Vandermonde matrix

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***Abstract*—This document calculates the determinant of a vandermonde matrix.**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Derive an expression for the determinant of a Vandermonde matrix

$$\mathbf{V} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} \tag{1.0.1}$$

2 construction

The given matrix is of the order $n \times n$.The determinant is given as

$$\det \mathbf{V} = \det \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} \tag{2.0.1}$$

By doing Row operations we can reduce it to the form

$$\det \mathbf{V} = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} \xleftrightarrow[i\neq 1]{R_i=R_i-R_1} \tag{2.0.2}$$

$$= \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 0 & \alpha_2-\alpha_1 & \alpha_2^2-\alpha_1^2 & \cdots & \alpha_2^{n-1}-\alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_n-\alpha_1 & \alpha_n^2-\alpha_1^2 & \cdots & \alpha_n^{n-1}-\alpha_1^{n-1} \end{vmatrix} \tag{2.0.3}$$

By doing column operations we can reduce it to the form

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 0 & \alpha_2-\alpha_1 & \alpha_2^2-\alpha_1^2 & \cdots & \alpha_2^{n-1}-\alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_n-\alpha_1 & \alpha_n^2-\alpha_1^2 & \cdots & \alpha_n^{n-1}-\alpha_1^{n-1} \end{vmatrix} \xleftrightarrow[i>1]{C_i=C_i-C_{i-1}} \tag{2.0.4}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2-\alpha_1 & (\alpha_2-\alpha_1)\alpha_2 & \cdots & (\alpha_2-\alpha_1)\alpha_2^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_n-\alpha_1 & (\alpha_n-\alpha_1)\alpha_n & \cdots & (\alpha_n-\alpha_1)\alpha_n^{n-2} \end{vmatrix} \tag{2.0.5}$$

$$= \prod_{n\geq j>1} (\alpha_j-\alpha_1) \begin{vmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} \end{vmatrix} \tag{2.0.6}$$

3 explanation

By using equation (2.0.6) the determinant can be reduced to $(n-1) \times (n-1)$, which is also of the form vandermonde matrix.The determinant of the reduced vandermonde matrix will be

$$\begin{vmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} \end{vmatrix} \tag{3.0.1}$$

$$= \prod_{n\geq j>2} (\alpha_j-\alpha_2) \begin{vmatrix} 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-3} \\ 1 & \alpha_4 & \alpha_4^2 & \cdots & \alpha_4^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-3} \end{vmatrix} \tag{3.0.2}$$

So in each calculation the dimension is reduced.Continuing the similar reduction, the determinant of the matrix $\mathbf{V}$,we can write it as

$$\det \mathbf{V} \tag{3.0.3}$$

$$= \prod_{1<j\leq n} (\alpha_j-\alpha_1) \prod_{2<j\leq n} (\alpha_j-\alpha_2) \cdots \prod_{(n-1)<j\leq n} (\alpha_j-\alpha_{n-1}) \tag{3.0.4}$$

$$= \prod_{(1\leq i<n)} \prod_{(i<j\leq n)} (\alpha_j-\alpha_i) \tag{3.0.5}$$