Task 2

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

$$B = A^T \cdot A = \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix}$$

$\chi_\lambda(B)$ – characteristic polynomial of matrix B.

$$\chi_\lambda(B) = \begin{vmatrix} 28-\lambda & 14 & 28 \\ 14 & 7-\lambda & 14 \\ 28 & 14 & 28-\lambda \end{vmatrix} = (28-\lambda)\cdot\begin{vmatrix} 7-\lambda & 14 \\ 14 & 28-\lambda \end{vmatrix} - 14\cdot\begin{vmatrix} 14 & 14 \\ 28 & 28-\lambda \end{vmatrix} + 28\cdot\begin{vmatrix} 14 & 7-\lambda \\ 28 & 14 \end{vmatrix} =$$

$$= (28-\lambda)\left((7-\lambda)(28-\lambda)-14^2\right) - 14\cdot\left(14\cdot(28-\lambda)-14\cdot 28\right) + 28\cdot\left(14^2-28(7-\lambda)\right) =$$

$$= (28-\lambda)\left(\lambda^2-35\lambda+\cancel{7\cdot 28}-\cancel{14^2}\right) - 14\left(\cancel{14\cdot 28}-14\lambda-\cancel{14\cdot 28}\right) + 28\cdot\left(\cancel{14^2}-\cancel{28\cdot 7}+28\lambda\right) =$$

$$= (28-\lambda)(\lambda^2-35\lambda) + 14^2\lambda + 28^2\lambda = \lambda\left((28-\lambda)(\lambda-35)+14^2+4\cdot 14^2\right) =$$

$$= \lambda\left(-\lambda^2+63\lambda-28\cdot 35+5\cdot 14^2\right) = \lambda\left(-\lambda^2+63\lambda-5\cdot 14^2+5\cdot 14^2\right) = \lambda\left(-\lambda^2+63\lambda\right) =$$

$$= -\lambda^2(\lambda-63)$$

Eigenvalues of matrix B are $\lambda_1 = 63$, $\lambda_2 = \lambda_3 = 0$, hence singular values of A are $\sigma_1 = \sqrt{63}$; $\sigma_2 = \sigma_3 = 0$.

At this step we can acquire matrix Σ:

$$\Sigma = \begin{pmatrix} \sqrt{63} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Next we can find eigenvectors of B.

$$B v_i = \lambda_i v_i \Rightarrow (B - \lambda_i I)v_i = 0$$

- For $\lambda_1 = 63$: $(B - 63\cdot I)\cdot v_1 = 0 \Rightarrow \left[\begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} - \begin{pmatrix} 63 & 0 & 0 \\ 0 & 63 & 0 \\ 0 & 0 & 63 \end{pmatrix}\right]\cdot v_1 = 0$

$\begin{pmatrix} -35 & 14 & 28 \\ 14 & -54 & 14 \\ 28 & 14 & -35 \end{pmatrix} \cdot v_1 = 0.$ We can solve this SLAE via Gaussian elimination.

$$\left(\begin{array}{ccc|c} -35 & 14 & 28 & 0 \\ 14 & -54 & 14 & 0 \\ 28 & 14 & -35 & 0 \end{array}\right) \xrightarrow{:7} \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 10 & -40 & 10 & 0 \\ 20 & 10 & -25 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 5 & -2 & -4 & 0 \\ 0 & -36 & 18 & 0 \\ 0 & 18 & -9 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 5 & -2 & -4 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 5 & 0 & -5 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = x_3 \\ 2x_2 = x_3 \\ x_3 \in \mathbb{R} \end{cases} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

After normalizing it we get $v_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$.

$\lambda_2=\lambda_3=0: \quad (B-0I)\cdot v=0 \Rightarrow B\cdot v=0$

$$\left(\begin{array}{ccc|c} 28 & 14 & 28 & 0 \\ 14 & 7 & 14 & 0 \\ 28 & 14 & 28 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} 2x_1=-x_2-2x_3 \\ x_2,x_3\in\mathbb{R} \end{cases} \Rightarrow v_2=\begin{pmatrix}-1\\2\\0\end{pmatrix},\ v_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

We can normalize v_3 and get $v_3=\begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$.

Vectors v_2 and v_3 are orthogonal to v_1, as they correspond to different eigenvalues of matrix B. But we still need to orthonormalize vector v_2 relative to v_3. For that we can use Gram-Schmidt orthogonalization algorithm: $v_{2(new)}=v_2-\frac{v_2\cdot v_3}{v_3\cdot v_3}\cdot v_3$

$v_2=\begin{pmatrix}-1\\2\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}-1/2\\2\\-1/2\end{pmatrix}$. After normalizing it we get $v_2=\begin{pmatrix}-1/\sqrt{18}\\4/\sqrt{18}\\-1/\sqrt{18}\end{pmatrix}$

Finally we get matrix $V=(v_1,v_2,v_3)=\begin{pmatrix}2/3 & -1/\sqrt{2} & -1/\sqrt{18}\\ 1/3 & 0 & 4/\sqrt{18}\\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18}\end{pmatrix}$

- Next we have to find matrix U (4x4 orthogonal matrix)

We can find it's first column by using singular values. Sadly we only have one such non-zero value.

$$u_1=\frac{1}{\sigma_1}\cdot Av_1=\frac{1}{\sqrt{63}}\cdot\begin{pmatrix}2&1&2\\-2&-1&-2\\4&2&4\\2&1&2\end{pmatrix}\cdot\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}=\frac{1}{3\sqrt{7}}\begin{pmatrix}3\\-3\\6\\3\end{pmatrix}=\begin{pmatrix}1/\sqrt{7}\\-1/\sqrt{7}\\2/\sqrt{7}\\1/\sqrt{7}\end{pmatrix}$$

As matrix U should be orthogonal, we have to pick orthonormal vectors u_2,u_3,u_4.

It's easy to find $u_2=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$ and $u_3=\begin{pmatrix}0\\2\\1\\0\end{pmatrix}$, which are orthogonal to each other and to vector u_1.

Next we can use properties of vectors' inner product to find the last vector u_4.

Let $u_4=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$. As v_4 should be orthogonal, we can get a system of equations:

$$\begin{cases} u_1\cdot u_4=0 \\ u_2\cdot u_4=0 \\ u_3\cdot u_4=0 \end{cases} \Leftrightarrow \begin{cases} 1/\sqrt{7}\cdot x_1-1/\sqrt{7}\cdot x_2+2/\sqrt{7}\cdot x_3+1/\sqrt{7}\cdot x_4=0 \\ -1\cdot x_1+0+0+1\cdot x_4=0 \\ 0+2x_2+1\cdot x_3+0=0 \end{cases} \Leftrightarrow \begin{pmatrix}1/\sqrt{7}&-1/\sqrt{7}&2/\sqrt{7}&1/\sqrt{7}\\-1&0&0&1\\0&2&1&0\end{pmatrix}\cdot u_4=0$$

We can solve this SLAE via Gaussian elimination:

$$\begin{pmatrix}1&-1&2&1\\-1&0&0&1\\0&2&1&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-1\\0&1&-2&-2\\0&2&1&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-1\\0&1&-2&-2\\0&0&5&4\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-1\\0&5&-10&-10\\0&0&10&8\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-1\\0&5&0&-2\\0&0&5&4\end{pmatrix}\Rightarrow\begin{cases}x_1=x_4\\5x_2=2x_4\\5x_3=x_4\\x_4\in\mathbb{R}\end{cases}$$

$u_4=\begin{pmatrix}5\\2\\-4\\5\end{pmatrix}$.

After normalizing vectors $u_1,\dots,u_4$ we obtaim orthonormal matrix U.

$$U=(u_1,u_2,u_3,u_4)=\begin{pmatrix}1/\sqrt{7}&-1/\sqrt{2}&0&5/\sqrt{70}\\-1/\sqrt{7}&0&2/\sqrt{5}&2/\sqrt{70}\\2/\sqrt{7}&0&1/\sqrt{5}&-4/\sqrt{70}\\1/\sqrt{7}&1/\sqrt{2}&0&5/\sqrt{70}\end{pmatrix}$$

$$A=U\cdot\Sigma\cdot V^T=\begin{pmatrix}1/\sqrt{7}&-1/\sqrt{2}&0&5/\sqrt{70}\\-1/\sqrt{7}&0&2/\sqrt{5}&2/\sqrt{70}\\2/\sqrt{7}&0&1/\sqrt{5}&-4/\sqrt{70}\\1/\sqrt{7}&1/\sqrt{2}&0&5/\sqrt{70}\end{pmatrix}\cdot\begin{pmatrix}3\sqrt{7}&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}2/3&1/3&2/3\\-1/\sqrt{2}&0&1/\sqrt{2}\\-1/\sqrt{18}&4/\sqrt{18}&-1/\sqrt{18}\end{pmatrix}$$