Assignment 2

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \tag{2}$$

2 construction

When two lines are not intersecting the distance between them is non-zero.The equation of above mentioned lines in symmetric form is

$$L_1 : x - 1 = 2 - y = z - 1 \tag{3}$$

$$L_2 : \frac{x-2}{2} = y + 1 = \frac{z+1}{2} \tag{4}$$

The above line equations have no point of intersection as for no value of λ_1, λ_2 both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \tag{5}$$

$$\lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} - \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{6}$$

$$\begin{pmatrix}1 & -2\\-1 & -1\\1 & -2\end{pmatrix} \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{7}$$

The Augmented matrix will be

$$\tag{8}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{9}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix}0 & 0 & 3\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{10}$$

The above matrix has a $rank = 3$ .Hence the lines do not intersect

3 solution

Let $\mathbf{A}$ be a point on line L_1 and $\mathbf{B}$ be point on the line L_2.Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through $\mathbf{A}$ and $\mathbf{B}$.

The vector passing through $\mathbf{A}$ and $\mathbf{B}$ will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2}\end{pmatrix} \begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} \tag{11}$$

The vectors $\mathbf{v_1}$,$\mathbf{v_2}$ are both perpendicular to the line $\mathbf{AB}$.So the dot product of $\mathbf{v_1}$,$\mathbf{v_2}$ with the line $\mathbf{AB}$ is zero.

The dot product of $\mathbf{v_1}$ with the line $\mathbf{AB}$ is

$$\mathbf{v_1^T}(\mathbf{A} - \mathbf{B}) = 0$$

$$\mathbf{v_1^T} \begin{pmatrix}x_1 & \mathbf{v_1}\end{pmatrix} \begin{pmatrix}1\\\lambda_1\end{pmatrix} - \mathbf{v_1^T} \begin{pmatrix}x_2 & \mathbf{v_2}\end{pmatrix} \begin{pmatrix}1\\\lambda_2\end{pmatrix} = 0 \tag{12}$$

The dot product of $\mathbf{v_2}$ with the line $\mathbf{AB}$ is

$$\mathbf{v_2^T}(\mathbf{A} - \mathbf{B}) = 0$$

$$\mathbf{v_2^T} \begin{pmatrix}x_1 & \mathbf{v_1}\end{pmatrix} \begin{pmatrix}1\\\lambda_1\end{pmatrix} - \mathbf{v_2^T} \begin{pmatrix}x_2 & \mathbf{v_2}\end{pmatrix} \begin{pmatrix}1\\\lambda_2\end{pmatrix} = 0 \tag{13}$$

Rearranging the equations (12) and (13) in matrix form we get

$$\begin{pmatrix}\mathbf{v_1^T}x_1 & \mathbf{v_1^T}\mathbf{v_1} & -\mathbf{v_1^T}x_2 & -\mathbf{v_1^T}\mathbf{v_2}\\\mathbf{v_2^T}x_1 & \mathbf{v_2^T}\mathbf{v_1} & -\mathbf{v_2^T}x_2 & -\mathbf{v_2^T}\mathbf{v_2}\end{pmatrix} \begin{pmatrix}1\\\lambda_1\\1\\\lambda_2\end{pmatrix} = 0 \tag{14}$$

simplifying it further

$$\begin{pmatrix} \boldsymbol{v_1^T v_1} & -\boldsymbol{v_1^T v_2} \\ \boldsymbol{v_2^T v_1} & -\boldsymbol{v_2^T v_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v_1^T (x_2 - x_1)} \\ \boldsymbol{v_2^T (x_2 - x_1)} \end{pmatrix} \tag{15}$$

To find the points on the lines which make up the shortest distance we need to find λ_1 and λ_2 using the following expression

$$\begin{pmatrix} \boldsymbol{v_1^T v_1} & -\boldsymbol{v_1^T v_2} \\ \boldsymbol{v_2^T v_1} & -\boldsymbol{v_2^T v_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v_1^T (x_2 - x_1)} \\ \boldsymbol{v_2^T (x_2 - x_1)} \end{pmatrix} \tag{16}$$

we know that

$$\boldsymbol{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \boldsymbol{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} and \boldsymbol{v_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Using the above expression, we get the points as $\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix}$ and $\frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}$ on the line L_1,L_2 respectively.
The shortest distance between the lines is the absolute value of projection of the vector $\boldsymbol{AB}$ on to the unit vector $\boldsymbol{u}$.

$$\|(\boldsymbol{B} - \boldsymbol{A})\| = \|\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}\| = \frac{3}{\sqrt{2}} \tag{17}$$

Therefore the shortest distance between the given lines is $\frac{3}{\sqrt{2}}$
The unit vector perpendicular to lines

$$Line_1 : \boldsymbol{x} = \boldsymbol{x_1} + \lambda_1 \boldsymbol{v_1} \tag{18}$$
$$Line_2 : \boldsymbol{x} = \boldsymbol{x_2} + \lambda_1 \boldsymbol{v_2} \tag{19}$$

can be found by

$$\frac{\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}}{\|\frac{1}{12}\begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12}\begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}\|} \tag{20}$$

So the unit vector perpendicular to both L_1 and L_2 is

$$\boldsymbol{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{21}$$