Random numbers simulations

The Inverse Transformation Method

by Matteo M. Costa

Random simulations

In different financial applications, the simulation of random numbers is an important issue.
In particular, an interesting approach to simulate a continuous random variable is the well known ***Inverse Transformation Method***.
Before showing the result, it is important making some considerations about uniform random distributions. These can be simulated using an ***iterative congruential method***:

$$x_i = (ax_{i-1} + b)mod(m),$$

where a and b are fixed numbers and m is a prime number.

Simulation in python

```
import numpy as np
import matplotlib.pyplot as plt

x0, a, b, m = 0, 1140671485, 128201163, 2**24
n = 3000
U = np.zeros(n)
U[0] = x0

for i in range(1, n):
    U[i] = ((a*U[i-1]+b) % m) / m

plt.hist(U, bins=100)
plt.show()
```

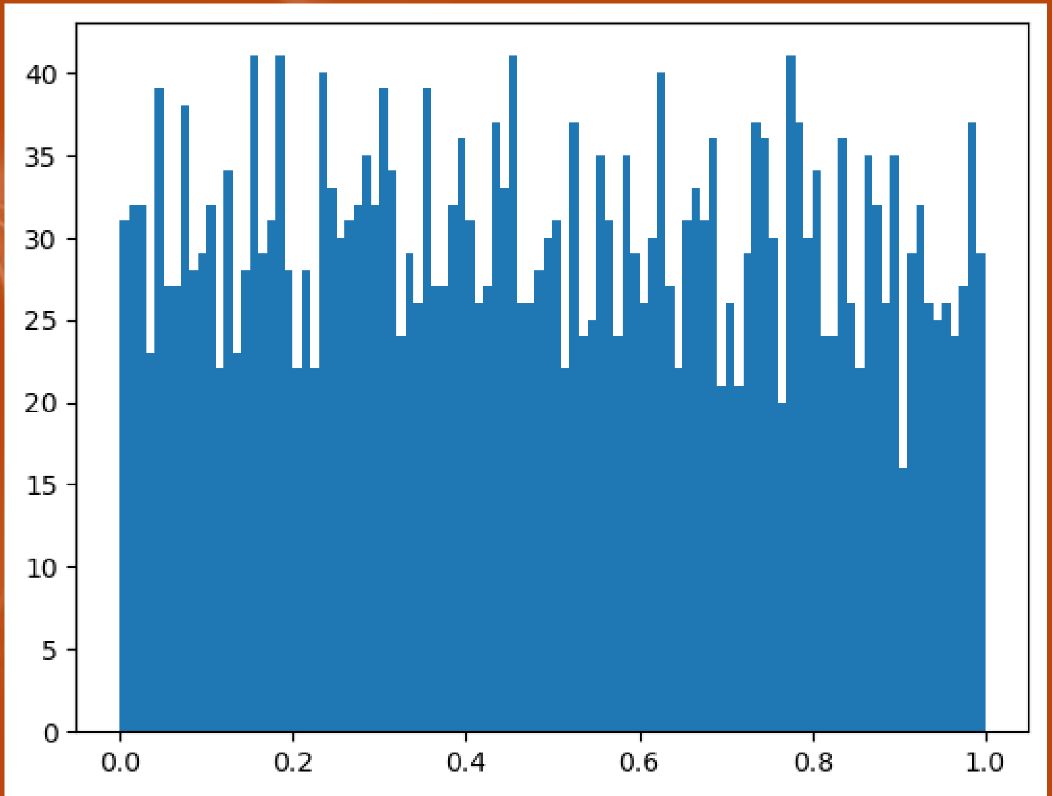

Figure 1: histogram with a sample of 3000 uniformly distributed numbers.

The theorem for continuous laws

Let F be an invertible real cumulative distribution function (i.e. cdf) and let $U \sim U(0,1)$ be a uniform random variable. Then the random variable

$$X = F^{-1}(U),$$

is distributed as F, that is $\mathbb{P}(X \leq z) = F(z)$ for each $z \in \mathbb{R}$.

This result is also true if we relax the invertibility assumption on F. In this case the proof adopts the ***pseudo-inverse*** of F. In the following proof we consider the invertible case.

Proof. For any fixed $z \in \mathbb{R}$, the cumulative distribution function of the transformation $X = F^{-1}(U)$ can be derived using F invertibility as follows:

$$\mathbb{P}(X \leq z) = \mathbb{P}(F^{-1}(U) \leq z) = \mathbb{P}\big(U \leq F(z)\big).$$

By hypothesis $U \sim U(0,1)$, so $F_U(u) = \mathbb{P}(U \leq u)$ for each $u \in \mathbb{R}$ and so we conclude that

$$\mathbb{P}\big(U \leq F(z)\big) = F_U(F(z)) = \frac{F(z) - 0}{1 - 0} = F(z),$$

which means that X is distributed as F. □

Example

Let $exp(\lambda)$ be an exponential law, so $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. We have

$$F^{-1}(y) = -\frac{1}{\lambda} ln(1-y),$$

and so, given $U \sim U(0,1)$, we obtain

$$X = F^{-1}(U) = -\frac{1}{\lambda} ln(1-U) \sim exp(\lambda).$$

So, if we can simulate a uniform random variable U on $(0,1)$, then the random variable X follows an exponential distribution.

Simulation in python

According to the theorem, we can simulate an exponential distribution as follows. Let's start by simulating a uniform random variable as shown on page 2 and then we apply the last formula:

```
X = -(1/5)*(np.log(1-U))
plt.hist(X, bins=100)
plt.show()
```

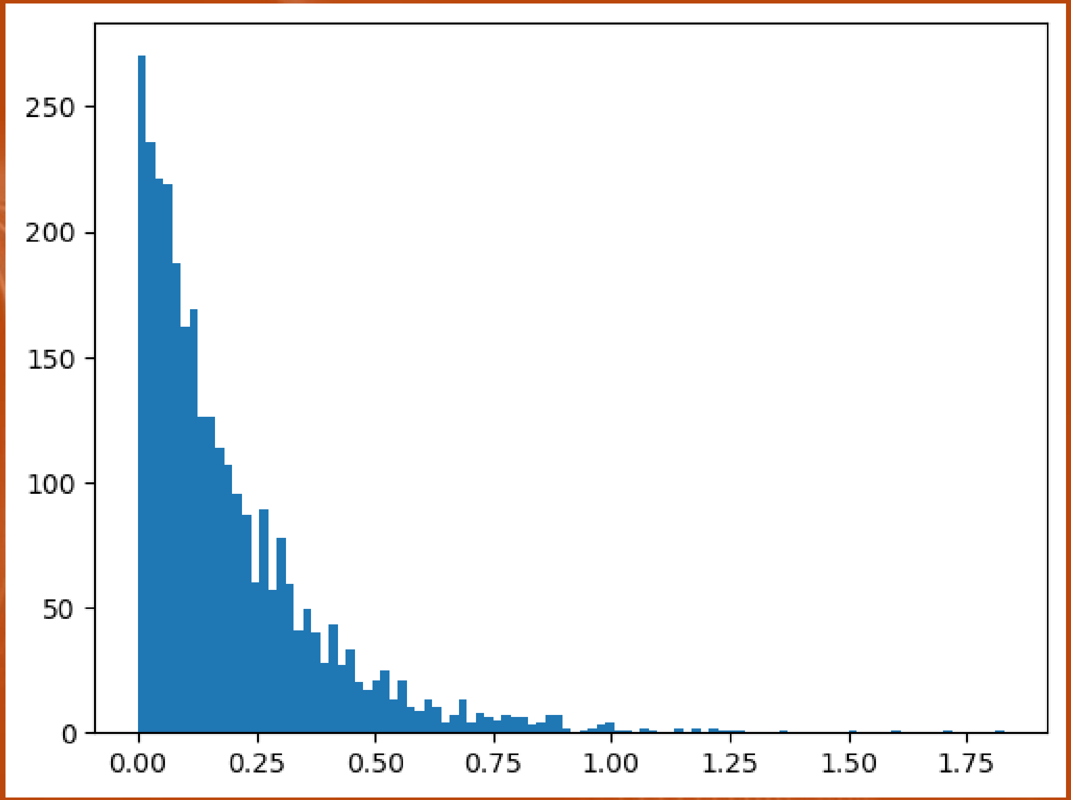

Figure 2: exponential simulation; the distribution of numbers in vector X remembers the exponential density with parameter $\lambda = 5$.

Conclusion

- The Inverse Transformation Method can also be generalized for discrete distributions.

- The method is very interesting when it is easy finding the inverse of F. In general, if F is invertible then the expression of F^{-1} could be very difficult.