Face Recognition using Eigenface, A Simple Example

The size of face image used in this program is 92 pixels x 112 pixels. But in this example, assume that the size of face image is 4 x 4 pixels.

Training dataset

$$Face_1 = \begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} \quad Face_2 = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \quad Face_3 = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$$

Test dataset

$$UnknownFace_1 = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

STEP 1: Convert the face images from training dataset into face vectors

Let matrix $A = \begin{array}{c} \begin{array}{ccc} Face_1 & Face_2 & Face_3 \end{array} \\ \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix} \end{array}$

STEP 2: Find Ψ , the mean face from training dataset

$$\Psi = \begin{bmatrix} (3+2+7)/3 \\ (1+1+4)/3 \\ (9+5+5)/3 \\ (5+2+3)/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 19/3 \\ 10/3 \end{bmatrix} \xrightarrow{\text{Duplicate columns}} \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix}$$

STEP 3: Find Φ , the unique features of the training dataset faces

$$\Phi = A - \Psi$$

$$= \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 4 \\ 9 & 5 & 5 \\ 5 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 19/3 & 19/3 & 19/3 \\ 10/3 & 10/3 & 10/3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix}$$

STEP 4: Construct Matrix L with size $M \, x \, M$, where M is the total number of faces in training dataset

$$
\begin{aligned}
L &= \Phi^T \Phi \\
&= \begin{bmatrix} -1 & -1 & 8/3 & 5/3 \\ -2 & -1 & -4/3 & -4/3 \\ 3 & 2 & -4/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 8/3 & -4/3 & -4/3 \\ 5/3 & -4/3 & -1/3 \end{bmatrix} \\
&= \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}
\end{aligned}
$$

STEP 5: Find eigenvalues of the Matrix L

$$det(\lambda I - L) = 0$$

$$det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 107/9 & -25/9 & -82/9 \\ -25/9 & 77/9 & -52/9 \\ -82/9 & -52/9 & 134/9 \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix}\right) = 0$$

$$\begin{aligned} (\lambda - 107/9)[(\lambda - 77/9)(\lambda - 134/9) - (52/9)(52/9)] - \\ 25/9[(25/9)(\lambda - 134/9) - (52/9)(82/9)] + \\ 82/9[(25/9)(52/9) - (\lambda - 77/9)(82/9)] &= 0 \end{aligned}$$

$$\begin{aligned} (\lambda - 107/9)[\lambda^2 - 211/9\,\lambda + 94] - \\ 25/9[25/9\,\lambda - 94] + \\ 82/9[-82/9\,\lambda + 94] &= 0 \end{aligned}$$

$$\begin{aligned} (\lambda^3 - 211/9\,\lambda^2 + 94\,\lambda - 107/9\,\lambda^2 + 22577/81\,\lambda - 10058/9) - \\ (625/81\,\lambda - 2350/9) + \\ (-6724/81\,\lambda + 7708/9) &= 0 \end{aligned}$$

$$\begin{aligned} (\lambda^3 - 106/3\,\lambda^2 + 30191/81\,\lambda - 10058/9) - \\ (625/81\,\lambda - 2350/9) + \\ (-6724/81\,\lambda + 7708/9) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda^3 - 106/3\,\lambda^2 + 30191/81\,\lambda - 625/81\,\lambda - 6724/81\,\lambda - \\ 10058/9 + 2350/9 + 7708/9 &= 0 \end{aligned}$$

$$\lambda^3 - 106/3\,\lambda^2 + 282\,\lambda + 0 = 0$$

$$\begin{aligned} \lambda_1 &= 23.1540 \\ \lambda_2 &= 0 \\ \lambda_3 &= 12.1793 \end{aligned} \xrightarrow{\text{Sort the eigenvalues in descending order}} \begin{aligned} \lambda_1 &= 23.1540 \\ \lambda_2 &= 12.1793 \\ \lambda_3 &= 0 \end{aligned}$$

STEP 6: Use Gaussian Elimination method to find the eigenvectors of the Matrix L

$$Let\ B=\begin{bmatrix} \lambda-107/9 & 25/9 & 82/9 \\ 25/9 & \lambda-77/9 & 52/9 \\ 82/9 & 52/9 & \lambda-134/9 \end{bmatrix}$$

To find the eigenvectors, we need to solve $Bx=0$ for every eigenvalues.

Case 1: $\lambda = 23.1540$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 23.1540 - 107/9 & 25/9 & 82/9 \\ 25/9 & 23.1540 - 77/9 & 52/9 \\ 82/9 & 52/9 & 23.1540 - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11.2651 & 25/9 & 82/9 \\ 25/9 & 14.5984 & 52/9 \\ 82/9 & 52/9 & 8.2651 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 11.2651 & 25/9 & 82/9 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_1/11.2651} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 25/9 & 14.5984 & 52/9 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_2 - 25/9\, R_1} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 82/9 & 52/9 & 8.2651 & 0 \end{array}\right]$$

$$\underline{R_3 - 82/9\, R_1} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 13.9134 & 3.5311 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_2/13.9134} \left[\begin{array}{ccc|c} 1 & 0.2466 & 0.8088 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_1 - 0.2466\, R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0.6836 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 3.5310 & 0.8960 & 0 \end{array}\right]$$

$$\underline{R_3 - 3.5310\, R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0.6836 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & -0.0002 & 0 \end{array}\right]$$

$$\underline{R_3/-0.0002} \left[\begin{array}{ccc|c} 1 & 0 & 0.6836 & 0 \\ 0 & 1 & 0.2538 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$
\begin{aligned}
x_1 + 0.6836\,x_3 &= 0\\
x_2 + 0.2538\,x_3 &= 0\\
x_3 \text{ is free, Let } x_3 &= 1
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= -0.6836(1)\\
x_2 &= -0.2538(1)\\
x_3 &= 1
\end{aligned}
$$

Therefore, the eigenvector for eigenvalue $\lambda = 23.1540$ is $\begin{bmatrix} -0.6836 \\ -0.2538 \\ 1.0000 \end{bmatrix}$

Case 2: $\lambda = 12.1793$

$$Bx = 0$$

$$\begin{bmatrix} \lambda - 107/9 & 25/9 & 82/9 \\ 25/9 & \lambda - 77/9 & 52/9 \\ 82/9 & 52/9 & \lambda - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12.1793 - 107/9 & 25/9 & 82/9 \\ 25/9 & 12.1793 - 77/9 & 52/9 \\ 82/9 & 52/9 & 12.1793 - 134/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2904 & 25/9 & 82/9 \\ 25/9 & 3.6237 & 52/9 \\ 82/9 & 52/9 & -2.7096 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0.2904 & 25/9 & 82/9 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$

$$\underline{R_1/0.2904} \left[\begin{array}{ccc|c} 1 & 9.5654 & 31.3743 & 0 \\ 25/9 & 3.6237 & 52/9 & 0 \\ 82/9 & 52/9 & -2.7096 & 0 \end{array}\right]$$