

PAL: Parallel Algorithms

Important Notes About These Notes

These notes were written by me, Mark Ormesher, during my revision for Computer Science exams at King's College London. I am publishing them here to help my classmates and other students in the years below me. To the best of my knowledge they are correct.

- These notes are **not endorsed** by King's College London, the module lecturers, or any member of College staff.
- These notes are **not checked** by any qualified individual, and I offer no guarantee as to how accurate or complete these notes are. They are offered for free, as-is, and I accept no responsibility for how you choose to use them.
- These notes were relevant for my year (2016/17) but **the content for your course may be different**. Check your lecture slides, syllabi, tutorial guides, etc.
- These notes were produced for my own personal use (it's how I like to study and revise). That means that some annotations may be irrelevant to you, and some topics might be skipped entirely.
- Feel free to **share** these notes, however please only share a link to the repo (see the link below), not individual files.

Notes are originally from <https://github.com/markormesher/CS-Notes> and remain the copyright-protected work of Mark Ormesher.

Contents

1	Important Notes About These Notes	1
2	Recap: Algorithms	4
2.1	Runtime of Algorithms	4
3	Introduction	5
3.1	Studying Parallel Algorithms	5
3.2	Programming Models	5
3.3	Classification of Parallel Models by Communication	6
3.4	Sequential, Parallel, Distributed?	6
3.4.1	Sequential vs. Parallel	6
3.4.2	Parallel vs. Distributed	6
4	Multi-Threading Model	7
4.1	Example: Parallelised Fibonacci Calculation	7
4.2	Complexity Measures for Multi-Threading	8
4.3	Example: Adding 1, n Times	8
4.4	Example: Computing Squares	9
4.5	Speed-up Bounds for p Processors	10
4.5.1	Greedy Scheduling Principle	10
5	PRAM Model	11
5.1	Example: Binary Fan-In Array Sum on EREW-PRAM	11
5.1.1	Analysis	11
5.2	Example: Array Sum CRCW-PRAM with Priority-CW	12
5.2.1	Analysis	12
5.3	Greedy Work-Time Scheduling Principle	12
5.3.1	Example 1	13
5.3.2	Example 2	13
6	PRAM Algorithms	14
6.1	Matrix * Vector Multiplication	14
6.2	Matrix * Matrix Multiplication	15
6.3	Binary Tree Algorithms	15
6.3.1	Recap: Binary Tree Basics	15
6.3.2	Array Membership Algorithm EREW (Intro)	16
6.3.3	Broadcast (Array Filling) EREW	16
6.3.4	Parallel Array Membership EREW	16
6.3.5	Binary Fan-in Array Minimum EREW	17
6.3.6	Array Membership Algorithm EREW (Algorithm)	17
6.3.7	Binary Fan-in Array Maximum EREW	18
6.3.8	Parallel Prefix Sums EREW	18
6.4	Array Maximum CRCW	19
6.5	Array Membership CRCW	20
6.6	PRAM-CRCW Sorting	21
6.7	Is PRAM-CRCW More Powerful than PRAM-EREW?	21
6.7.1	Simulation	21

7	Modelling Other Data Structures	22
7.1	Parent Labelling	22
7.1.1	Example: Tree	22
7.1.2	Example: List	22
7.2	List Ranking	23
7.3	Forest Root Finding	23
8	Evaluating Parallel Algorithms	24
8.1	Analysis of Algorithms	24
8.2	Work/Time and Work/Span	24
8.3	Other Efficiency Metrics	25
8.3.1	Speed-Up	25
9	Graph Interconnected Model (Message Passing)	26
9.1	Common Structures	26
9.2	Message Passing Costs	26
9.2.1	Store-and-Forward (SF) Routing	27
9.3	Graph-Based Network Quality Measures	27
10	Graph Interconnected Algorithms	28
10.1	Notation	28
10.2	Indexing	28
10.3	Array Minimum, 2D Mesh (Pesudocode)	28
10.4	Array Minimum, 2D Mesh	29
10.5	Broadcast, 2D Mesh	29
10.6	Membership Query, 2D Mesh	30
10.7	Array Prefix, 2D Mesh	30
10.8	Matrix Multiplication, 1D Path, Pipelining	31
10.9	Even/Odd Transposition Sort, 1D Mesh	32
10.10	Snake Order Sort, 2D Mesh	32

Recap: Algorithms

An algorithm is a **well-defined finite set** of rules that describe a **sequence of primitive operations** that can be applied to some **input** to produce some **output** in a finite amount of time.

Any algorithmic solution to a problem will include **designing** an algorithm and then **analysing its performance**. For some algorithms, parallelism can be used to improve performance.

Runtime of Algorithms

The runtimes and memory consumptions of algorithms are usually stated with **Big-O** notation, but **Big-Θ** and **Big-Ω** may also be used. These are the same notations that has been covered in other notes and as such will not be covered here, other than a short summary:

- If f and g are functions of n , then $f = O(g)$ means that there is a constant $c > 0$ such that $f(n) \leq c \times g(n)$ (i.e. f grows no faster than g).
- **Big-O** (big-oh) is an **upper** bound ('grows no faster than...').
- **Big-Θ** (big-theta) is a **tight** bound ('grows at...').
- **Big-Ω** (big-omega) is a **lower** bound ('grows at least as fast as...').
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$ (or they are roughly within a constant of each other).

Introduction

Parallelism is everywhere and can be exploited to improve performance:

- Global scale: huge computational grids, such as Folding@Home
- Super computers: simulations, modelling, etc.
- Desktop scale: multi-core CPUs and GPUs
- Specialised hardware for processes like encryption, hashing, etc.

Studying Parallel Algorithms

To study and consider parallel computing we will need **four things**:

- **Machine models** to define what operations are available at a fairly abstract level.
- **Cost models** to define the costs of these operations in terms of resources we care about (time and memory, usually).
- **Analysis techniques** to help us to map from algorithms to costs with reasonable accuracy.
- **Metrics** that let us discriminate between costs (e.g. speed vs. accuracy).

Programming Models

There are different programming models for parallel algorithms that reflect differences in underlying architectures:

- The **shared address space** model allows threads to interact directly through shared memory spaces. Care is needed to avoid race conditions and other problems. We will consider two variations of this model: **multi-threading** and **PRAM**.
- The **message passing** model gives each process its own address space and allows results to be sent between processes using messages. We will consider a **graph interconnection network** model.

We consider the following three models:

- [See more: Multi-Threading Model, page 7.](#)
- [See more: PRAM Model, page 11.](#)
- [See more: Graph Interconnected Model \(Message Passing\), page 26.](#)

Classification of Parallel Models by Communication

All of the models we consider are **data parallel** (SIMD). Models can be classified by the methods that are used by parallel processes to communicate with each other.

- **Implicit interaction** (multi-threading) - process interaction is invisible to the programmer; the compiler and/or runtime scheduler handle parallelism.
- **Shared memory** (PRAM) - parallel processes share a global address space that they can all read/write to/from asynchronously.
- **Message passing** (graph interconnected network) - parallel processes explicitly pass messages to each other.

Sequential, Parallel, Distributed?

Sequential vs. Parallel

Some things are inherently sequential (like a train journey), whereas some things are inherently parallel (like a marathon). **Divide and conquer** algorithms can often be parallelised, but not always.

Parallel vs. Distributed

- **Parallel** systems have some central controller.
- **Distributed** systems have no central controller and cooperation is needed.

Multi-Threading Model

The multi-threading model uses the operations **spawn** and **sync** without worrying about the underlying hardware that enables them.

- **spawn** creates an instance of some process to run in parallel.
- **sync** waits for spawned processes to stop and allows results to be collated.

Many languages support these functions already and allow the creation of separately runnable processes called threads. Exact scheduling and parallelism is left to the OS - we consider this model at a **higher, logical level**.

Example: Parallelised Fibonacci Calculation

The 'dumb' method is the slow, recursive version:

```

1 fun recFib(n) :
2   if (n <= 1) return n
3   return recFib(n - 1) + recFib(n - 2)

```

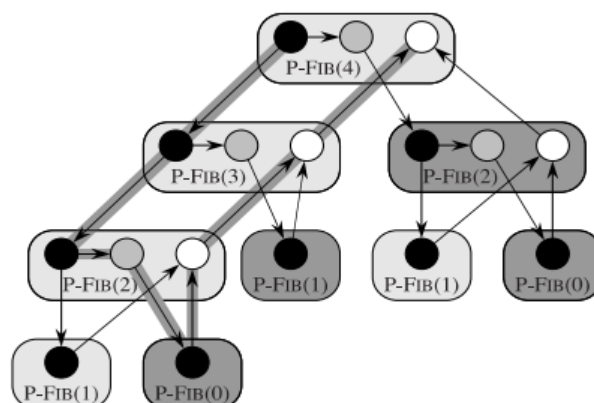
This can be parallelised by running the two recursive calls in parallel, so they can work side by side. This **does not reduce the work** to be done, it just gets it done faster.

```

1 fun parFib(n) :
2   if (n <= 1) return n
3   x = spawn parFib(n - 1)
4   y = parFib(n - 2)
5   sync
6   return x + y

```

The process call diagram for pFib(4) is shown below:



The diagram forms a **directed, acyclic graph** (a.k.a. a **computation DAG**) with a **single source** and **single sink** node at the root.

- Black to black edges: a spawned process
- Grey to black edges: work done within a process
- White nodes: synced results

The **critical path** is the longest path through the whole system (shaded in the diagram above) and represents the necessary amount of sequential work in the solution.

Complexity Measures for Multi-Threading

T_p = the time to execute on p processors.

The **span** S or T_∞ = the number of vertices on the critical path.

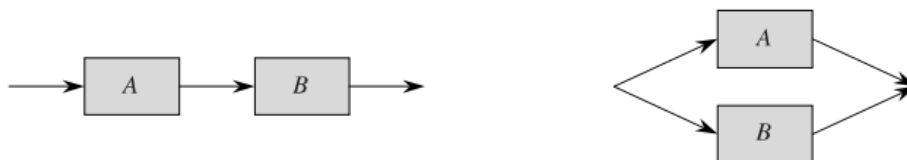
The **work** W or T_1 = the time to execute on one processor.

Speed up = T_1/T_p (i.e. how much faster it runs on p processors).

Parallelism = T_1/T_∞ (i.e. the maximum possible speed up).

Note: the **span** represents the unavoidable sequential work that must be done. The **work** is defined as the number of vertices in the **computation DAG**.

Work vs. span: when parallelised, work remains the same, but the span will reduce.



In the first case, work is $W = A + B$ and the span is $S = A + B$. In the second case, work is the same, but the span is now $S = \max(A, B)$.

Example: Adding 1, n Times

This trivial example reduces to $S(n) = n$. We can **assume n will be some power of 2**.

The sequential approach is obvious, and the recursive approach is as follows:

```

1 | fun recAdd(n) :
2 |     if (n <= 1) return n
3 |     return recAdd(n / 2) + recAdd(n / 2)
  
```


The recursive approach can be parallelised as before:

```

1 | fun recAdd(n) :
2 |     if (n <= 1) return n
3 |     x = spawn recAdd(n / 2)
4 |     y = recAdd(n / 2)
5 |     sync
6 |     return x + y

```

Assuming $n = 2^m$...

$Work = T_1(n) = \Theta(n)$ because the actual work will comprise n additions.

$$\begin{aligned}
 Span = T_\infty(n) &= \Theta(1) + \max(T_\infty(n/2), T_\infty(n/2)) \\
 &= \Theta(1) + T_\infty(n/2) \\
 &= (m - 1) \times \Theta(1) + T_\infty(n/2^m) \\
 &= m \times \Theta(1) \\
 &= m
 \end{aligned}$$

Parallelism is $T_1(n)/T_\infty(n) = \Theta(n/\log_2(n))$. This is an almost linear speed-up over the initial sequential algorithm.

Example: Computing Squares

$$\begin{aligned}
 S(n) &= 1^2 + 2^2 + 3^2 + \dots + n^2 \\
 &= n^2 + (n - 1)^2 + \dots + 1
 \end{aligned}$$

```

1 | fun parSqaure(n) :
2 |     if (n == 1) return 1
3 |     x = spawn parSquare(n - 1)
4 |     y = n * n
5 |     sync
6 |     return x + y

```

Here, the spawned task does almost all of the work. On the **computation DAG**, all nodes are on the critical path, so this won't speed anything up. This is therefore **not parallel computing**; it is sequential computing *disguised* as parallel.

$$T_1(n) = n$$

$$\begin{aligned}
 T_\infty(n) &= \Theta(1) + \max(1, T_\infty(n - 1)) \\
 &= \Theta(n)
 \end{aligned}$$

Parallelism is $n/n = 1$ (i.e. useless parallelisation).

Speed-up Bounds for p Processors

Reminder: speed-up = T_1/T_p for p processors.

How much faster can something be made to run? What are the bounds on $T_p(n)$?

A very crude bound is $T_p \geq T_1/p$ if the **work is split perfectly** between p processors and no additional overhead is incurred. Any better than this would imply that a processor is running faster than the one used for T_1 , making the comparison pointless.

In reality, it's **hard to split work perfectly**. This gives the bound $p \times T_p \geq T_1$ because there will always be some **overhead** and **non-perfect sharing** of work.

If p is **very large**, this lower bound is inaccurate. In any problem there will be some sequential work (indicated by the **span**), which affects the real lower bound. We need better accuracy.

Greedy Scheduling Principle

A greedy scheduler allocates work to any processor **as soon as it becomes free**.

If a computation is ran on a greedy scheduler, T_p is bounded by $T_p \leq (W/p) + S$. The lower bound is $T_p \geq \max(W/p, S)$.

W/p assumes work can be allocated so that all p processors finish at the same time, therefore:

$$\max\left(\frac{W}{p}, S\right) \leq T_p \leq \frac{W}{p} + S$$

Increasing p reduces W/p , but S does not change. If $W/p < S$, then S dominates the equation and we are wasting resources because some processors will lie empty, waiting to be used. **There is no value in reducing W/p below S .**

PRAM Model

In the PRAM model, **multiple processors act synchronously** to execute the same command on **different sets of data** (SIMD = Same Instruction, Multiple Data).

There are **two options** each for an algorithm's **read/write modes**: exclusive (ER or EW) and concurrent (CR or CW). An EREW algorithm can be very different to a CRCW algorithm. Various **CW protocols** exist:

- The simplest is *priority* - the lowest ID process has priority.
- Another is *arbitrary* - a process is picked at random.
- Another is *common* - the write is allowed only if the values are the same.

CW can create a problem: if all processes are waiting to write, the program is sequential.

Example: Binary Fan-In Array Sum on EREW-PRAM

$A[1..n]$ = n numbers. We want to put the sum of the n numbers into $A[1]$.

We can halve the array size in **one parallel step** by adding the values at locations $2i$ and $2i - 1$, then putting them into location i . Overwriting data isn't a problem, because this is a **SIMD** system - every process reads their cells, performs the addition and write the result at the same time (read, read, add, write). This pattern is known as a **binary fan-in** and works for **any associative binary operation**.

```

1 fun arrSum-PRAM-EREW:
2   Input:  $n$  numbers stored in the array  $A[1..n]$ 
3   Output: sum of  $A[1..n]$ , written into  $A[1]$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6   for  $i$  from 1 to  $n$ , in parallel do:
7      $B[i] = A[i]$ 
8
9   for  $h$  from 1 to  $k$ , do:
10    for  $i$  from 1 to  $(n / 2^h)$ , in parallel do:
11       $B[i] = B[2i] + B[2i - 1]$ 
12
13   $A[1] = B[1]$ 

```

On line 10, the i runs from 1 to $n/2$, then $n/4$, then $n/8$, etc.

Analysis

- Line 7: one round of parallel $\Theta(1)$ time

- Line 11: $k = \log_2(n)$ rounds of parallel $\Theta(1)$ time
- Line 13: $\Theta(1)$ time

Example: Array Sum CRCW-PRAM with Priority-CW

```

1 fun arrSum-PRAM-EREW:
2   Input:  $n = 2^k$  numbers stored in the array  $A[1..n]$ 
3   Output: sum of  $A[1..n]$ , written into  $A[1]$ 
4   Note:  $k = \log_2(n)$ 
5   Note: CW rule is PRIORITY-CW
6
7    $S = 0$ 
8   for  $i$  from 1 to  $n$ , in parallel do:
9      $S = S + A[i]$ 
10   $A[1] = S$ 

```

This works if you consider each processor to execute each parallel line in one step. After $S = 0$, this gives:

- $S = S + A[1] = A[1]$
- $S = S + A[2] = A[1] + A[2]$
- $S = S + A[3] = A[1] + A[2] + A[3]$
- etc.

Analysis

- Line 7: $\Theta(1)$ time
- Line 9: one round of parallel $\Theta(1)$ time
- Line 10: $\Theta(1)$ time

Greedy Work-Time Scheduling Principle

This is the PRAM equivalent of the greedy scheduling principle defined for the multi-threading model ([see more: Greedy Scheduling Principle, page 10](#)).

We know that $T_\infty(n) \leq T_p(n)$ and we assume that p processors can do p units of work in one time step. Therefore:

$$T_\infty(n) \leq T_p(n) \leq \frac{W(n)}{p} + T_\infty(n)$$

There are $W(n)$ operations to be done, and we assume that p processors can do p units of work in one step, giving $W(n)/p$ time steps, plus $T_\infty(n)$ unavoidable sequential steps.

Example 1

An algorithm has $T_\infty = 50$ and $W = 10,000$ on some input. How do increasing p values affect the runtime bounds?

- $50 \leq T_{200} \leq 100$
- $50 \leq T_{1000} \leq 60$
- $50 \leq T_{10000} \leq 51$
- $50 \leq T_{100000} \leq 51$

Once $W(n)/p$ drops below $T_\infty(n)$, there is no value to increasing p .

Example 2

An algorithm has $T_\infty(n) = \Theta(n^{2/3})$ and $W(n) = \Theta(n)$. What is the maximum useful p ?

$$T_\infty(n) \leq T_p(n) \leq \frac{W(n)}{p} + T_\infty(n)$$

$$\Theta(n^{2/3}) \leq T_p(n) \leq \frac{\Theta(n)}{p} + \Theta(n^{2/3})$$

The maximum useful p is $n^{1/3}$. When p is this large, $\frac{\Theta(n)}{p} = \Theta(n^{2/3})$, which gives the best possible upper bound; any more processors and resources would be wasted, because the constant $T_\infty(n) = \Theta(n^{2/3})$ begins to dominate the equation.

PRAM Algorithms

Matrix * Vector Multiplication

Matrix and vector multiplications are good candidates for parallel computation because each element of the result can be computed independently and only associative operations are used.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

The sequential algorithm has a runtime of $\Theta(n^2)$:

```

1 fun matrixVectorMult-SEQ:
2   Input:  $n \times n$  matrix  $A$ ,  $n \times 1$  vector  $B$ 
3   Output: vector  $Y = A * B$ 
4
5   for  $r$  from 1 to  $n$ , do:
6      $Y[r] = 0$ 
7     for  $c$  from 1 to  $n$ , do:
8        $Y[r] = Y[r] + A[r, c] * B[c]$ 

```

This can be parallelised by computing all of the multiplications in one parallel step, using the parallel addition algorithm ([see more: Example: Binary Fan-In Array Sum on EREW-PRAM, page 11](#)) to add up each row ($\log_2(n)$ parallel steps), then copying into the result (one parallel step).

```

1 fun matrixVectorMult-PRAM-CREW:
2   Input:  $n \times n$  matrix  $A$ ,  $n \times 1$  vector  $B$ 
3   Output:  $n \times 1$  vector  $Y = A * B$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6   for  $r, c$  from 1 to  $n$ , in parallel do:
7      $C[r, c] = A[r, c] * B[c]$ 
8
9   for  $i$  from 1 to  $n$ , in parallel do:
10    for  $h$  from 1 to  $k$ , do:
11      for  $j$  from 1 to  $(n / 2^h)$ , in parallel do:
12         $C[i, 2j] = C[i, 2j] + C[i, 2j - 1]$ 
13
14   for  $i$  from 1 to  $n$ , in parallel do:
15      $Y[i] = C[i, 1]$ 

```

With enough processors, this reduces the runtime from $\Theta(n^2)$ to $\Theta(\log_2(n))$.

Matrix * Matrix Multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj + bm + cp & ak + bn + cq & al + bo + cr \\ dj + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}$$

```

1 fun matrixMatrixMult-PRAM-CREW:
2   Input: n * n matrix A, n * n matrix B
3   Output: n * n matrix Y = A * B
4   Note: n = 2^k; k = log_2(n)
5
6   for i, j, p from 1 to n, in parallel do:
7     C[i, j, p] = A[i, p] * B[p, j]
8
9   for i, j from 1 to n, in parallel do:
10    for h from 1 to k, do:
11      for p from 1 to (n / 2^h), in parallel do:
12        C[i, j, p] = C[i, j, 2p] + C[i, j, 2p - 1]
13
14   for i, j from 1 to n, in parallel do:
15     Y[i, j] = C[i, j, 1]

```

This still has a runtime of $\Theta(\log_2(n))$ with n^3 processors or more.

Binary Tree Algorithms

Recap: Binary Tree Basics

- The **height** h of a binary tree is the maximum distance from the root to any leaf.
- If the tree is **complete** then h will be the same for all leaves.
- A complete binary tree of height h has $2^{h+1} - 1$ nodes, of which 2^h are leaves and $2^h - 1$ are internal nodes.
- If there are T leaves then the total tree size is $2T - 1$.
- Tree nodes are labelled as follows:
 - The root node has index 1.
 - For any node at index i , its left child has index $2i$ and its right child has index $2i + 1$.

Array Membership Algorithm EREW (Intro)

To achieve a complete PRAM-EREW array membership checking method, we will use the following components:

- Broadcast (array filling)
- Parallel Array Membership
- Binary Fan-in Array Minimum

The resulting algorithm will take an input array and a single element, and return the lowest index at which that element occurs in the array (or ∞ if it doesn't). The trivial solution runs in $O(n)$; this solution will run in $\Theta(\log_2(n))$.

Broadcast (Array Filling) EREW

Because we are building an **EREW** algorithm, the single element we are searching for must be replicated n times, so that all parallel processes can read it at the same time. Typically this would take $\Theta(n)$, defeating the purpose of the parallelisation, but it can be done in $\Theta(\log_2(n))$.

```

1 fun arrBroadcast-PRAM-EREW:
2   Input: an  $n$ -item array  $A$ , a number  $X$  to be broadcast to the array  $A$ 
3   Output:  $A[i] = X$  for  $i = 1..n$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6    $A[1] = X$ 
7   for  $i$  in 0 to  $k-1$ , do:
8     for  $j$  in  $2^i + 1$  to  $2^{i+1}$ , in parallel do:
9        $A[j] = A[j - 2^i]$ 

```

In this method the first element of the array is set manually, then one parallel thread copies the value to the next element, then two parallel threads copy those values to the next two elements, then four parallel threads copy those values to the next four elements, etc.

Parallel Array Membership EREW

This method will use n parallel threads to compare two n -length arrays: the first is the input array and the second is the array of the element we are searching for (created with the broadcast method above). For each process, if the elements in corresponding cells match, the ID of that process is written back into the second array; if they do not match, ∞ is written.


```

1 fun arrMembership-PRAM-EREW:
2   Input: an  $n$ -item array  $A$ ,
3         an  $n$ -item array  $B$  containing the search number
4   Output:  $B[i]$  = process ID if  $A[i]$  = the search number,
5            $\text{inf.}$  otherwise
6
7   for  $i$  in 1 to  $n$ , in parallel do:
8     if  $A[i] == B[i]$ 
9        $B[i] = i$ 
10    else
11       $B[i] = \text{inf}$ 

```

This algorithm runs in one parallel step.

Binary Fan-in Array Minimum EREW

A binary fan-in can apply any binary associative operator to reduce an array to one element. [See more: Example: Binary Fan-In Array Sum on EREW-PRAM, page 11.](#) Here, we use it to find the array minimum:

```

1 fun arrMin-PRAM-EREW:
2   Input:  $n$  numbers stored in the array  $A[1..n]$ 
3   Output: min of  $A[1..n]$ , written into  $A[1]$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6   for  $i$  from 1 to  $n$ , in parallel do:
7      $B[i] = A[i]$ 
8
9   for  $h$  from 1 to  $k$ , do:
10    for  $i$  from 1 to  $(n / 2^h)$ , in parallel do:
11       $B[i] = \min(B[2i] + B[2i - 1])$ 
12
13    $A[1] = B[1]$ 

```

Array Membership Algorithm EREW (Algorithm)

1. **arrBroadcast-PRAM-EREQ**
to create the temporary n -array of the searched-for number
2. **arrMembership-PRAM-EREQ**
to isolate the positions that contain the searched-for number
3. **arrMin-PRAM-EREQ**
to return the lowest position from the previous step

Binary Fan-in Array Maximum EREW

This algorithm is very similar to the binary fan-in array minimum seen earlier, but it uses a double-sized array to avoid overwriting values, and **indexes 'backwards'** for no particular reason (other than to show that it can be done).

```

1 fun arrMax-PRAM-EREW:
2   Input: n numbers stored in the array A[n+1..2n]
3   Output: max of A[n+1..2n], written into A[1]
4   Note: n = 2k; k = log2(n)
5
6   for i from k - 1 to 0, do:
7     for j from 2i to 2(i+1) - 2, in parallel do:
8       A[j] = max(A[2j], A[2j + 1])

```

In this $\Theta(\log_2(n))$ algorithm, each cell is written to only once, and no data is overwritten.

Parallel Prefix Sums EREW

Input:	1	2	3	4	5
Output:	1	3	6	10	15

For an input array of size n (where $n = 2^k$), the trivial sequential approach to this has a runtime of $\Theta(n)$. It can be parallelised to achieve $\Theta(\log_2(n))$ as follows:

1. Construct a complete binary tree with n leaves (i.e. $2n - 1$ nodes in total).
2. Place the array values on the leaves.
3. Work up through the tree, putting the sum of each pair of child nodes into the parent node, ending with the full array sum in the root.
 - This can be done in $\log_2(n)$ parallel steps.
4. Work down through the tree, and for each non-leaf node:
 - Set the left child to equal the parent node's value minus the right child's value.
 - Set the right child to equal the parent node's value.
 - This can be done in $\log_2(n)$ parallel steps.
5. The output can now be read from the leaves.

```

1 fun prefixSum-PRAM-EREW:
2   Input:  $n$  numbers, stored in entries  $n..2n-1$  of array  $A[1..2n]$ 
3   Output: prefix sums, stored in entries  $n..2n-1$  of array  $A[1..2n]$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6   for  $m$  from  $k - 1$  to  $0$ , do:
7     for  $i$  from  $2^m$  to  $2^{(m+1)}-1$ , in parallel do:
8        $A[i] = A[2i] + A[2i + 1]$ 
9
10   $B[1] = A[1]$ 
11  for  $m$  from  $1$  to  $k$ , do:
12    for  $i$  from  $2^m$  to  $2^{(m+1)}-1$ , in parallel do:
13      if  $i$  is odd
14         $B[i] = B[(i - 1) / 2]$ 
15      else
16         $B[i] = B[(i / 2)] - A[i + 1]$ 

```

Array Maximum CRCW

This algorithm pairwise compares all elements and writes 1 into a secondary array M for all 'losing' (smaller) elements. After duplicate 0-values from M , the last remaining 0-value indicates the maximum element.

```

1 fun arrMax-PRAM-CRCW:
2   Input:  $n$  numbers stored in the array  $A[1..n]$ 
3   Output: max of  $A[1..n]$ , written into  $A[1]$ 
4   Note:  $n = 2^k$ ;  $k = \log_2(n)$ 
5
6   // initialise  $n$ -size temp array with broadcast
7    $M = [0, 0, \dots]$ 
8
9   // pairwise compare
10  for all pairs  $(i, j)$ ,  $1 \leq i, j \leq n$ , in parallel do:
11    if  $A[i] > A[j]$ 
12       $M[j] = 1$  //  $j$  has "lost"
13
14  // smallest max index
15  for all pairs  $(i, j)$ ,  $1 \leq i, j \leq n$ ,  $i < j$ , in parallel do:
16    if  $M[i] == 0$  and  $i < j$ 
17       $M[j] = 1$  // higher index has "lost"
18
19  // store max
20  for  $i$  from  $1$  to  $n$ , in parallel do:
21    if  $M[i] == 0$ 
22       $Max = A[i]$ 
23       $IndexOfMax = i$ 

```

This algorithm runs in $\Theta(1)$, assuming infinite processors (it has $O(n^2)$ parallel steps).

The **CW model doesn't matter**, because only the value 1 is ever written into **M**.

Array Membership CRCW

```

1  fun arrMembership-PRAM-CRCW:
2      Input: n numbers stored in the array A[1..n],
3             a number X to check
4      Output: true if X is in A
5      Note: n =  $2^k$ ; k =  $\log_2(n)$ 
6
7      Index = -1
8      for i from 1 to n, in parallel do:
9          if A[i] == X
10             Index = i
11
12      XIsInArray = (Index != -1)

```

This $\Theta(1)$ algorithm (with $\Theta(n)$ parallel steps) works for priority and arbitrary CW modes, but not common.

PRAM-CRCW Sorting

```

1 fun arrSort-PRAM-CRCW:
2   Input: n unsorted numbers in the array A[1..n]
3   Output: n sorted numbers in the array B[1..n]
4   Note: n =  $2^k$ ; k =  $\log_2(n)$ 
5         uses the work array W[1..n]
6
7   for i from 1 to n, in parallel do
8     W[i] = 1
9
10  // check all pairs and +1 to the position of the "winner"
11  for all pairs (i, j),  $1 \leq i, j \leq n, i < j$ , in parallel do:
12    if A[i] > A[j]
13      W[i] = W[i] + 1
14    else
15      W[j] = W[j] + 1
16
17  // for i from 1 to n, W[i] is the number of items that
18  // are smaller than or equal to A[i]
19
20  // copy items into position
21  for i from 1 to n, in parallel do:
22    B[W[i]] = A[i]

```

Is PRAM-CRCW More Powerful than PRAM-EREW?

Yes. **CRCW-priority** is the most powerful model; **EREW** is the least powerful model.

When computing boolean functions of size n in the form $(a_1|a_2|\dots|a_n)$ CRCW gives performance of $O(1)$, whereas EREW gives $\Theta(\log_2(n))$ using binary fan-in.

- [See more: Example: Binary Fan-In Array Sum on EREW-PRAM, page 11](#)

Simulation

CRCW algorithms can be simulated on EREW with a $\log_2(n)$ overhead.

- Concurrent (CRCW $O(1)$) read/write operations can be implemented on an n -processor EREW system to run in $O(\log_2(n))$ time.
- CR: use broadcasting to implement on ER.
 - [See more: Broadcast \(Array Filling\) EREW, page 16](#)
- CW: the processor IDs that want to write go into an array and binary fan-in is used to find the minimum (simulating CW-priority), which is then allowed to write.

Modelling Other Data Structures

All data in these parallel algorithms is **represented as an array**, which is an abstraction of a block of memory, but **other data structures can be represented** within the array with the right indexing.

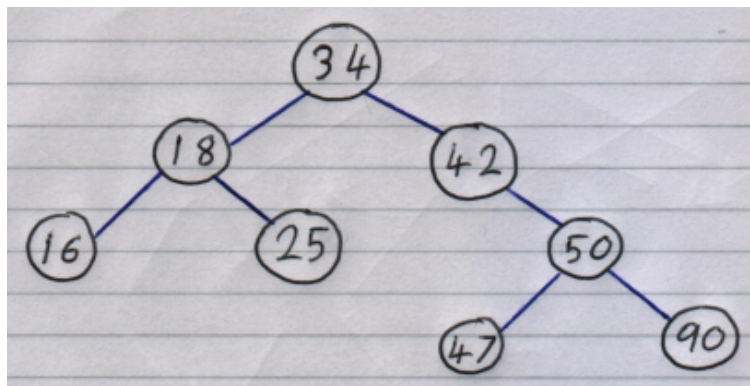
Parent Labelling

Parent labelling can be used to **represent lists or trees**. Every item in the data array is a node or item within the list/tree, each of which has a corresponding value in the same position in the **parent array** P , indicating the index of that node/item's parent.

- If i is someone's child, then $P[i]$ is the index of i 's parent.
- If i is a tree root or list head, then $P[i] = i$.

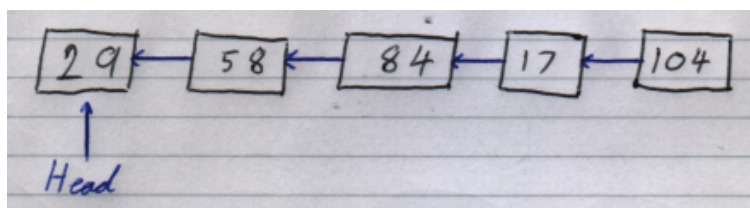
Example: Tree

Index	1	2	3	4	5	6	7	8
Data	34	18	42	50	47	16	90	25
Parent	1	1	1	3	4	2	4	2



Example: List

Index	1	2	3	4	5
Data	84	104	58	29	17
Parent	3	6	4	4	1



List Ranking

The rank of an item in a list is its **distance from the head/root of the list**. It can be determined efficiently by a parallel algorithm using **pointer jumping**.

Note: pointer jumping will **corrupt the parent array**. Make a duplicate first if needed.

```

1 fun listRank-PRAM-CREW:
2   Input: parent array P[1..n]
3   Output: distance array D[1..n] giving the distance of each
4           node i from the head/root of the list
5   Note: n = 2^k; k = log_2(n)
6
7   // initialise distance
8   for i from 1 to n, in parallel do
9     if P[i] == i
10      D[i] = 0 // for the head/root
11    else
12      D[i] = 1 // for everything else
13
14   for i from 1 to log_2(n), do:
15     for j from 1 to n, in parallel do:
16       if P[j] != P[P[j]]
17         D[j] = D[j] + D[P[j]]
18         P[j] = P[P[j]]

```

This algorithm determines the distance/rank value for n items in $\Theta(\log_2(n))$.

It works by initialising everything at 1 away from the root (except for the root itself) and then repeatedly ‘jumping’ everything to its parent’s pointer until the root is reached. At each jump, it counts the distance it already was from the root, plus the distance of the parent it just ‘jumped’ past.

Forest Root Finding

```

1 fun forestRoots-PRAM-CREW:
2   Input: parent array P[1..n]
3   Output: root array R[1..n] giving the root of each node i in the forest
4   Note: n = 2^k; k = log_2(n)
5
6   for i from 1 to n, in parallel do:
7     for j from i to log_2(n), do:
8       P[i] = P[P[i]]
9     R[i] = P[i]

```

Evaluating Parallel Algorithms

There are many things to consider when understanding parallel algorithms:

- Pseudocode with some parallelisation syntax.
- Algorithm design ('why is something done a certain way?') - often linked to *where* the data needs to go.
- Flow charts ('what is being done?').
- The number of sequential steps (i.e. the **span**).
- The **best sequential** algorithm, for comparison (this might not be known).
- Analysis of work (multi-threading)/time (PRAM).
- Efficiency metrics (like **speed-up** and **parallelism**).

Analysis of Algorithms

We apply '**Big-O thinking**' and a **uniform cost model** to evaluate algorithms, wherein we charge 1 unit for each **basic operation**. What we charge for depends on the computation model:

- Multi-Threading: we charge for recursive calls, including base cases, spawn, sync, etc.
- PRAM: we charge for every basic operation.
- Graph Interconnected Network: we charge for communication/messaging.

Work/Time and Work/Span

In the multi-threading and PRAM models, work W is the number of basic operations required in the sequential execution.

- **Span** $S = T_\infty$ refers to the multi-threading model.
- **Time** $T = T_\infty$ refers to the PRAM model, assuming we have as many processors as we need (i.e. $p = \infty$).

W , S and T are all functions of n , which we can assume is a power of two (i.e. $n = 2^k$).

In both models, **parallelism** is a measurement of the maximum number of processors p that can be efficiently used.

Other Efficiency Metrics

Cost and **efficiency** can be used to compare algorithms, based on the number of processors available.

- **Cost** $C_p(n) = p \times T_p(n)$ (this may be much higher than $W(n)$).
- **Efficiency** $E_p(n) = \frac{W(n)}{C_p(n)}$.
- $T_p(n)$ is normally obtained experimentally.

Speed-Up

What should speed-up measure, exactly? This is tricky.

For **parallelism** we compared against T_1 to see how well the algorithm scaled beyond a single processor. For **speed-up** though, comparing to T_1 may be misleading because there **may be a sequential solution that is faster** than forcing a parallel solution to run sequentially.

In PRAM we can use T^* to denote the **best known sequential algorithm runtime**; this may be very different from the T_1 runtime. In this case, speed-up is computed as $S_p(n) = \frac{T^*(n)}{T_p(n)}$.

For a lot of problems, T^* is not known, so we use T_1 instead.

Graph Interconnected Model (Message Passing)

A typical message passing architecture includes many isolated units, each with their own **processor and memory**, which are connected together and allowed to communicate. We consider communication to be possible **only between adjacent nodes** (i.e. directly connected). This is because the network topology can have a significant impact as the size grows, both in terms of execution speed and cost to create.

Formally, a network can be viewed as a **graph**, where each node represents a processor with its own memory. We consider **synchronous, static (point-to-point) communication**.

Common Structures

- **Complete graph** - all nodes are connected to all other nodes.
 - Full connectivity, but very expensive to create.
- **Meshes**
 - **1D mesh** is a path or cycle.
 - **2D mesh** is a grid.
 - **3D mesh** is a cube.
 - The ends of the grid are sometimes wrapped around to the other side, creating a **torus** or **wrapped grid**.
- **Hypercubes**
 - Each node in a d -dimension hypercube stores its ID as a binary vector, (b_1, b_2, \dots, b_d)
 - A node's neighbours are the nodes for which the ID differs by one bit
 - Example: the node $(0, 1, 0)$ has neighbours at $(\mathbf{1}, 1, 0)$, $(0, \mathbf{0}, 0)$ and $(0, 1, \mathbf{1})$.
 - 0D, 1D, 2D and 3D hypercubes are points, lines, squares and cubes; 4D and upwards get more interesting.

Symmetric layouts are preferred, as they make nodes easier to connect.

This course will consider 1D meshes as **paths**, 1D meshes as **cycles**, and 2D meshes (**grids**).

Message Passing Costs

Precise modelling of costs is impossible in practical terms, so standard approximations respect two factors for messages passed between **directly-connected processors**:

- Every message incurs some fixed start-up costs.
- The larger the message, the longer it takes.

For messages that are passed along several 'hops' there are two standard models:

- **Store-and-forward (SF)** routing (**This is the model we consider in this course**).
- **Cut-through (CT)** routing.

Store-and-Forward (SF) Routing

We assume that the message is **passed in its entirety between each pair** of processors on the route before continuing.

The connectivity of the interconnection network influences algorithm design and performance. We used **graph-based measures of network quality**, which are equally valid for parallel and distributed algorithms.

Graph-Based Network Quality Measures

- **Diameter** - maximum shortest distance between any pair of nodes.
 - Very important for runtime.
- **Node or Arc Connectivity** - minimum number of links that must be cut to split the network into two disjoint parts.
- **Bisection Width** - minimum number of links that must be cut to split the network into two *equal* parts.

For networks with p nodes:

Network	Diameter	# Links
Complete	1	$p(p-1)/2$
Star	2	$p-1$
Binary tree	$2\log_2((p+1)/2)$	$p-1$
Linear	$p-1$	$p-1$
Ring	$\lfloor p/2 \rfloor$	p
2D mesh	$2(\sqrt{p}-1)$	$2(p-\sqrt{p})$
Mesh (wrapped)	$\lfloor \sqrt{p}/2 \rfloor$	$2p$
Hypercube	$\log_2(p)$	$(p\log_2(p))/2$

Graph Interconnected Algorithms

Notation

- $P(i)$ is the processor at position i in a 1D mesh.
- $P(r, c)$ is the processor at row r and column c in a 2D mesh.
- $P(\dots).Name$ is the value labelled as $Name$ in $P(\dots)$'s memory.

Indexing

The mesh $M(h, w)$ has $h * w$ processors. Processor indexing is **0-based**.

To use the array $A[0..n-1]$, the index k corresponds to processor $P(r, c)$ where $k = r * w + c$.

Array Minimum, 2D Mesh (Pesudocode)

```

1 | Model: Mesh M(q, q) with n = q*q processors
2 | Input: array X[0..n-1]
3 | Output: min(X)
4 |
5 | 1. Compute column minimums in parallel.
6 |     Pairwise compare rows starting at the bottom.
7 |     Write the minimum into the higher node.
8 |     At the end, the minimum of each column will be in the top row.
9 |
10 | 2. Compute the minimum of the top row.
11 |     Pairwise compare nodes, starting at the end.
12 |
13 | 3. Min(X) is in the node (0, 0) (top-left).
```

Array Minimum, 2D Mesh

```

1 | Model: Mesh  $\mathbf{M}(\mathbf{q}, \mathbf{q})$  with  $\mathbf{n} = \mathbf{q} * \mathbf{q}$  processors
2 | Input: array  $\mathbf{X}[0..\mathbf{n}-1]$ 
3 | Output:  $\min(\mathbf{X})$ 
4 |
5 | for  $\mathbf{c}$  from 0 to  $\mathbf{q}-1$ , in parallel do:
6 |     for  $\mathbf{r}$  from  $\mathbf{q}-2$  to 0, do:
7 |          $\mathbf{P}(\mathbf{r}, \mathbf{c}).\mathbf{Temp} = \mathbf{P}(\mathbf{r} + 1, \mathbf{c}).\mathbf{X}$ 
8 |          $\mathbf{X} = \min(\mathbf{X}, \mathbf{Temp})$ 
9 |
10 | for  $\mathbf{c}$  from  $\mathbf{q}-2$  to 0, do:
11 |      $\mathbf{P}(0, \mathbf{c}).\mathbf{Temp} = \mathbf{P}(0, \mathbf{c} + 1).\mathbf{X}$ 
12 |      $\mathbf{X} = \min(\mathbf{X}, \mathbf{Temp})$ 
13 |
14 | return  $\mathbf{P}(0, 0).\mathbf{X}$ 

```

Running time is $\Theta(n + m)$ for an $n * m$ grid.

Broadcast, 2D Mesh

```

1 | Model: Mesh  $\mathbf{M}(\mathbf{q}, \mathbf{q})$  with  $\mathbf{n} = \mathbf{q} * \mathbf{q}$  processors
2 | Input:  $\mathbf{P}(0, 0).\mathbf{X}$  with data to broadcast
3 | Output: All processors have the data  $\mathbf{X}$ 
4 |
5 | // propagate across the top row
6 | for  $\mathbf{c}$  from 0 to  $\mathbf{q}-2$ , do:
7 |      $\mathbf{P}(0, \mathbf{c} + 1).\mathbf{X} = \mathbf{P}(0, \mathbf{c}).\mathbf{X}$ 
8 |
9 | // propagate down the columns in parallel
10 | for  $\mathbf{c}$  from 0 to  $\mathbf{q}-1$ , in parallel do:
11 |     for  $\mathbf{r}$  from 0 to  $\mathbf{q}-2$ , do:
12 |          $\mathbf{P}(\mathbf{r} + 1, \mathbf{c}).\mathbf{X} = \mathbf{P}(\mathbf{r}, \mathbf{c}).\mathbf{X}$ 

```

Running time is $\Theta(n + m)$ for an $n * m$ grid.

Membership Query, 2D Mesh

```

1 | Model: Mesh  $M(q, q)$  with  $n = q \times q$  processors
2 | Input: Array  $A[0..n-1]$  held in the mesh, value  $Y$  to search for
3 | Output: Minimum index of processor containing  $Y$ 
4 |
5 | // share the value  $Y$ 
6 | 2d-mesh-broadcast( $M, Y$ )
7 |
8 | // check membership and store matching indexes
9 | for each  $r, c$  where  $0 \leq r, c < q$ , in parallel do:
10 |     if  $P(r, c).L == P(r, c).Y$ 
11 |          $P(r, c).S = r * q + c$ 
12 |     else
13 |          $P(r, c).S = \text{infinity}$ 
14 |
15 | // determine minimum index
16 | 2d-mesh-minimum( $M$ ) on each processor's  $S$ -value

```

Running time is $\Theta(n + m)$ for an $n * m$ grid.

Array Prefix, 2D Mesh

```

1 | Model: Mesh  $M(q, q)$  with  $n = q \times q$  processors
2 | Input: Array  $X[0..n-1]$  held in the mesh
3 | Output:
4 |
5 | // initialise rows in parallel
6 | for  $r$  from 0 to  $q-1$ , in parallel do:
7 |      $P(r, 0).S = P(r, 0).X$ 
8 |     for  $c$  from 1 to  $q-1$ , do:
9 |          $P(r, c).S = P(r, c-1).S + P(r, c).X$ 
10 |
11 | // add up last column
12 | for  $r$  from 1 to  $q-1$ , do:
13 |      $P(r, q-1).S = P(r, q-1).S + P(r-1, q-1).S$ 
14 |
15 | // back-factor last column across next row in parallel
16 | for  $r$  from 1 to  $q-1$ , in parallel do:
17 |     for  $c$  from 0 to  $q-2$ , do:
18 |          $P(r, c).S = P(r, c).S + P(r-1, q-1).S$ 

```

Running time is $\Theta(n + m)$ for an $n * m$ grid.

Matrix Multiplication, 1D Path, Pipelining

The columns of a matrix A are fed synchronously into the processor of the path in a skewed fashion, as shown below. At each step, the value held by processor $P(i)$ is the value held by $P(i - 1)$ in the previous step, plus the i^{th} item from the current input matrix row multiplied by the i^{th} item in the vector. Example:

Matrix:	1	2	3	4	Vector:	3
	0	2	2	0		3
	5	3	6	4		4
	1	1	1	0		2

This is computed as follows:

				4	Step 7
			3	0	Step 6
		2	2	4	Step 5
Matrix:	1	2	6	0	Step 4
	0	3	1		Step 3
	5	1			Step 2
	1				Step 1
<hr/>					
Vector:	3	3	4	2	
<hr/>					
Step 1:	3	0	0	0	
Step 2:	15	3 + 3	0	0	
=	15	6	0	0	
Step 3:	0	15 + 9	6 + 4	0	
=	0	24	10	0	
Step 4:	3	0 + 6	24 + 24	10 + 0	
=	3	6	48	10	
Step 5:	0	3 + 6	6 + 8	48 + 8	
=	0	9	14	56	
Step 6:	0	0	9 + 12	14 + 0	
=	0	24	21	14	
Step 7:	0	0	0	21 + 8	
=	0	0	0	29	

Even/Odd Transposition Sort, 1D Mesh

```

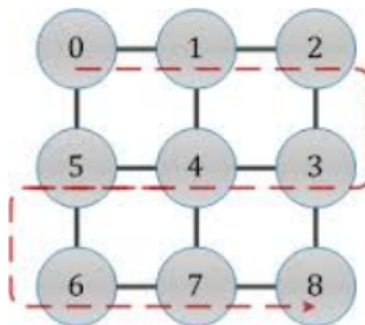
1 | Model: Mesh  $M(n)$  with  $n$  processors
2 | Input: Array  $X[0..n-1]$  held in the mesh
3 | Output: Sorted array  $X[0..n-1]$  held in the mesh
4 |
5 | for  $s$  from 0 to  $n - 1$ , do:
6 |     if  $s$  is even
7 |         for all even processor labels  $i$ , in parallel do:
8 |             compareExchange( $P(i).X$ ,  $P(i + 1).X$ )
9 |     else
10 |        for all odd processor labels  $i$ , in parallel do:
11 |            compareExchange( $P(i).X$ ,  $P(i + 1).X$ )

```

Coloured pairs show the **result** of compareExchange operations:

Processor	P_0	P_1	P_2	P_3	P_4	P_5	P_6
Input	4	2	8	5	1	3	0
Step 0 (E)	2	4	5	8	1	3	0
1 (O)	2	4	5	1	8	0	3
2 (E)	2	4	1	5	0	8	3
3 (O)	2	1	4	0	5	3	8
4 (E)	1	2	0	4	3	5	8
5 (O)	1	0	2	3	4	5	8
6 (E)	0	1	2	3	4	5	8
Output	0	1	2	3	4	5	8

Snake Order Sort, 2D Mesh




```
1 | Model: Mesh M(q, q) with n = q * q processors
2 | Input: Array X[0..n-1] held in the mesh
3 | Output: Snake-order sorted array X[0..n-1] held in the mesh
4 |
5 | for s from 0 to ceil(log2(n)), do:
6 |     if s is even
7 |         in parallel, sort all rows with 1d-mesh-sort
8 |     else
9 |         in parallel, sort all columns with 1d-mesh-sort
```