

# Assignment 2

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**Abstract**—This document explains how to find the shortest distance between two lines if and when the two lines are not intersecting with each other.

Download all python codes from

<https://github.com/Zeehan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeehan-IITH/IITH-EE5609>

## 1 PROBLEM

Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (1.0.1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (1.0.2)$$

## 2 CONSTRUCTION

When two lines are not intersecting the distance between them is non-zero. The equation of above mentioned lines in symmetric form is

$$L_1: x - 1 = 2 - y = z - 1 \quad (2.0.1)$$

$$L_2: \frac{x - 2}{2} = y + 1 = \frac{z + 1}{2} \quad (2.0.2)$$

The above line equations have no point of intersection as for no value of  $\lambda_1, \lambda_2$  both the equations (2.0.1) and (2.0.2) are equal.

If the two line intersect then (2.0.1)=(2.0.2) i.e.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (2.0.3)$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (2.0.4)$$

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (2.0.5)$$

The Augmented matrix will be

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (2.0.6)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (2.0.7)$$

The above matrix has a  $rank = 3$ . Hence the lines do not intersect.

## 3 SOLUTION

Let  $A$  be a point on line  $L_1$  and  $B$  be point on the line  $L_2$ . Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines  $L_1, L_2$  and passing through  $A$  and  $B$ .

The vector passing through the points  $A$  and  $B$  will be

$$\mathbf{A} - \mathbf{B} = \mathbf{x}_1 - \mathbf{x}_2 + (\mathbf{m}_1 - \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (3.0.1)$$

The vectors  $\mathbf{m}_1, \mathbf{m}_2$  are both perpendicular to the line  $AB$ . So the dot product of  $\mathbf{m}_1, \mathbf{m}_2$  with the line  $AB$  is zero.

The dot product of  $\mathbf{m}_1$  with the line  $AB$  is

$$\mathbf{m}_1^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.0.2)$$

$$\mathbf{m}_1^T (\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{m}_1^T (\mathbf{m}_1 - \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \quad (3.0.3)$$

The dot product of  $\mathbf{m}_2$  with the line  $AB$  is

$$\mathbf{m}_2^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.0.4)$$

$$\mathbf{m}_2^T (\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{m}_2^T (\mathbf{m}_1 - \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \quad (3.0.5)$$

Let the matrix  $\mathbf{M}$  be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \end{pmatrix} \quad (3.0.6)$$

Combining the equations (3.0.3) and (3.0.5) in matrix form, using equation (3.0.6), we get

$$\mathbf{M} \mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} + \mathbf{M} (\mathbf{x}_1 - \mathbf{x}_2) = 0 \quad (3.0.7)$$

simplifying it further

$$\mathbf{M} \mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M} (\mathbf{x}_2 - \mathbf{x}_1) \quad (3.0.8)$$

To find the points on the lines which make up the shortest distance we need to find  $\lambda_1$  and  $\lambda_2$  using the above expression to get the augmented form

$$\begin{pmatrix} \mathbf{m}_1^T \mathbf{m}_1 & \mathbf{m}_1^T \mathbf{m}_2 \\ \mathbf{m}_2^T \mathbf{m}_1 & \mathbf{m}_2^T \mathbf{m}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{m}_1^T (\mathbf{x}_2 - \mathbf{x}_1) \\ \mathbf{m}_2^T (\mathbf{x}_2 - \mathbf{x}_1) \end{pmatrix} \quad (3.0.9)$$

$$\begin{pmatrix} \mathbf{m}_1^T \mathbf{m}_1 & \mathbf{m}_1^T \mathbf{m}_2 & \mathbf{m}_1^T (\mathbf{x}_2 - \mathbf{x}_1) \\ \mathbf{m}_2^T \mathbf{m}_1 & \mathbf{m}_2^T \mathbf{m}_2 & \mathbf{m}_2^T (\mathbf{x}_2 - \mathbf{x}_1) \end{pmatrix} \quad (3.0.10)$$

we know that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{m}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

so the augmented matrix will be

$$\begin{pmatrix} 23 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \quad (3.0.11)$$

Using row reduction method to get row-reduced echelon form, we get

$$\begin{pmatrix} 23 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.0.12)$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=2R_1-R_2} \begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.0.13)$$

$$\begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{6}, R_2=\frac{R_2}{6}} \begin{pmatrix} 1 & 0 & \frac{11}{6} \\ 0 & 1 & -\frac{7}{6} \end{pmatrix} \quad (3.0.14)$$

$$\text{So the values are } \lambda_1 = \frac{11}{6} \text{ and } \lambda_2 = \frac{7}{6} \quad (3.0.15)$$

Using the equation (1.0.1) and (1.0.2), we get the points as

$$\frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} \text{ and } \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \text{ on the line } L_1, L_2 \text{ respectively}$$

The shortest distance between the lines is the distance between the points  $A$  and  $B$ .

$$\|(\mathbf{B} - \mathbf{A})\| = \left\| \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix} \right\| = \frac{3}{\sqrt{2}} \quad (3.0.16)$$

Therefore the shortest distance between the given

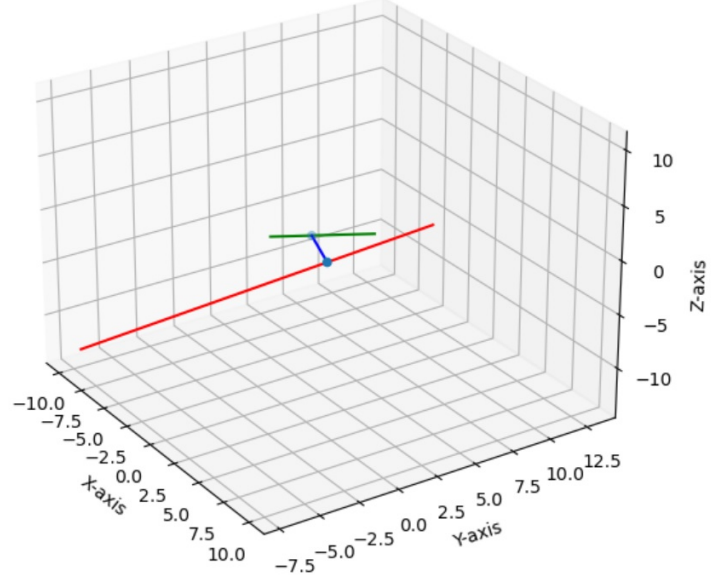


Fig. 1: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

lines is  $\frac{3}{\sqrt{2}}$ .

The unit vector perpendicular to lines

$$\text{Line}_1: \mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3.0.17)$$

$$\text{Line}_2: \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (3.0.18)$$

can be found by

$$\frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} \quad (3.0.19)$$

$$\frac{\frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}}{\frac{3}{\sqrt{2}}}$$

So the unit vector perpendicular to both  $L_1$  and  $L_2$  is

$$\mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad (3.0.20)$$