Problem 1

Let M be a module over the integral domain R.

- (a) Suppose that M has rank n and that $x_1, x_2, \ldots, x_n$ is any maximal set of linearly independent elements of M.

  Let $N = Rx_1 + \ldots + Rx_n$ be the submodule generated by $x_1, x_2, \ldots, x_n$.

  Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module (equivalently, the elements $x_1, \ldots, x_n$ are linearly independent and for any $y \in M$ there is a nonzero element $r \in R$ such that ry can be written as a linear combination $r_1x_1 + \ldots + r_nx_n$ of the x_i).

- (b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \cong R^n$) such that the quotient M/N is a torsion R-module then M has rank n.

  [Let $y_1, y_2, \ldots, y_{n+1}$ be any $n+1$ elements of M.

  Use the fact that M/N is torsion to write r_iy_i as a linear combination of a basis for N for some nonzero elements $r_1, \ldots, r_{n+1}$ of R.

  Use an argument as in the proof of Proposition 3 to see that the r_iy_i, and hence also the y_i, are linearly dependent.]

Problem 2

Let R be any ring, let $A_1, A_2, \ldots, A_m$ be R-modules and let B_i be a submodule of A_i, $1 \leq i \leq m$. Prove that

$$\left(A_1 \oplus A_2 \oplus \cdots \oplus A_m\right) / \left(B_1 \oplus B_2 \oplus \cdots \oplus B_m\right) \cong \left(A_1/B_1\right) \oplus \left(A_2/B_2\right) \oplus \cdots \oplus \left(A_m/B_m\right).$$

Problem 3

Let R be a P.I.D., let a be a nonzero element of R and let $M = R/(a)$. For any prime p of R prove that

$$p^{k-1}M/p^kM \cong \begin{cases} R/(p) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

where n is the power of p dividing a in R.

Problem 4

Let R be a Euclidean Domain and let M be an R-module.

Prove that M is finitely generated if and only if there is a surjective R-homomorphism $\varphi : R^n \to M$ for some integer n (this is true for any ring R).

Problem 5

Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $x^2\left(x^2+1\right)^2$.