

# Assignment 12

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Abstract

This document is about the linear operator and minimal polynomials.

Download all python codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

## 1 PROBLEM

Let  $\mathbf{T}$  be a linear operator on  $\mathbb{R}^2$ , the matrix of which in the standard ordered basis is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \quad (1.0.1)$$

- 1) Prove that the only subspaces of  $\mathbb{R}^2$  invariant under  $\mathbf{T}$  are  $\mathbb{R}^2$  and the zero subspace.
- 2) If  $\mathbf{U}$  is the linear operator on  $\mathbb{C}^2$ , the matrix of which in the standard ordered basis is  $\mathbf{A}$ , show that  $\mathbf{U}$  has 1-dimensional invariant subspaces.

## 2 PROOF

Definition Of linear operator	If $\mathbf{V}$ is a vector space over a field $\mathbb{F}$ , a linear operator on $\mathbf{V}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ . So for two vectors $\alpha$ and $\beta$ in $\mathbf{V}$ , the transformation will be $\mathbf{T}(c\alpha + \beta) = c(\mathbf{T}\alpha) + \mathbf{T}\beta$ where $\mathbf{T}\alpha$ and $\mathbf{T}\beta$ are in $\mathbf{V}$ and $c$ in $\mathbb{F}$ .
Invariant subspaces	If $\mathbf{W}$ is a subspace of $\mathbf{V}$ , we say that $\mathbf{W}$ is invariant under $\mathbf{T}$ if for each vector $\alpha$ in $\mathbf{W}$ the vector $\mathbf{T}\alpha$ is in $\mathbf{W}$ , i.e, if $\mathbf{T}(\mathbf{W})$ is contained in $\mathbf{W}$ .
Given	<p><math>\mathbf{T}</math> is a linear operator in <math>\mathbb{R}^2</math>, the matrix of which in the standard basis is</p> $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ <p>So by definition <math>\mathbb{R}^2</math> is invariant under <math>\mathbf{T}</math>, since <math>\mathbf{T}</math> is a linear operator.</p>

TABLE 1: construction

Characteristic equation of $\mathbf{A}$	$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1 - \lambda & -1 \\ 2 & 2 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 4.$ $\lambda_1 = \frac{3+\sqrt{7}i}{2} \text{ and } \lambda_2 = \frac{3-\sqrt{7}i}{2} \text{ are the eigenvalues of } \mathbf{A}.$
Proof for 1	<p>The null space of <math>\mathbf{T}</math> can be calculated as</p> $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \text{ by augmenting we get } \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix}$ <p>applying row reduction we get</p> $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \xrightarrow{R_2=R_2-2R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow{R_2=\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ <p>Therefore the nullspace of <math>\mathbf{T}</math> contains only the zero vector.</p> <p>Assume that there is a 1-dimensional subspace that is invariant under <math>\mathbf{T}</math>, then</p> $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \text{ is the eigen vector and } c \text{ is eigen value.}$ <p>Since the field of vector space is <math>\mathbb{R}</math>, there are no eigen values and hence no eigen vectors</p> <p>Since there are no eigen values in the field <math>\mathbb{R}</math>, there are no 1-dimensional vectors which can be invariant under <math>\mathbf{T}</math>.</p> <p>Since the range of the linear operator is <math>\mathbb{R}^2</math>, the vector space <math>\mathbb{R}^2</math> is invariant under <math>\mathbf{T}</math>.</p> <p>And as the nullspace maps to zero vector and zero vector is invariant under <math>\mathbf{T}</math>, the only subspaces that are invariant under <math>\mathbf{T}</math> are the vector space <math>\mathbb{R}</math> and the zero vector.</p>
Proof for 2	<p>If <math>\mathbf{U}</math> is the linear operator on <math>\mathbb{C}^2</math>, the matrix of which in the standard ordered basis is <math>\mathbf{A}</math></p> <p>The eigen vectors will be, for <math>\lambda_1 = \frac{3+\sqrt{7}i}{2}</math>, the nullspace of <math>\mathbf{A} - \lambda \mathbf{I}</math> will be the eigen vector.</p> $\begin{pmatrix} \frac{-1-\sqrt{7}i}{2} & -1 & 0 \\ 2 & \frac{1-\sqrt{7}i}{2} & 0 \end{pmatrix} \xrightarrow{R_2=R_2-\frac{(-1+\sqrt{7}i)}{2}R_1} \begin{pmatrix} \frac{-1-\sqrt{7}i}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

	<p>Therefore one of the eigen vectors is <math>\mathbf{e}_1 = \begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix}</math></p> <p>This eigen vector <math>\mathbf{e}_1</math> is a subspace of <math>\mathbb{C}^2</math>.</p>
	<p>Applying the linear operator <math>\mathbf{U}</math> on <math>\mathbf{e}_1</math>, we get</p> $\mathbf{U}(\mathbf{ce}_1) = \mathbf{Ae}_1 = c \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix} = c^{\frac{3+\sqrt{7}i}{2}} \begin{pmatrix} -1 \\ \frac{1+\sqrt{7}i}{2} \end{pmatrix} \Rightarrow \mathbf{U}(\mathbf{ce}_1) = c' \mathbf{e}_1$ <p>Therefore the vector space <math>\mathbb{C}^2</math> has 1-dimensional invariant subspaces which are the two subspaces along each eigen vectors containing the zero vector.</p>

TABLE 2: Proof