Face Recognition using Eigenface, A Simple Example

The size of face image used in this program is 92 pixels x 112 pixels. But in this example, assume that the size of face image is 2 x 2 pixels.

Training dataset

$$Face_1 = \begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} \quad Face_2 = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \quad Face_3 = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$$

Test dataset

$$UnknownFace_1 = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

STEP 1: Convert the face images from training dataset into face vectors

$$\text{Let matrix} \quad A = \begin{array}{c} \begin{array}{ccc} Face_1 & Face_2 & Face_3 \end{array} \\ \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix} \end{array}$$

STEP 2: Find Ψ, the mean face from training dataset

$$\Psi = \begin{bmatrix} (3+2+7)/3 \\ (1+1+4)/3 \\ (9+5+5)/3 \\ (5+2+3)/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \xrightarrow{\text{Duplicate columns}} \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix}$$

STEP 3: Find Φ, the unique features of the training dataset faces

$$\begin{aligned} \Phi &= A - \Psi \\ &= \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix} \end{aligned}$$

STEP 4: Construct Matrix L with size $M \, x \, M$, where M is the total number of faces in training dataset

$$L = \Phi^T \Phi$$

$$= \begin{bmatrix} -1 & -1 & 8/3 & 5/3 \\ -2 & -1 & -4/3 & -4/3 \\ 3 & 2 & -4/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}$$

STEP 5: Find eigenvalues of the Matrix L

$$det(\lambda I - L) = 0$$

$$det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix}\right) = 0$$

$$\begin{aligned} &(\lambda - 107/9)[(\lambda - 77/9)(\lambda - 134/9) - (52/9)(52/9)] - \\ &25/9[(25/9)(\lambda - 134/9) - (52/9)(82/9)] + \\ &82/9[(25/9)(52/9) - (\lambda - 77/9)(82/9)] = 0 \end{aligned}$$

$$\begin{aligned} &(\lambda - 107/9)[\lambda^2 - 211/9\,\lambda + 94] - \\ &25/9[25/9\,\lambda - 94] + \\ &82/9[-82/9\,\lambda + 94] = 0 \end{aligned}$$

$$\begin{aligned} &(\lambda^3 - 211/9\,\lambda^2 + 94\,\lambda - 107/9\,\lambda^2 + 22577/81\,\lambda - 10058/9) - \\ &(625/81\,\lambda - 2350/9) + \\ &(-6724/81\,\lambda + 7708/9) = 0 \end{aligned}$$

$$\begin{aligned} &(\lambda^3 - 106/3\,\lambda^2 + 30191/81\,\lambda - 10058/9) - \\ &(625/81\,\lambda - 2350/9) + \\ &(-6724/81\,\lambda + 7708/9) = 0 \end{aligned}$$

$$\begin{aligned} &\lambda^3 - 106/3\,\lambda^2 + 30191/81\,\lambda - 625/81\,\lambda - 6724/81\,\lambda - \\ &10058/9 + 2350/9 + 7708/9 = 0 \end{aligned}$$

$$\lambda^3 - 106/3\,\lambda^2 + 282\,\lambda + 0 = 0$$

$$\lambda_1 = 23.1540 \qquad \lambda_2 = 0 \qquad \lambda_3 = 12.1793$$

Sort the eigenvalues in descending order

$$\lambda_1 = 23.1540 \qquad \lambda_2 = 12.1793 \qquad \lambda_3 = 0$$

STEP 6: Use Gaussian Elimination method to find the eigenvectors of the Matrix L

$$Let\ B=\begin{bmatrix} \lambda-107/9 & 25/9 & 82/9 \\ 25/9 & \lambda-77/9 & 52/9 \\ 82/9 & 52/9 & \lambda-134/9 \end{bmatrix}$$

To find the eigenvectors, we need to solve $Bx=0$ for every eigenvalues.

Case 1: $\lambda = 23.1540$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 23.1540 - 107/9 & 25/9 & 82/9 \\ 25/9 & 23.1540 - 77/9 & 52/9 \\ 82/9 & 52/9 & 23.1540 - 134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11.2651 & 25/9 & 82/9 \\ 25/9 & 14.5984 & 52/9 \\ 82/9 & 52/9 & 8.2651 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 11.2651 & 25/9 & 82/9 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_1/11.2651}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_2 - 25/9\,R_1}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_3 - 82/9\,R_1}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_2/13.9134}\left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_1 - 0.2466\,R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_3 - 3.5310\,R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & -0.0002 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0002}\left[\begin{array}{ccc|c} 1 & 0 & 0.7462 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + 0.7462\, x_3 &= 0 \\ x_2 + 0.2538\, x_3 &= 0 \\ x_3 \text{ is free, Let } x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= -0.7462(1) \\ x_2 &= -0.2538(1) \\ x_3 &= 1 \end{aligned}$$

Therefore, the eigenvector for eigenvalue $\lambda = 23.1540$ is $\begin{bmatrix} -0.7462 \\ -0.2538 \\ 1.0000 \end{bmatrix}$

Case 2: $\lambda = 12.1793$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12.1793 - 107/9 & 25/9 & 82/9 \\ 25/9 & 12.1793 - 77/9 & 52/9 \\ 82/9 & 52/9 & 12.1793 - 134/9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2904 & 25/9 & 82/9 \\ 25/9 & 3.6237 & 52/9 \\ 82/9 & 52/9 & -2.7096 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.2904 & 25/9 & 82/9 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_1/0.2904}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_2 - 25/9\,R_1}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_3 - 82/9\,R_1}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & -22.9469 & -81.3731 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_2/-22.9469}\left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_1 - 9.5654\,R_2}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & -81.3736 & -288.5643 & 0 \end{array}\right]$$

$$\underline{R_3 + 81.3736\,R_2}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & -0.0054 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0054}\left[\begin{array}{ccc|c} 1 & 0 & -2.5456 & 0 \\ 0 & 1 & 3.5461 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2.5456\,x_3 &= 0 \\ x_2 + 3.5461\,x_3 &= 0 \\ x_3 \text{ is free, Let } x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 2.5456(1) \\ x_2 &= -3.5461(1) \\ x_3 &= 1 \end{aligned} \xrightarrow{\text{Normalize}} \begin{bmatrix} 2.5456/3.5461 \\ -3.5461/3.5461 \\ 1.0000/3.5461 \end{bmatrix} = \begin{bmatrix} 0.7179 \\ -1.0000 \\ 0.2820 \end{bmatrix}$$

Therefore, the eigenvector for eigenvalue $\lambda = 12.1793$ is $\begin{bmatrix} 2.5456 \\ -3.5461 \\ 1.0000 \end{bmatrix}$

Case 3: $\lambda = 0$

Eigenvalue with 0 value will produce a useless eigenvector, so this case is discarded.

Therefore, the eigenvector of Matrix L is

$$\begin{array}{cc} \lambda_1 = 23.1540 & \lambda_2 = 12.1793 \\ \multicolumn{2}{c}{\begin{bmatrix} -0.7462 & 0.7179 \\ -0.2538 & -1.0000 \\ 1.0000 & 0.2820 \end{bmatrix}} \end{array}$$

STEP 7: Find the Eigenface value, u

Let v = The eigenvector of Matrix L

$$v = \begin{bmatrix} -0.7462 & 0.7179 \\ -0.2538 & -1.0000 \\ 1.0000 & 0.2820 \end{bmatrix}$$

Use the following equation to find the Eigenface:

$$u_l = \sum_{k=1}^{M} V_{lk} \Phi_k$$, where M = 3, the number faces in training dataset

$$u_1 = (-0.7462)\begin{bmatrix} -1 \\ -1 \\ 8/3 \\ 5/3 \end{bmatrix} + (-0.2538)\begin{bmatrix} -2 \\ -1 \\ -4/3 \\ -4/3 \end{bmatrix} + (1.0000)\begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7462 \\ 0.7462 \\ -1.9899 \\ -1.2437 \end{bmatrix} + \begin{bmatrix} 0.5076 \\ 0.2538 \\ 0.3384 \\ 0.3384 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4.2538 \\ 3.0000 \\ -2.9848 \\ -1.2386 \end{bmatrix}$$

$$u_2 = (0.7179)\begin{bmatrix} -1 \\ -1 \\ 8/3 \\ 5/3 \end{bmatrix} + (-1.0000)\begin{bmatrix} -2 \\ -1 \\ -4/3 \\ -4/3 \end{bmatrix} + (0.2820)\begin{bmatrix} 3 \\ 2 \\ -4/3 \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} -0.7179 \\ -0.7179 \\ 1.9144 \\ 1.1965 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 4/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 0.8460 \\ 0.5640 \\ -0.3760 \\ -0.0940 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1281 \\ 0.8461 \\ 2.8717 \\ 2.4358 \end{bmatrix}$$

Therefore, $u=\begin{bmatrix} 4.2538 & 2.1281 \\ 3.0000 & 0.8461 \\ -2.9848 & 2.8717 \\ -1.2386 & 2.4358 \end{bmatrix}$