conjugate hyperbola

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***Abstract*—This document is about matrix representation of lines and the bisectors of angles between them.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 PROBLEM

Find the conjugate hyperbola to

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.0.1}$$

where

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.0.2}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{1.0.3}$$

2 CONSTRUCTION

The general quadratic equation of the form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

is a hyperbola when

$$\mathbf{V} < 0 \tag{2.0.2}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} \neq 0 \tag{2.0.3}$$

the center of the hyperbola will be c,where

$$c = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.4}$$

So,the matrix representation can be written as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} - 2c^T\mathbf{V}\mathbf{x} + f = 0 \tag{2.0.5}$$

3 AFFINE TRANSFORMATION

The normalized eigen vector are used along with the center of the hyperbola to perform the affine transformation which brings the hyperbola into standard form.The affine transformation will be

$$(\mathbf{P}\mathbf{y} + c)^T \mathbf{V} (\mathbf{P}\mathbf{y} + c) - 2c^T\mathbf{V}(\mathbf{P}\mathbf{y} + c) + f = 0 \tag{3.0.1}$$

Simplifying it further we get

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = c^T\mathbf{V}c - f \tag{3.0.2}$$

$$\frac{1}{c^T\mathbf{V}c - f}\left(\mathbf{y}^T\mathbf{D}\mathbf{y}\right) = 1 \tag{3.0.3}$$

where

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.0.4}$$

Here λ_1, λ_2 are the eigen values of $\mathbf{V}$.

4 EXPLANATION

The Standard equation of hyperbola is

$$\mathbf{y}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{y} = 1 \tag{4.0.1}$$

$$\frac{x}{a^2} - \frac{y}{b^2} = 1 \tag{4.0.2}$$

The conjugate hyperbola will be

$$-\frac{x}{a^2} + \frac{y}{b^2} = 1 \tag{4.0.3}$$

$$\mathbf{y}^T \begin{pmatrix} -\frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{y} = 1 \tag{4.0.4}$$

$$\mathbf{y}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{y} = -1 \tag{4.0.5}$$

Therefore the conjugate hyperbola for the hyperbola in equation (3.0.3) will be

$$\frac{1}{c^T\mathbf{V}c - f}\left(\mathbf{y}^T\mathbf{D}\mathbf{y}\right) = -1 \tag{4.0.6}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = c^T\mathbf{V}c - f \tag{4.0.7}$$

Doing the inverse affine transformation we get,$\mathbf{y} = \mathbf{P}^{-1}(\mathbf{x} - c)$

$$\left(\mathbf{P}^{-1}(\mathbf{x} - c)\right)^T \mathbf{D}\left(\mathbf{P}^{-1}(\mathbf{x} - c)\right) = f - c^T\mathbf{V}c \tag{4.0.8}$$

$$(\mathbf{x} - c)^T \mathbf{P}\mathbf{D}\mathbf{P}^T (\mathbf{x} - c) = f - c^T\mathbf{V}c \tag{4.0.9}$$

$$(\mathbf{x} - c)^T \mathbf{V}(\mathbf{x} - c) = f - c^T\mathbf{V}c \tag{4.0.10}$$

$$\mathbf{x}^T\mathbf{V}\mathbf{x} - 2c^T\mathbf{V}\mathbf{x} + c^T\mathbf{V}c = f - c^T\mathbf{V}c \tag{4.0.11}$$

$$\mathbf{x}^T\mathbf{V}\mathbf{x} - 2c^T\mathbf{V}\mathbf{x} + 2c^T\mathbf{V} - f = 0 \tag{4.0.12}$$

Therefore the conjugate hyperbola of the $\mathbf{x}^T\mathbf{V}\mathbf{x} - 2c^T\mathbf{V}\mathbf{x} + f = 0$ is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} - 2c^T\mathbf{V}\mathbf{x} + 2c^T\mathbf{V}c - f = 0 \tag{4.0.13}$$

5 Example

A hyperbola of the $\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$ is

$$\mathbf{x}^T\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 & 4 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{5.0.1}$$

The center of the hyperbola is at

$$c = -\mathbf{V}^{-1}\mathbf{u} \tag{5.0.2}$$

$$= -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & -\frac{8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.0.3}$$

$$= \begin{pmatrix} -\frac{1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{5.0.4}$$

The value of the constant will be

$$c^T\mathbf{V}c - f = \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} -\frac{1}{7} \\ \frac{3}{7} \end{pmatrix} + 2 \tag{5.0.5}$$

$$= \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2 \tag{5.0.6}$$

$$= -2 + 2 \tag{5.0.7}$$

$$= 0 \tag{5.0.8}$$

therefore the conjugate hyperbola is

$$\mathbf{x}^T\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 & 4 \end{pmatrix}\mathbf{x} = 0 \tag{5.0.9}$$

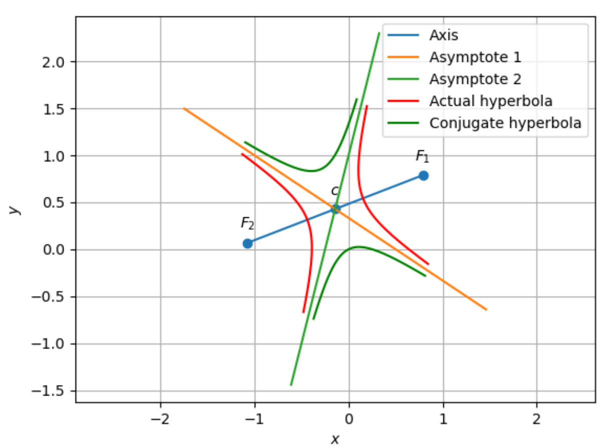

Fig. 1: Conjugate hyperbola