

Assignment 5

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AI20MTECH11001

Abstract—This document is about matrix representation of lines and the bisectors of angles between them.

Download all python codes from

<https://github.com/ZeeShan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/ZeeShan-IITH/IITH-EE5609>

1 PROBLEM

Show that the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \mathbf{x} + (-11 \ 31) \mathbf{x} - 10 = 0 \quad (1.0.1)$$

represents two straight lines, and find the equations of the bisectors of the angles between them.

2 CONSTRUCTION

Any quadratic equation in terms of x, y of the form $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$, can be written as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

$$\text{where, } \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.0.3)$$

The equation (1.0.1) represents two intersecting straight lines when

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.4)$$

$$|\mathbf{V}| < 0 \quad (2.0.5)$$

3 EXPLANATION

From equation (1.0.1) we get

$$\mathbf{V} = \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ 31 \end{pmatrix} \quad (3.0.2)$$

$$f = -10 \quad (3.0.3)$$

calculating the equation (2.0.4), we get

$$\begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ \frac{-11}{2} & \frac{31}{2} & -10 \end{pmatrix} \xrightarrow{R_3=R_3+R_2+R_1} \begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.0.4)$$

Therefore the determinant 0. And also the determinant of \mathbf{V} is

$$|\mathbf{V}| = \begin{vmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{vmatrix} \quad (3.0.5)$$

$$= -90.25 \quad (3.0.6)$$

$$< 0 \quad (3.0.7)$$

Therefore the given equation represents the equation of two straight lines which intersect.

4 POINT OF INTERSECTION

Let the two lines be

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.0.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.0.2)$$

The equation of two lines in quadratic form will be

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (4.0.3)$$

comparing equations (1.0.1) and (4.0.3), we get

$$-(c_2 \mathbf{n}_1^T + c_1 \mathbf{n}_2^T) = (-11 \ 31) \quad (4.0.4)$$

$$c_1 c_2 = -10 \quad (4.0.5)$$

slopes of the two lines are the roots of the equation

$$cm^2 + 2bm + a = 0 \quad (4.0.6)$$

$$15m^2 + m - 6 = 0 \quad (4.0.7)$$

$$m_1 = \frac{-2}{3}, m_2 = \frac{3}{5} \quad (4.0.8)$$

therefore the normal vectors will be

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.0.9)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (4.0.10)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-3}{5} \\ 1 \end{pmatrix} \quad (4.0.11)$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.0.12)$$

$$k_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} \quad (4.0.13)$$

$$k_1 k_2 = -15 \quad (4.0.14)$$

Choosing the values of $k_1 = 3$ and $k_2 = -5$, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (4.0.15)$$

For verifying the values of \mathbf{n}_1 and \mathbf{n}_2 , we compute the convolution by representing \mathbf{n}_1 as a toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (4.0.16)$$

$$= \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.0.17)$$

Using the equations (4.0.4) and (4.0.5), we get

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 11 \\ -31 \end{pmatrix} \quad (4.0.18)$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -5 & -31 \end{pmatrix} \xrightarrow{R_2=2R_2-3R_1} \begin{pmatrix} 2 & 3 & 11 \\ 0 & -19 & -95 \end{pmatrix} \quad (4.0.19)$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1=R_1-3R_2} \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 5 \end{pmatrix} \quad (4.0.20)$$

The values are $c_2 = 5$ and $c_1 = -2$. Therefore the equation of the lines are

$$(2 \ 3)\mathbf{x} = 5 \quad (4.0.21)$$

$$(3 \ -5)\mathbf{x} = -2 \quad (4.0.22)$$

$$(2x + 3y - 5)(3x - 5y + 2) = 0 \quad (4.0.23)$$

The point of intersection will be

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (4.0.24)$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & -5 & -2 \end{pmatrix} \xrightarrow{R_2=2R_2-3R_1} \begin{pmatrix} 2 & 3 & 5 \\ 0 & -19 & -19 \end{pmatrix} \quad (4.0.25)$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-3R_2} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.0.26)$$

Therefore the lines intersect at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5 EIGENVECTORS

The characteristic equation of the matrix \mathbf{V} is

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.0.1)$$

$$\begin{vmatrix} 6 - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15 - \lambda \end{vmatrix} = 0 \quad (5.0.2)$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \quad (5.0.3)$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2} (9 + \sqrt{442}) \quad (5.0.4)$$

$$\lambda_2 = \frac{-1}{2} (9 - \sqrt{442}) \quad (5.0.5)$$

The eigen vectors will be in the nullspace of $\mathbf{V} - \lambda_1 \mathbf{I}$ and $\mathbf{V} - \lambda_2 \mathbf{I}$. The eigen vector corresponding to eigen value λ_1 will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}(9 + \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}(9 + \sqrt{442}) \end{pmatrix} \quad (5.0.6)$$

$$= \begin{pmatrix} \frac{1}{2}(21 + \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}(-21 + \sqrt{442}) \end{pmatrix} \quad (5.0.7)$$

$$\xrightarrow{R_2=(21+\sqrt{442})R_2+R_1} \begin{pmatrix} \frac{1}{2}(21 + \sqrt{442}) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (5.0.8)$$

The above reduced matrix has one free variable. Let it be 1, then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (5.0.9)$$

normalizing p_1 , we get

$$p_1 = \frac{1}{\sqrt{884 + 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (5.0.10)$$

the eigen vector corresponding to eigen value λ_2 will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}(9 - \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}(9 - \sqrt{442}) \end{pmatrix} \quad (5.0.11)$$

$$= \begin{pmatrix} \frac{1}{2}(21 - \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}(-21 - \sqrt{442}) \end{pmatrix} \quad (5.0.12)$$

$$\xrightarrow{R_2 = (21 - \sqrt{442})R_2 + R_1} \begin{pmatrix} \frac{1}{2}(21 - \sqrt{442}) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (5.0.13)$$

The above reduced matrix has one free variable. Let it be 1, then the eigen vector will be

$$p_2 = \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (5.0.14)$$

normalizing p_2 , we get

$$p_2 = \frac{1}{\sqrt{884 - 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (5.0.15)$$

So the transformation matrix will be

$$\mathbf{P} = (p_1 \ p_2) = \begin{pmatrix} \frac{1}{\sqrt{884+42\sqrt{442}}} & \frac{1}{\sqrt{884-42\sqrt{442}}} \\ \frac{21+\sqrt{442}}{\sqrt{884+42\sqrt{442}}} & \frac{21-\sqrt{442}}{\sqrt{884-42\sqrt{442}}} \end{pmatrix} \quad (5.0.16)$$

6 AFFINE TRANSFORMATION

Doing the affine transformation on given quadratic equation, we get pair of intersecting straight lines passing through origin.

Let the affine transformation be $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$. The transformation will be

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0 \quad (6.0.1)$$

$$\mathbf{y}^T (\mathbf{P}^T \mathbf{V} \mathbf{P}) \mathbf{y} + 2(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (6.0.2)$$

if the point \mathbf{c} is taken as the point of intersection of the two lines.

$$\mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (6.0.3)$$

$$\mathbf{c}^T \mathbf{V} + \mathbf{u}^T = 0 \quad (6.0.4)$$

So the affine transformation of the given lines will be

$$\mathbf{y}^T (\mathbf{P}^T \mathbf{V} \mathbf{P}) \mathbf{y} = 0 \quad (6.0.5)$$

$$\mathbf{y}^T \begin{pmatrix} -1.5 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 0 \quad (6.0.6)$$

$$(x - 2y)(x + 2y) = 0 \quad (6.0.7)$$

Since the two lines are symmetric with respect to both X -axis and Y -axis, the axes themselves are the bisectors of the transformed pair of lines. So the bisectors will be $x = 0$ and $y = 0$. Matrix notation

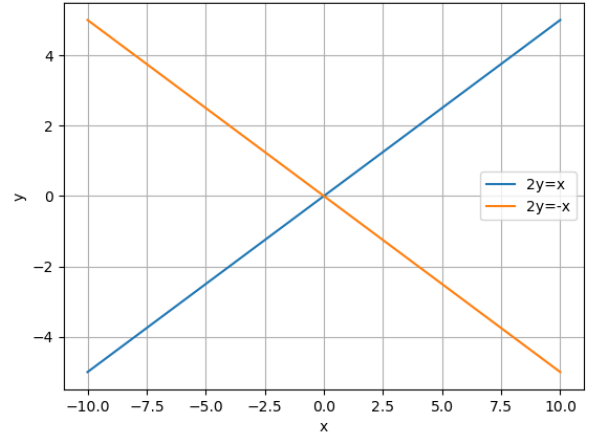


Fig. 1: straight lines after affine transformation passing through origin

will be of the form

$$\mathbf{y}^T \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \mathbf{y} = 0 \quad (6.0.8)$$

$$\mathbf{y}^T \mathbf{K} \mathbf{y} = 0 \quad (6.0.9)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \quad (6.0.10)$$

7 BISECTORS

Taking the inverse of the affine transformation of the equation $xy = 0$, will give the angle bisectors.

$$(\mathbf{P}^{-1} \mathbf{x} - \mathbf{P}^{-1} \mathbf{c})^T \mathbf{K} (\mathbf{P}^{-1} \mathbf{x} - \mathbf{P}^{-1} \mathbf{c}) = 0 \quad (7.0.1)$$

$$\mathbf{x}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{x} + \mathbf{c}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{c} = 0 \quad (7.0.2)$$

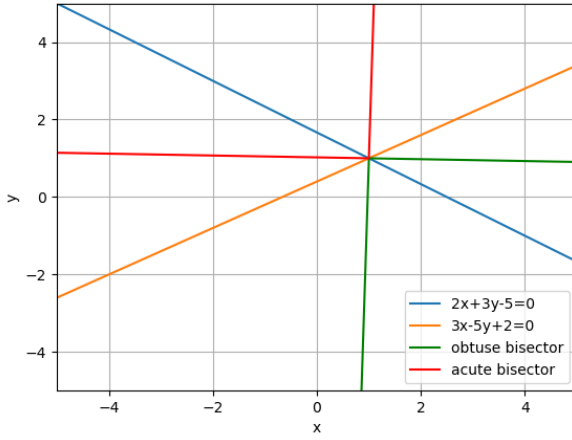


Fig. 2: Par of straight lines and their angular bisectors

Substituting the values we get

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{2\sqrt{442}} & \frac{21}{2\sqrt{442}} \\ \frac{21}{2\sqrt{442}} & -\frac{1}{2\sqrt{442}} \end{pmatrix} \mathbf{x} - \left(\frac{22}{\sqrt{442}} \quad \frac{20}{\sqrt{442}} \right) \mathbf{x} + \frac{21}{\sqrt{442}} = 0 \quad (7.0.3)$$

$$\frac{x^2}{2\sqrt{442}} + \frac{21}{\sqrt{442}}xy - \frac{y^2}{2\sqrt{442}} - \frac{22x}{\sqrt{442}} - \frac{20y}{\sqrt{442}} + \frac{21}{\sqrt{442}} = 0 \quad (7.0.4)$$

$$x^2 + 42xy - y^2 - 44x - 40y + 42 = 0 \quad (7.0.5)$$

Therefore the equation of bisectors of the given line in quadratic form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 21 \\ 21 & -1 \end{pmatrix} \mathbf{x} - (44 \quad 40) \mathbf{x} + 42 = 0 \quad (7.0.6)$$