Challenge 1

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AI20MTECH11001

***Abstract*—This document explains how to find points on two skew lines where the distance is shortest.**

Download all python codes from

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https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan-IITH/IITH-EE5609
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1 Problem

Find the points on two skew lines where the distance between the lines is shortest

$$L_1: \mathbf{x} = x_1 + \lambda_1 \mathbf{v_1} \tag{1}$$

$$L_2: \mathbf{x} = x_2 + \lambda_2 \mathbf{v_2} \tag{2}$$

2 Construction

Let $\mathbf{a}$,$\mathbf{b}$ be two points on the lines L_1,L_2 respectively,where the distance $\|\mathbf{a}-\mathbf{b}\|$ is shortest. The vector along the line $(\mathbf{a}-\mathbf{b})$ will be parallel to $\mathbf{v_1} \times \mathbf{v_2}$

$$\mathbf{a} = x_1 + \lambda_1 \mathbf{v_1} \tag{3}$$

$$\mathbf{b} = x_2 + \lambda_2 \mathbf{v_2} \tag{4}$$

The vector along the line $\mathbf{ab}$ will be

$$\mathbf{ab} = x_1 + \lambda_1 \mathbf{v_1} - x_1 - \lambda_1 \mathbf{v_1}$$

$$\mathbf{ab} = \begin{pmatrix} x_1 & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \begin{pmatrix} x_2 & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{5}$$

3 Explanation

The vectors $\mathbf{v_1}$,$\mathbf{v_2}$ are both perpendicular to the line $\mathbf{ab}$.So the dot product of $\mathbf{v_1}$,$\mathbf{v_2}$ with the line $\mathbf{ab}$ is zero.
The dot product of $\mathbf{v_1}$ with the line $\mathbf{ab}$ is

$$\mathbf{v_1}^T \begin{pmatrix} x_1 & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \mathbf{v_1}^T \begin{pmatrix} x_2 & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = 0 \tag{6}$$

The dot product of $\mathbf{v_2}$ with the line $\mathbf{ab}$ is

$$\mathbf{v_2}^T \begin{pmatrix} x_1 & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \mathbf{v_2}^T \begin{pmatrix} x_2 & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = 0 \tag{7}$$

Rearranging the equations (5) and (6) in matrix form we get

$$\begin{pmatrix} \mathbf{v_1}^T x_1 & \mathbf{v_1}^T \mathbf{v_1} & -\mathbf{v_1}^T x_2 & -\mathbf{v_1}^T \mathbf{v_2} \\ \mathbf{v_2}^T x_1 & \mathbf{v_2}^T \mathbf{v_1} & -\mathbf{v_2}^T x_2 & -\mathbf{v_2}^T \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \\ 1 \\ \lambda_2 \end{pmatrix} = 0 \tag{8}$$

simplifying it further

$$\begin{pmatrix} \mathbf{v_1}^T \mathbf{v_1} & -\mathbf{v_1}^T \mathbf{v_2} \\ \mathbf{v_2}^T \mathbf{v_1} & -\mathbf{v_2}^T \mathbf{v_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v_1}^T(\mathbf{x_2} - \mathbf{x_1}) \\ \mathbf{v_2}^T(\mathbf{x_2} - \mathbf{x_1}) \end{pmatrix} \tag{9}$$

Solving the equation (9) we get the values of λ_1 and λ_2.Substituting the values of λ_1 and λ_2 in the equations (3) and (4) we get the values of the point $\mathbf{a}$ and $\mathbf{b}$.

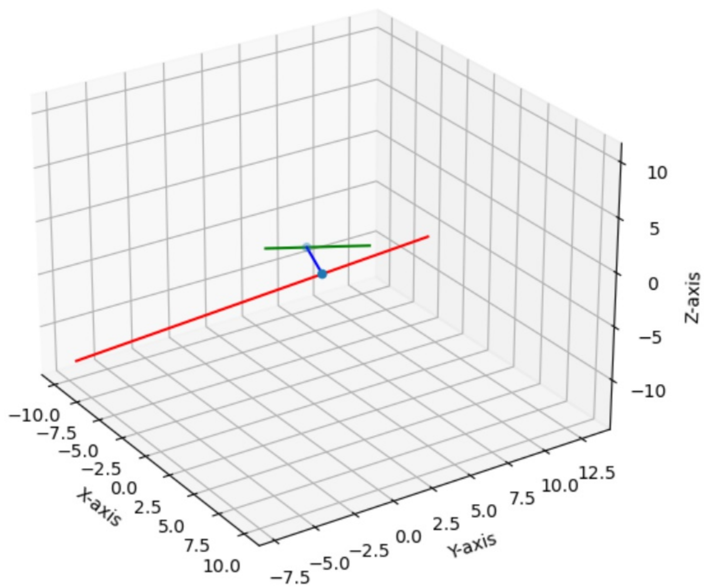
4 Example

Find the points where the distance is shortest between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{10}$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{11}$$

Using the above equation (9) to solve we get the points as $\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix}$ and $\frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}$ for the lines represented by equations (10) and (11) as shown in the figure below


10
5
0
−5
−10
Z-axis
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
X-axis
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
Y-axis