Assignment 2

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***Abstract*—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \tag{2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1 : x - 1 = 2 - y = z - 1 \tag{3}$$

$$L_2 : \frac{x-2}{2} = y + 1 = \frac{z+1}{2} \tag{4}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$\lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} - \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$$\begin{pmatrix}1 & -2\\-1 & -1\\1 & -2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

The Augmented matrix will be

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix}0 & 0 & 3\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The shortest distance between the lines will be the projection of any line between the points on L_1,L_2 on to the unit vector which is perpendicular to both L_1,L_2.

The unit vector perpendicular to lines

$$Line_1 : \mathbf{x} = x_1 + \lambda_1 \mathbf{b_1}$$

$$Line_2 : \mathbf{x} = x_2 + \lambda_1 \mathbf{b_2}$$

can be found by calculating

$$\frac{\mathbf{b_1} \times \mathbf{b_2}}{\|\mathbf{b_1} \times \mathbf{b_2}\|}$$

In our question the value of $\mathbf{b_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\mathbf{b_2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$

So the unit vector perpendicular to both L_1 and L_2

is

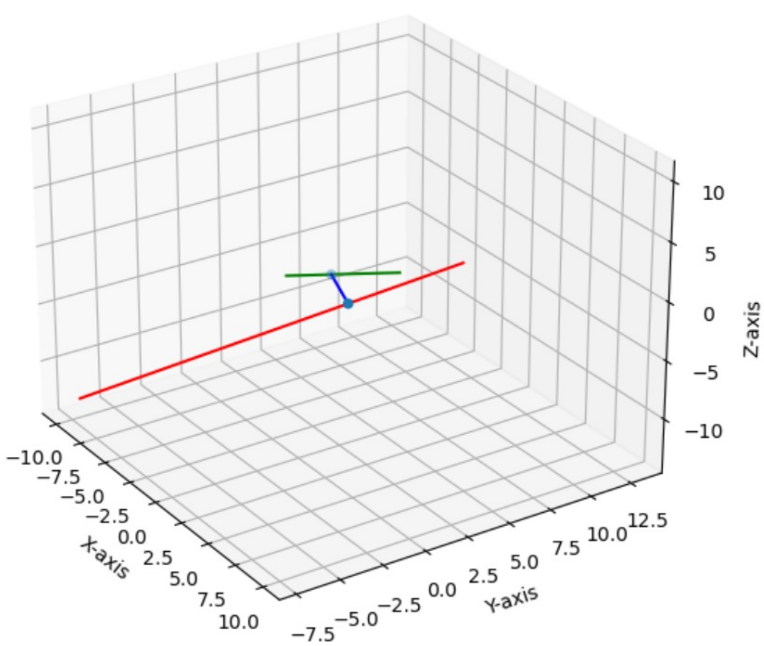
$$\boldsymbol{u} = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{\| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The points $\boldsymbol{A} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ lie on the line L_1,L_2 respectively.

The shortest distance between the lines is the absolute value of projection of the vector $\boldsymbol{B} - \boldsymbol{A}$ on to the unit vector $\boldsymbol{u}$.

$$\|(\boldsymbol{B} - \boldsymbol{A})^T \boldsymbol{u}\| = \| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \| = \frac{3}{\sqrt{2}}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$



To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the following expression

$$\begin{pmatrix} \boldsymbol{b_1^T b_1} & -\boldsymbol{b_1^T b_2} \\ \boldsymbol{b_2^T b_1} & -\boldsymbol{b_2^T b_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b_1^T}(x_2 - x_1) \\ \boldsymbol{b_2^T}(x_2 - x_1) \end{pmatrix}$$

we know that

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \boldsymbol{b_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} and \boldsymbol{b_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Using the above expression, we get the points as $\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix}$ and $\frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}$ on the line L_1,L_2 respectively