

Challenge 6

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AI20MTECH11001

Abstract—This document is to prove that convolution is a unique map.

Download all python codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

1 PROBLEM

Given two signals (x_0, \dots, x_{n-1}) and (h_0, \dots, h_{m-1}) , the (linear) convolution of the two is an $m+n-1$ -length signal defined as

$$y(t) = (h * x)_t = \sum_{\tau=0}^{\tau=n-1} x_{\tau} h_{(t-\tau)} \quad (1.0.1)$$

$$0 \leq t < m + n - 1$$

If

$$y(n) = (h_1 * x) \quad (1.0.2)$$

$$y(n) = (h_2 * x) \quad (1.0.3)$$

then prove that $h_1 = h_2$

2 CONSTRUCTION

Writing the convolution operation in matrix form

$$Y = \begin{pmatrix} h_0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_0 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & h_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_1 & h_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{m-1} & h_{m-2} & h_{m-3} & \dots & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & \dots & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & \dots & h_{m-n+3} & h_{m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} \quad (2.0.1)$$

3 EXPLANATION

Therefore we can write equation (1.0.1) in matrix form as $Y = HX$ where

$$Y = \begin{pmatrix} h_0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_0 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & h_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_1 & h_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{m-1} & h_{m-2} & h_{m-3} & \dots & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & \dots & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & \dots & h_{m-n+3} & h_{m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} \quad (3.0.1)$$

So the equations (1.0.2) and (1.0.3) can be written in matrix form where

$$H_1 = \begin{pmatrix} h_{10} & 0 & 0 & \dots & 0 & 0 \\ h_{11} & h_{10} & 0 & \dots & 0 & 0 \\ h_{12} & h_{11} & h_{10} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1n-1} & h_{1n-2} & h_{1n-3} & \dots & h_{11} & h_{10} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1m-1} & h_{1m-2} & h_{1m-3} & \dots & h_{1m-n+1} & h_{1m-n} \\ 0 & h_{1m-1} & h_{1m-2} & \dots & h_{1m-n+2} & h_{1m-n+1} \\ 0 & 0 & h_{1m-1} & \dots & h_{1m-n+3} & h_{1m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{1m-1} \end{pmatrix} \quad (3.0.2)$$

$$H_2 = \begin{pmatrix} h_{20} & 0 & 0 & \dots & 0 & 0 \\ h_{21} & h_{20} & 0 & \dots & 0 & 0 \\ h_{22} & h_{21} & h_{20} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2n-1} & h_{2n-2} & h_{2n-3} & \dots & h_{21} & h_{20} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2m-1} & h_{2m-2} & h_{2m-3} & \dots & h_{2m-n+1} & h_{2m-n} \\ 0 & h_{2m-1} & h_{2m-2} & \dots & h_{2m-n+2} & h_{2m-n+1} \\ 0 & 0 & h_{2m-1} & \dots & h_{2m-n+3} & h_{2m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_{2m-1} \end{pmatrix} \quad (3.0.3)$$

So

$$(h_1 * x) - (h_2 * x) = y(n) - y(n) = 0 \quad (3.0.4)$$

$$(H_1 - H_2)X = 0 \quad (3.0.5)$$

For the sake of simplicity let's assume that $m = n$, then Toeplitz matrix is of the form

$$\mathbf{H} = \begin{pmatrix} L \\ U \end{pmatrix} \quad (3.0.6)$$

Because $\mathbf{H}_1, \mathbf{H}_2$ are in toeplitz form, their difference is also in toeplitz form.

$$\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2 \quad (3.0.7)$$

$$\mathbf{H} = \begin{pmatrix} h_0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ h_1 & h_0 & 0 & \cdot & \cdot & 0 & 0 \\ h_2 & h_1 & h_0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{m-2} & h_{m-3} & \cdot & \cdot & h_1 & h_0 & 0 \\ h_{m-1} & h_{m-2} & h_{m-3} & \cdot & \cdot & h_1 & h_0 \\ 0 & h_{m-1} & h_{m-2} & \cdot & \cdot & h_2 & h_1 \\ 0 & 0 & h_{m-1} & \cdot & \cdot & h_3 & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & h_{m-1} \end{pmatrix} \quad (3.0.8)$$

Suppose

$$L = \begin{pmatrix} h_0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ h_1 & h_0 & 0 & \cdot & \cdot & 0 & 0 \\ h_2 & h_1 & h_0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{n-2} & h_{n-3} & \cdot & \cdot & h_1 & h_0 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdot & \cdot & h_1 & h_0 \end{pmatrix} \quad (3.0.9)$$

$$LX = 0 \quad (3.0.10)$$

For $LX = 0$ to have a non-trivial solution i.e nullity $\neq 0$, the rank of the matrix $R(L) < n$, which implies $h_0 = 0$. So, $\text{rank}(L) = 0$. Similarly if you consider the submatrix of L

$$L = \begin{pmatrix} h_1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ h_2 & h_1 & 0 & \cdot & \cdot & 0 & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{n-2} & h_{n-3} & \cdot & \cdot & h_2 & h_2 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdot & \cdot & h_2 & h_1 \end{pmatrix} \quad (3.0.11)$$

where

$$\begin{pmatrix} h_1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ h_2 & h_1 & 0 & \cdot & \cdot & 0 & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{n-2} & h_{n-3} & \cdot & \cdot & h_2 & h_2 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdot & \cdot & h_2 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \end{pmatrix} = 0 \quad (3.0.12)$$

By using the similar logic, $h_1 = 0$. And all the elements of the lower triangular matrix are zero i.e $L = 0$ which implies $H = 0$.

The solution for $LX = 0$ or $UX = 0$ when $X \neq 0$, is that each of the elements of the matrices L, U is zero irrespective of X because X , the input signal does not depend on the system. So, $H = 0$, $\mathbf{H}_1 - \mathbf{H}_2 = 0$ which means $\mathbf{H}_1 = \mathbf{H}_2$