Assignment 15

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Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Let f be a non-zero symmetric bilinear form on $\mathbb{R}^3$.Suppose that there exist linear transformations $T_i : \mathbb{R}^3 \to \mathbb{R}, i = 1, 2$ such that for all $\alpha, \beta \in \mathbb{R}^3$, $f(\alpha, \beta) = T_1(\alpha) T_2(\beta)$. Then

1) rank $f = 1$
2) dim $\{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0 \text{ for all } \alpha \in \mathbb{R}^3\} = 2$
3) f is positive semi-definite or negative semi-definite
4) $\{\alpha : f(\alpha, \alpha) = 0\}$ is a linear subspace of dimension 2

2 Construction

Definition of bilinear form	A bilinear form on a vector space $\mathbf{V}$ is a function f, which assigns to each ordered pair of vectors α, β in $\mathbf{V}$ a scalar $f(\alpha, \beta)$ in field $\mathbf{F}$ which satisfies $$i)\ f(c\alpha_1 + \alpha_2, \beta) = cf(\alpha_1, \beta) + f(\alpha_2, \beta)$$ $$ii)\ f(\alpha, c\beta_1 + \beta_2) = cf(\alpha, \beta_1) + f(\alpha, \beta_2)$$
Symmetric bilinear form	A bilinear form on the vector space $\mathbf{V}$ is symmetric if $$f(\alpha, \beta) = f(\beta, \alpha)$$ for all vectors $\alpha, \beta \in \mathbf{V}$
Matrix of bilinear form	Let $\alpha, \beta \in \mathbb{R}^3$ be two vectors, which are represented in standard basis as $\alpha = \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2} + \alpha_3\mathbf{e_3}$ and $\beta = \beta_1\mathbf{e_1} + \beta_2\mathbf{e_2} + \beta_3\mathbf{e_3}$,therefore $f(\alpha, \beta)$ can be represented in matrix form as $$f(\alpha, \beta) = f(\alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2} + \alpha_3\mathbf{e_3}, \beta_1\mathbf{e_1} + \beta_2\mathbf{e_2} + \beta_3\mathbf{e_3})$$ $$= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} f(\mathbf{e_1}, \mathbf{e_1}) & f(\mathbf{e_1}, \mathbf{e_2}) & f(\mathbf{e_1}, \mathbf{e_3}) \\ f(\mathbf{e_2}, \mathbf{e_1}) & f(\mathbf{e_2}, \mathbf{e_2}) & f(\mathbf{e_2}, \mathbf{e_3}) \\ f(\mathbf{e_3}, \mathbf{e_1}) & f(\mathbf{e_3}, \mathbf{e_2}) & f(\mathbf{e_3}, \mathbf{e_3}) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

<table>
<tr><td>Given</td><td>Given a non-zero symmetric bilinear form f such that $f(\alpha,\beta) = T_1(\alpha)\,T_2(\beta)$ where $\alpha,\beta \in \mathbb{R}^3$. So the symmetric bilinear form can be represented on matrix form as
$$f(\alpha,\beta) = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix} f(\mathbf{e_1},\mathbf{e_1}) & f(\mathbf{e_1},\mathbf{e_2}) & f(\mathbf{e_1},\mathbf{e_3}) \\ f(\mathbf{e_2},\mathbf{e_1}) & f(\mathbf{e_2},\mathbf{e_2}) & f(\mathbf{e_2},\mathbf{e_3}) \\ f(\mathbf{e_3},\mathbf{e_1}) & f(\mathbf{e_3},\mathbf{e_2}) & f(\mathbf{e_3},\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$$
$$f(\alpha,\beta) = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix} T_1(\mathbf{e_1})T_2(\mathbf{e_1}) & T_1(\mathbf{e_1})T_2(\mathbf{e_2}) & T_1(\mathbf{e_1})T_2(\mathbf{e_3}) \\ T_1(\mathbf{e_2})T_2(\mathbf{e_1}) & T_1(\mathbf{e_2})T_2(\mathbf{e_2}) & T_1(\mathbf{e_2})T_2(\mathbf{e_3}) \\ T_1(\mathbf{e_3})T_2(\mathbf{e_1}) & T_1(\mathbf{e_3})T_2(\mathbf{e_2}) & T_1(\mathbf{e_3})T_2(\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$$
$$f(\alpha,\beta) = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix} T_1(\mathbf{e_1}) \\ T_1(\mathbf{e_2}) \\ T_1(\mathbf{e_3})\end{pmatrix}\begin{pmatrix} T_2(\mathbf{e_1}) & T_2(\mathbf{e_2}) & T_2(\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \alpha^T\mathbf{T_1}\mathbf{T_2}^T\beta$$
where $\mathbf{T_1} = \begin{pmatrix} T_1(\mathbf{e_1}) \\ T_1(\mathbf{e_2}) \\ T_1(\mathbf{e_3})\end{pmatrix}$ and $\mathbf{T_2} = \begin{pmatrix} T_2(\mathbf{e_1}) \\ T_2(\mathbf{e_2}) \\ T_2(\mathbf{e_3})\end{pmatrix}$ are the matrix representation of the linear transformations T_1, T_2.So, the matrix representation of f is $\mathbf{T_1}\mathbf{T_2}^T$ or $\mathbf{T_2}\mathbf{T_1}^T$ since f is symmetric.
note : Since f is non-zero symmetric bilinear form $rank(\mathbf{T_1}) = rank(\mathbf{T_2}) = 1$</td></tr>
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TABLE 1: Construction

3 Answer

Option 1	By using the property of rank of product of two matrices, we get $rank(f) = rank\left(\mathbf{T_1}\mathbf{T_2}^T\right) \leq min\left(rank(\mathbf{T_1}), rank(\mathbf{T_2})\right) \leq 1.$ Since f is non-zero the $rank(f) \neq 0$. Hence the $rank(f) = 1$
Option 2	$\beta \in \mathbb{R}^3 : f(\alpha,\beta) = 0$ for all $\alpha \in \mathbb{R}^3 \implies \beta \in \mathbb{R}^3 : T_2(\beta) = 0$ for all $\alpha \in \mathbb{R}^3$ because $T_1(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}^3$. By using rank nullity theorem

	$rank\{T_2\} + dim\{Nullspace\,(T_2)\} = 3 \implies dim\{Nullspace\,(T_2)\} = 2$. Similarly for T_1, we get dim{Nullspace(T_1)}=2. Therefore dim $\{\beta \in \mathbb{R}^3 : f\,(\alpha,\beta) = 0$ for all $\alpha \in \mathbb{R}^3\} = dim\{Nullspace\,(T_1)\} = dim\{Nullspace\,(T_2)\} = 2$
Option 3	By using rank nullity theorem we get $rank\,(f) + dim\{nullspace\,(f)\} = 3$. We know that $rank\,(f) = 1 \implies dim\{nullspace\,(f)\} = 2$. Therefore two eigen values of f will be 0. Since the matrix is a symmetric matrix the eigen values are real. So, the third eigen value can be either positive or negative. So, the matrix will be either positive semi-definite or negative semi-definite accordingly.This option is correct.
Option 4	$\{\alpha : f\,(\alpha,\alpha) = 0\}$ is a linear subspace of dimension 2. Since the $dim\{nullspace\,(f)\} = 2$, and f diagonalizable the two eigen vectors corresponding to 0 eigen values form a subspace of dimension 2.

TABLE 2: Answer

4 Example

Construction	Consider the non-zero symmetric bilinear form $f\,(\alpha,\beta) = T_1\,(\alpha)\,T_2\,(\beta)$ on $\mathbb{R}^3$ where Where the matrix of linear transformations are $\mathbf{T_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{T_2} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. The matrix of symmetric bilinear form is $f = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$.The $rank\,(f) = 1$. $f\,(\alpha,\beta) = \alpha^T \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} \beta$ The characteristic equation is $\lvert \mathbf{f} - \lambda \mathbf{I} \rvert = \lambda^2\,(\lambda - 4)$. So the eigen values are $0, 0, 4$

	Therefore f is positive semi-definite. $f(\alpha,\beta) = 0$ for all $\alpha \in \mathbb{R}^3$, then $\beta = xe_1 + ye_2$ where $e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.Therefore $\dim \{\beta \in \mathbb{R}^3 : f(\alpha,\beta) = 0 \text{ for all } \alpha \in \mathbb{R}^3\} = 2$ $\alpha : f(\alpha,\alpha) = 0$ also has a dimension of 2 which forms the nullspace of f, where nullspace of f is the $span\{e_1, e_2\}$

TABLE 3: Example