Assignment 2

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Abstract—**This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

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1 PROBLEM

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad (1.0.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \qquad (1.0.2)$$

2 CONSTRUCTION

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1 : x - 1 = 2 - y = z - 1 \qquad (2.0.1)$$

$$L_2 : \frac{x - 2}{2} = y + 1 = \frac{z + 1}{2} \qquad (2.0.2)$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (2.0.1) and (2.0.2) are equal.

If the two line intersect then,equation (2.0.1)=(2.0.2) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \qquad (2.0.3)$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \qquad (2.0.4)$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \qquad (2.0.5)$$

The Augmented matrix will be

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \qquad (2.0.6)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xleftarrow{R_1 = R_1 - R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \qquad (2.0.7)$$

The above matrix has a *rank* = 3 .Hence the lines do not intersect.

3 SOLUTION

Let A be a point on line L_1 and B be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through A and B.

The vector passing through the points A and B will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \begin{pmatrix} \mathbf{m_1} & -\mathbf{m_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \qquad (3.0.1)$$

The vectors $\mathbf{m_1},\mathbf{m_2}$ are both perpendicular to the line AB.So the dot product of $\mathbf{m_1},\mathbf{m_2}$ with the line AB is zero.

The dot product of $\mathbf{m_1}$ with the line AB is

$$\mathbf{m_1}^T (\mathbf{A} - \mathbf{B}) = 0 \qquad (3.0.2)$$

$$\mathbf{m_1^T}(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_1^T}(\mathbf{m_1} - \mathbf{m_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \qquad (3.0.3)$$

The dot product of $\mathbf{m_2}$ with the line AB is

$$\mathbf{m_2}^T (\mathbf{A} - \mathbf{B}) = 0 \qquad (3.0.4)$$

$$\mathbf{m_1^T}(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_2^T}(\mathbf{m_1} - \mathbf{m_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \qquad (3.0.5)$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix} \qquad (3.0.6)$$

Combining the equations (3.0.3) and (3.0.5) in matrix form,using equation (3.0.6), we get

$$\mathbf{MM}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} + \mathbf{M}(\mathbf{x_1} - \mathbf{x_2}) = 0 \qquad (3.0.7)$$

simplifying it further

$$\mathbf{MM}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}(\mathbf{x_2} - \mathbf{x_1}) \qquad (3.0.8)$$

To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the above expression to get the augmented form

$$\begin{pmatrix} \mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2} \\ \mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{m_1}^T(\mathbf{x_2} - \mathbf{x_1}) \\ \mathbf{m_2}^T(\mathbf{x_2} - \mathbf{x_1}) \end{pmatrix}$$
(3.0.9)

$$\begin{pmatrix} \mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2} & \mathbf{m_1}^T(\mathbf{x_2} - \mathbf{x_1}) \\ \mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2} & \mathbf{m_2}^T(\mathbf{x_2} - \mathbf{x_1}) \end{pmatrix}$$
(3.0.10)

we know that

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} and \mathbf{m_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so the augmented matrix will be

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix}$$
(3.0.11)

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - R_1} \begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix}$$
(3.0.12)

$$\begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \xleftrightarrow{R_1 = 2R_1 - R_2} \begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix}$$
(3.0.13)

$$\begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \xleftrightarrow{R_1 = \frac{R_1}{6}, R_2 = \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & \frac{11}{6} \\ 0 & 1 & \frac{-7}{6} \end{pmatrix}$$
(3.0.14)

$$\lambda_1 = \frac{11}{6}, \lambda_2 = \frac{7}{6}$$
(3.0.15)

Using the equation (1.0.1) and (1.0.2), we get the points as $A = \frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix}$ and $B = \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}$ on the line L_1, L_2 respectively. The shortest distance between the lines is the distance between the points A and B.

$$\|(\mathbf{B} - \mathbf{A})\| = \left\| \frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \right\| = \frac{3}{\sqrt{2}}$$
(3.0.16)

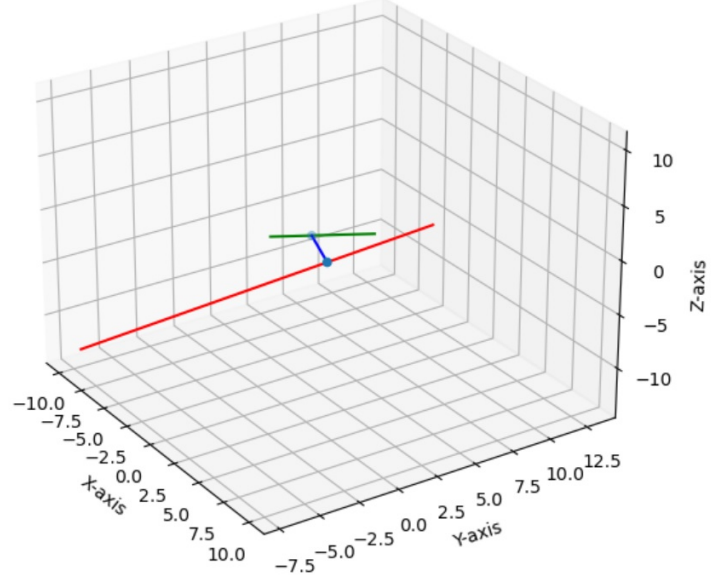Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$.

Fig. 1: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

The unit vector perpendicular to lines

$$Line_1: \mathbf{x} = \mathbf{x_1} + \lambda_1\mathbf{m_1}$$
(3.0.17)
$$Line_2: \mathbf{x} = \mathbf{x_2} + \lambda_1\mathbf{m_2}$$
(3.0.18)

can be found by

$$\mathbf{n} = \frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|}$$
(3.0.19)

$$\mathbf{n} = \frac{\frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}}{\frac{3}{\sqrt{2}}}$$
(3.0.20)

$$\mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
(3.0.21)

So the unit vector perpendicular to both L_1 and L_2 is

$$\mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$