Assignment 5

Shaik Zeeshan Ali

AI20MTECH11001

Abstract—**This document is about matrix representation of lines and the bisectors of angles between them.**

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1 PROBLEM

Show that the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -11 & 31 \end{pmatrix} \mathbf{x} - 10 = 0 \quad (1.0.1)$$

represents two straight lines,and find the equations of the bisectors of the angles between them.

2 CONSTRUCTION

Any quadratic equation in terms of x, y of the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$,can be written as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

$$\text{where,} \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.0.3)$$

The equation (1.0.1) represents two intersecting straight lines when

$$\begin{vmatrix} \mathbf{V} & u \\ u^T & f \end{vmatrix} = 0 \quad (2.0.4)$$

$$|\mathbf{V}| < 0 \quad (2.0.5)$$

3 EXPLANATION

From equation (1.0.1) we get

$$\mathbf{V} = \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (3.0.2)$$

$$f = -10 \quad (3.0.3)$$

calculating the equation (2.0.4),we get

$$\begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ \frac{-11}{2} & \frac{31}{2} & -10 \end{pmatrix} \xleftarrow{R_3 = R_3 + R_2 + R_1} \begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ 0 & 0 & 0 \end{pmatrix}$$
$$(3.0.4)$$

Therefore the determinant 0.And also the determinant of $\mathbf{V}$ is

$$|\mathbf{V}| = \begin{vmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{vmatrix} \quad (3.0.5)$$

$$= -90.25 \quad (3.0.6)$$

$$< 0 \quad (3.0.7)$$

Therefore the given equation represents the equation of two straight lines which intersect.

4 POINT OF INTERSECTION

Let the two lines be

$$\mathbf{n_1}^T \mathbf{x} = c_1 \quad (4.0.1)$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \quad (4.0.2)$$

The equation of two lines in quadratic form will be

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_2}^T \mathbf{x} - c_2\right) = 0 \quad (4.0.3)$$

comparing equations (1.0.1) and (4.0.3),we get

$$-\left(c_2 \mathbf{n_1}^T + c_1 \mathbf{n_2}^T\right) = \begin{pmatrix} -11 & 31 \end{pmatrix} \quad (4.0.4)$$

$$c_1 c_2 = -10 \quad (4.0.5)$$

slopes of the two lines are the roots of the equation

$$cm^2 + 2bm + a = 0 \quad (4.0.6)$$

$$15m^2 + m - 6 = 0 \quad (4.0.7)$$

$$m_1 = \frac{-2}{3}, m_2 = \frac{3}{5} \quad (4.0.8)$$

therefore the normal vectors will be

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.0.9)$$

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (4.0.10)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} \frac{-3}{5} \\ 1 \end{pmatrix} \quad (4.0.11)$$

We know that

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.0.12)$$

$$k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} \quad (4.0.13)$$

$$k_1 k_2 = -15 \quad (4.0.14)$$

Choosing the values of $k_1 = 3$ and $k_2 = -5$,we get

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (4.0.15)$$

For verifying the values of $\mathbf{n_1}$ and $\mathbf{n_2}$,we compute the convolution by representing $\mathbf{n_1}$ as a toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (4.0.16)$$

$$= \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.0.17)$$

Using the equations (4.0.4) and (4.0.5), we get

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 11 \\ -31 \end{pmatrix} \quad (4.0.18)$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -5 & -31 \end{pmatrix} \xrightarrow{R_2 = 2R_2 - 3R_1} \begin{pmatrix} 2 & 3 & 11 \\ 0 & -19 & -95 \end{pmatrix} \quad (4.0.19)$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 5 \end{pmatrix} \quad (4.0.20)$$

The values are $c_2 = 5$ and $c_1 = -2$.Therefore the equation of the lines are

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 5 \quad (4.0.21)$$

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = -2 \quad (4.0.22)$$

$$(2x + 3y - 5)(3x - 5y + 2) = 0 \quad (4.0.23)$$

The point of intersection will be

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (4.0.24)$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & -5 & -2 \end{pmatrix} \xrightarrow{R_2 = 2R_2 - 3R_1} \begin{pmatrix} 2 & 3 & 5 \\ 0 & -19 & -19 \end{pmatrix} \quad (4.0.25)$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.0.26)$$

Therefore the lines intersect at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5 EIGENVECTORS

The characteristic equation of the matrix $\mathbf{V}$ is

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.0.1)$$

$$\begin{vmatrix} 6 - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15 - \lambda \end{vmatrix} = 0 \quad (5.0.2)$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \quad (5.0.3)$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2}\left(9 + \sqrt{442}\right) \quad (5.0.4)$$

$$\lambda_2 = \frac{-1}{2}\left(9 - \sqrt{442}\right) \quad (5.0.5)$$

The eigen vectors will be in the nullspace of $\mathbf{V} - \lambda_1 \mathbf{I}$ and $\mathbf{V} - \lambda_2 \mathbf{I}$.The eigen vector corresponding to eigen value λ_1 will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}\left(9 + \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}\left(9 + \sqrt{442}\right) \end{pmatrix} \quad (5.0.6)$$

$$= \begin{pmatrix} \frac{1}{2}\left(21 + \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21 + \sqrt{442}\right) \end{pmatrix} \quad (5.0.7)$$

$$\xrightarrow{R_2 = (21 + \sqrt{442})R_2 + R_1} \begin{pmatrix} \frac{1}{2}\left(21 + \sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (5.0.8)$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (5.0.9)$$

normalizing p_1,we get

$$p_1 = \begin{pmatrix} 0.0238 \\ 1 \end{pmatrix} \quad (5.0.10)$$

the eigen vector corresponding to eigen value λ_2 will be

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}\left(9 - \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}\left(9 - \sqrt{442}\right) \end{pmatrix} \tag{5.0.11}$$

$$= \begin{pmatrix} \frac{1}{2}\left(21 - \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21 - \sqrt{442}\right) \end{pmatrix} \tag{5.0.12}$$

$$\xrightarrow{R_2=(21-\sqrt{442})R_2+R_1} \begin{pmatrix} \frac{1}{2}\left(21 - \sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \tag{5.0.13}$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_2 = \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \tag{5.0.14}$$

normalizing p_2,we get

$$p_2 = \begin{pmatrix} 1 \\ -0.0238 \end{pmatrix} \tag{5.0.15}$$

So the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} 0.0238 & 1 \\ 1 & -0.0238 \end{pmatrix} \tag{5.0.16}$$

6 AFFINE TRANSFORMATION

Doing the affine transformation on given quadratic equation, we get pair to intersecting straight lines passing through origin.

Let the affine transformation be $\mathbf{x} = \mathbf{P}\mathbf{y} + c$.The transformation will be

$$(\mathbf{P}\mathbf{y} + c)^T \mathbf{V}(\mathbf{P}\mathbf{y} + c) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + c) + f = 0 \tag{6.0.1}$$

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y} + 2\left(c^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{P}\mathbf{y} + c^T\mathbf{V}c + 2\mathbf{u}^Tc + f = 0 \tag{6.0.2}$$

if the point c is taken as the point of intersection of the two lines.

$$c^T\mathbf{V}c + 2\mathbf{u}^Tc + f = 0 \tag{6.0.3}$$

$$c^T\mathbf{V} + \mathbf{u}^T = 0 \tag{6.0.4}$$

So the affine transformation of the given lines will be

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y} = 0 \tag{6.0.5}$$

$$\mathbf{y}^T\begin{pmatrix} -1.5 & 0 \\ 0 & 6 \end{pmatrix}\mathbf{y} = 0 \tag{6.0.6}$$

$$(x - 2y)(x + 2y) = 0 \tag{6.0.7}$$

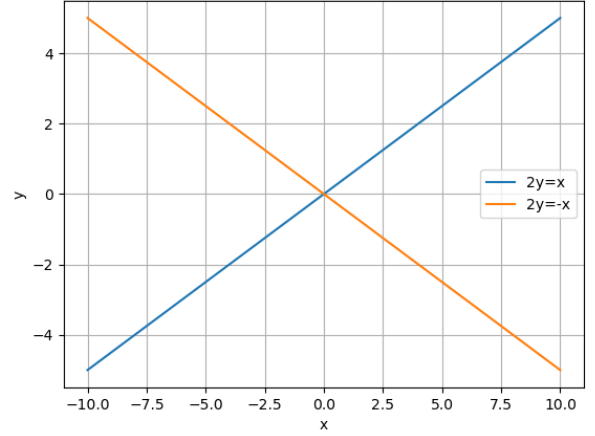This line will pass through origin,whose bisectors

Fig. 1: straight lines after affine transformation passing through origin

will be the $x-$axis and $y-$axis.The bisectors wil be of the form

$$\mathbf{y}^T\begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}\mathbf{y} = 0 \tag{6.0.8}$$

$$\mathbf{y}^T\mathbf{K}\mathbf{y} = 0 \tag{6.0.9}$$

$$\mathbf{K} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \tag{6.0.10}$$

7 BISECTORS

Taking the inverse of the affine transformation of the equation $xy = 0$,will give the angle bisectors.

$$\left(\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}^{-1}c\right)^T \mathbf{K}\left(\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}^{-1}c\right) = 0 \tag{7.0.1}$$

$$\mathbf{x}^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x} - 2c^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x} + c^T\mathbf{P}\mathbf{K}\mathbf{P}^Tc = 0 \tag{7.0.2}$$
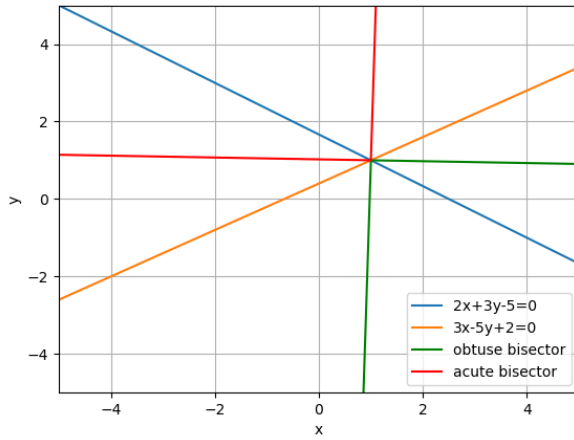
Fig. 2: Par of straight lines and their angular bisectors

Substituting the values we get

$$\mathbf{x}^T \begin{pmatrix} 0.0238 & 0.5 \\ 0.5 & -0.0238 \end{pmatrix} \mathbf{x}$$
$$- \begin{pmatrix} 1.046 & 0.951 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{7.0.3}$$

$$0.0238x^2 + xy - 0.0238y^2$$
$$- 1.046x - 0.951y + 1 = 0 \tag{7.0.4}$$

$$x^2 + 42xy - y^2 - 44x - 40y + 42 = 0 \tag{7.0.5}$$

Therefore the equation of bisectors of the given line in quadratic form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 21 \\ 21 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 44 & 40 \end{pmatrix} \mathbf{x} + 42 = 0 \tag{7.0.6}$$