About Stochastic Differential Equation

Guanyu Chen

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1 Fokker-Planck-Kolmogorov equation

Problem 1. *Assume we have a Stochastic Differential Equation like:*

$$dX_t = f(X_t, t)dt + G(X_t, t)dW_t \tag{1}$$

where $X_t \in \mathbf{R}^d, f \in \mathcal{L}(\mathbf{R}^{d+1}, \mathbf{R}^d)$*, and* W_t *is m-dim Brownian Motion with diffusion matrix* Q*,* $G(X_t, t) \in \mathcal{L}(\mathbf{R}^{m+1}, \mathbf{R}^d)$*, with initial condition* $X_0 \sim p(X_0)$*.*

Definition 1 (Generator)**.** *The infinitesimal generator of a stochastic process* $X(t)$ *for function* $\phi(x)$*, i.e.* $\phi(X_t)$ *can be defined as*

$$\mathcal{A}\phi(X_t) = \lim_{s\to 0^+} \frac{E[\phi(X(t+s)] - \phi(X(t))}{s} \tag{2}$$

Where ϕ *is a suitable regular function.*

This leads to Dynkin's Formula very naturally.

Theorem 1 (Dynkin's Formula)**.**

$$E[f(X_t)] = f(X_0) + E\left[\int_0^t \mathcal{A}(f(X_s))ds\right] \tag{3}$$

Theorem 2. *If* $X(t)$ *s.t. 1, then the generator is given:*

$$\mathcal{A}(\cdot) = \sum_i \frac{\partial(\cdot)}{\partial x_i} f_i(X_t, t) + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2(\cdot)}{\partial x_i \partial x_j}\right)\left[G(X_t, t)QG^\top(X_t, t)\right]_{ij} \tag{4}$$

Proof. See P119 of SDE by Oksendal. □

Example 1. *If* $dX_t = dW_t$*, then* $\mathcal{A} = \frac{1}{2}\Delta$*, where* Δ *is the Laplace operator.*

Definition 2 (Generalized Generator)**.** *For* $\phi(x, t)$*, i.e.* $\phi(X_t, t)$*, the generator can be defined as:*

$$A_t\phi(x, t) = \lim_{s\to 0^+} \frac{E[\phi(X(t+s), t+s)] - \phi(X(t), t)}{s} \tag{5}$$

Theorem 3. *Similarly if* $X(t)$ *s.t. 1, then the generalized generator is given:*

$$\mathcal{A}_t(\cdot) = \frac{\partial(\cdot)}{\partial t} + \sum_i \frac{\partial(\cdot)}{\partial x_i} f_i(X_t, t) + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2(\cdot)}{\partial x_i \partial x_j}\right)\left[G(X_t, t)QG^\top(X_t, t)\right]_{ij} \tag{6}$$

We want to consider the density distribution of $X_t, P(x, t)$

Theorem 4 (Fokken-Planck-Kolmogorov equation)**.** *The density function* $P(x, t)$ *of* X_t *s.t. 1 solves the PDE:*

$$\frac{\partial P(x,t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left[f_i(x,t)p(x,t)\right] + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GQG^\top\right)_{ij} P(x,t)\right] \tag{7}$$

The PDE is called FPK equation / forward Kolmogorov equation.

Proof. Consider the function $\phi(x)$, let $x = X_t$ and apply Ito's Formula:

$$\begin{aligned} d\phi &= \sum_i \frac{\partial\phi}{\partial x_i} dx_i + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right) dx_i dx_j \\ &= \sum_i \frac{\partial\phi}{\partial x_i}\left(f_i\left(X_t, t\right)dt + \left(G\left(X_t, t\right)dW_t\right)\right) + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)\left[G(X_t, t)QG^\top(X_t, t)\right]_{ij} dt. \end{aligned} \tag{8}$$

Take expectation of both sides:

$$\frac{dE[\phi]}{dt} = \sum_i E\left[\frac{\partial\phi}{\partial x_i} f_i(X_t, t)\right] + \frac{1}{2}\sum_{ij} E\left[\frac{\partial^2\phi}{\partial x_i \partial x_j}\left[GQG^\top\right]_{ij}\right] \tag{9}$$

So

$$\begin{cases} \frac{dE[\phi]}{dt} = \frac{d}{dt}\left[\int \phi(x)P(X_t = x,t)dx\right] = \int \phi(x)\frac{\partial P(x,t)}{\partial t}dx \\ \sum_i E\left[\frac{\partial \phi}{\partial x_i}f_i\right] = \sum_i \int \frac{\partial \phi}{\partial x_i}f_i(X_t = x,t)Pdx = -\sum_i \int \phi \cdot \frac{\partial}{\partial x_i}\left[f_i(x,t)p(x,t)\right]dx. \\ \frac{1}{2}\sum_{ij} E\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}\left[GQG^\top\right]_{ij}\right] = \frac{1}{2}\sum_{ij}\int \frac{\partial^2 \phi}{\partial x_i \partial x_j}\left[GQG^\top\right]_{ij}Pdx = \frac{1}{2}\sum_{ij}\int \phi(x)\frac{\partial^2}{\partial x_i \partial x_j}\left(\left[GQG^\top\right]_{ij}P\right)dx. \end{cases} \tag{10}$$

then

$$\int \phi \frac{\partial P}{\partial t}dX = -\sum_i \int \phi \frac{\partial}{\partial x_i}\left(f_i P\right)dX + \frac{1}{2}\sum_{ij}\int \phi \frac{\partial^2}{\partial x_i x_j}\left(\left[GQG^\top\right]_{ij}P\right)dx$$

Hence

$$\int \phi \cdot \left[\frac{\partial P}{\partial t} + \sum_i \frac{\partial}{\partial x_i}\left(f_i P\right) - \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\left[GQG^\top\right]_{ij}P\right)\right]dX = 0$$

Therefore P s.t.

$$\frac{\partial P}{\partial t} + \sum_i \frac{\partial}{\partial x_i}\left(f_i(x,t)P(x,t)\right) - \frac{1}{2}\sum_{i=1}\frac{\partial^2}{\partial X_i \partial X_j}\left(\left[GQG^\top\right]_{ij}P(x,t)\right) = 0 \tag{11}$$

Which gives the FPK Equation. □

Remark 1. *When SDE is time independent:*

$$dX_t = f(X_t)dt + G(X_t)dW_t \tag{12}$$

then the solution of FPK often converges to a stationary solution s.t. $\frac{\partial P}{\partial t} = 0$.

Here is an another way to show FPK equation: Since we have inner product $\langle \phi, \psi \rangle = \int \phi(x)\psi(x)dx$. Then $E[\phi(x)] = \langle \phi, P \rangle$.

As the equation 9 can be written as

$$\frac{d}{dt}\langle \phi, P \rangle = \langle \mathcal{A}\phi, P \rangle \tag{13}$$

Where $\mathcal{A}$ has been mentioned above. If we note the adjoint operator of $\mathcal{A}$ as $\mathcal{A}^*$, then we have

$$\langle \phi, \frac{dP}{dt} - \mathcal{A}^*(P) \rangle = 0, \forall \phi(x) \tag{14}$$

Hence we have

Theorem 5 (FPK Equation)**.**

$$\frac{dP}{dt} = \mathcal{A}^*(P), \text{where}\, \mathcal{A}^*(\cdot) = -\sum_i \frac{\partial}{\partial x_i}\left(f_i(x,t)(\cdot)\right) + \frac{1}{2}\sum_{i=1}\frac{\partial^2}{\partial X_i \partial X_j}\left(\left[GQG^\top\right]_{ij}(\cdot)\right) \tag{15}$$

Theorem 6 (Transition Density(Forward Komogorov Equation))**.** *The transition density* $P_{t|s}(x_t|x_s), t \geq s$, *which means the propability of transition from* $X(s) = x_s$ *to* $X(t) = x_t$*, satisfies the FPK equation with initial condition* $P_{s|s}(x|x_s) = \delta(x - x_s)$ *i.e. for* $P_{t|s}(x|y)$*, it solves*

$$\frac{\partial P_{t|s}(x|y)}{\partial t} = \mathcal{A}^*(P_{t|s}(x|y)), \text{with}\, P_{s|s}(x|y) = \delta(x - y) \tag{16}$$

Theorem 7 (Backward Komogorov Equation)**.** $P_{s|t}(y|x)$ *for* $t \geq s$ *solves:*

$$\frac{\partial P_{s|t}(y|x)}{\partial s} + \mathcal{A}(P_{s|t}(y|x)) = 0, \text{with}\, P_{s|t}(y|x) = \delta(x - y) \tag{17}$$

2 Feynman-Kac Formula

The Feynman-Kac Formula bridges PDE and certain stochastic value of SDE solutions.

Consider $u(x,t)$ satisfied the following PDE:

$$\frac{\partial u}{\partial t}+f(x)\frac{\partial u}{\partial x}+\frac{1}{2}L^2(x)\frac{\partial^2 u}{\partial x^2}=0. \quad u(x,T)=\psi(x). \tag{18}$$

Then we define a stochastic process $X(t)$ on $[t',T]$ as

$$dX=f(X)dt+L(X)dW_t \quad X(t')=x' \tag{19}$$

By Ito formula:

$$\begin{aligned} du &= \frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2 \\ &= \frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}\left(f(x)dt+L(x)dW_t\right)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}L^2(x)dt \\ &= \left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}f(x)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}L^2(x)\right)dt+\frac{\partial u}{\partial x}L(x)dW_t. \\ &= \frac{\partial u}{\partial x}L(x)dW_t. \end{aligned} \tag{20}$$

Integrating both sises from t' to T:

$$\begin{aligned} \int_{t'}^{T}\frac{\partial u}{\partial x}L(x)dW_t &= u(X(T),T)-u(X(t'),t') \\ &= \psi(X(T))-u(x',t') \end{aligned} \tag{21}$$

Take expectation of both sides:

$$u(x',t')=E[\Psi(X(T))] \tag{22}$$

This can be generalized to PDE like:

$$\frac{\partial u}{\partial t}+f(x)\frac{\partial u}{\partial x}+\frac{1}{2}L^2(x)\frac{\partial^2 u}{\partial x^2}-ru=0. \quad u(x,T)=\psi(x). \tag{23}$$

By consider the Ito formula of $e^{-rt}u(x,t)$, we can similarly compute the resulting Feynman-Kac equation as

$$u(x',t')=e^{-r(T-t')}E\left[\psi(X(T))\right] \tag{24}$$

This means we can get the value of PDE at (x',t') by simulating SDE paths beginning at (x',t'), and compute corresponding $E\left[\psi(X(T))\right]$. We can get more generalized conclusion:

Theorem 8 (Solve Backward PDE). *To compute the backward PDE:* $(\mathcal{A}_t-r)(u)=0$, *i.e.*

$$\frac{\partial u}{\partial t}+\sum_i\frac{\partial u(x,t)}{\partial x_i}f_i(x,t)+\frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 u(x,t)}{\partial x_i\partial x_j}\right)\left[G(x,t)QG^\top(x,t)\right]_{ij}-ru(x,t)=0 \tag{25}$$

with boundary condition $u(x,T)=\psi(x)$. *Then for any fixed points* (x',t') *where* $t'\le T, x'\in D$, $u(x',t')$ *can be computed as:*
Step1. Simulate N sample paths of SDE from t' *to* T*:*

$$dX_t=f(X_t,t)dt+G(X_t,t)dW_t \,\text{with}\, X(t')=x' \tag{26}$$

Step2. Estimate $u(x',t')=e^{-r(T-t')}E\left[\psi(X(T))\right]$

Theorem 9 (Solve Forward PDE). *Consider the solution* $u(x,t)$ *of forward PDE:* $\frac{\partial u}{\partial t}=(\mathcal{A}-r)(u)$, *i.e.*

$$\frac{\partial u}{\partial t}=\sum_i\frac{\partial u(x,t)}{\partial x_i}f_i(x,t)+\frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 u(x,t)}{\partial x_i\partial x_j}\right)\left[G(x,t)QG^\top(x,t)\right]_{ij}-ru(x,t) \tag{27}$$

with initial condition $u(x,0)=\psi(x)$. *Then for any fixed points* (x',t') *where* $t'\le T, x'\in D$, $u(x',t')$ *can be computed as:*
Step1. Simulate N sample paths of SDE from 0 *to* t'*:*

$$dX_t=f(X_t,t)dt+G(X_t,t)dW_t \,\text{with}\, X(0)=x' \tag{28}$$

Step2. Estimate $u(x',t')=e^{-rt'}E\left[\psi(X(t'))\right]$

Theorem 10 (Solve Boundary Value Problem). *For solution $u(x)$ to the following elliptic PDE defined on some domain D:*

$$\sum_i \frac{\partial u(x)}{\partial x_i} f_i(x) + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right) \left[G(x) Q G^\top(x) \right]_{ij} - r u(x) = 0 \tag{29}$$

with boundary condition $u(x) = \psi(x)$ on ∂D. Then for any fixed points in D can be computed as:
Step1. Simulate N sample paths of SDE from t' to the first exit time T_e:

$$dX_t = f(X_t)dt + G(X_t)dW_t \,\text{with}\, X(t') = x' \tag{30}$$

Step2. Estimate $u(x') = e^{-r(T_e - t')} E\left[\psi(X(T_e))\right]$

3 Linear Filtering Problem

4 Parameter Estimation in SDE