Exercises and Solutions for **Partial Differential Equations**

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Omitted

2 Chapter 2: Four Important Linear Differential Equations

In all the following exercises, all given functions are assumed smooth, unless otherwise stated.

Problem 2.1 Write down an explicit formula for a function u solving the initial-value problem.

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution of Problem 2.1 Consider the following equation: $v = \exp(ct) \cdot u(t, x)$, then:

$$v_t = \exp(ct) \cdot u_t + c \exp(ct) \cdot u = \exp(ct) \cdot (-b \cdot Du)$$

and $Dv = \exp(ct)Du$. Therefore:

$$v_t + b \cdot Dv = 0$$

holds in $\mathbb{R}^n \times (0, \infty)$, which becomes a transport equation. Denote $w(s) = v(x + sb, t + s)$, then:

$$w'(s) = b \cdot Dv(x + sb, t + s) + v_t(x + sb, t + s) \equiv 0,$$

which means that $w(s)$ is a constant function for all given (x, t).

$$v(x, t) = v(x - tb, 0) = u(x - tb, 0) = g(x - tb).$$

To sum up, the solution is:

$$u(x, t) = \exp(-ct) \cdot g(x - tb).$$

Problem 2.2 Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Solution of Problem 2.2 Notice that for all $i \in [n]$, we have:

$$\frac{\partial^2}{\partial x_i^2} v = O^\top \frac{\partial^2}{\partial x_i^2} u(Ox) \cdot O.$$

After adding up $i = 1, 2, \ldots, n$:

$$\Delta v(x) = O^\top \Delta u(Ox) \cdot O = 0,$$

which shows that v is also Laplacian.

Problem 2.4 Give a direct proof that if $u \in C^2(U) \cap C(\bar{U})$ is harmonic within a bounded open set U, then:

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

Solution of Problem 2.4 Define $u_\varepsilon := u + \varepsilon |x|^2$ for $\varepsilon > 0$. Then:

$$\Delta u_\varepsilon = \Delta u + 2n\varepsilon = 2n\varepsilon > 0,$$

which shows that u_ε can not attain its maximum over $\bar{U}$ at an interior point:

$$\max_{\bar{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon.$$

Since U is bounded, we can assume that $|x| \leqslant D$ holds for $\forall x \in \bar{U}$, which leads to:

$$0 \leqslant \max_{\bar{U}} u_\varepsilon - \max_{\bar{U}} u \leqslant \varepsilon D^2 \quad \text{and} \quad 0 \leqslant \max_{\partial U} u_\varepsilon - \max_{\partial U} u \leqslant \varepsilon D^2.$$

Therefore:

$$|\max_{\bar{U}} u - \max_{\partial U} u| \leqslant \varepsilon D^2.$$

Since the inequality above holds for $\forall \varepsilon > 0$, we can conclude that

$$\max_{\bar{U}} u = \max_{\partial U} u.$$