

Vandermonde matrix

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Abstract—This document calculates the determinant of a vandermonde matrix.

Download all python codes from

<https://github.com/ZeeShan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/ZeeShan-IITH/IITH-EE5609>

1 PROBLEM

Derive an expression for the determinant of a Vandermonde matrix

$$\mathbf{V} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} \quad (1.0.1)$$

2 CONSTRUCTION

The given matrix is of the order $n \times n$. The determinant is given as

$$\det \mathbf{V} = \det \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} \quad (2.0.1)$$

By doing Row operations we can reduce it to the form

$$\det \mathbf{V} = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} \xrightarrow[i \neq 1]{R_i = R_i - R_1} \quad (2.0.2)$$

$$= \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \cdots & \alpha_2^{n-1} - \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & \cdots & \alpha_n^{n-1} - \alpha_1^{n-1} \end{vmatrix} \quad (2.0.3)$$

By doing column operations we can reduce it to the form

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \cdots & \alpha_2^{n-1} - \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & \cdots & \alpha_n^{n-1} - \alpha_1^{n-1} \end{vmatrix} \xrightarrow[i > 1]{C_i = C_i - C_{i-1}} \quad (2.0.4)$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 - \alpha_1 & (\alpha_2 - \alpha_1)\alpha_2 & \cdots & (\alpha_2 - \alpha_1)\alpha_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_n - \alpha_1 & (\alpha_n - \alpha_1)\alpha_n & \cdots & (\alpha_n - \alpha_1)\alpha_n^{n-2} \end{vmatrix} \quad (2.0.5)$$

$$= \prod_{n \geq j > 1} (\alpha_j - \alpha_1) \begin{vmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} \end{vmatrix} \quad (2.0.6)$$

3 EXPLANATION

By using equation (2.0.6) the determinant can be reduced to $(n-1) \times (n-1)$, which is also of the form vandermonde matrix. The determinant of the reduced vandermonde matrix will be

$$\begin{vmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} \end{vmatrix} \quad (3.0.1)$$

$$= \prod_{n \geq j > 2} (\alpha_j - \alpha_2) \begin{vmatrix} 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-3} \\ 1 & \alpha_4 & \alpha_4^2 & \cdots & \alpha_4^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-3} \end{vmatrix} \quad (3.0.2)$$

So in each calculation the dimension is reduced. Continuing the similar reduction, the determinant of the matrix \mathbf{V} , we can write it as

$$\det \mathbf{V} \quad (3.0.3)$$

$$= \prod_{n \geq j > 1} (\alpha_j - \alpha_1) \prod_{n \geq j > 2} (\alpha_j - \alpha_2) \cdots \prod_{n \geq j > n-1} (\alpha_j - \alpha_{n-1}) \quad (3.0.4)$$

$$= \prod_{n > i \geq 1} \prod_{(n \geq j > i)} (\alpha_j - \alpha_i) \quad (3.0.5)$$