2024.11.1 第三次SDE分享.

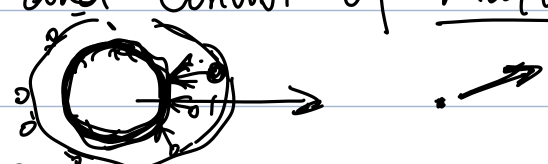
SODE $\begin{cases} \text{Euler-Maruyama} \\ \text{Milstein} \end{cases}$

Strong convergence

Weak convergence.

Code

Stochastic Modeling and Control of Multiscale System.

Micro → atom  B.M.

↓

Mesos → particle SODE

↓ $N \to \infty$

clusters SPDE $\begin{cases} \text{B.C. / I.C.} \\ \text{source term.} \end{cases}$  $u_0(x)$ $l \to 0$

↓ $l \to 0$

分布

Macro u PDE

解

Numerical methods for Itô SODE

$$\int_0^t f(u(s))\,dW(s) = \lim_{N\to 0} \underbrace{\sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} f(u(s))\,dW(s)}_{S_N(t)}$$

⇓

$$f(u(t_i))\left(W(t_{i+1}) - W(t_i)\right)$$

$$\frac{f(u(t_i)) + f(u(t_{i+1}))}{2}$$ Stratonovich. Integral

1. Review

1.1. Generally. SODE can be written as

$$du = f(u,t)\,dt + G(u,t)\,dW(t).$$

$u \in \mathbb{R}^d$ $\quad f(u,t): \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ $\quad W(t) \in \mathbb{R}^m.$ $\quad W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_m(t) \end{pmatrix}$

$W_i(t)$ i.i.d. $\underline{G(u,t)}: \mathbb{R}^d \times \mathbb{R} \to \underline{\mathbb{R}^{d\times m}}$

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & & \vdots \\ g_{d1} & g_{d2} & \cdots & g_{dm} \end{pmatrix}$$

$g_{ij}: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$

$f(u,t)$

自治系统 $\underline{f(u), \; G(u)}$

$$\underline{du = f(u)dt + G(u)dW(t)}$$

E.g.1. (Geometric Brownian Motion)

$$du = r u dt + \delta u \, dW(t) \qquad u(0) = u_0 \qquad \xi(t) \sim N(0, \cancel{dt})$$

$$\frac{du}{dt} = (r + \delta \xi(t)) u. \qquad \underline{dW(t)} \sim \sqrt{\Delta t}\, \xi \sim N(0,1)$$

E.g.2. (Ornstein-Uhlenbeck process) $P(t)$ $\qquad dW(t)$

$$\frac{dP(t)}{dt} = -\lambda P(t) + \delta \xi(t). \qquad P(0) = P_0 \qquad t_n$$

1.2. Itô Integral

$$\int_0^t X(s)\, dW(s) = \lim_{N\to\infty} S_N^X(t).$$

(Martingale property)

$$E\left[\int_0^t X(s)\, dW(s) \,\middle|\, \underline{\mathcal{F}_r}\right] = \int_0^r X(s)\, dW(s) \qquad (r < t)$$

(Itô Isometry)

$$E\left[\left\|\int_0^t X(s)\, dW(s)\right\|_2^2\right] = \int_0^t E\left[\|X(s)\|_2^2\right] dt.$$

2. Numerical method.

$$\begin{cases} u(t_{n+1}) = u(0) + \int_0^{t_{n+1}} f(u(s))ds + \int_0^{t_{n+1}} G(u(s))dW(s) \\ u(t_n) = u(0) + \int_0^{t_n} f(u(s))ds + \int_0^{t_n} G(u(s))dW(s). \end{cases}$$

$$\Rightarrow u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} f(u(s))\,ds + \int_{t_n}^{t_{n+1}} G(u(s))\,dW(s).$$

$$= f(u(t_n)) \int_{t_n}^{t_{n+1}} ds + G(u(t_n)) \int_{t_n}^{t_{n+1}} dW(s),$$

(Taylor expansion) $u \in C^{n+1}(\mathbb{R}^d, \mathbb{R}^m)$ $x \in \mathbb{R}^d$ $h \in \mathbb{R}^d$.

$$u(x+h) = u(x) + Du(x)\cdot h + \cdots + \frac{1}{n!} D^{(n)} u(x) h^{(n)} + R_n.$$

$$\Big(D^{(n)} u(x) \in \mathcal{L}(\underbrace{\mathbb{R}^d \times \mathbb{R}^d \cdots \times \mathbb{R}^d}_{n}, \mathbb{R}^m),\ h \in \mathbb{R}^d.$$

$$R_n = \frac{1}{n!} \int_0^1 (1-s)^n D^{(n+1)} u(x+sh) h^{(n+1)} ds. \Big).$$

For drift term f.

$$f(u(r)) = f(u(s)) + Df(u(s))(u(r) - u(s))$$
$$+ \int_0^1 (1-h) D^2 f(u(s) + h(u(r) - u(s)))(u(r) - u(s))^2\, dh.$$

For diffusion term G.

$$G(u(r)) = G(u(s)) + DG(u(s))(u(r) - u(s))$$
$$+ \int_0^1 (1-h) D^2 G(u(s) + h(u(r) - u(s)))(u(r) - u(s))^2\, dh.$$

$$R_f = f(u(r)) - f(u(s))$$
$$R_G = \int_0^1 (1-h) D^2 G(u(s) + h(u(r) - u(s)))(u(r) - u(s))^2\, dh.$$

$$\Rightarrow u(t) = u(s) + \int_s^t f(u(r))\,dr + \int_s^t G(u(r))\,dW_r \qquad (*)$$

$$= u(s) + \int_s^t [f(u(s)) + R_f]\,dr$$

$$+ \int_s^t [G(u(s)) + DG(u(s))(u(r) - u(s)) + R_G]\,dW_r,$$

$$= u(s) + f(u(s)) \int_s^t dr + G(u(s)) \int_s^t dW_r$$

$$\boxed{+ \int_s^t R_f\,dr + \int_s^t DG(u(s))(u(r) - u(s))\,dW_r + \int_s^t R_G\,dW_r.} = R_E.$$

Euler-Maruyama. Drop R_E.

$$u(t) = u(s) + f(u(s))(t-s) + G(u(s))\Delta W(s) \quad \left(\Delta W(s) = \int_s^t dW(r)\right)$$

Milstein $\int_s^t DG(u(s))\,(u(r)-u(s))\,dW(r)$ $\quad dt\,dW$.

$$u(t) - u(s) = G(u(s))\int_s^t dW(r) + R_1(t,s,u(s))$$

$$\Rightarrow \int_s^t \left[DG(u(s))\left(G(u(s))\int_s^r dW(p) + R_1\right)\right] dW(r).$$

(*) can be rewrited as.

$$u(t) = u(s) + f(u(s))(t-s) + G(u(s))\int_s^t dW(r)$$

$$\boxed{+\int_s^t DG(u(s))\,R_1\,dW(r) + \int_s^t R_G\,dW(r) + \int_s^t R_f\,dW(r)} = R_M.$$

$\Rightarrow$

$$u(t) = u(s) + f(u(s))(t-s) + G(u(s))\int_s^t dW(r)$$

$$+\int_s^t DG(u(s))\,G(u(s))\int_s^r dW(p)\,dW(r) \quad (**)$$

$\mathbb{R}^d$. K -

$$W(t) \in \mathbb{R}^m = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_m(t) \end{pmatrix}$$

$$\int_s^t \int_s^r dW(p)\,dW(r). = \frac{1}{2}\left(\Delta W_s^2 - (t-s)\right)$$

$$I_i(s,t) = \int_s^t dW_i(r).$$

Prop 1. $\int_s^t \int_s^r dW_i(p)\,dW_j(r) + \int_s^t \int_s^r dW_j(p)\,dW_i(p)$ $\quad (i \neq j)$

$$= \int_s^t dW_i(t) \cdot \int_s^t dW_j(t). = I_i \cdot I_j.$$

$W_i(t)$ $W_j(t)$

Pf: $LHS = \int_s^t (W_i(r) - W_i(s))\,dW_j(r) + \int_s^t (W_j(r) - W_j(s))\,dW_i(r)$

$$= \int_s^t W_i(r)\,dW_j(r) + \int_s^t W_j(r)\,dW_i(r) - W_i(s)\int_s^t dW_j(r)$$

$$- W_j(s)\int_s^t dW_i(r).$$

$$= \int_s^t d(W_i(r)W_j(r)) - W_i(s)(W_j(t)-W_j(s)) - W_j(s)(W_i(t)-W_i(s))$$

$$= W_i(t)W_j(t) - W_i(s)W_j(s) - \cdots$$

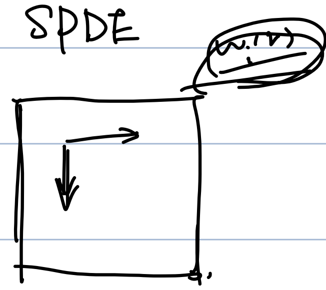
$$= [W_i(t)-W_i(s)][W_j(t)-W_j(s)] = \int_s^t dW_i(r)\cdot\int_s^t dW_j(r) = RHS.$$

prop2. $(i=j)$ $\int_s^t\int_s^r dW_i(p)\,dW_i(r) = \frac{1}{2}\left(\left(\int_s^t dW_i(r)\right)^2 - (t-s)\right)$

$$= \frac{1}{2}\left(I_i^2(s,t) - (t-s)\right).$$

Pf: $LHS = \int_s^t (W_i(r)-W_i(s))\,dW_i(r)$

SPDE



$$= \int_s^t W_i(r)\,dW_i(r) - W_i(s)\int_s^t dW_i(r).$$

$$\int_s^t W_i(r)\,dW_i(r) = \frac{1}{2}\int_s^t d(W_i^2(r)) - \frac{1}{2}\int_s^t dt.$$

$$= \frac{1}{2}\left(W_i^2(t) - W_i^2(s)\right) - \frac{1}{2}(t-s)$$

$$LHS = \frac{1}{2}\left(W_i^2(t) - W_i^2(s)\right) - \frac{1}{2}(t-s) - W_i(s)\left(W_i(t)-W_i(s)\right)$$

$$= \frac{1}{2}\left([W_i(t)-W_i(s)]^2\right) - \frac{1}{2}(t-s)$$

$$= \frac{1}{2}\left(\left(\int_s^t dW_i(r)\right)^2 - (t-s)\right) = \frac{1}{2}\left(I_i^2(s,t) - (t-s)\right) = RHS.$$

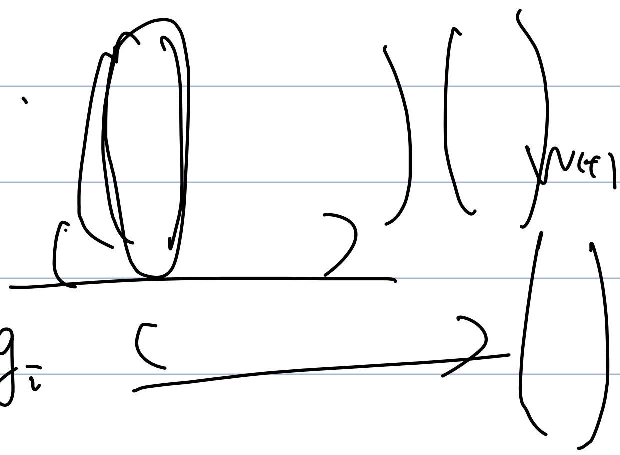
□

Note $A_{ij}(s,t) = \underbrace{\int_s^t\int_s^r dW_i(p)\,dW_j(r)}_{①} - \underbrace{\int_s^t\int_s^r dW_j(p)\,dW_i(r)}_{②}.$

$$\Rightarrow 2\int_s^t\int_s^r dW_i(p)\,dW_j(r) = I_iI_j + A_{ij}$$

$$2\int_s^t\int_s^r dW_j(p)\,dW_i(r) = I_iI_j - A_{ij}.$$

dW_1 $\quad$ G $\quad$ g_i



$$\int_s^t \underbrace{DG(u(s))}_{[d,m,d]} \left(\underbrace{G(u(s))}_{[d,m]} \underbrace{\int_s^r dW(p)}_{[m,1]} \right) \underbrace{dW(r)}_{[m,1]} \quad [d,1].$$

$[d,1]$

G:
$$\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots \\ g_{d1} & \cdots & g_{dm} \end{bmatrix}$$

DG
$$\begin{bmatrix} \left[\frac{\partial g_{11}}{\partial u_1} \cdots \frac{\partial g_{11}}{\partial u_d}\right], & \left[\frac{\partial g_{12}}{\partial u_1} \cdots \frac{\partial g_{12}}{\partial u_d}\right] & \cdots & [\ ] \\ & \frac{\partial g_{k\cdot}}{\partial u} & & \\ [\ ] & [\ ] & \cdots & [\ ] \end{bmatrix}$$

k

$$\left[\frac{\partial g_{kj}}{\partial u_1}\ \frac{\partial g_{kj}}{\partial u_2} \cdots \frac{\partial g_{kj}}{\partial u_d}\right]$$

the kth component.

$$\sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \cdot \sum_{i=1}^m g_{li} \int_s^r dW_i(p)\, dW_j(r) \quad (**)$$

$W(t) \in \mathbb{R}$

If $j=i$

$I_i(s,t) = \Delta W_i$

$$\sum_{i=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{ki}}{\partial u_l} g_{li} \int_s^r dW_i(p)\, dW_i(r)$$

$$= \sum_{i=1}^m \sum_{l=1}^d \frac{\partial g_{ki}}{\partial u_l} g_{li} \int_s^t \int_s^r dW_i(p)\, dW_i(r) \quad \text{by prop2.}$$

$$= \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} g_{li} \left(I_i^2(s,t) - (t-s) \right).$$

If $i \neq j$.

$$\sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \cdot \sum_{i\neq j} g_{li} \int_s^r dW_i(p)\, dW_j(r)$$

$$= \sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \left[\sum_{i<j} \cdots + \sum_{i>j} \cdots \right].$$

for $i<j$. $$\sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \sum_{i<j} g_{li} \int_s^r dW_i(p)\, dW_j(p)$$

$$= \sum_{j=1}^{m} \sum_{i<j}^{m} \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} g_{li} \int_s^t \int_s^r dW_i(p) dW_j(p).$$

$$= \frac{1}{2} \sum_{j=1}^{m} \sum_{i<j}^{m} \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} g_{li} \left(I_i I_j + A_{ij} \right)$$

for $i>j$.
$\Downarrow$
$i<j$.

$$\frac{\partial g_{ki}}{\partial u_l} g_{lj} \int_s^t \int_s^r dW_j(p) dW_i(p).$$

$$= \frac{1}{2} \sum \sum \sum \frac{\partial g_{ki}}{\partial u_l} g_{lj} \cdot (I_i I_j - A_{ij})$$

$$(**) = \frac{1}{2} \sum_{i=1}^{m} \sum_{l=1}^{d} \frac{\partial g_{ki}}{\partial u_l} \cdot g_{li} (I_i^2 - (t-s))$$

g G.

$$+ \frac{1}{2} \sum_{j=1}^{m} \Bigg[\sum_{i<j} \sum_{l=1}^{d} \left(\frac{\partial g_{kj}}{\partial u_l} g_{li} + \frac{\partial g_{ki}}{\partial u_l} \cdot g_{lj} \right) I_i I_j$$

$$- \sum_{i<j} \sum_{l=1}^{d} \left(\frac{\partial g_{kj}}{\partial u_l} g_{li} - \frac{\partial g_{ki}}{\partial u_l} \cdot g_{lj} \right) A_{ij} \Bigg] = R_k.$$

$$u_k(t_{n+1}) = u_k(t_n) + f_k(u(t_n)) \Delta t + \sum_{j=1}^{m} g_{kj}(u(t_n)) \Delta W_j + R_k$$

$G: \mathbb{R}^d \to \mathbb{R}^{d\times m}$, 1° G is independent of u. addieive

$$\begin{pmatrix} u_1 & 0 \\ 0 & 2u_2 \end{pmatrix}$$

$g_{ij} = C$. $R_k \equiv 0$.

Milstein is equal to Euler.

$W(t) \in \mathbb{R}$

$$u(t_{n+1}) = u(t_n) + f(u(t_n)) \Delta t + g(u(t_n)) \Delta W_n + \frac{1}{2} g'(u(t_n)) \cdot g(u(t_n)) \cdot (\Delta W^2 - \Delta t)$$

2° G is diagonal $g_{ij} \equiv 0 \quad (i \neq j)$.

$$u_k(t_{n+1}) = u_k(t_n) + f_k(u(t_n)) \Delta t + g_{kk}(u_n) \Delta W_k(t_n)$$

$$+ \frac{1}{2}\frac{\partial g_{kk}}{\partial u_k}(u_n)\, g_{kk}(u_n)\left((\Delta W_k^2(t_n)) - \Delta t\right)$$

□.

$$f(u(t)) = f(u(s)) + \underline{Df(u(s))\,(u(t)-u(s))} + R_f$$

$$\int_s^t \int_s^r dp\, dr \quad \times \qquad \int_s^t \int_s^r dW(p)\, dr \quad \times \qquad s \to t.$$

$$dr \cdot dW(p) \quad \times \qquad dW(p)\, dW(r)$$

Δt needs control.

Stability.

Geometric B.M.

$$du = ru\,dt + \delta u\, dW(t). \qquad u(0) = u_0,$$

$$u = u_0 e^{(r - \frac{\delta^2}{2})t + \delta W(t)}.$$

EM. $$u_{n+1} = u_n + r u_n \Delta t + \delta u_n \Delta W_n$$

$$= u_n (1 + r\Delta t + \delta \Delta W_n).$$

$$\Rightarrow t_n: \quad u_n = \prod_{j=0}^{n-1} \underline{(1 + r\Delta t + \delta \Delta W_j)}\, u_0$$

$$E[u_n^2] = \prod_{j=0}^{n-1} E[\underline{(1 + r\Delta t + \delta\Delta W_j)^2}]\, u_0^2$$

$$= \prod_{j=0}^{n-1} \left[(1 + r\Delta t)^2 + \delta^2 \Delta t\right] u_0^2$$

$$\Rightarrow \left|(1 + r\Delta t)^2 + \delta^2 \Delta t\right| = 1 + 2\Delta t\left(r + \frac{\delta^2}{2} + \frac{\Delta t \cdot r^2}{2}\right) < 1$$

$$\Rightarrow r + \frac{\delta^2}{2} + \frac{\Delta t}{2} \cdot r^2 < 0 \Rightarrow 0 < \Delta t < -\frac{2(r + \frac{\delta^2}{2})}{r^2}$$

Implicit method.

$$u_{n+1} = u_n + \underline{f(u_n)}\,\Delta t + G(u_n)\,\Delta W_n ,$$

$$= u_n + \left[(1-\theta) f(u_n) + \theta f(u_{n+1})\right]\Delta t + G(u_n)\,\Delta W_n$$

$$u$$

G.B.M.

$$u_{n+1} = u_n + r\left[(1-\theta)u_n + \theta u_{n+1}\right]\Delta t + \sigma u_n \Delta W_n$$

$$\Rightarrow u_n = \prod_{j=0}^{n-1} \left[(r(1-\theta)\Delta t + 1) + \sigma \Delta W_j\right] \Big/ (1 - r\theta\Delta t) \cdot u_0$$

$$\Rightarrow E[u_n^2] = \prod_{j=0}^{n-1} \frac{\left[(r(1-\theta)\Delta t + 1)^2 + \sigma^2 \Delta t\right]}{(1 - r\theta\Delta t)^2} \cdot u_0^2 .$$

$$1 + 2r(1-\theta)\Delta t + (r(1-\theta)\Delta t)^2 + \sigma^2 \Delta t < 1 - 2r\theta\Delta t + r^2\theta^2\Delta t^2$$

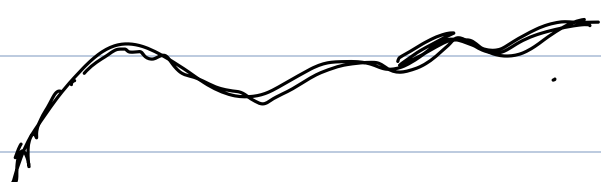
$$\Rightarrow \quad r + \frac{\sigma^2}{2} + \Delta t \cdot \frac{1-2\theta}{2} < 0 ,$$

Take $\theta = \frac{1}{2}$ $\quad \boxed{r + \frac{\sigma^2}{2} < 0}$. $\quad \Delta t$.

$\theta \neq \frac{1}{2} \Rightarrow \Delta t < \boxed{}$.

Strong Convergence.

path w. $\quad u_n(\cdot, w) \to u(\cdot, w) \quad (n \to \infty)$



$$\sup_{0\le t\le T} \| u(t_n) - u_n \|_{L^2(\Omega, \mathbb{R}^2)}.$$

$$= \sup_{0\le t\le T} E\left[\| u(t_n) - u_n \|_2^2\right]^{\frac{1}{2}} \longrightarrow 0.$$

E.M.M. $O(\Delta t^{\frac{1}{2}})$

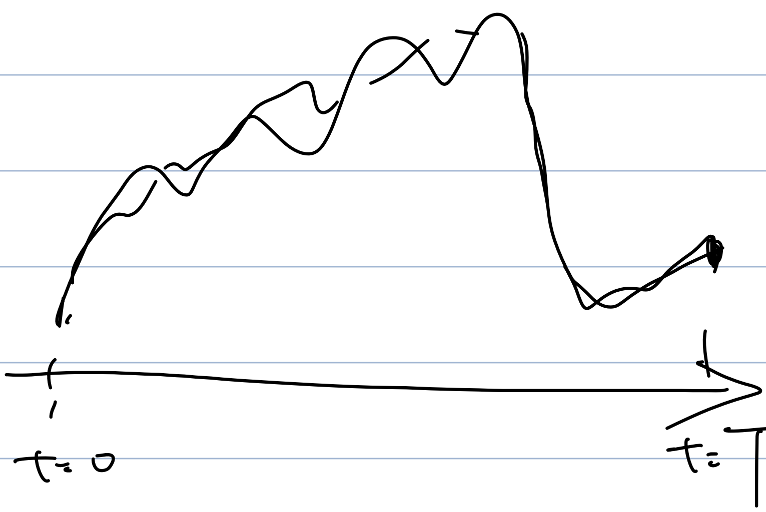
Milstein $O(\Delta t)$

R_M. R_1 R_f $\| \int R_f \, dt \| \le$ ______

R_G Δt^2.

Weak convergence. w. Milstein.

$$\lim_{\Delta t\to 0} E[\phi(u_n(t))] = E[\phi(u(t))]$$



$E[\phi(u(T))]$

MC: $\frac{1}{M}\sum_{j=1}^{M} \phi(u^j(T))$

$$E[\phi(u(T))] - \mu_M = \big(E[\phi(u(T))] - E[\phi(u_N)]\big).$$

$$+ \underbrace{E[\phi(u_N)] - \mu_M}_{\text{MC error}}.$$

Data. KL. expansion. $\longrightarrow W(t)$ B.M.

$t_n \longrightarrow t_{n+1}$.

$$W(t) = \sum_{j=0}^{\infty} \frac{\sqrt{2}}{(j+\frac{1}{2})\pi} \xi_j \sin\left((j+\tfrac{1}{2})\pi t\right) \qquad N(0,1),$$

$W(t)$ 連續. $\quad \downarrow$ truncated $\quad \frac{dW(t)}{dt} = W'(t)$

$$\int_0^1 W(t)\, W'(t)\, dt. \Longrightarrow \text{Stratonovich}.$$

$$\text{Itô} \quad \int_0^1 W(t)\, dW(t) = \frac{1}{2} W^2(1) - \frac{1}{2}$$

2道.

$$\frac{1}{2} W^2(1)$$

(Wong – Zakai.) Thm.

$$V_j \text{ s.t. } \frac{dV_j}{dt} = f(V_j) + g(V_j)\, W_j'(t) \qquad |V(t) - V_j(t)| \to 0.$$

$$V \text{ s.t. } dV = \tilde{f}(V) + g(V)\, dW(t).$$

$$\tilde{f} = f(V) + \frac{1}{2} g(V) g'(V).$$

Itô $\Longleftrightarrow$ Stratonovich.