Problem 1

Let A be a local ring, M and N finitely generated A-modules.

Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

[Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field.

Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.

By Nakayama's lemma, $M_k = 0 \Rightarrow M = 0$.

But $M \otimes_A N = 0 \Rightarrow (M \otimes_A N)_k = 0 \Rightarrow M_k \otimes_k N_k = 0 \Rightarrow M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Problem 2

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\operatorname{Ker}(\phi)$ is finitely generated.

[Let $e_1, \ldots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ $(1 \leqslant i \leqslant n)$. Show that M is the direct sum of $\operatorname{Ker}(\phi)$ and the submodule generated by $u_1, \ldots, u_n$.]

Problem 3

Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$.

Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants.

[If $M = \mathfrak{a}M$, then $S^{-1}M = \left(S^{-1}\mathfrak{a}\right)\left(S^{-1}M\right)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Problem 4

The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A.

Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14).

Show that every minimal prime ideal of A is contained in D. [Use Exercise 6.]

The ring $S_0^{-1}A$ is called the *total ring of fractions* of A.

Prove that

- i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.
- ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.
- iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Problem 5

Let $\varphi : A \to B$ be a commutative ring homomorphism.

Let E be an A-module and F a B-module.

Let F_A be the A-module obtained from F via the operation of A on F through φ, that is for $y \in F_A$ and $a \in A$ this operation is given by

$$(a, y) \mapsto \varphi(a)y.$$

Show that there is a natural isomorphism

$$\operatorname{Hom}_B(B \otimes_A E, F) \approx \operatorname{Hom}_A(E, F_A)$$

Problem 6

[The norm]

Let B be a commutative algebra over the commutative ring R and assume that B is free of rank r.
Let A be any commutative R-algebra.
Then $A \otimes B$ is both an A-algebra and a B-algebra.
We view $A \otimes B$ as an A-algebra, which is also free of rank r.
If $\{e_1, \ldots, e_r\}$ is a basis of B over R, then

$$1_A \otimes e_1, \ldots, 1_A \otimes e_r$$

is a basis of $A \otimes B$ over A.
We may then define the norm

$$N = N_{A \otimes B, A} : A \otimes B \to A$$

as the unique map which coincides with the determinant of the regular representation.
In other words, if $b \in B$ and b_B denotes multiplication by b, then

$$N_{B,R}(b) = \det(b_B)$$

and similarly after extension of the base. Prove:

- (a) Let $\varphi : A \to C$ be a homomorphism of R-algebras. Then the following diagram is commutative:

$$\begin{tikzcd} A \otimes B \arrow[r, "\varphi \otimes 1"] \arrow[d, "N"'] & C \otimes B \arrow[d, "N"] \\ A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

- (b) Let $x, y \in A \otimes B$. Then $N(x \otimes_B y) = N(x) \otimes N(y)$. [Hint: Use the commutativity relations $e_i e_j = e_j e_i$ and the associativity.]