Assignment 10

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***Abstract*—This document is about inverse of the given matrices.**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
```

and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over the field F.Prove that if $(\mathbf{I}-\mathbf{AB})$ is invertibe then $(\mathbf{I}-\mathbf{BA})$ is invertible and

$$(\mathbf{I}-\mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I}-\mathbf{AB})^{-1}\mathbf{A} \tag{1.0.1}$$

2 Invertible

Given $(\mathbf{I}-\mathbf{AB})$ is invertible.This implies that the nullspace of $(\mathbf{I}-\mathbf{AB})$ contains only zero.

$$(\mathbf{I}-\mathbf{AB})\mathbf{x} = 0 \implies \mathbf{x} = 0 \tag{2.0.1}$$

Consider the matrix $(\mathbf{I}-\mathbf{BA})$ and $\mathbf{x}$ such that

$$(\mathbf{I}-\mathbf{BA})\mathbf{x} = 0 \tag{2.0.2}$$
$$\mathbf{x} = \mathbf{BAx} \tag{2.0.3}$$

Left multiplying with $\mathbf{A}$ on both sides of (2.0.3), we get

$$\mathbf{Ax} = \mathbf{ABAx} \tag{2.0.4}$$
$$(\mathbf{I}-\mathbf{AB})\mathbf{Ax} = 0 \implies \mathbf{Ax} = 0 \tag{2.0.5}$$

From equation (2.0.3)and (2.0.5),we get

$$\mathbf{x} = \mathbf{B}(\mathbf{Ax}) \tag{2.0.6}$$
$$\mathbf{x} = 0 \tag{2.0.7}$$

So the matrix $(\mathbf{I}-\mathbf{BA})$ is invertible.

3 Proof

Expanding $(\mathbf{I}-\mathbf{AB})^{-1}$, we get

$$(\mathbf{I}-\mathbf{AB})^{-1} = \mathbf{I} + (\mathbf{AB}) + (\mathbf{AB})^2 + (\mathbf{AB})^3 + \ldots \tag{3.0.1}$$

Right multiplying with $\mathbf{A}$ and left multiplying with $\mathbf{B}$ on both sides we get

$$\mathbf{B}(\mathbf{I}-\mathbf{AB})^{-1}\mathbf{A}$$
$$= \mathbf{BA} + \mathbf{B}(\mathbf{AB})\mathbf{A} + \mathbf{B}(\mathbf{AB})^2\mathbf{A} + \mathbf{B}(\mathbf{AB})^3\mathbf{A} + \ldots \tag{3.0.2}$$
$$= (\mathbf{BA}) + (\mathbf{BA})^2 + (\mathbf{BA})^3 + \ldots \tag{3.0.3}$$
$$= (\mathbf{I}-\mathbf{BA})^{-1} - \mathbf{I} \tag{3.0.4}$$

Therefore we can say that

$$(\mathbf{I}-\mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I}-\mathbf{AB})^{-1}\mathbf{A} \tag{3.0.5}$$