

Problem 1

Let M be a module over the integral domain R .

- (a) Suppose that M has rank n and that x_1, x_2, \dots, x_n is any maximal set of linearly independent elements of M .

Let $N = Rx_1 + \dots + Rx_n$ be the submodule generated by x_1, x_2, \dots, x_n .

Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R -module (equivalently, the elements x_1, \dots, x_n are linearly independent and for any $y \in M$ there is a nonzero element $r \in R$ such that ry can be written as a linear combination $r_1x_1 + \dots + r_nx_n$ of the x_i).

- (b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \cong R^n$) such that the quotient M/N is a torsion R -module then M has rank n .

[Let y_1, y_2, \dots, y_{n+1} be any $n+1$ elements of M .

Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements r_1, \dots, r_{n+1} of R .

Use an argument as in the proof of Proposition 3 to see that the $r_i y_i$, and hence also the y_i , are linearly dependent.]

Problem 2

Let R be any ring, let A_1, A_2, \dots, A_m be R -modules and let B_i be a submodule of A_i , $1 \leq i \leq m$. Prove that

$$(A_1 \oplus A_2 \oplus \dots \oplus A_m) / (B_1 \oplus B_2 \oplus \dots \oplus B_m) \cong (A_1/B_1) \oplus (A_2/B_2) \oplus \dots \oplus (A_m/B_m).$$

Problem 3

Let R be a P.I.D., let a be a nonzero element of R and let $M = R/(a)$.

For any prime p of R prove that

$$p^{k-1}M/p^kM \cong \begin{cases} R/(p) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

where n is the power of p dividing a in R .

Problem 4

The following set of exercises outlines a proof of Theorem 5 in the special case where R is a Euclidean Domain using a matrix argument involving row and column operations.

This applies in particular to the cases $R = \mathbb{Z}$ and $R = F[x]$ of interest in the applications and is computationally useful.

Let R be a Euclidean Domain and let M be an R -module.

Prove that M is finitely generated if and only if there is a surjective R -homomorphism $\varphi : R^n \rightarrow M$ for some integer n (this is true for any ring R).

Problem 5

Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $x^2(x^2 + 1)^2$.