Assignment 2

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***Abstract*—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{1}$$

$$L_2: \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1: x-1 = 2-y = z-1 \tag{3}$$

$$L_2: \frac{x-2}{2} = y+1 = \frac{z+1}{2} \tag{4}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{5}$$

$$\lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} - \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{6}$$

$$\begin{pmatrix}1 & -2\\-1 & -1\\1 & -2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{7}$$

The Augmented matrix will be

$$\tag{8}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{9}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix}0 & 0 & 3\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{10}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The vector passing through $\mathbf{A}$ and $\mathbf{B}$ will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \begin{pmatrix}\mathbf{m_1} & -\mathbf{m_2}\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \tag{11}$$

The vectors $\mathbf{m_1}$,$\mathbf{m_2}$ are both perpendicular to the line $\mathbf{AB}$.So the dot product of $\mathbf{m_1}$,$\mathbf{m_2}$ with the line $\mathbf{AB}$ is zero.

The dot product of $\mathbf{m_1}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_1}^T(\mathbf{A} - \mathbf{B}) = 0$$
$$\mathbf{m_1}^T(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_1}^T(\mathbf{m_1} - \mathbf{m_2})\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{12}$$

The dot product of $\mathbf{m_2}$ with the line $\mathbf{AB}$ is

$$\mathbf{m_2}^T(\mathbf{A} - \mathbf{B}) = 0$$
$$\mathbf{m_1}^T(\mathbf{x_1} - \mathbf{x_2}) + \mathbf{m_2}^T(\mathbf{m_1} - \mathbf{m_2})\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{13}$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1}^T\\\mathbf{m_2}^T\end{pmatrix} \tag{14}$$

Combining the equations (12) and (13) in matrix form,using equation (14), we get

$$\boldsymbol{M}\boldsymbol{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} + \boldsymbol{M}(\boldsymbol{x_1} - \boldsymbol{x_2}) = 0 \tag{15}$$

simplifying it further

$$\boldsymbol{M}\boldsymbol{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \boldsymbol{M}(\boldsymbol{x_2} - \boldsymbol{x_1}) \tag{16}$$

To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the above expression to get the augmented form

$$\begin{pmatrix} \boldsymbol{m_1^T m_1} & \boldsymbol{m_1^T m_2} \\ \boldsymbol{m_2^T m_1} & \boldsymbol{m_2^T m_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{m_1^T(x_2 - x_1)} \\ \boldsymbol{m_2^T(x_2 - x_1)} \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} \boldsymbol{m_1^T m_1} & \boldsymbol{m_1^T m_2} & \boldsymbol{m_1^T(x_2 - x_1)} \\ \boldsymbol{m_2^T m_1} & \boldsymbol{m_2^T m_2} & \boldsymbol{m_2^T(x_2 - x_1)} \end{pmatrix} \tag{18}$$

we know that

$$\boldsymbol{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \boldsymbol{m_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} and \boldsymbol{m_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so the augmented matrix will be

$$\tag{19}$$

$$\left(\begin{array}{cc|c} 3 & 3 & 2 \\ 3 & 9 & -5 \end{array}\right) \tag{20}$$

Using row reduction $\boldsymbol{R_2} = \boldsymbol{R_2} - R_1$, we get

$$\left(\begin{array}{cc|c} 3 & 3 & 2 \\ 0 & 6 & -7 \end{array}\right) \tag{21}$$

So the values of $\lambda_1 = \frac{11}{6}$ and $\lambda_2 = \frac{7}{6}$

Using the equation (1) and (2), we get the points as $\frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix}$ and $\frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}$ on the line L_1,L_2 respectively.

The shortest distance between the lines is the

absolute value of projection of the vector $\boldsymbol{AB}$ on to the unit vector $\boldsymbol{n}$.

$$\|(\boldsymbol{B} - \boldsymbol{A})\| = \|\frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}\| = \frac{3}{\sqrt{2}} \tag{22}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$

The unit vector perpendicular to lines

$$Line_1 : \boldsymbol{x} = \boldsymbol{x_1} + \lambda_1 \boldsymbol{m_1} \tag{23}$$

$$Line_2 : \boldsymbol{x} = \boldsymbol{x_2} + \lambda_1 \boldsymbol{m_2} \tag{24}$$

can be found by

$$\frac{\boldsymbol{A} - \boldsymbol{B}}{\|\boldsymbol{A} - \boldsymbol{B}\|} \tag{25}$$

$$\frac{\frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}}{\|\frac{1}{6}\begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}\|}$$

So the unit vector perpendicular to both L_1 and L_2 is

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{26}$$