第四次分享

1. Stationary Process { Definitions / How to generate. Code.
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2. Ramdom Field { Definitions / How to generate.

$\mathbb{C}$

Section 1. Stationary Gaussian Process. (Fourier tranform).

Def 1.1. $\{X(t): t\in\mathbb{R}\}$ is stationary if $\mu(t)$ is independent of t. Covariance function $C(s,t)$ only depends on $s-t$.

i.e. $C(s,t) = c(s-t)$ ($c(t)$ called stationary covariance func).

$C(s,t) = C(t,s) = c(t-s) = c(s-t) \Rightarrow c(t)$ is even.

$c(t) = c(-t)$.

Def 1.2. (non-negative definite) $c: \mathbb{R}\to\mathbb{C}$ is non-negative defi. if $\forall\, a_i, a_j \in \mathbb{C}$, $t_i\in\mathbb{R}$ s.t. $\sum_{i,j}^{n} a_i c(t_i - t_j)\overline{a_j} \geq 0$.

Thm 1.3. (Bochner). A function $c: \mathbb{R}\to\mathbb{C}$ is non-negative definite. and continuous. iff $c(t) = \int_{\mathbb{R}} e^{ivt}\, dF(v)$ where F is some finite measure on $\mathbb{R}$ ($F(\mathbb{R})<\infty$)

Thm 1.4 (Wiener - Khintchine). 1)$\Leftrightarrow$2)

1) $X(t)$ mean-square continuous. real-valued. stationary process. with covariance $c(t)$.

2) $C: \mathbb{R} \to \mathbb{R}$ s.t. $C(t) = \int_{\mathbb{R}} e^{ive} \, dF(v)$. for some finite measure F on $\mathbb{R}$

F called spectral distribution. If $\exists f$ s.t. $dF(v) = f(v)\,dv$.

$\Rightarrow f$ called spectral density distribution.

[Rmk] $C(t) = \sqrt{2\pi}\, \check{f}(t)$

If given $C(t): \mathbb{R} \to \mathbb{R} \Rightarrow f$ is even.

$$\Rightarrow f(v) = \frac{1}{\sqrt{2\pi}} \hat{C}(v).$$

$\Rightarrow$ If f is non-negative and integrable $\overset{\text{Thm 1.4}}{\Rightarrow}$ C is valid cov.

E.g. 1.5 (Exponential covariance). $C(t) = e^{-\frac{|t|}{\ell}}$ $(\ell > 0)$.

$$\Rightarrow f(v) = \frac{1}{\sqrt{2\pi}} \hat{C}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itv} \cdot e^{-\frac{|t|}{\ell}} \, dt.$$

$$= \frac{\ell}{\pi(1+\ell^2 v^2)}. \Rightarrow \text{Cauchy}\left(0, \frac{1}{\ell}\right).$$

$f(v)$. non-negative. $F(\mathbb{R}) = C(0) = 1 < \infty$

$\Rightarrow C(t)$. valid. $X(t)$.

E.g. 1.6 (Gaussian covariance) $C(t) = e^{-\left(\frac{t}{\ell}\right)^2}$ $(\ell > 0)$.

$$\Rightarrow f(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itv} e^{-\left(\frac{t}{\ell}\right)^2} dt = \frac{\ell}{\sqrt{4\pi}} e^{-\frac{\ell^2 v^2}{4}} \qquad N\left(0, \frac{2}{\ell^2}\right).$$

$\Rightarrow F(\mathbb{R}) = C(0) = 1 < \infty$. $\Rightarrow$ $C(t)$ valid.

E.g. 1.7. (Whittle-Matern covariance).

$$C_q(t) = \frac{|t|^q}{2^{q-1}\Gamma(q)} K_q(|t|). \qquad \text{(Bessel modified order } q\text{)}.$$

$$\Rightarrow f(v) = \frac{\Gamma(q+\frac{1}{2})}{\Gamma(q)\Gamma(\frac{1}{2})} \cdot \frac{1}{[1+v^2]^{q+\frac{1}{2}}}$$ valid!

when $q=\frac{1}{2}$ $\quad f(v) = \underbrace{\frac{1}{\pi(1+v^2)}}$. $\quad C_{\frac{1}{2}}(t) = e^{-t}$. (exponential $\ell = 1$).

Def 1.8 Mean-squre ~~di~~differentiability.

$\{X(t): t\in T\}$ is mean-square differentiable

if $\forall t \in T$, as $h \to 0$.

$$\left\| \frac{X(t+h)-X(t)}{h} - \frac{dX(t)}{dt} \right\|_{L^2(\Omega)} = E\left[\left| \frac{X(t+h)-X(t)}{h} - \frac{dX(t)}{dt} \right|^2 \right] \to 0$$

$\frac{dX(t)}{dt}$ called mean-square derivative.

Thm 1.9. (smooth) Consider $X(t)$, mean zero, spectral density $f(v)$.

if $\exists\, q,\ L_q > 0$ s.t. $f(v) \le \frac{L_q}{|v|^{2q+1}}$.

$\Rightarrow X(t)$ is n times mean-square differentiable. for $n<q$.

(W.M.C)

Section 2. Complex-valued R.V. and stochastic process.

Def 2.1 (Complex-valued R.V).

Z: a measurable function: $(\Omega, \mathcal{F}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

$Z = X + iY$. $\Rightarrow \mu(Z) = \mu(X) + i\mu(Y)$

$$Cov(Z_1, Z_2) = E[(Z_1 - \mu(Z_1))\overline{(Z_2 - \mu(Z_2))}].$$

$$Var(Z) = Var(X) + Var(Y).$$

Z is $\mathbb{C}^d$-valued R.V.

$Cov(Z_1, Z_2) = E[(Z_1 - \mu(Z_1))(Z_2 - \mu(Z_2))^*]$. complex-valu...

Thm 2.2. If X, Y uncorrelated. $\Rightarrow C_Z$ is $d \times d$ real-valued ma...

and $C_Z = C_X + C_Y$, Z (mean zero).

pf: $C_Z = E[(X+iY)(X+iY)^*]$.

$= E[XX^T] + E[YY^T] + iE[-XY^T + YX^T]$.

$\Rightarrow E[XY^T] = E[YX^T] = 0 \Rightarrow X, Y$ are uncorrelated.

$\Rightarrow C_Z = C_X + C_Y$. □

Def. 2.3. (Complex Gaussian $CN(\mu_Z, C_Z)$)

A $\mathbb{C}^d$-valued r.v. $Z = X + iY \sim CN(\mu_Z, C_Z)$. ($\mu_Z \in \mathbb{C}^d$, $C_Z \in \mathbb{R}^{d \times d}$)

if $Re(Z - \mu_Z)$ $Im(Z - \mu_Z)$ independent.

each has distribution $N(0, \frac{C_Z}{2})$.

$\Rightarrow X \sim N(Re(\mu_Z), \frac{C_Z}{2})$ $\quad Y \sim N(Im(\mu_Z), \frac{C_Z}{2})$.

$$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} Re(\mu_Z) \\ Im(\mu_Z) \end{pmatrix}, \begin{pmatrix} \frac{C_Z}{2} & 0 \\ 0 & \frac{C_Z}{2} \end{pmatrix}\right).$$

Def 2.4. (Complex-valued Gaussian process).

$\{Z(t) : t \in T\}$ $\forall t_1 \cdots t_N \in T$. $(Z(t_1) \cdots Z(t_N))^T \sim CN$.

$\mu_Z : T \to \mathbb{C}$, $C_Z : T \times T \to \mathbb{R} \Rightarrow Z \sim CN(\mu_Z, C_Z)$.

If $Z(t) \in L^2(\Omega, \mathbb{C})$ $\forall t$ $\quad Z(t)$ is second-order.

Section 3. How to generate? DKL. C.

(Circulant Embedding method)

DFT. $\quad C \Rightarrow \tilde{C} \xrightarrow{\text{DFT}} \square$ $\xi \sim CN(0, I)$

$X(t)$. $(0,T)$. uniformly. $t_n = n\Delta t$.

Def 3.1. (Toeplitz matrix). $\mathbb{R}^{N\times N}$ T.M.

$$C = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{1-N} \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ c_{N-1} & \cdots & \cdots & c_0 \end{pmatrix}.$$

determined by $\underline{(c_{1-N}\ c_{2-N} \cdots c_0 \cdots c_{N-1})^T}$ $\in \mathbb{R}^{2N-1}$.

If C is symmetric. $c_i = c_{-i}$ $i = 1 \cdots N-1$.

Def 3.2. (Circulant matrix). C.M

$$C = \begin{pmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & & c_2 \\ \vdots & \vdots & \vdots & & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{pmatrix}.$$

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-2} \\ c_{N-1} \end{pmatrix}.$$

If C is symmetric $\Rightarrow$ $c_i = c_{N-i}$ $(i = 1 \cdots N-1)$?

$\Rightarrow$ at most $\lfloor \frac{N}{2} \rfloor + 1$ unique entries.

Def 3.3.
$\mathbf{c}$ called $\boxed{\text{Hermitian vector}}$.

$\mathbf{c} \in \mathbb{R}^N$ $c_i = c_{N-i}$

$\mathbf{c} \in \mathbb{C}^N$ $c_i = \overline{c_{N-i}}$.

$WDW^* \Rightarrow WD^{\frac{1}{2}}D^{\frac{1}{2}}W^*$.

$\boxed{WD^{\frac{1}{2}}}$

☆ Thm 3.4 $\forall$ $\underline{C}$ is $N\times N$ real-valued circulant matrix. deter--d by the first column $\mathbf{c}_1$ $\Rightarrow$ $\boxed{C = WDW^*}$, $\underline{W}$ is the $N\times N$ Fourier matrix. D is a diagonal matrix $\underline{d = \sqrt{N}\, W^* \mathbf{c}_1}$

DFT $C \Rightarrow \tilde{C}$ DKL $\Rightarrow$ $O(N\log N)$

Def 3.5 (Fourier Matrix). $\underline{W}$ $d\times d$ $W_{lm} = w^{(l-1)(m-1)}/\sqrt{d}$.

Def 3.6. (Minimal circulant extension) S.T.M. C.M.

Given S.T.M. $C \in \mathbb{R}^{N\times N}$. $C_1 = (c_0 \ c_1 \cdots c_{N-1})^T$.

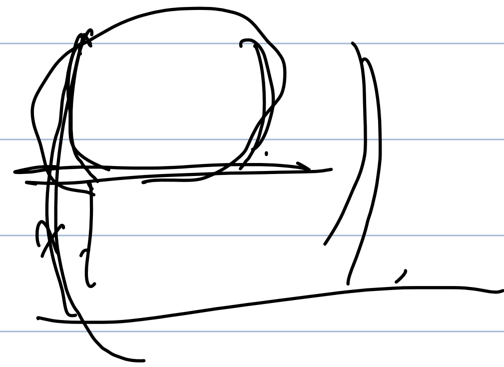
$\Rightarrow \tilde{C} \in \mathbb{R}^{2N' \times 2N'}$ $(N' = N-1)$.

$\underline{\tilde{C}_1} = [c_0 \ c_1 \cdots c_{N-2} \ c_{N-1} \ c_{N-2} \cdots c_1]^T \in \mathbb{R}^{2N'}$

E.g. 3.7. $C = \begin{pmatrix} 5 & 2 & 3 & 4 \\ 2 & 5 & 2 & 3 \\ 3 & 2 & 5 & \\ 4 & 3 & 2 & 5 \end{pmatrix}$

$$\Rightarrow \tilde{C} = \left(\begin{array}{cccc|cc} 5 & 2 & 3 & 4 & 3 & 2 \\ 2 & 5 & 2 & 3 & 4 & 3 \\ 3 & 2 & 5 & & & \\ 4 & 3 & 2 & 5 & 3 & 4 \\ \hline 3 & 4 & 3 & 2 & 5 & 3 \\ 2 & 3 & 4 & 3 & 2 & 5 \end{array}\right)$$

S.T.M. $\Rightarrow \tilde{C}$



Def 3.7. (padding) $C \Rightarrow C_1 = [c_0 \cdots c_{N-1}]^T$

(Circulant extension with padding) $x = [x_1 \cdots x_M]^T$.

$\underline{C_1^*} = [c_0 \cdots c_{N-1} \ x_1 \cdots x_M]^T \in \mathbb{R}^{N+M}$.

$\Rightarrow C^*$ S.T.M. $\Rightarrow \tilde{C}^* \in \mathbb{R}^{2N' \times 2N'}$ $N' = N \times M - 1$.

Case 1. $\underline{N(0, C)}$ $\underline{C}$ $\underline{\text{symmetric}}$, non-negative definite. S.C.M.

$\Rightarrow d_j$ real and non-negative. $\tilde{C}$

$\Rightarrow \underline{Z = WD^{\frac{1}{2}}\xi}$ $\xi \sim \underline{CN(0, 2I_N)}$

$\underline{Z \sim CN(0, 2C)}$.

$\Rightarrow$ two independent sample paths.

Case 2. C is S.T.M.

minimal. circulant extension $\Rightarrow \tilde{C}$

Case 3. $\tilde{C}$ indefinite. $D +$ $-$. $D^{\frac{1}{2}}$

padding. $N+M-1$ x.

$M \to 7N$. $\boxed{\rho(D_-) \to 0}$

$\tilde{Z}_1 = W D_+^{\frac{1}{2}} \xi$. $\tilde{\xi}_1 \sim CN(0, 2I_{2N'})$.

$\tilde{Z}_1 \sim CN(0, 2(\tilde{C} + \tilde{C}_-))$. $\tilde{C}_- = W D_- W^*$.

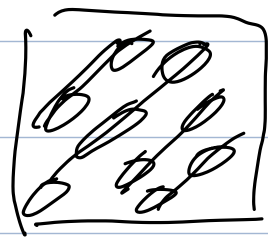
PDE with Random Data.

Random term.

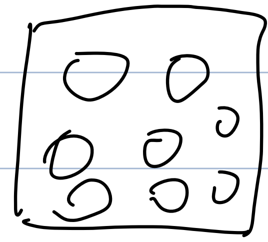
$\nabla \cdot (a(x) \nabla u(x)) = f(x)$ SPDE $a(x)$

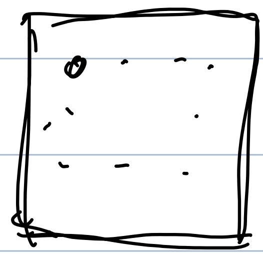
$a(x)$ Ramdom Field $\boxed{u(x)}$

Section 4. Ramdom Field. R.F.



- Stationary R.F. $(x-y)$ $x, y \in \mathbb{R}^d$.
- Isotropic R.F. $c(\|x-y\|_2)$
- anisotropic R.F (directionally dependent).





R.F. properties. $E[u(x)]$

$Var[u(x)]$.

Def 4.1. (R.F). $D \subset \mathbb{R}^d$ R.F $\{u(x) : x \in D\}$, a set of real-valued R.V on $(\Omega, \mathcal{F}, \mathbb{P})$. $(u : D \times \Omega \to \mathbb{R})$.

$X(t)$. $\forall \omega \in \Omega$. sample path.

$u(x)$ $\forall \omega \in \Omega$, realization.

Fix w $u(x)$ is determinate. $u(\cdot, w)$. $X(\cdot, w) \in L^2(\Omega)$

Def 4.2. (Second-order) $\forall x \in D$. $u(x) \in L^2(\Omega)$. second-order

$\Rightarrow E[u(x)]$ $C(x,y) = Cov(u(x), u(y))$,

$= E[- - -]$

Thm 4.3. (Mean-square regularity).

Assume $u(x)$ is second-order.

1) If $C \in C(D \times D) \Rightarrow u(x)$ is mean-square continuous,

i.e. $\| u(x+h) - u(x) \|_{L^2(\Omega)} \to 0$.

2). If $C \in C^2(D \times D) \Rightarrow u(x)$ is mean-square differentiable

i.e $\exists \frac{\partial u(x)}{\partial x_i}$ s.t. $\left\| \frac{u(x+he_i) - u(x)}{h} - \frac{\partial u(x)}{\partial x_i} \right\|_{L^2(\Omega)} \to 0$. as $h \to 0$.

$\frac{\partial u(x)}{\partial x_i}$ has $C_i(x,y) = \frac{\partial^2 C(x,y)}{\partial x_i \partial y_i}$.

Def 4.4. $u(x)$ $\forall x_1 \cdots x_m$.

$u = [u(x_1) \cdots u(x_m)]^T \sim N(\mu, C)$. $\mu(x_i) = \mu_i$ $C_{ij} = C(x_i, x_j)$

E.g. 4.5. (Brownian Sheet) $\beta(x)$ $(\mathbb{R}^+)^d = D$. $\mu(x) = 0$

$C(x,y) = \prod_{i=1}^{d} \min\{x_i, y_i\}$. $\to$ $\boxed{\text{Gaussian R.F}}$.

||.

$u(x) = W_1(x_1) W_2(x_2)$ Mean Covariance

p.d.f

Def 4.6. (Stationary R.F). $\{u(x) : x \in \mathbb{R}^d\}$. with mean $\mu(x)$ is independent of x. Covariance. $C(x,y) = C(x-y)$

$\Rightarrow$ $c(x)$ called stationary covariance function.

Thm 4.7 (Wiener-Khintchine) $1) \Leftrightarrow 2)$

1) $\exists$. S.R.F $\{u(x)\}$ with s.c.f $c(x)$ that is mean-square conti

2) $c: \mathbb{R}^d \to \mathbb{R}$ can be written as $c(x) = \int_{\mathbb{R}^d} e^{i\nu^T x}\, dF(\nu)$ for some finite measure on $\mathbb{R}^d$. $F(\mathbb{R}^d) < \infty$.

$$c(x) = (2\pi)^{\frac{d}{2}} \check{f}(x)$$

$$f(\nu) = (2\pi)^{-\frac{d}{2}} \hat{c}(\nu).$$

E.g 4.8 (Separable exponential covariance)

$$c(x) = \prod_{i=1}^{d} e^{-\frac{|x|}{\ell_i}} \qquad f(\nu) \text{ -- -}$$

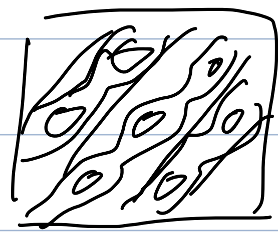
(Gaussian covariance)

$$c(x) = e^{-xA^Tx} \quad x \in \mathbb{R}^d. \qquad f(\nu).$$

$$\Downarrow$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \prod e^{-x_i^2} = e^{-r^2} \qquad \text{Isometry R.F.}$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

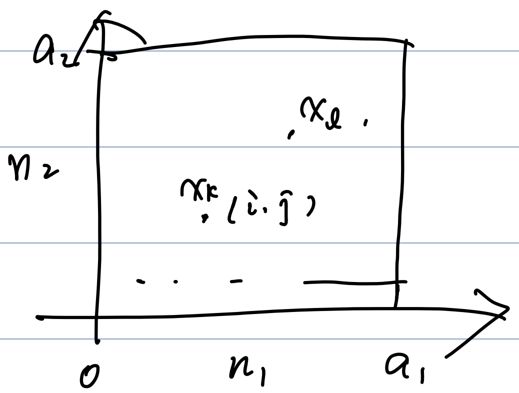


Def 4.9. (Isometry R.F).

$c(x)$ is invariant to rotation. $c(x) = c^{\circ}(r)$ $\quad r = \|x\|_2$

(W.K. thm) (Hankel transform).

Section 5 How to generate.



uniformly sample.

$$\Delta x_1 = \frac{a_1}{n_1 - 1} \qquad \Delta x_2 = \frac{a_2}{n_2 - 1}$$

$$x_{\underline{k}} = x_{\underline{i + j n_1}} = \begin{pmatrix} i \Delta x_1 \\ j \Delta x_2 \end{pmatrix}$$

$\Rightarrow u = [u(x_0) \cdots u(x_{N-1})]^T \quad N = n_1 n_2, \quad c(x).$

$$\Rightarrow c(x_k - x_l) = c\left(\begin{pmatrix} \underline{(i-r)\Delta x_1} \\ \underline{(j-s)}\Delta x_2 \end{pmatrix}\right)$$

$k = i + j n_1$ $\qquad$ $i, r = 0 \cdots n_1 - 1$

$l = r + s n_1$ $\qquad$ $j, s = 0 \cdots n_2 - 1$

$$(C_{js})_{ir} = c\left(\begin{pmatrix} \underline{(i-r)\Delta x_1} \\ \underline{(j-s)}\Delta x_2 \end{pmatrix}\right)$$

$$\underline{C} = \begin{pmatrix} \underline{C_{00}} & C_{01} & \cdots & C_{0, n_2-1} \\ C_{10} & C_{11} & \cdots & C_{1, n_2-1} \\ \vdots & \vdots & & \vdots \\ C_{n_2-1, 0} & C_{n_2-1, 1} & \cdots & C_{n_2-1, n_2-1} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{C_0} & C_{-1} & \cdots & C_{1-n_2} \\ C_1 & C_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{n_2-1} & \cdots & \cdots & C_0 \end{pmatrix}.$$

$C_{10} \quad C_{01} = c\begin{pmatrix} (i-r)\Delta x_1 \\ -\Delta x_2 \end{pmatrix}$

$= c\begin{pmatrix} (i-r)\Delta x_1 \\ \Delta x_2 \end{pmatrix}$

Note $m = j - s$.

T.M. $\qquad N = n_1 n_2$.

$$C \in \mathbb{R}^{n_1 n_2 \times n_1 n_2} = \mathbb{R}^{N \times N}$$

$$(C_m)_{ir} = c\left(\begin{pmatrix}(i-r)\Delta x_1 \\ m\,\Delta x_2\end{pmatrix}\right) \Rightarrow \text{T.M.}$$

Def. (BTTB M). (Block Toeplitz with Toeplitz Block matrix)

$$(C_{-m})_{ir} = c\left(\begin{pmatrix}(i-r)\Delta x_1 \\ -m\,\Delta x_2\end{pmatrix}\right) = c\left(-\begin{pmatrix}(r-i)\Delta x_1 \\ m\,\Delta x_2\end{pmatrix}\right) = (C_m)_{ri}$$

$\Rightarrow C_m^T = C_{-m}$,

$\Rightarrow$ C is Symmetric BTTB. M.

$\Downarrow$.

(Circulant Matrix),

BTTBM.

BCCB. M.

C_m. T.M.

2.

$\Downarrow$.

$\tilde{C}_m$ C.M.

B.T

$\Rightarrow$ B.C.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \\ 0 & 3 & 2 \\ 3 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

$\mathbb{R}^{4N\times 4N}$.

BCCB. M.

2D DFT.

$WD^{i}\xi$