第五次分享

一些前置知识 $\begin{cases} \text{Hilbert space.} \\ \text{FEM.} \\ \text{Spectral theory} \end{cases}$

1. Hilbert space $L^2(D)$

Thm 1.1 For $\forall$ domain D, $L^2(D)$ is a real Hilbert space, with inner product $\langle u, v\rangle_{L^2(D)} = \int_D u(x)v(x)dx$

Thm 1.2. Let $(X, \mathcal{F}, \mu)$ be a measure space and H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{L^2(X,H)}$. Then $L^2(X,H)$ is a Hilbert space with

$$\langle u, v\rangle_{L^2(X,H)} = \int_X \langle u(x), v(x)\rangle d\mu(x)$$

Eg. 1.3. We have complex Hilbert space $L^2(\mathbb{R}^d, \mathbb{C})$ with inner product $\langle u, v\rangle_{L^2(\mathbb{R},\mathbb{C})} = \int_{\mathbb{R}^d} u(x)\overline{v(x)}dx$. For SPDE., we have measure space $(\Omega, \mathcal{F}, \mathbb{P})$. Note the square integrable functions $u(x,\omega)$ on $D\times\Omega$ space as $L^2(\Omega, L^2(D))$ s.t. $u(\cdot,\omega)\in L^2(D)$ for almost $\forall\ \omega\in D$:

$$\langle u, v\rangle_{L^2(\Omega, L^2(D))} = \int_\Omega\int_D u(x,\omega)\, v(x,\omega)\,dx\,d\mathbb{P}(\omega)$$

Since we have inner product, then we define orthonormal projections and bases. Consider H as Hilbert space, then G be a subspace of H

Def 1.4 (orthogonal component).

$$G^{\perp} = \{u \in H : \langle u, v \rangle = 0, \forall v \in G\}$$

$\Rightarrow G^{\perp} \cap G = \{0\}$

Thm 1.5 If G is a closed subspace of a Hilbert space H.
$\forall u \in H$ can be uniquely written as $u = p^* + q$ $(p^* \in G, q \in G^{\perp})$
and $\|u - p^*\| = \inf_{p \in G} \|u - p\|$

Def 1.5 (Orthogonal projection) $P: H \to G$ is defined by
$Pu = p^* \Rightarrow P^2 = P$.

Def 1.6 (Orthonormal basis) Let H be a Hilbert space. $\{\phi_j : j \in \mathbb{N}\}$
is orthonormal if $\langle \phi_i, \phi_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$
If $\mathrm{span}\{\phi_i\}$ is additionally dense in H, then we have a
complete orthonormal set, call $\{\phi_i\}$ orthonormal basis

$$\Rightarrow \quad u = \sum_{j=1}^{\infty} \langle u, \phi_j \rangle \phi_j. \quad \forall u \in H.$$

$$\|u\|^2 = \sum_{j=1}^{\infty} |\langle u, \phi_j \rangle|^2 = \sum_{j=1}^{\infty} |\langle \phi_j, u \rangle|^2$$

Thm 1.7. Consider a seperable Hilbert space H with orthonormal
basis $\{\phi_j : j \in \mathbb{N}\}$. Let $G_N = \mathrm{span}\{\phi_1 \cdots \phi_N\}$. Denote the projection
from H to G_N by P_N, then $P_N u = \sum^{N} \langle u, \phi_j \rangle \phi_j$.

Def 1.8 (Weak derivative) Let Y be a Banach space. We say a measurable function $D^\alpha u : D \to \mathbb{R}$ is αth weak derivative if

$$\int_D D^\alpha u(x)\,\phi(x)\,dx = (-1)^{|\alpha|}\int_D u(x)\,D^\alpha\phi(x)\,dx \quad \forall \phi \in C_c^\infty(D)$$

Def 1.9 (Sobolev space) Let D be domain, Y be Banach space $\forall p \geq 1$. Sobolev space $W^{r,p}(D,Y)$ is the set of functions whose weak derivatives up to $r \in \mathbb{N}$ are in $L^p(D,Y)$. That is

$$W^{r,p}(D,Y) = \{u : D^\alpha u \in L^p(D,Y) \text{ if } |\alpha| \leq r\}$$

where $p=2$, then Y is a Hilbert space H.
then note $W^{r,2}(D,H)$ as $H^r(D,H)$

(Consider the sobolev space cause the good properties)

Thm 1.10 (1) $W^{r,p}(D,Y)$ is a Banach Space with norm

$$\|u\|_{W^{r,p}(D,Y)} = \left(\sum_{0\leq|\alpha|\leq r}\|D^\alpha u\|^p_{L^p(D,Y)}\right)^{\frac{1}{p}}$$

(2) $H^r(D,H)$ is a Hilbert space with inner product.

$$\langle u,v\rangle_{H^r(D,H)} = \sum_{0\leq|\alpha|\leq r}\langle D^\alpha u, D^\alpha v\rangle_{L^2(D,H)}$$

$\Rightarrow$ the norm is

$$\|u\|_{H^r(D)} = \left(\sum_{0\leq|\alpha|\leq r}\|D^\alpha u\|^2_{L^2(D)}\right)^{\frac{1}{2}}$$

Also we can define semi-norm: $|u|_{H^r(D)} = \left(\sum_{|\alpha|=r}\|D^\alpha u\|^2_{L^2(D)}\right)^{\frac{1}{2}}$

2. Linear operator and spectral theory.

Def 2.1 (Linear operator) Let X, Y be two vector spaces on F

A function $L: X \to Y$ is said ~~to~~ be linear operator if

$\forall u, v \in X$ and any scalar $a \in F$: $L(u+v) = L(u) + L(v)$.

Def 2.2. (Bounded linear operator) $(X, \|\cdot\|_X)$ $(Y, \|\cdot\|_Y)$ be Banach Space.

Linear operator $L: X \to Y$ is bounded if $\exists K > 0$ s.t.

$$\|Lu\|_Y \leq K\|u\|_X, \quad \forall u \in X.$$

Note $\mathcal{L}(X, Y)$: bounded linear operators: $L: X \to Y$.

If $X = Y \Rightarrow \mathcal{L}(X) = \mathcal{L}(X, X)$.

Then we have the norm: $\|L\|_{\mathcal{L}(X,Y)} = \sup\limits_{u \neq 0} \dfrac{\|Lu\|_Y}{\|u\|_X}$.

Def 2.3. (Bounded Linear functional): a memeber of $\mathcal{L}(X, \mathbb{R})$.

$\Rightarrow$ If X is banach, the set of B.L.F on X is called dual sp.

and itself is a Banach space with norm $\|\cdot\|_{\mathcal{L}(X,\mathbb{R})}$

Thm 2.4 (Rietz Representation). H (Hilbert space with inner product $\langle \cdot, \cdot \rangle$

Let l be bounded linear functional on H. There exists a unique

$u_l \in H$ st. $\langle u_l, x \rangle = l(x)$ $\forall x \in H$.

Thm 2.5. (Lax - Milgram).

We consider Hilbert-Schmidt operators.

Def 2.6. Let H, U be seperable Hilbert Spaces with norms $\|\cdot\|$, $\|\cdot\|_U$ respectively. $\{\phi_j\}$ orthonormal basis of U. Define Hilbert-Schmidt norm $\|L\|_{HS(U,H)} := \left(\sum_{j=1}^{\infty} \|L\phi_j\|_U^2\right)^{\frac{1}{2}}$

$HS(U,H) = \{L \in \mathcal{L}(U,H) : \|L\|_{HS(U,H)} < \infty\}$ is Banach.

$L \in HS(U,H)$ is a Hilbert-Schmitz operator.

If $U=H$, then $\|L\|_{HS} = \|L\|_{HS(H,H)}$

E.g. 2.7. $\mathbb{R}^{d\times d}$ Matrix. $: \mathbb{R}^d \to \mathbb{R}^d$.

$$\|A\|_{HS} = \left(\sum_{ij}^{d} a_{ij}^2\right)^{\frac{1}{2}}$$ which is Frobenius norm.

E.g. 2.8 $L \in \mathcal{L}(L^2(0,1))$ defined by $(Lu)(x) = \int_0^x u(y)\,dy$ $u \in L^2(0,1)$.

Consider the orthonormal basis $\phi_j(x) = \sqrt{2}\sin(j\pi x)$.

$\Rightarrow (L\phi_j)(x) = \sqrt{2}\,(1-\cos(j\pi x))\frac{1}{j\pi} \Rightarrow \|L\phi_j\|_{L^2(0,1)} \le \frac{2\sqrt{2}}{j\pi}$.

$\Rightarrow \|L\|_{HS}^2 = \sum \|L\phi_j\|_{L^2(0,1)}^2 \le \sum_{j=1}^{\infty} \frac{8}{j^2\pi^2} < \infty$.

$\Rightarrow L$ is a valid HS operator.

We focus on integral operators and relationship with HS operator on $L^2(D)$.

Def 2.9. (Integral operator with kernel G). For a domain D. and $G \in L^2(D\times D)$, the integral operator L on $L^2(D)$ with kernel G is defined by

$$(Lu)(x) := \int_D G(x,y)\, u(y)\, dy \qquad x \in D,\ u \in L^2(D). \quad (*)$$

Thm 2.10. Any integral operator with kernel $G \in L^2(D\times D)$ is a HS operator on $L^2(D)$.

Conversely, any HS operator L on $L^2(D)$ can be written as $(*)$.

and $\|L\|_{HS} = \|G\|_{L^2(D\times D)}$

Def 2.11 Symmetric operator $L \in \mathcal{L}(H)$ is symmetric on Hilbert H if $\langle Lu, v\rangle = \langle u, Lv\rangle$ for $\forall u, v \in H$.

Thm 2.12. The integral operator L with kernel G is symmetric.

Thm 2.13. (HS spectral). Let H be countably infinite-dim Hilbert space and $L \in \mathcal{L}(H)$ be symmetric and compact. with Eigenvalue $|\lambda_j| > |\lambda_{j+1}|$ with corresponding eigenfunction $\phi_j \in H$.

Then (i) $\forall \lambda_j \in \mathbb{R}$ and $\lambda_j \to 0$ as $j \to \infty$.

(ii) ϕ_j can be chosen to form an orthonormal basis for L

(iii) $\forall u \in H$, $Lu = \sum_{j=1}^{\infty} \lambda_j \langle u, \phi_j\rangle \phi_j$

Def 2.14 (Non-negative definite operators).

Prepare.

Section 1. Semigroups of linear operators.

The space $\mathbb{R}^d$ where $u(t)$ exists called the phase space

$S(t) : \mathbb{R}^d \to \mathbb{R}^d$ is a family of operators on the phase space $\mathbb{R}^d$.

phase space.

Def 1.1 (Semigroup) Let X be a vector space. $S(t) : X \to X$

Is a semigroup if 1) $S(0) = I$ (The identity operator)

2) $S(t+s) = S(t)S(s)$ for $s, t \geq 0$.

Specially, if $S(t)$ is linear, then $S(t)$ is a semigroup of linear operat

Def 1.2. (Infinitesimal generator) Let $S(t)$ be a semigroup of linear operators on Banach space X.

The infinitesimal generator $-A$ is defined by

$$(-A)u := \lim_{h \to 0^+} \frac{S(h)u - u}{h}.$$

The domain of A is denoted by $D(A) \subset X$ (the set of $u \in X$ where the limit exists.

E.g. 1.3. (linear ODE). $\frac{du}{dt} = -\lambda u$. $u_0 \in \mathbb{R}$ $\lambda > 0$.

$\Rightarrow$ phase space is $\mathbb{R}$ and $S(t)u_0 := u(t) = e^{-\lambda t} u_0$

$\Rightarrow S(t) = e^{-\lambda t} \Rightarrow$ generator is $-\lambda$

$-\lambda$ is said to generate the semigroup.

Rmk 1.4. $u(t,x)$ is interpreted as functions $u(t): t \to X$.
X is a function space like $L^2(D)$. Then PDE can be written as ODEs.

Assumption 1.5. (*) Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the linear operator $-A: D(A) \subset H \to H$ has a complete orthonormal set of eigenfunctions $\{\phi_j : j \in \mathbb{N}\}$ and eigenvalue $\lambda_j > 0$ $(\lambda_{j+1} > \lambda_j)$

Def 1.6. If Assumption 1.5 is satisfied.

$$e^{-tA} u = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle u, \phi_j \rangle \phi_j$$

We can prove that e^{-tA} is a semigroup of linear operators on H. with generator $-A$

Then we talk about semilinear evolution equations,
Many PDEs have semilinear structure like reaction-diffusion eq.
Def 1.7 (semilinear ODEs).

$$\frac{du}{dt} = -Mu + f(u) \qquad u(0) = u_0 \in \mathbb{R}^d.$$

where M needs to be positive-definite.

$$\Rightarrow u(t) = e^{-tM} u_0 + \int_0^t e^{-(t-s)M} f(u(s))\, ds.$$

Eg 1.8. For $f: \mathbb{R} \to \mathbb{R}$. consider $u_t = \Delta u + f(u)$. $x \in D$, $t \in [0, +\infty)$.
with homogenous boundary conditions; the initial condition $u_0(x) = u(0,x)$
The solution $u(t,x)$. by semigroup theory, can be interpreted as
a function $u: [0,+\infty) \to L^2(D)$

$$\Rightarrow \quad \frac{du}{dt} = -Au + f(u)$$

where $A = -\Delta$ with $D(A) = H^2(D) \cap H_0^1(D)$.

$-A$ is the generator of semigroup $S(t)$ on $L^2(D)$

Rmk: we find that the boundary conditions are incorporated into A.

A

Def 1.9. (Semilinear evolution equation)

a Semilinear evolution equation on Hilbert space H:

$$\frac{du}{dt} = -Au + f(u) \qquad u(0) = u_0 \in H.$$

where A needs to satisfy Assumption (*). $f: H \to H$.

$\Rightarrow$ variation of constant formula.

$f(u)$.

$$u(t) = \underbrace{e^{-tA} u_0}_{\text{the variation of } u_0} + \underbrace{\int_0^t e^{-(t-s)A} f(u(s))\, ds}_{\text{the accumulation of source term } f.}$$

Def. 1.10. (Mild solution) A mild solution of semilinear evolution equation on $[0,T]$ is a function $u \in C([0,T], H)$ s.t. variation of constant formula holds.

Then just as ODE, we need some conditions to construct the existence and uniqueness of mild solution.

~~First~~ 1.11

Assumption: Assumption $(*)$ holds, then let $u_0 \in H$. $f: H \to H$. s.t.

$$\|f(u)\| \le L(1+\|u\|)$$

$$\|f(u_1) - f(u_2)\| \le L\|u_1 - u_2\|.$$

$\Rightarrow \exists!$ mild solution $u(t)$. And $\exists\, K_T > 0$, s.t. $\forall\, u_0 \in H$

$$\|u(t)\| \le K_T(1+\|u_0\|) \qquad 0 \le t \le T.$$

Thm 1.12. Let Assumption 1.11 holds, then $u(t)$ be mild solution then $u(t) \in D(A^{\frac{1}{2}})$ $\quad \frac{du}{dt}(t) \in D(A^{-\frac{1}{2}})$. $\forall\, t > 0$.

Then we develop the weak solution.

$$\Rightarrow \begin{cases} \langle \frac{du}{dt}, v \rangle = \langle A^{-\frac{1}{2}} \frac{du}{dt}, A^{\frac{1}{2}} v \rangle \\ \langle Au, v \rangle = \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v \rangle \end{cases} \quad \text{for } v \in D(A^{\frac{1}{2}}),\ u \in D(A)\ \frac{du}{dt} \in H.$$

Def. 1.13. Let $V = D(A^{\frac{1}{2}})$ $\quad V^* = D(A^{-\frac{1}{2}})$. $u: [0,T] \to V$ is a weak solution of $\frac{du}{dt} = -Au + f(u)$. if almost all $s \in [0,T]$.

we have $\frac{du(s)}{dt} \in V^*$ and

$$\left\langle \frac{du(s)}{dt}, v \right\rangle = -a(u(s), v) + \langle f(u(s)), v \rangle \quad \forall v \in V.$$

where $a(u(s), v) = \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v \rangle$

If take $\tilde{V} = \mathrm{span}\{\psi_1 \cdots \psi_J\} \subset V$. seek $\tilde{u} = \sum \hat{u}_j(t) \psi_j$.

$\Rightarrow$
$$\left\langle \frac{d\tilde{u}}{dt}, v \right\rangle = -a(\tilde{u}, v) + \langle f(\tilde{u}), v \rangle \quad \forall v \in \tilde{V}$$

That is $\left\langle \frac{d\tilde{u}}{dt}, \psi_j \right\rangle = -a(\tilde{u}, \psi_j) + \langle f(\tilde{u}), \psi_j \rangle$ for $\forall j$.

Section 2. Normal numerical method.

Spectral Galerkin Approximation.

Assume. A satisfy Assumption $(*)$, then we have $\{\phi_j\}$ for H.

$\Rightarrow$ we can define $\tilde{V} = V_J = \mathrm{span}\{\phi_1 \cdots \phi_J\}$. Note $P_J : H \to V_J$

$$P_J u = \sum^{J} \hat{u}_j(t) \phi_j \quad \text{where} \quad \hat{u}_j = \frac{1}{\|\phi_j\|^2} \langle u, \phi_j \rangle$$

$$\Rightarrow \frac{du_J}{dt} = -A_J u_J + P_J f(u_J) \qquad u_J(0) = P_J(u_0)$$

$$\Rightarrow \text{For } j = 1, \cdots, J. \quad \frac{d\hat{u}_j(t)}{dt} = -\lambda_j \hat{u}_j(t) + \hat{f}_j(u_J) \qquad \hat{u}_j(0) = \hat{u}_{0,j}$$

Let $\hat{u}_J(t) = [\hat{u}_1(t), \cdots, \hat{u}_J(t)]^T \qquad \hat{f}_J(u_J) = [\hat{f}_1(u_J), \cdots \hat{f}_J(u_J)]^T$

we have $$\frac{d\hat{u}_J(t)}{dt} = -M\hat{u}_J(t) + \hat{f}_J(u_J) \quad \left(M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_J \end{pmatrix} \right).$$

Then How to compute? We need to approximate $\hat{u}_J(t)$.

Note $\hat{u}_{J,n} \to \hat{u}_J(t_n)$

where $\hat{u}_J(t_n) = [\hat{u}_1(t_n) \cdots \hat{u}_J(t_n)]^T$

$\Rightarrow$ $u_{J,n} = \sum^{J} \hat{u}_{j,n} \phi_j$

$\Rightarrow$ Semi-implicit Euler method.

$$\frac{u_{J,n+1} - u_{J,n}}{\Delta t} = -A_J u_{J,n+1} + P_J f(u_{J,n})$$

$$\Rightarrow \quad \frac{\hat{u}_{J,n+1} - \hat{u}_{J,n}}{\Delta t} = -M \hat{u}_{J,n+1} + \hat{f}_J(\hat{u}_{J,n})$$

$$\Rightarrow \quad (I + \Delta t M)\hat{u}_{J,n+1} = \hat{u}_{J,n} + \Delta t \cdot \hat{f}_J(\hat{u}_{J,n})$$

that is $\quad (1 + \Delta t\, \lambda_j)\, \hat{u}_{j,n+1} = \hat{u}_{j,n} + \Delta t \cdot \hat{f}_j(u_{J,n})$ $\square$

Galerkin Finite elements Method.

$$\frac{du}{dt} = -Au + f(u). \qquad -A = \varepsilon \frac{d^2}{dx^2}$$

We formulate PDE as ODE on $H = L^2(0,a)$. with $-A = \varepsilon \frac{d^2}{dx^2}$

with $D(A) = H^2(0,a) \cap H_0^1(0,a)$

$$\Rightarrow \quad \left\langle \frac{du_h}{dt}, v \right\rangle_H = -a(u_h, v) + \langle f(u_h), v \rangle_H.$$

$$u_h(t,x) = \sum_{j=1}^{J} u_j(t)\phi_j(x)$$

Let $u_h(t) = [u_1(t) \cdots u_J(t)]^T$. This satisfies ODEs:

$$M \frac{du_h(t)}{dt} = -K u_h + f(u_h).$$

where $f(u_h) \in \mathbb{R}^J$. $f_j(u_h) = \langle f(u_h), \phi_j \rangle_H$

$$(M)_{ij} = \langle \phi_i, \phi_j \rangle_H \qquad K_{ij} = a(\phi_i, \phi_j)$$

$$\Rightarrow \quad \frac{du_h}{dt} = -M^{-1}K u_h + M^{-1} f(u_h).$$

$dW(t,x)$

Rieview. Random Fields.

FEM

$$u_t = -\Delta u + \xi(t,x)$$ 噪声

$$\boxed{\int dW(t,x)}$$ Itô

We mainly talk about Elliptic PDEs and semilinear SPDEs.

Elliptic PDEs with Random Data.

E.g.

$$\begin{cases} -\nabla\cdot(a(x,w)\nabla u(x)) = f(x,w) & x\in D \\ u(x,w) = g(x) & x\in\partial D. \end{cases} \quad (*)$$

where $\{a(x)\}$ and $\{f(x)\}$ are second-order random fields.

$\Rightarrow u(x)$ is a random field. $\quad X(t)$ sample ~~path~~ $\forall w$. $X(t,w)$

$v(\cdot, w) \in L^2(D)$ $u(x)$. realization

Hence we work with real-valued random fields $\{v(x): x\in D\}$ that are $L^2(D)$-valued random variables. i.e. $v: D\times\Omega \to \mathbb{R}$ $\quad \Omega \ \boxed{\{w\}}$

$v\in L^2(\Omega, \mathcal{F}, L^2(D))$ s.t.

$$\|v\|^2_{L^2(\Omega, L^2(D))} = \int_\Omega \int_D v^2(x,w)\,dx\,dP(w) = E\left[\|v\|^2_{L^2(D)}\right] < \infty.$$

For simplicity. we take $f\in L^2(D)$, instead of $L^2(\Omega, L^2(D))$, $g=0$

Variational formulation on D.

Problem setting: Given realization of $a(\cdot, w)$. Consider the random field $u(x)$ with realizations $u(\cdot,w)\in V := H_0^1(D)$ s.t.

$$\int_D a(x,w)\nabla u(x,w)\cdot\nabla v(x)\,dx = \int_D f(x)v(x)\,dx \quad \forall v\in V.$$

(The well-poseness of $u(\cdot,w)$ is the some as FEM).

Here we use FEM to approximate $u(\cdot,\omega) \Longrightarrow u_h(\cdot,\omega) \in V^h$ where $V^h \subset H_0^1(D)$:

$$\int_D a(x,\omega)\nabla u_h(x,\omega)\cdot\nabla v(x)\,dx = \int_D f(x)v(x)\,dx \quad \forall v \in V$$

Suppose we only have the realization $\tilde{a}(\cdot,\omega)$ of an approximation $\tilde{a}(x)$ to $a(x)$.

We just need to consider the approximation $\tilde{u}_h(x) \in V^h$ s.t.

$$\int_D \tilde{a}(x,\omega)\cdot\nabla\tilde{u}_h(x,\omega)\cdot\nabla v(x)\,dx = \int_D f(x)v(x)\,dx \quad \forall v \in V^h.$$

If we want to estimate $E[\tilde{u}_h]$, the basic idea is MC.

Monte Carlo Method.

Given i.i.d samples $\tilde{a}_r = \tilde{a}(\cdot,\omega_r)$ $r = 1\cdots Q$.

We can get Q variational problems to get $\tilde{u}_h^r(x) = \tilde{u}_h(x,\omega_r)$ with Galerkin Matrix $A \in \mathbb{R}^{N\times N}$, $b \in \mathbb{R}^N$.

$$\begin{cases} a_{ij} = \int_D \tilde{a}_r(x)\nabla\phi_i(x)\nabla\phi_j(x)\,dx. \\ b_i = \int_D f(x)\phi_i(x)\,dx. \end{cases}$$

So once we get $\tilde{u}_h^r(x)$. then $E[\tilde{u}_h] = \frac{1}{Q}\sum^{Q}\tilde{u}_h^r(x) = \mu_{Q,h}(x)$

$$\delta_{Q,h}^2(x) = \frac{1}{Q-1}\left(\sum \tilde{u}_h^r(x)^2 - Q\mu_{Q,h}^2(x)\right)$$

E.g. Review how we get realization of $a(\cdot,\omega)$.

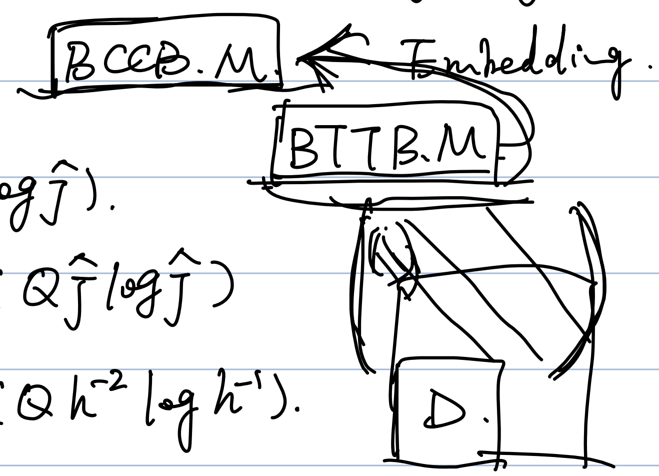
If we have explicit eigenfunction and eigenvalues, then by KL

$$a(x,\omega) = \mu + \sum_{k=1}^{P} \frac{\sigma}{k^2\pi^2}\cos(\pi k x)\,\xi_k(\omega) \qquad \xi_k \sim U(-1,1) \text{ i.i.d.}$$

Then apply each $a(\cdot,\omega)$. we can get $E[\tilde{u}_h]$ and $Var(\tilde{u}_h)$.
But here we find that the cost of MCFEM is $O(\varepsilon^{-4})$, and the convergence is very slow.

E.g. C_{all} = Cost (generating Q samples of $a(x)$.)
+ Cost (Solving Q FEM problems.)

The size of A is $J \times J$. $O(h^{-2})$ (Take 2D problem).
The circulant embedding generate a matrix $4\hat{J} \times 4\hat{J}$
$(\hat{J} = (1+\alpha)^2(ns+1)^2,\ h = \frac{1}{ns})$



Use FFT. the cost is $O(Q\hat{J}\log\hat{J})$.
$\Rightarrow$ Cost_all is $O(Qh^{-2}) + O(Q\hat{J}\log\hat{J})$
$= O(Qh^{-2}) + O(Qh^{-2}\log h^{-1})$.

After we estimate the error of $E[\tilde{u}_h]$ with $E[u]$

$$\|E[u] - \mu_{Q,h}\|_{H_0^1(D)} = O(h) + O(Q^{-\frac{1}{2}}).$$

That is, if we control the error down to ε, then.
$Q = O(\varepsilon^{-2})$ and $h = O(\varepsilon)$.
$\Rightarrow$ The cost is $O(\varepsilon^{-4})$!

$\phi_j(x)$ $\quad x \in D$.

$\psi_j(\xi_j)$. $\xi_j \in \Gamma$.

That is we need to develop new methods. We discuss the variational formulation on $D \times \Omega$. That is we try to

Seek weak solutions $u: D\times\Omega \to \mathbb{R}$. v. $L^2(D)$-valued r.v.

Def (Weak solution on $D\times\Omega$). A weak solution to $(*)$ with $g=0$ is a function $u\in V = L^2(\Omega, L^2(D))$ that satisfies

$$a(u, v) = l(v) \quad \forall\, v\in V.$$

where $a(\cdot,\cdot): V\times V \to \mathbb{R}$ and $l: V\to\mathbb{R}$:

$$\begin{cases} a(u,v) = E\left[\int_D a(x,\cdot)\,\nabla u(x,\cdot)\cdot\nabla v(x,\cdot)\,dx\right] \\ l(v) = E\left[\int_D f(x,\cdot)\,v(x,\cdot)\,dx\right]. \end{cases}$$

So if $g\neq 0$ the extension is natural. just find $u\in W = L^2(\Omega, H_g^1(D))$.

When we have the approximate random fields $\tilde{a}, \tilde{f}: D\times\Omega\to\mathbb{R}$ to $a(x)$ and $f(x)$, the problem leads to find $\tilde{u}: D\times\Omega\to\mathbb{R}$ s.t. $\mathbb{P}$. a.s.

$$\begin{cases} -\nabla\cdot(\tilde{a}(x,\omega)\cdot\nabla\tilde{u}(x,\omega)) = \tilde{f}(x,\omega) & x\in D \\ \tilde{u}(x,\omega) = g(x) & x\in\partial D \end{cases}$$

The corresponding weak form is find $\tilde{u}\in W$ st.

$$\tilde{a}(\tilde{u}, v) = \tilde{l}(v) \quad \forall\, v\in V.$$

where $\tilde{a}: W\times V\to\mathbb{R}$, $\tilde{l}: V\to\mathbb{R}$:

$$\begin{cases} \tilde{a}(\tilde{u},v) = E\left[\int_D \tilde{a}(x,\cdot)\cdot\nabla\tilde{u}(x,\cdot)\cdot\nabla\tilde{v}(x,\cdot)\,dx\right] \\ \tilde{l}(v) = E\left[\int_D \tilde{f}(x,\cdot)\,v(x,\cdot)\,dx\right]. \end{cases}$$

So after we have the approximation, the weak form still hard to solve. Since the integrals involve the abstract set Ω

i.e.

$$\begin{cases} \tilde{a}(u,v) = \int_{\Omega}\int_{D} \tilde{a}(x,\omega)\cdot\nabla u(x,\omega)\cdot\nabla v(x,\omega)\, dx\, d\mathbb{P}(\omega). \\ \tilde{\ell}(v) = \int_{\Omega}\int_{D} \tilde{f}(x,\omega)\cdot v(x,\omega)\cdot dx\, d\mathbb{P}(\omega) \end{cases}$$

If we can let $\tilde{a}(x)$. $\tilde{f}(x)$ only depend on finite number of random variables. $\xi_k : \Omega \to \Gamma_k \subset \mathbb{R}$, We can make a change of variable and get an equivalent weak form on $D\times\Gamma$. where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_M \subset \mathbb{R}^M$.

Def: Let $\xi_k : \Omega \to \Gamma_k$ for $k=1,\cdots,M$. be real-valued r.v. Then $v \in L^2(\Omega, L^2(D))$ of the form $v(x,\xi(\omega))$ for $x\in D$, $\omega\in\Omega$. Where $\xi(\omega) = [\xi_1(\omega), \xi_2(\omega), \cdots, \xi_M(\omega)]^T : \Omega \to \Gamma \subset \mathbb{R}^M$. is called finite-dimensional (M-dimensional noise).

E.g. We can take the truncated KL expansion.

$$\begin{cases} a(x,\omega) = \mu_a(x) + \sum_{k=1}^{\infty} \sqrt{v_k^a}\, \phi_k^a(x)\, \xi_k(\omega). \\ f(x,\omega) = \mu_f(x) + \sum_{k=1}^{\infty} \sqrt{v_k^f}\, \phi_k^f(x)\, \eta_k(\omega) \end{cases}$$

Where $\{v_k^a, \phi_k^a\}$ $\{v_k^f, \phi_k^f\}$ are eigenpairs of C_a and C_f. resp. ξ_k. η_k are random variables Here for similisity. assume ξ_k and η_k have the same distribution. Now truncated the expansion:

$$\begin{cases} a(x,\omega) = \mu_a(x) + \sum_{k=1}^{P} \sqrt{v_k^a}\, \phi_k^a(x)\, \xi_k(\omega). \\ f(x,\omega) = \mu_f(x) + \sum_{k=1}^{N} \sqrt{v_k^f}\, \phi_k^f(x)\, \xi_k(\omega) \end{cases}$$

ξ_k i.i.d.

$\underline{M} = \max\{P, N\}$.

ξ_k needs control. Assume $\xi_k \in [-\gamma, \gamma]$

Then we can define the weak solution on $D \times \Gamma$

Let $\tilde{a}(x)$ and $\tilde{f}(x)$ be the truncated KL expansion. we define $L_p^2(\Gamma; H_g^1(D)) = \{v : D \times \Gamma \to \mathbb{R} : \int_\Gamma p(y) \|v(\cdot, y)\|^2_{H^1(D)} dy < \infty$. and $v|_{\partial D} = g\}$

similarly we can define $L_p^2(\Gamma; H_0^1(D))$

Def: An equivalent weak solution on $D \times \Gamma$ is a solution $\tilde{u} \in W = L_p^2(\Gamma, H_g^1(D))$ s.t. $\tilde{a}(\tilde{u}, v) = \tilde{l}(v)$, $\forall v \in V = L_p^2(\Gamma, H_0^1(D))$

where $\tilde{a} : W \times V \to \mathbb{R}$. $\tilde{l} : V \to \mathbb{R}$:

$$\tilde{a}(u, v) = \int_\Gamma p(y) \int_D \tilde{a}(x, y) \nabla u(x, y) \cdot \nabla v(x, y) \, dx \, dy.$$

$$\tilde{l}(v) = \int_\Gamma p(y) \int_D \tilde{f}(x, y) \cdot v(x, y) \, dx \, dy.$$

The $p : \Gamma \to \mathbb{R}^+$. Normally, we take as the joint density of ξ.

And we find that $\tilde{a}(u, v) = \tilde{l}(v)$ is an $(M+2)$ dim deterministic problem on $D \times \Gamma$.

The strong formulation is find $\tilde{u} : \bar{D} \times \Gamma \to \mathbb{R}$ s.t.

$$\begin{cases} -\nabla \cdot (\tilde{a}(x, y) \cdot \nabla \tilde{u}(x, y)) = \tilde{f}(x, y) & (x, y) \in D \times \Gamma. \\ \tilde{u}(x, y) = g(x) & (x, y) \in D \times \Gamma. \end{cases}$$

Working on Γ instead of Ω. we have.

$$\begin{cases} \tilde{a}(x, y) = \mu_a(x) + \sum_{k=1}^{P} \sqrt{\nu_k^a} \, \phi_k^a(x) \, y_k & x \in D. \ y \in \Gamma. \ y_k \in \Gamma_k. \\ \tilde{f}(x, y) = \mu_f(x) + \sum_{k=1}^{P} \sqrt{\nu_k^f} \, \phi_k^f(x) \, y_k \end{cases}$$

ψ_j ϕ_j ✓

After we construct the variational formula on $D \times \Gamma$, we can apply Galerkin approximation on $D\times\Gamma$. First we replace $H_g^1(D)$ and $H_0^1(D)$ with the finite element space W^h and V^h. and study weak solutions $\tilde{u}_h \in L_p^2(\Gamma, W^h)$.

Def: A semi-discrete weak solution on $D\times\Gamma$ is $\tilde{u}_h \in L_p^2(\Gamma, W^h)$ s.t.

$$\tilde{a}(\tilde{u}_h, v) = \tilde{l}(v). \quad \forall v \in L_p^2(\Gamma, V^h) \qquad W^h \subset H_g^1(D),\ V^h \subset H_0^1(D)$$

$$L_p^2(\Gamma) \Leftarrow S^k. \qquad W^h \longrightarrow H_g^1(D).$$

We can derive a fully discrete, finite-dimensional problem on $D\times\Gamma$

Def: A stochastic Galerkin solution is a function $\tilde{u}_{hk} \in W^{hk}$, where $W^{hk} \subset W = L_p^2(\Gamma, H_g^1(D))$ s.t.

$$\tilde{a}(\tilde{u}_{hk}, v) = \tilde{l}(v) \qquad \forall v \in V^{hk} \subset V = L_p^2(\Gamma, H_0^1(D))$$

where $\tilde{a}: W\times V \to \mathbb{R}$, $\tilde{l}: V \to \mathbb{R}$:

$$\begin{cases} \tilde{a}(u,v) = \int_\Gamma p(y) \int_D \tilde{a}(x,y)\, \nabla u(x,y) \cdot \nabla v(x,y)\, dx\, dy. \\ \tilde{l}(v) = \int_\Gamma p(y) \int_D \tilde{f}(x,y) \cdot v(x,y)\, dx\, dy. \end{cases}$$

V basis function $\boxed{S^k \otimes V^h}$

Hence we need to select finite-dimensional subspaces of W and V. Let $S^k \subset L_p^2(\Gamma)$: $S^k = \text{span}\{\psi_1 \cdots \psi_Q\}$

Then we can construct a finite-dimensional subspace of V by tensorising ψ_j with finite element basis ϕ_i for V^h.

That is $V^h = \text{span}\{\phi_1 \cdots \phi_J\}$. $\quad T_i \ \xi_i \ P_i \quad L_{P_i}^2(\Gamma_i)$

$\Rightarrow V^h \otimes S^k = \text{span}\{\phi_i \psi_j : i = 1\cdots J,\ j = 1\cdots Q\} = V^{hk}$ $\Downarrow$ ψ.

gPC

So when $g \neq 0$, we need to construct W^{hk}.

$W^{hk} = V^{hk} \oplus \mathrm{span}\{\phi_{J+1}, \cdots \phi_{J+J_b}\}$.

$\Rightarrow \forall w \in W^{hk}$,

$$w(x,y) = \sum_i^J \sum_j^Q w_{ij}\,\phi_i(x)\,\psi_j(y) + \sum_{i=J+1}^{J+J_b} w_i\,\phi_i(x) = w_0(x,y) + w_g(x)$$

V^h

So. here we only have the finite element basis func.

Then how can I choose the basis for S^k. That is stochastic basis functions.

Let S_i^k denote the set of univariate polynomials of degree k or less in y_i on Γ_i : $S_i^k = \mathrm{span}\{P_{\alpha_i}^i(y_i) : \alpha_i = 0, \cdots, k\}$.

where $P_{\alpha_i}^i(y_i)$ is some polynomial of degree α_i on y_i,

Then $S^k = \boxed{S_1^k \otimes S_2^k \otimes \cdots \otimes S_M^k}$

Γ_i $k+1$

$$S^k = \mathrm{span}\Big\{\prod_{i=1}^M P_{\alpha_i}^i(y_i) : \alpha_i = 0, 1, \cdots, k,\ i = 1, \cdots, M\Big\}.$$

$\Rightarrow \dim(S^k) = Q = (k+1)^M$.

Γ_i k_i

If the index of $k = (k_1\ k_2 \cdots, k_M)$, then

Γ_i $\{\psi_i\}$

$$S^k = \mathrm{span}\Big\{\prod_{i=1}^M P_{\alpha_i}^i(y_i) : \alpha_i = 0, 1, \cdots, k_i,\ i = 1, \cdots, M\Big\}.$$

Additionally, ~~if~~ the ~~total order of~~ polynomial is k or less.

$<k$. xy. x^2 y^2

$\Rightarrow |\alpha| \le k$, then.

$$S^k = \mathrm{span}\Big\{\prod_{i=1}^M P_{\alpha_i}^i(y_i) : \alpha_i = 0, 1, \cdots, k,\ i = 1, \cdots, M,\ |\alpha| \le k\Big\}.$$

where $|\alpha| = \sum \alpha_i$

$\Rightarrow Q = \dim(S^k) = C_{M+k}^k = \frac{(M+k)!}{M!\,k!}$

$\boxed{\text{Dong bin Xiu.}}$

SPDE

UQ.

Then we need to think about how to generate $P^i_{\alpha_i}(y_i)$.

Here we introduce the $\boxed{\text{generalized polynomial chaos}}$ (gPC).

First we define the inner product. $y \sim N(0,1)$ $U(-1,1)$.

$$\langle v, w \rangle_p = \int_\Gamma p(y) v(y) w(y) dy \quad \text{for } v, w : \Gamma \to \mathbb{R}.$$

Let $P_{-1} = 0$, $P_0 = 1$. We can construct a sequence of polys. P_j on Γ that are orthonormal with respect to $\langle \cdot, \cdot \rangle_p$, and satisfies $\boxed{P_j(y) = (a_j y + b_j) P_{j-1}(y) - C_j P_{j-2}(y).}$ $j = 1, 2, \cdots$

Hence a_j, b_j, C_j depend on $p(y)$

a_j b_j C_j $\{a_j\}$

$$\begin{cases} \langle P_j, P_{j+1} \rangle_p = 0 \\ \langle P_j, P_j \rangle_p = 1 \end{cases}$$

E.g. (Legendre polynomial) Let $\Gamma = [-\sqrt{3}, \sqrt{3}]$ and $p = \frac{1}{2\sqrt{3}}$

$$\Rightarrow \quad L_j(y) = \hat{L}_j\left(\frac{y}{\sqrt{3}}\right) \qquad \hat{L}_j(y) = \frac{\sqrt{2j+1}}{2^j \cdot j!} \frac{d^j}{dy^j}(y^2-1)^j$$

$\xi \sim U(-\sqrt{3}, \sqrt{3})$

$$\Rightarrow \langle L_i, L_j \rangle_p = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} L_i(y) L_j(y) dy = \delta_{ij}$$

$L^2_p(\Gamma)$ $\{\Psi_j\}$.

$S^k \leftarrow (\cdot) \leftarrow$ gPC.

Thm.

Hence we can chose $\{P^i_{\alpha_i}(y_i)\}$ are orthonormal with respect to $\langle \cdot, \cdot \rangle_{p_i}$ on Γ_i for $i = 1 \cdots M$. then the poly $\Psi_j = \prod^M P^i_{\alpha_i}(y_i)$ are orthonormal respect to $\langle \cdot, \cdot \rangle_p$ on Γ where $\boxed{p(y) = p_1(y_1) p_2(y_2) \cdots p_M(y_M)}$.

i.e. $\langle \Psi_r, \Psi_s \rangle_p = \delta_{rs}$, $\quad r = (\alpha_1 \cdots \alpha_M)$ $\sum \alpha_i = k$.

$\quad s = (\beta_1 \cdots \beta_M)$. $\sum \beta_i = k$.

Sum up: the Ψ is chosen by the distribution on ξ_i.

Normally, ξ_i i.i.d. $\Rightarrow$ note $P^i_{\alpha_i}$ as P_{α_i}

Then apply the basis for $S^k \subset L^2_p(\Gamma)$. $\forall\, \tilde{u}_{h,k} \in W^{hk}$

$$\boxed{\tilde{u}_{hk}(x,y) = \sum_{i}^{J} \sum_{j}^{Q} u_{ij}\, \phi_i(x)\, \psi_j(y) + w_g(x).}$$

Assume $\psi_1(y) = 1$

$$= \sum_{j}^{Q} \left(\sum_{i}^{J} u_{ij}\, \phi_i(x) \right) \psi_j(y) + w_g(x).$$

FEM. u_{ij} ψ_j

$$= \sum_{j}^{Q} \boxed{u_j(x)}\, \psi_j(y) + w_g(x)$$

$$= (u_1(x) + w_g(x))\, \psi_1 + \sum_{j=2}^{Q} u_j(x)\, \psi_j(y).$$

Since ψ_j are orthonormal, then.

$$E[\tilde{u}_{hk}] = \int_\Gamma p(y)\, \tilde{u}_{h,k}(\cdot, y) \cdot 1\, dy = \langle \tilde{u}_{h,k}, \psi_1 \rangle_p$$

$$= \boxed{u_1 + w_g}.$$

$$Var[\tilde{u}_{hk}] = E[\tilde{u}_{hk}^2] - (u_1 + w_g)^2$$

$$= (u_1 + w_g)^2 + \sum_{2}^{Q} u_j^2 - (u_1 + w_g)^2$$

$$= \sum_{2}^{Q} u_j^2.$$

$Ax = b$.

So. like FEM. we have the linear equation. $Au = b$.

where A has block structure like

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1Q} \\ A_{21} & A_{22} & \cdots & A_{2Q} \\ \vdots & \vdots & & \vdots \\ A_{Q1} & A_{Q2} & \cdots & A_{QQ} \end{pmatrix} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_Q \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_Q \end{pmatrix}.$$

$u_j = (u_{1j} \ u_{2j} \ \cdots \ u_{Jj})^T.$ $\qquad \langle \psi_j, \psi_s \rangle_p = G_{j,s} K_0.$

$$A_{sj} = \boxed{\langle \psi_j, \psi_s \rangle_p} K_0 + \sum_{l=1}^{P} \langle y_l \psi_j, \psi_s \rangle_p K_l \qquad G \quad Q\times Q$$

$K_0 \cdot G \quad J\times J$

$$[K_0]_{ir} = \int_D \mu_a(x) \nabla \phi_i(x) \cdot \nabla \phi_r(x)\, dx \qquad i, r = 1 \cdots J.$$

$$[K_l]_{ir} = \int_D (\sqrt{v_l^a}\, \phi_l^a(x)) \nabla \phi_i(x) \cdot \nabla \phi_r(x)\, dx. \qquad G \otimes K_0.$$

similarly we can get $\frac{\psi}{D}$. $\qquad G_0 \otimes K_0 + \sum G_l \otimes K_l.$

Pass the solver. $\qquad k=3$ can fit well. $\qquad E \quad Var.$

gPC

Stochastic collocation method. $\quad$ UQ

$\xi(t,x)$

The second class of SPDE is the semilinear SPDE.
which include the random fluctuations like

$$u_t = \Delta u + \xi(t,x). \qquad t>0.\ x\in D.$$

So the $\xi(t,x)$ depends on both t and X. (space and time)
Rewritten as $du = \Delta u dt + dW(t,x)$.
More generally, we consider semilinear SPDES like.

$$du = [\Delta u + f(u)]dt + G(u)\, dW(t,x). \qquad L^2(D)$$

We prefer to think $u(t)$ as an H-valued stochastic process
$u(t,x,\omega)$. For fixed t, $u(t,\cdot,\cdot)$ is a R.F.
Called $u(\cdot,\cdot,\omega)$ a sample path. $\qquad u(t,\cdot,\omega)$ a realization.

Here we need to define the Q-Wiener process. $W(t)$

Before this, we need some assumptions. Consider a separatable Hilbert space U with $\|\cdot\|_U$, $\langle\cdot,\cdot\rangle_U$. We define a Q-Wiener process $\{W(t): t\geq 0\}$ as a U-valued process.

That is, $\forall$ $W(t)$ is a U-valued r.v.

When $t=1$, note the covariance operator as Q.

$\boxed{\text{Assumption:}}$ $Q\in\mathcal{L}(U)$ is non-negative definite and symmetric.

Further, Q has an orthonormal basis $\{\chi_j, j\in\mathbb{N}\}$ of eigenfunc with corresponding eigenvalue $q_j\geq 0$, s.t. $\sum q_j<\infty$.

$Qu=\sum u_j q_j \phi_j(x)$. $u=\sum u_j\phi_j(x)$. $Q\in\mathcal{L}(U)$. ϕ_j q_j

Def: A U-valued stochastic process $\{W(t): t\geq 0\}$ is a $\boxed{\text{Q-Wiener process}}$ if

$W(t)$ $W(t,x)$ $x\in D$

$W(t): t\longmapsto U$. $L^2(D)$

1) $W(0)=0$ a.s.

$W(t,x)$

2) $W(t)$ is a continuous function $\mathbb{R}^+\to U$ for each $\omega\in\Omega$.

3). $W(t)$ is $\mathcal{F}_t$-adapted and $W(t)-W(s)$ is independent of $\mathcal{F}_s$ ($\forall s<t$).

U. $L^2(D)$.

4) $W(t)-W(s)\sim \boxed{N(0,(t-s)Q)}$ for $\forall$ $0\leq s\leq t$.

Q χ_j q_j.

Thm. $W(t)$ is a Q-Wiener process iff

D.

$$W(t)=\sum_{j=1}^{\infty}\sqrt{q_j}\,\chi_j\,\beta_j(t)$$

($\beta_j(t)$ i.i.d $\mathcal{F}_t$ Brownian M.)

and it converges in $L^2(\Omega,U)$

$W(t):\mathbb{R}^+\to U$.

truncated form: $\sum^{J}\sqrt{q_j}\,\chi_j\,\beta_j(t)$.

$L^2(D)$

$\{u: D\to\mathbb{R}\}$

So. Let's consider the $L^2(D)$-valued Q-Wiener process $W(t)$

We want to approximate the sample path of $W(t)$.

Since the eigenvalue / func of Q is unknown.

If we have a finite set of sample points. $x_1 \cdots x_K \in D$.

try to seek $W(t, x_1) \cdots W(t, x_K)$ numerically.

Take time step as Δt_{ref} $\quad \Delta W(t) = \sqrt{\Delta t}\, \xi \sim N(0,1)$.

$$\Rightarrow W^J(t_{n+1}) - W^J(t_n) = \sqrt{\Delta t_{ref}} \sum_j^J \sqrt{q_j}\, \chi_j \xi_j^n .$$

$$\xi_j^n = \frac{\beta_j(t_{n+1}) - \beta_j(t_n)}{\sqrt{\Delta t_{ref}}} \sim N(0,1)$$

$W(t)$ - U-valued. $L^2(D)$.

E.g. Consider $H_0^r(0,a)$-valued process. We can choose

$K = J-1$ sample points $x_k = \frac{ka}{J}$. $k = 1 \cdots J-1$.

$\chi_j = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi j x}{a}\right)$ $\quad j = 1 \cdots J-1$.

$$\Rightarrow W^{J-1}(t_{n+1}, x_k) - W^{J-1}(t_n, x_k) = \sum_j^{J-1} b_j \sin\left(\frac{j\pi k}{J}\right) \xi_j^n .$$

$\left(b_j = \sqrt{\frac{2 q_j \Delta t_{ref}}{a}} \right)$.

$\int du = \int (Au\, dt + B\, dW)$ $\quad U$-valued.

Then we need to develop the stochastic integral like

$$\boxed{\int_0^t B(s)\, dW(s)} . \qquad \{B(s)\}$$

where $W(s)$ is a Q-Wiener process, take values in U.

Then first we need to identify a suitable class of $B(s)$.

We treat SPDE in Hilbert space $\boxed{H}$ and with Itô integral

take values in H. Hence we consider B that are $\mathcal{L}(U_0, H)$

valued process. $U_0 \subset U$ called "Cameron-Martin space".

Def: Let $U_0 = \{Q^{\frac{1}{2}}u : u \in U\}$. L_0^2 is the set of linear operators $B : U_0 \to H$ s.t.

$\int_0^t B(s) dW(s)$. U, $\boxed{L^2(D)}$.

$$\|B\|_{L_0^2} := \left(\sum_{j=1}^{\infty} \|BQ^{\frac{1}{2}} \chi_j\|_H^2 \right)^{\frac{1}{2}} = \|BQ^{\frac{1}{2}}\|_{HS(U,H)} < \infty.$$

L_0^2 is a Banach space with $\|\cdot\|_{L_0^2}$

$$Q^{\frac{1}{2}}u = \sum \lambda_i^{\frac{1}{2}} u_i \phi_i(x).$$

Rmk: Hilbert-Schmidt operators.

Let H, U be seperable Hilbert spaces with norms $\|\cdot\|$, $\|\cdot\|_U$ respectively. $\{\phi_j\}$ orthonormal basis of U. Define Hilbert-Schmidt norm $\|L\|_{HS(U,H)} := \left(\sum_{j=1}^{\infty} \|L\phi_j\|_U^2 \right)^{\frac{1}{2}}$

$HS(U,H) = \{L \in \mathcal{L}(U,H) : \|L\|_{HS(U,H)} < \infty\}$ is Banach.

$L \in HS(U,H)$ is a Hilbert-Schmitz operator.

If $U = H$, then $\|L\|_{HS} = \|L\|_{HS(H,H)}$

$$\|A\|_{HS(\mathbb{R}^d)} = \left(\sum \|Ae_j\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i,j} a_{ij}^2 \right)^{\frac{1}{2}}$$

~~Fro~~ x.

Then we build the H-valued Itô integral:

$$\int_0^t B(s) dW^J(s) = \sum_{j=1}^{J} \int_0^t B(s) \boxed{\sqrt{q_j}\, \chi_j} d\beta_j(s).$$

$\leftarrow Q$

exists a limit as $J \to \infty$. noted as $\boxed{\int_0^t B(s) dW(s)}$

Thm: Suppose $\{B(s) : s \in [0,T]\}$ is a L_0^2-valued predictable process s.t. $\int_0^T E[\|B(s)\|_{L_0^2}^2] ds < \infty$.

Then (1) for $\forall t \in [0,T]$. $\to$ H-valued.

$$\boxed{\int_0^t B(s) dW(s)} = \sum_{j=1}^{\infty} \int_0^t B(s) \sqrt{q_j}\, \chi_j d\beta_j(s).$$

is well-defined in $L^2(\Omega, H)$.

(12) And we have the Itô isometry:

$$E\left[\left\| \int_0^t B(s)\,dW(s) \right\|_H^2\right] = \int_0^t E[\|B(s)\|_{L_0^2}^2]\,ds.$$

(13) $\{\int_0^t B(s)\,dW(s)\}$ is an H-valued predictable process.

$L^2(D)$

$\nabla\cdot(a(x)\nabla\ )$. Δ.

Back to semilinear evolution equations.

$$du = [-Au + f(u)]dt + G(u)\,dW(t) \qquad u(0) = u_0 \quad (u_0 \in H).$$

Assume $A: D(A) \subset H \to H$ satisfy assumption (*). (**)

$f: H \to H \qquad G: H \to L_0^2$

A: non-negative.

$\exists\ \phi_j \in H.\ q_j > 0$,

$W(t)$.

E.g. (Stochastic heat equation).

$$du = \Delta u\,dt + \gamma\,dW(t), \qquad u(0) = u_0 \in L^2(D).$$

$\Rightarrow$ $H = L^2(D) = U$, $f = 0$ and $G(u) = \gamma I$.

$u(t,\cdot)$ $\quad W(t,\cdot)$.

$du = (-Au + f(u))dt + G(u)\,dW$.

Def: strong solution: A predictable H-valued process $\{u(t): t \in [0,T]\}$ is called a strong solution of (**):

$$u(t) = u_0 + \int_0^t [-Au(s) + f(u(s))]ds + \int_0^t G(u(s))\,dW(s). \quad \forall t \in [0,T]$$

(The restiction is strong. since $u(t)$ requires: $u(t) \in D(A)$)

$A = \Delta$

$dW(t) = \quad d\beta_j$.

Def: Weak solution:

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t [-\langle u(s), Av\rangle + \langle f(u(s)), v\rangle]ds$$
$$+ \int_0^t \langle G(u(s))\,dW(s), v\rangle \qquad \forall v \in D(A).$$

$$\left(\int_0^t \langle G(u(s))\,dW(s), v\rangle = \sum_{j=1}^{\infty} \int_0^t \langle G(u(s))\sqrt{q_j}\,\chi_j, v\rangle\, d\beta_j(s).\right.$$

Def: Mild solution:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))\,ds + \boxed{\int_0^t e^{-(t-s)A} G(u(s))\,dW(s)}.$$

where e^{-tA} is the semigroup generated by $-A$.

$\boxed{\text{SPDE}}$ $\quad u_j(t)\varphi_j(x) \quad$ $\boxed{\phi_j(x)\psi_j(\xi)}$.

$D\times\Gamma$.

ODEs.