OME: Optimisation Methods

Important Notes About These Notes

These notes were written by me, Mark Ormesher, during my revision for Computer Science exams at King's College London. I am publishing them here to help my classmates and other students in the years below me. To the best of my knowledge they are correct.

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These notes were relevant for my year (2016/17) but **the content for your course may be different**. Check your lecture slides, syllabi, tutorial guides, etc.

These notes were produced for my own personal use (it's part of how I study and revise). That means that some annotations may by irrelevant to you, and **some topics might be skipped** entirely.

Feel free to **share** these notes, however please only share a link to the repo (see the link below), not individual files.

Notes are originally from **https://github.com/markormesher/CS-Notes**.

Contents

Single-Source Shortest Paths

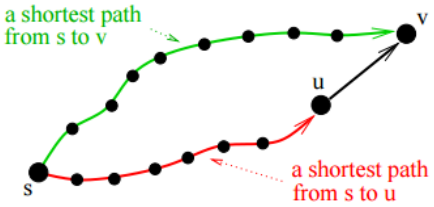
Objective: given a weighted directed graph, find the **minimum-weight paths** from a single vertex s to **all other vertices**. Algorithms that solve this problem can also be adapted to solve **point-to-point** problems.

Notation

- $G = (V, E)$ is a **directed graph** G with vertices V and edges E.
  - Vertices may also be referred to as nodes or sites.
  - $|V| = n$, $|E| = m$.
- $w(u, v)$ is the **weight of the edge** (u, v).
- $s \in E$ is the **source** vertex.
- $p = \langle v_1, v_2, v_3, ..., v_k \rangle$ is a **path** from vertex v_1 to v_k via v_2, v_3, etc.
- $w(p) = w(v_1, v_2) + w(v_2, v_3) + ... + w(v_{k-1}, v_k)$ is the **weight of a path**, equal to the sum of the weights of all edges in the path.
- $\delta(u, v)$ is the **minimum of** $w(p)$ for all p from u to v, or $+\infty$ if no such p exists.
  - Multiple minimum-weight paths may exist.
  - A shortest path from u to v is any path $p = \langle u, ..., v \rangle$ such that $w(p) = \delta(u, v)$.

Useful Facts

- A sub-path of a shortest path is also a shortest path.
  - If a shortest path from a to e is $\langle a, b, c, d, e \rangle$, a shortest path from b to d is $\langle b, c, d \rangle$.
- For each edge $(u, v) \in E$, it holds that $\delta(s, v) \leq \delta(s, u) + w(u, v)$.
  - This states that the weight of the shortest path between s and v is **at most** the weight from s to u, plus the weight of the edge from u to v.
  - This is an example of the **triangle inequality** and is shown in the following diagram:



Negative Weights

Negative weights can be useful/essential in solving a problem - the most obvious example is that of finding the **maximum-weight** paths. Rather than create new algorithms, it is possible to negate every edge and then find the minimum-weight paths as normal.

The **minimum-weight paths for negated edges** are equivalent to the **maximum-weight paths for non-negated edges**.

Exchange Rate Example

Taking the example of a graph showing exchange rates between currencies, the most profitable path **maximises the product** of its edge weights. This is unsuitable, because the algorithms we study **minimise the sum** of edges along a path.

This can be resolved by taking the negative logarithm of every edge. If $\gamma(u, v)$ is the exchange rate from u to v, $w(u, v) = -ln(\gamma(u, v))$. [1] This works because the sum of logarithms of a set of numbers is equal to the logarithm of their products [2].

However, we **cannot avoid negative weights** now: if $\gamma(u, v) > 1$ then $w(u, v) < 0$.

Negative Cycles

If a negative cycle exists on a path from s to v, repeated journeys around this cycle will continually reduce the weight of the path. Therefore, if a negative cycle exists on a path from s to v, $w(s, v) = -\infty$.

We consider only shortest-paths algorithms that:

- Compute the shortest paths for graphs where **no negative cycles are reachable** from the source.
- **Correctly detect when negative cycles are reachable** from the source and are not required to compute anything else.

Why not look for the **shortest simple paths** (i.e. avoid cycles). It's a valid question, but it's NP-hard and we do not cover it in this section. *See more: Optimisations for NP-Hard Problems, page 25*.

Shortest-Path Trees

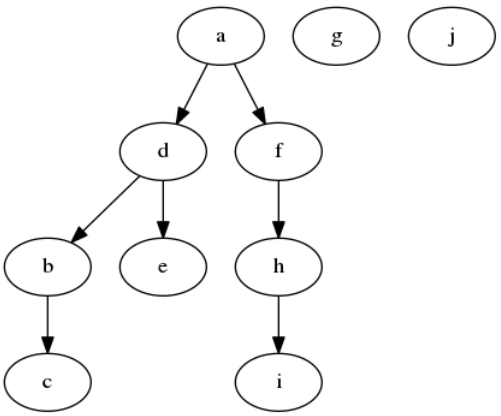
For each vertex v reachable from the source s, we want to find any one of the minimum-weight paths that exist. If there are no negative cycles, these paths can be easily represented with a **shortest-paths tree**. Such a tree contains exactly one shortest path from s to each v reachable from s.

[1] Any fixed logarithm base greater than 1 would work.

[2] $ln(a) + ln(b) + ln(c) = ln(abc)$

The tree T can be represented by an array $parent[v], v \in V$ in $\Theta(n)$ space. $parent[v]$ is the parent of the vertex v in the tree T. An optional array $d[]$ can store the weight of the shortest paths to each vertex. For example, from the source vertex a:

v	a	b	c	d	e	f	g	h	i	j
$parent[v]$	nil	d	b	a	d	a	nil	f	h	nil
$d[v]$	0	6	9	2	4	3	∞	5	6	∞



A disadvantage of this representation is that only one path for each reachable vertex can be stored. Listing every path explicitly leads to $O(n^2)$ space requirements.

Relaxation Technique

This is the most common approach for solving single-source shortest paths problems. It uses an **initialisation** step followed by a sequence of **relaxation** steps. Algorithms following this pattern differ in their **termination condition(s)** and the **order that edges are relaxed**.

At each vertex v, the following is maintained:

- $d[v]$ is the shortest-path **estimate** for reaching v from the source s. It is an upper-bound.
- $parent[v]$ is the current predecessor of v.

Initialisation

This stage sets the upper-bound for all non-source vertices at infinity and creates no parent relationships.

```
1 fun INITIALISE(G, s):
2     d[s] = 0
3     parent[s] = nil
4     for each vertex v in V − {s}, do:
5         d[v] = inf
6         parent[v] = nil
```

Initialisation is a $\Theta(n)$ operation. For large graphs this can be too slow, so an alternative is to only initialise s at the start of the algorithm and then **initialise other vertices as they are discovered**.

Relaxation

Applied to an edge (u, v), this stage checks whether a path to v via u would have a lower weight than the current estimate of reaching v, $d[v]$. If so, $d[v]$ is updated with the weight of the path via u and the parent relationship is updated so that $parent[v] = u$. This process stems from the triangle inequality (*see more: Useful Facts, page 4*).

```
1 fun RELAX(u, v, w):
2     if (d[v] > d[u] + w(u, v)):
3         d[v] = d[u] + w(u, v)
4         parent[v] = u
```

Relaxation is a $\Theta(1)$ operation.

Relaxation Properties

- For every vertex v, throughout the computation...
  - $d[v]$ will only **decrease**.
  - $d[v]$ is either ∞ or the weight of some path from s to v.
  - $d[v] \geq \delta(s, v)$.
- An **effective relaxation operation** is defined as an operation that decreases $d[v]$.
- During computation, there is an edge $(u, v) \in E$ such that $d[v] > d[u] + w(u, v)$ (i.e. there is an effective relaxation operation) if and only if there is a vertex x such that $d[x] > \delta(s, x)$.
  - From this, it follows that if no effective relaxation operations remain then all shortest paths have been found.
- If there **is no negative cycle** reachable from s...
  - There will be a **finite number of effective relaxation operations**.
  - The $parent$ pointers will form a tree rooted at s.
  - When no effective relaxation operations remain, then for each vertex v, $d[v] = \delta(s, v)$ and $parent[v]$ points to v's predecessor on a shortest path from s.
- If there **is a negative cycle** reachable from s...
  - There will always be an effective relaxation operation that can be applied.
  - The $parent$ pointers will eventually form a cycle. This suggests that negative cycles in the graph can be detected by periodically checking $parent$ for cycles.

Checking for Cycles

If the $parent$ array contains a cycle then the graph contains a negative cycle, so computation should indicate this and terminate.

A cycle can be detected by following parent pointers from all vertices, marking vertices as visited during traversal, and reducing work by avoiding re-visiting any vertices. On each traversal, if a vertex that was already seen is encountered then a cycle exists.

```
fun CHECK—CYCLE(parent):
    visited = [false, false, ...]
    for each v in V, do:
        if (!visited[v]):
            seen = { }
            while (true):
                visited[v] = true
                seen.add(v)

                next = parent[v]
                if (next == nil):
                    break

                if (seen.contains(next)):
                    return CYCLE

                if (visited[next]):
                    break

                v = next

    return NO_CYCLE
```

Bellman-Ford Algorithm

The Bellman-Ford algorithm is a **relaxation-based algorithm** for finding shortest single-sources paths of a graph. It **permits negative-weight edges**.

```
fun BELLMAN–FORD(G, w, s):

    // run initialisation and relaxation:
    INITIALISE(G, s)
    for i from 1 to n − 1, do:
        for each edge (u, v) in G, do:
            RELAX(u, v, w)

    // determine outcome:
    for each edge (u, v) in G, do:
        if (d[v] > d[u] + w(u, v)):
            return FALSE // negative cycle reachable from s
        else:
            return TRUE // no negative cycle, d[] now holds shortest path weight
```

Correctness

If a **negative cycle is reachable** from s then the algorithm will always **correctly** return `FALSE` because an effective relaxation will always be possible.

If **no negative cycles are reachable** from s then the algorithm will always **correctly** return `TRUE` because at the end of the first section no more effective relaxations will be possible. This is guaranteed, because of two lemmas:

- If the shortest simple path from s to v is of length k, then after k iterations of the first loop $d[v] = \delta(s, v)$.
  - Assume the k-length path is $\langle s, x_1, x_2, ..., v \rangle$.
  - After the first iteration $d[x_1] = \delta(s, x_1)$.
  - After the second iteration $d[x_2] = \delta(s, x_2)$.
  - ...
  - After k iterations $d[k] = \delta(s, k)$.
- The shortest path from s to any vertex contains at most $n-1$ edges, because the longest non-cyclic path will visit every vertex once.

Running Time

This algorithm **runs the** $\Theta(1)$ **relaxation step exactly** $n - 1$ **times** for each edge in the first section, then in the second section it checks to see if any further relaxations are possible. If more effective relaxations are possible, a negative cycle must exist and `FALSE` is returned; if no effective relaxations are possible then the shortest path weights have been found and `TRUE` is returned.

The running time is $\Theta(nm)$. The worst case for **any** single-source shortest paths problem with negative weights is $\Omega(nm)$.

Improving the Running Time

The $\Theta(nm)$ running time comes from the first section of the algorithm, so we focus our efforts there:

- We could try to **decrease the number of iterations** of the outer loop by terminating it early in certain conditions:
  - Not every graph will require $n - 1$ iterations to find the solution. If **no effective relaxation operations took place** in a given iteration the loop can **terminate early** because the shortest paths were already computed.
  - Some iteration may create a cycle in the $parent$ array. It can be checked periodically (after every iteration?); if a $parent$ **cycle is found** then a negative loop is reachable from s and the entire algorithm can **terminate early** with `FALSE`.
- We could try to **reduce the work done in each iteration** of the outer loop by considering only edges that will give effective relaxations.
  - A vertex u is **active** if its outgoing edges have not been relaxed since the last time $d[u]$ was decreased.
  - We perform relaxations only on the edges out of active vertices.
  - We store active vertices in a **FIFO** queue.

Optimisations: Early Termination and Queue of Active Vertices

This optimisation keeps track of active vertices and **only considers relaxations of their outgoing edges**, as described above.

```
fun BELLMAN—FORD—FIFO(G, w, s):

    // initialisation
    INITIALISE(G, s)
    Q = empty queue // active vertices
    Q.enqueue(s)

    // relaxation of active vertices
    while (Q is not empty), do:
        u = Q.dequeue()
        for each v in adj[u], do:
            RELAX(u, v, w)

        if (CHECK—CYCLE(parent)):
            return FALSE

    return TRUE
```

The relaxation method is augmented to **enqueue the vertex** v if $d[v]$ is updated when considering the edge (u, v) (i.e. if it is 'active').

```
fun RELAX(u, v, w):
    if (d[v] > d[u] + w(u, v)):
        d[v] = d[u] + w(u, v)
        parent[v] = u

        // enqueue v if it was updated
        if (v is not in Q):
            Q.enqueue(v)
```

Note that a mechanism for detecting negative cycles has been added, because Q will never be empty if a negative cycle is reachable from s. This could be in the form of checking for cycles in $parent$ after each set of relaxations.

The running time is now $O(nm)$, but still $\Theta(nm)$ in the worst case.

Dijkstra's Algorithm

- Critical assumption: **all weights are non-negative**.
  - As a consequence, there are also no negative cycles.
- We assume that all vertices are reachable from s.

A **set** S is maintained of all vertices for which $d[v] = \delta(s, v)$ (i.e. vertices that are 'finished'). A **min-priority queue** Q is maintained of all non-finished vertices and their current path weight estimate.

```
fun DIJKSTRA(G, w, s):
    INITIALISE(G, s)
    S = []
    Q = V

    while (Q is not empty), do:
        u = Q.removeMin()
        S.add(u)
        for each v in adj[u], do:
            RELAX(u, v)
```

The relaxation method is augmented to **update the min-priority queue** value of the vertex v if $d[v]$ is updated when considering the edge (u, v).

```
fun RELAX(u, v, w):
    if (d[v] > d[u] + w(u, v)):
        d[v] = d[u] + w(u, v)
        parent[v] = u
        Q.decreaseKey(v, d[v])
```

Running Time

- The data structures will be instantiated $\Theta(1)$ times.
- The main **`while`** loop will run $\Theta(n)$ times, because all vertices start in the queue, each vertex is removed, and no vertex is replaced.
- A total of $\Theta(n)$ calls to `Q.removeMin()` will be made.
- A total of $\Theta(m)$ relaxations will take place, each of which may include a call to `Q.decreaseKey()`.

It is clear that the priority queue implementation will be the main factor in the algorithm's running time. There are three implementations to consider, each with different running times:

- **Unsorted array**:
  - Instantiation is $\Theta(n)$.
  - `removeMin()` requires an $O(n)$ search and will be done $\Theta(n)$ times.
  - `decreaseKey()` requires a $\Theta(1)$ update and will be done $O(m)$ times.
  - The total running time is $\Theta(n) + O(n^2) + \Theta(m) = O(n^2)$.
- **Sorted array**:
  - Instantiation is $\Theta(n)$.
  - `removeMin()` requires a $\Theta(1)$ removal and will be done $\Theta(n)$ times (this assumes that values are not shifted to fill the gap created).
  - `decreaseKey()` requires an $O(n)$ re-ordering and will be done $O(m)$ times.
  - The total running time is $\Theta(n) + \Theta(1) + O(mn) = O(mn)$.
    - Note: $n - 1 \leq m \leq n(n-1)$, so the total is $O(n^3)$ for a dense graph.
- **Heap**:
  - Instantiation is $\Theta(n)$.
  - `removeMin()` requires an $O(log_2(n))$ removal and will be done $\Theta(n)$ times.
  - `decreaseKey()` requires an $O(log_2(n))$ re-ordering and will be done $O(m)$ times.
  - The total running time is $\Theta(n) + O(n \cdot log_2(n)) + O(m \cdot log_2(n))) = O(m \cdot log_2(n))$.

If the input graph is **dense**, such that $m = \Omega(n^2/log_2(n))$, then the **unsorted array** implementation of the priority queue gives a better worst-case running time. If the graph is **not dense**, the **heap implementation** gives a better worst-case running time.

For most applications the input graphs are **not dense**, so Dijkstra's algorithm is **assumed to use a heap-based priority queue**.

Correctness - Invariants

We establish **invariants** for the algorithm, based on each vertex's membership in S or Q.

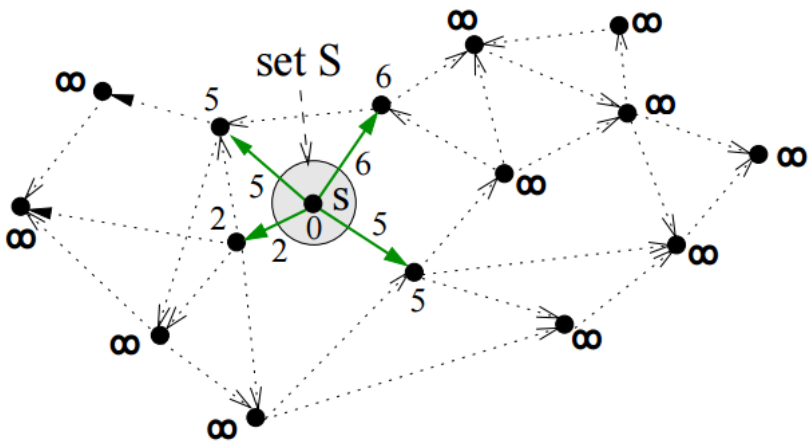
1. For each $v \in S$:
   (a) v is in the current tree.
   (b) $d[v]$ is the shortest-path weight from s to v, i.e. $d[v] = \delta(s, v)$.
   (c) The path from s to v in the current tree is a shortest path from s to v.
2. For each $z \notin S$ such that there **is an edge** (v, z) for some vertex $v \in S$:
   (a) z is a leaf in the current tree.
   (b) The path from s to z in the current tree is a shortest path from s to v where all intermediate nodes are in S.
   (c) $d[z]$ is the weight of this path (the tree path from s to z).
3. For each $y \notin S$ such that there **is no edge** (v, y) for some vertex $v \in S$:
   (a) y is not in the current tree.
   (b) $parent[y] = nil$
   (c) $d[y] = \infty$

At the end of the algorithm's execution $S = V$, so only the first set of invariants apply for each vertex. These state that $d[v] = \delta(s, v)$ for all vertices v and that the current three is the shortest path three from s to all other vertices. Therefore, if these invariants hold, the algorithm is correct.

Correctness - Induction on Invariants

Basis step: prove that the invariants hold for the first iteration ($i = 0$).

At the first iteration, s is taken from Q, added to S, and all outgoing edges are relaxed. This produces the state pictured below, for which all invariants hold.



Inductive step: prove that if the invariants hold for iteration i, they also hold for $i + 1$.

- Assume that the invariants hold at the end of some iteration i (possibly the first iteration, described above).
- Let u denote the vertex selected in the iteration $i + 1$.
- Show that the invariants hold at the end of iteration $i + 1$ when u has moved from Q to S.

Proving Invariant 1b

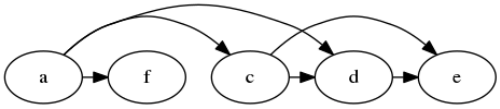
- We have selected vertex u to add to S, so we must prove that at the end of the iteration $d[u] = \delta(s, u)$.
- We already know that $d[u] \geq \delta(s, u)$ because it is a basic property of the relaxation operation, so we must prove $d[u] \leq \delta(s, u)$.
- Select any path R from s to u and show that $w(R) \geq d[u]$.
  - Split R into $R1$ and $R2$, such that $R1$ ends at the first vertex y outside of S.
  - $w(R) = w(R1) + w(R2) \geq d[y] + w(R2) \geq d[u] + w(R2) \geq d[u]$
  - Therefore, no path from s to u has a smaller weight than $d[u]$, so $d[y] \leq \delta(s, u)$.
- This works because u has the lowest d-value, therefore $d[y] \geq d[u]$, and all edge weights are non-negative, therefore $d[y] + w(R2) \geq d[u]$.

Directed Acyclic Graphs (DAGs)

The objective is the same as before: find the **minimum-weight paths** from a single vertex s to **all other vertices**. Dijkstra's algorithm won't work because negative-edges are possible; the Bellman-Ford algorithm could work in $O(mn)$, but the restricted graph format (DAG) allows for a $\Theta(n + m)$ algorithm.

Topological Order

The algorithm relies on creating a **topological order** for the vertices in the graph. A topological order rearranges vertices (without changing edges) so that all edges point in the same direction, as pictured:



Once the topological order has been created, the algorithm is simple:

```
fun DAG—SHORTEST—PATHS(G, w, s):
    TOPO—SORT(G)
    INITIALISE(G)
    for each vertex u in topological order, do:
        for each vertex v in adj[u], do:
            RELAX(u, v, w)
```

The running time is made up of one $\Theta(n+m)$ topological order generation, one $\Theta(n)$ initialisation and $\Theta(m)$ relaxations, giving $\Theta(n+m)$.

The algorithm works because when a vertex is considered all of its incoming edges have already been relaxed, therefore $d[u] = \delta(s, u)$ when u is considered.

Geographic Networks

When looking for a **single destination** d, one approach is to run Dijkstra's algorithm from the source and stop when d is considered (i.e. removed from Q). This may require checking the entire network, wasting a lot of effort.

Another approach is to **run two Dijkstra computations**, one searching 'forwards' from the source and another searching 'backwards' from the destination. When a shortest path from s to d is found both computations can stop. This **bi-directional** search can result in a smaller part of the networking being examined, but is very tricky to implement with an optimal stopping condition.

When we have a geographic network we know the **coordinates** of each vertex. With this extra information we can compute the **straight-line distance between vertices**, which is a lower-bound on the shortest paths between them. Straight-line distances to the destination can be used to **re-weight** edges.

Re-Weighting Edges

Graph edges can be re-weighted so that edges leading geographically *towards* d become cheaper and edges leading *away* from d get more expensive. In a similar fashion to Johnson's Algorithm (*see more: Johnson's Algorithm, page 16*), edges can re-weighted as follows:

$$\hat{w}(u, v) = w(u, v) - straight\_line(u, d) + straight\_line(v, d)$$

Values in $\hat{w}$ are always ≥ 0, satisfying the requirements of Dijkstra's algorithm.

Note that this is a **heuristic** that will improve on the average-case running time, but **will not improve the worst-case** scenario.

All-Pairs Shortest Paths

Objective: given a weighted directed graph, find the **minimum-weight path** from **all vertices to all other vertices**. This will create $n(n-1)$ paths and weights.

The output is two $n * n$ matrices:

- D, such that $D[i,j] = \delta(i,j)$.
- P, such that $P[i,j]$ holds the parent of j in a path from i to j.

For simplicity, we will only consider computation of the first matrix.

Existing Algorithms

- If all edge weights are **non-negative** we can run **Dijkstra's algorithm** from each vertex.
  - $n \cdot O(min\{m \cdot log_2(n), n^2\}) = O(min\{mn \cdot log_2(n), n^3\})$.
  - This is the best worst-case running time that is known.
- In the general case where negative edges are allowed we can run the **Bellman-Ford** algorithm from each vertex.
  - $n \cdot O(mn) = O(mn^2)$, up to $O(n^4)$.

Floyd-Warshall Algorithm

This algorithm runs in $\Theta(n^3)$ and is not covered in this course.

Johnson's Algorithm

This algorithm runs in $O(min\{mn \cdot log_2(n), n^3\})$, making it **suitable for sparse graphs** but **unsuitable for dense graphs**.

It works by **re-weighting edges** to remove negatives, then applying **Dijkstra's algorithm** at every vertex.

Re-Weighting Edges

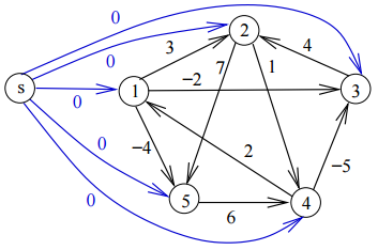
We must change the weight of edges to **remove negatives without changing shortest paths**. Adding some large value to each edge won't work because the shortest paths may change (because the path length becomes a major component in its weight).

Johnson's algorithm re-weights edges by first assigning a value $h(v)$ to each vertex v and then creating an updated set of weights $\hat{w}$ such that $\hat{w}(i,j) = w(i,j) + h(i) - h(j)$. That is, each edge weight is increased by its source vertex h-value and decreased by its destination vertex h-value.

For any path the h-values of all intermediate nodes **cancel each other out**. For a path $P = \langle v_1, v_2, ..., v_k \rangle$ its new weight is calculated as $\hat{w}(P) = w(P) + h(v_1) - h(v_k)$. All paths between v_1 and v_k are offset by the same amount regardless of their length, so shortest paths are not affected.

Calculating h-values

The **Bellman-Ford** algorithm can be used to calculate the h-value for each vertex. An 'imaginary vertex' s is created with a 0-weight edge to all vertices of the graph, and then the shortest path weights are found from s. Once complete, $h(v) = d[v] = \delta(s, v)$ for all vertices in the 'real' graph.



Full Algorithm

```
fun JOHNSON(G, w):
    D = output matrix

    // add imaginary vertex
    V' = V + { s }
    E' = E + { (s, v) for all v in V }
    w(s, v) = 0 for all v in V
    G' = (V', E')

    // detect negative cycle or find h—values
    if (BELLMAN—FORD(G', w, s) == false):
        return false // negative cycle
    else:
        // re—weight edges
        for each vertex v in V, do:
            h(n) = d[v]
        for each edge (u, v) in E, do:
            w'(u, v) = w(u, v) + h(u) — h(v)

        // run Dijkstra and record (adjusted) result in D
        for each vertex u in V, do:
            d' = DIJKSTRA(G, w', u)
            for each vertex v in d', do:
                D[u, v] = d'[v] — h(u) + h(v)

    return D
```

Running Time

The running time is made up of one iteration of the Bellman-Ford algorithm and n iterations of Dijkstra's algorithm.

$$O(mn) + n \cdot O(min\{m \cdot log_2(n), n^2\}) = O(min\{mn \cdot log_2(n), n^3\})$$

Correctness

- **Detecting negative cycles**: we know that the Bellman-Ford algorithm can do this correctly and that the 'imaginary vertex' s cannot create a cycle (because it has no incoming edges), so therefore this algorithm can correctly detect negative cycles.
- **Computing shortest path weights**: we know that Dijkstra's algorithm can do this correctly and we apply it at every vertex, so therefore this algorithm can correctly compute the shortest path weights.

Network Flow Problems

Flow Network Notation

- $G = (V, E, c, s, t)$ is a **flow network** G with vertices V and edges E.
  - Vertices may also be referred to as nodes or sites.
  - $|V| = n$, $|E| = m$.
- $c(u, v)$ is the **capacity of the edge** (u, v).
- $s \in E$ is the **source** vertex.
- $t \in E$ is the **sink/destination** vertex.
- $s \neq t$.

Types of Flow Problem

- **Maximum Flow**: find a **maximum flow** from the **source** to the **sink** of a flow network. A maximum flow sends the **maximum amount of the underling commodity** from the source to the sink without exceeding the capacity of any edge.
- **Flow Feasibility** (Transshipment Problem): find a flow which satisfies edge capacities *and* the specified supply/demand values at various vertices.
  - This can be reduced to the maximum flow problem.
- **Minimum-cost Flow**: *covered later*.
- **Multi-commodity Flow**: *covered later*.

Flow Formalisation

We assume that if $(u, v) \in E$ then $(v, u) \in E$. Edges with zero capacity are added when required, but usually not shown on diagrams.

A **flow is a function** $f : E \mapsto \mathbb{R}$ where $f(u, v) \geq 0$ is the flow on the edge (u, v) with the following properties:

- **Capacity constraints**: for each edge $(u, v) \in E$, $0 \leq F(u, v) \leq c(u, v)$.
- **Flow conservation**: for each vertex $v \in V - \{s, t\}$, the total flow in to v is equal to the total flow out of v (i.e. the net flow through v is zero).
- **Single-direction flow**: for each edge $(u, v) \in E$, if $f(u, v) > 0$ then $f(v, u) = 0$.

The **value of a flow** is the net flow into the sink (which is the same as the net flow out of the source, because of flow conservation):

$$|f| = \sum_{(x,t)\in E} f(x,t) - \sum_{(t,z)\in E} f(t,z)$$

$$|f| = \sum_{(s,x)\in E} f(s,x) - \sum_{(z,s)\in E} f(z,s)$$

For a given flow f, if $f(u,v) = c(u,v)$ then f is said to **saturate** the edge (u,v) and the edge (u,v) is said to be a **saturated edge**.

From Flows to Paths

Given a network $G = (V, E, c, s, t)$ and a flow $f : E \mapsto \mathbb{R}$, the flow f can be **decomposed** into at most m paths from s to t, where $m = |E|$.

Algorithm: **select and remove maximal** $\langle s, ..., t\rangle$ **flow paths** from f until f is empty. Each path removes all remaining flow from at least one edge, so at most m paths are created.

Each path can be found in $O(n)$ and there are $O(m)$ paths, so a flow can be decomposed into paths in $O(mn)$. This is less than the time needed to find a maximum flow, so this is okay.

Note that this algorithm does not work when **flow cycles** exist, but can be modified to accommodate them.

Flow Feasibility Problem

This variation adds **supply and demand** at various vertices, such that:

- $G = (V, E, c, d)$ is a **flow network** G with vertices V and edges E.
- $c(u,v)$ is the **capacity of the edge** (u,v).
- $d(v)$ is the supply/demand at the vertex v.
  - $d(v) > 0$ indicates a supply of $d(v)$ units at v.
  - $d(v) < 0$ indicates a demand for $d(v)$ units at v.
  - $d(v) = 0$ indicates a 'transitional' vertex.
- We assume that supply matches demand, i.e. $\sum_{v\in V} d(v) = 0$.

The objective is to find a flow f that 'moves' all supply to meet all demand. This means that each vertex should have a net flow of zero, i.e. for each vertex v,

$$\sum_{(v,x)\in E} f(v,x) - \sum_{z,v} f(z,v) = d(v)$$

Reduction to Maximum Flow Problem

For a given $G = (V, E, c, d)$ for a flow feasibility problem, $G' = (V', E', c', s, t)$ can be constructed to solve it as a maximum flow problem:

- Create an **artificial source vertex** s with edges **to** all 'supply' vertices v, each with capacity equal to the supply, $d(v)$.
- Create an **artificial sink vertex** t with edges **from** all 'demand' vertices u, each with capacity equal to the negative of the demand, $-d(u)$.

Formally:

- $V' = V + \{s, t\}$
- $E' = E + \{(s, v) : v \in V, d(v) > 0\} + \{(u, t) : u \in V, d(u) < 0\}$
- $c'(u, v) = c(u, v)$
- $c'(s, v) = d(v)$
- $c'(u, t) = -d(u)$

A maximum flow in G' saturates all outgoing edges from s (and therefore saturates all incoming edges to t) if and only if there is a feasible flow in G.

- If a maximum flow f' in G' saturates all edges outgoing from s, then remove the artificial vertices s and t to get a feasible flow for G.
- If f is a feasible flow in G, then saturate all edges outgoing from s and all edges incoming to t to get a maximum flow in G'.

Solving Maximum Flow Problems

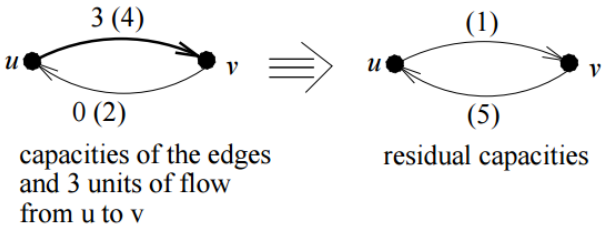
Maximum flow problems are solved by continually adding flow to a network. The amount that can be added (i.e. the network's remaining capacity) is the **residual capacity**.

Residual Capacity

If f is a flow defined in the network $G = (V, E, c, s, t)$, then residual edges c_f are defined as follows:

- $c_f(u, v) = c(u, v) - f(u, v)$ if $f(u, v) \geq 0$ and $f(v, u) = 0$.
  - When there is flow from u to v, the residual capacity of (u, v) decreases by the amount of that flow.
- $c_f(u, v) = c(u, v) + f(u, v)$ if $f(u, v) = 0$ and $f(v, u) \geq 0$.
  - Where there is flow from v to u, the residual capacity of (u, v) increases by the amount of that flow.

In the example below, $c(u, v) = 4$, $c(v, u) = 2$ and $f(u, v) = 2$. The residual edges $c_f(u, v) = 1$ and $c_f(v, u) = 5$ are created.



This shows that we can send 1 more unit of flow from u to v, or we can send 5 unit from v to u by 'sending back' the 3 that are already being sent in the other direction, plus saturating the edge (v, u).

If $c_f(u, v) > 0$ then (u, v) is a **residual edge**.

The **residual network** of G induced by the flow f is the flow network $G_f(V, E_f, c_f, s, t)$ where c_f are the residual capacities and E_f is the set of residual edges.

Adding Flow Using the Residual Network

If f is a flow in the network G and f' is a flow in the residual network G_f, then f and f' can be combined to generate a new flow h in G.

The new flow $h = f \bigoplus f'$ in G is defined as follows:

For each (u, v) such that $f(u, v) \geq 0$ and $f(v, u) = 0$:

- If $f'(u, v) \geq 0$ and $f'(v, u) = 0$, then:
  - $h(u, v) = f(u, v) + f'(u, v)$
  - $h(v, u) = 0$
  - i.e. when the flows go in the **same direction** just add them together and leave the opposite direction as zero.
- If $f'(u, v) = 0$ and $f'(v, u) > 0$, then:
  - $h(u, v) = max\{f(u, v) - f'(v, u), 0\}$
  - $h(v, u) = max\{f'(v, u) - f(u, v), 0\}$
  - i.e. when the flows go in **opposite directions**, cancel out the smaller flow and add whatever is left.

The value of h is defined as $|h| = |f| + |f'|$.

General Approach for Finding Maximum Flow

- Start with f as a zero flow (i.e. $f(u, v) = 0$ for all $(u, v) \in E$).
- Loop forever:
  - Construct the residual network G_f.
  - Find a non-zero flow f' in G_f.
    - If no such flow exists, exit loop.
  - $f = f \bigoplus f'$
- Return f as the maximum flow.

Most maximum flow algorithms use this approach.

Linear Programming

TODO: Lecture 5.

Optimisations for NP-Hard Problems

TODO: Lectures 6, 7, 8.