Chapter 4 keywords:

Integer programming, Cutting Plane

▷ Fundamental Theorem of Integer Programming:

for $P=\{x\in\mathbb{R}^n : Ax\le b\}$, where all entries of A and b are rational numbers. Let S be the set of all integer points in P. Then the convex hull of S is a polyhedron Q described by a matrix and a vector with all rational entries.

▷ IP relaxation:

Consider IP

$$\begin{aligned} \max\ & c^Tx \\ \text{s.t. } & Ax\le b \\ & \text{int } x \end{aligned}$$

Then for the following LP:

$$\begin{aligned} \max\ & c^Tx \\ \text{s.t } & A'x\le b' \end{aligned}$$

($A'x\le b'$ represent the convex hull of all feasible solution of the IP)

Then:

1. IP is feasible iff LP is feasible
2. IP is unbounded iff LP is unbounded
3. if $\bar{x}$ is optimal to IP, $\bar{x}$ is optimal to LP.
4. if $\bar{x}$ is optimal to LP, $\bar{x}$ is optimal to IP if it is an extreme point.

▷ Cutting planes:

Suppose we wish to solve the following IP:

$$\begin{aligned} \max\ & (2,5)x \\ \text{s.t.}\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad \text{(IP)} \\ & \text{int } x \geq 0 \end{aligned}$$

we first ignore the integrality requirements, and solve for the optimal solution. We get the optimal solution $\bar{x} = (\frac{8}{3}, \frac{4}{3})^T$. But this is not a feasible solution to IP. We wish to find a valid constraint (*) so that:

1. (*) is valid for IP: every feasible solution to IP satisfies (*)
2. $\bar{x}$ doesn't satisfy (*).

Such an inequality is called a cutting plane for $\bar{x}$.

▷ LP relaxation:

for $\max\{c^T x : x \in P\}$ (P1) and $\max\{c^T x : x \in Q\}$ (P2)

if $P \subseteq Q$, P2 is a relaxation of P1.

▷ Relaxation:

If P2 is a relaxation of P1:

a) P2 is feasible $\Rightarrow$ P1 is feasible

b) $\bar{x}$ is optimal for P2, $\bar{x}$ is feasible for P1 $\Rightarrow$ $\bar{x}$ is optimal for P1

c) $\bar{x}$ is an optimal solution for P2, $c^T x$ is an upper bound for P1.

▷ Cutting plane algorithm: Input: an IP Output: IP solved

1. loop
2. let LP be $\max\{c^T x : Ax \le b\}$
3. if LP is infeasible then
   1. stop IP is infeasible

   end if
4. let $\bar{x}$ be the optimal solution to LP.
5. if $\bar{x}$ is integral then

   stop $\bar{x}$ is optimal to IP

   end if
6. Find a cutting plane $a^T x \le \beta$ for $\bar{x}$.

   Append $a^T x \le \beta$ to $Ax \le b$
7. End loop

▷ Find an cutting plane:

Let go back to the same example:

$$\max \ (2\ 5)x$$
$$\text{s.t.} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$
$$\text{int } x \ge 0$$

By introducing slack variables, we get

$$\max \ (2\ 5\ 0\ 0)x$$
$$\text{s.t.} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$
$$\text{int } x \ge 0.$$

We rewrite the LP in canonical with respect to its optimal basis, $\{1, 2\}$. Denote $\bar{x}$ the BFS / optimal solution.

$$\max \quad 12 + (0\ 0\ -1\ -1)x$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} x = \begin{pmatrix} \frac{8}{3} \\ \frac{4}{3} \end{pmatrix} \quad ①$$

$$x \geq 0$$

Consider the first constraint:

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{8}{3} \quad \text{then}$$

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

then if we replace the left half with a smaller value, this inequality still holds:

$$x_1 + \lfloor -\tfrac{1}{3} \rfloor x_3 + \lfloor \tfrac{4}{3} \rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$ Now the left hand side is an integer.

we can reduce the right side to

$$x_1 - x_3 + x_4 \leq \lfloor \tfrac{8}{3} \rfloor$$

$$x_1 - x_3 + x_4 \leq 2 \quad (*)$$

$(*)$ is a cutting plane for LP.

we thus obtain:

$$\max \quad (2\ 5\ 0\ 0\ 0)x$$

$$(LP') \quad \text{s.t.} \quad \begin{pmatrix} 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix}$$

$$x \geq 0$$

We can then use the two phase simplex to obtain a feasible basis, and solve it. Which gives:

$$\max \quad 11 + (0\ 0\ 0\ -\tfrac{1}{2}\ -\tfrac{3}{2})x$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$x \geq 0$$

Since $(3,1,1,0,0)$ is integral, $(3,1)$ is an optimal solution to LP.

▷ Finding Cutting Planes:

for $\max\{c^T x : Ax = b, \text{ int } x \geq 0\}$, write it in canonical form for the optimal basis to obtain: $\max\{\bar{z} + \bar{c}_N^T x_N : x_B + \bar{A}_N x_N = \bar{b}, x \geq 0\}$ (P)

For the ith constraint, let l denote basic variables. Then:

$$x_l + \sum_{j \in N} \lfloor \bar{A}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \quad (*)$$

is a cutting plane for the basic solution to (P).