Implicit Function Theorem

Suppose we are interested in the solutions to the equation $F(x,y)=c$. For instance, perhaps $F(x,y)=x^2+y^2$ and $c=1$. It would be nice if choosing a value for x in the equation $F(x,y)=c$ would immediately determine the value of y — that is, if $F(x,y)=c$ determined y as a function of x. But we know that this isn't generally true. In the case of the unit circle, fixing a value x leaves both $\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$ as possibilities for the value of y. Graphically, this obstruction is represented by the fact that $x^2+y^2=1$ fails the familiar vertical line test, as can be seen in Figure 1.

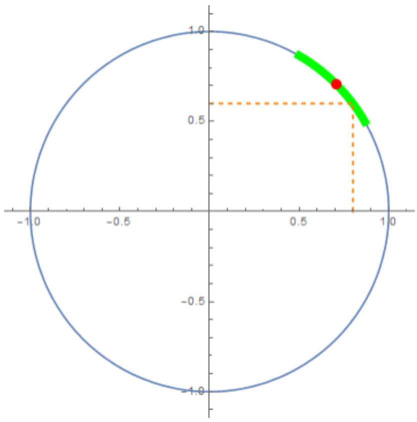

Figure 1. The level curve $x^2+y^2=1$. The green segment represents a neighborhood of the red point on which y is determined by x.

The Implicit Function Theorem requires the function to have a nonzero partial derivative in the y direction. Loosely speaking, it ensures that if you are on the curve and slightly nudge x to the left or right, there is always another slight nudge of y up or down such that you are still on the curve.

Now consider the point $(1,0)$. If we nudge x ever so slightly to the right, let's say to the point $1+h$, with $h\to 0$. Can we find a value of y such that we are still on the curve? That means that we would need $(1+h)^2+y^2-1=0$ to hold, which (if we expand the brackets) means that $1+2h+h^2+y^2-1=0$, or $y^2=-2h-h^2=-h(h+2)$. However, since h is positive, y^2 must be negative, which is not possible in the real numbers. Therefore we cannot find a y such that we are still on the curve if we slightly nudge x to the right.

The reason why this is the case, is because the curve that satisfies $f(x,y)=0$ at the point $(1,0)$ is vertical. Another way of saying the curve is vertical is the statement that the partial derivative $\frac{\partial f}{\partial y}(1,0)=0$, which is why the Implicit Function Theorem requires that $\frac{\partial f}{\partial y}(x,y)\neq 0$ at the point of interest.

Definition. The Implicit Function Theorem for $\mathbb{R}^2$. Consider a continuously differentiable function $F\colon\mathbb{R}^2\to\mathbb{R}^2$ and a point $(x_0,y_0)\in\mathbb{R}^2$ so that $F(x_0,y_0)=c$. If $\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$, then there is a neighborhood of (x_0,y_0) so that whenever x is sufficiently close to x_0 there is a unique y so that $F(x,y)=c$. Moreover, this assignment is makes y a continuous function of x.