# Face Recognition using Eigenface, A Simple Example

The size of face image used in this program is 92 pixels x 112 pixels. But in this example, assume that the size of face image is 2 x 2 pixels.

**Training dataset**

$$Face_1 = \begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} \quad Face_2 = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \quad Face_3 = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$$

**Test dataset**

$$UnknownFace_1 = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

STEP 1: Convert the face images from training dataset into face vectors

$$\text{Let matrix } A = \begin{array}{c} \begin{matrix} Face_1 & Face_2 & Face_3 \end{matrix} \\ \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix} \end{array}$$

STEP 2: Find $\Psi$, the mean face from training dataset

$$\Psi = \begin{bmatrix}(3+2+7)/3\\(1+1+4)/3\\(9+5+5)/3\\(5+2+3)/3\end{bmatrix}=\begin{bmatrix}4\\2\\19/3\\10/3\end{bmatrix}\xrightarrow{\text{Duplicate columns}}\begin{bmatrix}4&4&4\\2&2&2\\19/3&19/3&19/3\\10/3&10/3&10/3\end{bmatrix}$$

$$\Psi = \begin{bmatrix}4&4&4\\2&2&2\\19/3&19/3&19/3\\10/3&10/3&10/3\end{bmatrix}$$

STEP 3: Find $\Phi$, the unique features of the training dataset faces

$$\Phi = A-\Psi$$

$$= \begin{bmatrix}3&2&7\\1&1&4\\9&5&5\\5&2&3\end{bmatrix}-\begin{bmatrix}4&4&4\\2&2&2\\19/3&19/3&19/3\\10/3&10/3&10/3\end{bmatrix}$$

$$= \begin{bmatrix}-1&-2&3\\-1&-1&2\\8/3&-4/3&-4/3\\5/3&-4/3&-1/3\end{bmatrix}$$

STEP 4: Construct Matrix $L$ with size $M \, x \, M$, where $M$ is the total number of faces in training dataset

$$
\begin{aligned}
L &= \Phi^T \Phi \\
&= \begin{bmatrix} -1 & -1 & 8/3 & 5/3 \\ -2 & -1 & -4/3 & -4/3 \\ 3 & 2 & -4/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix} \\
&= \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}
\end{aligned}
$$

STEP 5: Find eigenvalues of the Matrix $L$

$$\det(\lambda I - L) = 0$$

$$\det\left(\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}107/9 & -25/9 & -82/9\\ -25/9 & 77/9 & -52/9\\ -82/9 & -52/9 & 134/9\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}\lambda-107/9 & 25/9 & 82/9\\ 25/9 & \lambda-77/9 & 52/9\\ 82/9 & 52/9 & \lambda-134/9\end{bmatrix}\right) = 0$$

$$\begin{aligned}(\lambda-107/9)[(\lambda-77/9)(\lambda-134/9)-(52/9)(52/9)]-\\ 25/9[(25/9)(\lambda-134/9)-(52/9)(82/9)]+\\ 82/9[(25/9)(52/9)-(\lambda-77/9)(82/9)] &= 0\end{aligned}$$

$$\begin{aligned}(\lambda-107/9)[\lambda^2-211/9\,\lambda+94]-\\ 25/9[25/9\,\lambda-94]+\\ 82/9[-82/9\,\lambda+94] &= 0\end{aligned}$$

$$\begin{aligned}(\lambda^3-211/9\,\lambda^2+94\,\lambda-107/9\,\lambda^2+22577/81\,\lambda-10058/9)-\\ (625/81\,\lambda-2350/9)+\\ (-6724/81\,\lambda+7708/9) &= 0\end{aligned}$$

$$\begin{aligned}(\lambda^3-106/3\,\lambda^2+30191/81\,\lambda-10058/9)-\\ (625/81\,\lambda-2350/9)+\\ (-6724/81\,\lambda+7708/9) &= 0\end{aligned}$$

$$\begin{aligned}\lambda^3-106/3\,\lambda^2+30191/81\,\lambda-625/81\,\lambda-6724/81\,\lambda-\\ 10058/9+2350/9+7708/9 &= 0\end{aligned}$$

$$\lambda^3-106/3\,\lambda^2+282\,\lambda+0 = 0$$

$$\begin{aligned}\lambda_1 &= 23.1540\\ \lambda_2 &= 0\\ \lambda_3 &= 12.1793\end{aligned}$$

Sort the eigenvalues in descending order

$$\begin{aligned}\lambda_1 &= 23.1540\\ \lambda_2 &= 12.1793\\ \lambda_3 &= 0\end{aligned}$$

STEP 6: Use Gaussian Elimination method to find the eigenvectors of the Matrix $L$

$$Let\ B=\begin{bmatrix} \lambda-107/9 & 25/9 & 82/9 \\ 25/9 & \lambda-77/9 & 52/9 \\ 82/9 & 52/9 & \lambda-134/9 \end{bmatrix}$$

To find the eigenvectors, we need to solve $Bx=0$ for every eigenvalues.

Case 1: $\lambda = 23.1540$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 23.1540 - 107/9 & 25/9 & 82/9 \\ 25/9 & 23.1540 - 77/9 & 52/9 \\ 82/9 & 52/9 & 23.1540 - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11.2651 & 25/9 & 82/9 \\ 25/9 & 14.5984 & 52/9 \\ 82/9 & 52/9 & 8.2651 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 11.2651 & 25/9 & 82/9 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_1/11.2651} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_2 - 25/9\, R_1} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_3 - 82/9\, R_1} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_2/13.9134} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_1 - 0.2466\, R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_3 - 3.5310\, R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & -0.0002 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0002} \left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned}
x_1 + 0.7462\,x_3 &= 0 \\
x_2 + 0.2538\,x_3 &= 0 \\
x_3 \text{ is free, Let } x_3 &= 1
\end{aligned}$$

$$\begin{aligned}
x_1 &= -0.7462(1) \\
x_2 &= -0.2538(1) \\
x_3 &= 1
\end{aligned}$$

Therefore, the eigenvector for eigenvalue $\lambda = 23.1540$ is $\begin{bmatrix} -0.7462 \\ -0.2538 \\ 1.0000 \end{bmatrix}$

Case 2: $\lambda = 12.1793$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12.1793 - 107/9 & 25/9 & 82/9 \\ 25/9 & 12.1793 - 77/9 & 52/9 \\ 82/9 & 52/9 & 12.1793 - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2904 & 25/9 & 82/9 \\ 25/9 & 3.6237 & 52/9 \\ 82/9 & 52/9 & -2.7096 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.2904 & 25/9 & 82/9 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_1/0.2904} \left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_2 - 25/9\, R_1} \left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_3 - 82/9\, R_1} \left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_2/-22.9469} \left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_1 - 9.5654\, R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_3 + 81.3736 R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & -0.0054 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0054} \left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2.5456\,x_3 &= 0 \\ x_2 + 3.5461\,x_3 &= 0 \\ x_3 \text{ is free, Let } x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 2.5456(1) \\ x_2 &= -3.5461(1) \\ x_3 &= 1 \end{aligned} \xrightarrow{\text{Normalize}} \begin{bmatrix} 2.5456/3.5461 \\ -3.5461/3.5461 \\ 1.0000/3.5461 \end{bmatrix} = \begin{bmatrix} 0.7179 \\ -1.0000 \\ 0.2820 \end{bmatrix}$$

Therefore, the eigenvector for eigenvalue $\lambda = 12.1793$ is $\begin{bmatrix} 2.5456 \\ -3.5461 \\ 1.0000 \end{bmatrix}$

Case 3: $\lambda = 0$

Eigenvalue with 0 value will produce a useless eigenvector, so this case is discarded.

Therefore, the eigenvector of Matrix $L$ is

$$\begin{array}{cc} \lambda_1 = 23.1540 & \lambda_2 = 12.1793 \\ \multicolumn{2}{c}{\begin{bmatrix} -0.7462 & 0.7179 \\ -0.2538 & -1.0000 \\ 1.0000 & 0.2820 \end{bmatrix}} \end{array}$$

STEP 7: Find the Eigenface value, $u$

Let $v$ = The eigenvector of Matrix $L$

$$v = \begin{bmatrix} -0.7462 & 0.7179 \\ -0.2538 & -1.0000 \\ 1.0000 & 0.2820 \end{bmatrix}$$

Use the following equation to find the Eigenface:

$$u_l = \sum_{k=1}^{M} V_{lk} \Phi_k$$ , where M = 3, the number faces in training dataset