SGA 2 week

1.

Consider two random events A and B defined for the same random experiment.

1. Is it possible that A and B are independent and mutually exclusive (disjoint) at the same time? Explain your answer.

2. Does the answer change if given that $P(A) > 0$ and $P(B) > 0$? Explain your answer.

<u>Solution:</u>

a)

A and B are independent $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

If A and B are disjoint : $P(A \cap B) = 0$

So they can be disjoint and independent if

$P(A) = 0$ or $P(B) = 0$ or both.

b) If $P(A) > 0$ and $P(B) > 0$, A and B events cannot be disjoint and independent at the same time.

$P(A) \cdot P(B) \neq 0$, при $P(A) > 0$, $P(B) > 0$

In other words, $P(A \cap B)$ cannot be zero and non-zero at the same time.

2.

The main prize at some TV show is a car. At the beginning the car is placed behind one door randomly chosen out of three (with equal probabilities). Participant has to guess the door with a car. If the guess is correct, the participant wins the car, otherwise she wins nothing. After the participant picks the door and announces her choice, host opens one of the two remaining doors and shows that there are no car there. Then the participant can either switch her decision and pick the remaining closed door or keep the door she picked initially. After the final decision is made, the chosen door opens and the participant get her prize (if any). Assume that the host never opens a door that is picked by the participant initially and never opens a door with a car. If the host can chose between several doors, the choice is random (with equal probabilities). Let us enumerate doors in such a way that the door initially picked by the participant has number 1.

Consider events H_1: *the car is behind door number 1*, H_2: *the car is behind door number 2* and H_3: *the car is behind door number 3*. Consider also event A: *the host opened door number 2*.

1. What can you say about probability of H_3 before any door is opened?
2. Assume that the host opened door number 2. What would you say about probability of H_3 after you observe that?
3. Are events A and H_1 independent? A and H_2? A and H_3?
4. Use Bayes' rule to find $P(H_1 \mid A)$ and $P(H_3 \mid A)$ (find all necessary probabilities that are used in Bayes' rule first). Compare with your previous answers.
5. Should the participant change the initial decision to increase probability of winning?

Please, provide full explanations of your answers, with proper references to the facts that were discussed in the videos and all necessary calculations and derivations.

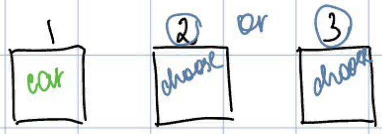
Solution:

1) $P(H_1) = P(H_2) = P(H_3) = 1/3$

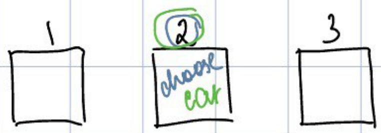
2) A – the host opened door number 2.

$P(A|H_3)$ – ?

$P(A|H_1) = 1/2$ (host choose door 2 or 3 if car behind door 1.)



$P(A|H_2) = 1$ (host open door 3 if the car is behind door 2)



$P(A|H_3) = 0$ (host won't open door with car cause he's already found car behind 2d door)

3) a) $P(A \cap H_1) \stackrel{?}{=} P(A) \cdot P(H_1)$

b) $P(A \cap H_2) \stackrel{?}{=} P(A) \cdot P(H_2)$

c) $P(A \cap H_3) \stackrel{?}{=} P(A) \cdot P(H_3)$

Let's first find P(A) by using formula of total probability:

$P(A) = P(A|H_1) \cdot P(H_1) + P(A|H_2) \cdot P(H_2) + P(A|H_3) \cdot P(H_3)$

$(1/2 \cdot 1/3) + (1 \cdot 1/3) + (0 \cdot 1/3) = 1/6 + 1/3 = 1/6 + 2/6 = 3/6 = 1/2$

$P(A) = 1/2$ (host choose only 2d door)

$P(A \cap H_1) = P(A|H_1) \cdot P(H_1) = 1/6$

$P(A \cap H_2) = P(A|H_2) \cdot P(H_2) = 1/3$

$P(A \cap H_3) = P(A|H_3) \cdot P(H_3) = 0$

a) $P(A \cap H_1) = P(A) \cdot P(H_1) = 1/2 \cdot 1/3 = 1/6 \Rightarrow$ A and H_1 are independent

b) $P(A \cap H_2) = P(A) \cdot P(H_2) = 1/2 \cdot 1/3 = 1/6 \Rightarrow$ A and H_2 are dependent

c) $P(A \cap H_3) = P(A) \cdot P(H_3) = 1/2 \cdot 1/3 = 1/6 \Rightarrow$ A and H_3 are dependent

4) a) $P(H_1|A) = \frac{P(A|H_1) \cdot P(H_1)}{P(A)} = \frac{1/2 \cdot 1/3}{1/2} = 1/3$

б) $P(H_3|A) = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{0 \cdot 1/3}{1/2} = 0$

c) $P(H_2|A) = \frac{P(A|H_2) \cdot P(H_2)}{P(A)} = \frac{1 \cdot 1/3}{1/2} = 2/3$

5) Yes, the participant has to change their decision from 1 door to 2d door because the probability will increase from 1/3 to 2/3.

3.

I can buy one lottery ticket out of two available. In the first lottery I can win $\$100$ with probability 0.1, and the price of ticket is $\$10$. In the second lottery, I can win $\$50$ with probability 0.1 and $\$500$ with probability 0.01. The price of ticket is $\$20$. To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise. Let X be random variable that denotes my net payout (taking into account price of a ticket). Find probability mass function of X (hint: use law of total probability). Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

Solution:

1 lottery (1 ticket):
win 100 prob 0.1
price 10

2d lottery (2 ticket):
win 50 prob. 0.1
and
win 500 prob. 0.01
price 20

fair coin $P(1T) = P(2T) = 1/2$

H_1 – get H ⇒ choose 1st ticket

H_2 – get T ⇒ choose 2d ticket

X – net payout (r.v)

PMF_X – ?

$$EX = \frac{EX_1 + EX_2}{2}$$

1 lottery:

X_1	90	−10
$P(X_1=x)$	0.1	0.9

2 lottery:

X_2	30	480	−20
$P(X_2=x)$	0.1	0.01	0.89

So according to the law of total probability:

$$p'(X=x) = \frac{1}{10} \cdot \frac{1}{2} = 1/20 = 0.05$$

$$p'(X=x) = 9/10 \cdot 1/2 = 9/20 = 0.45$$

$$p'(X=x) = 1/10 \cdot 1/2 = 0.05$$

$$p'(X=x) = 1/100 \cdot 1/2 = 1/200 = 0.005$$

$$p'(X=x) = 0.89 \cdot 1/2 = 0.445$$

X_1	90	-10
$P(X_1=x)$	0.1	0.9
$p'(X_1=x)$	0.05	0.45

X_2	30	480	-20
$P(X_2=x)$	0.1	0.01	0.89
$p'(X_2=x)$	0.05	0.005	0.445

The total expected value:

$$\mathbb{E}X = 90 \cdot 0.05 + 0.45(-10) + 30 \cdot 0.05 + 480 \cdot 0.005 - 20 \cdot 0.445$$
$$= 4.5 - 4.5 + 1.5 + 2.4 - 8.9 = 3.9 - 8.9 = -5$$

Expected value of 1st lottery:

$$\mathbb{E}X_1 = 90 \cdot 0.1 - 10 \cdot 0.9 = 0$$

Expected value of 2d lottery:

$$\mathbb{E}X_2 = 30 \cdot 0.1 + 480 \cdot 0.01 - 20 \cdot 0.89 = 3 + 4.8 - 17.8 = -10$$

$$\mathbb{E}X = \frac{\mathbb{E}X_1 + \mathbb{E}X_2}{2} = \frac{-10+0}{2} = -5$$

$-5 = -5 \Rightarrow$ true, we have same probabilities of choosing a lottery.

This rule will hold if lotteries has different payouts and same probabilities and won't hold if lotteries will have different probabilities of choosing. Let's prove it:

Let probabilities of choosing be $p = 1/2$

Let probabilities of r.v. x_i be p_1, p_2, p_3, p_4, p_5.

Let outcomes be x_1, x_2, x_3, x_4, x_5

So $E'X = pp_1x_1 + pp_2x_2 + \dots + pp_5x_5 = p(p_1x_1 + \dots + p_5x_5)$ -

- EX of 1st lottery with chance p of choosing and 2d lottery with chance p also

$\mathbb{E}_1X = p_1x_1 + p_2x_2$ - expected value of 1st lottery

$\mathbb{E}_2X = p_3x_3 + p_4x_4 + p_5x_5$ - ex.v. of 2d lottery

$$E'X = \frac{\mathbb{E}_1X + \mathbb{E}_2X}{2} = \frac{1}{2}(\mathbb{E}_1X + \mathbb{E}_2X)$$

$$p(p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5) = \frac{1}{2}(\underbrace{(p_1x_1 + p_2x_2)}_{\mathbb{E}X_1} + \underbrace{(p_3x_3 + p_4x_4 + p_5x_5)}_{\mathbb{E}X_2})$$

$p = 1/2$

So, we can see that equation we recieved in previous step doesn't depend on random variable distribution and will hold only if $p = 1/2$, which means that choosing of lottery must have same probabilities.

4.

Space Company is preparing to launch a spaceship. If the launch is successful, the company earns $100 million in profit. In case of failure, the company loses $200 million (i.e. negative profit). Probability of failure is 1/10. The company can buy insurance for this launch. Cost of insurance is $30 million (paid before the launch). In case of failure the insurer will pay Space Company $200 million (thus compensating all the damages). Consider two cases: 1. Space Company decided not to buy insurance. 2. Space Company decided to buy insurance. Denote its profit in the first case by X and in the second case by Y. (In the second case profit includes payments to/from the insurer taken with appropriate sign.) Find expected values and variances of X and Y. Describe how buying of insurance affects the profit, its expected value and variance? Does buying insurance is cost-efficient in the long run? In which case Space Company can decide to buy insurance and why?

④

I - company decided not to buy insurance

X	100	-200
$P(X=x)$	0.9	0.1
$X-\mathbb{E}X$	30	-270
$(X-\mathbb{E}X)^2$	900	72900

$\mathbb{E}X = 100 \cdot 0.9 - 200 \cdot 0.1 = 90 - 20 = 70$ million (profit)

$\mathbb{E}(X-\mathbb{E}X)^2 = 0.9 \cdot 900 + 0.1 \cdot 72900 = 810 + 7290 = 8100$

II - company decided to buy insurance:

y	$100-30=70$	$-200-30+200=-30$
$P(Y=y)$	0.9	0.1
$X-\mathbb{E}X$	10	-90
$(X-\mathbb{E}X)^2$	100	8100

$\mathbb{E}X = 70 \cdot 0.9 - 30 \cdot 0.1 = 63 - 3 = 60$ million (profit)

$\mathbb{E}(X-\mathbb{E}X) = 0.9 \cdot 100 + 0.1 \cdot 8100 = 90 + 810 = 900$

1. Buying insurance reduce expected value and variance.
2. Buying insurance is not profitable in long term because company earn 70 million per launch in average which more then with insurance (60 million)
3. The space company can decide to buy insurance if its cost is less then potential increase in expected profit in average.