Assignment 2

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***Abstract*—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{1.0.2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1 : x - 1 = 2 - y = z - 1 \tag{2.0.1}$$

$$L_2 : \frac{x-2}{2} = y + 1 = \frac{z+1}{2} \tag{2.0.2}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (2.0.1) and (2.0.2) are equal.

If the two line intersect then (2.0.1)=(2.0.2) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{2.0.3}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{2.0.4}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{2.0.5}$$

The Augmented matrix will be

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \tag{2.0.6}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \xleftrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{array}\right) \tag{2.0.7}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect.

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The vector passing through $\mathbf{A}$ and $\mathbf{B}$ will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \begin{pmatrix} \mathbf{m_1} & -\mathbf{m_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{3.0.1}$$

The vectors $\mathbf{m_1}$,$\mathbf{m_2}$ are both perpendicular to the line $\mathbf{AB}$.So the dot product of $\mathbf{m_1}$,$\mathbf{m_2}$ with the line $\mathbf{AB}$ is zero.

The dot product of $\mathbf{m_1}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_1}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.0.2}$$

$$\mathbf{m_1^T}(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_1^T}(\mathbf{m_1} - \mathbf{m_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.0.3}$$

The dot product of $\mathbf{m_2}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_2}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.0.4}$$

$$\mathbf{m_1^T}(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_2^T}(\mathbf{m_1} - \mathbf{m_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.0.5}$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix} \tag{3.0.6}$$

Combining the equations (3.0.3) and (3.0.5) in matrix form,using equation (3.0.6), we get

$$\mathbf{M}\mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} + \mathbf{M}(\mathbf{x_1} - \mathbf{x_2}) = 0 \tag{3.0.7}$$

simplifying it further

$$\mathbf{M}\mathbf{M}^T\begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} = \mathbf{M}(\mathbf{x_2}-\mathbf{x_1}) \tag{3.0.8}$$

To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the above expression to get the augmented form

$$\begin{pmatrix}\mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2}\\\mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2}\end{pmatrix}\begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} = \begin{pmatrix}\mathbf{m_1}^T(\mathbf{x_2}-\mathbf{x_1})\\\mathbf{m_2}^T(\mathbf{x_2}-\mathbf{x_1})\end{pmatrix} \tag{3.0.9}$$

$$\left(\begin{array}{cc|c}\mathbf{m_1}^T\mathbf{m_1} & \mathbf{m_1}^T\mathbf{m_2} & \mathbf{m_1}^T(\mathbf{x_2}-\mathbf{x_1})\\\mathbf{m_2}^T\mathbf{m_1} & \mathbf{m_2}^T\mathbf{m_2} & \mathbf{m_2}^T(\mathbf{x_2}-\mathbf{x_1})\end{array}\right) \tag{3.0.10}$$

we know that

$$\mathbf{x_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \mathbf{m_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix} and \mathbf{m_2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

so the augmented matrix will be

$$\left(\begin{array}{cc|c}3 & 3 & 2\\3 & 9 & -5\end{array}\right) \tag{3.0.11}$$

Using row reduction $\mathbf{R_2} = \mathbf{R_2} - \mathbf{R_1}$, we get

$$\left(\begin{array}{cc|c}3 & 3 & 2\\0 & 6 & -7\end{array}\right) \tag{3.0.12}$$

So the values are $\lambda_1 = \frac{11}{6}$ and $\lambda_2 = \frac{7}{6}$. Using the equation (1.0.1) and (1.0.2), we get the points as $\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix}$ and $\frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}$ on the line L_1,L_2 respectively.

The shortest distance between the lines is the absolute value of projection of the vector **AB** on to the unit vector **n**.

$$\|(\mathbf{B}-\mathbf{A})\| = \left\|\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix} - \frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}\right\| = \frac{3}{\sqrt{2}} \tag{3.0.13}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$.

The unit vector perpendicular to lines

$$Line_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1\mathbf{m_1} \tag{3.0.14}$$

$$Line_2 : \mathbf{x} = \mathbf{x_2} + \lambda_1\mathbf{m_2} \tag{3.0.15}$$

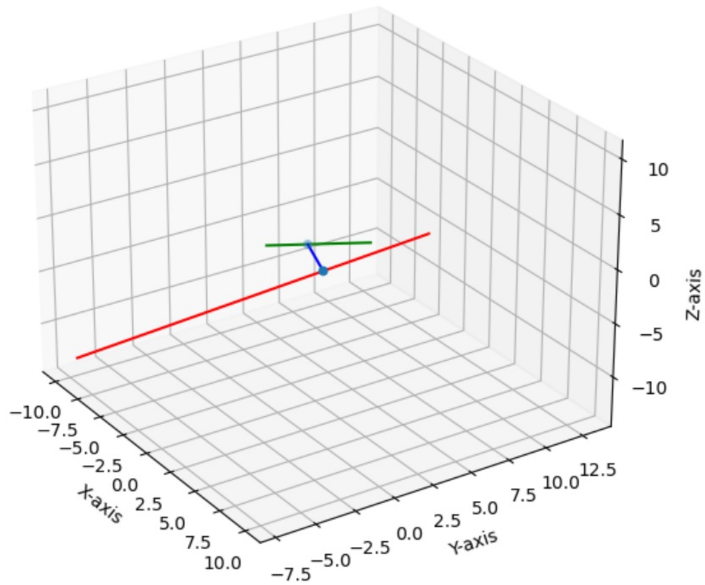

Fig. 1: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

can be found by

$$\frac{\mathbf{A}-\mathbf{B}}{\|\mathbf{A}-\mathbf{B}\|} \tag{3.0.16}$$

$$\frac{\frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix} - \frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}}{\frac{3}{\sqrt{2}}}$$

So the unit vector perpendicular to both L_1 and L_2 is

$$\mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} \tag{3.0.17}$$