

Problem 1

Let A be a local ring, M and N finitely generated A -modules.

Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

[Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field.

Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.

By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.

But $M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0 \Rightarrow M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Problem 2

Let M be a finitely generated A -module and $\phi : M \rightarrow A^n$ a surjective homomorphism. Show that $\text{Ker}(\phi)$ is finitely generated.

[Let e_1, \dots, e_n be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ ($1 \leq i \leq n$). Show that M is the direct sum of $\text{Ker}(\phi)$ and the submodule generated by u_1, \dots, u_n .]

Problem 3

Let \mathfrak{a} be an ideal of a ring A , and let $S = 1 + \mathfrak{a}$.

Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

[If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Problem 4

The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A .

Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14).

Show that every minimal prime ideal of A is contained in D . [Use Exercise 6.]

The ring $S_0^{-1}A$ is called the *total ring of fractions* of A .

Prove that

- i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \rightarrow S_0^{-1}A$ is injective.
- ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.
- iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \rightarrow S_0^{-1}A$ is bijective).

Problem 5

[Nakayama's lemma]

Let R be any ring and M a finitely generated module.

Let N be the radical of R .

If $NM = M$ show that $M = 0$.

[Hint: Observe that the proof of Nakayama's lemma still holds.]

Problem 6

- (a) Let J be a two-sided nilpotent ideal of R . Show that J is contained in the radical.
- (b) Conversely, assume that R is Artinian. Show that its radical is nilpotent, i.e., that there exists an integer $r \geq 1$ such that $N^r = 0$. [Hint: Consider the descending sequence of powers N^r , and apply Nakayama to a minimal finitely generated left ideal $L \subset N^\infty$ such that $N^\infty L \neq 0$.]