Task 3

$A = \begin{pmatrix}1\\1\\1\end{pmatrix} \in R^3$;

$L \subset R^3$; $b_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $b_2 = \begin{pmatrix}3\\5\\x\end{pmatrix}$; $b_1, b_2 \in L$; $B = \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix}$.

$\{b_1, b_2\}$ - basis of L.

Let $P = pr_L A = B(B^T \cdot B)^{-1} \cdot B^T \cdot A$ – orthogonal projection of A onto L.

1) $B^T \cdot B = \begin{pmatrix}1 & 0 & -1\\3 & 5 & x\end{pmatrix} \cdot \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix} = \begin{pmatrix}2 & 3-x\\3-x & x^2+34\end{pmatrix}$

2) $\Delta(B^T \cdot B) = 2x^2+68-(3-x)^2 = 2x^2+68-x^2+6x-9 = x^2+6x+59$

$$(B^T \cdot B)^{-1} = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix}x^2+34 & x-3\\x-3 & 2\end{pmatrix}$$

3) $$P = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix}\begin{pmatrix}x^2+34 & x-3\\x-3 & 2\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & -1\\3 & 5 & x\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} =$$

$$= \frac{1}{x^2+6x+59} \cdot \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix}\begin{pmatrix}x^2+34 & x-3\\x-3 & 2\end{pmatrix}\begin{pmatrix}0\\x+8\end{pmatrix} = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix} \cdot \begin{pmatrix}(x+8)(x-3)\\2\cdot(x+8)\end{pmatrix} =$$

$$= \frac{x+8}{x^2+6x+59} \cdot \begin{pmatrix}1 & 3\\0 & 5\\-1 & x\end{pmatrix}\begin{pmatrix}x-3\\2\end{pmatrix} = \frac{x+8}{x^2+6x+59} \cdot \begin{pmatrix}x+3\\10\\x+3\end{pmatrix}$$

Let $d = ort_L A = A - pr_L A = A - P$. Then distance between A and L is $|d|$.

$$d = A - P = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{x+8}{x^2+6x+59} \cdot \begin{pmatrix}x+3\\10\\x+3\end{pmatrix} = \begin{pmatrix}1 - \frac{(x+3)(x+8)}{x^2+6x+59}\\ 1 - \frac{10x+80}{x^2+6x+59}\\ 1 - \frac{(x+3)(x+8)}{x^2+6x+59}\end{pmatrix} = \begin{pmatrix}\frac{x^2+6x+59-(x^2+11x+24)}{x^2+6x+59}\\ \frac{x^2+6x+59-10x-80}{x^2+6x+59}\\ \frac{x^2+6x+59-(x^2+11x+24)}{x^2+6x+59}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{-5x+35}{x^2+6x+59}\\ \frac{x^2-4x-21}{x^2+6x+59}\\ \frac{-5x+35}{x^2+6x+59}\end{pmatrix} = \begin{pmatrix}\frac{-5(x-7)}{x^2+6x+59}\\ \frac{(x-7)(x+3)}{x^2+6x+59}\\ \frac{-5(x-7)}{x^2+6x+59}\end{pmatrix} = \frac{x-7}{x^2+6x+59}\begin{pmatrix}-5\\x+3\\-5\end{pmatrix}$$

Next we can find an expression for squared distance between A and L and find its maximum.

$$|d|^2 = d^2 = \frac{(x-7)^2}{(x^2+6x+59)^2} \cdot \begin{pmatrix}-5\\x+3\\-5\end{pmatrix}^2 = \frac{(x-7)^2 \cdot (25+(x+3)^2+25)}{(x^2+6x+59)^2} = \frac{(x-7)^2(x^2+6x+59)}{(x^2+6x+59)^2} =$$

$$= \frac{(x-7)^2}{x^2+6x+59} = f(x)$$

$$f'(x) = \frac{2(x-7)(x^2+6x+59)-(x-7)^2(2x+6)}{(x^2+6x+59)^2} = \frac{2(x-7)(x^2+6x+59-(x-7)(x+3))}{(x^2+6x+59)^2} =$$

$$= \frac{2(x-7)(x^2+6x+59-(x^2-4x-21))}{(x^2+6x+59)^2} = \frac{2(x-7)(10x+80)}{(x^2+6x+59)^2} = \frac{20(x-7)(x+8)}{(x^2+6x+59)^2}$$

$f'(x) = 0$ at $x = 7$ and $x = -8$.

f'	$+$		$-$		$+$	x
		-8		7		
		max		min.		

$f(x)$ has a maximum at $x = -8$, hence

$|d|^2$ reaches its maximal value at $x = -8$.

$$|d|^2 = \frac{(-8-7)^2}{(-8)^2 + 6\cdot(-8) + 59} = \frac{15^2}{75} = 3$$