Second Order Vortex Panel Method for NACA 2412

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This paper walks through the calculation of the aerodynamic characteristics of the NACA 2412 airfoil in uniform flow using the second order vortex panel method as explained in Kuethe and Chow's *Foundations of Aerodynamics*. 12, 48, and 120 panel airfoils were each generated in two ways: the standard method and the closed trailing edge method. The lift, drag, and leading edge moment coefficients and center of pressure at various angles of attacks were then calculated and then used to determine the aerodynamic center and its moment about the quarter chord.

I. Nomenclature

a_0	=	lift slope
m_0	=	moment slope at quarter chord
C_p	=	pressure coefficient
c_l	=	2D lift coefficient
c_d	=	2D drag coefficient
c_n	=	2D normal force coefficient
c_a	=	2D axial force coefficient
c_{mle}	=	2D moment coefficient about the leading edge
$c_{m@c/4}$	=	2D moment coefficient about the quarter chord
LE	=	leading edge
TE	=	trailing edge
Re	=	Reynolds number
x_{ac}	=	aerodynamic center from LE to TE as a proportion of the chord
x_{cp}	=	center of pressure from LE to TE as a proportion of the chord
x	=	position along the chord from LE to TE as a proportion of the chord
y	=	position above the chord as a proportion of the chord

II. Introduction

Due to the complexity of the differential equations involved in fluid flows, there are no known analytical solutions for general aerodynamic problems. As a result, the modern industry places great emphasis on numerical methods, as they provide reasonable and practical estimates for the true flows. One such method—the vortex panel method—was explored to determine the aerodynamic characteristics of the NACA 2412 airfoil.

III. Method

The camber line of the general 4-digit NACA airfoil—with the first, second, and last two digits equal to $100m$, $10p$, and $100t$ respectively—is defined as

$$y_c(x) = \begin{cases} \frac{m}{p^2}(2px - x^2) & 0 \le x \le p \\ \frac{m}{(1-p)^2}\left((1-2p) + 2px - x^2\right) & p \le x \le 1. \end{cases} \tag{1}$$

The upper and lower airfoil coordinates (x_u, y_u) and (x_l, y_l) are defined as

$$\begin{aligned} x_u &= x - y_t \sin(\theta) \quad & y_u &= y_c + y_t \cos(\theta) \\ x_l &= x + y_t \sin(\theta) \quad & y_l &= y_c - y_t \cos(\theta), \end{aligned} \tag{2}$$

*Student, Department of Aerospace Engineering where $\theta = \arctan(\frac{dy_c}{dx})$ and y_t is the thickness of the symmetric airfoil at a given chord position [1]. The thickness of the airfoil was modelled in two ways: the standard way

$$y_t(x) = 5t(.2969\sqrt{x} - .1260x - .3516x^2 + .2843x^3 - .1015x^4, \tag{3}$$

and the closed trailing edge (TE) way

$$y_t(x) = 5t(.2969\sqrt{x} - .1260x - .3516x^2 + .2843x^3 - .1036x^4. \tag{4}$$

Because the former's thickness is not 0 at the TE, y_u(TE) and y_l(TE) was set to 0. To obtain the airfoil points for the finite panel approximation of the airfoil, cosine spacing was utilized along the profile of the airfoil clockwise from the TE. Note that each m panel approximation has $m + 1$ boundary points.

For the remainder of the paper, the i^{th} panel's center will be denoted as (x_i^c, y_i^c), boundary points as (X_i, Y_j) and (X_{i+1}, Y_{j+1}), angle as

$$\theta_i = \arctan(Y_{i+1} - Y_i, X_{i+1} - X_i), \tag{5}$$

and length as

$$S_i = \sqrt{(X_{i+1} - X_i)^2 + (Y_{i+1} - Y_i)^2}. \tag{6}$$

Note that $\arctan(y, x)$ returns the counterclockwise angle that the Cartesian point (x, y) makes with the $+x$ axis.

To determine the circulation and coefficient of pressure of each panel for an m panel approximation, the following $m \times m$ matrices with (i, j) entry corresponding to the i^{th} and j^{th} panel were defined as according to [2]:

$$\begin{aligned}
A_{i,j} &= (X_j - x_i^c)\cos(\theta_j) + (Y_j - y_i^c)\sin(\theta_j)\\
B_{i,j} &= (X_j - x_i^c)^2 + (Y_j - y_i^c)^2\\
C_{i,j} &= \sin(\theta_i - \theta_j)\\
D_{i,j} &= \cos(\theta_i - \theta_j)\\
E_{i,j} &= (Y_j - y_i^c)\cos(\theta_j) - (X_j - x_i^c)\sin(\theta_j)\\
F_{i,j} &= \ln(1 + (S_j + 2A_{i,j})S_j/B_{i,j})\\
G_{i,j} &= \arctan(E_{i,j}S_j, B_{i,j} + A_{i,j}S_j)\\
P_{i,j} &= (x_i^c - X_j)\sin(\theta_i - 2\theta_j) + (y_i^c - Y_j)\cos(\theta_i - 2\theta_j)\\
Q_{i,j} &= (x_i^c - X_j)\cos(\theta_i - 2\theta_j) - (y_i^c - Y_j)\sin(\theta_i - 2\theta_j)
\end{aligned} \tag{7}$$

Note that although Kuethe and Chow's expresses G as

$$G_{i,j} = \arctan(E_{i,j}S_j/(B_{i,j} + A_{i,j}S_j)),$$

such definition gives nonsensical results, as the range of the standard one-argument arctan function is $[-\pi/2, \pi/2]$ and does not capture an entire 2π radians. These matrices were then used to define other $m \times m$ matrices:

$$\begin{aligned}
C_{n2i,j} &= D_{i,j} + (.5Q_{i,j}F_{i,j} - (A_{i,j}C_{i,j} + D_{i,j}E_{i,j})G_{i,j})/S_j\\
C_{n1i,j} &= .5D_{i,j}F_{i,j} + C_{i,j}G_{i,j} - C_{n2i,j}\\
C_{t2i,j} &= C_{i,j} + (.5P_{i,j}F_{i,j} + (A_{i,j}D_{i,j} - C_{i,j}E_{i,j})G_{i,j})/S_j\\
C_{t1i,j} &= .5C_{i,j}F_{i,j} - D_{i,j}G_{i,j} - C_{t2i,j}
\end{aligned} \tag{8}$$

where $i \neq j$, while for each i^{th} entry,

$$\begin{aligned}
C_{n2i,i} &= 1\\
C_{n1i,i} &= -1\\
C_{t2i,i} &= C_{t1i,i} = \pi/2.
\end{aligned} \tag{9}$$

Finally the $m + 1 \times m + 1$ matrix A_n, $m \times m + 1$ matrix A_t, and $m + 1$ vector RHS were defined such that

$$\begin{aligned}
A_{nm+1,1} &= A_{nm+1,m+1} = 1\\
RHS_{m+1} &= 0
\end{aligned} \tag{10}$$

and for each entry $1 \leq i \leq m$ and $2 \leq j \leq m$,

$$\begin{aligned} A_{n\,i,1} &= C_{n1\,i,1} \\ A_{n\,i,j} &= C_{n1\,i,j} + C_{n2\,i,j-1} \\ A_{n\,i,m+1} &= C_{n2\,i,m} \\ A_{n\,m+1,j} &= 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} A_{t\,i,1} &= C_{t1\,i,1} \\ A_{t\,i,j} &= C_{t1\,i,j} + C_{t2\,i,j-1} \\ A_{t\,i,m+1} &= C_{t2\,im}. \end{aligned} \tag{12}$$

With these defined constants, the vortex of the i^{th} panel corresponds to the i^{th} entry of the vector

$$\gamma = {A_n}^{-1} RHS \tag{13}$$

according to [2], and the velocity to that of the freestream ratio of the i^{th} panel corresponds to the i^{th} entry of vector

$$V = \cos(\theta - \alpha) + A_t \gamma. \tag{14}$$

The C_p of the i^{th} panel can then be readily determined as

$$C_{p_i} = 1 - V_i^2. \tag{15}$$

To determine the various desired coefficients, the trapezoid rule was utilized to evaluate the integrals

$$c_n = \int_0^c C_{p_l}(x) - C_{p_u}(x)\, dx, \tag{16}$$

$$c_a = \int_0^c C_{p_u}(x)\frac{dy_u}{dx} - C_{p_l}(x)\frac{dy_l}{dx}\, dx, \tag{17}$$

$$c_{mle} = \int_0^c -(C_{p_l}(x) - C_{p_u}(x))x + C_{p_u}(x) y_u(x)\frac{dy_u}{dx} - C_{p_l}(x) y_l(x)\frac{dy_l}{dx}\, dx, \tag{18}$$

where C_{p_u} and C_{p_l} denote C_p on the upper and lower surfaces respectively. Trapezoid rule was chosen as it is a second order of convergence method, as the utilized vortex panel method is second order. Any integration method of higher order such as Simpson's Rule would thus be pointless, as the overall method to determine the coefficients is a second order method. To determine c_l and c_a, the following relationship was utilized:

$$\begin{bmatrix} c_l \\ c_d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} c_n \\ c_a \end{bmatrix}. \tag{19}$$

To determine the aerodynamic center, the following formula was utilized:

$$x_{ac} = .25 - \frac{\partial c_{m@c/4}}{\partial c_l} = .25 - \frac{m_0}{a_0} \tag{20}$$

IV. Solution

The generated NACA 2412 is plotted in Figure 1.

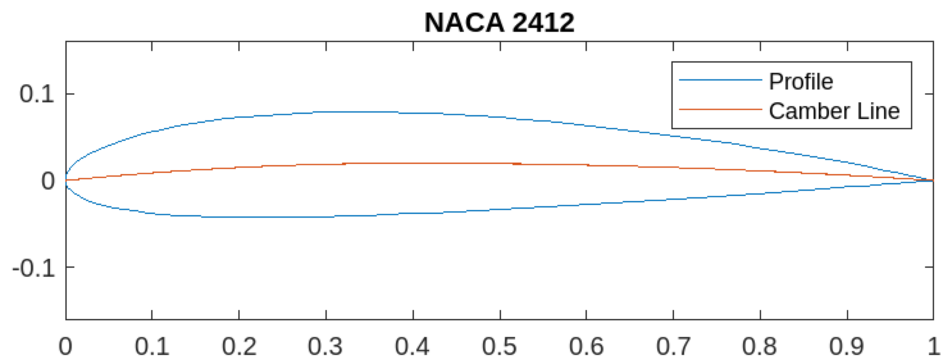

Fig. 1 The profile and camber line of the NACA 2412 airfoil of 120 panels generated with profile cosine spacing.

Figures 2 and 3 plot the coefficient of pressures for 12, 48, and 120 panel approximations for the standard and closed thickness profiles respectively at $\alpha = 8°$. Note the center of pressure's failure to converge at the TE for the standard thickness method.

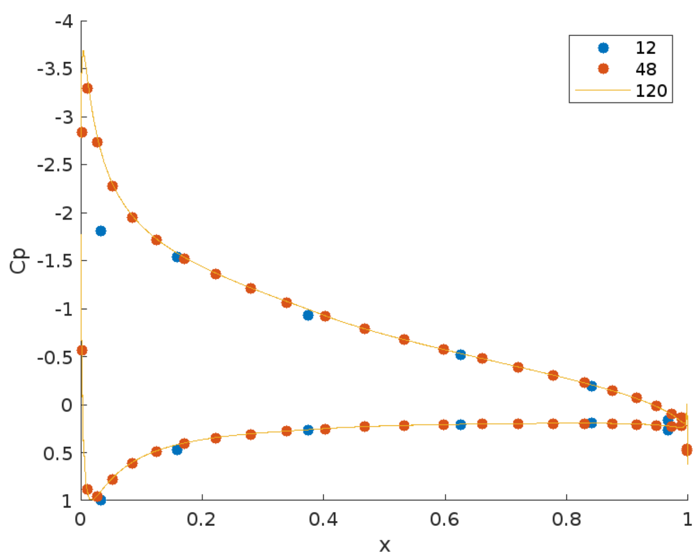

Fig. 2 The coefficients of pressure at the panel centers for 12, 48, and 120 panel approximations generated via the standard thickness method.

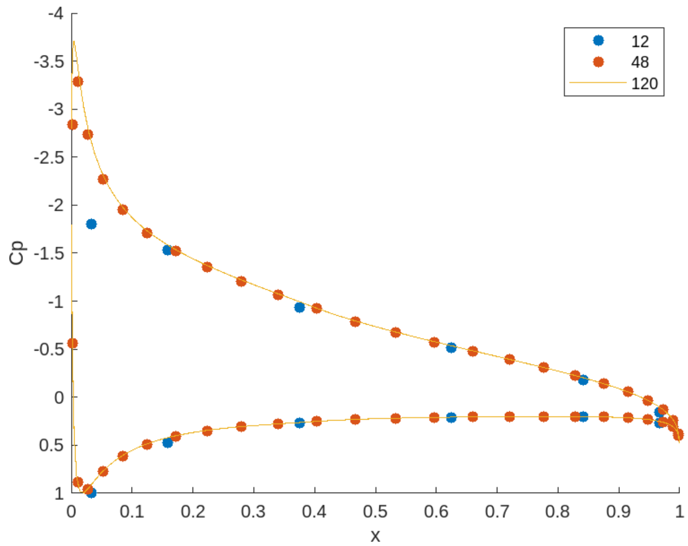

Fig. 3 The coefficients of pressure at the panel centers for 12, 48, and 120 panel approximations generated via the closed thickness method.

The calculated c_l, c_d, c_{mle}, and x_{cp} are tabulated in Figure 4 along with the interpolated NACA 824 experimental values at $Re = 3.1 \cdot 10^6$ [3].

Parameter	Calculated	Experimental
c_l	1.2107	1.0461
c_d	.0066	.0113
c_{mle}	$-.3685$	$-.3019$
x_{cp}	.3044	.2886

Fig. 4 The calculated and experimental values of c_l, c_d, c_{mle}, and x_{cp} at $\alpha = 8°$.

The c_l, $c_{m@c/4}$, and x_{cp} at various angles of attack are plotted with their curve fits in Figure 5.

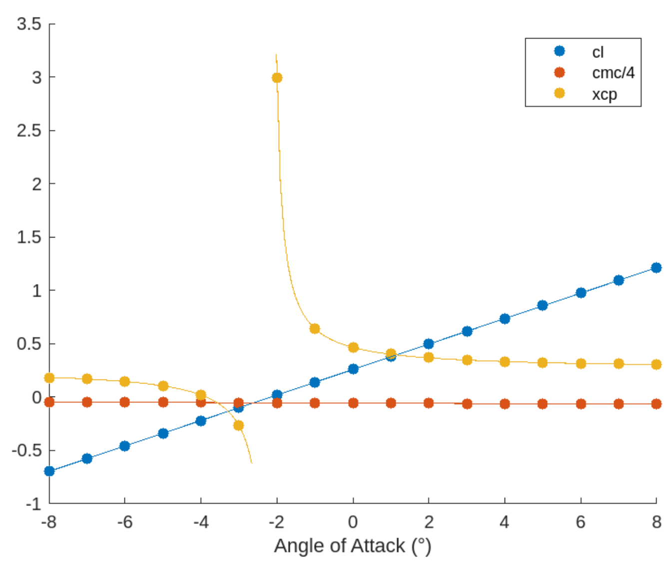

Fig. 5 The aerodynamic characteristics of the NACA 2412 airfoil at various angles of attacks.

A curve was then fitted through the points to determine a_0 and m_0, which are tabulated along with x_{ac} and c_{mac} and their experimental values and percent differences from the experimental value in Figure 6.

Parameter	Calculated	Experimental	Percent Difference
a_0	.1194	.0983	21.5%
m_0	.0013	.0044	−70.%
x_{ac}	.2606	.239	9.04%
c_{mac}	−.0534	−.0503	6.16%

Fig. 6 The calculated values, experimental values, and the percent differences from experiment.

V. Conclusion

Because the standard thickness method yielded a C_p graph with abruptly different values at the edges while the closed thickness method yield a smooth C_p graph, it is likely that the abruptness is a consequence of changing the trailing edge's y ordinance to 0. Even though the vortex panel method is based off of inviscid flow theory—which predicts that there should be no drag—the described vortex panel method predicts that the airfoil will have a non-negligible drag component of 66 drag counts. This will likely be mitigated as the number of panels increases. Our determined aerodynamic values ultimately differ, likely because the vortex panel method is based off of incompressible and inviscid flow theory. Of course, in the physical world, density and viscosity are non-negligible. Most notably, the calculated c_l is much higher than the experimental, and the calculated x_{ac} is above the quarter chord.

References

[1] Ladson, C. L., Brooks, C. W., Hill, A. S., and Sproles, D. W., "Computer Program to Obtain Ordinates for NACA Airfoils," Technical Memorandum NASA-TM-4741, National Aeronautics and Space Administration, Washington, D.C., 1996.

[2] Arnold M. Kuethe, C.-Y. C., *Foundations of Aerodynamics: Bases of Aerodynamic Design (5th Ed.)*, 5th ed., John Wiley Sons, Inc., 1998. URL `http://gen.lib.rus.ec/book/index.php?md5=9a982cc5d1d13307a0100670f55bea73`.

[3] Abbott, I. H., Doenhoff, A. E. V., and Stivers, L., "Theory of Wing Sections Including a Summary of Airfoil Data," NACA Technical Report NACA-TR-824, National Advisory Committee for Aeronautics, Washington, D.C., 1945.