Task 2

$$\begin{cases} x + 2y + 3z = 8 \\ x + 3z = -2 \\ 2x + 4z = 0 \\ x - y + 2z = 16 \end{cases} \iff A \cdot v = b, \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix}, v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 8 \\ -2 \\ 0 \\ 16 \end{pmatrix}.$$

- By Rouche-Capelli theorem system of linear equations is consistent only if $\operatorname{rank}(A) = \operatorname{rank}(A|b)$.

Let's find rank of matrix A first:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\operatorname{rank}(A) = 3.}$$

Now let's find rank of $(A|b)$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 1 & 0 & 3 & -2 \\ 2 & 0 & 4 & 0 \\ 1 & -1 & 2 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 2 & 0 & 10 \\ 0 & 0 & 2 & -4 \\ 1 & -1 & 2 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & 20 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 21 \end{array}\right) \Rightarrow \underline{\operatorname{rank}(A|b) = 4.}$$

$$\boxed{\text{As } \operatorname{rank}(A) \neq \operatorname{rank}(A|b) \text{ the system is inconsistent.}}$$

- Next let's find a least squares solution of the system.

Let $\hat{x}$ be such a solution. Then

$$\hat{x} = (A^T A)^{-1} \cdot A^T \cdot b.$$

$$A^T \cdot A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 3 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 16 \\ 1 & 5 & 4 \\ 16 & 4 & 38 \end{pmatrix}.$$ We can find inverse of $(A^T \cdot A)$ via

Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 7 & 1 & 16 & 1 & 0 & 0 \\ 1 & 5 & 4 & 0 & 1 & 0 \\ 16 & 4 & 38 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 5 & 4 & 0 & 1 & 0 \\ 7 & 1 & 16 & 1 & 0 & 0 \\ 16 & 4 & 38 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & 34 & 12 & -1 & 7 & 0 \\ 0 & 76 & 26 & 0 & 16 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & 1 & 6/17 & -1/34 & \frac{7}{34} & 0 \\ 0 & 76 & 26 & 0 & 16 & -1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & 5 & 30/17 & \frac{-5}{34} & \frac{35}{34} & 0 \\ 0 & 76 & 26 & 0 & 16 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{38}{17} & \frac{5}{34} & -\frac{1}{34} & 0 \\ 0 & 1 & 6/17 & -\frac{1}{34} & \frac{7}{34} & 0 \\ 0 & 76 & 26 & 0 & 16 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 38/17 & 5/34 & -1/34 & 0 \\ 0 & 76 & 456/17 & -76/34 & 532/34 & 0 \\ 0 & 76 & 26 & 0 & 16 & -1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 38/17 & 5/34 & -1/34 & 0 \\ 0 & 1 & 6/17 & -1/34 & 7/34 & 0 \\ 0 & 0 & 14/17 & -76/34 & -12/34 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 38/17 & 5/34 & -1/34 & 0 \\ 0 & 1 & 6/17 & -1/34 & 7/34 & 0 \\ 0 & 0 & 1 & -76/28 & -12/28 & 17/14 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 38/17 & 5/34 & -1/34 & 0 \\ 0 & 1 & 6/17 & -1/34 & 7/34 & 0 \\ 0 & 0 & 1 & -19/7 & -3/7 & 17/14 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 87/14 & 13/14 & -38/14 \\ 0 & 1 & 0 & 13/14 & 5/14 & -6/14 \\ 0 & 0 & 1 & -\frac{19}{7} & -3/7 & 17/14 \end{array}\right) \Rightarrow (A^T \cdot A)^{-1} = \begin{pmatrix} 87/14 & 13/14 & -38/14 \\ 13/14 & 5/14 & -6/14 \\ -38/14 & -6/14 & 17/14 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 87 & 13 & -38 \\ 13 & 5 & -6 \\ -38 & -6 & 17 \end{pmatrix}$$

Now we can calculate $\hat{x}$:

$$\hat{x} = (A^TA)^{-1}A^T b = \frac{1}{14}\begin{pmatrix} 87 & 13 & -38 \\ 13 & 5 & -6 \\ -38 & -6 & 17 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 4 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -2 \\ 0 \\ 16 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 87 & 13 & -38 \\ 13 & 5 & -6 \\ -38 & -6 & 17 \end{pmatrix} \cdot \begin{pmatrix} 22 \\ 0 \\ 50 \end{pmatrix} =$$

$$= \frac{1}{7}\begin{pmatrix} 87 & 13 & -38 \\ 13 & 5 & -6 \\ -38 & -6 & 17 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 0 \\ 25 \end{pmatrix} = \frac{1}{7} \cdot \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Least squares solution is } \hat{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.}$$