Task 3

A fair coin is tossed 400 times. Let X be number of heads. Prove that $P(X > 240) \leq 1/32$.
(Hint: use Chebyshev's inequality and symmetry considerations.)

Let X_i be random variables with Bernoulli distribution, such that $X_i = 1$ if i-th toss is a "Head" and $X_i = 0$ otherwise.

So $X_i \sim B\left(p = \frac{1}{2}\right)$ for $i \in [1, 400]$.

Then we can say that X is a sum of X_i for i in $[1..400]$.

$$X = \sum_{i=1}^{400} X_i$$

As X_i are independent variables that have same Bernoulli distribution we can say that their sum X has Binomial distribution.

- $X \sim Binomial\left(n = 400, p = \frac{1}{2}\right)$

$EX = np = 200, \quad VarX = np(1-p) = 100.$

PDF_X of binomially distributed variable X is symmetric about its estimated value EX, so we can say, that $P(X > EX + \alpha) = P(X < EX - \alpha)$.

As such we get

$$P(|X - 200| > 40) = P(X < 200 - 40 \;\; or \; X > 200 + 40) =$$
$$= P(X < 160) + P(X > 240) = 2 \cdot P(X > 240)$$

$$2 \cdot P(X > 240) = P(|X - 200| > 40)$$
$$P(X > 240) = \frac{1}{2} P(|X - 200| > 40)$$

- According to Chebyshev's inequality

$$P(|X - EX| > \alpha) \leq \frac{VarX}{\alpha^2}.$$

Hence

$$P(X > 240) = \frac{1}{2} P(|X - 200| > 40) \leq \frac{1}{2} \cdot \frac{VarX}{\alpha^2}$$

$$\frac{1}{2} \cdot \frac{VarX}{\alpha^2} = \frac{100}{2 \cdot 40^2} = \frac{100}{2 \cdot 1600} = \frac{1}{32}$$

$$P(X > 240) \leq \frac{1}{32}, q.e.d$$