

# Challenge 6

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## 3 EXPLANATION

**Abstract**—This document is to prove that convolution is a unique map.

Download all python codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

### 1 PROBLEM

Given two signals  $(x_0, \dots, x_{n-1})$  and  $(h_0, \dots, h_{m-1})$ , the (linear) convolution of the two is an  $m+n-1$ -length signal defined as

$$y(t) = (h * x)_t = \sum_{\tau=0}^{\tau=n-1} x_{\tau} h_{(t-\tau)} \quad (1.0.1)$$

$$0 \leq t < m + n - 1$$

If

$$y(n) = (h_1 * x) \quad (1.0.2)$$

$$y(n) = (h_2 * x) \quad (1.0.3)$$

then prove that  $h_1 = h_2$

### 2 CONSTRUCTION

Writing the convolution operation in matrix form

$$Y = \begin{pmatrix} h_0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_0 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & h_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_1 & h_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{m-1} & h_{m-2} & h_{m-3} & \dots & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & \dots & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & \dots & h_{m-n+3} & h_{m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} \quad (2.0.1)$$

Therefore we can write equation (1.0.1) in matrix form as  $\mathbf{Y} = \mathbf{H}\mathbf{X}$  where

$$Y = \begin{pmatrix} h_0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_0 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & h_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_1 & h_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{m-1} & h_{m-2} & h_{m-3} & \dots & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & \dots & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & \dots & h_{m-n+3} & h_{m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} \quad (3.0.1)$$

So the equations (1.0.2) and (1.0.3) can be written in matrix form where

$$\mathbf{H}_1 = \begin{pmatrix} h_{10} & 0 & 0 & \dots & 0 & 0 \\ h_{11} & h_{10} & 0 & \dots & 0 & 0 \\ h_{12} & h_{11} & h_{10} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1n-1} & h_{1n-2} & h_{1n-3} & \dots & h_{11} & h_{10} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1m-1} & h_{1m-2} & h_{1m-3} & \dots & h_{1m-n+1} & h_{1m-n} \\ 0 & h_{1m-1} & h_{1m-2} & \dots & h_{1m-n+2} & h_{1m-n+1} \\ 0 & 0 & h_{1m-1} & \dots & h_{1m-n+3} & h_{1m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{1m-1} \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{H}_2 = \begin{pmatrix} h_{20} & 0 & 0 & \dots & 0 & 0 \\ h_{21} & h_{20} & 0 & \dots & 0 & 0 \\ h_{22} & h_{21} & h_{20} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2n-1} & h_{2n-2} & h_{2n-3} & \dots & h_{21} & h_{20} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2m-1} & h_{2m-2} & h_{2m-3} & \dots & h_{2m-n+1} & h_{2m-n} \\ 0 & h_{2m-1} & h_{2m-2} & \dots & h_{2m-n+2} & h_{2m-n+1} \\ 0 & 0 & h_{2m-1} & \dots & h_{2m-n+3} & h_{2m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{2m-1} \end{pmatrix} \quad (3.0.3)$$

So

$$(h_1 * x) - (h_2 * x) = y(n) - y(n) = 0 \quad (3.0.4)$$

$$(\mathbf{H}_1 - \mathbf{H}_2)\mathbf{X} = 0 \quad (3.0.5)$$

For the sake of simplicity lets assume that  $m = n$ , then Toeplitz matrix is of the form

$$\mathbf{H} = \begin{pmatrix} L \\ U \end{pmatrix} \quad (3.0.6)$$

Because  $\mathbf{H}_1, \mathbf{H}_2$  are in toeplitz form, their difference is also in toeplitz form.

$$\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2 \quad (3.0.7)$$

$$\mathbf{H} = \begin{pmatrix} h_0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ h_1 & h_0 & 0 & \cdot & \cdot & 0 & 0 \\ h_2 & h_1 & h_0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{m-2} & h_{m-3} & \cdot & \cdot & h_1 & h_0 & 0 \\ h_{m-1} & h_{m-2} & h_{m-3} & \cdot & \cdot & h_1 & h_0 \\ 0 & h_{m-1} & h_{m-2} & \cdot & \cdot & h_2 & h_1 \\ 0 & 0 & h_{m-1} & \cdot & \cdot & h_3 & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & h_{m-1} \end{pmatrix} \quad (3.0.8)$$

The solution for  $LX = 0$  or  $UX = 0$  when  $X \neq 0$ , is that each of the elements of the matrices  $L, U$  is zero. So,  $\mathbf{H}_1 - \mathbf{H}_2 = 0$  which means  $\mathbf{H}_1 = \mathbf{H}_2$