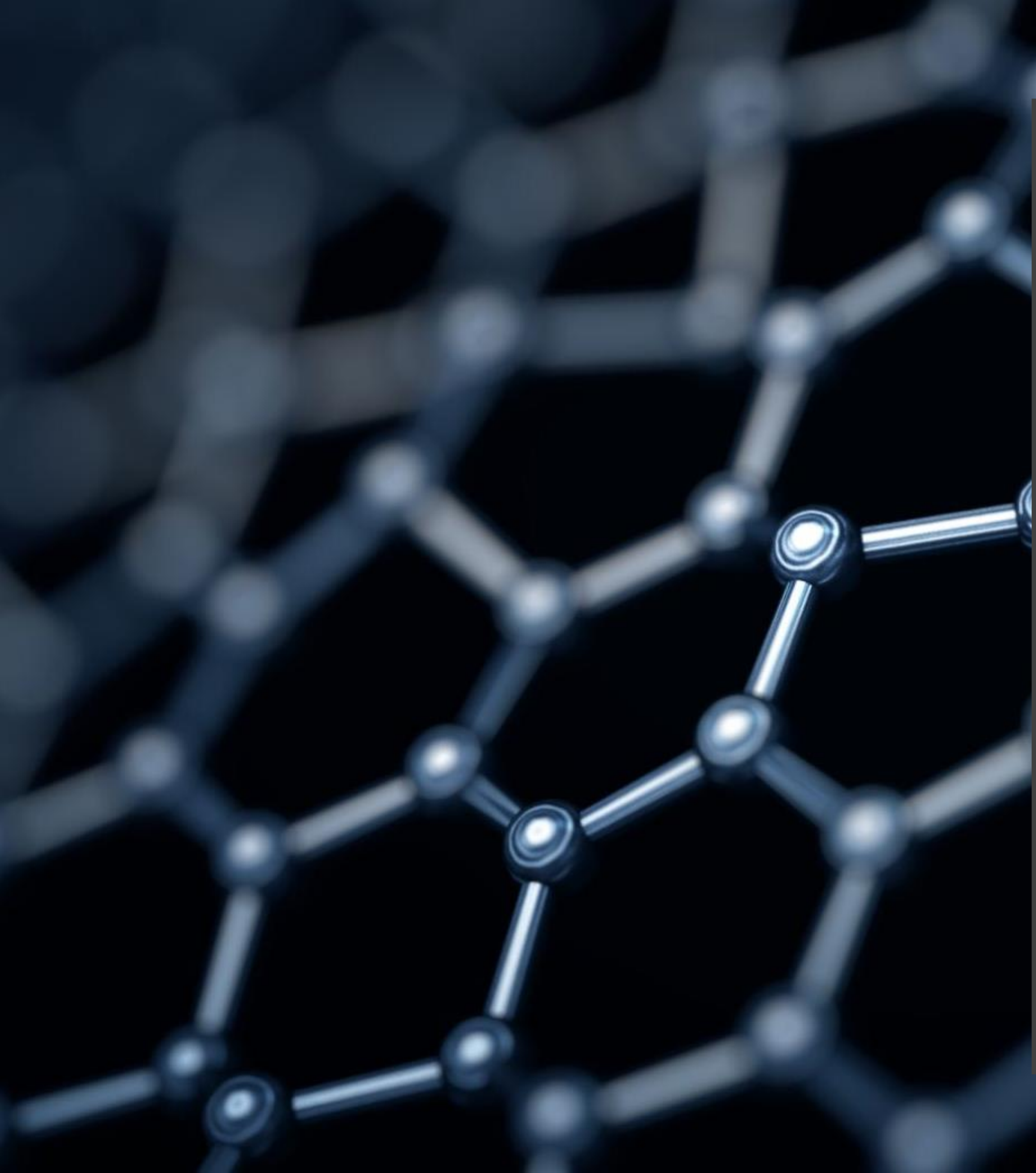
Walrasian equilibrium

Market models and Tatonnment



TOPICS

MARKET SETTING

WALRASIAN EQUILIBRIUM IN PURE EXCHANGE

EXISTANCE OF EQUILIBRIUM

PRICE TATONNMENT

THE MARKET SETTING

1. I > 0 consumers
2. J > 0 firms
3. L > 0 commodities
4. Consumption sets and preference relations
5. Prices and initial goods

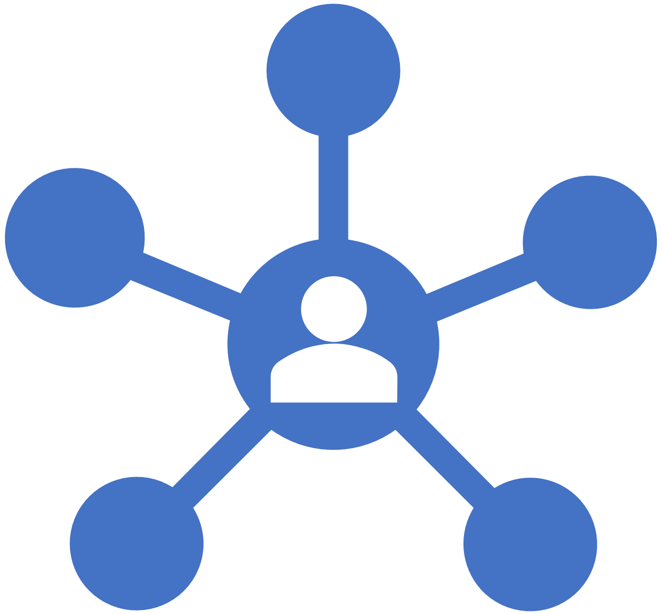
CONSUMPTION SETS AND PREFERENCE RELATIONS

A consumption set is a subset $X_i \subset \mathbb{R}^L$ whose elements are the consumption bundles that a consumer can reach under some physical conditions given by his environment.

A preference relation is a mathematical relation $\geq_i \subset X_i \times X_i$ where $x \geq_i y$ means that a consumer prefers the choice x than y. We assume that consumers are rational, so:

1. $\forall\ x, y \in X_i$ we have $x \geq_i y$ or $y \geq_i x$

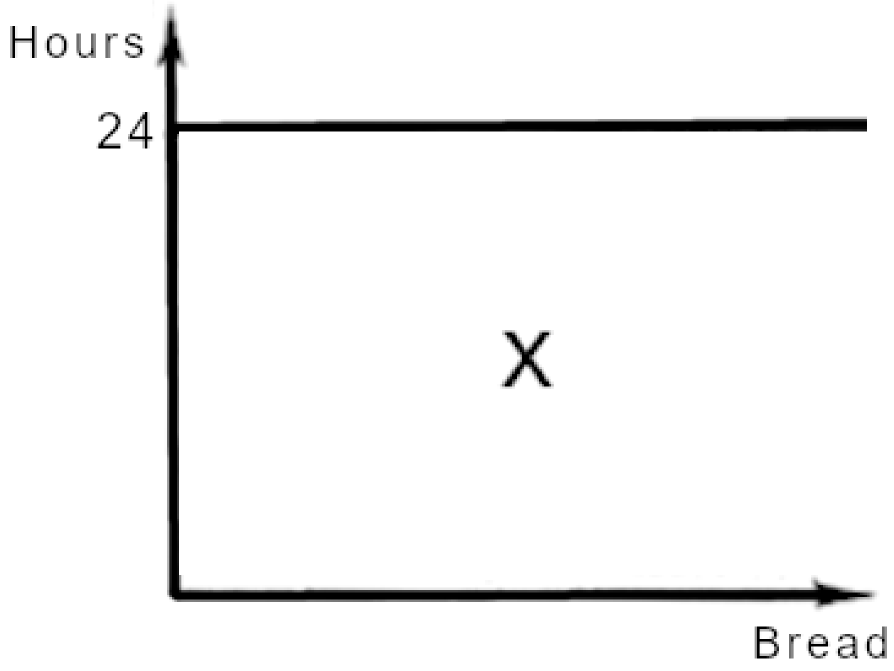
2. $\forall\ x, y, z \in X_i$ if $x \geq_i y, y \geq_i z$ then $x \geq_i z$



EATEN BREAD

This example shows the levels of bread consumption in a day. In this case we have a bounded set in hours, so from the topological point of view we have:

$$\mathrm{X} = \mathbb{R} \times [0, 24] \subset \mathbb{R}^2$$

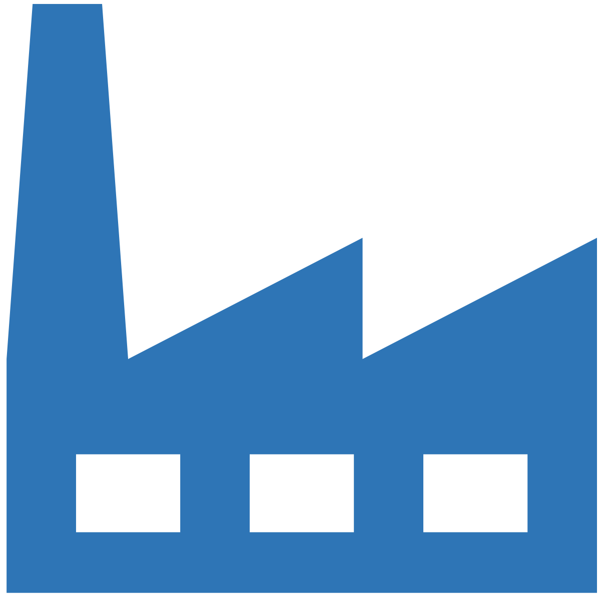


PRODUCTION SET

Every firms j has a certain technology to produce new goods. A production vectors

$$(y_1^j, \ldots, y_L^j) \in \mathbb{R}^L$$

is a vector where y_i^j is a certain produced quantity of a certain good. If $y_i^j > 0$ we have produced quantities, if $y_i^j < 0$ we have quantities used during the production. The set of all production vector for the j-firm is the set of all production vectors, called $Y_i \subset \mathbb{R}^L$.



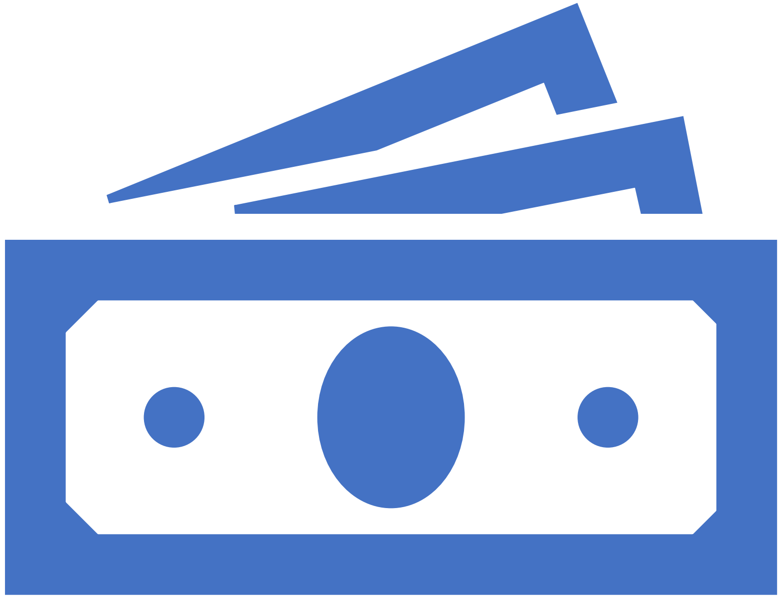
THE BUDGET SET

If we call with W the value (for example in dollars) of the wealth-level of a consumer, an element $x \in \mathbb{R}^{L+}$ is convenient if

$$p \cdot x = \sum_{i=1}^{L} p_i x_i \leq W$$

So if W is the wealth-level for a consumer, a certain choice $x \in \mathbb{R}^{L+}$ is favorable if it is an element of the budget set:

$$B_{p,W} = \{x \in \mathbb{R}^{L+} : p \cdot x \leq W\}$$



The budget set is a convex subset in $\mathbb{R}^L$

Proof. Let be $p = (p_1, \dots, p_L) \in \mathbb{R}^{L+}$ a vector of prices and W the wealth-level of a consumer. So for every $x, y \in B_{p,W}$ and $t \in [0,1]$ we have

$$p \cdot [tx + (1-t)y] = tp \cdot x + (1-t)p \cdot y \leq tW + (1-t)W = W$$

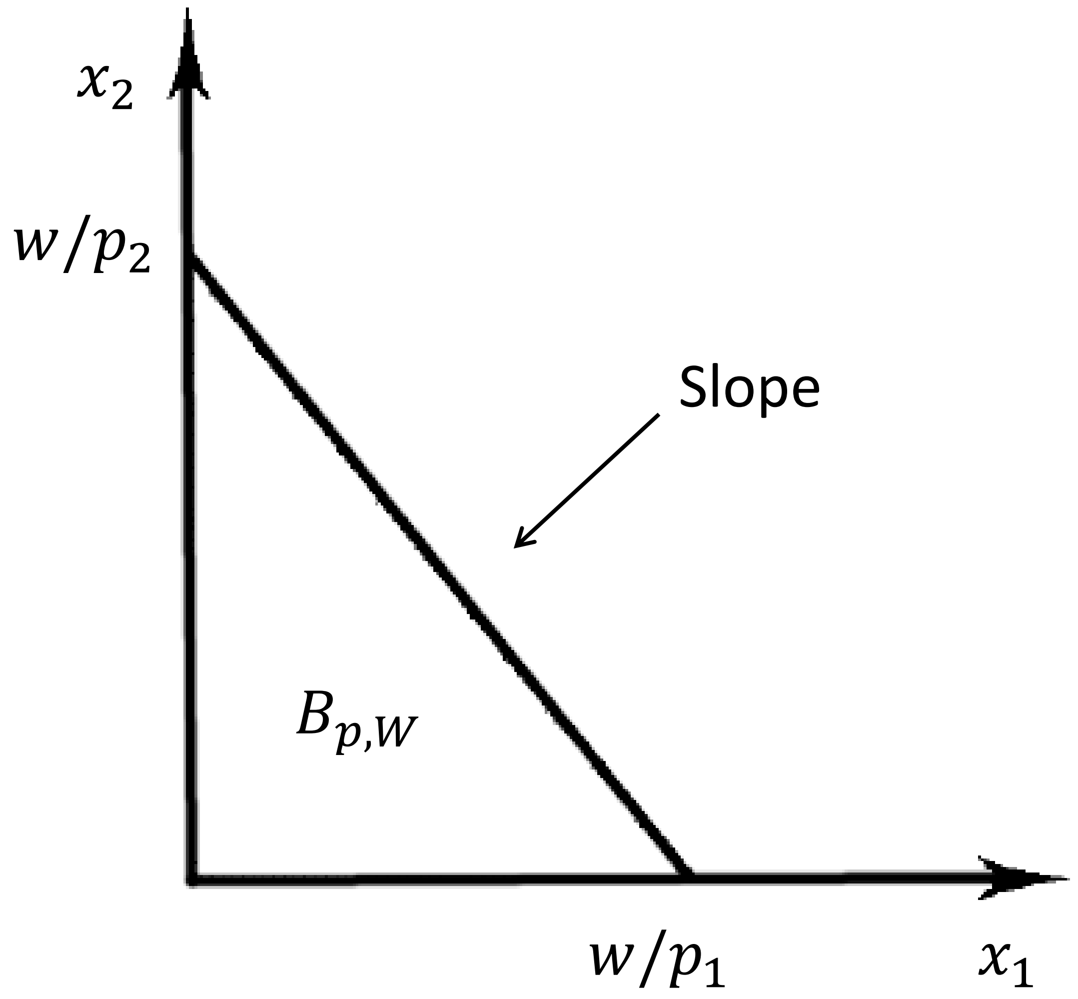
and so every convex combination is an element of $B_{p,W}$. ■

EXAMPLE

The budget set permits to have a vision how prices could variate. In this case L = 2 and so the slope of the line

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{w}{p_2}$$

tell us how p_1 and p_2 variate.



Walrasian equilibrium: private ownership economies (POE)

Walras' equilibrium has a great predictive power to analyze different economical model. The idea is to define it using system of equations. We start the analysis into a POE where:

1. prices are public
2. the wealth of a consumer is from his goods and by taking parts of firms that trade in the market

Mathematically, consumer *i* has an initial endowment vector of commodities $(w_1, \dots, w_L) \in \mathbb{R}^L$ and a claim to a share $\theta_{ij} \in [0, 1]$ of the profits of firm *j*.

WALRASIAN EQUILIBRIUM IN A POE

DEFINITION In a POE

$$\left(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(w_i, \theta_{i1}, \dots, \theta_{iJ})\}_{i=1}^{I}\right)$$

an allocation (x^*, y^*) and a price vector $p \in \mathbb{R}^{L+}$ are a Walras' equilibrium if:

1. market equilibrium: $\sum_{i=1}^{I} x_i^* = w + \sum_{j=1}^{J} y_j^*$
2. maximal production: $\forall\, j,\ p \cdot y_j \le p \cdot y_j^*\ \forall\, y_j \in Y_j$
3. $\forall\, i, x_i^*$ is a maximal element for $\succeq_i$ in the budget set:

$$\left\{x \in X_i : p \cdot x_i \le p \cdot w_i + \sum_{i} \theta_{ij}\, p \cdot y_j^*\right\}$$

Walrasian equilibrium: POE in pure exchange

We assume another hypothesis: every agents are consumers, there isn't production and every agents can trade their initial commodities. So, in this case, we have J = 1 and so $Y_1 = \mathbb{R}^{L+}$. In this case an allocation $(x^*, y^*) = (x^*, y_1^*)$ and a price vector $p \in \mathbb{R}^{L+}$ are a Walras' equilibrium if:

1. $y_1^* \leq 0, p \cdot y_1^* = 0$
2. $x_i^* = x_i(p, p \cdot w_i)\ \forall\, i$
3. $\sum_i x_i^* = \sum_i w_i + y_1^*$

In a POE in pure exchange which consumers preferences are continuous and strictly convex, a vector of prices $p \in \mathbb{R}^{L+}$ is a Walras' equilibrium if and only if $\sum_i (x_i(p, p \cdot w_i) - w_i) \leq 0$.

Proof. If (x^*, y_1^*) and $p \in \mathbb{R}^{L+}$ is a Walras' equilibrium, using (2) in (3), we have

$$\sum_i x_i(p, p \cdot w_i) = \sum_i w_i + y_1^*$$

and by (1) we have $\sum_i (x_i(p, p \cdot w_i) - w_i) = y_1^* \leq 0$. Instead if we assume the inequality for a fixed p and w, we define $x_i^* := x_i(p, p \cdot w_i)$ and $y_1^* := \sum_i (x_i(p, p \cdot w_i) - w_i)$, with these choices the three properties are all satisfied. ■

EXCESS DEMAND FUNCTION

DEFINITION The excess demand function is

$$z(p) = \sum_i z_i(p)$$

where $z_i(p) = x_i(p, p \cdot w_i) - w_i$. This function:

1. is continuous
2. is of 0-degree so $z_i(ap) = z_i(p)$ for every $a \in \mathbb{R}$
3. satisfies Walras' law: $p \cdot z(p) = 0$ for every $p \in \mathbb{R}^{L+}$

We can rewrite last result: $p \in \mathbb{R}^{L+}$ is an equilibrium price vector if and only if $z(p) \leq 0$.

IS IT POSSIBLE TO FIND EQUILIBRIUM?

EXISTANCE THEOREM

Suppose that $z(\cdot)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^{L+}$ and satisfying last properties. Then the system of equation

$$z(p) = 0$$

has a solution. Hence a Walrasian equilibrium exists in any pure exchange private ownership economy in which $\sum_i w_i > 0$ and every *i*-consumer has continuous, strictly convex and strongly monotone preferences.

CORRESPONDENCES AND HUPPER-HEMICONTINUITY

DEFINITION Given $A \subset \mathbb{R}^N$ a correspondence is a multifunction $f : A \to P(\mathbb{R}^k)$ such that $f(x) \subset \mathbb{R}^k$ is a not empty subset for all $x \in A$. If for all $x \in A \mid f(x) \mid = 1$, then $f(\cdot)$ is a function.

DEFINITION Given $\mathrm{A} \subset \mathbb{R}^N$ a not empty subset and $\mathrm{Y} \subset \mathbb{R}^k$ a closed subset, then $f : A \to Y$ is a UH -correspondence if:

1. forall $\forall \{x_n\}_n \in A$ such that $x_n \to x \in A$ and $\{y_n\}_n \in f(x_n)$ such that $y_n \to y$, then $y \in f(x)$

2. the set $f(B) = \{y \in Y : y \in (x), x \in B\}$ is bounded forall compact subset $\mathrm{C} \subset A$

KAKUTANI'S FIXED-POINT THEOREM

THEOREM Given a convex UH-correspondence

$$f : A \to A$$

with $A \subset \mathbb{R}^N$ a not empty compact and convex subset such that $f(x) \subset A \; \forall \, x \in A$, then $f(\cdot)$ has a fixed point, so exists $x \in A$ such that $x \in f(x)$.

EXISTANCE OF WALRASIAN EQUILIBIRUM: suppose that $z(\cdot)$ *is a function defined for all strictly positive price vectors p* $\in \mathbb{R}^{L+}$ *and satisfying last properties. Then the system of equation* $z(p) = 0$ *has a solution. Hence a Walrasian equilibrium exists in any pure exchange private ownership economy in which* $\sum_i w_i > 0$ *and every i-consumer has continuous, strictly convex and strongly monotone preferences.*

Proof. Because $z(\cdot)$ is a function of degree zero, we can restrict our research for an equilibrium to

$$\Delta = \left\{ p \in \mathbb{R}^{L+} \colon \sum_i p_i = 1 \right\}$$

Note that $z(\cdot)$ is well defined only for price vectors in

$$Int(\Delta) = \{p \in \Delta : \; p_i > 0 \; \forall \, i\}$$

Step 1 We start to define the following correspondence; for every $p \in \mathbb{R}^{L+}$ we define:

$$f(p) = \begin{cases} \{ q \in \Delta : q_i = 0 \; if \; z_i(p) < max_{i=1\ldots L}\big(z_i(p)\big)\}, & if \; p \in Int(\Delta) \\ \{ q \in \Delta : q \cdot p = 0 \}, & if \; p \in \partial\Delta \end{cases}$$

In words: given the current "proposal" $p \in Int(\Delta)$, the "counterproposal" assigned by f is any price vector q that, among permissible price in Δ, maximizes the value of the excess demand vector $z(p)$. Moreover, we observe that: if $z(p) \neq 0$ then exists some i such that $z_i(p) < 0$ and some i such that $z_i(p) > 0$. So, for such p, any $q \in f(p)$ has $q_i = 0$ for some i. So, if $z(p) \neq 0$, then $f(p) \in \partial\Delta = \Delta \backslash \mathrm{Int}(\Delta)$ and in contrast, if $z(p) = 0$ then $f(p) = \Delta$.

Step 2 This correspondence is convex and UH.

Step 3 Being Δ a compact set, Kakutani's Fixed-point theorem tell us that a convex, UH-correspondence from a nonempty, compact, convex set into itself has a fixed point. So exists $p^* \in \Delta$ such that $p^* \in f(p^*)$.

Step 4 The fixed point p^* is our Walras' equilibrium. In fact, suppose $p^* \in f(p^*)$. By first step, p^* is not in $\partial\Delta$ and by hypothesis $p^* > 0$. Suppose by absurd that p^* is not an equilibrium, so $z(p^*) \neq 0$. In the first step we saw that $f(p^*) \in \partial\Delta$, which is incompatible with $p^* \in f(p^*) \subset \partial\Delta$. So it must be $z(p^*) = 0$ if $p^* \in f(p^*)$. ■

HOW TO PREDICT EQUILIBRIUM?

DYNAMICS: TATONNMENT STABILITY

From the economical point of view, finding equilibrium is interesting if it is possible to write it as system of some equations. In economy, the most difficult problem is not recognize equilibrium but how to predict it. The idea is to try to translate equilibrium in mathematical laws.

In 1874, the French economist Leon Walras, introduced the concept of Tatonnment: this French word means "groping". So the idea is to arrive to the correct solution using a trial-and-error approach.

PRICE TATONNMENT

Suppose to have an economy modeled by a excess demand function z. If a vector price p is not an equilibrium, $z(p) \neq 0$. In 1947, Samuelson proposed to study the reaching of equilibrium using differential equations:

$$\frac{dp_i}{dt} = c_i z_i(p)$$

This equation represents the variation of price *i* where c_i is a positive real constant. So we assume a certain regularity on every p_i.

PROBLEMS?

Samuelson’s equations are interesting but:

1. which possible economic agents could influence prices?
2. what sort of time does *t* represent?

The hope is that the analysis of equations will provide to give more properties to find equilibrium.

SYSTEM STABILITY: 2-DIMENSIONAL CASE

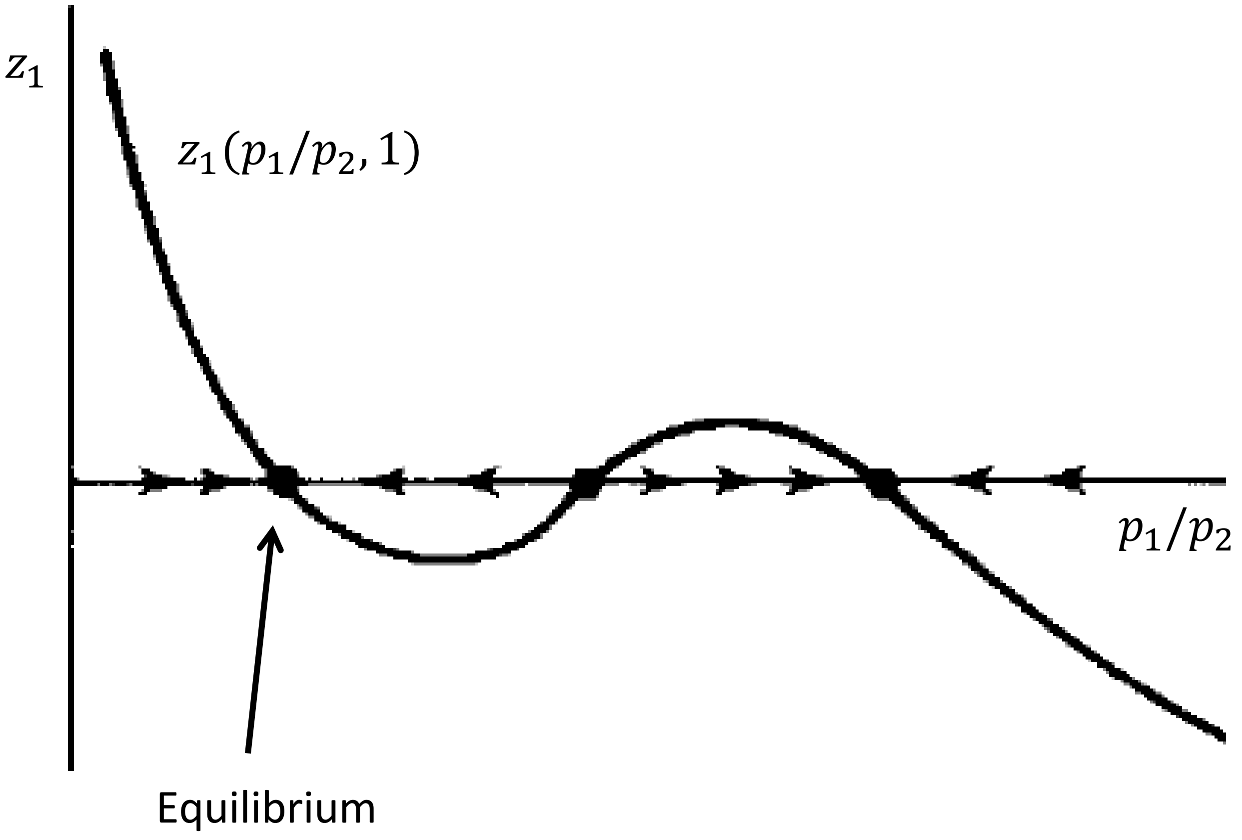
If L = 2 there is system stability if for any initial point

$$p(0) = \big(p_1(0), p_2(0)\big)$$

if the corresponding trajectory of relative prices

$$R(t) = \frac{p_1(t)}{p_2(t)}$$

converges to some equilibrium for $t \to +\infty$.



IN HIGHER DIMENSION?

If we consider *L > 2* we have a very difficult mathematical point of view. To solve this situation, we can assume useful hypothesis to research equilibrium. If $c_1 = \ldots = c_L = 1$ and if we suppose to work in the normalized set of prices

$$S \coloneqq \left\{ p = (p_1, \ldots, p_L) \in \mathbb{R}^{L+} \colon \sum_{i=1}^{n} p_i^2 = 1 \right\}$$

for any choice of $z(\cdot)$ the dynamic flow $p(t)$, generated by Samuelson’s equation, remains in *S*. In other words we have the following result.

If $p(0) \in S$ then $p(t) \in S$ for every t

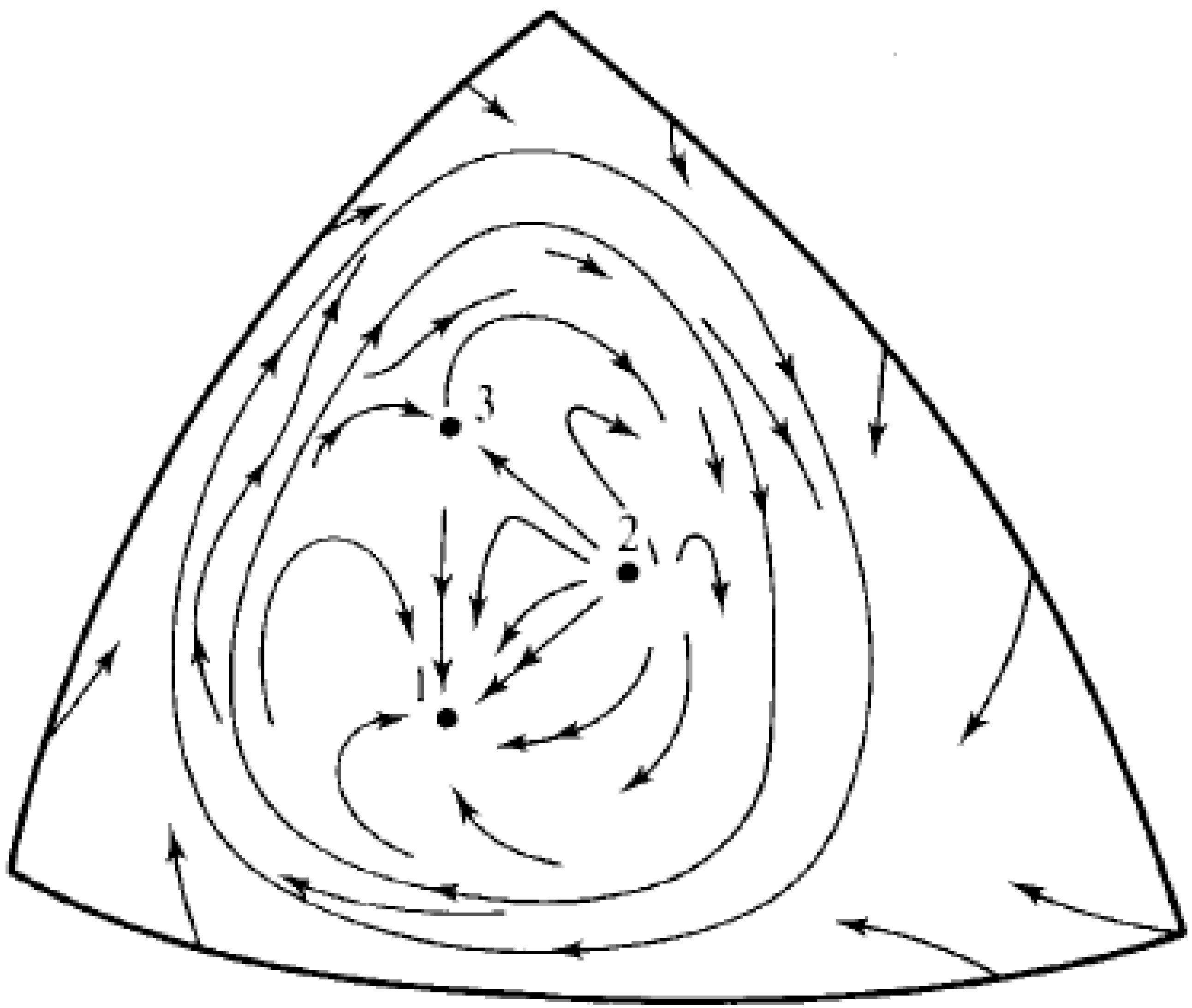
Proof. If $p(0) \in S$ then $\sum_{i=1}^{n} p_i(0)^2 = 1$. Moreover $P(t) = \sum_{i=1}^{n} p_i(t)^2$ is derivable because sum of derivable functions, so

$$\frac{dP}{dt} = \sum_{i=1}^{n} 2\, p_i(t) \frac{dp_i}{dt} = \sum_{i=1}^{n} 2\, p_i(t) z_i(p_i(t)) = 0$$

where zero is consequence of Walras law. So these means that $P(t)$ is a constant function, but $P(0) = 1$ by hypothesis and so it must be $P(t) = \sum_{i=1}^{n} p_i(t)^2 = 1$ and so $p(t) \in S$. ■

EXAMPLE ON THE SPHERE

An example of Tatonnment trajectories for L = 3. In this case S is a curve triangle on the sphere and trajectories are bounded in this region.



SCARF'S ECONOMY: NEWTON STABILITY (1960)

It is an interesting example of economy proposed in 1960 by the American economist Herbert Scarf. In this economy we have 3 kind of individuals such that:

	Type A	Type B	Type C
Utility function	$\min(q_A, q_B)$	$\min(q_B\ q_C)$	$\min(q_C, q_A)$
Initial endowment	1 Apple	1 Banana	1 Coconut

Consider a type A agent who is endowed with one apple and has utility function $\min(q_A, q_B)$. For given prices p_A and p_B this agent's demands for apples and bananas are $q_A = q_B = p_A/(p_A + p_B)$. The demands for type B and C agents can be derived similarly, and it is readily varied that the equilibrium prices for which demand equals supply satisfy $p_A = p_B = p_C$.

How does the economy arrive at competitive equilibrium prices?

Consider any set of prices p_A and p_B for apples and bananas respectively expressed in terms of the numeraire p_C = 1. According to Samuelson's approach we have

$$\frac{dp_A}{dt} = n\left(\frac{-p_B}{p_A + p_B} + \frac{1}{1 + p_A}\right)$$

$$\frac{dp_B}{dt} = n\left(\frac{-1}{1 + p_B} - \frac{p_A}{p_A + p_B}\right)$$

where $n \geq 1$ is the number of replicas of each type of agent in the economy. In the first line, the first term between parentheses on the right side is the supply of apples by type A and the second term is the demand for apples by type C. Likewise, in the second line, the first term between parentheses represents the supply of bananas by type B and the second term is the demand for bananas by agent A.

NEWTON DYNAMICS

In a Scarf's economy with n≥1 agents of each type, the Newtonian dynamic

$$\frac{dp}{dt} = -\left(\nabla z(p)\right)^{-1} z(p)$$

is globally stable where $\nabla z(p)$ is the Jacobian matrix of $z(\cdot)$.

SKETCH OF THE PROOF According to Lyapunov's theorem, if an equilibirum isolated point admits a Lyapunov functional, then the point is stable. In our case we can consider

$$L(p_A, p_B) = \left(\frac{1}{1+p_A} - \frac{1}{1+p_A/p_B}\right)^2 + \left(\frac{1}{1+p_B} - \frac{1}{1+p_B/p_A}\right)^2$$

and according to general theory we have that $0 \leq L \leq 1$ and $L = 0$ if and only if $p_A = p_B = 1$. So $(p_A, p_B) = (1{,}1)$ is the candidate point of equilibirum and now, to conclude, we have to show that L is a decrising function. ■

CONCLUSIONS

1. From the theorical point of view, equlibrium exists.
2. Difficulties how to predict equilibria are solvable using mathematical tecniques (for example Tattonnment or Lyapunov's theorem).

Thankyou for your attention!