

Advanced Algorithms & Data Structures

Assignment 1

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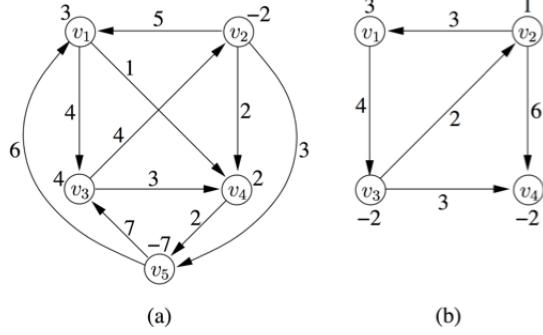
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Contents

1	<i>b</i> -flow	3
2	Rectilinear Planar Embedding	4
2.1	Exercise 2.1	4
2.2	Exercise 2.2	5
2.3	Exercise 2.3	5
2.4	Exercise 2.4	6
2.5	Exercise 2.5	6
3	Exercise 3	8
3.1	Exercise 3.1	8
3.2	Exercise 3.2	8
3.3	Exercise 3.3	8
3.4	Exercise 3.4	9

1 b -flow

Find for each of the two graphs in Figure 1, a b -flow for the graph or argue that the graph has no b -flow



For a directed graph $G = (V, E)$. For each vertex $v \in V$ let $\delta^+(v)$ be the set of outgoing edges from v and $\delta^-(v)$ be the set of incoming edges to v .

Given is that each a b -flow under the following constraints

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v, \forall v \in V \quad (1)$$

$$0 \leq x_e \leq u_e, \forall e \in E \quad (2)$$

Given these constraints we find that the following b -flows exist for graph (a)

$$\begin{aligned} x_{(v_2, v_4)} &= 2 \\ x_{(v_5, v_3)} &= 4 \\ x_{(v_5, v_1)} &= 3 \end{aligned}$$

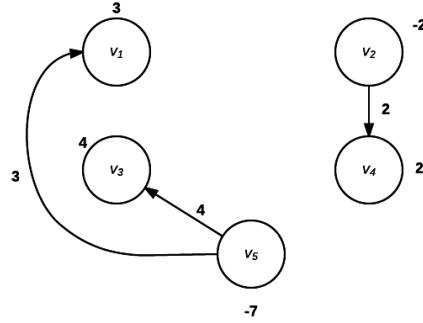


Figure 1: Graph 1A b -flow

We find that there is no b -flow in graph (b) as there is no outgoing edge from v_4 and we do not allow for negative flows. Thus only some of the nodes can satisfy equations (1) and (2) resulting in no existing b -flow.

2 Rectilinear Planar Embedding

2.1 Exercise 2.1

Table 1 shows the values for x_{vf} that correspond to Figure 3 from the Assignment Document.

x_{vf}	a	b	c	d	e
v_1	0	1	1	0	0
v_2	0	0	1	1	0
v_3	1	0	1	1	1
v_4	0	0	0	-1	1
v_5	1	0	0	0	-1
v_6	1	1	0	1	1
v_7	0	0	0	0	0

Table 1: x_{vf} values for Figure 3 in Assignment Document

Table 2 shows the values for z_{fg} that correspond to Figure 3 from the Assignment Document. There are 13 break-points in this graph.

z_{fg}	a	b	c	d	e
a	-	0	0	0	0
b	2	-	1	1	0
c	1	1	-	0	0
d	0	1	0	-	1
e	4	0	0	0	-

Table 2: z_{fg} values for Figure 3 in Assignment Document

We are asked to draw a rectilinear layout of graph 2(a) from the assignment description

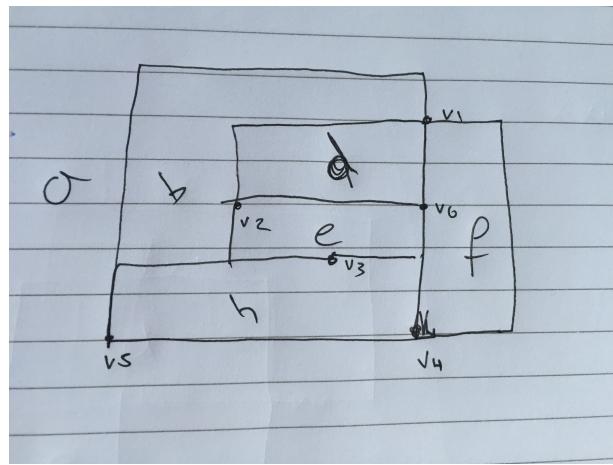


Figure 2: Rectilinear drawing of Graph 2(a) in assignment desc

2.2 Exercise 2.2

Let f_e be the external boundary cycle and F be the set of all boundary cycles. Given two boundary cycles x and y , inner turns (from x to y) are denoted by z_{xy} and outer turns (from y to x) are denoted by zyx .

$$\sum_v x_{vf_e} + \sum_{b \in F \setminus f_e} z_{f_e b} - z_{bf_e} = -4 \quad (3)$$

$$\forall b \in F \setminus f_e : \sum_v x_{vf} + \sum_{b \in F \setminus f} z_{fb} - z_{bf} = 4 \quad (4)$$

Boundary cycle a is the external boundary cycle so Equation 3 must hold.

$$\begin{aligned} & \sum_v x_{va} + \sum_{b \in B \setminus \{a\}} z_{ab} - z_{ba} \\ &= 3 + z_{ab} - z_{ba} + z_{ac} - z_{ca} + z_{ad} - z_{da} + z_{ae} - z_{ea} \\ &= 3 + 0 - 2 + 0 - 1 + 0 - 0 + 0 - 4 \\ &= 3 - 2 - 1 - 4 = -4 \end{aligned}$$

Boundary cycle e is an internal boundary cycle so Equation 4 must hold.

$$\begin{aligned} & \sum_v x_{vf} + \sum_{b \in B \setminus f} z_{fb} - z_{bf} \\ &= 2 + z_{ea} - z_{ae} + z_{eb} - z_{be} + z_{ec} - z_{ce} + z_{ed} - z_{de} \\ &= 2 + 4 - 0 + 0 - 0 + 0 - 0 + 0 - 2 \\ &= 2 + 4 - 2 = 4 \end{aligned}$$

2.3 Exercise 2.3

Since G is a graph that contains only straight and horizontal edges, it is impossible for G to have more than 4 edges: a horizontal edge going left; a horizontal edge going right; a vertical edge going up; a vertical edge going down. To show that Equation 1 from the Assignment Document holds we will show that it is true for each of the 3 cases presented, ie v has degree $\in [2, 3, 4]$.

Firstly, the sum of the angles for a vertex v must equal 360° . Secondly, as we only allow horizontal and vertical edges the only possible turn angles $\in [90, 180, 270]$ for an x_{vf} . We will use these fact to show that the conditions hold for each of the three cases presented.

Case 1: v has degree 2.

Then there are exactly 2 boundary cycles that pass through v , say h and g . Then there are only two possible combinations of the inner and outer turns from the point of view from each boundary cycle. Either both are 180° , and therefore $x_{vh} = 0$ and $x_{vg} = 0$. So $\sum_f x_{vf} = 0$. Or one boundary cycle has a turn of 90° (an inner turn, $x_{vf'} = 1$) and the other a turn of 270° (an outer turn, $x_{vf''} = -1$), then $\sum_f x_{vf} = 1 - 1 = 0$

Case 2: v has degree 3.

Then there are exactly 3 boundary cycles that pass through v , then since the angles around v must add to 360° and we can only have horizontal or vertical

edges then there must be one angle of 180° and two 90° turns, thus there can only be two inner turns and $\sum_f x_{vf} = 0 + 1 + 1 = 2$

Case 3: v has degree 4.

Then there are exactly 4 boundary cycles that pass through v , as the angles around v must add to 360° , we can only have four 90° turns, thus there can only be four inner turns and $\sum_f x_{vf} = 1 + 1 + 1 + 1 = 4$

2.4 Exercise 2.4

The objective function is given by

$$\min \sum_{\forall f \in G, f \neq g} z_{fg}$$

given that the aim is to minimise the number of break points. The constraints that the problem is subjected to are:

$$z_{fg}, z_{gf} \geq 0 \quad (5)$$

$$\sum_v x_{vfe} + \sum_{b \in F \setminus f_e} z_{f_e b} - z_{b f_e} = -4 \quad (6)$$

$$\forall b \in F \setminus f_e : \sum_v x_{vf} + \sum_{b \in F \setminus f} z_{fb} - z_{bf} = 4 \quad (7)$$

$$\sum_f x_{vf} = \begin{cases} 0 & \text{if } v \text{ has degree 2} \\ 2 & \text{if } v \text{ has degree 3} \\ 4 & \text{if } v \text{ has degree 4} \end{cases} \quad (8)$$

2.5 Exercise 2.5

Given a rectilinear graph $G = (V, E)$ and a MCFP $G' = (V', E')$. V' is the set of vertices which contains all vertices $v \in G$, and also a vertex for every face in G . E' is the set of edges which contains edges (f, g) for f, g faces in G , if f and g share a vertex, and edges (v, f) for $v \in V$ if v was a vertex in f . As we are dealing with turns and breakpoints, and these are relative to faces and vertices and faces and faces, there is no information stored on connections from vertices to vertices in G and therefore these connections are not present in G' . The transformation from a rectilinear graph to a MCFP is given by the following conditions:

- (v, f) where vertex v lies on the boundary of face f in G .
The capacity of these edges is lower bounded by 1 and upper bounded by 4. They have zero cost.
- (f, g) and (g, f) for every pair of adjacent faces, f and g in G .
The capacity of these edges is lower bounded by 0 and upper bounded by infinity. They have unit cost.

(Tomassia, 1993)

Given the above our graph solution to this question is based on the assumption that two faces f and g are adjacent if they share a vertex v . This assumption yields the following sets of connected vertices:

$$\begin{aligned}
a : & \quad v_1, v_4, v_5, b, f, h & (9) \\
b : & \quad v_1, v_2, v_5, a, c, d, e & (10) \\
c : & \quad v_2, v_3, v_5, b, e, h & (11) \\
d : & \quad v_1, v_2, v_6, b, e, f & (12) \\
e : & \quad v_2, v_3, v_6, c, d, g & (13) \\
f : & \quad v_1, v_4, v_6, a, d, g & (14) \\
g : & \quad v_3, v_4, v_6, e, f, h & (15) \\
h : & \quad v_3, v_4, v_5, a, c, g & (16)
\end{aligned}$$

The demands of vertices are calculated from the constraints as described before. All vertices v_i are sources and produce 4 units of flow,i.e demand of -4. Face nodes are sink nodes and have demand 4, except external face nodes which have demand -4. Edges $e_{f,g}$ have unit cost and infinite capacity. Edges $e_{f,v}$ have unit capacity and zero cost.

3 Exercise 3

3.1 Exercise 3.1

We start here with I_0 , where there is the constraint $l_e \leq x_e \leq u_e, \forall e \in E$
 $l_e \in \mathbb{R} \cup -\infty, u_e \in \mathbb{R} \cup +\infty$

If we take an edge e from u to v , then if $x_e < 0$ we can say that flow is going in the other direction, i.e. along (v, u) . As a flow is defined in a graph that does not have anti-parallel edges we cannot have flow going in both directions, and then we introduce a new vertex, say w . To ensure we are not altering the possible max-flow of the network, we say that b_w has value 0 and the cost from $(u, w) -> (w, v)$ will have the same cost as the edge (u, v) . Now one of 3 possibilities exists.

First, the flow with value u_e cancels out the flow from l_e and l_e becomes zero. Second, the flow with value l_e cancels out the flow from u_e and u_e becomes zero. Finally, the forward flow and backward flow cancel each other out and then both become zero.

Doing this we ensure that there is at least one of l_e or u_e bounded and finite.

3.2 Exercise 3.2

The solution to this exercise is analogous exercise 3.1, the difference being that instead of both $l_e \in \mathbb{R} \cup -\infty$ and $u_e \in \mathbb{R} \cup +\infty$ only l_e is finite. The solution can be achieved using the same strategy as in 3.1, namely by adding a node and two edges, as to create a cycle through a node with no cost, which is necessary for the same reason as above, being that there can be no anti-parallel edges.

3.3 Exercise 3.3

We start here with I_2 , where there is the constraint

$$l_e \leq x_e \leq u_e, \forall e \in E, l_e \in \mathbb{R}, \text{finite. } u_e \in \mathbb{R} \cup +\infty$$

Now we have two situations. Either $l_e < 0$ or $l_e > 0$.

If $l_e < 0$: Then we can modify the flow f in the original problem to f' in the new augmented problem with constraint:

$$0 \leq x_e + l_e \leq u_e + l_e$$

In the augmented problem $b'_u = b_u + l_e$ and $b'_v = b_v - l_e$

If $l_e > 0$: Then we can modify the flow f in the original problem to f' in the new augmented problem with constraint:

$$0 \leq x_e - l_e \leq u_e - l_e$$

In the augmented problem $b'_u = b_u - l_e$ and $b'_v = b_v + l_e$

These new augmented problems can then be solved using an algorithm to solve MCFP and will have the same flow as the original problem I_2

3.4 Exercise 3.4

We were given I_0 and instance of a GMCFP, with a directed graph $G = (V, E)$, in Exercises 3.1 and 3.2 we introduced at most one node and two edges to the modified problem in each of them. This is an increase of $2 * (O(2E) + O(V))$ edges and vertices to the original graph G .

References

Tomassia, R. (1993). Computer vision with python. Seen on 30th april 2015,
<http://cs.brown.edu/courses/cs252/misc/resources/lectures/pdf/notes13.pdf>.