Task 4

$$A = \begin{pmatrix} \alpha & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & \alpha & 2 \end{pmatrix}$$

1) First we need to get A into ~~row~~ row echelon form.

$$\begin{pmatrix} 3 & 3 & 6 \\ 5 & 5 & -2 \\ \alpha & 1 & 2 \\ 1 & \alpha & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 5 & 5 & -2 \\ \alpha & 1 & 2 \\ 1 & \alpha & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 12 \\ \alpha-1 & 0 & 0 \\ 0 & \alpha-1 & 0 \end{pmatrix}.$$

If $\alpha = 1$, then $\text{rank } A = 2$.

Otherwise we can continue Gaussian elimination (divide 3^{rd} and 4^{th} rows by $\alpha-1$)

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{\text{rank}(A) = \begin{cases} 2, & \text{if } \alpha = 1 \\ 3, & \text{if } \alpha \neq 1. \end{cases}}$$

2) For $\alpha = 7$ we get $p = \text{rank}(A) = 3$.

So we need to find such $B_{[m \times p]}$ and $C_{[p \times n]}$ that $A_{[4\times 3]} = B \cdot C$.

B has to be 4 by 3 matrix, while C has to be 3 by 3.

There is a trivial solution: $B = A$, $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In general if $A = \begin{pmatrix} 7 & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & 7 & 2 \end{pmatrix}$ we can take $B = \frac{1}{k} \cdot A$ and $C = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, $k \in \mathbb{R}, k \neq 0$.

For example $\begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.5 & 0.5 & -0.2 \\ 0.3 & 0.3 & 0.6 \\ 0.1 & 0.7 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = A$