Assignment 6

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***Abstract*—This document is about tracing a parabola**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \tag{1.0.1}$$

2 Construction

The given quadratic equation can be written in the matrix form as

3 Construction

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{3.0.1}$$

Calculating the parameters,we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \tag{3.0.2}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \tag{3.0.3}$$

Therefore the given parabola equation is a degenerate.The quadratic equation corresponds to a pair of coincident straight lines.

The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \tag{3.0.4}$$

$$= \lambda^2 - 5\lambda \tag{3.0.5}$$

$$\lambda_1 = 0, \lambda_2 = 5 \tag{3.0.6}$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda\mathbf{I}$.For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \tag{3.0.7}$$

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.0.8}$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.0.9}$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \tag{3.0.10}$$

$$p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.11}$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.12}$$

Therefore the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.13}$$

4 Equation of the coincident line

The general equation of coincident lines in quadratic form can be written as

$$(mx - y + c)^2 = 0 \tag{4.0.1}$$

$$\mathbf{x}^T \begin{pmatrix} m^2 & -m \\ -m & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} cm & -c \end{pmatrix} \mathbf{x} + c^2 = 0 \tag{4.0.2}$$

The degenerate equation in matrix form will be

$$\begin{pmatrix} \mathbf{x}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{4.0.3}$$

$$\begin{pmatrix} m^2 & -m & cm \\ -m & 1 & -c \\ cm & -c & c^2 \end{pmatrix} \xleftrightarrow{R_3 = mR_3 - cR_1} \begin{pmatrix} m^2 & -m & cm \\ -m & 1 & -c \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.4}$$

$$\xleftrightarrow{R_2 = mR_2 + R_1} \begin{pmatrix} m^2 & -m & cm \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{R_1}{m}} \begin{pmatrix} m & -1 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.5}$$

So the solution to nullspace will be

$$\begin{pmatrix} m & -1 & c \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = mx - y + c = 0 \tag{4.0.6}$$

So the 3×3 matrix can also be written as

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} = \begin{pmatrix} m \\ -1 \\ c \end{pmatrix} \begin{pmatrix} m & -1 & c \end{pmatrix} \tag{4.0.7}$$

This matrix has a rank of 1.

Applying this on the given problem we get

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} = \begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{pmatrix} \tag{4.0.8}$$

$$\xleftrightarrow{R_3 = \frac{2R_2}{3} + R_1} \begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.9}$$

Therefore the equation of the line is $2x - y - 3 = 0$,which can be expressed in vector form as

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}^T \mathbf{x} = 3 \tag{4.0.10}$$