

Assignment 9

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Abstract—This document is about positive definite properties of real symmetric non-singular matrix.

Download all python codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

1 PROBLEM

For every 4×4 real symmetric non-singular matrix \mathbf{A} , prove if there exists a positive integer p such that

- 1) $p\mathbf{I} + \mathbf{A}$ is positive definite
- 2) \mathbf{A}^p is positive definite
- 3) \mathbf{A}^{-p} is positive definite
- 4) $\exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

2 CONSTRUCTION

Definition	An $n \times n$ symmetric real matrix \mathbf{M} is said to be positive definite if $\mathbf{x}^T \mathbf{M} \mathbf{x} > 0$ for all non-zero \mathbf{x} in \mathbb{R}^n
Properties	An $n \times n$ symmetric real matrix always have real eigen vectors and the set of eigen vectors can be selected such that they form an orthonormal basis $\mathbf{M} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T$ and $\mathbf{Q} \mathbf{Q}^T = \mathbf{I}$
Implication	An $n \times n$ symmetric real matrix \mathbf{M} is said to be positive definite if <ol style="list-style-type: none">i) Eigen values of \mathbf{M} are all positiveii) The pivot elements of \mathbf{M} are all positive

TABLE I: Properties of positive definite matrix

Proof for 1	$p\mathbf{I} + \mathbf{A} = \mathbf{Q}p\mathbf{I}\mathbf{Q}^T + \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T = \mathbf{Q}(p\mathbf{I} + \mathbf{\Lambda})\mathbf{Q}^T$ <p>if the eigen values of \mathbf{A} are $\lambda_1, \lambda_2, \dots, \lambda_4$, then the eigen values of $p\mathbf{I} + \mathbf{A}$ will be $\lambda_1 + p, \lambda_2 + p, \dots, \lambda_4 + p$. If we choose p to be $> \min(\lambda_1, \lambda_2, \dots, \lambda_4)$ then $p\mathbf{I} + \mathbf{A}$ will be positive definite.</p>
Proof for 2	$\mathbf{A}^p = (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T)^p = \mathbf{Q}\mathbf{\Lambda}^p\mathbf{Q}^T$ <p>if the eigen values of \mathbf{A} are $\lambda_1, \lambda_2, \dots, \lambda_4$, then the eigen values of \mathbf{A}^p will be $\lambda_1^p, \lambda_2^p, \dots, \lambda_4^p$. If p is even then the eigen values are positive.</p>
Proof for 3	$\mathbf{A}^{-1} = (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T)^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^T$ <p>if the eigen values of \mathbf{A} are $\lambda_1, \lambda_2, \dots, \lambda_4 \neq 0$ (For inverse to exist), then the eigen values of \mathbf{A}^{-p} will be $\frac{1}{\lambda_1^p}, \frac{1}{\lambda_2^p}, \dots, \frac{1}{\lambda_4^p}$. If p is even then the eigen values are positive.</p>
Proof for 4	$\exp(p\mathbf{A}) - \mathbf{I} = \sum_{k=0}^{\infty} \frac{1}{k!} (p\mathbf{A})^k - \mathbf{I} = p\mathbf{A} + \frac{1}{2!} (p\mathbf{A})^2 + \frac{1}{3!} (p\mathbf{A})^3 + \dots$ $= \mathbf{Q} \left(p\mathbf{\Lambda} + \frac{1}{2!} (p\mathbf{\Lambda})^2 + \frac{1}{3!} (p\mathbf{\Lambda})^3 + \dots \right) \mathbf{Q}^T$ <p>if the eigen values of \mathbf{A} are $\lambda_1, \lambda_2, \dots, \lambda_4$, then the eigen values of $\exp(p\mathbf{A}) - \mathbf{I}$ will be $e^{p\lambda_1} - 1, e^{p\lambda_2} - 1, \dots, e^{p\lambda_4} - 1$. So if the eigen value of \mathbf{A} is negative then the corresponding eigen value of $\exp(p\mathbf{A}) - \mathbf{I}$ is also negative for every positive integer p. So such a positive integer p does not exist.</p>

TABLE II: PROOF