

Assignment 6

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Abstract—This document is about tracing a parabola

Download all python codes from

[https://github.com/Zeeshan-IITH/IITH-EE5609/
new/master/codes](https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes)

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

1 PROBLEM

Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \quad (1.0.1)$$

2 CONSTRUCTION

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \quad (2.0.1)$$

Calculating the parameters

$$|\mathbf{V}| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \quad (2.0.2)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \quad (2.0.3)$$

Therefore the given parabola equation is a degenerate.

The characteristic equation of \mathbf{V} will be

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{vmatrix} \quad (2.0.4)$$

$$= \lambda^2 - 5\lambda \quad (2.0.5)$$

$$\lambda_1 = 0, \lambda_2 = 5 \quad (2.0.6)$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda \mathbf{I}$. For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_2=2R_2+R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \quad (2.0.7)$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.8)$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xrightarrow{R_2=R_2-2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \quad (2.0.9)$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.0.10)$$

Therefore the transformation matrix will be

$$\mathbf{P} = (p_1 \ p_2) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.0.11)$$

The value of η will be

$$\eta = 2p_1^T \mathbf{u} \quad (2.0.12)$$

$$= 2 \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} \quad (2.0.13)$$

$$= 0 \quad (2.0.14)$$

3 EQUATION OF THE COINCIDENT LINE

The vertex of the degenerate hyperbola can be calculated as

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2} p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \frac{\eta}{2} p_1 - \mathbf{u} \end{pmatrix} \quad (3.0.1)$$

$$\begin{pmatrix} -6 & 3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} c = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix} \quad (3.0.2)$$

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix} \xrightarrow{R_3=3R_3-R_1} \begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.3)$$

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2=\frac{3}{2}R_2+R_1} \begin{pmatrix} -6 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.4)$$

Therefore the vertex is $c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Applying affine transformation on equation (2.0.1), we get

$$\mathbf{x}^T \mathbf{P} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{P}^T \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{P} \mathbf{x} + 9 = 0 \quad (3.0.5)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & 3\sqrt{5} \end{pmatrix} \mathbf{x} + 9 = 0 \quad (3.0.6)$$

$$5y^2 + 6\sqrt{5}y + 9 = 0 \quad (3.0.7)$$

$$(\sqrt{5}y + 3)^2 = 0 \quad (3.0.8)$$

So the line is $\sqrt{5}y + 3 = 0$.

Applying inverse affine transformation on the line we get

$$(0 \quad \sqrt{5}) \mathbf{P} \mathbf{x} + 3 = 0 \quad (3.0.9)$$

$$(0 \quad \sqrt{5}) \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} + 3 = 0 \quad (3.0.10)$$

$$(2 \quad -1) \mathbf{x} + 3 = 0 \quad (3.0.11)$$

$$2x - y - 3 = 0 \quad (3.0.12)$$

Therefore the equation of coincident lines is $(2x - y - 3)^2 = 0$.

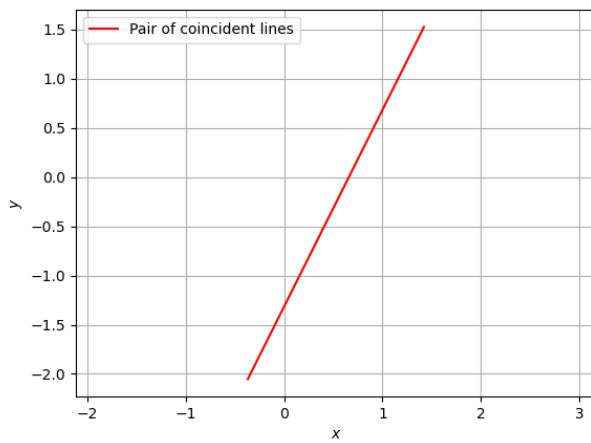


Fig. 1: Pair of coincident lines