COMP3411/9814: Artificial Intelligence

Propositions and Inference

Lecture Outline

- Knowledge Representation and Logic
- Logical Arguments
- Propositional Logic
  - Syntax
  - Semantics
- Validity, Equivalence, Satisfiability, Entailment

Knowledge Bases

- A knowledge base is a set of sentences in a formal language.
- Declarative approach to building an agent:
  - Tell the system what it needs to know,
    then it can ask itself what it needs to do
  - Answers should follow from the knowledge based.
- How do you formally specify how to answer questions?

Knowledge Based Agent

The agent must be able to:

- represent states, actions, etc.
- incorporate new percepts
- update internal representations of the world
- deduce hidden properties of the world
- determine appropriate actions

Why Formal Languages (not English, or other natural language)?

- Natural languages are ambiguous: “The fisherman went to the bank” (lexical)
- “The boy saw a girl with a telescope” (structural)
- “The table won’t fit through the doorway because it is too [wide/narrow]” (co-reference)
- Ambiguity makes it difficult to interpret meaning of phrases/sentences
  - But also makes inference harder to define and compute
- Symbolic logic is a syntactically unambiguous language

Syntax vs Semantics

Syntax - legal sentences in knowledge representation language
(e.g. in the language of arithmetic expressions $x < 4$)

Semantics - meaning of sentences.
Refers to a sentence's relationship to the "real world" or to some model of the world.

- Semantic properties of sentences include truth and falsity
  (e.g. $x < 4$ is true for $x = 3$ and false when $x = 5$).
- Semantic properties of names and descriptions include referents.
- The meaning of a sentence is not intrinsic to that sentence.
  - An interpretation is required to determine sentence meanings.
  - Interpretations are agreed amongst a linguistic community.

Propositions

- Propositions are entities (facts or non-facts) that can be true or false

Examples:

- "The sky is blue" - the sky is blue (here and now).
- "Socrates is bald" (assumes 'Socrates', 'bald' are well defined)
  "The car is red" (requires 'the car' to be identified)
- "Socrates is bald and the car is red" (complex proposition)

- Use single letters to represent propositions, e.g. *P*: Socrates is bald
- Reasoning is independent of definitions of propositions

Logical Arguments

An argument relates a set of premises to a conclusion

- valid if the conclusion necessarily follows from the premises

  All humans have 2 eyes
  Jane is a human
  Therefore Jane has 2 eyes

  All humans have 4 eyes
  Jane is a human
  Therefore Jane has 4 eyes

- Both are (logically) correct valid arguments
- Which statements are true/false? Why?

Logical Arguments

An argument relates a set of premises to a conclusion

- invalid if the conclusion can be false when the premises are all true

All humans have 2 eyes
Jane has 2 eyes
Therefore Jane is human

No human has 4 eyes
Jane has 2 eyes
Therefore Jane is not human

- Both are (logically) incorrect invalid arguments
- Which statements are true/false? Why?

Propositional Logic

- Letters stand for "basic" propositions
- Combine into more complex sentences using operators not, and, or, implies, iff
- Propositional connectives:

$\neg$	negation	$\neg P$	"not P"
$\wedge$	conjunction	$P \wedge Q$	"P and Q"
$\vee$	disjunction	$P \vee Q$	"P or Q"
$\rightarrow$	implication	$P \rightarrow Q$	"If P then Q"
$\leftrightarrow$	bi-implication	$P \leftrightarrow Q$	"P if and only if Q"

From English to Propositional Logic

- "It is not the case that the sky is blue": $\neg B$
  (alternatively "the sky is not blue")
- "The sky is blue and the grass is green": $B \wedge G$
- "Either the sky is blue or the grass is green": $B \vee G$
- "If the sky is blue, then the grass is not green": $B \rightarrow \neg G$
- "The sky is blue if and only if the grass is green": $B \leftrightarrow G$
- "If the sky is blue, then if the grass is not green, the plants will not grow":
  $B \rightarrow (\neg G \rightarrow \neg P)$

Improving Readability

- $(P \to (Q \to (\neg(R)))$ vs $P \to (Q \to \neg R)$
- Rules for omitting parentheses
  - Omit parentheses where possible
  - Precedence from highest to lowest is: $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$
  - All binary operators are left associative
    – so $P \to Q \to R$ abbreviates $(P \to Q) \to R$
- Sometimes parentheses can't be removed:
  - Is $(P \vee Q) \vee R$ (always) the same as $P \vee (Q \vee R)$?
  - Is $(P \to Q) \to R$ (always) the same as $P \to (Q \to R)$? **NO!**
- https://web.stanford.edu/class/cs103/tools/truth-table-tool/

P	Q	R	((P → Q) → R)	(P → (Q → R))
F	F	F	**F**	**T**
F	F	T	**T**	**T**
F	T	F	**F**	**T**
F	T	T	**T**	**T**
T	F	F	**T**	**T**
T	F	T	**T**	**T**
T	T	F	**F**	**F**
T	T	T	**T**	**T**

Truth Table Semantics

- The semantics of the connectives can be given by truth tables

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
True	True	False	True	True	True	True
True	False	False	False	True	False	False
False	True	True	False	True	True	False
False	False	True	False	False	True	True

- One row for each possible assignment of True/False to variables
- Important: P and Q are any sentences, including complex sentences

Example – Complex Sentence

R	S		$\neg R$	$R \wedge S$	$\neg R \vee S$	$(R \wedge S) \rightarrow (\neg R \vee S)$
True	True		False	True	True	True
True	False		False	False	False	True
False	True		True	False	True	True
False	False		True	False	True	True

Thus $(R \wedge S) \rightarrow (\neg R \vee S)$ is a tautology

https://web.stanford.edu/class/cs103/tools/truth-table-tool/

Definitions

- A sentence is valid if it is True under all possible assignments of True/False to its variables (e.g. $P \vee \neg P$)
- A tautology is a valid sentence
- Two sentences are equivalent if they have the same truth table, e.g. $P \wedge Q$ and $Q \wedge P$
  - So P is equivalent to Q if and only if $P \leftrightarrow Q$ is valid
- A sentence is satisfiable if there is some assignment of True/False to its variables for which the sentence is True
- A sentence is unsatisfiable if it is not satisfiable (e.g. $P \wedge \neg P$)
  - Sentence is False for all assignments of True/False to its variables
  - So P is a tautology if and only if $\neg P$ is unsatisfiable

Material Implication

- $P \rightarrow Q$ evaluates to False only when P is True and Q is False
- $P \rightarrow Q$ is equivalent to $\neg P \lor Q$: material implication
- English usage often suggests a causal connection between antecedent (P) and consequent (Q) – this is not reflected in the truth table
- All these are tautologies
  - $(P \land Q) \rightarrow Q$
  - $P \rightarrow (P \lor Q)$
  - $(P \land \neg P) \rightarrow Q$

Material Implication

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- English usage often suggests a causal connection between antecedent (P) and consequent (Q) – this is not reflected in the truth table
- All these are tautologies
  - $(P \wedge Q) \longrightarrow Q = \neg(P \wedge Q) \vee Q = \neg P \vee \neg Q \vee Q = T$
  - $P \longrightarrow (P \vee Q) = \neg P \vee P \vee Q = T$
  - $(P \wedge \neg P) \longrightarrow Q = \neg(P \wedge \neg P) \vee Q = \neg P \vee P \vee Q = T$

Logical Equivalences – All Valid

Commutativity:	$p \wedge q \leftrightarrow q \wedge p$	$p \vee q \leftrightarrow q \vee p$
Associativity:	$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$	$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$
Distributivity:	$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
Implication:	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	
Idempotent:	$p \wedge p \leftrightarrow p$	$p \vee p \leftrightarrow p$
Double negation:	$\neg\neg p \leftrightarrow p$	
Contradiction:	$p \wedge \neg p \leftrightarrow \text{FALSE}$	
Excluded middle:		$p \vee \neg p \leftrightarrow \text{TRUE}$
De Morgan:	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

Proof of Equivalence

Let $P \Leftrightarrow Q$ mean "P is equivalent to Q" ($P \Leftrightarrow Q$ is not a formula)

Then $P \wedge (Q \rightarrow R) \Leftrightarrow \neg(P \rightarrow Q) \vee (P \wedge R)$

$$
\begin{aligned}
P \wedge (Q \rightarrow R) &\Leftrightarrow P \wedge (\neg Q \vee R) && \text{[Implication]} \\
&\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge R) && \text{[Distributivity]} \\
&\Leftrightarrow (\neg\neg P \wedge \neg Q) \vee (P \wedge R) && \text{[Double negation]} \\
&\Leftrightarrow \neg(\neg P \vee Q) \vee (P \wedge R) && \text{[De Morgan]} \\
&\Leftrightarrow \neg(P \rightarrow Q) \vee (P \wedge R) && \text{[Implication]}
\end{aligned}
$$

Assumes substitution: if $A \Leftrightarrow B$, replace A by B in any subformula

Assumes equivalence is transitive: if $A \Leftrightarrow B$ and $B \Leftrightarrow C$ then $A \Leftrightarrow C$

Interpretations and Models

- An interpretation is an assignment of values to all variables.
- A model is an interpretation that satisfies the constraints.
  - A model is a possible world in which a sentence (or set of sentences) is true, e.g.
  - $x + y = 4$ in a world where $x = 2$ and $y = 2$
  - May be more than one possible world (e.g. $x = 3$ and $y = 1$)
- Often want to know what is true in all models.
- A proposition is statement that is true or false in each interpretation.

Entailment

- Entailment means that one sentence follows logically from another sentence, or set of sentences (i.e. a knowledge base):

$$KB \models \alpha$$

- Knowledge base KB entails sentence α if and only if α is true in all models (possible worlds) where KB is true.

e.g. the KB containing "the Moon is full" and "the tide is high" entails "Either the Moon is full or the tide is high".

e.g. $x + y = 4$ entails $4 = x + y$

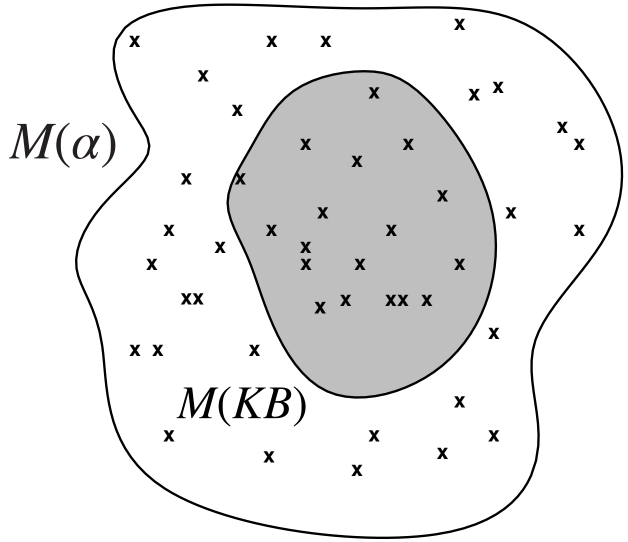
- Entailment is a relationship between sentences based on semantics.

Models

- For propositional logic, a model is one row of the truth table
- A model M is a model of a sentence α if α is True in M

Let $M(\alpha)$ be the set of all models of α

Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$



Entailment

- S entails P ($S \models P$) if whenever all formulae in S are True, P is True
  - Semantic definition – concerns truth (not proof)

- Compute whether $S \models P$ by calculating a truth table for S and P
  - Syntactic notion – concerns computation/proof
  - Not always this easy to compute (how inefficient is this?)

- A tautology is a special case of entailment where S is the empty set
  - All rows of the truth table are True

Entailment Example

P	Q	$P \rightarrow Q$		Q
True	True	True		True
True	False	False		False
False	True	True		True
False	False	True		False

- $\{P, P \rightarrow Q\} \models Q$ since when both P and $P \rightarrow Q$ are True (row 1), Q is also True

- $P \rightarrow Q$ is calculated from P and Q using the truth table definition, and Q is used again to check the entailment

Example – $S \vDash P$

Each row is an interpretation of S.
Only the first row is a model of S.

S = {p → q, q → p, p ∨ q}

P = p ∧ q

p	q	p → q	q → p	p ∨ q	S	p ∧ q
T	T	T	T	T	T	T
T	F	F	T	T	F	~~F~~
F	T	T	F	T	F	~~F~~
F	F	T	T	F	F	~~F~~

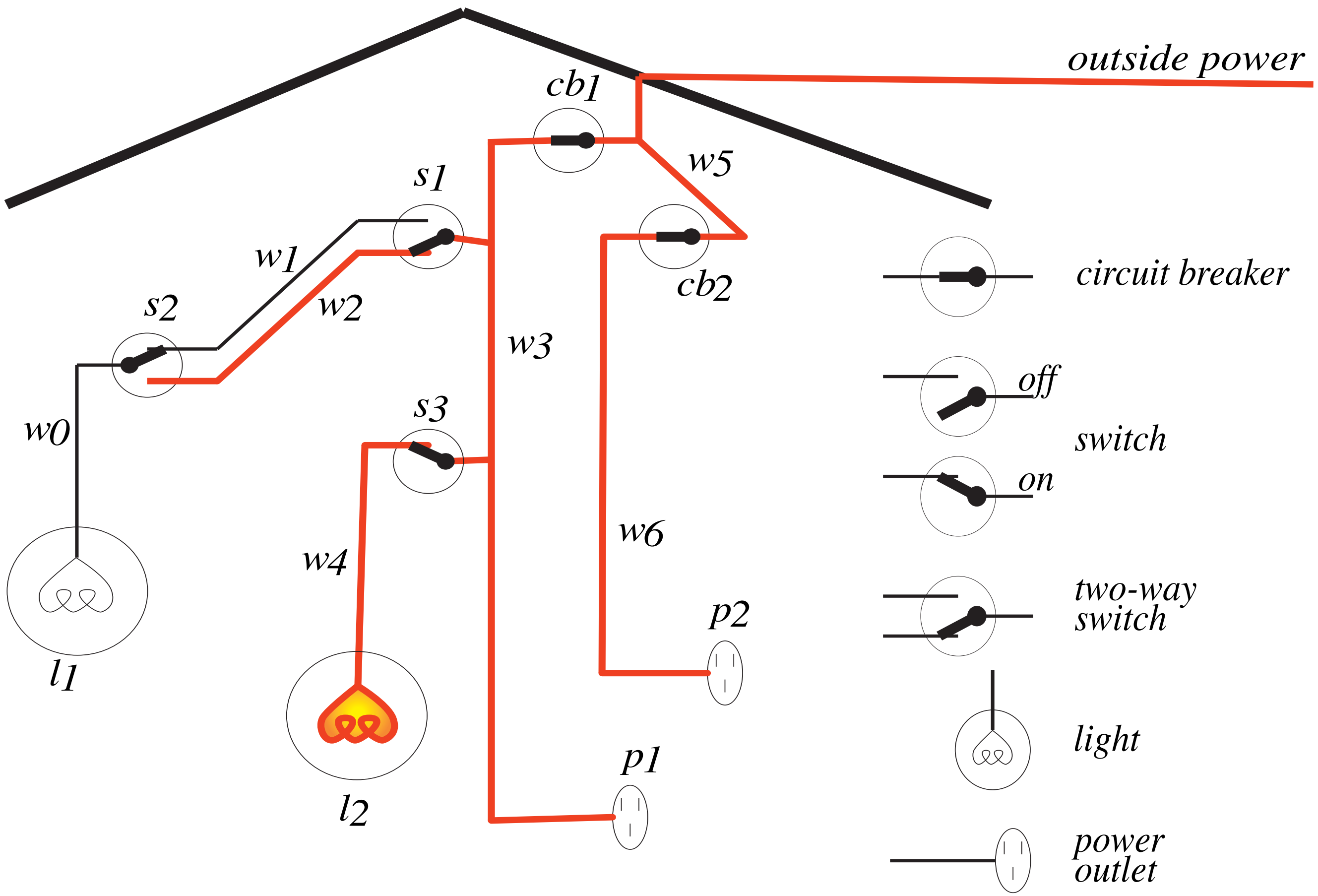
Example – $S \vDash P$

S = {q ∨ r, q → ~p, ¬(r ∧ p) }

P = ¬p

p	q	r	q ∨ r	q → ¬p	¬(r ∧ p)	S	¬p
T	T	T	T	F		F	
T	T	F	T	F		F	
T	F	T	T	T	F	F	
T	F	F	F			F	
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F			F	

Example - Modelling Electrical Circuits



Electrical Circuit in Proposition Logic

$light\_l_1.$

$light\_l_2.$

$down\_s_1.$

$up\_s_2.$

$up\_s_3.$

$ok\_l_1.$

$ok\_l_2.$

$ok\_cb_1.$

$ok\_cb_2.$

$live\_outside.$

$lit\_l_1 \leftarrow live\_w_0 \wedge ok\_l_1$

$live\_w_0 \leftarrow live\_w_1 \wedge up\_s_2.$

$live\_w_0 \leftarrow live\_w_2 \wedge down\_s_2.$

$live\_w_1 \leftarrow live\_w_3 \wedge up\_s_1.$

$live\_w_2 \leftarrow live\_w_3 \wedge down\_s_1.$

$lit\_l_2 \leftarrow live\_w_4 \wedge ok\_l_2.$

$live\_w_4 \leftarrow live\_w_3 \wedge up\_s_3.$

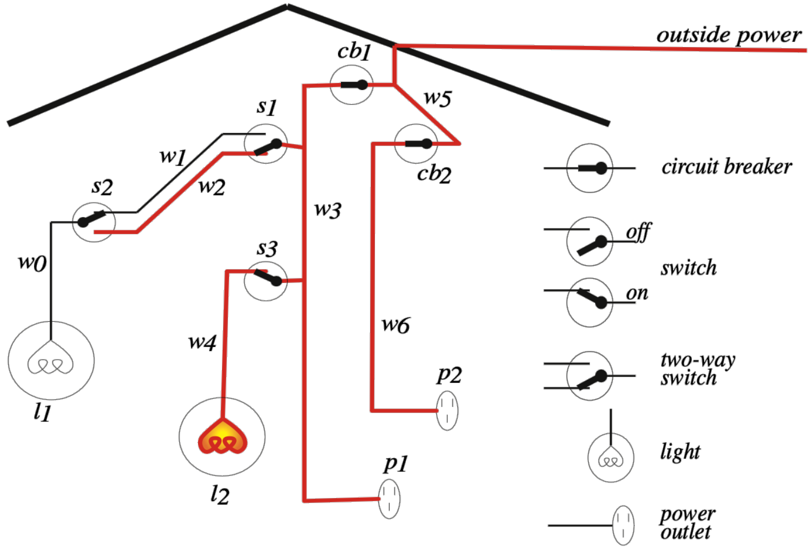
$live\_p_1 \leftarrow live\_w_3.$

$live\_w_3 \leftarrow live\_w_5 \wedge ok\_cb_1.$

$live\_p_2 \leftarrow live\_w_6.$

$live\_w_6 \leftarrow live\_w_5 \wedge ok\_cb_2.$

$live\_w_5 \leftarrow live\_outside.$



Conclusion

- Ambiguity of natural languages avoided with formal languages
- Enables formalisation of (truth preserving) entailment
- Propositional Logic: Simplest logic of truth and falsity
- Knowledge Based Systems: First-Order Logic
- Automated Reasoning: How to compute entailment (inference)
- Many many logics not studied in this course