# Assignment 9

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***Abstract*—This document is about positive definite properties of real symmetric non-singular matrix.**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
```

and latex-tikz codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609
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## 1 Problem

For every 4×4 real symmetric non-singular matrix $\mathbf{A}$,prove if there exists a positive integer $p$ such that

1) $p\mathbf{I} + \mathbf{A}$ is positive definite
2) $\mathbf{A}^p$ is positive definite
3) $\mathbf{A}^{-p}$ is positive definite
4) $exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

## 2 Construction

| | |
|---|---|
| Definition | An $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive definite if $\mathbf{x}^T\mathbf{M}\mathbf{x} > 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$ |
| Properties | An $n \times n$ symmetric real matrix always have real eigen vectors and the set of eigen vectors can be selected such that they form an orthonormal basis $\mathbf{M} = \mathbf{Q}\Lambda\mathbf{Q}^T$ and $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ |
| Implication | An $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive definite if i) Eigen values of $\mathbf{M}$ are all positive ii) The pivot elements of $\mathbf{M}$ are all positive |

TABLE I: Properties of positive definite matrix

| | |
|---|---|
| Proof for 1 | $p\mathbf{I}+\mathbf{A}=\mathbf{Q}p\mathbf{I}\mathbf{Q}^T+\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T=\mathbf{Q}\left(p\mathbf{I}+\boldsymbol{\Lambda}\right)\mathbf{Q}^T$ <br> if the eigen values of $\mathbf{A}$ are $\lambda_1,\lambda_2...\lambda_4$, then the eigen values of $p\mathbf{I}+\mathbf{A}$ will be $\lambda_1+p,\lambda_2+p...\lambda_4+p$.If we choose $p$ to be $>\lvert min\left(\lambda_1,\lambda_2...\lambda_4\right)\rvert$ then $p\mathbf{I}+\mathbf{A}$ will be positive definite. |
| Proof for 2 | $\mathbf{A}^p=\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T\right)^p=\mathbf{Q}\boldsymbol{\Lambda}^p\mathbf{Q}^T$ <br> if the eigen values of $\mathbf{A}$ are $\lambda_1,\lambda_2...\lambda_4$, then the eigen values of $\mathbf{A}^p$ will be $\lambda_1^p,\lambda_2^p...\lambda_4^p$. If p is even then the eigen values are positive. |
| Proof for 3 | $\mathbf{A}^{-1}=\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T\right)^{-1}=\mathbf{Q}\boldsymbol{\Lambda}^{-1}\mathbf{Q}^T$ <br> if the eigen values of $\mathbf{A}$ are $\lambda_1,\lambda_2...\lambda_4\neq 0$ (For inverse to exist), then the eigen values of $\mathbf{A}^{-p}$ will be $\frac{1}{\lambda_1^p},\frac{1}{\lambda_2^p}...\frac{1}{\lambda_4^p}$. If p is even then the eigen values are positive. |
| Proof for 4 | $exp\left(p\mathbf{A}\right)-\mathbf{I}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(p\mathbf{A}\right)^k-\mathbf{I}=p\mathbf{A}+\frac{1}{2!}\left(p\mathbf{A}\right)^2+\frac{1}{3!}\left(p\mathbf{A}\right)^3+...$ <br> $=\mathbf{Q}\left(p\boldsymbol{\Lambda}+\frac{1}{2!}\left(p\boldsymbol{\Lambda}\right)^2+\frac{1}{3!}\left(p\boldsymbol{\Lambda}\right)^3+...\right)\mathbf{Q}^T$ <br> if the eigen values of $\mathbf{A}$ are $\lambda_1,\lambda_2...\lambda_4$, then the eigen values of $exp\left(p\mathbf{A}\right)-\mathbf{I}$ will be $e^{p\lambda_1}-1,e^{p\lambda_2}-1,..e^{p\lambda_4}-1$.So if the eigen value of $\mathbf{A}$ is negative then the corresponding eigen value of $exp\left(p\mathbf{A}\right)-\mathbf{I}$ is also negative for every positive integer $p$.So such a positive integer $p$ does not exist for negative eigen values.If eigen values are positive then for any positive $p$ the matrix $exp\left(p\mathbf{A}\right)-\mathbf{I}$ is positive definite. |

TABLE II: PROOF