Assignment 5

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document is about matrix representation of lines and the bisectors of angles between them.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Show that the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -11 & 31 \end{pmatrix} \mathbf{x} - 10 = 0 \quad (1.0.1)$$

represents two straight lines,and find the equations of the bisectors of the angles between them.

2 Construction

Any quadratic equation in terms of x, y of the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$,can be written as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

$$\text{where,} \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.0.3)$$

The equation (1.0.1) represents two intersecting straight lines when

$$\begin{vmatrix} \mathbf{V} & u \\ u^T & f \end{vmatrix} = 0 \quad (2.0.4)$$

$$\left|\mathbf{V}\right| < 0 \quad (2.0.5)$$

3 Explanation

From equation (1.0.1) we get

$$\mathbf{V} = \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (3.0.2)$$

$$f = -10 \quad (3.0.3)$$

calculating the equation (2.0.4),we get

$$\begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ \frac{-11}{2} & \frac{31}{2} & -10 \end{pmatrix} \xleftrightarrow{R_3 = R_3 + R_2 + R_1} \begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.4)$$

Therefore the determinant 0.And also the determinant of $\mathbf{V}$ is

$$\left|\mathbf{V}\right| = \begin{vmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{vmatrix} \quad (3.0.5)$$

$$= -90.25 \quad (3.0.6)$$

$$< 0 \quad (3.0.7)$$

Therefore the given equation represents the equation of two straight lines which intersect.

4 Point of intersection

The point of intersection of the straight lines is given by

$$c = -\mathbf{V}^{-1}\mathbf{u} \quad (4.0.1)$$

The inverse of $\mathbf{V}$ can be found by using rref of augmented matrix of the matrices $\mathbf{V}$ and $\mathbf{I}$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -15 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 = 12R_2 + R_1} \begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{361}{2} & 1 & 12 \end{pmatrix} \quad (4.0.2)$$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \xleftrightarrow{R_1 = R_1 + \frac{R_2}{2}} \begin{pmatrix} 6 & 0 & \frac{360}{361} & -\frac{12}{361} \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (4.0.3)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (4.0.4)$$

$$c = -\begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \begin{pmatrix} -\frac{11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (4.0.5)$$

$$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.0.6)$$

Therefore the lines intersect at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5 Eigenvectors

The characteristic equation of the matrix $\mathbf{V}$ is

$$\left|\mathbf{V}-\lambda\mathbf{I}\right| = 0 \tag{5.0.1}$$

$$\begin{vmatrix} 6-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15-\lambda \end{vmatrix} = 0 \tag{5.0.2}$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \tag{5.0.3}$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2}\left(9+\sqrt{442}\right) \tag{5.0.4}$$

$$\lambda_2 = \frac{-1}{2}\left(9-\sqrt{442}\right) \tag{5.0.5}$$

The eigen vectors will be in the nullspace of $\mathbf{V}-\lambda_1\mathbf{I}$ and $\mathbf{V}-\lambda_2\mathbf{I}$.The eigen vector corresponding to eigen value λ_1 will be

$$\mathbf{V}-\lambda_1\mathbf{I} = \begin{pmatrix} 6+\frac{1}{2}\left(9+\sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15+\frac{1}{2}\left(9+\sqrt{442}\right) \end{pmatrix} \tag{5.0.6}$$

$$= \begin{pmatrix} \frac{1}{2}\left(21+\sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21+\sqrt{442}\right) \end{pmatrix} \tag{5.0.7}$$

$$\xleftrightarrow{R_2=\left(21+\sqrt{442}\right)R_2+R_1} \begin{pmatrix} \frac{1}{2}\left(21+\sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \tag{5.0.8}$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21+\sqrt{442} \end{pmatrix} \tag{5.0.9}$$

normalizing p_1,we get

$$p_1 = \frac{1}{\sqrt{884+42\sqrt{442}}}\begin{pmatrix} 1 \\ 21+\sqrt{442} \end{pmatrix} \tag{5.0.10}$$

the eigen vector corresponding to eigen value λ_2 will be

$$\mathbf{V}-\lambda_1\mathbf{I} = \begin{pmatrix} 6+\frac{1}{2}\left(9-\sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15+\frac{1}{2}\left(9-\sqrt{442}\right) \end{pmatrix} \tag{5.0.11}$$

$$= \begin{pmatrix} \frac{1}{2}\left(21-\sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21-\sqrt{442}\right) \end{pmatrix} \tag{5.0.12}$$

$$\xleftrightarrow{R_2=\left(21-\sqrt{442}\right)R_2+R_1} \begin{pmatrix} \frac{1}{2}\left(21-\sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \tag{5.0.13}$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_2 = \begin{pmatrix} 1 \\ 21-\sqrt{442} \end{pmatrix} \tag{5.0.14}$$

normalizing p_2,we get

$$p_2 = \frac{1}{\sqrt{884-42\sqrt{442}}}\begin{pmatrix} 1 \\ 21-\sqrt{442} \end{pmatrix} \tag{5.0.15}$$

So the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{884+42\sqrt{442}}} & \frac{1}{\sqrt{884-42\sqrt{442}}} \\ \frac{21+\sqrt{442}}{\sqrt{884+42\sqrt{442}}} & \frac{21-\sqrt{442}}{\sqrt{884-42\sqrt{442}}} \end{pmatrix} \tag{5.0.16}$$

6 Affine transformation

Doing the affine transformation on given quadratic equation, we get pair to intersecting straight lines passing through origin.

Let the affine transformation be $\mathbf{x} = \mathbf{P}\mathbf{y} + c$.The transformation will be

$$\left(\mathbf{P}\mathbf{y}+c\right)^T \mathbf{V}\left(\mathbf{P}\mathbf{y}+c\right) + 2\mathbf{u}^T\left(\mathbf{P}\mathbf{y}+c\right) + f = 0 \tag{6.0.1}$$

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y} + 2\left(c^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{P}\mathbf{y} + c^T\mathbf{V}c + 2\mathbf{u}^Tc + f = 0 \tag{6.0.2}$$

if the point c is taken as the point of intersection of the two lines.

$$c^T\mathbf{V}c + 2\mathbf{u}^Tc + f = 0 \tag{6.0.3}$$

$$c^T\mathbf{V} + \mathbf{u}^T = 0 \tag{6.0.4}$$

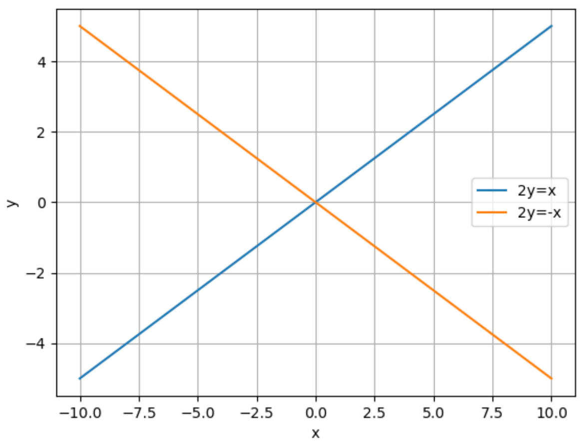

Fig. 1: straight lines after affine transformation passing through origin

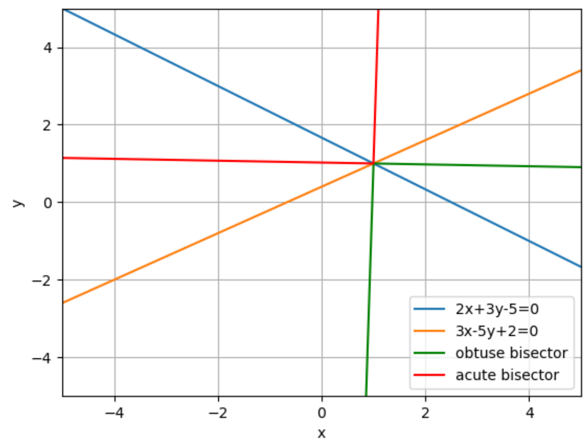

Fig. 2: Par of straight lines and their angular bisectors

So the affine transformation of the given lines will be

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y} = 0 \tag{6.0.5}$$

$$\mathbf{y}^T\begin{pmatrix}-1.5 & 0\\ 0 & 6\end{pmatrix}\mathbf{y} = 0 \tag{6.0.6}$$

$$(x-2y)(x+2y) = 0 \tag{6.0.7}$$

Since the two lines are symmetric with respect to both $X-axis$ and $Y-axis$,the axes themselves are the bisectors of the transformed pair of lines.So the bisectors will be $x = 0$ and $y = 0$. Matrix notation will be of the form

$$\mathbf{y}^T\begin{pmatrix}0 & 0.5\\ 0.5 & 0\end{pmatrix}\mathbf{y} = 0 \tag{6.0.8}$$

$$\mathbf{y}^T\mathbf{K}\mathbf{y} = 0 \tag{6.0.9}$$

$$\mathbf{K} = \begin{pmatrix}0 & 0.5\\ 0.5 & 0\end{pmatrix} \tag{6.0.10}$$

7 Bisectors

Taking the inverse of the affine transformation of the equation $xy = 0$,will give the angle bisectors.

$$\left(\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}^{-1}c\right)^T\mathbf{K}\left(\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}^{-1}c\right) = 0 \tag{7.0.1}$$

$$\mathbf{x}^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x} - 2c^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x} + c^T\mathbf{P}\mathbf{K}\mathbf{P}^Tc = 0 \tag{7.0.2}$$

Substituting the values we get

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{2\sqrt{442}} & \frac{21}{2\sqrt{442}}\\ \frac{21}{2\sqrt{442}} & -\frac{1}{2\sqrt{442}}\end{pmatrix}\mathbf{x} - \begin{pmatrix}\frac{22}{\sqrt{442}} & \frac{20}{\sqrt{442}}\end{pmatrix}\mathbf{x} + \frac{21}{\sqrt{442}} = 0 \tag{7.0.3}$$

$$\frac{x^2}{2\sqrt{442}} + \frac{21}{\sqrt{442}}xy - \frac{y^2}{2\sqrt{442}} - \frac{22x}{\sqrt{442}} - \frac{20y}{\sqrt{442}} + \frac{21}{\sqrt{442}} = 0 \tag{7.0.4}$$

$$x^2 + 42xy - y^2 - 44x - 40y + 42 = 0 \tag{7.0.5}$$

Therefore the equation of bisectors of the given line in quadratic form is

$$\mathbf{x}^T\begin{pmatrix}1 & 21\\ 21 & -1\end{pmatrix}\mathbf{x} - \begin{pmatrix}44 & 40\end{pmatrix}\mathbf{x} + 42 = 0 \tag{7.0.6}$$