Assignment 11

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***Abstract*—This document is about the linear operator and minimal polynomials.**

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Let $\mathbf{V}$ be the vector space of $n \times n$ matrices over field $\mathbf{F}$. Let $\mathbf{A}$ be a fixed $n \times n$ matrix. Let $\mathbf{T}$ be the linear operator on $\mathbf{V}$ defined by

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} \tag{1.0.1}$$

Show that the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.

2 Construction

Given	$\mathbf{A}$ is a fixed matrix from the vector space $\mathbf{V}$ of $n \times n$ matrices. A linear operator on the finite dimensional vector space $\mathbf{V}$, $\mathbf{T}$ is defined as $\mathbf{T}(\mathbf{B}) = \mathbf{AB}$.
Minimal polynomial	The minimal polynomial of a linear operator $\mathbf{T}$ is a monic polynomial which annihilates $\mathbf{T}$.
Matrix representation of $\mathbf{T}$	If we stack up the columns of the matrix $\mathbf{B}$,the linear operator $\mathbf{T}$ can be represented in the equivalent form as If $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & . & . & b_n \end{pmatrix}$,then the linear transformation of $\mathbf{B}$ will be $\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{A}b_1 & \mathbf{A}b_2 & . & .\mathbf{A}b_n \end{pmatrix}$ $\mathbf{M_T}(\mathbf{B}) = \begin{pmatrix} \mathbf{T}(b_1) \\ \mathbf{T}(b_2) \\ . \\ . \\ \mathbf{T}(b_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & & & \\ & \mathbf{A} & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$ here each element represents the elements of the matrix $\mathbf{AB}$

	$\mathbf{M_T} = \begin{pmatrix} \mathbf{A} & & & & \\ & \mathbf{A} & & \mathbf{O} & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & \mathbf{A} \end{pmatrix}$
Properties of minimal polynomial	The roots of the characteristic polynomial,eigen values and the minimal polynomial are same,except for multiplicities.The roots of the minimal polynomial of **A** are the roots of $\det(\mathbf{A} - \lambda\mathbf{I})$
The roots of minimal polynomial of **T**	The roots of the minimal polynomial of **T** are the roots of $\det(\mathbf{T} - \lambda\mathbf{I})$ $\det(\mathbf{T} - \lambda\mathbf{I}) = \begin{vmatrix} (\mathbf{A} - \lambda\mathbf{I}) & & & & \\ & (\mathbf{A} - \lambda\mathbf{I}) & & \mathbf{O} & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & (\mathbf{A} - \lambda\mathbf{I}) \end{vmatrix}$ $=(\det(\mathbf{A} - \lambda\mathbf{I}))^n$ Therfore we can see that the eigen values of **A** are also the eigen values of the linear operator **T**
Minimal polynomial of **T**	The minimal polynomial of **A** divides the characteristic polynomial of **A** and **T**. Let the minimal polynomial of **A** is of degree $p \leq n$ $f(x) = a_0 + a_1x + a_2x^2 ... a_px^p$ such that $f(\mathbf{A}) = 0$ $f(\mathbf{T}) = a_0\mathbf{I} + a_1\mathbf{T} + a_2\mathbf{T}^2 + .. + a_p\mathbf{T}^p$ $f(\mathbf{T}) = \begin{pmatrix} f(\mathbf{A}) & & & & \\ & f(\mathbf{A}) & & \mathbf{O} & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}_{n^2\times n^2}$ Therefore the minimal polynomial for **T** is the minimal polynomial for **A**.

Example	Consider $\mathbf{V}$ to be a vector space of 2×2 matrices and $\mathbf{A}=\begin{pmatrix}1&4\\0&2\end{pmatrix}$. So, the matrix of the linear operator $\mathbf{T}$ with respect to the basis $\mathbf{e_1}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \mathbf{e_2}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \mathbf{e_3}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \mathbf{e_4}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\mathbf{T}(\mathbf{e_1})=\begin{pmatrix}1&4\\0&2\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}=1.\mathbf{e_1}+0.\mathbf{e_2}+0.\mathbf{e_3}+0.\mathbf{e_4}$ $\mathbf{T}(\mathbf{e_2})=\begin{pmatrix}1&4\\0&2\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}4&0\\2&0\end{pmatrix}=4.\mathbf{e_1}+2.\mathbf{e_2}+0.\mathbf{e_3}+0.\mathbf{e_4}$ $\mathbf{T}(\mathbf{e_3})=\begin{pmatrix}1&4\\0&2\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}=0.\mathbf{e_1}+0.\mathbf{e_2}+1.\mathbf{e_3}+0.\mathbf{e_4}$ $\mathbf{T}(\mathbf{e_4})=\begin{pmatrix}1&4\\0&2\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&4\\0&2\end{pmatrix}=0.\mathbf{e_1}+0.\mathbf{e_2}+4.\mathbf{e_3}+2.\mathbf{e_4}$ So the matrix of the linear operator will be $\mathbf{M_T}=\begin{pmatrix}1&4&0&0\\0&2&0&0\\0&0&1&4\\0&0&0&2\end{pmatrix}=\begin{pmatrix}\mathbf{A}&\mathbf{O}\\\mathbf{O}&\mathbf{A}\end{pmatrix}$ The eigen values of $\mathbf{A}$ are $1,2$.So, the minimal polynomial is $f(x)=(x-1)(x-2)$ $f(\mathbf{M_T})=(\mathbf{T}-\mathbf{I})(\mathbf{T}-2\mathbf{I})=\begin{pmatrix}\mathbf{A}-\mathbf{I}&\mathbf{O}\\\mathbf{O}&\mathbf{A}-\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A}-2\mathbf{I}&\mathbf{O}\\\mathbf{O}&\mathbf{A}-2\mathbf{I}\end{pmatrix}$ $f(\mathbf{M_T})=\begin{pmatrix}(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I})&\mathbf{O}\\\mathbf{O}&(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I})\end{pmatrix}=\begin{pmatrix}f(\mathbf{A})&\mathbf{O}\\\mathbf{O}&f(\mathbf{A})\end{pmatrix}$ We know that the minimal polynomial $f(\mathbf{A})$ annihilates $\mathbf{A}$, i.e $f(\mathbf{A})=0$ We can see that $f(\mathbf{M_T})=\mathbf{O}$ The eigen values of $\mathbf{M_T}$ are roots of the characterstic equation $\det(\mathbf{T}-\lambda\mathbf{I})$ $\det(\mathbf{T}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda&4&0&0\\0&2-\lambda&0&0\\0&0&1-\lambda&4\\0&0&0&2-\lambda\end{vmatrix}=((1-\lambda)(2-\lambda))^2$ We know that eigen values of $\mathbf{T}$ should be roots of minimal polynomial of $\mathbf{T}$, thus minimal polynomial should be of the form $(x-1)^p(x-2)^q$ where $p,q\in\mathbb{N}..1\le p,q\le 2$ Therefore the minimal polynomial $f(\mathbf{A})$ of $\mathbf{A}$ annihilates $\mathbf{T}$, thus we can conclude that $f(x)$ is the minimal polynomial of linear operator $\mathbf{T}$