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Assignment 2

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Abstract—**This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{1.0.2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1 : x - 1 = 2 - y = z - 1 \tag{2.0.1}$$

$$L_2 : \frac{x-2}{2} = y + 1 = \frac{z+1}{2} \tag{2.0.2}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (2.0.1) and (2.0.2) are equal.

If the two line intersect then (2.0.1)=(2.0.2) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{2.0.3}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{2.0.4}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{2.0.5}$$

The Augmented matrix will be

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \tag{2.0.6}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \xleftrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \tag{2.0.7}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect.

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The vector passing through $\mathbf{A}$ and $\mathbf{B}$ will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \begin{pmatrix} \mathbf{m_1} & -\mathbf{m_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{3.0.1}$$

The vectors $\mathbf{m_1}$,$\mathbf{m_2}$ are both perpendicular to the line $\mathbf{AB}$.So the dot product of $\mathbf{m_1}$,$\mathbf{m_2}$ with the line $\mathbf{AB}$ is zero.

The dot product of $\mathbf{m_1}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_1}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.0.2}$$

$$\mathbf{m_1^T}(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_1^T}(\mathbf{m_1} - \mathbf{m_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.0.3}$$

The dot product of $\mathbf{m_2}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_2}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{3.0.4}$$

$$\mathbf{m_1^T}(\mathbf{x_1}-\mathbf{x_2}) + \mathbf{m_2^T}(\mathbf{m_1}-\mathbf{m_2})\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = 0 \tag{3.0.5}$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1}^T\\ \mathbf{m_2}^T\end{pmatrix} \tag{3.0.6}$$

Combining the equations (3.0.3) and (3.0.5) in matrix form,using equation (3.0.6), we get

$$\mathbf{MM}^T\begin{pmatrix}\lambda_1\\ -\lambda_2\end{pmatrix} + \mathbf{M}(\mathbf{x_1}-\mathbf{x_2}) = 0 \tag{3.0.7}$$

simplifying it further

$$\mathbf{MM}^T\begin{pmatrix}\lambda_1\\ -\lambda_2\end{pmatrix} = \mathbf{M}(\mathbf{x_2}-\mathbf{x_1}) \tag{3.0.8}$$

To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the above expression to get the augmented form

$$\begin{pmatrix}\mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2}\\ \mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2}\end{pmatrix}\begin{pmatrix}\lambda_1\\ -\lambda_2\end{pmatrix} = \begin{pmatrix}\mathbf{m_1}^T(\mathbf{x_2}-\mathbf{x_1})\\ \mathbf{m_2}^T(\mathbf{x_2}-\mathbf{x_1})\end{pmatrix} \tag{3.0.9}$$

$$\left(\begin{array}{cc|c}\mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2} & \mathbf{m_1}^T(\mathbf{x_2}-\mathbf{x_1})\\ \mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2} & \mathbf{m_2}^T(\mathbf{x_2}-\mathbf{x_1})\end{array}\right) \tag{3.0.10}$$

we know that

$$\mathbf{x_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \mathbf{m_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix} and \mathbf{m_2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

so the augmented matrix will be

$$\left(\begin{array}{cc|c}3 & 3 & 2\\ 3 & 9 & -5\end{array}\right) \tag{3.0.11}$$

Using row reduction $\mathbf{R_2} = \mathbf{R_2} - \mathbf{R_1}$, we get

$$\left(\begin{array}{cc|c}3 & 3 & 2\\ 0 & 6 & -7\end{array}\right) \tag{3.0.12}$$

So the values are $\lambda_1 = \frac{11}{6}$ and $\lambda_2 = \frac{7}{6}$. Using the equation (1.0.1) and (1.0.2), we get the points as $\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix}$ and $\frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}$ on the line L_1,L_2 respectively.

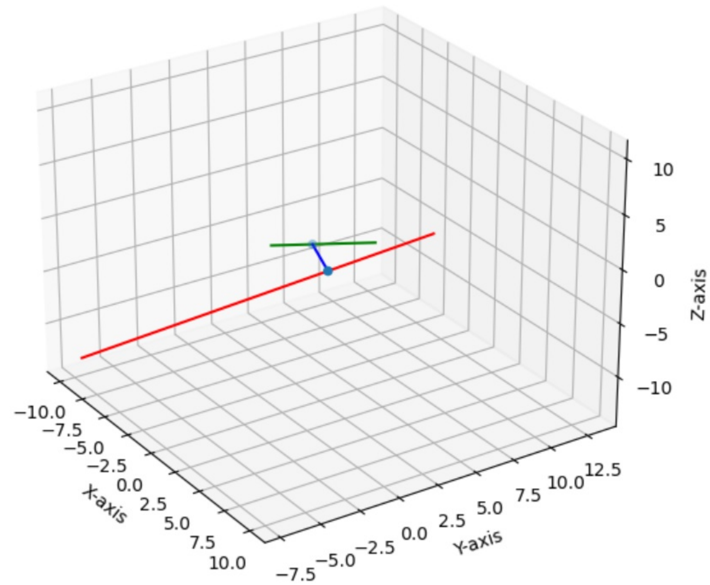

Fig. 0: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

The shortest distance between the lines is the absolute value of projection of the vector $\mathbf{AB}$ on to the unit vector $\mathbf{n}$.

$$\|(\mathbf{B}-\mathbf{A})\| = \left\|\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix} - \frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}\right\| = \frac{3}{\sqrt{2}} \tag{3.0.13}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$.

The unit vector perpendicular to lines

$$Line_1: \mathbf{x} = \mathbf{x_1} + \lambda_1\mathbf{m_1} \tag{3.0.14}$$

$$Line_2: \mathbf{x} = \mathbf{x_2} + \lambda_1\mathbf{m_2} \tag{3.0.15}$$

can be found by

$$\frac{\mathbf{A}-\mathbf{B}}{\|\mathbf{A}-\mathbf{B}\|} \tag{3.0.16}$$

$$\frac{\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix} - \frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}}{\frac{3}{\sqrt{2}}}$$

So the unit vector perpendicular to both L_1 and L_2

is

$$\mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad (3.0.17)$$