Task 3

I can buy one lottery ticket out of two available. In the first lottery I can win $\$100$ with probability 0.1, and the price of ticket is $\$10$. In the second lottery, I can win $\$50$ with probability 0.1 and $\$500$ with probability 0.01. The price of ticket is $\$20$. To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise. Let X be random variable that denotes my net payout (taking into account price of a ticket). Find probability mass function of X (hint: use law of total probability). Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

Let L_1 and L_2 be random variables that denote net payouts of the 1st and 2nd lotteries correspondingly.

L_1	90	0
P	0.1	0.9

$$EL_1 = 90 \cdot 0.1 + 0 \cdot 0.9 = 9$$

L_2	480	30	-20
P	0.01	0.1	0.89

$$EL_2 = 480 \cdot 0.01 + 30 \cdot 0.1 - 20 \cdot 0.89 = -10$$

Probability mass function of X:

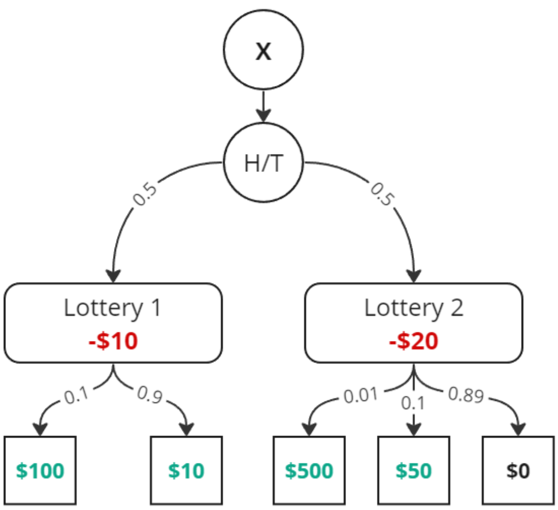
X	480	90	30	0	-20
P	0.005	0.05	0.05	0.45	0.445

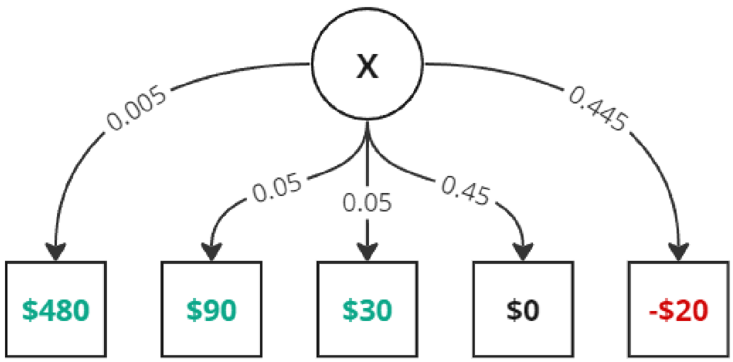
$$EX = 480 \cdot 0.005 + 90 \cdot 0.05 + 30 \cdot 0.05 + 0 \cdot 0.45 - 20 \cdot 0.445 = -0.5$$

$$\frac{EL_1 + EL_2}{2} = \frac{9 - 10}{2} = -0.5$$

Hence, $EX = \frac{EL_1 + EL_2}{2}$.

This can be explained by law of total probability and illustrated by following probability trees:





Now let's prove, that $EX = \frac{1}{2}(EL_1 + EL_2)$ for any probabilities and payouts in the lotteries.

Let L_1 take values $a_1, a_2, \dots, a_k$ with corresponding probabilities $p_1, p_2, \dots, p_k$ and L_2 take values $b_1, b_2, \dots, b_n$ with corresponding probabilities $q_1, q_2, \dots, q_n$.

Then their expected values are:
$EL_1 = a_1 \cdot p_1 + a_2 \cdot p_2 + \dots + a_k \cdot p_k$,
$EL_2 = b_1 \cdot q_1 + b_2 \cdot q_2 + \dots + b_n \cdot q_n$.

By the law of total probability variable X can have values $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_n$ with corresponding probabilities $\frac{p_1}{2}, \frac{p_2}{2}, \dots, \frac{p_k}{2}, \frac{q_1}{2}, \frac{q_2}{2}, \dots, \frac{q_n}{2}$.

Then we can calculate it's expected value:
$EX = a_1 \cdot \frac{p_1}{2} + a_2 \cdot \frac{p_2}{2} + \dots + a_k \cdot \frac{p_k}{2} + b_1 \cdot \frac{q_1}{2} + b_2 \cdot \frac{q_2}{2} + \dots + b_n \cdot \frac{q_n}{2}$.

We can group up some of the terms:

$$EX = \frac{1}{2} \cdot (a_1 \cdot p_1 + a_2 \cdot p_2 + \dots + a_k \cdot p_k) + \frac{1}{2}(b_1 \cdot q_1 + b_2 \cdot q_2 + \dots + b_n \cdot q_n) =$$

$= \frac{1}{2} EL_1 + \frac{1}{2} EL_2 = \frac{1}{2}(EL_1 + EL_2)$, q.e.d.