Last course, review of key points:

core : def of stochastic integral.

$$\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \xrightarrow{L^2(\Omega)} b-a \qquad \mathbb{P}^n = \{a = t_0 < t_1 \cdots < t_n = b\}$$

simple process. $G(t,w) := \sum_{i=1}^{n-1} e_i(w) 1_{(t_i, t_{i+1}]} + e_0(w) 1_{\{0\}}(t)$

$\rightarrow$ dense.

$\updownarrow$ Itô isometry.

progressively measurable process $\xrightarrow{\text{contain}}$ left, right continuous

Thm. 1.20 (Itô formula)

Itô process. $dX_t = u(s,w)dt + v(s,w)dB_t$ diffusion

$(dt)^{\frac{1}{2}}$

$\Updownarrow$

$$X_t - X_0 = \int_0^t u(s,w)ds + \int_0^t v(s,w)dB_s$$

$Y_t = g(t, X_t)$ $\qquad g(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$. $\qquad$ dt.

SS

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$$

$$(dX_t)^2 = dX_t \cdot dX_t = \langle X_t, X_t \rangle = v^2 \cdot dt + o(dt)$$

Example 1.21. Let's us turn to $I = \int_0^t B_s \, dB_s$

check well-defined. since.

$$\mathcal{L}^2(0,T) = \{ \varphi(t,w) \mid \mathbb{E}[\int_0^T \varphi^2(t,w)\, ds] < \infty \}.$$

$$\mathbb{E}[\int_0^T B^2(s,w)\, ds] = \int_0^T \mathbb{E}[B^2(s,w)]\, ds = \int_0^T s\, ds = \frac{T^2}{2} < \infty.$$

choose $X_t = B_t$, $g(t,x) = \frac{x^2}{2}$, $Y_t = \frac{X_t^2}{2} = \frac{B_t^2}{2}$.

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB_t)^2$$

$$= \cancel{\frac{\partial g}{\partial t} dt} + \frac{\partial g}{\partial x} dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} dt = B_t \, dB_t + \frac{1}{2} dt.$$

hence $\frac{1}{2} B_t^2 = \int_0^t B_s \, dB_s + \frac{1}{2} t.$

$$I = \frac{1}{2} B_t^2 - \frac{t}{2}.$$

Example 1.22. choose $X_t = B_t$, $g(t,x) = e^{\lambda x - \frac{\lambda^2 t}{2}}$

$Y_t = g(t, X_t)$ $\quad dY_t = \frac{\partial g(t,x)}{\partial t} dt + \frac{\partial g(t,x)}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 g(t,x)}{\partial x^2}\langle dX_t, dX_t\rangle$

$$= \cancel{-\frac{\lambda^2}{2} e^{\lambda B_t - \frac{\lambda^2 t}{2}} dt} + \lambda e^{\lambda B_t - \frac{\lambda^2 t}{2}} dB_t + \cancel{\frac{1}{2}\lambda^2 e^{\lambda B_t - \frac{\lambda^2 t}{2}} dt}$$

$$= \lambda Y_t \, dB_t.$$

solves

$$\begin{cases} dY_t = \lambda Y_t \, dB_t. \\ Y(0) = 1 \end{cases} \longleftrightarrow Y_t = e^{\lambda B_t - \frac{\lambda^2 t}{2}}$$

Example 1.23 $d(tB_t) = B_t\,dt + t\,dB_t$. $\leftrightarrow$ check.

Lemma 1.24. Let X and Y are Itô process, then X_tY_t is a Itô process and

$$X_tY_t - X_0Y_0 = \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \underline{\langle X,Y\rangle_t}.$$

(Rmk1. example 1.23 is a special case of Lemma 1.24)

$\langle t, B_t\rangle = 0$, $\langle B_t, B_t\rangle = t$.

$$d\,X_tY_t = X_t\,dY_t + Y_t\,dX_t + \underline{d\langle X,Y\rangle_t}.$$

Rmk 2. For independent B.M., B^i and B^j

$$\langle B^i, B^j\rangle = 0 \quad \text{and} \quad \langle B^i, B^i\rangle = t.$$

Thm 1.25 (Itô) Let $X = (X_t^1, \dots \underline{X_t^d})$ where X_t^i $(1\le i\le d)$ are Itô process and $f\in C^2(\mathbb{R}^d)$. Then

$$f(X_t) - f(X_0) = \underline{\sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dX_s^i}$$
$$\underline{+\frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\,d\langle X^i,X^j\rangle_s}$$

$f(x,y) = x\cdot y$

Rmk 3. For $f\in C^2(\mathbb{R}^d)$, $\dfrac{\partial^2 f}{\partial x_i\partial x_j} = \dfrac{\partial^2 f}{\partial x_j\partial x_i}$.

Prof: we only consider $d=1$.

Step 1. consider $f(x)=x^n$.

Induction with Lemma 1.24

Step 2. $\forall f \in C^2(\mathbb{R})$ $|x| \leq a$. can be approximated by polynomical sequence $\{f_n\}$. s.t. f_n, f_n', f_n'' uniformly converge to f, f', f'' on $[-a, a]$

Step 3. Localization skill.

§2. Stochastic Differential Equation

§2.1 Linear SDEs

In this section we study the following linear SDEs

$$X_t = \eta + \int_0^t [b_s^1 X_s + b_s^0]\, ds + \int_0^t [\sigma_s^1 X_s + \sigma_s^0]\, dB_s \quad 0 \le t \le T. \qquad X_t(\omega), \quad X_0 = \eta$$

(i.e. $\int_0^t f(s)\, dg(s)$) $\Leftrightarrow g \in$ B.V. B.M. $\in$ B.V.)

introduce the adjoint process $\Gamma_0 = 1$

$$\Gamma_t = \exp\left(\int_0^t \beta_s\, dB_s + \int_0^t [\alpha_s - \tfrac{1}{2}|\beta_s|^2]\, ds\right) \to \text{known}$$

$$\frac{d\Gamma_t}{\Gamma_t} = [\alpha_t - \tfrac{1}{2}\cancel{|\beta_t|^2}]\, dt + \beta_t\, dB_t + \tfrac{1}{2}\cancel{|\beta_t|^2}\, dt.$$

$$= \alpha_t\, dt + \beta_t\, dB_t.$$

$$\Rightarrow d\Gamma_t = \Gamma_t [\alpha_t\, dt + \beta_t\, dB_t]. \quad \Gamma_0 = 0 \qquad (*)$$

$$d(\Gamma_t X_t) \overset{\text{Itô}}{=} \Gamma_t\, dX_t + X_t\, d\Gamma_t + d\langle \Gamma_t, X_t \rangle.$$

$$= \Gamma_t [b_t^1 \cancel{X_t} + b_s^0 + \alpha_t \cancel{X_t} + \beta_t(\sigma_t^1 \cancel{X_t} + \sigma_t^0)]\, dt$$

$$+ \Gamma_t [\sigma_t^1 \cancel{X_t} + \sigma_t^0 + \beta_t \cancel{X_t}]\, dB_t.$$

set $\beta = -\sigma^1$ $\quad \alpha = -b^1 + (\sigma^1)^2$.

$$\Rightarrow d(\Gamma_t X_t) = \Gamma_t [b_t^0 - \sigma_t^0 \sigma_t^1]\, dt + \Gamma_t \sigma_t^0\, dB_t.$$

$$\Gamma_t X_t - \Gamma_0 X_0 = \int_0^t (*)\, ds + \int_0^t \square\, dB_s$$

$$X_t = \Gamma_t^{-1}\left[\eta + \int_0^t \Gamma_s [b_s^0 - \sigma_s^0 \sigma_s^1]\, ds + \int_0^t \Gamma_s \cdot \sigma_0^s\, dB_s\right]. \qquad \square$$

§2.2. An Existence and uniqueness Result.

We now turn to general case.

$\sigma = 0$

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \underline{\int_0^t \sigma(s, X_s)\,dB_s} \in \mathbb{R}^n$$

or

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t. \\ X_t|_{t=0} = X_0 \end{cases}$$

Thm 2.1. Let $T > 0$ and $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$

$\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$. be measurable satisfying.

$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|)$ $x \in \mathbb{R}^n$, $t \in [0,T]$. ①

$\Uparrow$

and $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|$ ✓ ②

② + $|b(t,0) + \sigma(t,0)| < C$. $\Rightarrow$ ①

Let $Z \in L^2(\Omega)$, the the SDEs

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \\ X_0 = Z. \end{cases}$$

has a unique continuous solution $X_t(\omega)$. s.t.

$$\mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty, \quad X_t \in \mathcal{F}_t = \sigma(B_s : s \leq t)$$

Rmk: a) if $\underline{\sigma = 0}$. $dX_t = b(t, X_t)\,dt$, $X_0 = x_0 \to$ r.v.

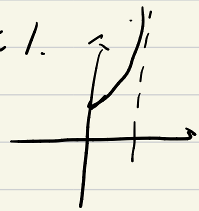
$$dX_t(\omega) = b(t, X_t(\omega))\,dt.$$

$$X_0(\omega) = x_0 \quad \downarrow$$

$$dX_t = AX_t\,dt \Rightarrow X_t = x_0 \cdot e^{At} \in \mathcal{F}_0 \subset \mathcal{F}_t.$$

b) The equation

$\begin{cases} \dfrac{dX_t}{dt} = X_t^2 \\ X_0 = 1 \end{cases}$ corresponding to $b(x) = x^2$ has the solution $X_t = \dfrac{1}{1-t}$, $0 \le t < 1$.



thus it's impossible to find a global solution.

c) The equation

$\begin{cases} \dfrac{dX_t}{dt} = 3X_t^{\frac{2}{3}} \\ X_0 = 0 \end{cases}$ has more than one solution.

In fact. $b = 3x^{\frac{2}{3}}$

$$X_t = \begin{cases} 0 & t \le a. \\ (t-a)^3 & t > a. \end{cases}$$

Proof of Thm 2.1.

Uniqueness. Consider the following two SDEs

$$\begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t \\ X_0 = Z \end{cases}$$

$$\begin{cases} d\hat{X}_t = b(t,\hat{X}_t)\,dt + \sigma(t,\hat{X}_t)\,dB_t \\ \hat{X}_0 = \hat{Z}. \end{cases} \qquad (\hat{Z} = Z)$$

Put $a(s,\omega) = b(s,X_s) - b(s,\hat{X}_s)$, $|a(s,\omega)| \le D|X_s - \hat{X}_s|$

$r(s,\omega) = \sigma(s,X_s) - \sigma(s,\hat{X}_s)$. $|r(s,\omega)| \le D|X_s - \hat{X}_s|$.

$$X_t - \hat{X}_t = Z - \hat{Z} + \int_0^t a(s,\omega)\,ds + \int_0^t r(s,\omega)\,dB_s.$$

$$\underline{\mathbb{E}[|X_t - \hat{X}_t|^2]} = \mathbb{E}\left[\left|Z - \hat{Z} + \int_0^t a(s,\omega)\,ds + \int_0^t r(s,\omega)\,dB_s\right|^2\right]$$

$$\le 3\,\underline{\mathbb{E}[|Z-\hat{Z}|^2]} + 3\mathbb{E}\Big[\underline{\Big(\int_0^t a(s,\omega)\,ds\Big)^2}\Big] + 3\mathbb{E}\Big[\underline{\Big(\int_0^t r(s,\omega)\,dB_s\Big)^2}\Big]$$

Itô

$$\le 3\mathbb{E}[|Z-\hat{Z}|^2] + 3\mathbb{E}\Big[\int_0^t 1\,ds\Big]\mathbb{E}\Big[\int_0^t a^2(s,\omega)\,ds\Big] + 3\mathbb{E}\Big[\int_0^t r^2(s,\omega)\,ds\Big]$$

$$\le 3\mathbb{E}[|Z-\hat{Z}|^2] + 3tD^2\,\mathbb{E}\Big[\int_0^t |X_s - \hat{X}_s|^2\,ds\Big] + 3D^2\,\mathbb{E}\Big[\int_0^t |X_s - \hat{X}_s|^2\,ds\Big]$$

$$= 3\mathbb{E}[|Z-\hat{Z}|^2] + 3(t+1)D^2\int_0^t \underline{\mathbb{E}[|X_s - \hat{X}_s|^2]\,ds}.$$

set $v(t) := \mathbb{E}[|X_t - \hat{X}_t|^2] \quad 0 \le t \le T$

$$v(t) \le E + A\int_0^t v(s)ds$$

By the Gronwall ineq. if $Z = \hat{Z}$ a.e $\Rightarrow X_t = \hat{X}_t$ a.e.

Existence.

we define $Y_t^0 = X_0$ where

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})\,ds + \int_0^t \sigma(s, Y_s^{(k)})\,dB_s \qquad (*)$$

similarly

$$\mathbb{E}[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le 3(1+T)D^2 \int_0^t \mathbb{E}[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\,ds$$

and $\mathbb{E}[|Y_t^{(1)} - Y_t^{(0)}|^2] \le A_1 t$.

by induction

$$\mathbb{E}[|Y_t^{(k+1)} - Y_t^{(k)}|^2] \le \frac{A_2^{k+1} t^{k+1}}{(k+1)!} \qquad 0 \le t \le T$$

therefore $\{Y_t^{(n)}\}$ is a Cauchy sequence in $\mathcal{L}^2[0,T]$.

For $(*)$, let $k \to +\infty$, we conclude

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s \qquad \text{a.s}$$

initial value

Thm 2.2. For any $\xi \in L^2_{\mathcal{F}_0}(\Omega, \mathbb{R}^n)$, we have

$$\mathbb{E}\big[\max_{0\le s\le T} |X_s|^2\big] \le K_T\,(1+\mathbb{E}|\xi|^2) \qquad 0\le t\le T$$

and
$$\mathbb{E}|X(t)-X(s)|^l \le \underline{K_T\,(1+\mathbb{E}|\xi|^l)}\,|t-s|^{\frac{l}{2}} \qquad (l\ge 1).$$

Moreover if $\hat{\xi} \in \mathcal{L}^2_{\mathcal{F}_0}(\Omega, \mathbb{R}^n)$ $\qquad \underline{|t-s|^{\frac{l}{2}} >> |t-s|^l}$

$\hat{X}(t)$ is the corresponding solution, we have

$$\mathbb{E}\max_{0\le s\le T} |X(s)-\hat{X}(s)|^2 \le \underline{K_T\,|\xi-\hat{\xi}|^2}.$$

Example 2.3 (Langevin equation)

$$dX_t = \alpha\, dB_t - \underline{\beta X_t\, dt}$$

$$d(e^{\beta t}X_t) = e^{\beta t}dX_t + \beta e^{\beta t}X_t\,dt = \alpha e^{\beta t}dB_t$$

$$e^{\beta t}X_t - X_0 = \alpha\int_0^t e^{\beta s}dB_s$$

$$X_t = e^{-\beta t}X_0 + \alpha e^{-\beta t}\int_0^t e^{\beta s}dB_s$$

Example 2.4. (Geometric B.M.).

Black-Scholes-Merton
1997 Nobel prize.

Consider Black-Scholes equation.

$$dX_t = b\underline{X_t}\,dt + \sigma \underline{X_t}\,dB_t$$

where b, σ are constant.

$$d(\ln X_t) = \frac{dX_t}{X_t} - \frac{1}{2X_t^2} d\langle X_t, X_t\rangle$$

$$= b\,dt + \sigma\,dB_t - \frac{\sigma^2 X_t^2}{2X_t^2} dt$$

$$= (b - \frac{\sigma^2}{2})\,dt + \sigma\,dB_t.$$

$$\ln X_t - \ln X_0 = (b - \frac{\sigma^2}{2})t + \sigma B_t$$

$$X_t = X_0 \exp[(b - \frac{\sigma^2}{2})t + \sigma B_t]$$