QR decomposition

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***Abstract*—This document analyzes QR decomposition of a non-singular square matrix**

Download all python codes from

https://github.com/Zeeshan−IITH/IITH−EE5609/new/master/codes

and latex-tikz codes from

https://github.com/Zeeshan−IITH/IITH−EE5609

1 Problem

What is the QR decomposition of a non-singular square matrix whose columns are mutually orthogonal.

2 Construction

The purpose of QR decomposition can be thought of as constructing a basis for the column space of a non-singular matrix.So the matrix Q represents the orthonormal basis of the columnn space and the matrix R represents the weights that each of the orthonormal basis vectors carry for each column vector.Let $\mathbf{A}$ be a square matrix of order $n \times n$ whose column vectors are mutually orthogonal.

$$\mathbf{A} = \begin{pmatrix} c_1 & c_2 & c_3 & . & . & . & c_n \end{pmatrix} \tag{2.0.1}$$

where

$$c_i^T c_j = 0$$
$$i \neq j \tag{2.0.2}$$

where every column vector is of dimension $n \times 1$.So the column space of the matrix $\mathbf{A}$ is a set of n linearly independent vectors

The orthogonal matrix Q will be

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 & . & . & . & q_n \end{pmatrix} \tag{2.0.3}$$
$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.0.4}$$

where q_i is a column vector of dimension $n \times 1$ and $q_i^T q_i = 1$ and $q_i^T q_j = 0$ if $i \neq j$.

we can define an upper triangular matrix R as

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & . & . & r_{1n} \\ 0 & r_{22} & . & . & r_{2n} \\ . & . & . & . & . \\ 0 & 0 & . & . & r_{nn} \end{pmatrix} \tag{2.0.5}$$

The matrix $\mathbf{A}$ can be written in terms of $\mathbf{Q}$ and $\mathbf{R}$ as

$$\mathbf{A} = \mathbf{QR} \tag{2.0.6}$$

3 Explanation

Because the column vectors of $\mathbf{A}$ are orthogonal to each other

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} c_1 & c_2 & . & . & . & c_n \end{pmatrix}^T \begin{pmatrix} c_1 & c_2 & . & . & . & c_n \end{pmatrix} \tag{3.0.1}$$

$$= \begin{pmatrix} c_1^T \\ c_2^T \\ c_3^T \\ . \\ . \\ . \\ c_n^T \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & . & . & . & c_n \end{pmatrix} \tag{3.0.2}$$

$$= \begin{pmatrix} ||c_1||^2 & 0 & . & . & . & 0 \\ 0 & ||c_2||^2 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & ||c_n||^2 \end{pmatrix} \tag{3.0.3}$$

which is a diagonal matrix because every column vector is mutually orthogonal.

$$\mathbf{A}^T\mathbf{A} = (\mathbf{QR})^T\mathbf{QR} \tag{3.0.4}$$
$$= \mathbf{R}^T\mathbf{Q}^T\mathbf{QR} \tag{3.0.5}$$
$$= \mathbf{R}^T\mathbf{IR} \tag{3.0.6}$$
$$= \mathbf{R}^T\mathbf{R} \tag{3.0.7}$$
$$= \begin{pmatrix} r_{11}^2 & r_{11}r_{12} & . & . & . & r_{11}r_{1n} \\ r_{11}r_{12} & r_{22}^2 & . & . & . & r_{22}r_{2n} \\ . & . & . & . & . & . \\ r_{11}r_{1n} & r_{22}r_{2n} & . & . & . & r_{nn}^2 \end{pmatrix} \tag{3.0.8}$$

Comparing equations (3.0.3) and (3.0.8),we get

$$r_{ii} = ||c_i|| \tag{3.0.9}$$
$$r_{ij} = 0 \tag{3.0.10}$$

The matrix $\mathbf{R}$ will be

$$\mathbf{R} = \begin{pmatrix} ||c_1|| & 0 & . & . & . & 0 \\ 0 & ||c_2|| & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & ||c_n|| \end{pmatrix} \tag{3.0.11}$$

Therefore the orthonormal vectors of **Q** will be the normalized column vectors of the matrix **A**.

$$\mathbf{Q} = \begin{pmatrix} \frac{c_1}{\|c_1\|} & \frac{c_2}{\|c_2\|} & \cdot & \cdot & \frac{c_n}{\|c_n\|} \end{pmatrix} \tag{3.0.12}$$