Reference:

L.C. Evans《An Introduction to stochastic differential equations》 ✓

B. OKsendal《Stochastic Differencetial Equations》 ✓

R. Durrett.《Probability Theory and Examples》 ✓

Shao Guojiang. Fudan. → Financial math..
Stochastic control → ....
→ ....

BSDE, SPDE, BSPDE, ... MFG.

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§1. A First Course.

§1.1. Conditional Expection.

Given a probability space $(\Omega, \mathcal{F}_0, \mathbb{P})$, a σ-field $\mathcal{F} \subset \mathcal{F}_0$ and a r.v. $X \in \mathcal{F}_0$ with $\mathbb{E}[X] < \infty$.

Def 1.1. We def the c.e. of X given $\mathcal{F}$, $(X \subset \mathcal{F}_0)$ to be any r.v. Y s.t.

(i) $Y \in \mathcal{F}$ i.e. $\mathcal{F}$ measurable.

(ii) $\int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P} \quad \forall A \in \mathcal{F}$.

$\mathbb{E}[Y] = \mathbb{E}[X]$

$\int_{\Omega_i} Y \, d\mathbb{P} = \int_{\Omega_i} X \, d\mathbb{P}$

$Y := \mathbb{E}[X|\mathcal{F}]$. (The representative element).

$\Rightarrow Y \cdot \mathbb{P}(\Omega_i)$

Example 1.2 Given $(\Omega, \mathcal{F}_0, \mathbb{P})$.

$\Omega = \biguplus_{i=1}^{n} \Omega_i$ with $\mathbb{P}(\Omega_i)$ $\quad \mathcal{F} = \sigma(\Omega_1, \cdots \Omega_n)$

$\mathcal{F} \subset \mathcal{F}_0$

$$\mathbb{E}[X|\mathcal{F}] = \frac{\int_{\Omega_i} X \, d\mathbb{P}}{\mathbb{P}(\Omega_i)} \quad \text{on } \Omega_i. \checkmark$$

A degenerate but important special case is $\mathcal{F} = \{\phi, \Omega\}$.

$$\mathbb{E}[X|\mathcal{F}] = \int_\Omega X \, d\mathbb{P} = \mathbb{E}[X] \text{ on } \Omega.$$

To start, ν is said to be absolutely continuous w.r.t μ ($\nu \ll \mu$). if $\mu(A)=0 \Rightarrow \nu(A)=0$

Radom-Nikodym. Theorem. Let μ and ν be σ-finite on $(\Omega, \mathcal{F})$. If $\nu \ll \mu$. there exists a function $f \in \mathcal{F}$. s.t. $\nu(A)=\int_A f \, d\mu$.

$(\mu(A)=0 \Rightarrow \nu(A)=0)$

$(\Omega, \mathcal{F}_0, \mathbb{P})$ Let $\mu=\mathbb{P}$ and $\nu(A):=\int_A X \, d\mathbb{P}$, $X \in \mathcal{F}_0$, $\underline{A \in \mathcal{F}}$.
$\nu \ll \mu$. R-N thm implies

$$\int_A X \, d\mathbb{P} = \nu(A) = \int_A f \, d\mathbb{P} \qquad f \in \mathcal{F}.$$, f denoted as $\frac{d\nu}{d\mu} \in \mathcal{F}$.

Thm 1.3. Assume $\mathbb{E}|X|$, $\mathbb{E}|Y| < \infty$, $X, Y \subset \mathcal{F}_0$
$\mathcal{F} \subset \mathcal{F}_0$. $\mathbb{E}[X|\mathcal{F}]$, $\mathbb{E}[X|\mathcal{F}]$

(a). $\mathbb{E}[aX+Y|\mathcal{F}] = a\,\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$. a.e.

check. take $A \subset \mathcal{F}$

$\int_A aX+Y \, d\mathbb{P} = \int_A \mathbb{E}[aX+Y|\mathcal{F}] \, d\mathbb{P}$ ✓

$\Downarrow$

$$a\int_A X \, d\mathbb{P} + \int_A Y \, d\mathbb{P} = a \cdot \int_A \mathbb{E}[X|\mathcal{F}] \, d\mathbb{P} + \int_A \mathbb{E}[Y|\mathcal{F}] \, d\mathbb{P}$$

$$= \int_A a\,\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}] \, d\mathbb{P}.$$

(b). If $X \le Y$, $\mathbb{E}[X|\mathcal{F}] \le \mathbb{E}[Y|\mathcal{F}]$ a.e.

(c). If $X \subset \mathcal{F} \Rightarrow \mathbb{E}[X|\mathcal{F}] = X$ a.s (d). If $X \in \mathcal{F}$. and $\mathbb{E}[|XY|] < \infty$ $\quad X, Y \in \mathcal{F}_0$ (☆)

$\Rightarrow \quad \mathbb{E}[XY|\mathcal{F}] = X\,\mathbb{E}[Y|\mathcal{F}]$.

(e). If X is independ of $\mathcal{F}$. (☆).

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

$$\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$$

take any $A \in \mathcal{F}$,

$$\int_A \mathbb{E}[X|\mathcal{F}]\, d\mathbb{P} = \underline{\int_A X\, d\mathbb{P} = \mathbb{P}(A)\, \mathbb{E}[X]}.$$

$$\Rightarrow \mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X].$$

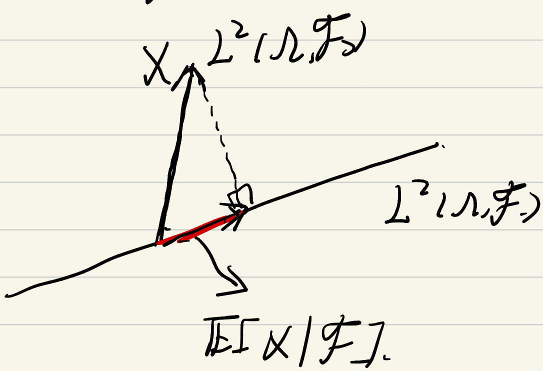
(f). If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_0$,

$$\mathbb{E}[\,\mathbb{E}[X|\mathcal{F}_1]\,|\,\mathcal{F}_2] = \mathbb{E}[\,\mathbb{E}[X|\mathcal{F}_2]\,|\,\mathcal{F}_1] = \mathbb{E}[X|\mathcal{F}_1].$$

$$L^2(\Omega, \mathcal{F}_0) \supset L^2(\Omega, \mathcal{F}) \qquad \mathcal{F} \subset \mathcal{F}_0$$

Thm 1.4. Suppose $\mathbb{E}X^2 < \infty$.
$\mathbb{E}[X|\mathcal{F}]$ is the r.v

$Y \in \mathcal{F}$ that minimizes the "error" $\mathbb{E}|X-Y|^2$.

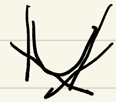


(It's an equivalent def).

$L^2(\Omega, \mathcal{F}, \mathbb{P})$ is dense in $L^1(\Omega, \mathcal{F}, \mathbb{P})$

Thm 1.6. If φ is convex. and $\mathbb{E}|X|, \mathbb{E}[|\varphi(X)|] < \infty$.

$$\varphi(\mathbb{E}[X|\mathcal{F}]) \le \mathbb{E}[\varphi(X)|\mathcal{F}]. \quad \text{a.e.} \checkmark$$



(Peng nonlinear expection. 2006). ✓

↳ bsde 1990. ⟷

nonlinear Feynman-Kac (PDE - BSDE - FBSDE)

nonlinear expectation. ICM

§ 1.2. Martingale and Brownian Motion

Def 1.7. Let $\mathcal{F}_n$ be a filtration, i.e. an increasing sequence of σ-field. If $\{X_n\}$ satisfies

(i) $\mathbb{E}[|X_n|] < \infty$ (ii) $X_n \subset \mathcal{F}_n$

(iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \quad \forall n$

"$\geq$" $\longleftrightarrow$ submartingale

"$\leq$" $\longleftrightarrow$ supermartingale.

Similarly. for a stochastic prozess $\{X_t\}$.

$$\mathcal{F}_t := \sigma\{X_s \mid 0 \le s \le t\} \qquad X_s^{-1}(B) \subset \mathcal{F}_0$$

intuitively find smallest σ-field

s.t. $X_s \subset \mathcal{F}_t$ ($\forall\, 0 \le s \le t$)

$\{X_t\}$ is a continuous martingale if

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \quad (t > s)$$

Def 1.8. $B(\cdot)$ is called Brownian motion if

(i) $B(0)=0$

(ii) $B(t)-B(s) \sim N(0, t-s)$

(iii) $\forall\ 0<t_1<t_2<\cdots<t_n$, the r.v.

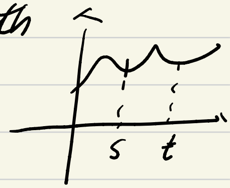
$$W(t_1),\ W(t_2)-W(t_1),\ \cdots\ W(t_n)-W(t_{n-1})$$

are independent.

(iv) For a.e. $w\in\Omega$, the sample path

$t\mapsto W(t,w)$ is continuous

$(\Omega, \mathcal{F}_0, \mathbb{P})$ $\qquad X_t(w)\in\mathbb{R}$, $(w\in\Omega)$



《Wiener 1923》 维纳

Def 1.9. f is uniformly Hölder continuous with r if

$$|f(t)-f(s)|\le k\,|t-s|^r \quad \forall\ s,t\in[0,T]$$

Thm 1.10. For $w\in\Omega$, $T>0$, the sample path

$t\mapsto B(t,w)$ is uniformly Hölder continuous

on $[0,T]$ for $0<r<\frac{1}{2}$. however

a.e. nowhere differentiable.

Example 1.11 Check B.M is a Martingle.

$$\mathbb{E}[B_t\,|\,\mathcal{F}_s]=\mathbb{E}[\underline{B_s}+B_t-B_s\,|\,\underline{\mathcal{F}_s}]=B_s+\mathbb{E}[B_t-B_s|\mathcal{F}_s]+$$

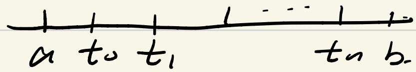
$$=B_s+\mathbb{E}[B_t-B_s]=B_s$$

$$\mathbb{E}[B_sB_t]=\mathbb{E}[\mathbb{E}[B_sB_t|\mathcal{F}_s]]=\mathbb{E}[B_s\,\mathbb{E}[B_t|\mathcal{F}_s]]$$

$$=\mathbb{E}[B_sB_s]=s$$

Prop 1.12. Suppose $\mathbb{P}^n := \{a = t_0 < t_1 < \cdots t_n = b\}$. and the mesh of $\mathbb{P}^n \longrightarrow 0$

$$\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \xrightarrow{L^2(\Omega)} b-a.$$



If $f(t) \equiv t$. $\quad \sum_{i=0}^{n-1} (f(t_{i+1}) - f(t_i)) = b-a$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (|f(t_{i+1}) - f(t_i)|^2) = 0$$

§1.3 Itô integral.

Def 1.13. A process is said to progressively measurable if $\forall t > 0$, the mapping

$$(w,s) \longmapsto X_s(w) \in \mathbb{R}$$

defined on $\Omega \times [0,t]$ is measurable for the $\mathcal{F}_t \otimes \mathcal{B}([0,t])$ $\longleftrightarrow$ $\mathcal{F}_\infty \times \mathcal{B}([0,t])$

Rmk: Specially, a right or left continuous adapted process is progressively measurable.

adapted $\Updownarrow$ $X_t \in \mathcal{F}_t$

Def 1.14. We denote by $\mathcal{L}^2(0,T)$. the space of P. measurable . s.t. $\mathbb{E}(\int_0^T G^2 dt) < \infty$.

$$L^2(\Omega) = \{X : \mathbb{E}|X|^2 < \infty\}.$$

Def 1.15. A process $G \in \mathcal{L}^2(0,T)$. is called a step process if $G(t,w) = \sum_{i=1}^{n-1} e_i(w) \underset{\sim}{1_{(t_i,t_{i+1}]}} + e_0 1_{\{0\}}(t)$

where $e_i \in \mathcal{F}_{t_i}$. (not $e_i \in \mathcal{F}_{t_{i+1}}$)

we define. Itô integral

$$\underline{\int_0^T G(t,w)\,dB_t \triangleq \sum_{i=1}^{n-1} e_i(w)(B_{t_{i+1}} - B_{t_i})} \to \text{a random variable}$$

Lemma 1.16 (Itô Isometry for a simple process)

For simple process $G(t,w)$, we have

$$\mathbb{E}\left|\int_0^T G(t,w)\,dB_t\right|^2 = \mathbb{E}\int_0^T |G(t,w)|^2 dt.$$

Proof.

$$LHS = \mathbb{E}\left|\sum_{i=1}^{n-1} e_i(w)(B_{t_{i+1}} - B_{t_i})\right|^2 \overset{?}{=} \mathbb{E}\left|\sum_{i=1}^{n-1} e_i^2(w)(B_{t_{i+1}} - B_{t_i})^2\right|$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left|(B_{t_{i+1}} - B_{t_i})^2 e_i^2(w)\right|$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left[\mathbb{E}\left[\underline{(B_{t_{i+1}} - B_{t_i})^2}\, \underline{e_i^2} \,\middle|\, \mathcal{F}_{t_i}\right]\right]$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left[e_i^2 (t_{i+1} - t_i)\right] = \mathbb{E}\int_0^T |G(t,w)|^2 dt.$$

Lemma 1.7. If $G \in \mathcal{L}^2[0,T]$, there exist a sequence of bounded step process $G^n \in \mathcal{L}^n[0,T]$ s.t. $\underline{\mathbb{E}[\int_0^T |G-G^n|^2 dt]} \to 0$

define

$$\int_0^T G(t,w)\, dB_t = \lim_{n\to\infty} \int_0^T G^n(t,w)\, dB_t$$

Thm 1.18. For $a,b \in \mathbb{R}$, $\forall\ \underline{G, H \in \mathcal{L}^2[0,T]}$ we have

(i) $\int_0^T (aG+bH)\, dB_t = a\int_0^T G\, dB_t + b\int_0^T H\, dB_t$.

G^n H^n G^n H^n

(ii) $\mathbb{E}(\int_0^T G(t,w)\, dB_t) = 0$ (*)

$\mathbb{E}(\int_0^T G^n(t,w)\, dB_t) = 0$

(iii) $\mathbb{E}[(\int_0^T G\, dB_t)^2] \overset{\text{Itô}}{=} \mathbb{E}\int_0^T G^2(t,w)\, dB_t$. (*)

G^n G^n

(iv) $\underline{\int_0^t G(t,w)\, dB_t}$ is a martingale. (*)

ξ_t.

G^n

Def 1.19. B_t 1-dim B.M. on $(\Omega, \mathcal{F}, \mathbb{P})$. Itô process

$$\underline{X_t} = X_0 + \int_0^t u(s,w)\,\underline{ds} + \int_0^t v(s,w)\,\underline{dB_s}$$

where $u, v \in \mathcal{L}^2(0,T)$.

Thm. (Itô formula) Let X_t be a Itô process

$$dX_t = u(t,w)\,dt + v(t,w)\,dB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$.

$dt \cdot dB_t = o(dt)$

$dt \cdot dt$

$\underline{dB_t \cdot dB_t \approx dt}$.

$$Y_t = g(t, X_t)$$

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t$$

$$+ \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\,(dX_t)^2$$

where $(dX_t)^2 = (u\,dt + v\,dB_t)^2 = v^2 (dB_t)^2 = v^2 dt$.