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Problem 4

Construct c-optimal designs.

Consider the cubic regression model,

$$y_i = \theta_0 + \theta_1 x_i + \theta_2 x_i^2 + \theta_3 x_i^3 + \varepsilon_i, \quad i = 1, 2, \cdots, n$$

where $x_i \in [a, b]$,

$\varepsilon_i \sim$ i.i.d $N(0, \sigma^2)$.

Let $\hat{\theta}$ be the LSE, and let $c = (c_1, c_2, c_3, c_4)^T$ be a vector of length 4. From Problem 1, we have

$$\mathrm{Cov}(\hat{\theta}) \propto \sigma^2 \left(D(w)\right)^{-1},$$

where $D(w) = \sum_{i=1}^{N} w_i A_i$, and

$$A_i = \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ x_i^3 \end{pmatrix} \begin{pmatrix} 1 & x_i & x_i^2 & x_i^3 \end{pmatrix}.$$

Now we look at

$$\mathrm{Cov}(c^T \hat{\theta}) = \mathrm{cov}(c_1 \hat{\theta}_0 + c_2 \hat{\theta}_1 + \cdots + c_4 \hat{\theta}_3)$$

$$\propto c^T \left(D(w)\right)^{-1} c.$$

We minimize $c^T \left(D(w)\right)^{-1} c$ over w to get c-optimal designs. Note that c is a given constant vector.

Construct c-optimal designs for the following cases:

① $[a, b] = [-1, 1]$, $c = (1, 1, 1, 1)^T$, $N = 1001$.

② $[a, b] = [-1, 1]$, $c = (1, -1, 1, -1)^T$, $N = 1001$.

③ $[a, b] = [-1, 1]$, $c = (0, 1, 0, -1)^T$, $N = 1001$.

④ $[a, b] = [0, 2]$, $c = (1, 1, 1, 1)^T$, $N = 1001$

⑤ $[a, b] = [0, 2]$, $c = (1, -1, 1, -1)^T$, $N = 1001$.

⑥ $[a, b] = [0, 10]$, $c = (1, -1, 1, -1)^T$, $N = 1001$.

Suppose there are two constant vectors

$c_1 = (c_{11}, c_{12}, c_{13}, c_{14})^T$ and $c_2 = (c_{21}, c_{22}, c_{23}, c_{24})^T$.

We can minimize

$$c_1^T (D(w))^{-1} c_1 + c_2^T (D(w))^{-1} c_2$$

over w to get c-optimal designs.

For example, $c_1 = (1, 1, 1, 1)$, $c_2 = (1, -1, 1, -1)$.

Use the two vectors to compute c-optimal designs for the following cases:

(i) $[a, b] = [-1, 1]$, $N = 1001$,

(ii) $[a, b] = [-2, 2]$, $N = 1001$,

(iii) $[a, b] = [0, 1]$, $N = 1001$,

(iv) $[a, b] = [0, 5]$, $N = 1001$.