Chapter 5 Keywords:

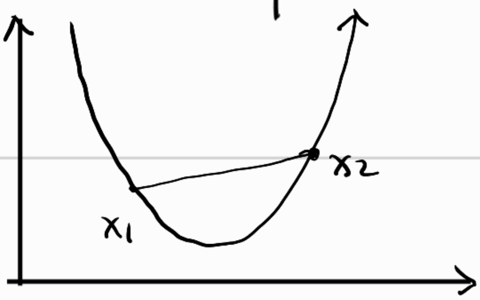
Non-linear program, Convex Function, Epigraph, Subgradients, Supporting halfspace, Karush-Kuhn-Tucker Theorems, KKT, Interior point

▷ Convex Functions:

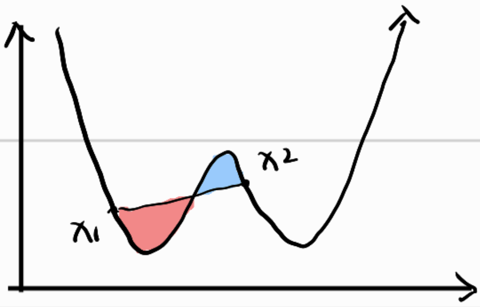
Recall that a function $f: \mathbb{R}^n - \mathbb{R}$ is convex iff $\forall x_1, x_2$, and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Convex:



Non convex:



▷ Epigraph:

the epigraph of a function is $epi(f) = \{(\mu, x) \in \mathbb{R}^{n+1} : f(x) \leq \mu\}$ or, the set of all points strictly above f.

▷ f is convex iff $epi(f)$ is convex.

▷ Level set:

Let $g: \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\beta \in \mathbb{R}$. We call the set $\{x \in \mathbb{R}^n : g(x) \leq \beta\}$ the level set of function g at β

Basically, the level set is the set of points above g and below β.

▷ If f is convex, then its level set is a convex set.

▷ The feasible region of a convex NLP is a convex set.

▷ Subgradient:

Let $g: \mathbb{R}^n \to \mathbb{R}$ be convex and let $\bar{x} \in \mathbb{R}^n$. $s \in \mathbb{R}^n$ is a subgradient of f at $\bar{x}$ if $\forall x \in \mathbb{R}^n$, the following holds:

$$h(x) = f(\bar{x}) + s^T(x - \bar{x}) \le f(x)$$

Note that $h(\bar{x}) = f(\bar{x})$

The subgradient provides a lower bound of the function f at point $\bar{x}$.

▷ Supporting Halfspaces:

let C be a convex set, $\bar{x} \in C$. The halfspace $F = \{x \in \mathbb{R}^n : s^T x \le \beta\}$ is a supporting halfspace of C at $\bar{x}$ if

1. $C \subseteq F$
2. $s^T \bar{x} = \beta$, or $\bar{x}$ is on the boundary of the halfspace

▷ Let $g: \mathbb{R}^n \to \mathbb{R}$ be a convex function, let $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) = 0$ and let $s \in \mathbb{R}^n$ be a subgradient of f at $\bar{x}$. Let C be the level set, $\{x \in \mathbb{R}^n : g(x) \le 0\}$ and F be the halfspace $\{x \in \mathbb{R}^n : g(\bar{x}) + s^T(x - \bar{x}) \le 0\}$.

Then F is a supporting halfspace of C at $\bar{x}$.

▷ Let NLP be $\min\{f(x) : g_i(x) \le 0, i \in \{1 \cdots m\}\}$. Let $\bar{x}$ be a feasible solution, and let $g_i(x) \le 0$ is tight for some $i \in \{1, \cdots, m\}$. If g_i is convex with subgradient s at $\bar{x}$, then the NLP obtained by replacing $g_i(x) \le 0$ by $s^T x \le s^T \bar{x} - g_i(\bar{x})$ is a relaxation of the original NLP.

▷ for $\min\{c^T x, g_i(x) \le 0, i \in \{1, \cdots, m\}\}$, if g_i are convex, $\bar{x}$ is feasible, and $\forall i \in J(\bar{x})$, $J(\bar{x}) = \{i : g_i(\bar{x}) = 0\}$ (tight constraint), we have a subgradient s_i at $\bar{x}$. If $-c \in \text{cone}\{s_i : i \in J(\bar{x})\}$, $\bar{x}$ is optimal.

▷ Let $f: \mathbb{R}^n \to \mathbb{R}$ be convex, let $\bar{x} \in \mathbb{R}^n$. If $\nabla f(\bar{x})$ exists, it is a subgradient of f at $\bar{x}$.

▷ KKT Theorem:

Let NLP be $\min\{f(x) : g_i(x) \le 0,\ i \in \{1, \cdots, m\}\}$, where all g are convex. Let there be a slater point where $g_i < 0\ \forall i$. Let $\bar{x}$ be a feasible solution to NLP, and assume that f, g are differentiable. Then $\bar{x}$ is optimal for NLP iff $-\nabla f(\bar{x}) \in \operatorname{cone}\{\nabla g_i(\bar{x}) : i \in J(\bar{x})\}$ (*)

or $-\nabla f(\bar{x})$ is in the cone of tight constraints at $\bar{x}$.

(*) can also be replaced with $\exists\, y \in \mathbb{R}^m_+$

$$-\nabla f(\bar{x}) = \sum_{i=1}^{m} \bar{y}_i \nabla g_i(\bar{x})$$

$$\bar{y}_i g_i(\bar{x}) = 0,\ \forall i$$