02180: Introduction to Artificial Intelligence

Lecture 8: Logical Agents

Nina Gierasimczuk

DTU

At a Restaurant

In a restaurant, your father has ordered fish, your mother ordered vegetarian, and you ordered meat. Out of the kitchen comes some new person carrying the three plates. What will happen?

At a Restaurant

In a restaurant, your father has ordered fish, your mother ordered vegetarian, and you ordered meat. Out of the kitchen comes some new person carrying the three plates. What will happen?

- The waiter asks two questions, say
  1. "Who has the meat?" and puts that plate.
  2. "Who has the fish?" and puts that plate.

  And, **without asking further**, he puts the remaining plate.

  What has happened here?

At a Restaurant

In a restaurant, your father has ordered fish, your mother ordered vegetarian, and you ordered meat. Out of the kitchen comes some new person carrying the three plates. What will happen?

- The waiter asks two questions, say
  1. "Who has the meat?" and puts that plate.
  2. "Who has the fish?" and puts that plate.

  And, **without asking further**, he puts the remaining plate.

  What has happened here?

The information in the two answers received allows the waiter to infer automatically where the third dish must go.

Sudoku

1		
		2

Sudoku

1		3
		2

Sudoku

1	2	3
		2

Three DTU students in a cafe

Three DTU students are sitting at a table.
The waiter asks: Does everyone want coffee?

- The first one says: I don't know.

Three DTU students in a cafe

Three DTU students are sitting at a table.
The waiter asks: Does everyone want coffee?

- The first one says: I don't know.
- Then the second one says: I don't know.

Three DTU students in a cafe

Three DTU students are sitting at a table.
The waiter asks: Does everyone want coffee?

- The first one says: I don't know.
- Then the second one says: I don't know.
- Then the third one says: No, not everyone wants coffee.

Three DTU students in a cafe

Three DTU students are sitting at a table.
The waiter asks: Does everyone want coffee?

- The first one says: I don't know.
- Then the second one says: I don't know.
- Then the third one says: No, not everyone wants coffee.
- The waiter gives the right people their coffees.

Three DTU students in a cafe

Three DTU students are sitting at a table.
The waiter asks: Does everyone want coffee?

- The first one says: I don't know.
- Then the second one says: I don't know.
- Then the third one says: No, not everyone wants coffee.
- The waiter gives the right people their coffees.

How?

Logic and Reasoning

- Logic studies inference and information update.
- Mathematics: already in Greek Antiquity (and in parallel, in other cultures), logical inference led to surprising new mathematical facts. Mathematical proof is a pure form of inference.
- Natural Sciences: ideally, combinations of inference with experiments.

Where is logic useful?

Mathematics Philosophy

Artificial Intelligence Psychology Computer Science

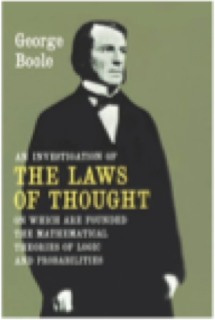
Linguistics Law

...

The Origins of Logic

- $+-$ 300 BC: Aristotle and the Stoic philosophers formulated explicit systems of reasoning in Greek Antiquity.
- In parallel: independent traditions in China and India.

Modern Logic



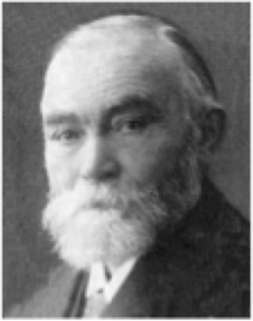


Figure: Boole and Frege

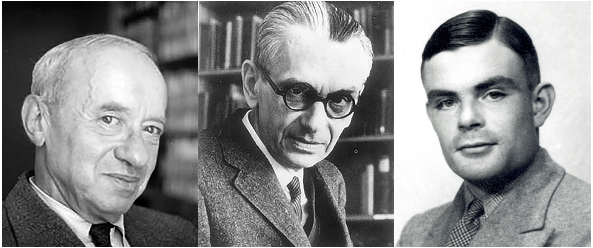
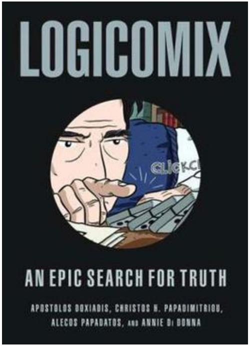


Figure: Tarski, Gödel and Turing

Modern Logic



Inference

Example

If you get Corona virus, you will get a cough.

You got a cough.

Therefore, you got Corona virus.

Inference

Example

If you get Corona virus, you will get a cough.

You got a cough.

Therefore, you got Corona virus.

What do you think about this inference?

Inference

Example

If you get Corona virus, you will get a cough.

You got a cough.

Therefore, you got Corona virus.

What do you think about this inference?

- You could be coughing even though you didn't get Corona virus!

Such reasoning can be dangerous in some situations!

Example

If I resist, the enemy will kill me.
But I am not resisting.

So, the enemy will not kill me.

Such reasoning can be dangerous in some situations!

Example

If I resist, the enemy will kill me.
But I am not resisting.

So, the enemy will not kill me.

- I might still get killed by the enemy for some other reason!

Validity

Example

If you eat these cookies you will get sick.
You did not get sick.

Therefore, you didn't eat the cookies.

Validity

Example

If you eat these cookies you will get sick.
You did not get sick.

Therefore, you didn't eat the cookies.

- This inference is **valid**!

Validity

Example

If you eat these cookies you will get sick.
You did not get sick.

Therefore, you didn't eat the cookies.

- This inference is **valid**!

Definition (Validity of Inference)

We call an inference *valid* if there is transmission of truth:
in every situation where all the premises are true,
the conclusion is also true.

Validity

Example

If you eat these cookies you will get sick.
You did not get sick.

Therefore, you didn't eat the cookies.

- This inference is **valid**!

Definition (Validity of Inference)

We call an inference *valid* if there is transmission of truth:
in every situation where all the premises are true,
the conclusion is also true.

In other words:

An inference is valid if it has no counter-examples (situations where the premises are all true while the conclusion is false).

Validity of Inference

- Is the following inference valid?

  If you take my medication, you will get better.
  You are not getting better.

  So, you have not taken my medication.

Validity of Inference

- Is the following inference valid?

  If you take my medication, you will get better.
  You are not getting better.

  So, you have not taken my medication.

  YES!!

Validity of Inference

- Is the following inference valid?

If you take my medication, you will get better.
But you are not taking my medication.

So, you will not get better.

Validity of Inference

- Is the following inference valid?

If you take my medication, you will get better.
But you are not taking my medication.

So, you will not get better.

No!! Can you give a counterexample?

What Validity really tells us

Premise 1
Premise 2

Therefore, Conclusion

- If a premise is false, nothing follows about the conclusion!
- Validity only rules out the situation where *all* the premises are true and the conclusion is false!

What validity of inference says:

- If *all* premises are *true*, then the conclusion is *true*.
- If the conclusion is *false*, then *at least one* premise is *false*

- important part of cognitive development:

  formal operational stage, where the mind is capable of reasoning according to classical propositional logic.
- the process of cognitive development:

  series of stages of enrichment of the logical apparatus of the child, enabling increasingly abstract reasoning.
- Human reasoning assumed to be logical and rational.
- But people turned out to be not so logical!

Figure: Jean Piaget

REASONING ABOUT A RULE

BY

P. C. WASON

From Psycholinguistics Research Unit, University College London

Two experiments were carried out to investigate the difficulty of making the contrapositive inference from conditional sentences of the form, "if P then Q." This inference, that not-P follows from not-Q, requires the transformation of the information presented in the conditional sentence. It is suggested that the difficulty is due to a mental set for expecting a relation of truth, correspondence, or match to hold between sentences and states of affairs. The elicitation of the inference was not facilitated by attempting to induce two kinds of therapy designed to break this set. It is argued that the subjects did not give evidence of having acquired the characteristics of Piaget's "formal operational thought."

A	D	3	4

If there is an A on one side, then there is a 3 on the other side.

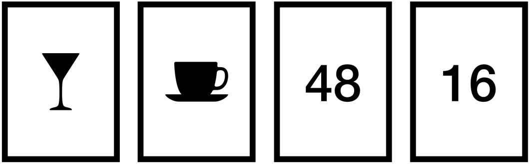
A	D	3	4

If there is an A on one side, then there is a 3 on the other side.

Which card would you turn?

Go to www.menti.com and use the code 6247 3098.

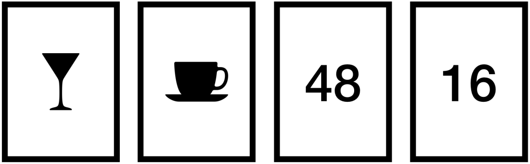
Wason Selection Task
Context sensitivity



If you drink alcohol here, you have to be over 18.

Wason Selection Task

Context sensitivity

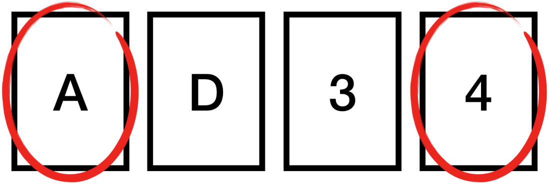


If you drink alcohol here, you have to be over 18.

Who would you check?

Wason Selection Task

‘Correct’ answer



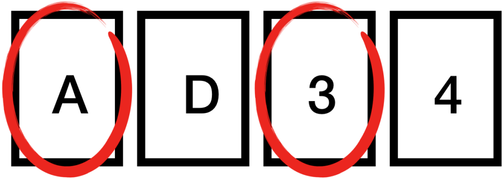
If there is an A on one side, then there is a 3 on the other side.

If P, then Q

Expected answer: P and $\neg Q$

Wason Selection Task

Typical answer

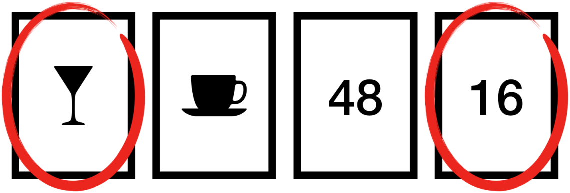


If there is an A on one side, then there is a 3 on the other side.

P	P, Q	P, ¬Q	P, Q, ¬Q
A	A, 3	A, 4	A, 3, 4
35 %	**45 %**	**5 %**	**7 %**

Wason Selection Task

Context sensitivity



If you drink alcohol here, you have to be over 18.

If P, then Q

Our focus: Knowledge-based agents

- Part of intelligence is **reasoning** operating on **internal representations**.
- **Logic** is a general apparatus for supporting knowledge-based agents.
- Knowledge-based agents are **flexible**: they accept new tasks in the form of explicitly formulated goals; they achieve competence by being learning new knowledge about the environment; they can adapt by updating knowledge.

Knowledge base (KB)

- Knowledge base is a set of **sentences**.
- Sentences are in a given **knowledge representation language**.
- Sentences represent assertions about the world.
- Some sentences have the status of **axioms** (they are given, must be and stay true).

Inference in KB

- Adding to the knowledge base (TELL).
- Querying the knowledge base (ASK).
- Inference must obey the following rule:
  when one ASKs a question of the knowledge base, the answer should follow from what it has been told before (TELLed).

A GENERIC KNOWLEDGE-BASED AGENT

function KB-AGENT(*percept*) **returns** an *action*
persistent: *KB* a knowledge base, initially background knowledge;
t, a counter, initially 0 (time)

TELL(*KB*,MAKE-PERCEPT-SENTENCE(*percept, t*))
action $\leftarrow$ ASK (*KB*, MAKE-ACTION-QUERY(*t*))
TELL(*KB*,MAKE-ACTION-SENTENCE(*action, t*))
t $\leftarrow$ t+1
return *action*

Basic Logical Notions

- Syntax (rules for building sentences).
- Semantics (rules determining the truth of sentences).
- Model (or a possible world):

  if a sentence φ is true in a model m, we say that the m satisfies φ, and that m is a model of φ.

  $M(\varphi)$ stands for the set of all models of φ.
- Entailment between sentences:

  φ follows from ψ: $\psi \models \varphi$ iff $M(\psi) \subseteq M(\varphi)$.
- Logical inference is carried out with the use of entailment.

- **Model-checking**: a procedure of logical inference by enumerating all models of KB and checking if the conclusion holds in all those models.
- If a logical inference algorithm can **derive** a sentence φ from KB, we write $KB \vdash \varphi$.
- A logical inference algorithm should be **sound** and **complete**.
- It should also be **grounded** in the underlying reality, e.g., KB should reflect the way the world really is, and adapt to the changing world.

From Natural Language to Propositional Notation

not ...	$\neg$	negation
... and ...	$\wedge$	conjunction
... or ...	$\vee$	disjunction
if ... then ...	$\rightarrow$	implication
... if and only if ...	$\leftrightarrow$	bi-implication

From Natural Language to Propositional Notation

not ...	$\neg$	negation
... and ...	$\wedge$	conjunction
... or ...	$\vee$	disjunction
if ... then ...	$\rightarrow$	implication
... if and only if ...	$\leftrightarrow$	bi-implication

A note on $\vee$: $\vee$ stands for the **inclusive** disjunction,
e.g.: "In order to pass the exam question 3 or question 4 must be answered correctly."
unlike "either ... or ..." in: "I will spend my summer in South America or in Asia."

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

Writing Natural Sentences in Logic

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade

Writing Natural Sentences in Logic

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\rightarrow$ (he has an Ace)

Formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

Step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\rightarrow$ (he has an Ace)
not (he has a King or a Spade) $\rightarrow$ (he has an Ace)

FORMALIZING A SENTENCE ABOUT A CARD GAME

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

STEP-BY-STEP

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\rightarrow$ (he has an Ace)
not (he has a King or a Spade) $\rightarrow$ (he has an Ace)
$\neg$(he has a King or a Spade) $\rightarrow$ (he has an Ace)

Writing Natural Sentences in Logic

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \to a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\to$ (he has an Ace)
not (he has a King or a Spade) $\to$ (he has an Ace)
$\neg$(he has a King or a Spade) $\to$ (he has an Ace)
$\neg$ ((he has a King) or (he has a Spade)) $\to$ (he has an Ace)

Writing Natural Sentences in Logic

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \rightarrow a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\rightarrow$ (he has an Ace)
not (he has a King or a Spade) $\rightarrow$ (he has an Ace)
$\neg$(he has a King or a Spade) $\rightarrow$ (he has an Ace)
$\neg$ ((he has a King) or (he has a Spade)) $\rightarrow$ (he has an Ace)
$\neg$((he has a King) $\vee$ (he has a Spade)) $\rightarrow$ (he has an Ace)

Writing Natural Sentences in Logic

formalizing a sentence about a card game

Sentence	"He has an Ace if he does not have a King or a Spade"
Formula	$\neg(k \vee s) \to a$
	k: "he has a King"
	s : "he has a Spade"
	a : "he has an Ace"

step-by-step

He has an Ace if he does not have a King or a Spade
if (he does not have a King or a Spade) then (he has an Ace)
(he does not have a King or a Spade) $\to$ (he has an Ace)
not (he has a King or a Spade) $\to$ (he has an Ace)
$\neg$(he has a King or a Spade) $\to$ (he has an Ace)
$\neg$ ((he has a King) or (he has a Spade)) $\to$ (he has an Ace)
$\neg$((he has a King) $\vee$ (he has a Spade)) $\to$ (he has an Ace)
$\neg(k \vee s) \to a$

Syntax: How to construct formulas

Definition (Language of Propositional logic)

1. Every proposition letter $(p, q, r, ...)$ is a formula.
2. If φ is a formula, then $\neg\varphi$ is also a formula.
3. If φ_1 and φ_2 are formulas, then $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \rightarrow \varphi_2)$ and $(\varphi_1 \leftrightarrow \varphi_2)$ are also formulas.
4. Nothing else is a formula.

Syntax: How to construct formulas

Definition (Language of Propositional logic)

1. Every proposition letter $(p, q, r, ...)$ is a formula.
2. If φ is a formula, then $\neg\varphi$ is also a formula.
3. If φ_1 and φ_2 are formulas, then $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \rightarrow \varphi_2)$ and $(\varphi_1 \leftrightarrow \varphi_2)$ are also formulas.
4. Nothing else is a formula.

Let Φ be a set of propositional letters, and let $p \in \Phi$ (that is: p is an element of the set Φ). Then the language of propositional logic is defined as follows:

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

WELL-FORMED FORMULAS?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

IS THIS A FORMULA OF PROPOSITIONAL LOGIC?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid (\varphi \to \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \lor (r \to (p \lor \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$

$\varphi \leftrightarrow \varphi :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \to (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$
$p \leftrightarrow (\varphi \vee \varphi) :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$
$p \leftrightarrow (\varphi \vee \varphi) :=$
$p \leftrightarrow (q \vee \varphi) :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$
$p \leftrightarrow (\varphi \vee \varphi) :=$
$p \leftrightarrow (q \vee \varphi) :=$
... $p \leftrightarrow (q \vee (r \rightarrow \varphi)) :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$
$p \leftrightarrow (\varphi \vee \varphi) :=$
$p \leftrightarrow (q \vee \varphi) :=$
... $p \leftrightarrow (q \vee (r \rightarrow \varphi)) :=$
... $p \leftrightarrow (q \vee (r \rightarrow (p \vee \varphi))) :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$
$\varphi \leftrightarrow \varphi :=$
$p \leftrightarrow \varphi :=$
$p \leftrightarrow (\varphi \vee \varphi) :=$
$p \leftrightarrow (q \vee \varphi) :=$
... $p \leftrightarrow (q \vee (r \rightarrow \varphi)) :=$
... $p \leftrightarrow (q \vee (r \rightarrow (p \vee \varphi))) :=$
$p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg\varphi))) :=$

Well-formed formulas?

Let p be any proposition symbol from a set Φ,

$$\varphi := \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Is this a formula of propositional logic?

$$(p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))))$$

We can argue for it by providing its **syntactic derivation**:

$\varphi :=$

$\varphi \leftrightarrow \varphi :=$

$p \leftrightarrow \varphi :=$

$p \leftrightarrow (\varphi \vee \varphi) :=$

$p \leftrightarrow (q \vee \varphi) :=$

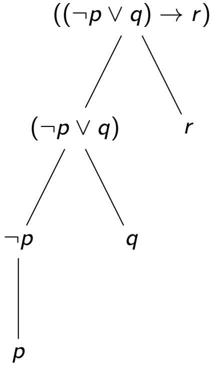
... $p \leftrightarrow (q \vee (r \rightarrow \varphi)) :=$

... $p \leftrightarrow (q \vee (r \rightarrow (p \vee \varphi))) :=$

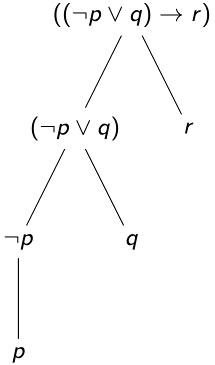
$p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg\varphi))) :=$

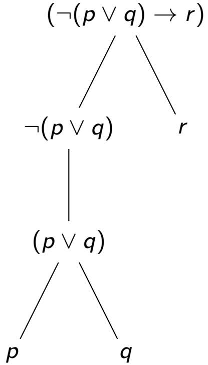
$p \leftrightarrow (q \vee (r \rightarrow (p \vee \neg q))$.

Construction Trees (Syntactic Trees)



Construction Trees (Syntactic Trees)





Valuations

Propositions generate different **possibilities**, ways the actual world might be.

The set $\{p, q, r\}$ will generate $2^3 = 8$ possibilities:

p	q	r
0	0	0

Valuations

Propositions generate different **possibilities**, ways the actual world might be.

The set $\{p, q, r\}$ will generate $2^3 = 8$ possibilities:

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Valuations

Propositions generate different **possibilities**, ways the actual world might be.

The set $\{p, q, r\}$ will generate $2^3 = 8$ possibilities:

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Each row is described by a function that assigns to each proposition a truth value: either 1 (="true") or 0 (="false"). We call such functions v, **valuations**.

Valuations

Propositions generate different **possibilities**, ways the actual world might be.

The set $\{p, q, r\}$ will generate $2^3 = 8$ possibilities:

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Each row is described by a function that assigns to each proposition a truth value: either 1 (="true") or 0 (="false"). We call such functions v, **valuations**.

For any proposition p, $v(p) = 1$ means that p is true, and $v(p) = 0$ means that p is false, in the situation represented by v.

Truth tables for Propositional logic

φ	$\neg\varphi$
0	
1	

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0				
0	1				
1	0				
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0				
0	1				
1	0				
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0				
0	1				
1	0				
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**			
0	1				
1	0				
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**			
0	1	**0**			
1	0				
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**			
0	1	**0**			
1	0	**0**			
1	1				

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**			
0	1	**0**			
1	0	**0**			
1	1	**1**			

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**		
0	1	**0**			
1	0	**0**			
1	1	**1**			

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**		
0	1	**0**	**1**		
1	0	**0**			
1	1	**1**			

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**		
0	1	**0**	**1**		
1	0	**0**	**1**		
1	1	**1**			

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**		
0	1	**0**	**1**		
1	0	**0**	**1**		
1	1	**1**	**1**		

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	
0	1	**0**	**1**		
1	0	**0**	**1**		
1	1	**1**	**1**		

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	
0	1	**0**	**1**	**1**	
1	0	**0**	**1**		
1	1	**1**	**1**		

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	
0	1	**0**	**1**	**1**	
1	0	**0**	**1**	**0**	
1	1	**1**	**1**		

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	
0	1	**0**	**1**	**1**	
1	0	**0**	**1**	**0**	
1	1	**1**	**1**	**1**	

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	**1**
0	1	**0**	**1**	**1**	
1	0	**0**	**1**	**0**	
1	1	**1**	**1**	**1**	

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	**1**
0	1	**0**	**1**	**1**	**0**
1	0	**0**	**1**	**0**	
1	1	**1**	**1**	**1**	

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	**1**
0	1	**0**	**1**	**1**	**0**
1	0	**0**	**1**	**0**	**0**
1	1	**1**	**1**	**1**	

Truth tables for Propositional logic

φ	$\neg\varphi$
0	**1**
1	**0**

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
0	0	**0**	**0**	**1**	**1**
0	1	**0**	**1**	**1**	**0**
1	0	**0**	**1**	**0**	**0**
1	1	**1**	**1**	**1**	**1**

Tautologies of Propositional Logic

A **tautology** is a statement that is always true, i.e.,
it is true under any valuation.

Tautologies of Propositional Logic

A **tautology** is a statement that is always true, i.e., it is true under any valuation.

Consider: $(p \to q) \lor (q \to p)$.

Tautologies of Propositional Logic

A **tautology** is a statement that is always true, i.e.,
it is true under any valuation.

Consider: $(p \to q) \lor (q \to p)$.

p	q	$(p \to q)$	$(q \to p)$	$(p \to q) \lor (q \to p)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

Satisfiability

A formula is **satisfiable** if there exists a valuation which makes the formula true.

Satisfiability

A formula is **satisfiable** if there exists a valuation which makes the formula true.

Consider: $\neg(p \wedge q) \vee (\neg r)$.

Satisfiability

A formula is **satisfiable** if there exists a valuation which makes the formula true.

Consider: $\neg(p \wedge q) \vee (\neg r)$.

p	q	r	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg r$	$\neg(p \wedge q) \vee (\neg r)$
0	0	0	0	1	1	1
0	0	1	0	1	0	1
0	1	0	0	1	1	1
0	1	1	0	1	0	1
1	0	0	0	1	1	1
1	0	1	0	1	0	1
1	1	0	1	0	1	1
1	1	1	1	0	0	0

Logical Consequence

A formula ψ is a logical consequence of the set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ if φ is always true when all of $\varphi_1, \ldots, \varphi_n$ are true.

In other words:
every valuation that makes all of $\varphi_1, \ldots, \varphi_n$ true, also makes ψ true.

When ψ is a logical consequence of $\varphi_1, \ldots, \varphi_n$ we write

$$\varphi_1, \ldots, \varphi_n \models \psi.$$

A formula ψ is a logical consequence of the set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ if φ is always true when all of $\varphi_1, \ldots, \varphi_n$ are true.

In other words:
every valuation that makes all of $\varphi_1, \ldots, \varphi_n$ true, also makes ψ true.

When ψ is a logical consequence of $\varphi_1, \ldots, \varphi_n$ we write

$$\varphi_1, \ldots, \varphi_n \models \psi.$$

You can determine if $\varphi_1, \ldots, \varphi_n \models \psi$ using the following method:

1. Construct a truth table for all of the formulas $\varphi_1, \ldots, \varphi_n, \psi$.
2. Check that in every row where all of $\varphi_1, \ldots, \varphi_n$ get the value 1, ψ also gets the value 1.

Logical Consequence (continued)

Consider: $p, q \models p \vee q$.

p	q	$(p \wedge q)$	$(p \vee q)$
0	0	0	0
0	1	0	1
1	0	0	1
1	**1**	**1**	**1**

Proof system

A **proof system** is a set of formulas called **axioms** and a set of **rules of inference**.

A **proof** of a formula ψ is a sequence of formulas $\varphi_1, \ldots, \varphi_n$, with $\varphi_n = \psi$, such that each φ_k is either an axiom or it is derived from previous formulas by rules of inference.

When such a proof exists, we say that ψ is a **theorem** (of the system) and that ψ is **provable** (in the system), denoted by:

$$\vdash \psi$$

Hilbert-style system $\mathcal{H}$

$\mathcal{H}$ is a proof system with three axiom schemes and one rule of inference.

For any formulas, φ, ψ and χ, the following formulas are axioms:

A1 $\vdash (\varphi \to (\psi \to \varphi))$

A2 $\vdash ((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$

A3 $\vdash ((\neg\psi \to \neg\varphi) \to (\varphi \to \psi))$

The single inference rule of $\mathcal{H}$ is modus ponens (MP for short):

$$\frac{\vdash \varphi \qquad \vdash (\varphi \to \psi)}{\psi}\ \mathrm{MP}$$

PROVING $\vdash (p \to p)$

Theorem. $\vdash (p \to p)$.

Proof.

1.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	(A2)
2.	$p \to ((p \to p) \to p)$	(A1)
3.	$(p \to (p \to p)) \to (p \to p)$	(MP 1,2)
4.	$p \to (p \to p)$	(A1)
5.	$p \to p$	(MP 3,4)

End of Lecture 8