Problem 3:

In problems 1 and 2, we have computed D-optimal designs, since we minimize

$$\det\left(\left(D(W)\right)^{-1}\right),$$

which is the same as minimizing

$$\log \det\left(\left(D(W)\right)^{-1}\right) = -\log\left(\det\left(D(W)\right)\right).$$

* In this problem, we minimize

$$\operatorname{trace}\left(\left(D(W)\right)^{-1}\right)$$

to obtain A-optimal designs.

We can start with the same model as in problem 1.

Here is the A-optimal design problem.

(i) Define $u_i = -1 + 2(i-1)/(N-1)$, $i = 1, 2, \dots, N$, and N is given. For example, $N = 101$.

(ii) For $i = 1, 2, \dots, N$, let

$$A_i = \begin{pmatrix} 1 \\ u_i \\ u_i^2 \end{pmatrix} (1, u_i, u_i^2).$$

(iii) Let $W = (w_1, w_2, \dots, w_N)$, where $w_i \geq 0$ for $i = 1, \dots, N$, and $\sum_{i=1}^{N} w_i = 1$. W is the unknown vector in the design problem.

(iv) Define

$$D(w) = \sum_{i=1}^{N} w_i A_i ,$$

(v) Convex optimization problem

$$\begin{cases} \min\limits_{w} & \operatorname{trace}\left((D(w))^{-1} \right) \\ \text{s.t.} & w_i \geq 0, \quad i = 1, 2, \cdots, N \\ & \sum_{i=1}^{N} w_i = 1 \end{cases}$$

(vi) use CVX to solve the optimization problem to get w.

To present the result as follows: for $N=101$

$$\begin{pmatrix} u_1 & w_1 \\ u_{51} & w_{51} \\ u_{101} & w_{101} \end{pmatrix}, \quad \text{if } w_1, w_{51}, w_{101} > 10^{-5}.$$

We can omit w_i if $w_i \leq 10^{-5}$.

We can also generalize the above problem to consider the following model.

$$y_i = \theta_0 + \theta_1 x_i + \theta_2 x_i^2 + \cdots + \theta_p x_i^p + \varepsilon_i \,, \quad i=1, 2, \ldots, n \,,$$

$$x_i \in [a, b] \,.$$

Try $p = 3, 4, 5, 6$,

and $[a, b] = [-2, 2], [-3, 3], [0, 1], [0, 5]$.

For each given p and $[a, b]$, how do we present the optimal designs graphically as N increases?