Assignment 11

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Abstract

This document is about the linear operator and minimal polynomials.

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Let $\mathbf{V}$ be the vector space of $n \times n$ matrices over field $\mathbf{F}$. Let $\mathbf{A}$ be a fixed $n \times n$ matrix. Let $\mathbf{T}$ be the linear operator on $\mathbf{V}$ defined by

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} \tag{1.0.1}$$

Show that the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.

2 Proof

Given	$\mathbf{A}$ is a fixed matrix from the vector space $\mathbf{V}$ of $n \times n$ matrices. A linear operator on the finite dimensional vector space $\mathbf{V}$, $\mathbf{T}$ is defined as $\mathbf{T}(\mathbf{B}) = \mathbf{AB}$.
Minimal polynomial	The minimal polynomial of a linear operator $\mathbf{T}$ is a monic polynomial which annihilates $\mathbf{T}$.
Matrix representation of $\mathbf{T}$	If we stack up the columns of the matrix $\mathbf{B}$,the linear operator $\mathbf{T}$ can be represented in the equivalent form as If $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & . & . & b_n \end{pmatrix}$,then the linear transformation of $\mathbf{B}$ will be $\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{A}b_1 & \mathbf{A}b_2 & . & .\mathbf{A}b_n \end{pmatrix}$ $\mathbf{M_T}(\mathbf{B}) = \begin{pmatrix} \mathbf{T}(b_1) \\ \mathbf{T}(b_2) \\ . \\ . \\ \mathbf{T}(b_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & & & \\ & \mathbf{A} & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$

	$\mathbf{M_T} = \begin{pmatrix} \mathbf{A} & & & & \\ & \mathbf{A} & \mathbf{O} & & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & \mathbf{A} \end{pmatrix}$

TABLE 1: Construction

Properties of minimal polynomial	The roots of the characteristic polynomial,eigen values and the minimal polynomial are same,except for multiplicities.The roots of the minimal polynomial of $\mathbf{A}$ are the roots of $\det(\mathbf{A} - \lambda\mathbf{I})$
The roots of minimal polynomial of $\mathbf{T}$	The roots of the minimal polynomial of $\mathbf{T}$ are the roots of $\det(\mathbf{T} - \lambda\mathbf{I})$ $\det(\mathbf{T} - \lambda\mathbf{I}) = \begin{vmatrix} (\mathbf{A} - \lambda\mathbf{I}) & & & & \\ & (\mathbf{A} - \lambda\mathbf{I}) & \mathbf{O} & & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & (\mathbf{A} - \lambda\mathbf{I}) \end{vmatrix}$ $=(\det(\mathbf{A} - \lambda\mathbf{I}))^n$ Therfore we can see that the eigen values of $\mathbf{A}$ are also the eigen values of the linear operator $\mathbf{T}$
Minimal polynomial of $\mathbf{T}$	The minimal polynomial of $\mathbf{A}$ divides the characteristic polynomial of $\mathbf{A}$ and $\mathbf{T}$. Let the minimal polynomial of $\mathbf{A}$ is of degree $p \leq n$ $f(x) = a_0 + a_1 x + a_2 x^2 ... a_p x^p$ such that $f(\mathbf{A}) = 0$ $f(\mathbf{T}) = a_0\mathbf{I} + a_1\mathbf{T} + a_2\mathbf{T}^2 + .. + a_p\mathbf{T}^p$

	$$f(\mathbf{T}) = \begin{pmatrix} f(\mathbf{A}) & & & & \\ & f(\mathbf{A}) & & \mathbf{O} & \\ & & \cdot & & \\ & \mathbf{O} & & \cdot & \\ & & & & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}_{n^2 \times n^2}$$ Therefore the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.

TABLE 2: Proof

Assuming matrix $\mathbf{A}$ as follows:	Let us Consider 2×2 matrix, $$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$$
Minimal polynomial of $\mathbf{A}$	The eigen values of $\mathbf{A}$ are $1, 2$. So, the minimal polynomial is $f(x) = (x-1)(x-2)$
Matrix of linear operator	So, the matrix of the linear operator $\mathbf{T}$ with respect to the basis $$\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ $$\mathbf{T}(\mathbf{e_1}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.\mathbf{e_1} + 0.\mathbf{e_2} + 0.\mathbf{e_3} + 0.\mathbf{e_4}$$ $$\mathbf{T}(\mathbf{e_2}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} = 4.\mathbf{e_1} + 2.\mathbf{e_2} + 0.\mathbf{e_3} + 0.\mathbf{e_4}$$ $$\mathbf{T}(\mathbf{e_3}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.\mathbf{e_1} + 0.\mathbf{e_2} + 1.\mathbf{e_3} + 0.\mathbf{e_4}$$

	$\mathbf{T}(\mathbf{e_4}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} = 0.\mathbf{e_1} + 0.\mathbf{e_2} + 4.\mathbf{e_3} + 2.\mathbf{e_4}$ So the matrix of the linear operator will be $\mathbf{M_T} = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}$
Characteristic equation of **T**	The characteristic equation of **T** is $(x-1)^2(x-2)^2$ So the eigen values are $1, 1, 2, 2$
Minimal polynomial of **T**	$f(\mathbf{M_T}) = (\mathbf{T}-\mathbf{I})(\mathbf{T}-\mathbf{2I}) = \begin{pmatrix} \mathbf{A}-\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}-\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{A}-\mathbf{2I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}-\mathbf{2I} \end{pmatrix}$ $f(\mathbf{M_T}) = \begin{pmatrix} (\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{2I}) & \mathbf{O} \\ \mathbf{O} & (\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{2I}) \end{pmatrix} = \begin{pmatrix} f(\mathbf{A}) & \mathbf{O} \\ \mathbf{O} & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}$ We know that eigen values of **T** should be roots of minimal polynomial of **T**, thus minimal polynomial should be of the form $(x-1)^p(x-2)^q$ where $p, q \in \mathbb{N}..1 \leq p, q \leq 2$ Therefore the minimal polynomial $f(\mathbf{A})$ of **A** annihilates **T**,thus we can conclude that $f(x)$ is the minimal polynomial of linear operator **T**

TABLE 3: Example