02180: Introduction to Artificial Intelligence

Lecture 9: Belief Revision

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The problem of Belief Revision

Belief revision is a topic of much interest in theoretical computer science and logic, and it forms a central problem in research into artificial intelligence. In simple terms: how do you update a database of knowledge in the light of new information? What if the new information is in conflict with something that was previously held to be true?

Gärdenfors, Belief Revision

History: Data Bases, Theories and Beliefs

- Computer science: updating databases (Doyle 1979 and Fagin et al. 1983)
- Philosophy (epistemology):
  - scientific theory change and revisions of probability assignments;
  - belief change (Levi 1977, 1980, Harper 1977) and its rationality.

Outline

Belief revision on belief sets

Belief revision on plausibility orders

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AGM Belief Revision Model

- Names: Carlos **A**lchourrón, Peter **G**ärdenfors, and David **M**akinson.
- 1985 paper in the Journal of Symbolic Logic.
- Starting point of belief revision theory.

Belief Representation in AGM

We are talking about **beliefs** rather than **knowledge**.

In this context, the difference is that beliefs are changeable and can be false.

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Language of Beliefs in AGM

Beliefs are expressed in propositional logic:

- propositions $p, q, r, \ldots$
- connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), and biconditional ($\leftrightarrow$).

Belief Sets

Example (What are the consequences of my beliefs?)

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Belief set is a set of formulas that is **deductively closed**.
As such it is an abstract object.

Why are Belief Sets important?

Example

Assume Bob tells you that their beliefs include:

$$p, q.$$

Upon further inquiry it turns out that Bob also believes:

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Upon further inquiry it turns out that Bob also believes:

$$p \to \neg q.$$

Bob's beliefs are inconsistent!

Even though Bob's expressed beliefs do not include two complementary literals, $\neg q$ can be deduced from p and $p \to \neg q$.

He is committed to both q and $\neg q$, and so both are in Bob's belief set.

Logical Consequence

Definition

For any set B of sentences, $Cn(B)$ is the set of **logical consequences** of B.

If φ can be derived from B by classical propositional logic, then $\varphi \in Cn(B)$.

Example (cntd)

Bob's expressed beliefs form the set: $B = \{p, q, p \rightarrow \neg q\}$, then $B \subset Cn(B)$, $p \wedge q \in Cn(B)$, but also $\neg q \in Cn(B)$.

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Why is it problematic to have inconsistent beliefs?

If B is an inconsistent set of formulas, then for any formula φ, $\varphi \in Cn(B)$.

In other words, we can deduce anything from inconsistent beliefs, as such they are uninteresting and uninformative!

Bob revises his beliefs

Example

Assume Bob believes: $Cn(\{p, q\})$

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He learns, from a reliable source: $\neg q$

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Bob's new belief set should be:

$$Cn(\{p, \neg q\})$$

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Assume Bob believes: $Cn(\{p, q, r\})$

He learns, from a reliable source:

What should his new belief set be?

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Assume Bob believes: $Cn(\{p, q, r\})$

He learns, from a reliable source: $\neg(q \lor r)$

What should his new belief set be?

Bob's new belief set should be:

Bob revises his beliefs

Example

Assume Bob believes: $Cn(\{p, q, r\})$

He learns, from a reliable source: $\neg(q \vee r)$

What should his new belief set be?

Bob's new belief set should be:

$$Cn(\{p, \neg(q \vee r)\})$$

Bob revises his beliefs

Example

Assume Bob believes: $Cn(\{p, q, p \to q\})$

He learns, from a reliable source:

What should his new belief set be?

Bob's new belief set should be:

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Assume Bob believes: $Cn(\{p, q, p \rightarrow q\})$

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Example

Assume Bob believes: $Cn(\{p, q, p \to q\})$

He learns, from a reliable source: $\neg q$

What should his new belief set be?

Bob's new belief set should be:
go to https://www.menti.com/ and enter code: 8734 9059

Option A: $Cn(\{p\})$
Option B: $Cn(\{p \to q, \neg q\})$
Option C: $Cn(\{p, q, p \to q\})$
Option D: $Cn(\{p, \neg q\})$
Option E: $Cn(\{p, p \to q\})$
Option F: $Cn(\{p, \neg q, p \to q\})$

Bob revises his beliefs

Example

Assume Bob believes:

He learns, from a reliable source:

What should his new belief set be?

Bob's new belief set should be:

Option B: $Cn(\{p \to q, \neg q\})$

Option D: $Cn(\{p, \neg q\})$

Three Parts of Taking in New Information

What can I do to my belief set?

1. **Revision**: $B * \varphi$; φ is added and other things are removed, so that the resulting new belief set B' is consistent.
2. **Contraction**: $B \div \varphi$; φ is removed from B giving a new belief set B'.
3. **Expansion**: $B + \varphi$; φ is added to B giving a new belief set B'.

Levi Identitiy

One formal way to combine those two is to use:

Levi identity

$$B * \varphi := (B \div \neg\varphi) + \varphi.$$

Belief revision can be defined as first removing any inconsistency with the incoming information and then adding the information itself.

Outline

Belief revision on belief sets

Belief revision on plausibility orders

Thinking in Terms of Plausibility Orders: Prior

Bob believes: $Cn(\{p, q, p \to q\})$, i.e., the state x is the most plausible.

But there are different ways in which the remaining options can be ordered.

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
			w
	y		
		z	
x			

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
	y		
		z	
			w
x			

In the above pictures, the lower the state the more plausible it is.

Bob believes: $Cn(\{p, q, p \to q\})$, i.e., the state x is the most plausible.

After revising with $\neg q$ his **posterior plausibility** changes differently depending on the **prior plausibility**.

We are looking for prior-minimal states that do not satisfy q.

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
			w
	y		
		z	
x			

TABLE: Option B: $Cn(\{p, \neg q\})$

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
	y		
		z	
			w
x			

TABLE: Option E: $Cn(\{p \to q, \neg q\})$

Thinking in Terms of Plausibility Orders

Contraction

Bob believes: $Cn(\{p, q, p \to q\})$, i.e., the state x is the most plausible.

After contracting with q, Bob has to expand his view.

We are looking for prior-minimal states that do not satisfy q.

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
			w
	y		
		z	
x			

Table: $Cn(\{p\})$

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
	y		
		z	
			w
x			

Table: $Cn(\{p \leftrightarrow q\})$

After contraction Bob's beliefs are specified by the union of his prior most plausible world and the prior most plausible word not-entailing q.

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

more plausible

Table: Plausibility order over valuations

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

more plausible

Table: Plausibility order over valuations

Definition

Let P be a set of propositions (e.g. above, $P = \{p, q\}$). A **plausibility order** is a total preorder $\leq$ over the possible truth assignments W on P. A total preorder on X is a binary relation that is:

- transitive: for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$;
- complete: for all $x, y \in X$, $x \leq y$ or $y \leq x$.

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

more plausible

Table: Plausibility order over valuations

Let B be a belief set, φ a formula, and let $|\varphi| := \{x \in W \mid \varphi \text{ is true in } x\}$.

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

more plausible

Table: B is determined by the most plausible world(s)

Let B be a belief set, φ a formula, and let $|\varphi| := \{x \in W \mid \varphi \text{ is true in } x\}$.

- $\varphi \in B$ iff $min_{\leq}(W) \subseteq |\varphi|$;

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

more plausible $\downarrow$

Table: $B * \neg p$ is determined by min world(s) with $\neg p$

Let B be a belief set, φ a formula, and let $|\varphi| := \{x \in W \mid \varphi \text{ is true in } x\}$.

- $\varphi \in B$ iff $min_{\leq}(W) \subseteq |\varphi|$;
- $\varphi \in B * \psi$ iff $min_{\leq}(|\psi|) \subseteq |\varphi|$;

Revision and Contraction on Plausibility Orders Formally

p, q	$p, \bar{q}$	$\bar{p}, q$	$\bar{p}, \bar{q}$
		z	
			w
	y		
x			

$\downarrow$ more plausible

TABLE: $B \div \neg p$ is the union of the previous two

Let B be a belief set, φ a formula, and let $|\varphi| := \{x \in W \mid \varphi \text{ is true in } x\}$.

- $\varphi \in B$ iff $min_{\leq}(W) \subseteq |\varphi|$;
- $\varphi \in B * \psi$ iff $min_{\leq}(|\psi|) \subseteq |\varphi|$;
- $\varphi \in B \div \psi$ iff $min_{\leq}(|\neg\psi|) \cup min_{\leq}(W) \subseteq |\varphi|$