposes of the cartridge case are to contain the proper amount of propellant, keep the propellant protected against the environment, help obturate the breech and, in the

case of a combustible cartridge case, provide additional propelling energy to the projectile. The breech plug must obturate when a combustible cartridge case is used.

**Cartridge Rim** The flange on the cartridge case that has several functions. It retains the cartridge when the bullet is loaded into the chamber. It allows the extractor a surface to interact with to remove the spent cartridge case from the weapon.

**Center of Gravity (CG)** The location on a body where, analytically, all the mass can be concentrated and the resultant force vector directed toward the center of the earth. The resultant force vector is equivalent to the distributed load.

**Center of Pressure (CP)** The location on a body in motion through a fluid where, analytically, all of the pressure force (integrated over the surface of the body) can be concentrated. The resultant force vector is equivalent to the distributed load.

**Centering Band** A band made of soft material attached ahead of the threaded region of a projectile for the purpose of maintaining concentricity of the parts. Centering bands have also been used on the exterior of projectiles to limit balloting or maintain a central position in the bore.

**Click** A military term for one kilometer.

**Clip** A device which contains several cartridges that is fed into the magazine of a weapon.

**Closing Plug** A threaded plug which seals the base end of a projectile (if base fuzed) or a Hi-Low cartridge case (q.v.).

**Commencement of Rifling** The point in a gun tube at which the lands have attained full size.

**Conical Ogive** An ogive that is conical in shape.

**Coppering** The deposition of copper from either rotating bands or jacketed projectiles along the bore of the weapon.

**Cradle** Device on a weapon platform that connects the sleigh to the trunnions and allows the sleigh to rotate about the trunnions (i.e., rotate in elevation).

**De-Coppering Agent** A material added to the propelling charge to react with the copper deposited by the projectile during firing to eliminate the buildup of copper or fouling of the gun tube.

**Down Bore** The direction from the breech toward the muzzle (in the direction of projectile travel).

**Elevation** The rotation of a gun about the trunnions (up and down).

**Equilibrators** Devices which overcome the effect of gravity when a weapon is elevated because the center of gravity of the weapon is usually ahead of the center of rotation (the trunnions).

**ET Fuze (Electronic Time Fuze)** A fuze that utilizes electrical energy and timing circuits to count time to initiation. This type of fuze is much more accurate than an MT (mechanical time) fuze.

**Eutectic Alloy** An alloy that has a physical state, under normal environmental conditions, at the Eutectic point on a phase diagram (near its melting point). These alloys are used in designs where the high temperature of a fire will melt them and allow some mechanism to drop or just open a vent hole.

**Expulsion Charge** A charge placed in a projectile whose purpose is to expel cargo.

**Fin Shroud** A ring-like structure used to tie fins on mortar rounds or rockets for structural support. These devices usually have an adverse effect on drag.

**Fins** Control surfaces mounted to an airframe or projectile aft of the center of gravity or center of pressure.

**Flash Hole** A hole through which hot gases may pass to ignite energetic material in a separate chamber or area.

**Flash Reducer** A mixture of material whose purpose is to reduce muzzle flash by either lowering the temperature of the combustion or inhibiting the reaction of the propellant combustion products with the air. Reduction of flash usually results in increased smoke.

**Flechette** Small dart used for antipersonnel rounds.

**Forcing Cone** The region immediately down bore of the chamber where the internal diameter of the tube tapers to the correct caliber.

**Fusible Lifting Plug** A lifting plug containing a eutectic alloy that melts out if the projectile is exposed to high temperature during storage allowing a pressure vent for the expulsion charge material.

**Gain Twist** A scheme of rifling where the twist increases with down bore distance. The intention is to minimize wear and angular acceleration of the projectile.

**Grommet** A device used with copper rotating bands to protect the soft copper from damage during rough handling. Removed before ramming the projectile.

**Grooves** The part of the rifling which is cut into the tube material. The internal diameter of the grooves is larger than the internal diameter of the lands.

**Guided Munition** A munition or projectile that has onboard guidance to steer it to the target.

**Head** The portion of the bolt which presses up against the rear face of the cartridge case through which the firing pin passes. The head obturates the breech with the assistance of the cartridge case.

**Headspace** The space between the head of the bolt and the forward lip of the chamber that accepts the rim of the cartridge. It is important that the headspace not be too large or small so that operation of the weapon can proceed smoothly.

**HEAT (High Explosive Anti-Tank)** A projectile which uses a shaped charge for terminal effects.

**HEP (High Explosive Plastic)** A projectile with a thin, soft shell that will mash upon impact with a target.

**HESH (High Explosive Squash Head)** Another name for a HEP projectile.

**High Explosive (HE)** An energetic material that detonates, given a proper stimulus, regardless of confinement.

**Hi-Low** A propelling charge configuration in which there are two chambers: a high-pressure chamber and a low-pressure chamber. The propellant burns in the high-pressure chamber and exits through vent holes to pressurize the low-pressure chamber. The gases in the low-pressure chamber actually push on the projectile to impart the proper velocity.

**Igniter Core** A cylinder of pyrotechnic material whose purpose is to ignite the propelling charge as uniformly as possible. The igniter core usually is initiated by an igniter pad or a primer.

**Igniter Pad** A cloth pouch containing a sensitive pyrotechnic mixture sewn to the rear of a bag charge. The function of the igniter pad is to accept the input flame from the primer, amplify it, and either ignite the propellant or begin the burning of the igniter core.

**Jacket** A hoop of metal assembled around a gun tube to increase its strength.

**Jet** [See Shaped Charge Jet](#).

**Laced Jacket** The outer casing of some bag charges.

**Lands** The part of the rifling with an internal diameter, i.e., the caliber of the weapon.

**Laser Designator** A device carried or mounted on a vehicle that can illuminate (sometimes called "paint") a target using laser energy, so that a Semi-Active Laser Guided projectile can ride the beam to the target.

**Lifting Plug** A device threaded into the fuze well of a nose-fuzed projectile to lift the projectile. It is removed before firing.

**Lifting Plug (Energy Absorbing)** A device threaded into the fuze well of a nose-fuzed projectile to lift the projectile. It is removed before firing. This device differs from a standard lifting plug in that it is designed to shear off if the ogive of the projectile is impacted, thereby preventing fusing of the round. Its design came about because HE projectiles would crack when dropped on the nose, the crack going unnoticed, and the projectile would detonate in-bore when fired due to structural failure.

**Liner** A conical, hemispherical, or other shape manufactured out of metal or glass that, when exposed to the properly conditioned detonation of an explosive, will form a jet which will penetrate armor plate.

**Loading Density** The ratio of the weight of the powder charge to the volume of the empty cartridge case or chamber. Also the density to which an explosive is consolidated.

**Lock Time** The amount of time between when a trigger of a weapon is pulled and the weapon discharges.

**Low Explosive** An energetic material that requires the proper stimulus and confinement to detonate. Gun propellants are low explosives.

**Lower Carriage** A platform-like structure on a field piece that contains the pintle and connects the trails to the wheels or axle.

**Lunette** Ring welded to the trails (or muzzle brake on some newer weapons) of a field piece that allows the weapon to be towed.

**Magazine** The device in a weapon that contains the cartridges.

**Man-in-the-Loop** A technique (frowned upon at one time by the U.S. Army) whereby a soldier is required to designate a target until the impact of the projectile.

**Meplat** The blunt forward end of a projectile.

**Mercy Mission (MRSI—Multiple Rounds, Simultaneous Impact)** A fire mission where the weapons fire multiple projectiles, varying the elevation and charge, so that all of the projectiles impact the target area simultaneously.

**Mil** Angular unit of elevation or deflection, 1/6400 of a circle approximately 1/1000 of the range. When used in a statement such as "The projectile had 5 mils of right deflection," means the projectile fell 5 me to the right of the line of fire for every kilometer of range.

**MT Fuze (Mechanical Time Fuze)** A fuze that utilizes stored mechanical energy in the form of springs and gearing to count time from firing until initiation.

**MTSQ Fuze (Mechanical Time, Super Quick Fuze)** A fuze that utilizes stored mechanical energy in the form of springs and gearing to count time from firing until initiation, and also has a point detonating mode that will initiate on contact with a surface. This allows a backup if the time setting is in error and will detonate before the projectile buries itself into the ground (which limits its effectiveness).

**Mushroom** Device mounted in the breech plug of weapons that use bag charges to seal (obturate) the breech upon pressurization of the chamber.

**Obturation** The sealing of propelling gases behind the projectile and in the chamber of a weapon.

**Obturator** Plastic band which seals propelling gases behind the projectile during gun launch (in spin-stabilized projectiles this device is used in conjunction with a rotating band).

**Ogive** (Pronounced Oh'-jive) Nose region of the projectile where the shape changes from cylindrical to curved or conical.

**Origin of Rifling** The point in a gun tube at which the lands begin to rise from the forcing cone.

**PD Fuze (Point Detonating Fuze)** A fuze which must impact an object to detonate.

**PIBD Fuze (Point Initiating, Base Detonating Fuze)** This type of fuze is used in HEAT, HEP, or HESH ammunition to ignite the rear of the explosive column thereby setting up the proper conditions for jet formation or target spall. It initiates upon impact of the projectile.

**PIMP (Permissible Individual Maximum Pressure)** Also called PMP (permissible maximum pressure). The three sigma upper limit on the pressure produced from a propelling charge conditioned to its maximum operating temperature. This is the charge used to proof a weapon.

**PIMP + 5%** The PIMP charge conditioned so as to produce 5% higher pressure.

**Pintle** Pin on a field piece that connects the lower carriage to the upper carriage and allows the weapon to traverse in azimuth.

**Precision Munition** A munition dispensed from a projectile or other device which uses a type of on-board or off-board electronics to improve its accuracy over standard munitions.

**Pressure Plate** A device used in a rifled-mortar projectile to press on and expand a rotating disc when the propellant burns and applies pressure to its face.

**Primer** A device containing small amounts of sensitive energetic material that is ignited first in a firing train. It may either be attached to the cartridge case or provided separately with bag loading ammunition. There are several types of primers: percussion primers rely on impact to begin the chemical reaction, stab primers rely on friction, and electric primers rely on the proper supply of electrical energy.

**Propellant Increment** A bag or C-shaped container of propellant that allows the range of a projectile to be altered by increasing or decreasing the amount of propellant.

**Proximity Fuze** See VT Fuze.

**Pusher Plate** A device used to transmit the pressure generated by an expulsion charge to a cargo stack. The pusher plate protects the cargo stack from damage during the expulsion event.

**Receiver** The portion of a small arm that comprises the interface between the barrel, the magazine, and the bolt.

**Recoil Cylinders** Cylinders filled with hydraulic fluid on a weapon platform that slow down and stop the rearward motion of the weapon during and immediately after firing.

**Recuperators** Devices which push a weapon back into battery after recoil.

**Rifling** Grooves cut into the bore of a weapon to impart spin to a projectile for stability.

**Rotating Band** A swaged, shrink-fit or welded metallic or plastic band attached to the projectile which is designed to engage the rifling of the bore and impart spin to the projectile.

**Rotating Disc** A disk of soft material used in rifled-mortar projectiles that is sub-caliber to allow a mortar round to drop down the tube initially, but when reacted upon by the pressure plate, it expands into the rifling of the mortar tube and imparts spin to the projectile.

**Sabot** (Pronounced Sa-bo') A device used to increase the diameter of a sub-caliber projectile to stabilize it in the bore of a weapon. These devices are usually discarded upon muzzle exit.

**Sear** The protrusion mechanically interfaced to the trigger which locks the bent. When the trigger is pulled, the sear moves off the bent allowing the firing pin to impact the primer in the cartridge.

**Secant Ogive** An ogive in which the radius is centered at a point behind the end of the cylindrical section of the projectile.

**Semi-Active Homing** A method of guidance whereby the device is guided by electronics that contain only a receiver (the transmitter being located off the munition), so that the munition trajectory can be corrected onto the target.

**Set Forward** The rapid unloading of the projectile as it leaves the muzzle, i.e., the un-springing of the compressed projectile structure when the base pressure drops off.

**Setback** The compressive reaction of the projectile mass to forward acceleration.

**Shaped Charge Jet** A stream of metal in a high state of strain resulting from proper detonation of an explosive encasing a liner. The jet has tremendous penetrating power and this form of terminal effect is utilized where kinetic energy of the projectile is limited.

**Sheathed Core** The central penetrator in some projectiles. It is usually a solid slug of material whose purpose is to penetrate a target by kinetic energy.

**Shell Splinters** Another name for fragments produced when a shell explodes.

**Shot Exit** Sometimes called muzzle exit. This is the moment at which the base of the projectile clears the muzzle or muzzle device attached to a weapon.

**Shot Start** The moment at which the rotating band of a projectile shears and the projectile begins moving into the rifled section of the weapon (separate loaded ammunition) or the moment at which the projectile moves from the cartridge case (fixed ammunition).

**Shrapnel** A projectile, invented by Lt. Henry Shrapnel in 1784 that contained 1-in. diameter steel balls for fragmentation effects. The name became synonymous to shell body fragments when a projectile detonates.

**Shroud Lines** The lines on a parachute connecting the body being supported to the canopy of the parachute.

**Sleigh** Device on a weapon platform that allows the gun tube to move axially during firing, recoil, and during transportation.

**Sling** The fabric or leather strap that allows the weapon to be carried on the back of a soldier.

**Smart Munition** A precision munition that can distinguish between targets and nontargets or countermeasures.

**Soft Recoil** Recoil system where the recoiling parts are accelerated forward to reduce the rearward momentum as the projectile leaves the weapon.

**Spades** Part of the trails on a field piece that dig in to the ground upon firing to arrest the rearward motion of the weapon during recoil.

**Split Rotating Band** A rotating band made up of multiple segments, either located on both the shell wall and base or simply separated by shell wall material.

**Stacking Swivel** A swivel located near the muzzle of a rifle which allows several weapons to be stacked in a pyramid shape, limiting the exposure of the weapons to dirt and corrosion.

**Standoff** The distance between the base of a liner and the intended target. There is an optimum value of the standoff where penetration of a particular shaped charge is optimum.

**Standoff Spike** A cylindrical protrusion at the nose of a HEAT projectile that impacts the target, thus, setting the proper standoff for the formation of a shaped charge jet.

**Stock** The portion of a rifle which supports the barrel and by which the weapon is held.

**Sub-Caliber** A term which describes anything with a diameter smaller than the bourrelet diameter of a projectile or sabot.

**Super-Caliber** A term which describes anything with a diameter larger than the bourrelet diameter of a projectile or sabot.

**Supplementary Charge** A charge added to HE rounds to further amplify the shock from a booster for added assurance that the main fill will detonate properly and completely.

**Swivel** A loop which can either be fixed or pivoted, through which the sling passes and allows the weapon to be carried on the back of a soldier.

**Tangent Ogive** An ogive whose radius begins exactly at the end of the cylindrical section.

**Torsional Impulse** The sudden rotation of a projectile as it engages the rifling after it has acquired some forward velocity (a common condition in worn gun tubes).

**Tracer** A device containing a pyrotechnic mixture which is inserted into the base of training projectiles and some tactical projectiles. The purpose of the tracer is to allow the firer to see where the projectiles are flying. The pyrotechnic composition in a tracer is usually initiated by the propelling charge.

**Trails** Part of a field piece that supports the weapon during firing and allows it to be towed.

**Treeburst** A technique where PD fuzes are fired into trees over an enemies head to maximize fragment lethality.

**Trigger** The device pulled by the finger of an operator to rotate the sear and fire the weapon.

**Tripod** A trio of supports to maintain a weapon such as a recoilless rifle or a machine gun steady. Used primarily when portability is essential.

**Trunnion** Pins on a weapon platform that connect the cradle to the upper carriage and allow the weapon to elevate.

**Undercut** Region of the projectile which separates the bourrelets and is sub-caliber to reduce friction and tube wear.

**Upper Carriage** A fork-like component on a field piece that contains the trunnions and connects to the lower carriage through the pintle.

**Volley Fire** When multiple guns fire simultaneously at the same target. It is called a "Salvo" in Navy parlance.

**VT Fuze (Variable Time Fuze)** An ET fuze that initiates in the vicinity of an object through use of a signal or other means. Sometimes called a Proximity Fuze.

**Wear Additive** A material added to the propelling charge to reduce the wear on the gun tube through either protectively coating the tube, flame temperature reduction, lubrication, or reduction of corrosive reactions.

**Wheel Base** Distance between forward and aft bourrelets, the size of the wheel base affects stability in the tube.

**White Phosphorous (WP)** Smoke producing compound used in smoke rounds which produces a very dense obscuring smoke. White Phosphorous reacts with air when exposed and tends to burn very hot, creating an updraft which tends to lift the smoke skyward which is not very desirable. Despite this, it is used frequently.

**Windshield** A device used to make a projectile more aerodynamically efficient by reducing drag. Sometimes called a ballistic cap.

**Wings** Lifting surfaces mounted near the center of gravity or center of pressure.

**Wooden Round** A projectile that does not require maintenance over its lifetime.

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## B. Tabulated Properties of Materials

The properties given in the following tables have been assembled from the references at the end of this section. Although not complete, these represent sufficient values to do the problems included in the text. Since this is not a thermodynamics or combustion text, the tables are coarse. None of the problems in this text requires interpolation between values in these tables. In fact, "never" interpolate with these tables. If the reader is performing an analysis that requires more refined tables, the authors suggest any of the texts in the references.

We have used the SI system for the tables since that was common among the references. The reader will also note that the specific internal energies and enthalpies contain an overbar—indicating that they are on a molar basis. This is reinforced in the units.

**TABLE B.1**  
Enthalpies of Formation for Select Materials

Material	Enthalpy of Formation ( $\bar{h}_f^0$ ) <sub>298</sub> (kJ/kg-mol)	Molecular Weight (MW) (kg/kg-mol)
Carbon monoxide (CO)	-110,541	28.010
Carbon dioxide (CO <sub>2</sub> )	-393,546	44.011
Hydrogen (H <sub>2</sub> )	0	2.016
Hydrogen, atomic (H)	217,997	1.008
Hydroxyl (OH)	38,985	17.007
Water (H <sub>2</sub> O)	-241,845	18.016
Nitrogen (N <sub>2</sub> )	0	28.013
Nitrogen, atomic (N)	472,629	14.007
Nitric oxide (NO)	90,297	30.006
Nitrogen dioxide (NO <sub>2</sub> )	33,098	46.006
Oxygen (O <sub>2</sub> )	0	31.999
Oxygen, atomic (O)	249,197	16.000
Carbon, solid (C)	0	12.010
Air	n/a	28.97

**TABLE B.2**  
Ideal Gas Properties of Carbon Monoxide (CO)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	110,541	0
500	116,484	5,943
1,000	132,238	21,697
1,500	149,388	38,847
2,000	167,278	56,737
2,500	185,577	75,036
3,000	204,103	93,562
3,500	222,776	112,235
4,000	241,573	131,032
4,500	260,489	149,948

**TABLE B.3**  
Ideal Gas Properties of Carbon Dioxide (CO<sub>2</sub>)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	393,546	0
500	401,847	8,301
1,000	426,971	33,425
1,500	455,227	61,681
2,000	484,966	91,420
2,500	515,490	121,944
3,000	546,437	152,891
3,500	577,666	184,120
4,000	609,159	215,613
4,500	640,919	247,373

**TABLE B.4**Ideal Gas Properties of Hydrogen ( $\text{H}_2$ )

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	0	0
500	5,874	5,874
1,000	20,664	20,664
1,500	36,307	36,307
2,000	52,968	52,968
2,500	70,492	70,492
3,000	88,733	88,733
3,500	107,566	107,566
4,000	126,897	126,897
4,500	146,672	146,672

**TABLE B.5**

Ideal Gas Properties of Atomic Hydrogen (H)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-217,997	0
500	-213,801	4,196
1,000	-203,408	14,589
1,500	-193,015	24,982
2,000	-182,622	35,375
2,500	-172,229	45,768
3,000	-161,836	56,161
3,500	-151,443	66,554
4,000	-141,050	76,947
4,500	-130,657	87,340

**TABLE B.6**Ideal Gas Properties of Hydroxyl ( $\text{OH}$ )

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-38,985	0
500	-32,984	6,001
1,000	-18,057	20,928
1,500	-2,125	36,860
2,000	14,791	53,776
2,500	32,435	71,420
3,000	50,605	89,590
3,500	69,152	108,137
4,000	87,977	126,962
4,500	107,023	146,008

**TABLE B.7**Ideal Gas Properties of Water ( $\text{H}_2\text{O}$ )

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	241,845	0
500	248,792	6,947
1,000	267,838	25,993
1,500	290,026	48,181
2,000	314,650	72,805
2,500	340,957	99,112
3,000	368,408	126,563
3,500	396,640	154,795
4,000	425,427	183,582
4,500	454,635	212,790

**TABLE B.8**Ideal Gas Properties of Nitrogen ( $\text{N}_2$ )

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	0	0
500	5,920	5,920
1,000	21,468	21,468
1,500	38,404	38,404
2,000	56,130	56,130
2,500	74,305	74,305
3,000	92,730	92,730
3,500	111,315	111,315
4,000	130,028	130,028
4,500	148,860	148,860

**TABLE B.9**

Ideal Gas Properties of Atomic Nitrogen (N)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-472,629	0
500	-468,433	4,196
1,000	-458,040	14,589
1,500	-447,644	24,985
2,000	-437,253	35,376
2,500	-426,858	45,771
3,000	-416,416	56,213
3,500	-405,857	66,772
4,000	-395,092	77,537
4,500	-384,016	88,613

**TABLE B.10**

Ideal Gas Properties of Nitric Oxide (NO)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-90,297	0
500	-84,218	6,079
1,000	-68,056	22,241
1,500	-50,565	39,732
2,000	-32,440	57,857
2,500	-13,966	76,331
3,000	4,698	94,995
3,500	23,487	113,784
4,000	42,383	132,680
4,500	61,384	151,681

**TABLE B.11**Ideal Gas Properties of Nitrogen Dioxide (NO<sub>2</sub>)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-33,098	0
500	-24,980	8,118
1,000	-723	32,375
1,500	26,197	59,295
2,000	54,149	87,247
2,500	82,581	115,679
3,000	111,211	144,309
3,500	139,940	173,038
4,000	168,763	201,861
4,500	197,685	230,783

**TABLE B.12**Ideal Gas Properties of Oxygen (O<sub>2</sub>)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	0	0
500	6,097	6,097
1,000	22,721	22,721
1,500	40,590	40,590
2,000	59,169	59,169
2,500	78,346	78,346
3,000	98,036	98,036
3,500	118,173	118,173
4,000	138,705	138,705
4,500	159,586	159,586

**TABLE B.13**

Ideal Gas Properties of Atomic Oxygen (O)

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	-249,197	0
500	-244,852	4,345
1,000	-234,336	14,861
1,500	-223,898	25,299
2,000	-213,485	35,712
2,500	-203,070	46,127
3,000	-192,623	56,574
3,500	-182,116	67,081
4,000	-171,519	77,678
4,500	-160,811	88,386

**TABLE B.14**

Ideal Gas Properties of Carbon (Graphite) (C)—in Solid Form

Temperature (K)	$\bar{h}(T)$ (kJ/kg-mol)	$\bar{h}(T) - (\bar{h}_{298}^0)$ (kJ/kg-mol)
298	0	0
500	2,365	2,365
1,000	11,795	11,795
1,500	23,253	23,253
2,000	35,525	35,525
2,500	48,289	48,289
3,000	61,427	61,427
3,500	74,889	74,889
4,000	88,646	88,646
4,500	102,685	102,685

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