Assignment 7

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document is about finding the Qr decomposition of a matrix and finding the solution using singular value decomposition.**

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Given the equation of a parabola is

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{1.0.1}$$

Find the QR-decomposition of the matrix $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Find the vertex of the parabola using singular value decomposition.

2 Construction

The vertex of the parabola can be found by using

$$\begin{pmatrix} \mathbf{u}^T + \eta p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \eta p_1 - \mathbf{u} \end{pmatrix} \tag{2.0.1}$$

where

$$\eta = p_1^T \mathbf{u} \tag{2.0.2}$$

3 QR-decomposition

The QR decomposition of the matrix will be done by using Gram-schmidt process of finding an orthogonal basis for the column space of the matrix.The matrix $\mathbf{V}$ can be written as a product of two matrices as $\mathbf{V} = \mathbf{QR}$ where

$$\mathbf{Q} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{3.0.2}$$

using gram-schmidt process,let $\mathbf{V} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$, we get

$$u_1 = a_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{3.0.3}$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{20}} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{3.0.4}$$

$$u_2 = a_2 - \frac{u_1^T a_2}{\|u_1\|^2} u_1 \tag{3.0.5}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{3.0.6}$$

$$= 0 \tag{3.0.7}$$

$$e_2 = 0 \tag{3.0.8}$$

The elements of the matrix $\mathbf{R}$ can be found by taking the projection of e_1, e_2 on to a_1, a_2.

$$r_1 = e_1^T a_1 = \frac{1}{\sqrt{20}} \begin{pmatrix} 4 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \sqrt{20} \tag{3.0.9}$$

$$r_2 = e_1^T a_2 = \frac{1}{\sqrt{20}} \begin{pmatrix} 4 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -\sqrt{5} \tag{3.0.10}$$

$$r_3 = e_2^T a_2 = 0 \tag{3.0.11}$$

Therefore the QR decomposition will be

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{20}} & 0 \\ -\frac{2}{\sqrt{20}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{20} & -\sqrt{5} \\ 0 & 0 \end{pmatrix} \tag{3.0.12}$$

Since the column vector $e_2 = 0$,we can write the QR decomposition as

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{20}} \\ -\frac{2}{\sqrt{20}} \end{pmatrix} \begin{pmatrix} \sqrt{20} & -\sqrt{5} \end{pmatrix} \tag{3.0.13}$$

4 Singular value decomposition

The characteristic equation of the matrix $\mathbf{V} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ is

$$\det(\mathbf{V} - \lambda \mathbf{I}) = \det \begin{pmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{pmatrix} \tag{4.0.1}$$

$$= \lambda^2 - 5\lambda = 0 \tag{4.0.2}$$

$$\lambda_1 = 0, \lambda_2 = 5 \tag{4.0.3}$$

The eigen vector corresponding to λ_1 is in the nullspace of $\mathbf{V} - 0\mathbf{I}$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + \frac{R_1}{2}} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \tag{4.0.4}$$

$$p_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.0.5}$$

The eigen vector corresponding to λ_2 is is in the nullspace of $\mathbf{V} - 5\mathbf{I}$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \tag{4.0.6}$$

$$p_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{4.0.7}$$

Diagonalising the matrix $\mathbf{V}$ using $\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix}$ we get

$$\mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \tag{4.0.8}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \tag{4.0.9}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \tag{4.0.10}$$

The standard equation of the parabola is

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \tag{4.0.11}$$

Where η can be calculated as

$$\eta = p_1^T \mathbf{u} \tag{4.0.12}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{4.0.13}$$

$$= 0 \tag{4.0.14}$$

The vertex of the parabola $\mathbf{c}$ can be calculated from

$$\begin{pmatrix} \mathbf{u}^T + 2\eta p_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ 2\eta p_1 - \mathbf{u} \end{pmatrix} \tag{4.0.15}$$

$$\begin{pmatrix} \mathbf{u}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ -\mathbf{u} \end{pmatrix} \tag{4.0.16}$$

$$\begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} \tag{4.0.17}$$

This equation (4.0.17) is of the form $\mathbf{A}\mathbf{c} = \mathbf{b}$,this can be calculated by using the singular value decomposition of $\mathbf{A}$,where

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{4.0.18}$$

The eigen vectors of $\mathbf{A}^T\mathbf{A}$ are columns of $\mathbf{V}$ and the eigen vectors of $\mathbf{A}\mathbf{A}^T$ are columns of $\mathbf{U}$ and the matrix $\mathbf{S}$ is a diagonal matrix with entries as the singular values of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{c} = \mathbf{b} \tag{4.0.19}$$

$$\mathbf{c} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{4.0.20}$$

where $\mathbf{A}_+ = \mathbf{V}\mathbf{S}_+\mathbf{U}^T$ is the moore-penrose pseudo-inverse of $\mathbf{S}$. Calculating $\mathbf{A}\mathbf{A}^T$,we get

$$= \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix}^T \tag{4.0.21}$$

$$= \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 4 & -2 \\ -3 & -2 & 1 \end{pmatrix} \tag{4.0.22}$$

$$= \begin{pmatrix} 45 & 30 & -15 \\ 30 & 20 & -10 \\ -15 & -10 & 5 \end{pmatrix} \tag{4.0.23}$$

The eigen values and eigen vectors for $\mathbf{A}\mathbf{A}^T$ are

$$\det\left(\mathbf{A}\mathbf{A}^T - \lambda\mathbf{I}\right) = 0 \tag{4.0.24}$$

$$\begin{vmatrix} 45 - \lambda & 30 & -15 \\ 30 & 20 - \lambda & -10 \\ -15 & -10 & 5 - \lambda \end{vmatrix} = 0 \tag{4.0.25}$$

$$\lambda^3 - 70\lambda^2 = 0 \tag{4.0.26}$$

$$\lambda_1 = 70, \lambda_2 = 0 \tag{4.0.27}$$

The eigen vectors corresponding to the eigen values in the normalized form are

$$\mathbf{u}_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \tag{4.0.28}$$

$$\mathbf{u}_2 = \frac{3}{\sqrt{13}} \begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} \tag{4.0.29}$$

$$\mathbf{u}_3 = \frac{3}{\sqrt{10}} \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \tag{4.0.30}$$

Thus we get

$$\mathbf{U} = \begin{pmatrix} -\frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{13}} & \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & 0 \\ \frac{1}{\sqrt{14}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{4.0.31}$$

The matrix $\mathbf{S}$ corresponding to the singular values is

$$\mathbf{S} = \begin{pmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.0.32}$$

Calculating $\mathbf{A}^T\mathbf{A}$,we get

$$= \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix}^T \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \tag{4.0.33}$$

$$= \begin{pmatrix} 6 & 4 & -2 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \tag{4.0.34}$$

$$= \begin{pmatrix} 56 & -28 \\ -28 & 14 \end{pmatrix} \tag{4.0.35}$$

The eigen values and eigen vectors for $\mathbf{A}^T\mathbf{A}$ are

$$\det\left(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}\right) = 0 \tag{4.0.36}$$

$$\begin{vmatrix} 56-\lambda & -28 \\ -28 & 14-\lambda \end{vmatrix} \tag{4.0.37}$$

$$\lambda^2 - 70\lambda = 0 \tag{4.0.38}$$

$$\lambda_1 = 70, \lambda_2 = 0 \tag{4.0.39}$$

The eigen vectors corresponding to the eigen values in the normalized form are

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{4.0.40}$$

$$\mathbf{v}_2 = \frac{2}{\sqrt{5}} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{4.0.41}$$

The matrix $\mathbf{V}$ will be

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{4.0.42}$$

The moore-penrose pseudo inverse of the matrix $\mathbf{A}_+ = \mathbf{V}\mathbf{S}_+\mathbf{U}^T$,where $\mathbf{S}_+$ is obtained taking the reciprocal of the non-zero entries of the matrix $\mathbf{S}$ be the

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{70}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.43}$$

The moore-penrose pseudo inverse will be

$$\mathbf{A}_+ = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{70}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{13}} & \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & 0 \\ \frac{1}{\sqrt{14}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix}^T \tag{4.0.44}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{350}} & 0 & 0 \\ \frac{1}{\sqrt{350}} & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{4.0.45}$$

$$= \begin{pmatrix} \frac{3}{35} & \frac{2}{35} & -\frac{1}{35} \\ -\frac{3}{70} & -\frac{2}{70} & \frac{1}{70} \end{pmatrix} \tag{4.0.46}$$

The value of $\mathbf{c}$ can now be calculated as

$$\mathbf{c} = \mathbf{A}_+\mathbf{b} \tag{4.0.47}$$

$$= \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{4.0.48}$$

$$= \begin{pmatrix} \frac{3}{35} & \frac{2}{35} & -\frac{1}{35} \\ -\frac{3}{70} & -\frac{2}{70} & \frac{1}{70} \end{pmatrix} \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix} \tag{4.0.49}$$

$$= \begin{pmatrix} \frac{12}{35} \\ -\frac{6}{35} \end{pmatrix} \tag{4.0.50}$$