Exercises and Solutions for **Partial Differential Equations**

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Omitted

2 Chapter 2: Four Important Linear Differential Equations

In all the following exercises, all given functions are assumed smooth, unless otherwise stated.

Problem 2.1 Write down an explicit formula for a function u solving the initial-value problem.

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution of Problem 2.1 Consider the following equation: $v = \exp(ct) \cdot u(t, x)$, then:

$$v_t = \exp(ct) \cdot u_t + c \exp(ct) \cdot u = \exp(ct) \cdot (-b \cdot Du)$$

and $Dv = \exp(ct) Du$. Therefore:

$$v_t + b \cdot Dv = 0$$

holds in $\mathbb{R}^n \times (0, \infty)$, which becomes a transport equation. Denote $w(s) = v(x + sb, t + s)$, then:

$$w'(s) = b \cdot Dv(x + sb, t + s) + v_t(x + sb, t + s) \equiv 0,$$

which means that $w(s)$ is a constant function for all given (x, t).

$$v(x, t) = v(x - tb, 0) = u(x - tb, 0) = g(x - tb).$$

To sum up, the solution is:

$$u(x, t) = \exp(-ct) \cdot g(x - tb).$$