02180: Introduction to Artificial Intelligence
Lecture 11: Belief Revision

Nina Gierasimczuk

Recall: The problem of Belief Revision

> Belief revision is a topic of much interest in theoretical computer science and logic, and it forms a central problem in research into artificial intelligence. In simple terms: how do you update a database of knowledge in the light of new information? What if the new information is in conflict with something that was previously held to be true?
>
> Gärdenfors, Belief Revision

Belief Sets

Belief set is a set of formulas that is **deductively closed**.

Definition

For any set B of sentences, $Cn(B)$ is the set of **logical consequences** of B.

If φ can be derived from B by classical propositional logic, then $\varphi \in Cn(B)$.

Three Parts of Taking in New Information

What can I do to my belief set?

1. **Revision**: $B * \varphi$; φ is added and other things are removed, so that the resulting new belief set B' is consistent.
2. **Contraction**: $B \div \varphi$; φ is removed from B giving a new belief set B'.
3. **Expansion**: $B + \varphi$; φ is added to B giving a new belief set B'.

AGM$^{\div}$ Rationality Postulates of Contraction

1. **Closure**: $B \div \varphi = Cn(B \div \varphi)$

   the outcome is logically closed
2. **Success**: If $\varphi \notin Cn(\emptyset)$, then $\varphi \notin Cn(B \div \varphi)$

   the outcome does not contain φ
3. **Inclusion**: $B \div \varphi \subseteq B$

   the outcome is a subset of the original set
4. **Vacuity**: If $\varphi \notin Cn(B)$, then $B \div \varphi = B$

   if the incoming sentence is not in the original set then there is no effect
5. **Extensionality**: If $\varphi \leftrightarrow \psi \in Cn(\emptyset)$, then $B \div \varphi = B \div \psi$.

   the outcomes of contracting with equivalent sentences are the same
6. **Recovery**: $B \subseteq (B \div \varphi) + \varphi$.

   contraction leads to the loss of as few previous beliefs as possible
7. **Conjunctive inclusion**: If $\varphi \notin B \div (\varphi \wedge \psi)$, then $B \div (\varphi \wedge \psi) \subseteq B \div \varphi$.
8. **Conjunctive overlap**: $(B \div \varphi) \cap (B \div \psi) \subseteq B \div (\varphi \wedge \psi)$.

AGM* Rationality Postulates of Revision

1. **Closure**: $B * \varphi = Cn(B * \varphi)$
2. **Success**: $\varphi \in B * \varphi$
3. **Inclusion**: $B * \varphi \subseteq B + \varphi$
4. **Vacuity**: If $\neg\varphi \notin B$, then $B * \varphi = B + \varphi$
5. **Consistency**: $B * \varphi$ is consistent if φ is consistent.
6. **Extensionality**: If $(\varphi \leftrightarrow \psi) \in Cn(\emptyset)$, then $B * \varphi = B * \psi$.
7. **Superexpansion**: $B * (\varphi \wedge \psi) \subseteq (B * \varphi) + \psi$
8. **Subexpansion**: If $\neg\psi \notin B * \varphi$, then $(B * \varphi) + \psi \subseteq B * (\varphi \wedge \psi)$.

Levi Identitiy

One formal way to combine those operations is to use:

Levi identity

$$B * \varphi := (B \div \neg\varphi) + \varphi.$$

Belief revision can be defined as first removing any inconsistency with the incoming information and then adding the information itself.

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- C is a subset of A,
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- there is no set C' such that C' does not imply φ and $C \subset C' \subseteq A$.

Contraction Remainders

Definition

For any set A and sentence φ the **remainder set** $A \bot \varphi$ is the set of inclusion-maximal subsets of A that do not imply φ.

Formally: a set C is an element of $A \bot \varphi$ just in case $C \subset A$, $C \not\models \varphi$, and there is no set C' such that $C' \not\models \varphi$ and $C \subset C' \subseteq A$.

Contraction Remainders

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For any set A and sentence φ the **remainder set** $A \perp \varphi$ is the set of inclusion-maximal subsets of A that do not imply φ.

Formally: a set C is an element of $A \perp \varphi$ just in case $C \subset A$, $C \not\models \varphi$, and there is no set C' such that $C' \not\models \varphi$ and $C \subset C' \subseteq A$.

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But:
For belief set A and any φ inconsistent with A, A revised with φ will be complete, i.e., for any formula ψ, $\psi \in A * \varphi$ or $\neg\psi \in A * \varphi$.
This means that any agent after revising their belief becomes omniscient.

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Assume our agent has a belief set A.
Incoming information is φ, such that $\neg\varphi \in A$.

Then, for every formula ψ both $\neg\varphi \vee \psi$ and $\neg\varphi \vee \neg\psi$ are in A.
Upon contracting $\neg\varphi$ according to the above-described way,
for any formula ψ, either ψ or $\neg\psi$ are in $A \div \neg\varphi$.

Solution: Let $B \div \varphi$ be the **intersection** of some of the remainders!

Definition

A **selection function** for B is a function γ such that if $B \perp \varphi$ is non-empty, then $\gamma(B \perp \varphi)$ is a non-empty subset of $B \perp \varphi$.

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Definition

A **selection function** for B is a function γ such that if $B \perp \varphi$ is non-empty, then $\gamma(B \perp \varphi)$ is a non-empty subset of $B \perp \varphi$.

The outcome of the **partial meet contraction** is equal to the intersection of the selected elements of $B \perp \varphi$, i.e., $B \div \varphi = \cap \gamma(B \perp \varphi)$.

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Discussion:

- Logical omniscience; belief sets are theories.
- 'Belief set is not what is believed, but what one is committed to believe' (Levi 1991).
- Computationally feasible: **belief base** instead of belief set.

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Changes are performed on the belief base. The underlying intuition is that the merely derived beliefs are not worth retaining for their own sake. If one of them loses the support that it had in basic beliefs, then it will be automatically and implicitly discarded.

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One belief set can be represented by different belief bases.
So, belief bases have more expressive power than belief sets.

Example

Alice's basic beliefs: p and q.
Bob's basic beliefs: p and $p \leftrightarrow q$.
Are Alice's and Bob's beliefs the same?
Go to: www.menti.com and enter code: 4560 3653

Example continued

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After that, Alice has the basic beliefs $\neg p$ and q, whereas Bob has the basic beliefs $\neg p$ and $p \leftrightarrow q$.

Now, their belief sets are no longer the same:
Alice believes that q whereas Bob believes that $\neg q$.

Belief Base Contraction

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$$\{p, p \wedge q, p \vee q, p \leftrightarrow q\} \perp p =$$

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Partial meet contraction, as defined before, is applicable to belief bases.

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Example

$$\{p, p \wedge q, p \vee q, p \leftrightarrow q\} \perp p = \{\{p \vee q\}, \{p \leftrightarrow q\}\}$$

Belief Revision Assignment: Highlights

- The assignment is now online.
- Work with belief bases.
- Implement your own method for generating a plausibility order and a selection function;
- Use AGM postulates to test your software.

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Contraction can be ruled by a binary relation of *epistemic entrenchment*.

Say p and q are in the belief set, q **is more entrenched than** p means that q is more useful in inquiry or deliberation, or has more "epistemic value" than p. So, the beliefs with the lowest entrenchment should be the ones that are most readily given up.

Entrenchment as an order

- $p \leq q$: p is at most as entrenched as q
- $p < q$: p is less entrenched than q, i.e., $p \leq q$ and $\neg(q \leq p)$
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The rationality postulates for entrenchment:

1. If $p \leq q$ and $q \leq r$, then $p \leq r$ (transitivity);
2. If $p \models q$, then $p \leq q$ (dominance);
3. Either $p \leq (p \wedge q)$ or $q \leq (p \wedge q)$ (conjunctiveness);
4. If the belief set K is consistent, then $p \notin K$ iff $p \leq q$, for all q (minimality);
5. if $q \leq p$ for all q, then $p \in Cn(\emptyset)$ (maximality).

Entrenchment-based contraction

An entrenchment relation $\leq$ gives rise to an operation $\div$ of **entrenchment-based contraction** according to the following definition:

$$q \in K \div p \text{ iff } q \in K \text{ and either } p < (p \vee q) \text{ or } p \in Cn(\emptyset).$$

End of Lecture 11