Assignment 1

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***Abstract*—This document is about the linear operator and minimal polynomials.**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Let $\mathbf{V}$ be the vector space of $n \times n$ matrices over field $\mathbf{F}$. Let $\mathbf{A}$ be a fixed $n \times n$ matrix. Let $\mathbf{T}$ be the linear operator on $\mathbf{V}$ defined by

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} \tag{1.0.1}$$

Show that the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.

2 Construction

Given	$\mathbf{A}$ is a fixed matrix from the vector space $\mathbf{V}$ of $n \times n$ matrices. A linear operator on the finite dimensional vector space $\mathbf{V}$, $\mathbf{T}$ is defined as $\mathbf{T}(\mathbf{B}) = \mathbf{AB}$.
Minimal polynomial	The minimal polynomial of a linear operator $\mathbf{T}$ is a monic polynomial which annihilates $\mathbf{T}$.
Matrix representation of $\mathbf{T}$	If we stack up the columns of the matrix $\mathbf{B}$,the linear operator $\mathbf{T}$ can be represented in the equivalent form as If $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & . & . & b_n \end{pmatrix}$,then the linear transformation of $\mathbf{B}$ will be $\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{A}b_1 & \mathbf{A}b_2 & . & .\mathbf{A}b_n \end{pmatrix}$ $\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{T}(b_1) \\ \mathbf{T}(b_2) \\ . \\ . \\ \mathbf{T}(b_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & & & \\ & \mathbf{A} & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$ here each element represents the elements of the matrix $\mathbf{AB}$

	$$\mathbf{T}=\begin{pmatrix}\mathbf{A} & & & & \\ & \mathbf{A} & \mathbf{O} & & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & \mathbf{A}\end{pmatrix}$$
Properties of minimal polynomial	The roots of the characteristic polynomial,eigen values and the minimal polynomial are same,except for multiplicities.The roots of the minimal polynomial of $\mathbf{A}$ are the roots of $\det(\mathbf{A}-\lambda\mathbf{I})$
The roots of minimal polynomial of $\mathbf{T}$	The roots of the minimal polynomial of $\mathbf{T}$ are the roots of $\det(\mathbf{T}-\lambda\mathbf{I})$ $$\det(\mathbf{T}-\lambda\mathbf{I})=\begin{vmatrix}(\mathbf{A}-\lambda\mathbf{I}) & & & & \\ & (\mathbf{A}-\lambda\mathbf{I}) & \mathbf{O} & & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & (\mathbf{A}-\lambda\mathbf{I})\end{vmatrix}$$ $$=(\det(\mathbf{A}-\lambda\mathbf{I}))^n$$ Therfore we can see that the eigen values of $\mathbf{A}$ are also the eigen values of the linear operator $\mathbf{T}$
Minimal polynomial of $\mathbf{T}$	The minimal polynomial of $\mathbf{A}$ divides the characteristic polynomial of $\mathbf{A}$ and $\mathbf{T}$. Let the minimal polynomial of $\mathbf{A}$ is of degree $p \le n$ $$f(x)=a_0+a_1x+a_2x^2...a_px^p \text{ such that } f(\mathbf{A})=0$$ $$f(\mathbf{T})=a_0\mathbf{I}+a_1\mathbf{T}+a_2\mathbf{T}^2+..+a_p\mathbf{T}^p$$ $$f(\mathbf{T})=\begin{pmatrix}f(\mathbf{A}) & & & & \\ & f(\mathbf{A}) & \mathbf{O} & & \\ & & . & & \\ & \mathbf{O} & & . & \\ & & & & f(\mathbf{A})\end{pmatrix}=\mathbf{O}_{n^2\times n^2}$$ Therefore the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.