Assignment 13

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Abstract

This document is about the linear operators which have all the vectors as eigen vectors.

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

Let $\mathbf{A}$ be an $n \times n$ matrix over $\mathbb{C}$ such that every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$.Then

1) All eigen values of $\mathbf{A}$ are equal.
2) All eigen values of $\mathbf{A}$ are distinct.
3) $\mathbf{A} = \lambda\mathbf{I}$ for some $\lambda \in \mathbb{C}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.
4) If $\chi_{\mathbf{A}}$ and $m_{\mathbf{A}}$ denote the characteristic polynomial and the minimal polynomial respectively, then $\chi_{\mathbf{A}} = m_{\mathbf{A}}$

2 Construction

Given	Every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$, where $\mathbf{A}$ is an $n \times n$ matrix over $\mathbb{C}$.
Determining $\mathbf{A}$	Since every vector is an eigen vector, the standard basis vectors are also eigen vectors $\implies \mathbf{A}\mathbf{e_i} = \lambda_i\mathbf{e_i} \implies \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \end{pmatrix}\mathbf{e_i} = \lambda_i\mathbf{e_i} \implies a_i = \lambda_i\mathbf{e_i}$ where $\lambda_i \in \mathbb{C}$ therefore $\mathbf{A} = \begin{pmatrix} \lambda_1\mathbf{e_1} & \lambda_2\mathbf{e_2} & . & . & \lambda_n\mathbf{e_n} \end{pmatrix}$ Any vector $\mathbf{b}$ can be represented in the standard basis as $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}$ where $b_i \in \mathbb{C}$ As every non-zero vector in $\mathbb{C}^n$ is an eigen vector $\mathbf{A}\mathbf{b} = \lambda\mathbf{b} \implies \mathbf{A}\left(b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}\right) = \lambda\left(b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}\right)$ $\implies b_1\lambda_1\mathbf{e_1} + b_2\lambda_2\mathbf{e_2} + ... + b_n\lambda_n\mathbf{e_n} = \lambda\left(b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}\right)$ $\implies b_1\left(\lambda_1 - \lambda\right)\mathbf{e_1} + b_2\left(\lambda_2 - \lambda\right)\mathbf{e_2} + ... + b_n\left(\lambda_n - \lambda\right)\mathbf{e_n} = 0$ since basis are linearly independent we get $\lambda_1 = \lambda_2 = .. = \lambda_n = \lambda$ Therefore the matrix $\mathbf{A}$ is $\mathbf{A} = \begin{pmatrix} \lambda_1\mathbf{e_1} & \lambda_2\mathbf{e_2} & . & . & \lambda_n\mathbf{e_n} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & . & . & \mathbf{e_n} \end{pmatrix} = \lambda\mathbf{I}_n$ where $\lambda \in \mathbb{C}$

3 Answers

option 1	Since $\mathbf{A} = \lambda\mathbf{I}_n$, all the eigen values are equal to λ. Therefore option 1 is correct as the matrix $\mathbf{A}$ is a scalar matrix.
option 2	since the matrix $\mathbf{A}$ is a scalar matrix, all the eigen values are equal. So this option is incorrect.
option 3	This option is correct. As proved in the construction the matrix $\mathbf{A} = \lambda\mathbf{I}$ for some $\lambda \in \mathbb{C}$
option 4	Since $\mathbf{A} = \lambda\mathbf{I}$ where $\lambda \in \mathbb{C}$, the characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x-\lambda)^n$ and $m_{\mathbf{A}} = (x-\lambda) \implies \chi_{\mathbf{A}} = m_{\mathbf{A}}^n$. Therefore this option is incorrect

TABLE 1: Answer