Assignment 6

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***Abstract*—This document is about tracing a parabola**

Download all python codes from

```
https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \tag{1.0.1}$$

2 Construction

The given quadratic equation can be written in the matrix form as

3 Construction

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{3.0.1}$$

Calculating the parameters,we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \tag{3.0.2}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \tag{3.0.3}$$

Therefore the given parabola equation is a degenerate.The quadratic equation corresponds to a pair of coincident straight lines.

The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \tag{3.0.4}$$

$$= \lambda^2 - 5\lambda \tag{3.0.5}$$

$$\lambda_1 = 0, \lambda_2 = 5 \tag{3.0.6}$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda\mathbf{I}$.For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xleftrightarrow{R_2=2R_2+R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \tag{3.0.7}$$

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.0.8}$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.0.9}$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \tag{3.0.10}$$

$$p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.11}$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.12}$$

Therefore the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.0.13}$$

The value of η will be

$$\eta = 2p_1^T\mathbf{u} \tag{3.0.14}$$

$$= 2\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{3.0.15}$$

$$= 0 \tag{3.0.16}$$

A point on the line can be found by using to following formula

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \frac{\eta}{2}p_1 - \mathbf{u} \end{pmatrix} \tag{3.0.17}$$

$$\begin{pmatrix} \mathbf{u}^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ -\mathbf{u} \end{pmatrix} \tag{3.0.18}$$

$$\begin{pmatrix} -6 & 3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} c = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix} \tag{3.0.19}$$

Writing it in augmented form, we get

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_3 = R_3 - \frac{R_1}{3}} \begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.20)$$

$$\xleftrightarrow{R_2 = \frac{3}{2}R_2 + R_1} \begin{pmatrix} -6 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.21)$$

Therefore we can see that the point $c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ lies on the line.

4 Equation of the straight line

Applying affine transformation we get

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (4.0.1)$$

$$\mathbf{y}^T \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{y} = 0 \quad (4.0.2)$$

$$5y^2 = 0 \quad (4.0.3)$$

Therefore the transformed line is $y = 0$,which in vector form will be $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{y} = 0$.

Taking the Inverse affine transformation we get

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(P^T (\mathbf{x} - c) \right) = 0 \quad (4.0.4)$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} (\mathbf{x} - c) = 0 \quad (4.0.5)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} (\mathbf{x} - c) = 0 \quad (4.0.6)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} - \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \quad (4.0.7)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} + \frac{3}{\sqrt{5}} = 0 \quad (4.0.8)$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \quad (4.0.9)$$

Therefore the equation of coincident lines is $(2x - y - 3) = 0$.