

# Assignment 2

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**Abstract**—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.

Download all python codes from

<https://github.com/Zeehan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeehan-IITH/IITH-EE5609>

## 1 PROBLEM

Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (2)$$

## 2 CONSTRUCTION

When two lines are not intersecting the distance between them is non-zero. The equation of above mentioned lines in symmetric form is

$$L_1: x - 1 = 2 - y = z - 1 \quad (3)$$

$$L_2: \frac{x-2}{2} = y+1 = \frac{z+1}{2} \quad (4)$$

The above line equations have no point of intersection as for no value of  $\lambda_1, \lambda_2$  both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (5)$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (7)$$

The Augmented matrix will be

(8)

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (10)$$

The above matrix has a  $rank = 3$ . Hence the lines do not intersect

## 3 SOLUTION

Let  $\mathbf{A}$  be a point on line  $L_1$  and  $\mathbf{B}$  be point on the line  $L_2$ . Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines  $L_1, L_2$  and passing through  $\mathbf{A}$  and  $\mathbf{B}$ .

The vector passing through  $\mathbf{A}$  and  $\mathbf{B}$  will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x}_1 - \mathbf{x}_2 + (\mathbf{m}_1 \quad -\mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (11)$$

The vectors  $\mathbf{m}_1, \mathbf{m}_2$  are both perpendicular to the line  $\mathbf{AB}$ . So the dot product of  $\mathbf{m}_1, \mathbf{m}_2$  with the line  $\mathbf{AB}$  is zero.

The dot product of  $\mathbf{m}_1$  with the line  $\mathbf{AB}$  is

$$\mathbf{m}_1^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\mathbf{m}_1^T (\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{m}_1^T (\mathbf{m}_1 - \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \quad (12)$$

The dot product of  $\mathbf{m}_2$  with the line  $\mathbf{AB}$  is

$$\mathbf{m}_2^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\mathbf{m}_1^T (\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{m}_2^T (\mathbf{m}_1 - \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \quad (13)$$

Let the matrix  $\mathbf{M}$  be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} \quad (14)$$

Combining the equations (12) and (13) in matrix form, using equation (14), we get

$$\mathbf{M}\mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} + \mathbf{M}(\mathbf{x}_1 - \mathbf{x}_2) = 0 \quad (15)$$

simplifying it further

$$\mathbf{M}\mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}(\mathbf{x}_2 - \mathbf{x}_1) \quad (16)$$

To find the points on the lines which make up the shortest distance we need to find  $\lambda_1$  and  $\lambda_2$  using the above expression to get the augmented form

$$\begin{pmatrix} \mathbf{m}_1^T \mathbf{m}_1 & \mathbf{m}_1^T \mathbf{m}_2 \\ \mathbf{m}_2^T \mathbf{m}_1 & \mathbf{m}_2^T \mathbf{m}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{m}_1^T (\mathbf{x}_2 - \mathbf{x}_1) \\ \mathbf{m}_2^T (\mathbf{x}_2 - \mathbf{x}_1) \end{pmatrix} \quad (17)$$

$$\left( \begin{array}{cc|c} \mathbf{m}_1^T \mathbf{m}_1 & \mathbf{m}_1^T \mathbf{m}_2 & \mathbf{m}_1^T (\mathbf{x}_2 - \mathbf{x}_1) \\ \mathbf{m}_2^T \mathbf{m}_1 & \mathbf{m}_2^T \mathbf{m}_2 & \mathbf{m}_2^T (\mathbf{x}_2 - \mathbf{x}_1) \end{array} \right) \quad (18)$$

we know that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{m}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

so the augmented matrix will be

$$(19)$$

$$\left( \begin{array}{cc|c} 3 & 3 & 2 \\ 3 & 9 & -5 \end{array} \right) \quad (20)$$

Using row reduction  $\mathbf{R}_2 = \mathbf{R}_2 - \mathbf{R}_1$ , we get

$$\left( \begin{array}{cc|c} 3 & 3 & 2 \\ 0 & 6 & -7 \end{array} \right) \quad (21)$$

So the values of  $\lambda_1 = \frac{11}{6}$  and  $\lambda_2 = \frac{7}{6}$

Using the equation (1) and (2), we get the points as

$$\frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} \text{ and } \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \text{ on the line } L_1, L_2 \text{ respectively.}$$

The shortest distance between the lines is the

absolute value of projection of the vector  $\mathbf{AB}$  on to the unit vector  $\mathbf{n}$ .

$$\|(\mathbf{B} - \mathbf{A})\| = \left\| \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \right\| = \frac{3}{\sqrt{2}} \quad (22)$$

Therefore the shortest distance between the given lines is  $\frac{3}{\sqrt{2}}$

The unit vector perpendicular to lines

$$\text{Line}_1: \mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (23)$$

$$\text{Line}_2: \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (24)$$

can be found by

$$\frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} \quad (25)$$

$$\frac{\frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}}{\left\| \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \right\|}$$

So the unit vector perpendicular to both  $L_1$  and  $L_2$  is

$$\mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (26)$$

