Task 4

Let X and Y be two independent normally distributed random variables with expected value 0 and variance 1. Find their joint PDF. Plot its level curves.

$X \sim N(0,1)$: $\quad PDF_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$Y \sim N(0,1)$: $\quad PDF_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

• As X and Y are independent $PDF_{X,Y}(x,y) = PDF_X(x) \cdot PDF_Y(y)$.

$$PDF_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{e^{-(x^2+y^2)/2}}{2\pi}$$

So joint PDF for X and Y is

$$PDF_{X,Y}(x,y) = \frac{e^{-(x^2+y^2)/2}}{2\pi}$$

• Next, we can draw its level curves. For that we need to draw graphs of $PDF(x,y) = C$ for different values of C.

First let's modify the equation $PDF(x,y) = C$ in general case.

$$\frac{e^{-(x^2+y^2)/2}}{2\pi} = C$$

$$e^{-(x^2+y^2)/2} = 2\pi C$$

$$-\frac{x^2+y^2}{2} = \ln(2\pi C)$$

$$x^2 + y^2 = -2\ln(2\pi C)$$

From this we can conclude, that level lines of joint PDF are concentric circles with center at point (0;0) and radius $R = \sqrt{-2\ln(2\pi C)}$.

Here we have some restrictions:

1) $2\pi C$ must be greater than 0, so $C > 0$;

2) $-2\ln(2\pi C) \geq 0 \leftrightarrow C \leq \frac{1}{2\pi}$.

Hence, $0 < C \leq \frac{1}{2\pi}$.

We can also see, that maximum value $\frac{1}{2\pi}$ is achieved at $(x,y) = (0,0)$.

Next let's calculate radius R of level lines for different values of C.

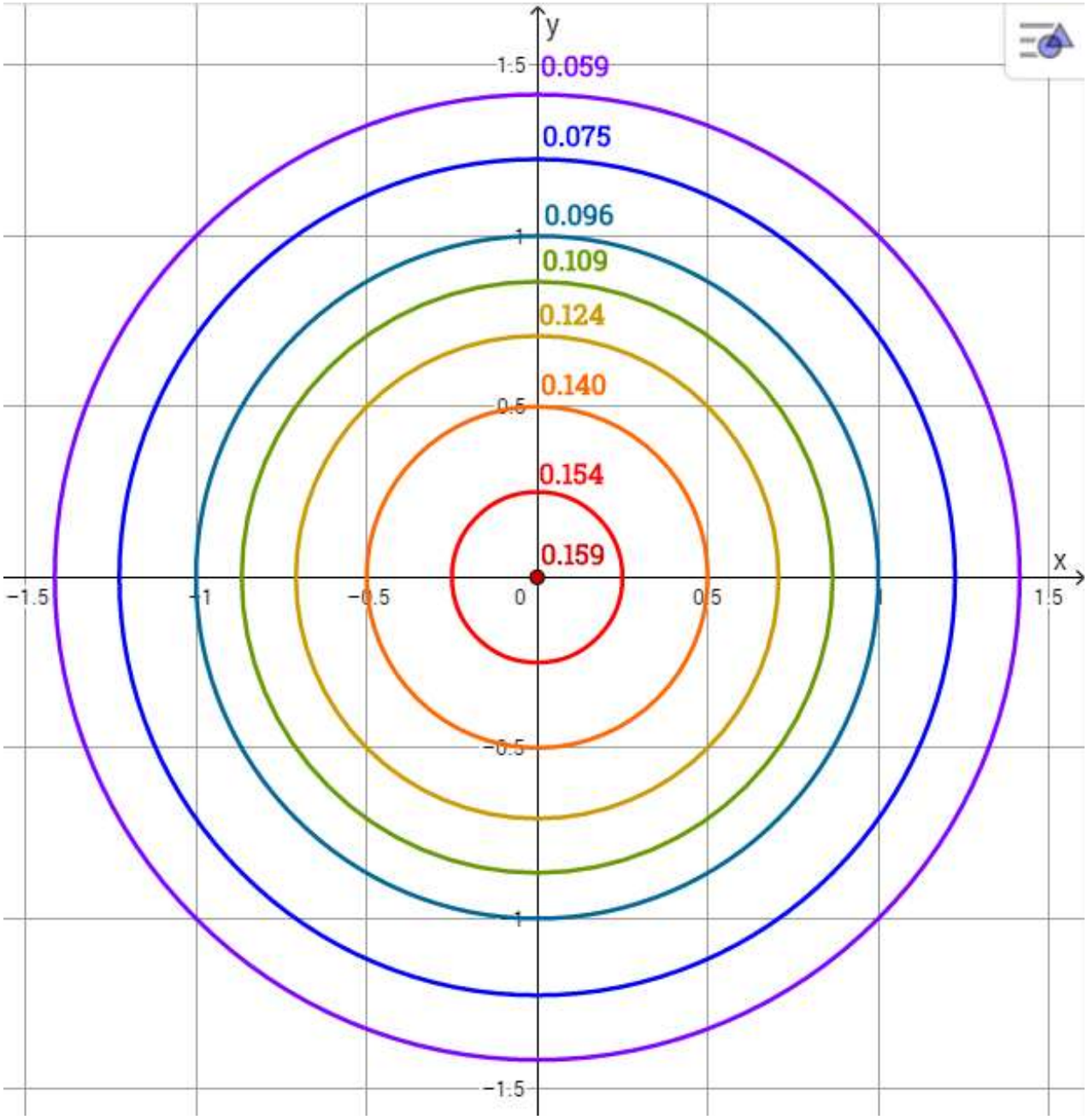
For simplicity I'll consider values of C in the form $\frac{1}{2\pi e^t}$.

$C = \frac{1}{2\pi e^t}$: $\quad x^2 + y^2 = -2\ln\left(\frac{2\pi}{2\pi e^t}\right) = -2\ln(e^{-t}) = 2t$

We can see that $R^2 = 2t$ for such values of C, so for every power t in $C = \frac{1}{2\pi e^t}$ we get a circle of radius $R = \sqrt{2t}$.

C	$C(approx.)$	t	R	Level curves
$\frac{1}{2\pi}$	0.15916	0	0	$(0;0)$
$\frac{1}{2\pi e^{1/32}}$	0.15426	1/32	1/4	$x^2 + y^2 = \frac{1}{16}$
$\frac{1}{2\pi e^{1/8}}$	0.14045	1/8	1/2	$x^2 + y^2 = \frac{1}{4}$
$\frac{1}{2\pi e^{1/4}}$	0.12395	1/4	$\frac{\sqrt{2}}{2}$	$x^2 + y^2 = \frac{1}{2}$
$\frac{1}{2\pi e^{3/8}}$	0.10939	3/8	$\frac{\sqrt{3}}{2}$	$x^2 + y^2 = \frac{3}{4}$
$\frac{1}{2\pi e^{1/2}}$	0.09653	1/2	1	$x^2 + y^2 = 1$
$\frac{1}{2\pi e^{3/4}}$	0.07518	3/4	$\frac{\sqrt{6}}{2}$	$x^2 + y^2 = \frac{3}{2}$
$\frac{1}{2\pi e}$	0.05855	1	$\sqrt{2}$	$x^2 + y^2 = 2$

Level curves of $PDF_{X,Y}(x, y)$ are shown in the graph below.



*Made and colored “by hand” in Geogebra app :)