Face Recognition using Eigenface, A Simple Example

The size of face image used in this program is 92 pixels x 112 pixels. But in this example, assume that the size of face image is 2 x 2 pixels.

| Training dataset | Test dataset |
| --- | --- |
| $T_1 = \begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix}$<br>$T_2 = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$<br>$T_3 = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ | $F_1 = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ |

STEP 1: Convert the training faces 2D-matrix into vector. The matrix $\boxed{\Gamma}$ will store these vectors.

$$\begin{aligned}\Gamma &= \left[T_1^T,\ T_2^T,\ T_3^T\right] \\ &= \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix}\end{aligned}$$

STEP 2: Find $\boxed{\Psi}$ , the mean face from training dataset.

$$\Psi = \frac{1}{M}\sum_{n=1}^{M}\Gamma_n = \frac{1}{3}\left(\begin{bmatrix}3\\1\\9\\5\end{bmatrix}+\begin{bmatrix}2\\1\\5\\2\end{bmatrix}+\begin{bmatrix}7\\4\\5\\3\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}12\\6\\19\\10\end{bmatrix} = \begin{bmatrix}4\\2\\19/3\\10/3\end{bmatrix}$$

STEP 3: Find $\boxed{\Phi_i}$, the unique features of the training faces.

Using the following equation to find the unique features:

$$\boxed{\Phi_i = \Gamma_i - \Psi}$$

$$\begin{aligned}
\Phi_1 &= \Gamma_1 - \Psi = \begin{bmatrix} 3 \\ 1 \\ 9 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 8/3 \\ 5/3 \end{bmatrix} \\
\Phi_2 &= \Gamma_2 - \Psi = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -4/3 \\ -4/3 \end{bmatrix} \\
\Phi_3 &= \Gamma_3 - \Psi = \begin{bmatrix} 7 \\ 4 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix}
\end{aligned}$$

STEP 4: Construct a Covariance Matrix $\boxed{L}$ with size $\boxed{M x M}$, where $\boxed{M}$ is the total number of faces in training dataset.

$$\begin{aligned}
L &= \Phi^T \Phi \\
&= \begin{bmatrix} -1 & -1 & 8/3 & 5/3 \\ -2 & -1 & -4/3 & -4/3 \\ 3 & 2 & -4/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix} \\
&= \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}
\end{aligned}$$

STEP 5: Find eigenvalues of the Matrix $\boxed{L}$ .

$$\det(\lambda I - L) = 0$$

$$\det\left(\lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}107/9 & -25/9 & -82/9\\-25/9 & 77/9 & -52/9\\-82/9 & -52/9 & 134/9\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}\lambda - 107/9 & 25/9 & 82/9\\25/9 & \lambda - 77/9 & 52/9\\82/9 & 52/9 & \lambda - 134/9\end{bmatrix}\right) = 0$$

$$(\lambda - 107/9)[(\lambda - 77/9)(\lambda - 134/9) - (52/9)(52/9)] -$$
$$25/9[(25/9)(\lambda - 134/9) - (52/9)(82/9)] +$$
$$82/9[(25/9)(52/9) - (\lambda - 77/9)(82/9)] = 0$$

$$(\lambda - 107/9)[\lambda^2 - 211/9\lambda + 94] -$$
$$25/9[25/9\lambda - 94] +$$
$$82/9[-82/9\lambda + 94] = 0$$

$$(\lambda^3 - 211/9\lambda^2 + 94\lambda - 107/9\lambda^2 + 22577/81\lambda - 10058/9) -$$
$$(625/81\lambda - 2350/9) +$$
$$(-6724/81\lambda + 7708/9) = 0$$

$$(\lambda^3 - 106/3\lambda^2 + 30191/81\lambda - 10058/9) -$$
$$(625/81\lambda - 2350/9) +$$
$$(-6724/81\lambda + 7708/9) = 0$$

$$\lambda^3 - 106/3\lambda^2 + 30191/81\lambda - 625/81\lambda - 6724/81\lambda -$$
$$10058/9 + 2350/9 + 7708/9 = 0$$

$$\lambda^3 - 106/3\lambda^2 + 282\lambda + 0 = 0$$

$$\lambda_1 = 23.1540$$
$$\lambda_2 = 0$$
$$\lambda_3 = 12.1793$$

Sort the eigenvalues in descending order

$$\lambda_1 = 23.1540$$
$$\lambda_2 = 12.1793$$
$$\lambda_3 = 0$$

STEP 6: Use Gaussian Elimination method to find $\boxed{v}$ , the eigenvectors of the Matrix $\boxed{L}$ .

$$\boxed{\begin{aligned} B &= [\lambda I - L] \\ &= \begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix} \end{aligned}}$$

To find the eigenvectors, we need to solve $\boxed{Bx = 0}$ for every eigenvalues.

Case 1: $\boxed{\lambda = 23.1540}$

$$Bx = 0$$

$$\begin{bmatrix} \lambda-107/9 & 25/9 & 82/9 \\ 25/9 & \lambda-77/9 & 52/9 \\ 82/9 & 52/9 & \lambda-134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 23.1540-107/9 & 25/9 & 82/9 \\ 25/9 & 23.1540-77/9 & 52/9 \\ 82/9 & 52/9 & 23.1540-134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11.2651 & 25/9 & 82/9 \\ 25/9 & 14.5984 & 52/9 \\ 82/9 & 52/9 & 8.2651 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 11.2651 & 25/9 & 82/9 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_1/11.2651}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_2-25/9R_1}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_3-82/9R_1}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_2/13.9134}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_1-0.2466R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_3-3.5310R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & -0.0002 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0002}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} x_1+0.7462\,x_3 &= 0 \\ x_2+0.2538\,x_3 &= 0 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= -0.7462(1) \\ x_2 &= -0.2538(1) \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} v_1 &= \frac{1}{\max|x_1, x_2, x_3|}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} -0.7462 \\ -0.2538 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.7462 \\ -0.2538 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, the eigenvector for eigenvalue $\boxed{\lambda = 23.1540}$ is

$$\boxed{v_1 = \begin{bmatrix} -0.7462 \\ -0.2538 \\ 1.0000 \end{bmatrix}}$$

Case 2: $\boxed{\lambda=12.1793}$

$$Bx = 0$$

$$\begin{bmatrix} \lambda-107/9 & 25/9 & 82/9 \\ 25/9 & \lambda-77/9 & 52/9 \\ 82/9 & 52/9 & \lambda-134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12.1793-107/9 & 25/9 & 82/9 \\ 25/9 & 12.1793-77/9 & 52/9 \\ 82/9 & 52/9 & 12.1793-134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2904 & 25/9 & 82/9 \\ 25/9 & 3.6237 & 52/9 \\ 82/9 & 52/9 & -2.7096 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.2904 & 25/9 & 82/9 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_1/0.2904}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_2-25/9R_1}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_3-82/9R_1}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_2/-22.9469}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_1-9.5654R_2}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_3+81.3736R_2}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & -0.0054 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0054}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$
\begin{aligned}
x_1 - 2.5456\,x_3 &= 0 \\
x_2 + 3.5461\,x_3 &= 0 \\
x_3 &= 1
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= 2.5456(1) \\
x_2 &= -3.5461(1) \\
x_3 &= 1
\end{aligned}
$$

$$
\begin{aligned}
v_2 &= \frac{1}{\max|x_1, x_2, x_3|}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \frac{1}{3.5461}\begin{bmatrix} 2.5456 \\ -3.5461 \\ 1.0000 \end{bmatrix} \\
&= \begin{bmatrix} 0.7179 \\ -1.0000 \\ 0.2820 \end{bmatrix}
\end{aligned}
$$

Therefore, the eigenvector for eigenvalue $\boxed{\lambda = 12.1793}$ is

$$
\boxed{v_2 = \begin{bmatrix} 0.7179 \\ -1.0000 \\ 0.2820 \end{bmatrix}}
$$

Case 3: $\boxed{\lambda=0}$

Eigenvalue with value 0 will produce a useless eigenvector, so this case is discarded.

Therefore, the eigenvector of Matrix $\boxed{L}$ is:

$$\boxed{v = \begin{bmatrix} -0.7462 & 0.7179 \\ -0.2538 & -1.0000 \\ 1.0000 & 0.2820 \end{bmatrix}}$$

STEP 7: Find the Eigenface, $\boxed{u}$ .

Using the following equation to find the Eigenface:

$\boxed{u_l = \sum_{k=1}^{M} v_{lk}\Phi_k}$ , where $\boxed{M = 3}$ , the number of training faces.

$$u_1 = (-0.7462)\begin{bmatrix}-1\\-1\\8/3\\5/3\end{bmatrix} + (-0.2538)\begin{bmatrix}-2\\-1\\-4/3\\-4/3\end{bmatrix} + (1.0000)\begin{bmatrix}3\\2\\-4/3\\-1/3\end{bmatrix}$$

$$= \begin{bmatrix}0.7462\\0.7462\\-1.9899\\-1.2437\end{bmatrix} + \begin{bmatrix}0.5076\\0.2538\\0.3384\\0.3384\end{bmatrix} + \begin{bmatrix}3\\2\\-4/3\\-1/3\end{bmatrix}$$

$$= \begin{bmatrix}4.2538\\3.0000\\-2.9848\\-1.2386\end{bmatrix}$$

$$u_2 = (0.7179)\begin{bmatrix}-1\\-1\\8/3\\5/3\end{bmatrix} + (-1.0000)\begin{bmatrix}-2\\-1\\-4/3\\-4/3\end{bmatrix} + (0.2820)\begin{bmatrix}3\\2\\-4/3\\-1/3\end{bmatrix}$$

$$= \begin{bmatrix}-0.7179\\-0.7179\\1.9144\\1.1965\end{bmatrix} + \begin{bmatrix}2\\1\\4/3\\4/3\end{bmatrix} + \begin{bmatrix}0.8460\\0.5640\\-0.3760\\-0.0940\end{bmatrix}$$

$$= \begin{bmatrix}2.1281\\0.8461\\2.8717\\2.4358\end{bmatrix}$$

Therefore, $$\boxed{u = \begin{bmatrix}4.2538 & 2.1281\\3.0000 & 0.8461\\-2.9848 & 2.8717\\-1.2386 & 2.4358\end{bmatrix}}$$

STEP 8: Find training weights $\boxed{\Omega_K}$ .

Using the following equations to find the weight:

$$\boxed{\Omega^T = [w_1, \quad w_2, \quad w_3 \quad \ldots \quad w_M]} \quad ,$$

where $\boxed{M}$ number of faces in the training dataset.

$$\boxed{w_k = u_k^T(\Gamma - \Psi)} \quad ,$$

where $\boxed{\Gamma}$ is a face vector from training dataset.

Find the weights for training face 1:

$$
\begin{aligned}
\Omega_1^T &= \begin{bmatrix} w_1, & w_2 \end{bmatrix} \\
w_1 &= u_1^T(\Gamma_1 - \Psi) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} 3 \\ 1 \\ 9 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} -1 \\ -1 \\ 8/3 \\ 5/3 \end{bmatrix} \right) \\
&= -17.2778 \\
w_2 &= u_2^T(\Gamma_1 - \Psi) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} 3 \\ 1 \\ 9 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} -1 \\ -1 \\ 8/3 \\ 5/3 \end{bmatrix} \right) \\
&= -14.6918
\end{aligned}
$$

Therefore,

$$
\boxed{\Omega_1^T = \begin{bmatrix} -17.2778, & -14.6918 \end{bmatrix}}
$$

Find the weights for training face 2:

$$
\begin{aligned}
\Omega_2^T &= \begin{bmatrix} w_1, & w_2 \end{bmatrix} \\
w_1 &= u_1^T(\Gamma_2 - \Psi) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} -2 \\ -1 \\ -4/3 \\ -4/3 \end{bmatrix} \right) \\
&= -5.8764 \\
w_2 &= u_2^T(\Gamma_2 - \Psi) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} -2 \\ -1 \\ -4/3 \\ -4/3 \end{bmatrix} \right) \\
&= 1.9744
\end{aligned}
$$

Therefore,

$$
\boxed{\Omega_2^T = \begin{bmatrix} -5.8764, & -1.9744 \end{bmatrix}}
$$

Find the weights for training face 3:

$$
\begin{aligned}
\Omega_3^T &= \begin{bmatrix} w_1, & w_2 \end{bmatrix} \\
w_1 &= u_1^T(\Gamma_3 - \Psi) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} 7 \\ 4 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 4.2538, & 3.0000, & -2.9848, & -1.2386 \end{bmatrix} \left( \begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix} \right) \\
&= 23.1540 \\
w_2 &= u_2^T(\Gamma_3 - \Psi) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} 7 \\ 4 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2.1281, & 0.8461, & -2.8717, & -2.4358 \end{bmatrix} \left( \begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix} \right) \\
&= 12.7174
\end{aligned}
$$

Therefore,

$$
\Omega_3^T = \begin{bmatrix} 23.1540, & 12.7174 \end{bmatrix}
$$

Thus, the weight for training faces in the training dataset is:

$$
\Omega = \begin{bmatrix} -17.2778 & -5.8764 & 23.1540 \\ -14.6918 & -1.9744 & 12.7174 \end{bmatrix}
$$