Assignment 12

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Abstract

This document is about the linear operator and minimal polynomials.

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Let $\mathbf{T}$ be a linear operator on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \tag{1.0.1}$$

1) Prove that the only subspaces of $\mathbb{R}^2$ invariant under $\mathbf{T}$ are $\mathbb{R}^2$ and the zero subspace.
2) If $\mathbf{U}$ is the linear operator on $\mathbb{C}^2$, the matrix of which in the standard ordered basis is $\mathbf{A}$, show that $\mathbf{U}$ has 1-dimensional invariant subspaces.

2 Proof

Definition Of linear operator	If $\mathbf{V}$ is a vector space over a field $\mathbb{F}$, a linear operator on $\mathbf{V}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{V}$. So for two vectors α and β in $\mathbf{V}$, the transformation will be $\mathbf{T}(c\alpha + \beta) = c(\mathbf{T}\alpha) + \mathbf{T}\beta$ where $\mathbf{T}\alpha$ and $\mathbf{T}\beta$ are in $\mathbf{V}$ and c in $\mathbb{F}$.
Invariant subspaces	If $\mathbf{W}$ is a subspace of $\mathbf{V}$, we say that $\mathbf{W}$ is invariant under $\mathbf{T}$ if for each vector α in $\mathbf{W}$ the vector $\mathbf{T}\alpha$ is in $\mathbf{W}$, i.e, if $\mathbf{T}(\mathbf{W})$ is contained in $\mathbf{W}$.
Given	$\mathbf{T}$ is a linear operator in $\mathbb{R}^2$, the matrix of which in the standard basis is $$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$ So by definition $\mathbb{R}^2$ is invariant under $\mathbf{T}$, since $\mathbf{T}$ is a linear operator.

TABLE 1: construction

Proof for 1	*i*) The null space of $\mathbf{T}$ can be calculated as $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, by augmenting we get $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix}$ applying row reduction we get $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 0 \end{pmatrix} \xleftrightarrow{R_1=R_1+\frac{R_2}{4}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Therefore the nullspace of $\mathbf{T}$ contains only the zero vector. *ii*) Assume that there is a 1-dimensional subspace that is invariant under $\mathbf{T}$,then $\mathbf{Ax} = c\mathbf{x} \implies \mathbf{x}$ is the eigen vector and c is eigen value of $\mathbf{A}$. $$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 4.$$ $\lambda_1 = \frac{3+\sqrt{7}i}{2}$ and $\lambda_2 = \frac{3-\sqrt{7}i}{2}$ are the eigenvalues of $\mathbf{A}$. Since the field of vector space is $\mathbb{R}$, there are no eigen values and hence no eigen vectors Since there are no eigen values in the field $\mathbb{R}$, there are no 1-dimensional vectors which can be invariant under $\mathbf{T}$. *iii*)If $\mathbf{T}(\mathbf{v}) = \mathbf{T}(\mathbf{u}) \implies \mathbf{T}(\mathbf{v}-\mathbf{u}) = 0 \implies \mathbf{v}-\mathbf{u} = 0 \implies \mathbf{v} = \mathbf{u}$ since the nullspace of $\mathbf{A}$ consists of only the zero vector. So the linear operator is one-to-one. Since the range of the linear operator is $\mathbb{R}^2$, the vector space $\mathbb{R}^2$ is invariant under $\mathbf{T}$. And as the nullspace maps to zero vector and zero vector is invariant under $\mathbf{T}$, the only subspaces that are invariant under $\mathbf{T}$ are the vector space $\mathbb{R}$ and the zero vector.
Proof for 2	If $\mathbf{U}$ is the linear operator on $\mathbb{C}^2$, the matrix of which in the standard ordered basis is $\mathbf{A}$ The eigen vectors will be, for $\lambda_1 = \frac{3+\sqrt{7}i}{2}$, the nullspace of $\mathbf{A} - \lambda\mathbf{I}$ will be the eigen vector. $\begin{pmatrix} \frac{-1-\sqrt{7}i}{2} & -1 & 0 \\ 2 & \frac{1-\sqrt{7}i}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2=R_2-\frac{(-1+\sqrt{7}i)R_1}{2}} \begin{pmatrix} \frac{-1-\sqrt{7}i}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

	Therefore one of the eigen vectors is $\mathbf{e_1} = \begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix}$ This eigen vector $\mathbf{e_1}$ is a subspace of $\mathbb{C}^2$. Applying the linear operator $\mathbf{U}$ on $\mathbf{e_1}$, we get $\mathbf{U}\left(\mathbf{ce_1}\right) = \mathbf{Ae_1} = c\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix} = c\frac{3+\sqrt{7}i}{2}\begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix} \implies \mathbf{U}\left(\mathbf{ce_1}\right) = c^{'}\mathbf{e_1}$ Therefore the vector space $\mathbb{C}^2$ has 1-dimensional invariant subspaces which are the two subspaces along each eigen vectors containing the zero vector.

TABLE 2: Proof