Assignment 6

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document is about tracing a parabola**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \tag{1.0.1}$$

2 Construction

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{2.0.1}$$

Calculating the parameters

$$\left|\mathbf{V}\right| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \tag{2.0.2}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \tag{2.0.3}$$

Therefore the given parabola equation is a degenerate.

The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \tag{2.0.4}$$

$$= \lambda^2 - 5\lambda \tag{2.0.5}$$

$$\lambda_1 = 0, \lambda_2 = 5 \tag{2.0.6}$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda\mathbf{I}$.For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \tag{2.0.7}$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.8}$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \tag{2.0.9}$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.10}$$

Therefore the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.11}$$

The value of η will be

$$\eta = 2p_1^T\mathbf{u} \tag{2.0.12}$$

$$= 2\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{2.0.13}$$

$$= 0 \tag{2.0.14}$$

3 Equation of the coincident line

The vertex of the degenerate hyperbola can be calculated as

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \frac{\eta}{2}p_1 - \mathbf{u} \end{pmatrix} \tag{3.0.1}$$

$$\begin{pmatrix} -6 & 3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} c = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix} \tag{3.0.2}$$

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_3 = 3R_3 - R_1} \begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.3}$$

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 = \frac{3}{2}R_2 + R_1} \begin{pmatrix} -6 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.4}$$

Therefore the vertex is $c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Applying affine transformation on equation (2.0.1),we get

$$\mathbf{x}^T\mathbf{P} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{P}^T\mathbf{x} + 2\begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{P}\mathbf{x} + 9 = 0 \tag{3.0.5}$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -3\sqrt{5} \end{pmatrix} \mathbf{x} + 9 = 0 \tag{3.0.6}$$

$$5y^2 - 6\sqrt{5}y + 9 = 0 \tag{3.0.7}$$

$$\left(\sqrt{5}y - 3\right)^2 = 0 \tag{3.0.8}$$

So the line is $\sqrt{5}y - 3 = 0$.

Applying inverse affine tranformation on the line we get

$$\begin{pmatrix} 0 & \sqrt{5} \end{pmatrix} \mathbf{P}^{-1}\mathbf{x} - 3 = 0 \tag{3.0.9}$$

$$\begin{pmatrix} 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} - 3 = 0 \tag{3.0.10}$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{3.0.11}$$

$$2x - y - 3 = 0 \tag{3.0.12}$$

Therefore the equation of coincident lines is $(2x - y - 3)^2 = 0$.

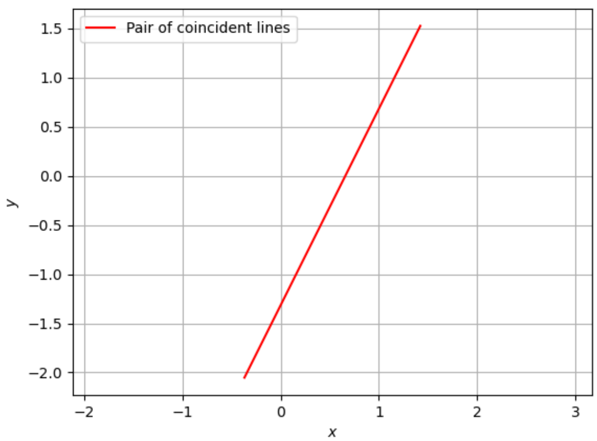

Fig. 1: Pair of coincident lines