

Assignment 6

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Abstract—This document is about tracing a parabola

Download all python codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609/new/master/codes>

and latex-tikz codes from

<https://github.com/Zeeshan-IITH/IITH-EE5609>

1 PROBLEM

Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \quad (1.0.1)$$

2 CONSTRUCTION

The given quadratic equation can be written in the matrix form as

3 CONSTRUCTION

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \quad (3.0.1)$$

Calculating the parameters, we get

$$|\mathbf{V}| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \quad (3.0.2)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \quad (3.0.3)$$

Therefore the given parabola equation is a degenerate. The quadratic equation corresponds to a pair of coincident straight lines.

The characteristic equation of \mathbf{V} will be

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{vmatrix} \quad (3.0.4)$$

$$= \lambda^2 - 5\lambda \quad (3.0.5)$$

$$\lambda_1 = 0, \lambda_2 = 5 \quad (3.0.6)$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda \mathbf{I}$. For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_2=2R_2+R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \quad (3.0.7)$$

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.0.8)$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (3.0.9)$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xrightarrow{R_2=R_2-2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \quad (3.0.10)$$

$$p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.11)$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.0.12)$$

Therefore the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.0.13)$$

4 EQUATION OF THE COINCIDENT LINE

The general equation of coincident lines in quadratic form can be written as

$$(mx - y + c)^2 = 0 \quad (4.0.1)$$

$$\mathbf{x}^T \begin{pmatrix} m^2 & -m \\ -m & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} cm & -c \end{pmatrix} \mathbf{x} + c^2 = 0 \quad (4.0.2)$$

The degenerate equation in matrix form will be

$$(\mathbf{x}^T \ 1) \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \quad (4.0.3)$$

So the solution to nullspace will be

$$\begin{pmatrix} m^2 & -m & cm \\ -m & 1 & -c \\ cm & -c & c^2 \end{pmatrix} \xrightarrow{R_3=mR_3-cR_1} \begin{pmatrix} m^2 & -m & cm \\ -m & 1 & -c \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.4)$$

$$\xrightarrow{R_2=mR_2+R_1} \begin{pmatrix} m^2 & -m & cm \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{m}} \begin{pmatrix} m & -1 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.5)$$

So the 3×3 matrix can also be written as

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} = \begin{pmatrix} m \\ -1 \\ c \end{pmatrix} \begin{pmatrix} m & -1 & c \end{pmatrix} \quad (4.0.6)$$

This matrix has a rank of 1.

Applying this on the given problem we get

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{pmatrix} = \begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{pmatrix} \quad (4.0.7)$$

$$\xleftrightarrow{R_3 = \frac{2R_2}{3} + R_1} \begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.8)$$

Therefore the equation of the line is $2x - y - 3 = 0$, which can be expressed in vector form as

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}^T \mathbf{x} = 3 \quad (4.0.9)$$