Task 1

$$\begin{cases} 5x_1 + x_2 - 2x_3 + 6x_4 = 0 \\ x_1 + 3x_2 - 2x_3 + 4x_4 = 0 \\ 3x_1 + 2x_2 - 2x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 - 4x_3 + 9x_4 = 0 \end{cases} \iff A \cdot x = b, \text{ where } A = \begin{pmatrix} 5 & 1 & -2 & 6 \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix},\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can get reduced echelon form of A via Gaussian elimination:

$$\begin{pmatrix} 5 & 1 & -2 & 6 \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix} \to \begin{pmatrix} 5 & 1 & -2 & 6 \\ 0 & 14 & -8 & 14 \\ 0 & 7 & -4 & 7 \\ 0 & 7 & -4 & 7 \end{pmatrix} \to \begin{pmatrix} 5 & 1 & -2 & 6 \\ 0 & 7 & -4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 35 & 0 & -10 & 35 \\ 0 & 7 & -4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 7 & 0 & -2 & 7 \\ 0 & 7 & -4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & -2/7 & 1 \\ 0 & 1 & -4/7 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Corresponding system of equations is $\begin{cases} x_1 - 2/7\, x_3 + x_4 = 0 \\ x_2 - 4/7\, x_3 + x_4 = 0 \end{cases}$

and it can be rewritten as $\begin{cases} x_1 = \frac{2}{7} x_3 - x_4 \\ x_2 = \frac{4}{7} x_3 - x_4 \end{cases}$ or in vector form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} x_3 - x_4 \\ 4/7\, x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 4/7 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} x_4$$

Hence solution space of the initial SLAE consists of all linear combinations of vectors $v_1 = \begin{pmatrix} 2/7 \\ 4/7 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. In other words v_1 and v_2 form a basis in the space of solutions. Therefore the solution space's dimension is 2.

- Let's check if rows of matrix $A = \begin{pmatrix} 4 & 8 & 14 & 0 \\ 2 & 4 & 7 & 0 \end{pmatrix}$ form FSS for the initial system. Obviously vectors $a_1 = (4\ 8\ 14\ 0)$ and $a_2 = (2\ 4\ 7\ 0)$ are linearly dependent ($a_1 = 2a_2$) so they can't form a basis of a 2-dimensional solution space, so they <u>don't form FSS</u> of the initial system.

- From matrix B we get vectors $b_1 = (-1\ -1\ 0\ 1)$ and $b_2 = (4\ 8\ 14\ 0)$.
  We can notice that $v_2 = b_1$ and $v_1 = \frac{1}{14} b_2$.
  As $\{v_1, v_2\}$ forms basis of solution space and there is a one-to-one correspondence between $\{v_1, v_2\}$ and $\{b_1, b_2\}$, then <u>$\{b_1, b_2\}$</u> is also <u>an FSS</u> of the initial SLAE.
  Each vector x from solution space can be expressed as a linear combination of b_1, b_2

$$x = \alpha \cdot v_1 + \beta \cdot v_2 = \frac{1}{14} \alpha \cdot b_2 + \beta \cdot b_1$$