Assignment 5

Shaik Zeeshan Ali
AI20MTECH11001

***Abstract*—This document is about matrix representation of lines and the bisectors of angles between them.**

Download all python codes from

```
https://github.com/Zeeshan-IITH/IITH-EE5609/
    new/master/codes
```

and latex-tikz codes from

```
https://github.com/Zeeshan-IITH/IITH-EE5609
```

1 Problem

Show that the equation

$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\ -\frac{1}{2} & -15\end{pmatrix}\mathbf{x}+\begin{pmatrix}-11 & 31\end{pmatrix}\mathbf{x}-10=0 \tag{1.0.1}$$

represents two straight lines,and find the equations of the bisectors of the angles between them.

2 Construction

Any quadratic equation in terms of x, y of the form $ax^2+2bxy+cy^2+2dx+2ey+f=0$,can be written as

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.0.1}$$

$$\text{where,}\mathbf{V}=\begin{pmatrix}a & b\\ b & c\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u}=\begin{pmatrix}d\\ e\end{pmatrix} \tag{2.0.3}$$

The equation (1.0.1) represents two intersecting straight lines when

$$\begin{vmatrix}\mathbf{V} & u\\ u^T & f\end{vmatrix}=0 \tag{2.0.4}$$

$$\left|\mathbf{V}\right|<0 \tag{2.0.5}$$

3 Explanation

From equation (1.0.1) we get

$$\mathbf{V}=\begin{pmatrix}6 & -\frac{1}{2}\\ -\frac{1}{2} & -15\end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u}=\begin{pmatrix}\frac{-11}{2}\\ \frac{31}{2}\end{pmatrix} \tag{3.0.2}$$

$$f=-10 \tag{3.0.3}$$

calculating the equation (2.0.4),we get

$$\begin{pmatrix}6 & -\frac{1}{2} & \frac{-11}{2}\\ -\frac{1}{2} & -15 & \frac{31}{2}\\ \frac{-11}{2} & \frac{31}{2} & -10\end{pmatrix}\xleftrightarrow{R_3=R_3+R_2+R_1}\begin{pmatrix}6 & -\frac{1}{2} & \frac{-11}{2}\\ -\frac{1}{2} & -15 & \frac{31}{2}\\ 0 & 0 & 0\end{pmatrix} \tag{3.0.4}$$

Therefore the determinant 0.And also the determinant of $\mathbf{V}$ is

$$\left|\mathbf{V}\right|=\begin{vmatrix}6 & -\frac{1}{2}\\ -\frac{1}{2} & -15\end{vmatrix} \tag{3.0.5}$$

$$=-90.25 \tag{3.0.6}$$

$$<0 \tag{3.0.7}$$

Therefore the given equation represents the equation of two straight lines which intersect.

4 Point of intersection

Let the two lines be

$$\mathbf{n_1}^T\mathbf{x}=c_1 \tag{4.0.1}$$

$$\mathbf{n_2}^T\mathbf{x}=c_2 \tag{4.0.2}$$

The equation of two lines in quadratic form will be

$$\left(\mathbf{n_1}^T\mathbf{x}-c_1\right)\left(\mathbf{n_2}^T\mathbf{x}-c_2\right)=0 \tag{4.0.3}$$

comparing equations (1.0.1) and (4.0.3),we get

$$-\left(c_2\mathbf{n_1}^T+c_1\mathbf{n_2}^T\right)=\begin{pmatrix}-11 & 31\end{pmatrix} \tag{4.0.4}$$

$$c_1c_2=-10 \tag{4.0.5}$$

slopes of the two lines are the roots of the equation

$$cm^2+2bm+a=0 \tag{4.0.6}$$

$$15m^2+m-6=0 \tag{4.0.7}$$

$$m_1=\frac{-2}{3}, m_2=\frac{3}{5} \tag{4.0.8}$$

therefore the normal vectors will be

$$\mathbf{n_i}=k_i\begin{pmatrix}-m_i\\ 1\end{pmatrix} \tag{4.0.9}$$

$$\mathbf{n_1}=k_1\begin{pmatrix}\frac{2}{3}\\ 1\end{pmatrix} \tag{4.0.10}$$

$$\mathbf{n_2}=k_2\begin{pmatrix}\frac{-3}{5}\\ 1\end{pmatrix} \tag{4.0.11}$$

We know that

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.12}$$

$$k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} \tag{4.0.13}$$

$$k_1 k_2 = -15 \tag{4.0.14}$$

Choosing the values of $k_1 = 3$ and $k_2 = -5$,we get

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{4.0.15}$$

For verifying the values of $\mathbf{n_1}$ and $\mathbf{n_2}$,we compute the convolution by representing $\mathbf{n_1}$ as a toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{4.0.16}$$

$$= \begin{pmatrix} 6 \\ -1 \\ -15 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.17}$$

Using the equations (4.0.4) and (4.0.5), we get

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 11 \\ -31 \end{pmatrix} \tag{4.0.18}$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -5 & -31 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 - 3R_1} \begin{pmatrix} 2 & 3 & 11 \\ 0 & -19 & -95 \end{pmatrix} \tag{4.0.19}$$

$$\begin{pmatrix} 2 & 3 & 11 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 5 \end{pmatrix} \tag{4.0.20}$$

The values are $c_2 = 5$ and $c_1 = -2$.Therefore the equation of the lines are

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 5 \tag{4.0.21}$$

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = -2 \tag{4.0.22}$$

$$(2x + 3y - 5)(3x - 5y + 2) = 0 \tag{4.0.23}$$

The point of intersection will be

$$\begin{pmatrix} 2 & 3 \\ 3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{4.0.24}$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & -5 & -2 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 - 3R_1} \begin{pmatrix} 2 & 3 & 5 \\ 0 & -19 & -19 \end{pmatrix} \tag{4.0.25}$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.0.26}$$

Therefore the lines intersect at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

5 Eigenvectors

The characteristic equation of the matrix $\mathbf{V}$ is

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \tag{5.0.1}$$

$$\begin{vmatrix} 6 - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15 - \lambda \end{vmatrix} = 0 \tag{5.0.2}$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \tag{5.0.3}$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2} \left(9 + \sqrt{442} \right) \tag{5.0.4}$$

$$\lambda_2 = \frac{-1}{2} \left(9 - \sqrt{442} \right) \tag{5.0.5}$$

The eigen vectors will be in the nullspace of $\mathbf{V} - \lambda_1 \mathbf{I}$ and $\mathbf{V} - \lambda_2 \mathbf{I}$.The eigen vector corresponding to eigen value λ_1 will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2} \left(9 + \sqrt{442} \right) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2} \left(9 + \sqrt{442} \right) \end{pmatrix} \tag{5.0.6}$$

$$= \begin{pmatrix} \frac{1}{2} \left(21 + \sqrt{442} \right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \left(-21 + \sqrt{442} \right) \end{pmatrix} \tag{5.0.7}$$

$$\xleftrightarrow{R_2 = \left(21 + \sqrt{442} \right) R_2 + R_1} \begin{pmatrix} \frac{1}{2} \left(21 + \sqrt{442} \right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \tag{5.0.8}$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \tag{5.0.9}$$

normalizing p_1,we get

$$p_1 = \begin{pmatrix} 0.0238 \\ 1 \end{pmatrix} \tag{5.0.10}$$

the eigen vector corresponding to eigen value λ_2 will be

$$\mathbf{V}-\lambda_1\mathbf{I}=\begin{pmatrix}6+\frac{1}{2}\left(9-\sqrt{442}\right) & -\frac{1}{2}\\ -\frac{1}{2} & -15+\frac{1}{2}\left(9-\sqrt{442}\right)\end{pmatrix} \tag{5.0.11}$$

$$=\begin{pmatrix}\frac{1}{2}\left(21-\sqrt{442}\right) & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\left(-21-\sqrt{442}\right)\end{pmatrix} \tag{5.0.12}$$

$$\xleftrightarrow{R_2=\left(21-\sqrt{442}\right)R_2+R_1}\begin{pmatrix}\frac{1}{2}\left(21-\sqrt{442}\right) & -\frac{1}{2}\\ 0 & 0\end{pmatrix} \tag{5.0.13}$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_2=\begin{pmatrix}1\\ 21-\sqrt{442}\end{pmatrix} \tag{5.0.14}$$

normalizing p_2,we get

$$p_2=\begin{pmatrix}1\\ -0.0238\end{pmatrix} \tag{5.0.15}$$

So the transformation matrix will be

$$\mathbf{P}=\begin{pmatrix}p_1 & p_2\end{pmatrix}=\begin{pmatrix}0.0238 & 1\\ 1 & -0.0238\end{pmatrix} \tag{5.0.16}$$

6 Affine Transformation

Doing the affine transformation on given quadratic equation, we get pair to intersecting straight lines passing through origin.

Let the affine transformation be $\mathbf{x}=\mathbf{P}\mathbf{y}+c$.The transformation will be

$$\left(\mathbf{P}\mathbf{y}+c\right)^T\mathbf{V}\left(\mathbf{P}\mathbf{y}+c\right)+2\mathbf{u}^T\left(\mathbf{P}\mathbf{y}+c\right)+f=0 \tag{6.0.1}$$

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y}+2\left(c^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{P}\mathbf{y}\\ +c^T\mathbf{V}c+2\mathbf{u}^Tc+f=0 \tag{6.0.2}$$

if the point c is taken as the point of intersection of the two lines.

$$c^T\mathbf{V}c+2\mathbf{u}^Tc+f=0 \tag{6.0.3}$$

$$c^T\mathbf{V}+\mathbf{u}^T=0 \tag{6.0.4}$$

So the affine transformation of the given lines will be

$$\mathbf{y}^T\left(\mathbf{P}^T\mathbf{V}\mathbf{P}\right)\mathbf{y}=0 \tag{6.0.5}$$

$$\mathbf{y}^T\begin{pmatrix}-1.5 & 0\\ 0 & 6\end{pmatrix}\mathbf{y}=0 \tag{6.0.6}$$

$$\left(x-2y\right)\left(x+2y\right)=0 \tag{6.0.7}$$

This line will pass through origin,whose bisectors

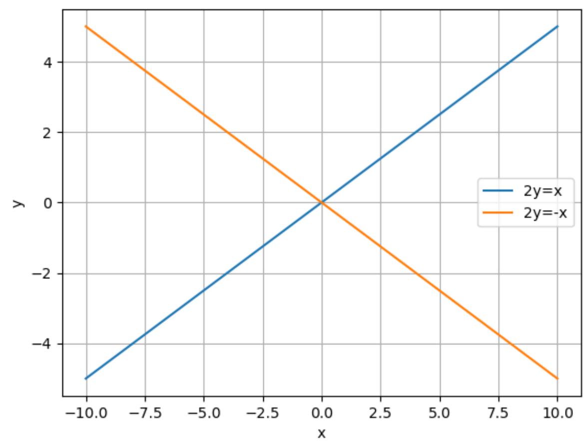

Fig. 1: straight lines after affine transformation passing through origin

will be the $x-$axis and $y-$axis.The bisectors wil be of the form

$$\mathbf{y}^T\begin{pmatrix}0 & 0.5\\ 0.5 & 0\end{pmatrix}\mathbf{y}=0 \tag{6.0.8}$$

$$\mathbf{y}^T\mathbf{K}\mathbf{y}=0 \tag{6.0.9}$$

$$\mathbf{K}=\begin{pmatrix}0 & 0.5\\ 0.5 & 0\end{pmatrix} \tag{6.0.10}$$

7 Bisectors

Taking the inverse of the affine transformation of the equation $xy=0$,will give the angle bisectors.

$$\left(\mathbf{P}^{-1}\mathbf{x}-\mathbf{P}^{-1}c\right)^T\mathbf{K}\left(\mathbf{P}^{-1}\mathbf{x}-\mathbf{P}^{-1}c\right)=0 \tag{7.0.1}$$

$$\mathbf{x}^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x}-2c^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x}+c^T\mathbf{P}\mathbf{K}\mathbf{P}^Tc=0 \tag{7.0.2}$$

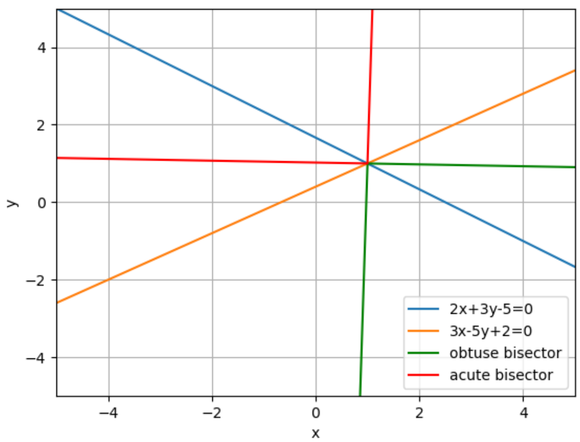

Fig. 2: Par of straight lines and their angular bisectors

Substituting the values we get

$$\mathbf{x}^T \begin{pmatrix} 0.0238 & 0.5 \\ 0.5 & -0.0238 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1.046 & 0.951 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{7.0.3}$$

$$0.0238x^2 + xy - 0.0238y^2 - 1.046x - 0.951y + 1 = 0 \tag{7.0.4}$$

$$x^2 + 42xy - y^2 - 44x - 40y + 42 = 0 \tag{7.0.5}$$