Assignment 13

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Abstract

This document is about the linear operators which have all the vectors as eigen vectors.

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 PROBLEM

Let $\mathbf{A}$ be an $n \times n$ matrix over $\mathbb{C}$ such that every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$.Then

1) All eigen values of $\mathbf{A}$ are equal.
2) All eigen values of $\mathbf{A}$ are distinct.
3) $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.
4) If $\chi_{\mathbf{A}}$ and $m_{\mathbf{A}}$ denote the characteristic polynomial and the minimal polynomial respectively, then $\chi_{\mathbf{A}} = m_{\mathbf{A}}$

2 CONSTRUCTION

Given	Every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$, where $\mathbf{A}$ is an $n \times n$ matrix over $\mathbb{C}$.
Determining $\mathbf{A}$	Since every vector is an eigen vector, the standard basis vectors are also eigen vectors $\implies \mathbf{A}\mathbf{e_i} = \lambda_i \mathbf{e_i} \implies \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \end{pmatrix} \mathbf{e_i} = \lambda_i \mathbf{e_i} \implies a_i = \lambda_i \mathbf{e_i}$ where $\lambda_i \in \mathbb{C}$ therefore $\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e_1} & \lambda_2 \mathbf{e_2} & . & . & \lambda_n \mathbf{e_n} \end{pmatrix}$ Any vector $\mathbf{b}$ can be represented in the standard basis as $\mathbf{b} = b_1 \mathbf{e_1} + b_2 \mathbf{e_2} + ... + b_n \mathbf{e_n}$ where $b_i \in \mathbb{C}$ As every non-zero vector in $\mathbb{C}^n$ is an eigen vector $\mathbf{A}\mathbf{b} = \lambda \mathbf{b} \implies \mathbf{A}(b_1 \mathbf{e_1} + b_2 \mathbf{e_2} + ... + b_n \mathbf{e_n}) = \lambda (b_1 \mathbf{e_1} + b_2 \mathbf{e_2} + ... + b_n \mathbf{e_n})$ $\implies b_1 \lambda_1 \mathbf{e_1} + b_2 \lambda_2 \mathbf{e_2} + ... + b_n \lambda_n \mathbf{e_n} = \lambda (b_1 \mathbf{e_1} + b_2 \mathbf{e_2} + ... + b_n \mathbf{e_n})$ $\implies b_1 (\lambda_1 - \lambda) \mathbf{e_1} + b_2 (\lambda_2 - \lambda) \mathbf{e_2} + ... + b_n (\lambda_n - \lambda) \mathbf{e_n} = 0$ since basis are linearly independent we get $\lambda_1 = \lambda_2 = .. = \lambda_n = \lambda$ Therefore the matrix $\mathbf{A}$ is $$\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e_1} & \lambda_2 \mathbf{e_2} & . & . & \lambda_n \mathbf{e_n} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & . & . & \mathbf{e_n} \end{pmatrix} = \lambda \mathbf{I}_n \text{ where } \lambda \in \mathbb{C}$$

3 Answers

option 1	Since $\mathbf{A} = \lambda\mathbf{I}_n$, all the eigen values are equal to λ. Therefore option 1 is correct as the matrix $\mathbf{A}$ is a scalar matrix.
option 2	since the matrix $\mathbf{A}$ is a scalar matrix, all the eigen values are equal. So this option is incorrect.
option 3	This option is correct. As proved in the construction the matrix $\mathbf{A} = \lambda\mathbf{I}$ for some $\lambda \in \mathbb{C}$
option 4	Since $\mathbf{A} = \lambda\mathbf{I}$ where $\lambda \in \mathbb{C}$, the characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x-\lambda)^n$ and $m_{\mathbf{A}} = (x-\lambda) \implies \chi_{\mathbf{A}} = m_{\mathbf{A}}^n$. Therefore this option is incorrect

TABLE 1: Answer

4 Examples

Scalar matrix	Consider a 3×3 scalar matrix $\mathbf{A} = (2+3i)\,\mathbf{I}$, for which the eigen values are $(2+3i)\,,(2+3i)\,,(2+3i)$ The eigen vectors will be the nullspace of $\mathbf{A} - \lambda\mathbf{I}$ $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2+3i & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2+3i \end{pmatrix} - (2+3i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ The nullspace consists of the entire vector space so every vector is an eigen vector The characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x-(2+3i))^3$ and $m_{\mathbf{A}} = (x-(2+3i)) \implies \chi_{\mathbf{A}} = m_{\mathbf{A}}^3$ Therefore options 1 and 3 are correct.
Diagonal matrix	Consider the matrix $\mathbf{A}$ as $\mathbf{A} = \begin{pmatrix} 2+3i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3i \end{pmatrix}$ The eigen values are $\lambda_1 = 2+3i$, $\lambda - 2 = 2$, $\lambda_3 = 3i$

	The eigen vector with respect to $\lambda_1 = 2 + 3i$ will be the nullspace of $\mathbf{A} - \lambda_1\mathbf{I}$ $\mathbf{A} - \lambda_1\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3i & 0 \\ 0 & 0 & -2 \end{pmatrix}$, so the eigen vector will be $\mathbf{e_1} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ where $x_1 \in \mathbb{C}$ The eigen vector with respect to $\lambda_2 = 2$ will be the nullspace of $\mathbf{A} - \lambda_2\mathbf{I}$ $\mathbf{A} - \lambda_2\mathbf{I} = \begin{pmatrix} 3i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3i - 2 \end{pmatrix}$, so the eigen vector will be $\mathbf{e_2} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ where $x_2 \in \mathbb{C}$ The eigen vector with respect to $\lambda_3 = 3i$ will be the nullspace of $\mathbf{A} - \lambda_3\mathbf{I}$ $\mathbf{A} - \lambda_3\mathbf{I} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 3i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so the eigen vector will be $\mathbf{e_3} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ where $x_3 \in \mathbb{C}$ Consider the vector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$ where $x_1 = x_2 = x_3 = 1$ $\mathbf{Ay} = \mathbf{Ae_1} + \mathbf{Ae_2} + \mathbf{Ae_3} = (2 + 3i)\,\mathbf{e_1} + 2\mathbf{e_2} + 3i\mathbf{e_3} = \begin{pmatrix} 2 + 3i \\ 2 \\ 3i \end{pmatrix}$ As $\mathbf{Ay}$ can not be written as $c\mathbf{y}$ where $c \in \mathbb{C}$, $\mathbf{y}$ is not an eigen vector which is a contradiction.

TABLE 2: Examples