1. INTRODUCTION

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Table of Contents

Prerequisites

- Understanding of fundamental data structures and algorithms
  - Arrays, trees, heaps, sorting, searching, etc.
- Written communication skills
  - No programming required - see COMP4121 and COMP4128
- Desirable (but not officially required)
  - MATH1081 Discrete Mathematics (proofs, graphs)
  - MATH1131/1141 (matrices, complex numbers, limits)

Classes

- Lectures (weeks 1–10)
  - Wednesday 16:00 - 18:00
  - Friday 14:00 - 16:00
  - In week 6 (flexibility week), we will hold a revision lecture. Attendance is not required.
  - Slides on Moodle.
- Consultation (weeks 1–10)
  - Thursday 11:00 - 12:00
  - Friday 17:00 - 18:00
  - Exam consultation TBA.
- Lecture and consultation recordings will be published on YouTube.

Getting Help

- Join the Ed forum! Link in Moodle welcome announcement.
- No tutorials or labs.
- Trial: peer tutorial
  - Every weekday from Thursday week 1
  - 11:00 - 12:00 and 17:00 - 18:00, except Wed and Fri PM
  - Primarily to discuss tutorial problems
  - Voluntary participation
  - Will be moderated but not recorded

Assessment

- Assignments
  - Four assignments, released approx bi-weekly
  - Each consists of 5 questions
  - Each weighted 10% of course mark
- Final Exam
  - Approx 5 questions, precise format TBA
  - Weighted 60% of course mark

Textbooks

Recommended textbook

Kleinberg and Tardos: *Algorithm Design*
paperback edition available at UNSW Bookshop

- excellent: very readable textbook (and very pleasant to read!);
- not so good: as a reference manual for later use.

An alternative textbook

Cormen, Leiserson, Rivest and Stein: *Introduction to Algorithms*
preferably the third edition, also available at UNSW Bookshop

- excellent: to be used later as a reference manual;
- not so good: somewhat formalistic and written in a rather dry style.

Table of Contents

Introduction

What is this course about?

It is about **designing algorithms** for solving practical problems.

What is an algorithm?

- An algorithm is a collection of precisely defined steps that are executable using certain specified mechanical methods.
- By "mechanical" we mean the methods that do not involve any creativity, intuition or even intelligence. Thus, algorithms are specified by detailed, easily repeatable "recipes".
- The word "algorithm" comes by corruption of the name of Muhammad ibn Musa al-Khwarizmi, a Persian scientist 780–850, who wrote an important book on algebra, *"Al-kitab al-mukhtasar fi hisab al-gabr wa'l-muqabala"*.

Introduction

In this course we will deal only with sequential deterministic algorithms which means that:

- they are given as sequences of steps, thus assuming that only one step can be executed at a time;
- the action of each step gives the same result whenever this step is executed for the same input.

Introduction

Why should you study algorithms design?

Can you find every algorithm you might need using Google?

Our goal:

To learn **techniques** which can be used to solve **new, unfamiliar** problems that arise in a rapidly changing field.

Course content:

- a survey of algorithm **design techniques**
- particular algorithms will be mostly used to illustrate design techniques
- emphasis on development of your algorithm design **skills**

Example: Two Thieves

Problem

Two thieves have robbed a warehouse and have to split a pile of items without price tags on them. Design an algorithm to split the pile so that each thief **believes** that they have got at least half the loot.

Solution

One of the two thieves splits the pile in two parts, so that they believe that both parts are of equal value. The other thief then chooses the part that they believe is no worse than the other.

- The hard part: how can a thief split the pile into two equal parts? Remarkably, there is no known algorithm that is more efficient than the brute force: we consider all partitions of the pile and see if there is one which results in two equal parts.

Example: Three Thieves

Problem

Three thieves have robbed a warehouse and have to split a pile of items without price tags on them. How do they do this in a way that ensures that each thief believes that they have got at least one third of the loot?

- The problem is much harder with 3 thieves!
- Let us try do the same trick as in the case of two thieves. Say the first thief splits the loot into three piles which they think are of equal value; then the remaining two thieves choose which pile they want to take.
- If they choose different piles, they can each take the piles they have chosen and the first thief gets the remaining pile; in this case clearly each thief thinks that they got at least one third of the loot.

Example: Three Thieves

- But what if the remaining two thieves choose the same pile?
- One might think that in this case the first thief can pick either of the other two piles, after which the remaining two piles are put together and the two remaining thieves split them as in Problem 1 with only two thieves.
- Unfortunately this does not work!

Example: Three Thieves

- Suppose that the first thief splits the loot into three piles A, B, C, and that the second thief thinks that

$$A = 50\%, B = 40\%, C = 10\%$$

of the total value, while the third thief thinks that

$$A = 50\%, B = 10\%, C = 40\%.$$

- Clearly both the second and third thieves choose pile A, so the first thief can choose pile B or C.
- However, if the first thief picks pile B, then the second thief will object that (in their eyes) only 60% of the loot remains, so they are not guaranteed to get at least one-third of the total.
- If the first thief picks pile C then the third thief will object for the same reason.
- What would be a correct algorithm?

Example: Three Thieves

Algorithm:

T_1 makes a pile P_1 which T_1 believes is $1/3$ of the whole loot;
T_1 proceeds to ask T_2 if they agree that $P_1 \leq 1/3$;

If T_2 says YES, **then** T_1 asks T_3 if T_3 agrees that $P_1 \leq 1/3$;

- **If** T_3 says YES, **then** T_1 takes P_1;
  T_2 and T_3 split the rest as in Problem 1.
- **Else** if T_3 says NO, **then** T_3 takes P_1;
  T_1 and T_2 split the rest as in Problem 1.

Else if T_2 says NO, **then** T_2 reduces the size of P_1 to $P_2 < P_1$ such that T_2 thinks $P_2 = 1/3$;
T_2 then proceeds to ask T_3 if they agree that $P_2 \leq 1/3$;

- **If** T_3 says YES **then** T_2 takes P_2;
  T_1 and T_3 split the rest as in Problem 1.
- **Else** if T_3 says NO then T_3 takes P_2;
  T_1 and T_2 split the rest as in Problem 1.

Example: n Thieves

Exercise

Try generalising this to n thieves! (a bit harder than with three thieves!)

Hint

There is a *nested recursion* happening even with 3 thieves!

Table of Contents

The role of proofs in algorithm design

Question

When do we need to give a **mathematical proof** that an algorithm we have just designed terminates and returns a solution to the problem at hand?

Answer

When this is not obvious by inspecting the algorithm using common sense!

Mathematical proofs are **NOT** academic embellishments; we use them to justify things which are not obvious to common sense!

Example: $\textsc{Merge-Sort}$

Algorithm

$\textsc{Merge-Sort}(A, \ell, r)$ *sorting $A[\ell..r]$*

1. **if** $\ell < r$
2. **then** $m \leftarrow \left\lfloor \frac{\ell + r}{2} \right\rfloor$
3. $\textsc{Merge-Sort}(A, \ell, m)$
4. $\textsc{Merge-Sort}(A, m + 1, r)$
5. $\textsc{Merge}(A, \ell, m, r)$

Example: Merge-Sort

- The depth of recursion in Merge-Sort is $\log_2 n$.
- On each level of recursion, merging all the intermediate arrays takes $O(n)$ steps in total.
- Thus, Merge-Sort always terminates, and in fact it terminates in $O(n \log_2 n)$ steps.
- Merging two sorted arrays always produces a sorted array, thus, the output of Merge-Sort will be a sorted array.
- The above is essentially a proof by induction, but we will never bother formalising proofs of (essentially) obvious facts.

The role of proofs in algorithm design

- However, sometimes it is **NOT** clear from a description of an algorithm that such an algorithm will not enter an infinite loop and fail to terminate.

- Sometimes it is **NOT** clear that an algorithm will not run in exponentially many steps (in the size of the input), which is usually almost as bad as never terminating.

- Sometimes it is **NOT** clear from a description of an algorithm why such an algorithm, after it terminates, produces a desired solution.

The role of proofs in algorithm design

- Proofs are needed for such circumstances; in a lot of cases they are **the only way** to know that the algorithm does the job.

- For that reason we will **NEVER** prove the obvious (the CLRS textbook sometimes does just that, by sometimes formulating and proving trivial little lemmas, being too pedantic!). We will prove only what is genuinely nontrivial.

- **However, BE VERY CAREFUL what you call trivial!!**

Table of Contents

Stable Matching Problem

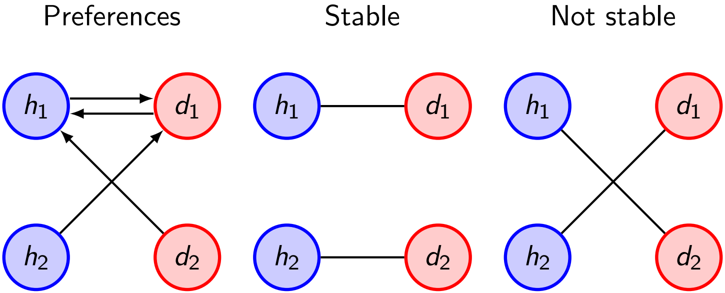
- Suppose there are n hospitals in the state, and n new doctors have graduated from university. Each hospital wants to hire exactly one new doctor.
- Every hospital submits a list of preferences, which ranks all the doctors, **and** every doctor submits a list of preferences, which ranks all the hospitals.
- Design an algorithm which produces a *stable matching*, which is: a set of n pairs $p = (h, d)$ of a hospital h and a doctor d so that the following situation never happens:

  for two pairs $p = (h, d)$ and $p' = (h', d')$:
  - hospital h prefers doctor d' to doctor d, **and**
  - doctor d' prefers hospital h to hospital h'.

Stable Matching Problem: Example 1

$h_1 : d_1, d_2$ $\qquad$ $d_1 : h_1, h_2$

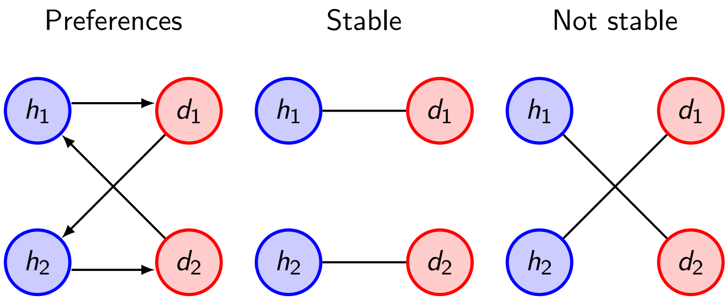
$h_2 : d_1, d_2$ $\qquad$ $d_2 : h_1, h_2$



Stable Matching Problem: Example 2

$h_1 : d_1, d_2$ $\quad$ $d_1 : h_2, h_1$

$h_2 : d_2, d_1$ $\quad$ $d_2 : h_1, h_2$



Stable Matching Problem: $n = 2$

Exercise

Up to symmetry, there are two more cases to consider. Solve them.

The following rules cover all situations with two hospitals and two doctors.

- If the hospitals prefer different doctors, assign each hospital their preferred doctor. Neither hospital wants to swap, so the matching is stable.
  - The same applies vice versa, i.e. if the doctors prefer different hospitals.
- However, if both hospitals prefer the same doctor d **and** both doctors prefer the same hospital h, pair h with d and the other hospital with the other doctor. Neither h nor d wants to swap, so the matching is stable.

Stable Matching Problem: Gale - Shapley algorithm

Question

Given n hospitals and n doctors, how many ways are there to match them, without regard for preferences?

Answer

$n! \approx (n/e)^n$ - more than exponentially many in n ($e \approx 2.71$).

Question

Is it true that for every possible collection of n lists of preferences provided by all hospitals, and n lists of preferences provided by all doctors, a stable matching always exists?

Answer

YES, but this is **NOT** obvious!

Stable Matching Problem: Gale - Shapley algorithm

Question

Can we find a stable matching in a reasonable amount of time?

Answer

YES, using the **Gale - Shapley algorithm**.

- Produces pairs in stages, with possible revisions
- A hospital which has not been paired with a doctor will be called *free*.
- Hospitals will be offering jobs to doctors. Doctors will decide if they accept a proposal or not.
- Start with all hospitals free.

Stable Matching Problem: Gale - Shapley algorithm

While there exists a free hospital which has not offered jobs to all doctors, pick such a free hospital h and have it offer a job to the highest ranking doctor d on its list to whom it has not offered a job yet;

If no one has proposed to d yet
they always accept and a pair $p = (h, d)$ is formed;

Else they are already in a pair $p' = (h', d)$;

If h is higher on her preference list than h':
the pair $p' = (h', d)$ is deleted,
h' becomes a free hospital, and
a new pair $p = (h, d)$ is formed;

Else h is lower on their preference list than h':
the job offer is rejected and h remains free.

Stable Matching Problem: Gale - Shapley algorithm

Claim 1

The algorithm terminates after $\leq n^2$ rounds of the *While* loop.

Proof

- In every round of the *While* loop one hospital offers a job to one doctor.
- Every hospital can make an offer to a doctor at most once.
- Thus, every hospital can make at most n offers.
- There are n hospitals, so in total they can make $\leq n^2$ offers.
- Thus the *While* loop can be executed no more than n^2 many times.

Stable Matching Problem: Gale - Shapley algorithm

Claim 2

The algorithm produces a matching, i.e., every hospital is eventually paired with a doctor (and thus also every doctor is paired to a hospital).

Proof

- Assume that the while *While* loop has terminated, but hospital h is still free.
- This means that h has already offered a job to every doctor.
- Thus, every doctor is paired with a hospital, because a doctor is not paired only if no hospital has offered them a job.
- But this would mean that n doctors are paired with all of n hospitals so h cannot be free. **Contradiction!**

Stable Matching Problem: Gale - Shapley algorithm

Claim 3

The matching produced by the algorithm is stable.

Proof

Note that during the *While* loop:

- a doctor is paired with hospitals of increasing ranks on their list, and
- a hospital is paired with doctors of decreasing ranks on its list.

Assume now the opposite, i.e. that the matching is not stable. Thus, there are two pairs $p = (h, d)$ and $p' = (h', d')$ such that:

$$h \text{ prefers } d' \text{ over } d;$$
$$d' \text{ prefers } h \text{ over } h'.$$

Stable Matching Problem: Gale - Shapley algorithm

Proof (continued)

- Since h prefers d' over d, it must have made an offer to d' before offering the job to d.
- Since h is paired with d, doctor d' must have either:
  - rejected h because they were already at a hospital they prefer to h, or
  - accepted h only to later rescind this and accept an offer from a hospital they prefer to h.
- In both cases d would now be at a hospital which they prefer over h. **Contradiction!**

Table of Contents

A Puzzle!!!

Why puzzles? It is a fun way to practice problem solving!

Problem

Tom and his wife Mary went to a party where nine more couples were present.

- Not every one knew everyone else, so people who did not know each other introduced themselves and shook hands.
- People who knew each other from before did not shake hands.
- Later that evening Tom got bored, so he walked around and asked all other guests (including his wife) how many hands they had shaken that evening, and got 19 different answers.
- How many hands did Mary shake?
- How many hands did Tom shake?

That's All, Folks!!