Challenge 6

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document is to prove that convolution is a unique map.**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
```

and latex-tikz codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609
```

1 Problem

Given two signals $(x_0, \ldots, x_n 1)$ and $(h_0, \ldots, h_m 1)$, the (linear) convolution of the two is an m+n1-length signal defined as

$$y(t) = (h * x)_t = \sum_{\tau=0}^{\tau=n-1} x_\tau h_{(t-\tau)} \tag{1.0.1}$$
$$0 \leq t < m+n-1$$

If

$$y(n) = (h_1 * x) \tag{1.0.2}$$
$$y(n) = (h_2 * x) \tag{1.0.3}$$

then prove that $h_1 = h_2$

2 Construction

Writing the convolution operation in matrix form

$$Y = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & n_{n-3} & . & . & h_1 & h_0 \\ . & . & . & . & . & . & . \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & . & . & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & . & . & h_{m-n+3} & h_{m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2. \\ . \\ . \\ x_{n-1} \end{pmatrix} \tag{2.0.1}$$

3 Explanation

Therefore we can write equation (1.0.1) in matrix form as $\mathbf{Y} = \mathbf{HX}$ where

$$Y = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & n_{n-3} & . & . & h_1 & h_0 \\ . & . & . & . & . & . & . \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & . & . & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & . & . & h_{m-n+3} & h_{m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2. \\ . \\ . \\ x_{n-1} \end{pmatrix} \tag{3.0.1}$$

So the equations (1.0.2) and (1.0.3) can be written in matrix form where

$$\mathbf{H_1} = \begin{pmatrix} h_{10} & 0 & 0 & . & . & 0 & 0 \\ h_{11} & h_{10} & 0 & . & . & 0 & 0 \\ h_{12} & h_{11} & h_{10} & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{1n-1} & h_{1n-2} & n_{1n-3} & . & . & h_{11} & h_{10} \\ . & . & . & . & . & . & . \\ h_{1m-1} & h_{1m-2} & h_{1m-3} & . & . & h_{1m-n+1} & h_{1m-n} \\ 0 & h_{1m-1} & h_{1m-2} & . & . & h_{1m-n+2} & h_{1m-n+1} \\ 0 & 0 & h_{1m-1} & . & . & h_{1m-n+3} & h_{1m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{1m-1} \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{H_1} = \begin{pmatrix} h_{20} & 0 & 0 & . & . & 0 & 0 \\ h_{21} & h_{20} & 0 & . & . & 0 & 0 \\ h_{22} & h_{21} & h_{20} & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{2n-1} & h_{2n-2} & n_{2n-3} & . & . & h_{21} & h_{20} \\ . & . & . & . & . & . & . \\ h_{2m-1} & h_{2m-2} & h_{2m-3} & . & . & h_{2m-n+1} & h_{2m-n} \\ 0 & h_{2m-1} & h_{2m-2} & . & . & h_{2m-n+2} & h_{2m-n+1} \\ 0 & 0 & h_{2m-1} & . & . & h_{2m-n+3} & h_{2m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{2m-1} \end{pmatrix} \tag{3.0.3}$$

So

$$(h_1 * x) - (h_2 * x) = y(n) - y(n) = 0 \tag{3.0.4}$$
$$(\mathbf{H_1} - \mathbf{H_2})\mathbf{X} = 0 \tag{3.0.5}$$

For the sake of simplicity lets assume that $m = n$,then Toeplitz matrix is of the form

$$\mathbf{H} = \begin{pmatrix} L \\ U \end{pmatrix} \tag{3.0.6}$$

Because $\mathbf{H_1}, \mathbf{H_2}$ are in toeplitz form,their difference is also in toeplitz form.

$$\mathbf{H} = \mathbf{H_1} - \mathbf{H_2} \tag{3.0.7}$$

$$\mathbf{H} = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{m-2} & h_{m-3} & . & . & h_1 & h_0 & 0 \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_1 & h_0 \\ 0 & h_{m-1} & h_{m-2} & . & . & h_2 & h_1 \\ 0 & 0 & h_{m-1} & . & . & h_3 & h_2 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \tag{3.0.8}$$

Suppose

$$L = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_1 & h_0 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_1 & h_0 \end{pmatrix} \tag{3.0.9}$$

$$LX = 0 \tag{3.0.10}$$

For $LX = 0$ to have a non-trivial solution i.e nullity$\neq 0$, the rank of the matrix $R(L) < n$, which implies $h_0 = 0$.So,rank(L)=0. Similarly if you consider the submatrix of L

$$L = \begin{pmatrix} h_1 & 0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & 0 & . & . & 0 & 0 \\ h_3 & h_2 & h_1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_2 & h_2 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_2 & h_1 \end{pmatrix} \tag{3.0.11}$$

where

$$\begin{pmatrix} h_1 & 0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & 0 & . & . & 0 & 0 \\ h_3 & h_2 & h_1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_2 & h_2 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_2 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2. \\ . \\ . \\ x_{n-1} \end{pmatrix} = 0 \tag{3.0.12}$$

By using the similar logic,$h_1 = 0$. And all the elements of the lower triangular matrix are zero i.e $L = 0$ which implies $H = 0$.

The solution for $LX = 0$ or $UX = 0$ when $X \neq 0$, is that each of the elements of the matrices L, U is zero.So,$H = 0$, $\mathbf{H_1} - \mathbf{H_2} = 0$ which means $\mathbf{H_1} = \mathbf{H_2}$