Assignment 16

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AI20MTECH11001

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https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

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https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Let $\mathbf{V}$ be a finite-dimensional inner product space, and let $\mathbf{W}$ be a subspace of $\mathbf{V}$. Then $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}^{\perp}$, that is, each α in $\mathbf{V}$ is uniquely expressible in the form $\alpha = \beta + \gamma$, with $\beta \in \mathbf{W}$ and $\gamma \in \mathbf{W}^{\perp}$. Define a linear operator $\mathbf{U}$ by $\mathbf{U}\alpha = \beta - \gamma$.

1) Prove that $\mathbf{U}$ is both self-adjoint and unitary.
2) If $\mathbf{V}$ is $\mathbb{R}^3$ with standard inner product and $\mathbf{W}$ is the subspace spanned by $(1, 0, 1)$, find the matrix of $\mathbf{U}$ in the standard ordered basis.

2 Construction

Given	$\mathbf{V}$ is a finite-dimensional inner product space, and $\mathbf{W}$ is a subspace of $\mathbf{V}$. Then $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}^{\perp}$ that is, each α in $\mathbf{V}$ is uniquely expressible in the form $\alpha = \beta + \gamma$, with $\beta \in \mathbf{W}$ and $\gamma \in \mathbf{W}^{\perp}$.
Inner product	An inner product on a vector space $\mathbf{V}$ is a function which assigns to each ordered pair of vecors $\alpha, \beta \in \mathbf{V}$ a scalar $(\alpha\|\beta)$ in $\mathbb{F}$ in such a way that for all $\alpha, \beta \in \mathbf{V}$ and all scalar $c \in \mathbb{F}$ $i)$ $(\alpha + \beta\|\gamma) = (\alpha\|\gamma) + (\beta\|\gamma)$ $ii)$ $(c\alpha\|\beta) = c\,(\alpha\|\beta)$ $iii)$ $(\beta\|\alpha) = \left(\overline{\alpha\|\beta}\right)$, where the bar denotes complex conjugation $iv)$ $(\alpha\|\alpha) > 0$ if $\alpha \neq 0$
Inner product space	Inner product space is a real or complex vector space together with a specified inner product on that space. If $\mathbf{V}$ is an inner product space, then for any vectors $\alpha, \beta \in \mathbf{V}$ and any scalar c $i)$ $\|\|\mathbf{c}\alpha\|\| = \|\mathbf{c}\|\ \|\|\alpha\|\|$

	$ii)$ $\|\|\alpha\|\| > 0$ for $\alpha \neq 0$ $iii)$ $\|(\alpha\|\beta)\| \leq \|\|\alpha\|\| \ \|\|\beta\|\|$ $iv)$ $\|\|\alpha + \beta\|\| \leq \|\|\alpha\|\| + \|\|\beta\|\|$
Orthogonal complement	Given that **W** is a subspace of **V**. The orthogonal complement of **W** is the set $\mathbf{W}^{\perp}$ of all vectors in **V** which are orthogonal to every vector in **W**.Given that $\beta \in \mathbf{W}$ and $\gamma \in \mathbf{W}^{\perp}$, then the inner product $(\beta\|\gamma)$ will be equal to 0.

TABLE 1: Construction

3 Proof

Theorem	For any linear operator **U** on a finite dimensional inner product space **V**, there exists a unique linear operator $\mathbf{U}^{*}$ on **V** such that $(\mathbf{U}\alpha\|\beta) = (\alpha\|\mathbf{U}^{*}\beta)$ for all $\alpha, \beta \in \mathbf{V}$. Then we say that **U** has an adjoint on **V**, which is $\mathbf{U}^{*}$
self adjoint	Let α_1, α_2 be any two arbitrary vectors in **V** such that $\alpha_1 = \beta_1 + \gamma_1$ and $\alpha_2 = \beta_2 + \gamma_2$ where $\beta_1, \beta_2 \in \mathbf{W}$ and $\gamma_1, \gamma_2 \in \mathbf{W}^{\perp}$. The linear operator **U** on the inner product space is $(\mathbf{U}\alpha_1\|\alpha_2) = (\beta_1 - \gamma_1\|\beta_2 + \gamma_2) = (\beta_1\|\beta_2) + (\beta_1\|\gamma_2) - (\gamma_1\|\beta_2) - (\gamma_1\|\gamma_2)$ since $\mathbf{W}^{\perp}$ is orthogonal complement of **W** $(\mathbf{U}\alpha_1\|\alpha_2) = (\beta_1\|\beta_2) - (\gamma_1\|\gamma_2)$ similarly $(\alpha_1\|\mathbf{U}\alpha_2) = (\beta_1 + \gamma_1\|\beta_2 - \gamma_2) = (\beta_1\|\beta_2) - (\beta_1\|\gamma_2) + (\gamma_1\|\beta_2) - (\gamma_1\|\gamma_2)$ since $\mathbf{W}^{\perp}$ is orthogonal complement of **W** $(\alpha_1\|\mathbf{U}\alpha_2) = (\beta_1\|\beta_2) - (\gamma_1\|\gamma_2)$ Since $(\mathbf{U}\alpha_1\|\alpha_2) = (\alpha_1\|\mathbf{U}\alpha_2)$, the linear operator **U** is self adjoint.
Theorem	Let **U** be a linear operator on the inner product space **V**. Then **U** is unitary if and only if the adjoint $\mathbf{U}^{*}$ of **U** exists and $\mathbf{U}\mathbf{U}^{*} = \mathbf{U}^{*}\mathbf{U} = \mathbf{I}$

Unitary	$(\mathbf{UU}^*)\,\alpha = (\mathbf{UU})\,\alpha = \mathbf{U}\,(\mathbf{U}\alpha) = \mathbf{U}^*\,(\mathbf{U})\,\alpha = (\mathbf{U}^*\mathbf{U})\,\alpha$ We know that $\mathbf{U}\alpha = \beta - \gamma$. Calculating $\mathbf{UU}\alpha$, we get $\mathbf{UU}\alpha = \mathbf{U}\,(\mathbf{U}\alpha) = \mathbf{U}\,(\beta - \gamma) = \mathbf{U}\beta - \mathbf{U}\gamma = \beta + \gamma = \mathbf{I}\alpha$ Therefore as $(\mathbf{UU}^*)\,\alpha = (\mathbf{U}^*\mathbf{U})\,\alpha = \mathbf{UU}\alpha = \mathbf{I}\alpha$ $(\mathbf{UU}^*) = (\mathbf{U}^*\mathbf{U}) = \mathbf{I}$ So the linear operator is unitary.
Part 2	Given $\mathbf{V}$ is $\mathbb{R}^3$ with standard inner product and $\mathbf{W}$ is the subspace spanned by $(1,0,1)$. Let the vector in $\mathbf{W}^\perp$, the orthogonal complement of $\mathbf{W}$ be $\mathbf{a} = (\gamma_1, \gamma_2, \gamma_3)$. Since $\mathbf{a}$ is orthogonal to the subspace $\mathbf{W}$, we get the inner product as $(\mathbf{a}\|(1,0,1)) = \gamma_1 + \gamma_3 = 0$ If $\gamma_2 = 1$ and $\gamma_1 = \gamma_3 = 0$ or $\gamma_2 = 0$ and $\gamma_1 = -\gamma_3 = 1$, both the vectors $(0,1,0)$ and $(1,0,-1)$ form the linearly independent basis of $\mathbf{W}^\perp$.Therefore $\mathbf{W} = span\{(1,0,1)\}$ $\mathbf{W}^\perp = span\{(0,1,0), (1,0,-1)\}$ Any vector $\alpha \in \mathbf{V}$ can be uniquely expressed as $\alpha = \beta + \gamma$ where $\beta \in \mathbf{W}$ and $\gamma \in \mathbf{W}^\perp$ So $\beta = x\,(1,0,1)$ and $\gamma = y\,(0,1,0) + z\,(1,0,-1)$. The standard basis e_1, e_2, e_3 can be represented in terms of $\mathbf{W}$ and $\mathbf{W}^\perp$ as $e_1 = (1,0,0) = \beta_1 + \gamma_1 = x_1\,(1,0,1) + y_1\,(0,1,0) + z_1\,(1,0,-1) = (x_1 + z_1, y_1, x_1 - z_1)$ Therefore $e_1 = \frac{1}{2}\,(1,0,1) + \frac{1}{2}\,(1,0,-1)$. Similarly $e_2 = (0,1,0)$ $e_3 = \frac{1}{2}\,(1,0,1) - \frac{1}{2}\,(1,0,-1)$ Representing the matrix of liner operator $\mathbf{U}$ with respect to standard basis $\mathbf{U} = \begin{pmatrix} \mathbf{Ue_1} & \mathbf{Ue_2} & \mathbf{Ue_3} \end{pmatrix}$ $\mathbf{Ue_1} = \mathbf{U}\left(\frac{1}{2}\,(1,0,1) + \frac{1}{2}\,(1,0,-1)\right) = \frac{1}{2}\,(1,0,1) - \frac{1}{2}\,(1,0,-1) = (0,0,1)$

	$\mathbf{Ue_2} = \mathbf{U}(0,1,0) = (0,-1,0)$ $\mathbf{Ue_3} = \mathbf{U}\left(\frac{1}{2}(1,0,1) - \frac{1}{2}(1,0,-1)\right) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,0,-1) = (1,0,0)$ Therefore the matrix of linear operator is $$\mathbf{U} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$ Verifying we get $(\mathbf{U}e_1 \| (1,1,1)) = ((0,0,1) \| (1,1,1)) = 1$ $(e_1 \| \mathbf{U}(1,1,1)) = ((1,0,0) \| (1,-1,1)) = 1$ $\mathbf{U}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$.Therefore the linear operator is unitary

TABLE 2: Proof