Lecture Notes (First introduce my work interest)

Theme: Numerical methods for Itô SODEs.

Notice the differences between different integral methods, like Itô, Stratonovich, etc.

Part 1. Riew.

Generally. SODE

$$du = f(u,t)\,dt + G(u,t)\,dW(t)$$

where $f: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ $\quad G: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d\times m}$.

We can write SODE as the form: 自治系统

$$du = f(u)\,dt + G(u)\,dW(t) \qquad u(0) = u_0$$

where $u_0 \in \mathbb{R}^d$, $u \in \mathbb{R}^d$ $\quad W(t) = [W_i(t)]_{i=1}^m \in \mathbb{R}^m$. ($W_i(t)$ is i.i.d B.M.) . So $G: \mathbb{R}^d \to \mathbb{R}^{d\times m}: u \mapsto G(u)$

Rmk. $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_d(t) \end{pmatrix}$ $\quad W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_m(t) \end{pmatrix}$ $\quad W_i(t)$ i.i.d.

$$G(u) = \begin{pmatrix} g_{11}(u) & \cdots & g_{1m}(u) \\ g_{21}(u) & \cdots & g_{2m}(u) \\ \vdots & \ddots & \vdots \\ g_{d1}(u) & \cdots & g_{dm}(u) \end{pmatrix}$$

1.1. Classical Model

E.g. 1. (Ornstein - Uhlenbeck process) Particle movement.

$$dP(t) = \underbrace{-\lambda P(t)\,dt}_{\text{dispative}} + \underbrace{\delta\, dW}_{\text{fluctuating}} \qquad P(0) = P_0$$

E.g. 2. (Geometric Brownian Motion) financial

$$du = ru\,dt + \delta u\, dW(t) \qquad u(0) = u_0$$

$$\Rightarrow \quad \frac{du}{dt} = (r + \delta \xi(t))\, dt$$

1.2. Itô Integral

$$\int_0^t X(s)\, dW(s) = \lim_{N\to\infty} S_N^X(t)$$

(Martingale property)

$$E\left[\int_0^t X(s)\, dW(s) \,\middle|\, \mathcal{F}_r\right] = \int_0^r X(s)\, dW(s)$$

in particular, $E\left[\int_0^t X(s)\, dW(s)\right] = \int_0^0 X(s)\, dW(s) = 0$

(Itô isometry)

$$E\left[\left\| \int_0^t X(s)\, dW(s) \right\|_2^2\right] = \int_0^t E\left[\|X(s)\|_2^2\right] dt.$$

Part 2. Numerical methods

Here 讨论自治系统.

Since

$$\begin{cases} u(t_{n+1}) = u_0 + \int_0^{t_{n+1}} f(u(s))\, ds + \int_0^{t_{n+1}} G(u(s))\, dW(s) \\ u(t_n) = u_0 + \int_0^{t_n} f(u(s))\, ds + \int_0^{t_n} G(u(s))\, dW(s) \end{cases}$$

$$\Rightarrow \quad u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} f(u(s))\, ds + \int_{t_n}^{t_{n+1}} G(u(s))\, dW(s)$$

$$\approx f(u(t_n))\Delta t + G(u(t_n))\Delta W_n$$

$$\left(\Delta t = \int_{t_n}^{t_{n+1}} dt , \quad \delta W_n = \int_{t_n}^{t_{n+1}} dW(t) \right)$$

Since we talk about Itô SODE, the $G(u(t))$ is chosen as $G(u(t_n))$, instead of $G(u(t))$ $t \in (t_n, t_{n+1}]$.

To obtain high-order numerical method, we identify the remainder term. Split $f(u)$ and $G(u)$ by Taylor Expansion.

(Taylor Expansion) If $u \in C^{n+1}(\mathbb{R}^d, \mathbb{R}^m)$, then

$$u(x+h) = u(x) + Du(x)h + \cdots + \frac{1}{n!} D^{(n)} u(x) h^{(n)} + R_n .$$

$$\left(D^{(n)} u(x) \in \mathcal{L}(\underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{n}, \mathbb{R}^m) \right), \quad h \in \mathbb{R}^d, \quad h^{(n)} = [\underbrace{h, h, \cdots, h}_{n}]$$

We have Integral Remainder:

$$R_n = \frac{1}{n!} \int_0^1 (1-s)^n D^{(n+1)} u(x+sh) h^{(n+1)} ds$$

Then we apply T.E. to f and G.

For drift term $f \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ 余项

$$f(u(r)) = f(u(s)) + \underline{Df(u(s))(u(r)-u(s))}$$
$$+ \underbrace{\int_0^1 (1-h) D^2 f(u(s)+h(u(r)-u(s)))(u(r)-u(s))^2 dh}. \quad (1)$$

Note $R_f(r,s;u(s)) = f(u(r)) - f(u(s)) =$

For diffusion $G \in C^2(\mathbb{R}^d, \mathbb{R}^{d\times m})$

$$G(u(r)) = G(u(s)) + DG(u(s))(u(r)-u(s)) + R_G(r,s;u(s)) \quad (2)$$

$$R_G(r,s;u(s)) = \int_0^1 (1-h) D^2 G(u(s)+h(u(r)-u(s)))(u(r)-u(s))^2 dh$$

Hense $u(t) = u(s) + \int_s^t \underline{f(u(r))} dr + \int_s^t \underline{G(u(r))} dW(r) \quad (*)$

$$= u(s) + f(u(s))(t-s) + G(u(s)) \int_s^t dW(r) + R_E(t,s,u(s)) \quad (3)$$

where $R_E(t,s,u(s)) = \int_s^t R_f(r,s,u(s))\,dr + \int_s^t R_G(r,s,u(s))\,dW(r)$

$$+ \int_s^t DG(u(s))(u(r)-u(s))\,dW(r) \quad \Delta \tag{4}$$

Q: ~~Rmk~~. Why we only expand f to $O(\Delta t)$? (Later).

2.1 Euler-Maruyama method

EM Method is found by dropping $R_E(t,s,u(s))$.

$$\Rightarrow u(t_{n+1}) = u(t_n) + f(u(t_n))\Delta t + G(u(t_n))\Delta W_n$$

2.2. Milstein method. (Add (Δ) term in (4))

Rewrite (3) as:

$$u(t) = \underbrace{u(s) + G(u(s))\int_s^t dW(r) + R_1(t,s,u(s))}$$

where $R_1(t,s,u(s)) = f(u(s))(t-s) + R_E(t,s,u(s))$

Hence $G(u(r)) = G(u(s)) + DG(u(s))\,(u(r)-u(s)) + R_G$

$$= G(u(s)) + DG(u(s))\,G(u(s))\int_s^r dW(p)$$

$$+ DG(u(s))R_1(r,s,u(s)) + R_G(r,s,u(s))$$

Apply $G(u(r))$ here and $f(u(r))$ to (*)

$$u(t) = u(s) + f(u(s))(t-s) + G(u(s))\int_s^t dW(r) \tag{**}$$

$$+ \int_s^t DG(u(s))\cdot G(u(s))\left(\int_s^r dW(p)\right)dW(r) + R_M(t,s,u(s))$$

where $R_M(t,s,u(s)) = \int_s^t DG(u(s))R_1(r,s,u(s))\,dW(r) + \int_s^t R_G\,dW(r)$

$$+ \int_s^t R_f(r,s,u(s))\,dr.$$

Then Milstein Method is found by dropping R_m.

Here we find the term $\int_s^t \int_s^r dW(p)\, dW(r)$. $\quad W(t) \in \mathbb{R}^m$.

Note $I_i(s,t) := \int_s^t dW_i(p)$, then we have some properties:

可以用于化简

prop1. $\displaystyle \int_s^t \int_s^r dW_i(p)\, dW_j(r) + \int_s^t \int_s^r dW_j(p)\, dW_i(r) = I_i(s,t)\, I_j(s,t)$

pf: Since W_i, W_j is independent, then.

$$LHS = \int_s^t (W_i(r) - W_i(s))\, dW_j(r) + \int_s^t (W_j(r) - W_j(s))\, dW_i(r)$$

$$= \int_s^t W_i(r)\, dW_j(r) + \int_s^t W_j(r)\, dW_i(r) - W_i(s)\int_s^t dW_j(r) - W_j(s)\int_s^t dW_i(r)$$

$$= \int_s^t d(W_i(r) W_j(r)) - W_i(s)(W_j(t) - W_j(s)) - W_j(s)(W_i(t) - W_i(s))$$

$$= W_i(t) W_j(t) - \cancel{W_i(s) W_j(s)} - W_i(s) W_j(t) + W_i(s) W_j(s)$$
$$- W_j(s) W_i(t) + \cancel{W_j(s) W_i(s)}$$

$$= [W_i(t) - W_i(s)][W_j(t) - W_i(s)] = \int_s^t dW_i(r) \cdot \int_s^t dW_j(r) = RHS. \quad \square$$

prop2. $\displaystyle \int_s^t \int_s^r dW_i(p)\, dW_i(r) = \frac{1}{2}\left(I_i(s,t)^2 - (t-s)\right)$

$$LHS = \int_s^t (W_i(r) - W_i(s))\, dW_i(r) = \int_s^t W_i(r)\, dW_i(r) - W_i(s)\int_s^t dW_i(r)$$

Since $\displaystyle \int_s^t W_i(r)\, dW_i(r) = \frac{1}{2}\int_s^t d(W_i^2(r)) - \frac{1}{2}\int_s^t dt$

$$= \frac{1}{2}(W_i^2(t) - W_i^2(s)) - \frac{1}{2}(t-s).$$

Hence $\displaystyle LHS = \frac{1}{2}(W_i^2(t) - W_i^2(s)) - \frac{1}{2}(t-s) - W_i(s) W_i(t) + W_i^2(s)$

$$= \frac{1}{2}\left([W_i(t) - W_i(s)]^2\right) - \frac{1}{2}(t-s)$$

$$= \frac{1}{2}\left(\int_s^t dW_i(r)\right)^2 - \frac{1}{2}(t-s) = \frac{1}{2}\left(I_i^2(s,t) - (t-s)\right) = RHS \quad \square$$

Define $A_{ij}(s,t) := \int_s^t \int_s^r dW_i(p)\, dW_j(r) - \int_s^t \int_s^t dW_j(p)\, dW_i(r)$

then $\begin{cases} I_i I_j + A_{ij} = 2\int_s^t \int_s^r dW_i(p)\, dW_j(r) \\ I_i I_j - A_{ij} = 2\int_s^t \int_s^r dW_j(p)\, dW_i(r) \end{cases}$

$\mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$

$G = [g_{ij}] \qquad g_{ij} : \mathbb{R}^d \to \mathbb{R}$

Consider the term $\int_s^t \underbrace{DG(u(s))}_{[d,m,d]} \underbrace{[G(u(s))}_{[d,m]} \underbrace{(\int_s^r dW(p))]}_{[m,1]} \underbrace{dW(r)}_{\mathbb{R}^m}$

$DG(u(s))$ $\begin{bmatrix} [\frac{\partial g_{11}}{\partial u_1} \cdots \frac{\partial g_{11}}{\partial u_d}], [\;], \cdots [\;] \\ \vdots \\ [\;], [\;] \cdots [\;] \end{bmatrix}$ $[d,m,d]$.

$g_{kj} \qquad \Delta_{li} \qquad W_j$

the kth component:

$$\sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \sum_{i=1}^m g_{li} \int_s^r dW_i(p)\, dW_j(p)$$

if $j=i$, $\sum_{i=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{ki}}{\partial u_l} g_{li} \int_s^r dW_i(p)\, dW_i(p)$

$$= \frac{1}{2} \sum_{i=1}^m \sum_{l=1}^d \frac{\partial g_{ki}}{\partial u_l} g_{li} \left(I_i^2 - (t-s) \right)$$

if $j \neq i$ $\sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \sum_{i \neq j} g_{li} \int_s^r dW_i(p)\, dW_j(p)$

$$= \sum_{j=1}^m \int_s^t \sum_{l=1}^d \frac{\partial g_{kj}}{\partial u_l} \left[\sum_{i<j} g_{li} \int_s^r dW_i(p)\, dW_j(p) + \sum_{i>j} g_{li} \int_s^r dW_i(p)\, dW_j(p) \right]$$

for $i<j$ $\sum_{j=1}^{m} \int_s^t \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} \overline{\sum_{i<j}} g_{li} \int_s^r dW_i(p)\, dW_j(p)$

$$= \sum_{j=1}^{m} \sum_{i<j}^{m} \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} g_{li} \int_s^t \int_s^r dW_i(p)\, dW_j(p)$$

$$= \frac{1}{2} \sum_{j=1}^{m} \overline{\sum_{i<j}} \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} g_{li} \left(I_i I_j + A_{ij}\right)$$

for $i>j$ $\sum_{j=1}^{m} \overline{\sum_{i>j}} \sum_{l=1}^{d} \frac{\partial g_{kj}}{\partial u_l} g_{li} \int_s^t \int_s^r dW_i(p)\, dW_j(p)$

$$= \sum_{j=1}^{m} \overline{\sum_{i<j}} \sum_{l=1}^{d} \frac{\partial g_{ki}}{\partial u_l} g_{lj} \int_s^t \int_s^r dW_j(p)\, dW_i(p)$$

$$= \sum_{j=1}^{m} \overline{\sum_{i<j}} \sum_{l=1}^{d} \frac{\partial g_{ki}}{\partial u_l} g_{lj} \left(I_i I_j - A_{ij}\right)$$

Hence $\frac{1}{2} \sum_{i=1}^{m} \sum_{l=1}^{d} \frac{\partial g_{ki}}{\partial u_l} \cdot g_{li} \left(I_i^2 - (t-s)\right)$

$$+ \frac{1}{2} \sum_{j=1}^{m} \Bigg[\overline{\sum_{i<j}} \sum_{l=1}^{d} \left(\frac{\partial g_{kj}}{\partial u_l} g_{li} + \frac{\partial g_{ki}}{\partial u_l} g_{lj} \right) I_i I_j$$

$$- \overline{\sum_{i<j}} \sum_{l=1}^{d} \left(\frac{\partial g_{kj}}{\partial u_l} g_{li} - \frac{\partial g_{ki}}{\partial u_l} g_{lj} \right) A_{ij} \Bigg] = R$$

$$\Rightarrow u_{k,n+1} = u_{k,n} + f_k(u_n)\Delta t + \sum_{j=1}^{m} g_{kj}(u_n)\Delta W_j + R \qquad \square$$

Rmk 1.

When the noise is additive, G is independent of u, then

$\frac{\partial g_{kj}}{\partial u_l} \equiv 0 \Rightarrow$ EM & Milstein are equivalent.

(Additive : G is independent of u)

Rmk2.

When G is diagonal, then $g_{ij}=0$ $(i\neq j)$, Hence.

Milstein reduced to:

$$u_{k,n+1}=u_{kn}+f_k(u_n)\Delta t+g_{kk}(u_n)\Delta W_{kn}$$
$$+\frac{1}{2}\frac{\partial g_{kk}}{\partial u_k}(u_n)g_{kk}(u_n)\left((\Delta W_{kn})^2-\Delta t\right).\ \star$$

#

Back to question mentioned, we have:

If we expand f to $O(\Delta t)$ as well, then

with $\begin{cases} f(u(r))=f(u(s))+Df(u(s))(u(r)-u(s))+R_f \\ G(u(r))=G(u(s))+DG(u(s))(u(r)-u(s))+R_G \end{cases}$

$$\Rightarrow u(t)=u(s)+f(u(s))\int_s^t dr+\underbrace{Df(u(s))\int_s^t(u(r)-u(s))dr}\ (*)$$
$$+G(u(s))\int_s^t dW(r)+\underbrace{DG(u(s))\int_s^t(u(r)-u(s))dW(r)}\ (**)$$
$$+\int_s^t R_f\,dr+\int_s^t R_G\,dW(r)$$

If $u(r)-u(s)=f(u(s))(r-s)+G(u(s))\int_s^r dW(p)$

then $(*)=Df(u(s))\int_s^t\left[f(u(s))(r-s)+G(u(s))\int_s^r dW(p)\right]dr$

$$=Df(u(s))\left[f(u(s))\underbrace{\int_s^t\int_s^r dp\,dr}_{=0}+G(u(s))\underbrace{\int_s^t\int_s^r dW(p)\,dr}_{=0}\right].$$
$$=0$$

$$(**)=DG(u(s))\left[f(u(s))\underbrace{\int_s^t\int_s^r dp\,dW(r)}_{=0}+G(u(s))\underbrace{\int_s^t\int_s^r dW(p)\,dW(r)}_{\neq 0}\right].$$

So, expand f to $O(1)$ is enough for Milstein Method. □

看实验，与例1.例2.

Part 3. (Implicit method)

3.1 Stability.

However, the time step needs to be controled for the stability of system.

E.g. Geometric B.M.

$$du = ru\,dt + \delta u\,dW(t) \qquad u(0) = u_0$$

We have the exact solution: $u = u_0 e^{(r - \delta^2/2)t + \delta W(t)}$

then $E[u^2(t)] = u_0^2 e^{(2r+\delta^2)t}$

E.M. method: $u_{n+1} = u_n + r u_n \Delta t + \delta u_n \Delta W_n$

$$\Rightarrow u_n = \prod_{j=0}^{n-1} (1 + r\Delta t + \delta \Delta W_j) u_0$$

$$\Rightarrow E[u_n^2] = \prod_{j=0}^{n-1} E[(1 + r\Delta t + \delta \Delta W_j)^2] u_0^2$$

$$= \prod_{j=0}^{n-1} \left[(1 + r\Delta t)^2 + \delta^2 \Delta t\right] u_0^2$$

$$\Rightarrow \left|(1 + r\Delta t)^2 + \delta^2 \Delta t\right| = 1 + 2\Delta t\left(r + \frac{\delta^2}{2} + \frac{\Delta t \cdot r^2}{2}\right) < 1$$

So $r + \frac{\delta^2}{2} + \frac{\Delta t}{2} \cdot r^2 < 0 \Rightarrow 0 < \Delta t < -\frac{2(r + \frac{\delta^2}{2})}{r^2}$

Then we consider (θ-Euler-Maruyama Method). It is an implicit method.

$$\Rightarrow u_{n+1} = u_n + \left[(1-\theta) f(u_n) + \theta f(u_{n+1})\right] \Delta t + G(u_n) \Delta W_n$$

Similiarly. for G.B.M.

$$u_{n+1} = u_n + r[(1-\theta)u_n + \theta u_{n+1}]\Delta t + \delta u_n \Delta W_n$$

$$(1 - r\theta\Delta t)u_{n+1} = [(r(1-\theta)\Delta t + 1 + \delta \Delta W_n] u_n.$$

$$\Rightarrow u_n = \prod_{j=0}^{n-1} [(r(1-\theta)\Delta t + 1) + \delta \Delta W_j] / (1 - r\theta\Delta t)\, u_0 .$$

$$E[u_n^2] = \prod_{j=0}^{n-1} \left[(r(1-\theta)\Delta t + 1)^2 + \delta^2 \Delta t\right] / (1 - r\theta\Delta t)^2 \cdot u_0^2$$

$$\Rightarrow \cancel{1} + 2r(1-\cancel{\theta})\Delta t + (r(1-\theta)\Delta t)^2 + \delta^2 \Delta t < \cancel{1} - 2\cancel{r\theta}\Delta t + r^2\theta^2\Delta t^2$$

$$\Rightarrow 2r\Delta t + [r^2(1-\theta)^2 - r^2\theta^2]\Delta t^2 + \delta^2 \Delta t < 0$$

$$\Rightarrow \Delta t \left[2r + \delta^2 + r^2(1-2\theta)\Delta t\right] < 0$$

$$\Rightarrow r + \frac{\delta^2}{2} + \Delta t \cdot \frac{1-2\theta}{2} < 0$$

So when $1-2\theta = 0$, i.e. $\theta = \frac{1}{2}$, the stability condition is independent of time step size.

Part 4. Strong approximation

the approximation about the sample path. i.e. for $\forall w \in \Omega$

$u_n(\cdot, w) \to u(\cdot, w)$ $(n \to \infty)$.

Define $\|\cdot\|_{L^2(\Omega, \mathbb{R}^d)} = E[\|\cdot\|_2^2]^{\frac{1}{2}}$

So we care about $\sup_{0 \le t_n \le T} \|u(t_n) - u_n\|_{L^2(\Omega, \mathbb{R}^d)}$

$$= \sup_{0 \le t_n \le T} E[\|u(t_n) - u_n\|_2^2]^{\frac{1}{2}}$$

We try to prove the strong convergence of EM M. and Milstein are $O(\Delta t^{\frac{1}{2}})$ and $O(\Delta t)$ respectively. $\sup_{0\le t_n \le T} \|u(t_n) - u_n\|_{L^2(\Omega, \mathbb{R}^d)} \le K\Delta t$

Review the last lecture. when we prove the existence and uniqueness of solution u.

We need ① the linear growth condition:

$$\begin{cases} \|f(u)\|_2^2 \le L(1+\|u\|_2^2) & \forall u \in \mathbb{R}^d. \\ \|G(u)\|_F^2 \le L(1+\|u\|_2^2) \end{cases}$$

② global Lipschitz condition:

$$\begin{cases} \|f(u_1) - f(u_2)\|_2 \le L\|u_1 - u_2\| & \forall u_1\ u_2 \in \mathbb{R}^d. \\ \|G(u_1) - G(u_2)\|_F \le L\|u_1 - u_2\| \end{cases}$$

Thm.

Assume ①② holds, $W(t)$ is an $\mathcal{F}_t$-adapted B.M. on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$

For $\forall T>0$, $u_0 \in \mathbb{R}^d$, then $\exists !\ u \in \mathcal{H}_{2,T}$ s.t. $t \in [0,T]$

$$u(t) = u_0 + \int_0^t f(u(s))\, ds + \int_0^t G(u(s))\, dW(s).$$

$\left(\mathcal{H}_{2,T} \text{ is } \{u(t): t\in[0,T]\} \text{ s.t. } \|u\|_{\mathcal{H}_{2,T}} := \sup_{t\in[0,T]} E\left[\|u(t)\|_2^2\right]^{\frac{1}{2}} < \infty \right)$

Recall that.

$$R_M(t,s,u(s)) = \int_s^t DG(u(s)) R_1(r,s,u(s))\, dW(r) + \int_s^t R_G\, dW(r)$$
$$+ \int_s^t R_f(r,s,u(s))\, dr.$$

We prove that $E\left[\left\|\sum_{t_j<t} R_M(t_{j+1}\wedge t; t_j; u(t_j))\right\|_2^2\right] \le K\Delta t^2$

pf:

$$LHS \le 3E\left[\left\|\sum_{t_j<t}\int_{t_j}^{t_{j+1}} R_f(r, t_j, u(t_j))\,dr\right\|_2^2\right]$$

$$+3E\left[\left\|\sum_{t_j<t}\int_{t_j}^{t_{j+1}} DG(u(t_j))R_1(r, t_j, u(t_j))\,dW(r)\right\|_2^2\right]$$

$$+3E\left[\left\|\sum_{t_j<t}\int_{t_j}^{t_{j+1}} R_G\,dW(r)\right\|_2^2\right]$$

Itô Isometry

$$= 3E\left[\left\|\sum_{t_j<t}\int_{t_j}^{t_{j+1}} R_f\,dr\right\|_2^2\right]$$

$$+3\sum_{t_j<t}\int_{t_j}^{t_{j+1}}\left(E\left[\left\|DG(u(t_j))\underbrace{R_1(r, t_j, u(t_j))}\right\|_F^2\right]+E\left[\left\|R_G(r, t_j, u(t_j))\right\|_F^2\right]\right)dr$$

provided $E\left[\left\|R_1(t, s, u(s))\right\|_2^2\right] \le K(t-s)^2$

$$E\left[\left\|R_G(t, s, u(s))\right\|_2^2\right] \le K(t-s)^2$$

$\sim \le K\Delta t^2$ $\square$

Part 5. Weak approximation.

$$\lim_{\delta t \to 0} E[\phi(u_n(t))] = E[\phi(u(t))]$$

We consider the approximation of $E[\varphi(u(T))]$ $(\varphi : \mathbb{R}^d \to \mathbb{R})$

E.g. We want to know the average quantities $E[\varphi(u(T))]$

then we generate M independent sample paths, then

$$\mu_M = \frac{1}{M} \sum_{j=1}^{M} \varphi(u_N^j)$$

Then

$$E[\varphi(u(T))] - \mu_M = \underbrace{\left(E[\varphi(u(T))] - E[\varphi(u_N)]\right)}_{\text{discretization error}} + \underbrace{\left(E[\varphi(u_N)] - \mu_M\right)}_{\text{MC error}}$$

Stratonovich Integrals and SODE.

Define $W'(t) = \frac{dW(t)}{dt}$ by K-L Expansion.

$$W(t) = \sum_{j=0}^{\infty} \frac{\sqrt{2}}{(j+\frac{1}{2})\pi} \xi_j \sin((j+\tfrac{1}{2})\pi t) \qquad \xi_j \sim N(0,1) \text{ i.i.d.}$$

then we can have the truncated form:

$$W_J(t) = \sum_{j=0}^{J} \frac{\sqrt{2}}{(j+\frac{1}{2})\pi} \xi_j \sin((j+\tfrac{1}{2})\pi t)$$

then we have $W_J'(t) = \sum_{j=0}^{J} \sqrt{2}\, \xi_j \cos((j+\frac{1}{2})\pi t)$

E.g. $\int_0^1 W(t) W'(t)\, dt \to \frac{1}{2} W(1)^2$ as $J \to \infty$.

Thm (Wong-Zakai)

$v_J(t)$ s.t. $\frac{dv_J}{dt} = f(v_J) + g(v_J) W_J'(t)$

$v(t)$ s.t. $dv = \tilde{f}(v)\, dt + g(v)\, dW(t)$

where the modified drift $\tilde{f}(v) = f(v) + \frac{1}{2} g(v) g'(v)$.

Then $\sup_{0 \le t \le 1} |v(t) - v_J(t)| \to 0$ as $J \to \infty$.