

# Assignment 2

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**Abstract**—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.

Download all python codes from

<https://github.com/Zeehan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeehan-IITH/IITH-EE5609>

## 1 PROBLEM

Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (2)$$

## 2 CONSTRUCTION

When two lines are not intersecting the distance between them is non-zero. The equation of above mentioned lines in symmetric form is

$$L_1: x - 1 = 2 - y = z - 1 \quad (3)$$

$$L_2: \frac{x-2}{2} = y+1 = \frac{z+1}{2} \quad (4)$$

The above line equations have no point of intersection as for no value of  $\lambda_1, \lambda_2$  both the equations (3) and (4) are equal.

If the two line intersect then (3)=(4) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (5)$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (7)$$

The Augmented matrix will be

(8)

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (10)$$

The above matrix has a  $rank = 3$ . Hence the lines do not intersect

## 3 SOLUTION

Let  $\mathbf{A}$  be a point on line  $L_1$  and  $\mathbf{B}$  be point on the line  $L_2$ . Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines  $L_1, L_2$  and passing through  $\mathbf{A}$  and  $\mathbf{B}$ .

The vectors  $\mathbf{v}_1, \mathbf{v}_2$  are both perpendicular to the line  $\mathbf{AB}$ . So the dot product of  $\mathbf{v}_1, \mathbf{v}_2$  with the line  $\mathbf{AB}$  is zero.

The dot product of  $\mathbf{v}_1$  with the line  $\mathbf{AB}$  is

$$\mathbf{v}_1^T (x_1 \quad \mathbf{v}_1) \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \mathbf{v}_1^T (x_2 \quad \mathbf{v}_2) \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = 0 \quad (11)$$

The dot product of  $\mathbf{v}_2$  with the line  $\mathbf{AB}$  is

$$\mathbf{v}_2^T (x_1 \quad \mathbf{v}_1) \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} - \mathbf{v}_2^T (x_2 \quad \mathbf{v}_2) \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = 0 \quad (12)$$

Rearranging the equations (11) and (12) in matrix form we get

$$\begin{pmatrix} \mathbf{v}_1^T x_1 & \mathbf{v}_1^T \mathbf{v}_1 & -\mathbf{v}_1^T x_2 & -\mathbf{v}_1^T \mathbf{v}_2 \\ \mathbf{v}_2^T x_1 & \mathbf{v}_2^T \mathbf{v}_1 & -\mathbf{v}_2^T x_2 & -\mathbf{v}_2^T \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda_1 \\ 1 \\ \lambda_2 \end{pmatrix} = 0 \quad (13)$$

simplifying it further

$$\begin{pmatrix} \mathbf{v}_1^T \mathbf{v}_1 & -\mathbf{v}_1^T \mathbf{v}_2 \\ \mathbf{v}_2^T \mathbf{v}_1 & -\mathbf{v}_2^T \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1^T (x_2 - x_1) \\ \mathbf{v}_2^T (x_2 - x_1) \end{pmatrix} \quad (14)$$

To find the points on the lines which make up the shortest distance we need to find  $\lambda_1$  and  $\lambda_2$  using

the following expression

$$\begin{pmatrix} v_1^T v_1 & -v_1^T v_2 \\ v_2^T v_1 & -v_2^T v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} v_1^T (x_2 - x_1) \\ v_2^T (x_2 - x_1) \end{pmatrix} \quad (15)$$

we know that

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Using the above expression, we get the points as

$$\frac{1}{12} \begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} \text{ and } \frac{1}{12} \begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix} \text{ on the line } L_1, L_2 \text{ respectively.}$$

The shortest distance between the lines is the absolute value of projection of the vector  $\mathbf{AB}$  on to the unit vector  $\mathbf{u}$ .

$$\|(\mathbf{B} - \mathbf{A})\| = \left\| \frac{1}{12} \begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12} \begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix} \right\| = \frac{3}{\sqrt{2}} \quad (16)$$

Therefore the shortest distance between the given lines is  $\frac{3}{\sqrt{2}}$

The unit vector perpendicular to lines

$$\text{Line}_1: \mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{v}_1 \quad (17)$$

$$\text{Line}_2: \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{v}_2 \quad (18)$$

can be found by

$$\frac{\frac{1}{12} \begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12} \begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix}}{\left\| \frac{1}{12} \begin{pmatrix} 27 \\ -3 \\ 27 \end{pmatrix} - \frac{1}{12} \begin{pmatrix} 10 \\ -19 \\ -26 \end{pmatrix} \right\|} \quad (19)$$

So the unit vector perpendicular to both  $L_1$  and  $L_2$  is

$$\mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (20)$$

