Formation and Structure of Dark Matter Halos

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- ① galaxies reside in extended halos of dark matter
- ② dark matter halos form from gravitational instability
- ③ density perturbation grows linearly util they reach a critical density, after which they turn around from the expansion of the Universe and collapse to form virialized dark matter halos.

We will discuss following topics:

A. the mass function of dark matter halos, its dependence on cosmology, and its evolution with redshift.

B. the matter distribution of progenitors of individual halos

C. the merger rate of dark matter halos.

D. intrinsic properties of dark matter halos, e.g. density profile, shape and angular momentum.

E. the mass function of dark matter subhalos, and it dependence on the mass of the host halo.

F. the large-scale environment and spatial clustering of dark matter halos.

A description of peaks in the cosmic density field:

Basic Assumption: the material which will collapse to form nonlinear objects of given mass can be identified in the initial density field by first smoothing it with a filter of the appropriate scale and then locating all peaks above some threshold.

- predict the number density of peaks as a function of peak height
- cannot be used to
  - obtain the mass function of non-linear objects
  - mathematically rigorous
  - provide a model for how dark matter halos grow with time.

$$N_{pk}(\nu)\,d\nu = \frac{1}{(2\pi)^2 R_*^3} e^{-\nu^2/2} G(\gamma, \gamma\nu)\, d\nu$$

$$R_* = \sqrt{3}\frac{\sigma_1(R)}{\sigma_2(R)} \qquad \gamma = \frac{\sigma_1^2(R)}{\sigma_2(R)\sigma_0(R)}$$

$$G(\gamma, y) = \frac{1}{\sqrt{2\pi(1-\gamma^2)}}\int_0^\infty \exp\left[-\frac{(x-y)^2}{2(1-\gamma^2)}\right] f(x)\, dx$$

$$R_* = \sqrt{3}\frac{\sigma_1(R)}{\sigma_2(R)} \qquad \gamma = \frac{\sigma_1^2(R)}{\sigma_2(R)\sigma_0(R)}$$

$$N_{pk}(\nu)\,d\nu = \frac{(\sigma_2^2/3\sigma_1^2)^{3/2}}{(2\pi)^2}(\nu^3-3\nu)e^{-\nu^2/2}\,d\nu \qquad \nu \gg 1$$

Peak-background split: $N_{pk}(\nu_s|\nu_b) \approx N_{pk}(\nu_p)$ $\quad \nu_p = (\delta_s-\delta_b)/\sigma_s$

As long as $R_b \gg R_s$, the effect of background field is just to shift the peak height from $\nu_s = \delta_s/\sigma_s$ to $\left(\frac{\delta_s-\delta_b}{\sigma_s}\right)$. This is called the peak-background split.

$$b_{pk}(\delta_s; R_s) = 1 + \frac{(\nu_s^2 - g_1)}{\sigma_s}$$

$\xi_{pk}(r) = b_{pk}^2(\delta, R)\xi(r)$, compared to the two-point correlation function of mass, $\xi(r)$, the two-point correlation function of the density peak is enhanced by a bias parameter.

Given a density perturbation field $\delta(x)$, we can filter it with some window function to get a smooth field

$$\delta(x;R) = \int \delta(x')\, W(x+x', R)\, d^3x' \qquad \int W(x,R)\, d^3x = 1$$

W: window function, spherically symmetric with characteristic radius R. $\quad \delta_k(R) = \delta_k \widetilde{W}(kR)$

two most commonly used functions:

① top-hat window function.

$$W_{th}(x;R) = \left(\frac{4}{3}\pi R^3\right)^{-1} = \begin{cases} 1 & \text{for } |x|<R \\ 0 & \text{for } |x|>R \end{cases}$$

$$\widetilde{W}_{th}(kR) = \frac{3(\sin kR - kR\cos kR)}{(kR)^3}$$

② Gaussian window function:

$$W_G(x;R) = \frac{1}{(2\pi)^{3/2}R^3}\exp\left(-\frac{|x|^2}{2R^2}\right)$$

$$\widetilde{W}_G(kR) = \exp\left(-\frac{(kR)^2}{2}\right)$$

③ sharp k-space filter: $\widetilde{W}_k(kR) = \begin{cases} 1 & \text{for } kR\le 1 \\ 0 & \text{for } kR>1 \end{cases}$

$$W_k(x;R) = \frac{1}{2\pi^2 R^3} y^{-3}(\sin y - y\cos y), \quad (y \equiv |x|/R)$$

$$V(R) = 6\pi^2 R^3$$

It's inappropriate to associate the number density of peaks in terms of a number density of collapsed objects of mass $M \propto \bar{\rho} R^3$.

— Cloud in cloud problem

Not quite sure about this.

Virialized object defined in astronomy:

The merging process won't go on forever, it stops when a cluster is virialized. Virialized means a system of gravitationaly interacting particles is stable. The small structures interact with each other, but the cluster doesn't expand or collapse. A system is virialized when the potential energy is twice the negative kinetic energy.

$$-V_{vir} = 2T_{vir}.$$

Virialized clusters: the merging process and the collpse of matter have finished, and the formation process of galaxies is down.

$R_{vir} = R_{max}/2$. wired.

How to connect smoothed density field with collapsed object? — Press & Schechter formalism

The ansatz of the Press & Schechter formalism.

$$\delta_s(x;R) = \int \delta_0(x') W(x+x';R)\, d^3x'$$

$$M = \tfrac{4}{3}\pi \bar{\rho} R^3$$

The probability that $\delta_s > \delta_c(t)$ is the same as the fraction of mass elements that at time t are contained in halos with mass greater than M.

$$F(>M) = \mathcal{P}(>\delta_c(t)]$$

$$\mathcal{P}(>\delta_c(t)] = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\delta_c(t)}{\sqrt{2}\,\sigma(M)}\right]$$

if $M \to 0$ $\sigma(M) \to +\infty$ $\mathcal{P}[>\delta_c(t)] = \frac{1}{2}$

linear theory. only overdense region can collapse

$$F(>M) = 2\,\mathcal{P}(>\delta_c)$$

$$n(M,t)\,dM = \frac{\bar{\rho}}{M}\frac{\partial F(>M)}{\partial M}\,dM$$
$$= 2\frac{\bar{\rho}}{M}\frac{\partial \mathcal{P}[>\delta_c(t)]}{\partial \sigma}\left|\frac{d\sigma}{dM}\right| dM.$$
$$= \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma}\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right)\left|\frac{d\ln\sigma}{d\ln M}\right| dM$$

define $\nu = \delta_c(t)/\sigma(M)$ $\quad n(M,t)\,dM = \frac{\bar{\rho}}{M^2} f_{PS}(\nu)\left|\frac{d\ln\nu}{d\ln M}\right| dM$

$$f_{PS}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\exp(-\nu^2/2)$$

characteristic mass $M^*(t)$: $\sigma(M^*) = \delta_c(t) = \delta_c/D(t)$

The Excursion Set Derivation of the Press-Schechter Formula.

$$\delta_s(x;R) = \int d^3k\, \tilde{W}_k(kR)\, \delta_{k,0}\, e^{ik\cdot x}$$
$$= \int_{k<k_c} d^3k\, \delta_{k,0}\, e^{ik\cdot x}$$

The advantage of using this particular filter is that the change $\Delta\delta_s$ corresponding to an increase from k_c to $k_c + \Delta k_c$ is a Gaussian Random variable with variance $\{\delta_s : S \in [0, +\infty)\}$

$$\langle(\Delta\delta_s)^2\rangle = \langle[\delta_s(x;k_c+\Delta k_c) - \delta_s(x;k_c)]^2\rangle$$
$$= \frac{1}{2\pi^2}\int_{k=k_c}^{k=k_c+\Delta k_c} P(k)\,k^2\,dk = \sigma^2(k_c+\Delta k_c) - \sigma^2(k_c)$$

where $\sigma^2(k_c) = \frac{1}{2\pi^2}\int_{k<k_c} P(k)\,k^2\,dk = S$

As such, $\Delta\delta_s$ is independent of the value of $\delta(x;k_c)$

So $\delta_s(S)$, the trajectory follows a Markovian random walk.

Derive halo mass function with Excursion Set formalism.

$M_1 > M_2$

$$F_{FU}(>S_1) = \int_{-\infty}^{\delta_c}[\mathcal{P}(\delta_s, S_1) - \mathcal{P}(2\delta_c - \delta_s, S_1)]\,d\delta_s$$
$$= \int_{-\infty}^{\nu_1}\frac{dx}{\sqrt{2\pi}}\exp(-x^2/2) - \int_{\nu_1}^{+\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}$$

$\nu_1 = \delta_c/\sqrt{S_1}$

$$n(M,t)\,dM = \frac{\bar{\rho}}{M}\frac{\partial F(>M)}{\partial M}\,dM = \frac{\bar{\rho}}{M} f_{FU}(S,\delta_c)\left|\frac{dS}{dM}\right| dM.$$

$$f_{FU}(S,\delta_c)\,dS = \frac{\partial F_{FU}}{\partial S}\,dS = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{S^{3/2}}\exp\left[-\frac{\delta_c^2}{2S}\right] dS.$$

A more elegant solution.

$$\langle \Delta\delta_s \rangle \equiv \langle \Delta\delta_s | \delta_s \rangle \qquad \sigma_0^2 \equiv \langle (\Delta\delta_s - \langle \Delta\delta_s \rangle)^2 | \delta_s \rangle$$

$$P(\Delta\delta_s|\delta_s) = \frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left[-\frac{(\Delta\delta_s - \langle\Delta\delta_s\rangle)^2}{2\sigma_0^2}\right] d(\Delta\delta_s)$$

$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2}$$

※ The mass associated with the first upcrossing of a mass element is only a lower limit on the actual mass of the collapsed object to which the mass element belongs.

※ This paradox is often interpreted as indicating that the excursion set formalism predicts, in a statistical sense, how much mass ends up in collapsed objects of a certain mass at a given time, but that it cannot be used to predict the halo mass in which a particular mass element ends up.

An interesting example: How to compute $\Omega_{m,0}$ and σ_8 using PS-formalism.

consider the mass of the clusters, estimated within the Abell radius: $M_A = 7.8 \times 10^{14} m_A h^{-1} M_\odot$

m_A is the mass parameter, defined through the mass overdensity.

$$\bar{\delta}(r_A) \equiv \frac{M_A}{\frac{4}{3}\pi \bar{\rho}_0 r_A^3} - 1 = 200 m_A/\Omega_{m,0}$$

How r_A is related to the total virial mass M?

$$M = \frac{4}{3}\pi r_{200}^3 \bar{\rho}_0 [1 + \bar{\delta}(r_{200})]$$

$$\bar{\delta}(r_{200}) = 200/\Omega_{m,0}$$

the density profile between r_A and r_{200} $\propto r^{-2}$ then

$$r_{200} = m_A^{1/2} r_A \qquad M \simeq m_A^{1/2} M_A$$

$$R = \left(\frac{3M}{4\pi\bar{\rho}_0}\right)^{1/3} \simeq 1.1\, \Omega_{m,0}^{-1/3} m_A^{1/2} r_8$$

$$\sigma(R) = \sigma_8 \left(\frac{R}{r_8}\right)^{-\beta} \qquad N(M_A) = \frac{4\pi r_8^3}{3}\int_{M_A}^{\infty} n(M, t_0)\, dM$$

Progenitors of Dark Matter Halos:

An advantage of the excursion set approach of the extended Press-Schechter (EPS) formalism — A neat way to calculate the progenitors which give rise to a collapsed object.

According to Markovian:

$$f_{FU}(S_1, \delta_1 | S_2, \delta_2) = \frac{1}{\sqrt{2\pi}} \frac{\delta_1 - \delta_2}{(S_1 - S_2)^{3/2}} \exp\left[-\frac{(\delta_1 - \delta_2)^2}{2(S_1 - S_2)}\right] dS_1$$

$$\boxed{\begin{aligned} n(M_1, t_1 | M_2, t_2)\, dM_1 &= \frac{M_2}{M_1} f_{FU}(S_1, \delta_1 | S_2, \delta_2) \left|\frac{dS_1}{dM_1}\right| dM_1 \\ &= \frac{M_2}{M_1^2} f_{PS}(\nu_2) \left|\frac{d\ln \nu_2}{d\ln M_1}\right| dM_1 \end{aligned}}$$

accurate as long as Δt is small.

underestimate the mass fraction in high mass progenitors for relatively large Δt.

Two principles for growing a merger tree:

① the number distribution of progenitor masses of many independent realizations needs to follow

② mass needs to be conserved.

正态过程：如果随机过程$\{X(t), t\in T\}$对任意$t_i \in T$ $(i=1,2,\dots,n)$，$X(t_1),\dots,X(t_n)$的联合分布为n维正态分布，则称$\{X(t), t\in T\}$为正态过程.

设$\{B(t), t\geq 0\}$为正态过程，轨道连续，$B(0)=0$，对$\forall s,t>0$有$E[B(t)]=0$，$E[B(s)B(t)]=t\wedge s$，则$\{B(t), t\geq 0\}$为布朗运动，反之亦然.

Main Progenitor Histories, the "trunk" of the tree.

Halo Merger Rate:

$$f_{FU}(S_2, \delta_2 | S_1, \delta_1)\, dS_2 = f_{FU}(S_1, \delta_1 | S_2, \delta_2) \frac{f_{FU}(S_2, \delta_2)}{f_{FU}(S_1, \delta_1)}$$

$$\mathcal{P}(\Delta M | M, t)\, d\ln\Delta M\, d\ln t = \frac{1}{\sqrt{2\pi}}\left[\frac{S_1}{S_1 - S_2}\right]^{3/2} \exp\left[-\frac{\delta_c^2(S_1 - S_2)}{2S_1S_2}\right] \times \left|\frac{d\ln\delta_c}{d\ln t}\right| \frac{\delta_c}{\sqrt{S_2}} \left|\frac{d\ln S_2}{d\ln\Delta M}\right| d\ln t\, d\ln\Delta M$$

Assembly bias: the dependence of halo clustering on halo assembly history at fixed halo mass is known as assembly bias. For example, low-mass halos in the high-redshift tail of the formation-time distribution are much more strongly clustered than halos of similar mass in the red-shift tail

Bias of dark matter halo:

$$\delta_h(X) = \hat{\delta}_h + \mathcal{E}(X), \quad \text{where } \hat{\delta}_h \equiv b_h(M_1, \delta_1; \underset{\hat{R}}{\delta})\, \delta(x)$$

Internal Structure of Dark Matter Halo.

$$\rho(r) \propto \begin{cases} r^{-2} & \text{for } n \le -1 \\ r^{-(3n+9)/(n+4)} & \text{for } n > -1 \end{cases}$$

$$\begin{cases} \text{truncated isothermal spheres: } \rho(r) \propto r^{-2} \text{ for } r \le r_h. \\ \\ \text{NFW profile: } \rho(r) = \rho_{crit} \dfrac{\delta_{char}}{(r/r_s)(1+r/r_s)^2} \end{cases}$$

Halo shape:

$$S = \frac{a_3}{a_1} \quad q = \frac{a_2}{a_1} \quad p = \frac{a_3}{a_2} \quad T = \frac{a_1^2 - a_2^2}{a_1^2 - a_3^2} = \frac{1-q^2}{1-s^2}$$

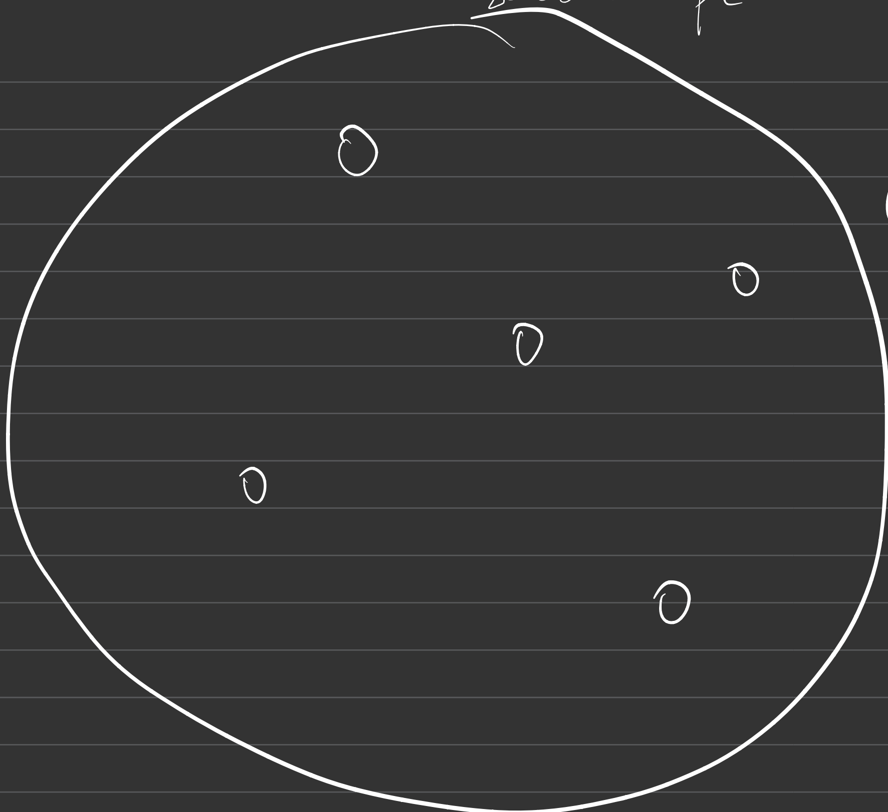
$0.5 \lesssim T \lesssim 0.85$ $\quad 0.5 \lesssim S \lesssim 0.75$ from simulation.

less massive halos are more spherical, and halos of a given mass become flatter with increasing redshift

Halo substrcuture: subhalo mass function.

Halo spin: log-normal

2000 cMpc



$z \sim 2$

① $\bar{\rho} = $ —

② $\rho = $ —

$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$

$B(t)$

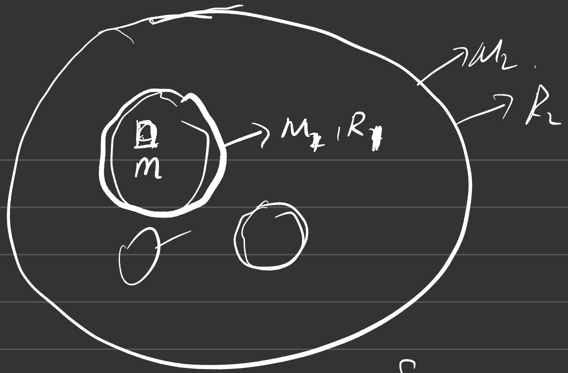
$X(t) \leftarrow$ 我是随机变量

$$E[B^2(t)] = t$$

$$E[B^2(\Delta t)] = \Delta t$$

$$\boxed{dX = S\,dt + \mu\,dB}$$ 因为我的二阶矩 dt.

$$\lim_{n\to\infty} E[(X_n - X)]^2 = X$$



$(\delta_1 \quad R_1) \to M_1,\ \delta_1 \qquad R_1 < R_2.$

$(\delta_2 \quad R_2) \to M_2 \qquad \delta_1 < \delta_2$