4.1 Helmholtz decomposition

Let $\Omega \subset \mathbb{R}^3$ be an open set. For any scalar function $u \in H^1(\Omega)$ and vector function $\mathbf{u} \in (H^1(\Omega))^3$, define

$$\begin{aligned}
\mathbf{grad}\, u &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)^\top, \\
\operatorname{div} \mathbf{u} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \\
\mathbf{curl}\, \mathbf{u} &= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)^\top.
\end{aligned}$$

For simplicity, we denote $\nabla u = \mathbf{grad}\, u, \nabla \cdot \mathbf{u} = \operatorname{div} \mathbf{u}, \nabla \times \mathbf{u} = \mathbf{curl}\, \mathbf{u}$.

Define corresponding Hilbert spaces

$$\begin{aligned}
H^1(\Omega) &= \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}, \\
H(\mathrm{div}, \Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3 : \nabla \cdot \mathbf{u} \in L^2(\Omega)\}, \\
H(\mathrm{curl}, \Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3 : \nabla \times \mathbf{u} \in (L^2(\Omega))^3\},
\end{aligned}$$

and subspaces with zero traces

$$\begin{aligned}
H_0^1(\Omega) &= \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}, \\
H_0(\mathrm{div}, \Omega) &= \{\mathbf{u} \in H(\mathrm{div}, \Omega) : \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \\
H_0(\mathrm{curl}, \Omega) &= \{\mathbf{u} \in H(\mathrm{curl}, \Omega) : \mathbf{n} \times \mathbf{u} = 0 \text{ on } \Gamma\}.
\end{aligned}$$

Define the differential operators in two dimensions

$$\begin{aligned}
\mathbf{Grad} u &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^\top, \quad \mathrm{Div}\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \\
\mathbf{Curl} u &= \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\right)^\top, \quad \mathrm{Curl}\mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.
\end{aligned}$$

Let Γ be a two-dimensional Lipschitz continuous manifold. Define the surface gradient and scalar curl as follows:

$$\nabla_\Gamma u = -\mathbf{n} \times (\mathbf{n} \times \nabla u) \quad \text{and} \quad \mathrm{Curl}_\Gamma \mathbf{u} = (\nabla \times \mathbf{u}) \cdot \mathbf{n}, \quad \text{on } \Gamma.$$

The surface divergence and vector curl can be defined by using the duality

$$\begin{aligned}
\int_\Gamma \mathrm{div}_\Gamma \mathbf{u}\, v &= -\int_\Gamma \mathbf{u} \cdot \nabla_\Gamma v \quad \text{for all } v \in C_0^\infty(\Gamma), \\
\int_\Gamma \mathbf{curl}_\Gamma u \cdot \mathbf{v} &= \int_\Gamma u \mathrm{curl}_\Gamma \mathbf{v}, \quad \mathbf{v} \in (C_0^\infty(\Gamma))^3.
\end{aligned}$$

Lemma 4.1.1. *Let K_1 and K_2 be two Lipschitz domains. Denote $\Sigma = \partial K_1 \cap \partial K_2$ and $D = K_1 \cup K_2 \Sigma$.*

1. *Let $u_1 \in H^1(K_1), u_2 \in H^1(K_2)$ and define $u \in L^2(D)$*

$$u = \begin{cases} u_1 & \text{in } K_1, \\ u_2 & \text{in } K_2. \end{cases}$$

If $u_1 = u_2$ on Σ, then we have $u \in H^1(D)$.

2. *Let $\mathbf{u}_1 \in H(\mathrm{curl}, K_1), \mathbf{u}_2 \in H(\mathrm{curl}, K_2)$ and define $\mathbf{u} \in L^2(D)$*

$$\mathbf{u} = \begin{cases} \mathbf{u}_1 & \text{in } K_1, \\ \mathbf{u}_2 & \text{in } K_2. \end{cases}$$

If $\mathbf{u}_1 \times \mathbf{n} = \mathbf{u}_2 \times \mathbf{n}$ on Σ, then we have $\mathbf{u} \in H(\mathrm{curl}, D)$.

3. *Let $\mathbf{u}_1 \in H(\mathrm{div}, K_1), \mathbf{u}_2 \in H(\mathrm{div}, K_2)$ and define $\mathbf{u} \in L^2(D)$*

$$\mathbf{u} = \begin{cases} \mathbf{u}_1 & \text{in } K_1, \\ \mathbf{u}_2 & \text{in } K_2. \end{cases}$$

If $\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}$ on Σ, then we have $\mathbf{u} \in H(\mathrm{div}, D)$.

Proof. 1. It suffices to prove $\nabla u \in (L^2(D))^3$ and

$$\nabla u = \begin{cases} \nabla u_1 & \text{in } K_1, \\ \nabla u_2 & \text{in } K_2. \end{cases} \tag{4.1}$$

For any $\mathbf{v} \in H_0(\mathrm{div}, D)$, it follows from the integration by parts that

$$\begin{aligned} \int_D u \nabla \cdot \mathbf{v} &= \int_{K_1} u_1 \nabla \cdot \mathbf{v} + \int_{K_2} u_2 \nabla \cdot \mathbf{v} \\ &= -\int_{K_1} \nabla u_1 \cdot \mathbf{v} - \int_{K_2} \nabla u_2 \cdot \mathbf{v} + \int_\Sigma (u_1 - u_2) \mathbf{v} \cdot \mathbf{n} \\ &= -\int_{K_1} \nabla u_1 \cdot \mathbf{v} - \int_{K_2} \nabla u_2 \cdot \mathbf{v}. \end{aligned}$$

Hence we have $\nabla u \in (L^2(D))^3$. Taking $\mathbf{v} \in H_0(\mathrm{div}, K_i), i = 1, 2$ yields (4.1.3).

2. If suffices to prove $\nabla \times \mathbf{u} \in (L^2(D))^3$ and

$$\nabla \times \mathbf{u} = \begin{cases} \nabla \times \mathbf{u}_1 & \text{in } K_1, \\ \nabla \times \mathbf{u}_2 & \text{in } K_2. \end{cases} \tag{4.2}$$

   Taking $\mathbf{v} \in (C_0^\infty(D))^3$, we have from the integration by parts that

$$\begin{aligned} \int_D \mathbf{u} \cdot (\nabla \times \mathbf{v}) &= \int_{K_1} \mathbf{u}_1 \cdot (\nabla \times \mathbf{v}) + \int_{K_2} \mathbf{u}_2 \cdot (\nabla \times \mathbf{v}) \\ &= \int_{K_1} \nabla \times \mathbf{u}_1 \cdot \mathbf{v} + \int_{K_2} \nabla \times \mathbf{u}_2 \cdot \mathbf{v} - \int_\Sigma (\mathbf{u}_1 - \mathbf{u}_2) \times \mathbf{n} \cdot \mathbf{v} \\ &= \int_{K_1} \nabla \times \mathbf{u}_1 \cdot \mathbf{v} + \int_{K_2} \nabla \times \mathbf{u}_2 \cdot \mathbf{v}. \end{aligned}$$

   Hence $\nabla \times \mathbf{u} \in (L^2(D))^3$. Taking $\mathbf{v} \in (C_0^\infty(K_i))^3, i = 1, 2$ yields (4.1.4).

3. Similar to the proof of 2.

□

Theorem 4.1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Omega \subset\subset O$. There exits a bounded linear operator $E : H(\mathrm{curl}, \Omega) \to H(\mathrm{curl}, \mathbb{R}^3)$ such that*

$$E\mathbf{u} = \mathbf{u} \textit{ in } \Omega, \quad \mathrm{supp}(E\mathbf{v}) \subset O, \quad \textit{for all } \mathbf{u} \in H(\mathrm{curl}, \Omega).$$

Theorem 4.1.3. *Let Ω be a simply connected Lipschitz domain. For $\mathbf{u} \in (L^2(\Omega))^3$, $\nabla \times \mathbf{u} = 0$ if and only if there exists $\varphi \in H^1(\Omega)/\mathbb{R}$ such that $\mathbf{u} = \nabla\varphi$.*

Theorem 4.1.4. *Let Ω be a bounded Lipschitz domain with boundary Γ. For $\mathbf{u} \in (L^2(\Omega))^3$ and satisfying*

$$\nabla \cdot \mathbf{u} = 0 \textit{ in } \Omega, \quad \int_\Gamma \mathbf{u} \cdot \mathbf{n} = 0,$$

if and only if there exists $\mathbf{w} \in (H^1(\Omega))^3$ such that $\mathbf{u} = \nabla \times \mathbf{w}$. Furthermore, $\mathbf{w}$ can be chose to satisfy $\nabla \cdot \mathbf{w} = 0$ and

$$\| \mathbf{w} \|_{(H^1(\Omega))^3} \leq C \| \mathbf{u} \|_{(L^2(\Omega))^3} .$$

It follows from Theorem 4.1.3 and Theorem 4.1.4 that we have the following Helmholtz decomposition theorem.

Theorem 4.1.5. *For any vector field $\mathbf{u} \in (L^2(\Omega))^3$, it has the following decomposition*

$$\mathbf{u} = \nabla q + \nabla \times \mathbf{w}, \; q \in H^1(\Omega)/\mathbb{R}, \; \mathbf{w} \in (H^1(\Omega))^3 \textit{ satisfying } \nabla \cdot \mathbf{w} = 0, \; (\nabla \times \mathbf{w}) \cdot \mathbf{n} = 0 \textit{ on } \Gamma.$$

Theorem 4.1.6. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain.*

1. *For any $\mathbf{u} \in H_0(\mathrm{curl}, \Omega)$, there exists $\mathbf{u}_0 \in (H_0^1(\Omega))^3$ and $\psi \in H_0^1(\Omega)$ such that*

$$\mathbf{u} = \mathbf{u}_0 + \nabla\psi, \quad \| \mathbf{u}_0 \|_{(H^1(\Omega))^3} + \| \psi \|_{H^1(\Omega)} \leq C \| \mathbf{u} \|_{H(\mathrm{curl},\Omega)} .$$

2. *For any $\mathbf{u} \in H(\mathrm{curl}, \Omega)$, there exists $\mathbf{u}_0 \in (H^1(\Omega))^3$ and $\psi \in H^1(\Omega)$ such that*

$$\mathbf{u} = \mathbf{u}_0 + \nabla\psi, \quad \| \mathbf{u}_0 \|_{(H^1(\Omega))^3} + \| \psi \|_{H^1(\Omega)} \leq C \| \mathbf{u} \|_{H(\mathrm{curl},\Omega)} .$$