Problem 1

Let A be a local ring, M and N finitely generated A-modules.
Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.
[Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field.
Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.
By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.
But $M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0 \Rightarrow M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Problem 2

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\operatorname{Ker}(\phi)$ is finitely generated.

[Let $e_1, \ldots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ $(1 \leqslant i \leqslant n)$. Show that M is the direct sum of $\operatorname{Ker}(\phi)$ and the submodule generated by $u_1, \ldots, u_n$.]

Problem 3

Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$.
Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.
Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

[If $M = \mathfrak{a}M$, then $S^{-1}M = \left(S^{-1}\mathfrak{a}\right)\left(S^{-1}M\right)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Problem 4

The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A.
Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14).
Show that every minimal prime ideal of A is contained in D. [Use Exercise 6.]
The ring $S_0^{-1}A$ is called the *total ring of fractions* of A.
Prove that

- i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.
- ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.
- iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Problem 5

Let A be a principal entire ring, and let K be its quotient field.
Let $\mathfrak{o}$ be a valuation ring of K containing A, and assume $\mathfrak{o} \neq K$.
Show that $\mathfrak{o}$ is the local ring $A_{(p)}$ for some prime element p.
[This applies both to the ring $\mathbb{Z}$ and to a polynomial ring $k[X]$ over a field k.]

Problem 6

Let A be an entire ring, and let K be its quotient field.
Assume that every finitely generated ideal of A is principal.
Let $\mathfrak{o}$ be a discrete valuation ring of K containing A.
Show that $\mathfrak{o} = A_{(p)}$ for some element p of A, and that p is a generator of the maximal ideal of $\mathfrak{o}$.