Assignment 2

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***Abstract*—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{1}$$

$$L_2: \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \tag{2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1: x-1 = 2-y = z-1 \tag{3}$$

$$L_2: \frac{x-2}{2} = y+1 = \frac{z+1}{2} \tag{4}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \tag{5}$$

$$\lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} - \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{6}$$

$$\begin{pmatrix}1 & -2\\-1 & -1\\1 & -2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{7}$$

The Augmented matrix will be

$$\tag{8}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{9}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix}0 & 0 & 3\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{10}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The shortest distance between the lines will be the projection of any line between the points on L_1,L_2 on to the unit vector which is perpendicular to both L_1,L_2.

The unit vector perpendicular to lines

$$Line_1: \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{b_1} \tag{11}$$

$$Line_2: \mathbf{x} = \mathbf{x_2} + \lambda_1 \mathbf{b_2} \tag{12}$$

can be found by calculating

$$\frac{\mathbf{b_1} \times \mathbf{b_2}}{\|\mathbf{b_1} \times \mathbf{b_2}\|} \tag{13}$$

In our question the value of $\mathbf{b_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\mathbf{b_2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$

So the unit vector perpendicular to both L_1 and L_2 is

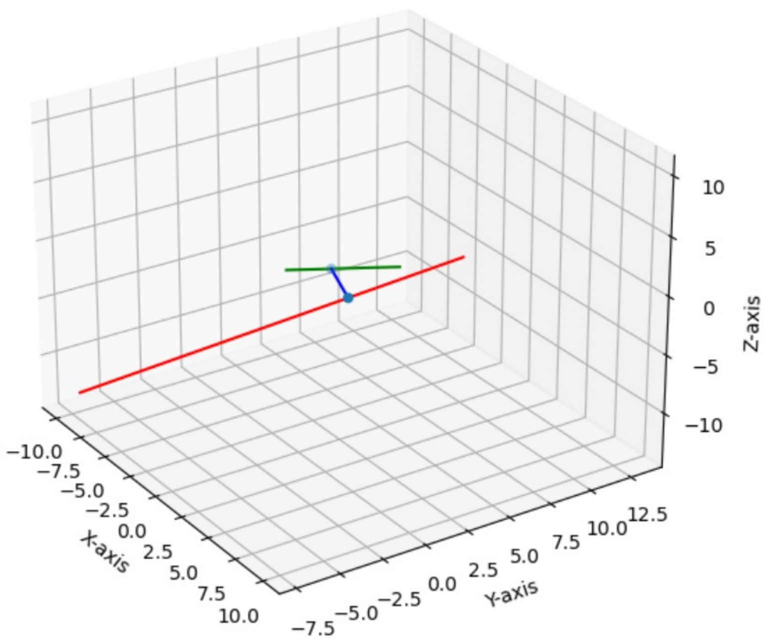
$$\mathbf{u} = \frac{\begin{pmatrix}1\\-1\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix}}{\|\begin{pmatrix}1\\-1\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix} \tag{14}$$

The points $\mathbf{A} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$ lie on the line L_1,L_2 respectively.

The shortest distance between the lines is the absolute value of projection of the vector $\vec{B} - \vec{A}$ on to the unit vector $\boldsymbol{u}$.

$$\|(\boldsymbol{B} - \boldsymbol{A})^T\boldsymbol{u}\| = \|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-3\\-2\end{pmatrix}^T\begin{pmatrix}-1\\0\\1\end{pmatrix}\| = \frac{3}{\sqrt{2}} \tag{15}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$



To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the following expression

$$\begin{pmatrix}\boldsymbol{b_1^T b_1} & -\boldsymbol{b_1^T b_2}\\ \boldsymbol{b_2^T b_1} & -\boldsymbol{b_2^T b_2}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{b_1^T(x_2 - x_1)}\\ \boldsymbol{b_2^T(x_2 - x_1)}\end{pmatrix} \tag{16}$$

we know that

$$\boldsymbol{x_1} = \begin{pmatrix}1\\2\\1\end{pmatrix}, \boldsymbol{x_2} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \boldsymbol{b_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix} and \boldsymbol{b_2} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

Using the above expression, we get the points as $\frac{1}{12}\begin{pmatrix}27\\-3\\27\end{pmatrix}$ and $\frac{1}{12}\begin{pmatrix}10\\-19\\-26\end{pmatrix}$ on the line L_1,L_2 respectively