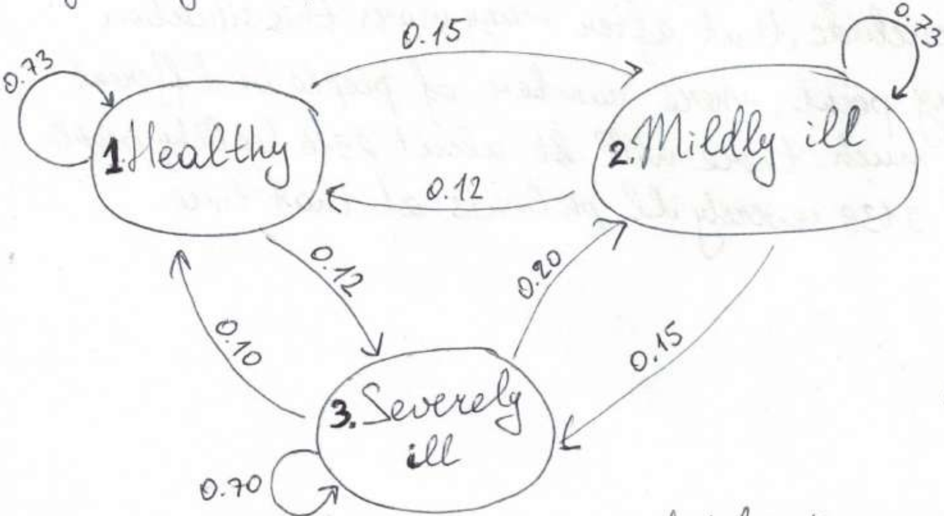
Task 4

Relations between groups of healthy, mildly ill and severely ill people may be represented in form of a directed weighted graph, where edge weights represent probabilities of moving from one group to another.



Amount of people in different groups may be represented by vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, where x_1 is the number of healthy people, x_2 – mildly ill patients, x_3 – severely ill patients.

Every day elements of x are updated with respect to the probabilities.

$x_0 = \begin{pmatrix} 9400 \\ 500 \\ 100 \end{pmatrix}$ - initial distribution of population between the groups.

$x_{new} = P \cdot x_{prev}$, where P is matrix of probabilities.

$P = \begin{pmatrix} 0.73 & 0.12 & 0.1 \\ 0.15 & 0.73 & 0.2 \\ 0.12 & 0.15 & 0.7 \end{pmatrix}$. Here P_{ij} represents the probability of a patient moving from group j to group i.

In this task we want to find out what will happen to the distribution of people between these 3 groups after a long period of time. Is there a stable distribution that won't change once reached?

To answer this quastion we need to check whether there exists such x, that $X = PX$. In other words we have to find out if matrix P has eigenvalue 1 and if it does, then we have to find it's eigenvector.

Coincidentally, matrix P is column-stochastic, which means 1 is it's eigenvalue.

Next we can find the corresponding eigenvector for $\lambda = 1$ by solving SLAE $(P - I) \cdot x = 0$.

That vector is $v = \begin{pmatrix} 170 \\ 230 \\ 183 \end{pmatrix}$. After dividing it by the sum of it's elements we can get vector v^*, which shows stationary distribution of the population between 3 groups: $v^* \approx \begin{pmatrix} 0.291595 \\ 0.394511 \\ 0.313894 \end{pmatrix}$, which in turn means that in long run ~29.2% of population will be healthy, ~39.4% will be mildly ill, and about 31.4%

of the population will be severely ill.

Taking into account the size of population we get vector x:

$$X = 10000 \cdot V^{*} = \begin{pmatrix} 2915.95 \\ 3945.11 \\ 3138.94 \end{pmatrix} \approx \begin{pmatrix} 2916 \\ 3945 \\ 3139 \end{pmatrix}$$

From this we can conclude, that after many years this situation will reach a stationary point, where number of people in different groups won't change much: there will be about 2916 healthy people, 3945 mildly ill and 3139 severely ill patients at that time.

Input interpretation

row reduce $\begin{pmatrix} -\frac{27}{100} & \frac{12}{100} & \frac{10}{100} \\ \frac{15}{100} & -\frac{27}{100} & \frac{20}{100} \\ \frac{12}{100} & \frac{15}{100} & -\frac{30}{100} \end{pmatrix}$

Result

$$\begin{pmatrix} 1 & 0 & -\frac{170}{183} \\ 0 & 1 & -\frac{230}{183} \\ 0 & 0 & 0 \end{pmatrix}$$

Here is the only computation I've done to find eigenvector v for matrix P (via Wolfram Alpha).
Here I solved aforementioned SLAE and found out that v = (170/183 x3, 230/183 x3, x3), which is just v = (170, 230, 183) for x3 = 183

There is an alternative solution via diagonalization on the next page, I've done it in two different ways, as I wasn't sure which way we were supposed to do this task

Alternatively we could look at this situation from another point of view.

$X_{new} = P \cdot X_{old} \Rightarrow X_{i+1} = P \cdot X_i$, so $X_1 = P \cdot X_0$, $X_2 = P X_1 = P^2 X_0$, and so on.

Basically $X_n = P^n \cdot X_0$

So if we want to see what happens after many years, we can try to find $\lim_{n \to \infty} X_n = \lim_{n \to \infty} P^n \cdot X_0$.

To calculate P^n we can use diagonalization.

$P = T A T^{-1}$, where A is matrix of eigenvalues of P in form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, and T is matrix of eigenvectors of P in form $\left(V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3} \right)$. Then we can find P^n as $T \cdot A^n \cdot T^{-1}$

$$\lim_{n \to \infty} P^n = \lim_{n \to \infty} T A^n T^{-1} = T \cdot \lim_{n \to \infty} A^n \cdot T^{-1}.$$

As $A \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.54 & 0 \\ 0 & 0 & 0.62 \end{pmatrix}$ we can conclude that $A^n \to \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 0.54^n & 0 \\ 0 & 0 & 0.62^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as $n \to \infty$,

$$\text{So } \lim_{n \to \infty} P^n = T \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot T^{-1} = \begin{pmatrix} 0.291596 & 0.291596 & 0.291596 \\ 0.394511 & 0.394511 & 0.394511 \\ 0.313894 & 0.313894 & 0.313894 \end{pmatrix} = P^*$$

$$\lim_{n \to \infty} X_n = \lim_{n \to \infty} P^n \cdot X_0 = P^* \cdot \begin{pmatrix} 9400 \\ 500 \\ 100 \end{pmatrix} = \begin{pmatrix} 2915.95 \\ 3945.11 \\ 3138.94 \end{pmatrix} \approx \begin{pmatrix} 2916 \\ 3945 \\ 3139 \end{pmatrix}.$$

WolframAlpha

diagonalize {{0.73, 0.12, 0.1}, {0.15, 0.73, 0.2}, {0.12, 0.15, 0.7}}

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

Input interpretation

$$\text{diagonalize} \quad \begin{pmatrix} 0.73 & 0.12 & 0.1 \\ 0.15 & 0.73 & 0.2 \\ 0.12 & 0.15 & 0.7 \end{pmatrix}$$

Result

$$\begin{pmatrix} 0.73 & 0.12 & 0.1 \\ 0.15 & 0.73 & 0.2 \\ 0.12 & 0.15 & 0.7 \end{pmatrix} = S.J.S^{-1}$$

where

$$S = \begin{pmatrix} 0.928962 & 0.290994 & -2.29099 \\ 1.25683 & -1.29099 & 1.29099 \\ 1 & 1 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.54127 & 0 \\ 0 & 0 & 0.61873 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 0.313894 & 0.313894 & 0.313894 \\ -0.00415332 & -0.391452 & 0.495847 \\ -0.30974 & 0.077558 & 0.19026 \end{pmatrix}$$

Main thing I had to compute is above.

- S is matrix of eigenvalues of matrix P from the task (in my notes it's matrix T),
- J is matrix of eigenvalues of matrix P (in my notes it's matrix A),
- S^{-1} is inverse matrix of S (in my notes it's matrix T^{-1})

After that I've substituted A by A^n and multiplied all these matrices back by WolframAlpha (result is written by hand in the main paper)