Quantum Field Theory Notes

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NOTE DESCRIPTION

This note is author's self-study notes of quantum field theory and what I had chose is the *Quantum Field Theory and the Standard Model* written by Matthew D. Schwartz. In this note, I shall use the same names of the chapters to fit with the book itself and almostly there will exist more things about my thoughts which may have some mistakes.

pS:*And the version is the cloud version which run on overleaf because of the hard work of setting automatic English spelling in my local texstudio. So this can be regarded as my first attmpt on the cloud.*

To those who taught me physics.
To human rationality.

Contents

Part I

Field Theory

Chapter 1

Microscopic Theory of Radiation

In this chapter, the Schwartz had shown to us that how the quantum theory was brought up especially the Plank's work in blackbody radiation also the Einstein's coefficients of spontaneous emission, and we shall use the first principle of quantum field theory to illustrate plenty of conclusions and phenomena. Also, we have to say, this is a period of history.

Let's go back to the classical time. We can derive the formula of frequencies from electrodynamics: the ω_n in a box of size L supported standing electromagnetic waves

$$\omega_n = \frac{2\pi}{L}|\vec{n}|c \tag{1.1}$$

By the classical equipartition theorem, we can come up the concept of blackbody, an object at fixed temperature whose internal structure we do not care about. And, how to say? Blackbodies should emit light equally in all modes with the intensity in volume of phase space:

$$I(\omega) = \frac{1}{V}\frac{d}{d\omega}E(\omega) = \text{constant} \times c^{-3}\omega^2 k_B T \tag{1.2}$$

But that will upset us, beacasue the I is the quadratic function of ω. That says, when the ω grows up, the I will diverge. That's the Ultra-violet catastrophe.

Experimentally, the distrubution looks more like a Maxwell-Boltzmann distribution which imply that there is no diverge from the frequency interval $[0, \infty]$. In order to deal with the ultraviolet disaster, Planck successfully matched the experimental data with phenomenological methods such as interpolation, but it required the assumption that energy was quantized and introduced a new constant to distinguish the classical bewteen quantum mechanics:

$$E_n = \hbar\omega_n = \frac{2\pi}{L}\hbar|\vec{n}| = |\vec{p}_n| \tag{1.3}$$

Later, the Albert Einstein interpreted this as implying that light is made up of particles, the photons. Anyway, quantizing light resolves the blackbody paradox and light having energy leads to the photoelectric effect which was proven by Compton scattering experiment.

Anyway, with Planck's energy hypothesis, we can compute the specturm functions from first principle: each mode of frequency ω_n can be excited an integer number j times giving energy

$jE_n = j\hbar\omega$ in that mode. The partition function from Boltzmann distribution is $\exp(-\text{energy}\cdot\beta)$, thus the mean value of energy can be computed by followings:

$$\begin{aligned}\langle E_n\rangle &= \frac{\sum_{j=0}^{+\infty} jE_n e^{-jE_n\beta}}{\sum_{j=0}^{+\infty} e^{-jE_n\beta}} \qquad \text{Convergent infinite series}\\ &= \frac{-\frac{d}{d\beta}\frac{1}{1-e^{-\hbar\omega_n\beta}}}{\frac{1}{1-e^{-\hbar\omega_n\beta}}}\\ &= \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta}-1}\end{aligned} \tag{1.4}$$

When we take the continue limit, namely $L\to\infty$, we turn the $\sum$ to $\int$:

$$\begin{aligned}E(\omega) &= \int^{\omega} d^3\vec{n}\frac{\hbar\omega_n}{e^{\hbar\omega_n\beta}-1} \quad \text{for all space}\\ &= 4\pi\hbar\frac{L^3}{8\pi^3}\int^{\omega} d\omega'\frac{(\omega')^3}{e^{\hbar\omega'\beta-1}}\end{aligned} \tag{1.5}$$

so our first question have been figured out. Thus, we have

$$I(\omega) = \frac{1}{V}\frac{dE(\omega)}{d\omega} = \frac{\hbar}{\pi^2}\frac{\omega^3}{e^{\hbar\omega\beta-1}} \tag{1.6}$$

So the UV catastrophe disappeared when we introduced quantization of energy!

> *What does this have to do with quantum field theory? In order for this derivation, which used equilibrium statistical mechanics, to make sence, light has to be able to equilibrate. For example, if we heat up a box with monochromatic light, eventually all frequencies must be excited. However, if different frequencies are different particles, equilibration must involve one kind of particle turning into another kind of particle. So, particles must be created and destoryed. Quantum field theory tells us how that happens!*

That's exciting rational analysis!

However, there's one thing worth thinking. When we talk about the UV, we will find that it is a statistical physics problem. Yeah, we used to tackle the spontaneous emission through the statistical method. But the most exciting part is that we tackle the spontaneous emission through the single atom from first principle developed from quantum field theory not the statistical mechanics. In other words, Einstein's spontaneous emission coefficient can be calculated from the first principles of quantum field theory without the use of statistical mechanical conditions that assume thermal equilibrium. So this is a new evidence and defense to the quantum field theory. Let's see his face! (We're just going to talk about how do we get a solution to this problem from quantum field theory)

Before the start, we plain the results from Einstein's A,B coefficients:

$$\begin{aligned}B' &= B\\ \frac{A}{B} &= \frac{\hbar}{\pi^2}\omega^3\end{aligned} \tag{1.7}$$

The $B = B'$, says that the coefficient of absorption must be the same as the coefficient for stimulated emission. The coefficient B and B' can be computed in quantum mechanics using time-dependent perturbation theory with an external electromagnetic field. The the ratio of A and B determines A. However, does spontaneous emission from an atom have anything to do with equilibrium of a gas since an atom radiates at the same rate no matter what is around it. That need the quantum field theory to figure out until 10 years after Einstein's calculation.

We know that, there exits some basic and profound motion mode, the $1/r$ and r^2. The first implies the motion of celestial bodies and the second implies an oscillator which often represents the cyclic motion. From the quantum mechanics, we can say that the easiest way to develop a quantum harmonic oscillator is with creation and annihilation operators, $a^\dagger$ and a, satisfying:

$$\left[a, a^\dagger\right] = 1 \tag{1.8}$$

and the number operator $\hat{N} = a^\dagger a$, which counts modes

$$\hat{N} \left|n\right\rangle = n \left|n\right\rangle \tag{1.9}$$

Notice that $a^\dagger \left|n\right\rangle$ is the eigenstate of the operator $\hat{N}$, proof are given:

$$\begin{aligned}
\hat{N} a^\dagger \left|n\right\rangle &= a^\dagger a a^\dagger \quad \text{use the commutator} \left[a, a^\dagger\right] = aa^\dagger - a^\dagger a = 1 \\
&= a^\dagger \left(1 + a^\dagger a\right) \left|n\right\rangle \\
&= a^\dagger \left|n\right\rangle + a^\dagger a^\dagger a \left|n\right\rangle \\
&= a^\dagger \left|n\right\rangle + a^\dagger a^\dagger \sqrt{n} \left|n-1\right\rangle \\
&= a^\dagger \left|n\right\rangle + \sqrt{n}\sqrt{n} a^\dagger \left|n\right\rangle \\
&= (n+1) a^\dagger \left|n\right\rangle
\end{aligned} \tag{1.10}$$

and the eigenvalue is $n + 1$. Also, we normalize the egienstates as $\langle n|n\rangle = 1$

Now, let's deal with the spontaneous emission. I think it is necessary to re-explain the concept of spontaneous emission quoted from Wikipedia.

> Spontaneous emission is the process in which a quantum mechanical system (such as a molecule, an atom or a subatomic particle) transits from an excited energy state to a lower energy state (e.g., its ground state) and emits a quantized amount of energy in the form of a photon. Spontaneous emission is ultimately responsible for most of the light we see all around us; it is so ubiquitous that there are many names given to what is essentially the same process. If atoms (or molecules) are excited by some means other than heating, the spontaneous emission is called luminescence. For example, fireflies are luminescent. And there are different forms of luminescence depending on how excited atoms are produced (electroluminescence, chemiluminescence etc.). If the excitation is affected by the absorption of radiation the spontaneous emission is called fluorescence. Sometimes molecules have a metastable level and continue to fluoresce long after the exciting radiation is turned

off; this is called phosphorescence. Figurines that glow in the dark are phosphorescent. Lasers start via spontaneous emission, then during continuous operation work by stimulated emission.

As we developed in quantum mechanics, the Fermi's Golden Rules

$$\Gamma \sim |\mathcal{M}|^2 \delta(E_f - E_i) \tag{1.11}$$

implies the transition rate between two state where δ function enforces the conservation of energy (We will derive the quantum field theory version of cross-section and decay rate in chapter 5). The matrix element (tip: perturbation theory) is

$$\mathcal{M} = \langle f | H_{int} | i \rangle \tag{1.12}$$

namely the schematic

$$\mathcal{M} = \left\{ \text{final state} \xleftarrow[\text{Interaction}]{\text{Hamiltonian}} \text{initial state} \right\} \tag{1.13}$$

No matter what the H_{int} is exactly, it should have the creation and annihilation operator in line with the case of spontaneous emission and it must be Hermitian. Thus, we can write it formally without losing the physical meaning as following:

$$H_{int} = H_I^\dagger a^\dagger + H_I a \tag{1.14}$$

As for the process of spontaneous emission, we note the initial excited atom state as $|\text{Atom 2}; n_\omega\rangle$ with n_ω photons of frequency $\omega = \Delta/\hbar$:

$$|i\rangle = |\text{Atom 2}; n_\omega\rangle \tag{1.15}$$

and the final state is a lower energy atom we note as Atom 1 with $n_\omega + 1$ photons of energy Δ:

$$|f\rangle = |\text{Atom 1}; n_\omega + 1\rangle \tag{1.16}$$

and

$$\left\{ |\text{Atom 1}; n_\omega + 1\rangle \xleftarrow[\Delta = \hbar\omega]{\text{A Photon}} |\text{Atom 2}; n_\omega\rangle \right\} \tag{1.17}$$

Then we can compute:

$$\begin{aligned}
\langle f | H_{int} | i \rangle &= \langle f | H_I^\dagger a^\dagger + H_I a | i \rangle \\
&= \langle \text{Atom 1}; n_\omega + 1 | H_I^\dagger a^\dagger + H_I a | \text{Atom 2}; n_\omega \rangle \\
&= \langle \text{Atom 1}; n_\omega + 1 | H_I^\dagger a^\dagger | \text{Atom 2}; n_\omega \rangle + \langle \text{Atom 1}; n_\omega + 1 | H_I a | \text{Atom 2}; n_\omega \rangle \\
&= \langle \text{Atom 1} | H_I^\dagger | \text{Atom 2} \rangle \langle n_\omega + 1 | a^\dagger | n_\omega \rangle + \langle \text{Atom 1} | H_I | \text{Atom 2} \rangle \langle n_\omega + 1 | a | n_\omega \rangle \\
&= \langle \text{Atom 1} | H_I^\dagger | \text{Atom 2} \rangle \sqrt{n_\omega + 1}
\end{aligned} \tag{1.18}$$

which we have used the algebraic eigen-equation of operat $a\dagger$, a and we decompose the eigen-states $|\text{Atom } 1; n_\omega + 1\rangle$, $|\text{Atom } 2; n_\omega\rangle$ and compound operators $H_I^\dagger a^\dagger$, $H_I a$ here. Further, we note that $\mathcal{M}_0^\dagger = \langle \text{Atom } 1| H_I^\dagger |\text{Atom } 2\rangle$. Thus, the process can be described as

$$|\mathcal{M}_{2\to1}|^2 = |\mathcal{M}_0|^2(n_\omega + 1) \tag{1.19}$$

Instead, we excite an atom, then the initial state is unexcited atom with n_ω photons. Then, the initial and final state can be written as

$$|i\rangle = |\text{Atom } 1; n_\omega\rangle \tag{1.20}$$

$$|f\rangle = |\text{Atom } 2; n_\omega - 1\rangle \tag{1.21}$$

Same as (1.18), we have:

$$\begin{aligned}\langle f| H_{int} |i\rangle &= \langle \text{Atom } 2| H_I^\dagger |\text{Atom } 1\rangle \langle n_\omega - 1| a^\dagger |n_\omega\rangle + \langle \text{Atom } 2| H_I |\text{Atom } 1\rangle \langle n_\omega - 1| a |n_\omega\rangle \\ &= \langle \text{Atom } 2| H_I |\text{Atom } 1\rangle \sqrt{n_\omega}\end{aligned} \tag{1.22}$$

and

$$\mathcal{M}_{1\to2} = \mathcal{M}_0\sqrt{n_\omega} \tag{1.23}$$

namely

$$|\mathcal{M}_{1\to2}|^2 = |\mathcal{M}_0|^2 n_\omega \tag{1.24}$$

According to Einstein's theory, the density of number satisfies (see on Wikipedia)

$$\begin{aligned}dn_2 = -dn_1 &= -|\mathcal{M}_{2\to1}|^2 n_2 + |\mathcal{M}_{1\to2}|^2 n_1 \\ &= -|\mathcal{M}_0|^2(n_\omega + 1)n_2 + |\mathcal{M}_0|^2 n_\omega n_1\end{aligned} \tag{1.25}$$

while the original form are

$$dn_2 = -dn_1 = -\big[A + BI(\omega)\big]n_2 + B'I(\omega)n_1 \tag{1.26}$$

and we need to relate the number of photon modes of frequency ω to the intensity $I(\omega)$ since the energies are quantized by $\Delta = \hbar\omega = \hbar 2\pi/L|\vec{n}|$. The total energy is

$$E(\omega) = \int^\omega d^3\vec{n}\hbar\omega n_\omega = 4\pi\hbar L^3 \int^\omega \frac{d\omega}{(2\pi)^3}\omega^3 n_\omega \tag{1.27}$$

But we should multiply this by 2 for the two polarizations of light. Including the 2 factor, we can get the intensity

$$I(\omega) = \frac{1}{L^3}\frac{dE(\omega)}{d\omega} = \frac{\hbar\omega^3}{\pi^2}n_\omega \tag{1.28}$$

Thus, we have the exciting results:

$$dn_2 = -dn_1 = -|\mathcal{M}_0|^2\big[1 + \frac{\pi^2}{\hbar\omega^3}I(\omega)\big]n_2 + |\mathcal{M}_0|^2\big[\frac{\pi^2}{\hbar\omega^3}I(\omega)\big]n_1 \tag{1.29}$$

where we can read

$$B' = B, \quad \frac{A}{B} = \frac{\hbar}{\pi^2}\omega^3 \tag{1.30}$$

without ant assumption of thermal equilibrium! That's one of the success of Quantum Field Theory!

In a word, that's a history or beginning of our quantum field theory journey. More details can be found on my own paper notebook which follows the online course of Yu Jia from UCAS on bilibili starting from the Dirac and K-G fields.

Chapter 2

Lorentz Invariance and the Second Quantization (A kind of overview of QFT)

In the last chapter, we already know that treating each mode of electromagnetic radiation in a cavity as a simple harmonic oscillator, we can derive Einstein's relation between the coefficients on induced and spontaneous emission without resorting to statistical mechanics. In this chapter, we gonna to talk about the scond quantization and free field used to describe the canonical quantization of relativistic fields.

In M.D. Schwartz's book, however, he devotes a larger part of the book to a review of special relativity and some tensor operations, a part I will omit in this note. In the course on general relativity (Canbin Liang's textbook), we did a lot of tensor operations and discussion of the differential geometry of spacetime geometry, so we will not go into some metrics, i.e. the so-called covariance and inversion, time-like and spacelike concepts, etc., and will just use them directly instead of elaborating on them again in this note.

Here, it is important to recall that there are two approaches to developing quantum field theory, canonical quantization and Feynman integrals. Here, we will mainly focus on the analogy of harmonic oscillators then slowly build up the concept of canonical quantization. As for the Feynman integral, we will briefly introduce its concepts as well as operations, which mainly refer to the books of A.Zee ,*Quantum field theory in a nutshell*[1](whose said that from our point of view, quantum mechanics is nothing more than 0+1 dimensional quantum field theory).

to be supplemented

Now, the followings are my personal overview of QFT (QFT textbook by M.D. Schwartz) Part I, and I make the Powerpoint of my reporting. And I put the content in this chapter, so you can have a entire view of the field theory so that you will not lost in the formulas.

Quantum field theory (QFT) is the framework that merges quantum mechanics and special relativity, describing particles as excitations of underlying fields. In classical mechanics we treat

[1]Very great book!

systems by a finite number of degrees of freedom, but many phenomena (like electromagnetic waves or sound) are better described by fields with infinitely many degrees of freedom. The transition from classical mechanics to classical field theory is the first step towards QFT. After introducing classical fields and their Lagrangian formulation, one sees that certain phenomena (the ultraviolet catastrophe, spontaneous emission of photons, etc.) cannot be explained without quantizing fields. Combining relativity with quantum principles requires allowing creation and annihilation of particles and demands a theory of quantized fields. This conceptual chapter reviews these motivations and outlines the basic steps: canonical quantization, second quantization (Fock space), perturbation theory and the S-matrix formalism, and the resulting Feynman rules in position and momentum space. Throughout we use natural units $\hbar = c = 1$.

Now we talk about the topic: Transition from classical mechanics to classical field theory. In classical mechanics, a system of particles is described by a finite number of coordinates $q_i(t)$ and is governed by Newton's law or a Lagrangaian $L(q, \dot{q})$. To describe a continuous medium or field, we replace the particle coordinate $q(t)$ by a field $\phi(\vec{x}, t)$ that has a value at each point in space. The Lagrangian becomes a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$, and the action is

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.1}$$

Varying S yields the Euler–Lagrange equations for the field (a partial differential equation). For example, the simplest relativistic free field is a real scalar $\phi(x)$ with

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 \tag{2.2}$$

whose Euler–Lagrange equation is the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0 \tag{2.3}$$

Another example is classical electromagnetism: the electromagenetic four-potential $A^\mu(x)$ has Lagrangaian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, yielding Maxwell's equations. Field theories like these have infinitely many degrees of freedom (one field value at each point in space), and their dynamics describe wave propagation and encode conservation laws via Noether's theorem.

In the Hamiltonian formulation, one defines a conjugate momentum density $\pi(\vec{x}, t) = \partial\mathcal{L}/\partial(\partial_0\phi)$ for each field. For the free scalar example, $\pi = \dot{\phi}$. THe Hamiltonian density is $\mathcal{H} = \pi\partial_0\phi - \mathcal{L}$, and the total Hamiltonian is

$$H = \int d^3x \mathcal{H} \tag{2.4}$$

. Classically, fiedls obey the relation such as

$$\left[\phi(\vec{x}, t), \pi(\vec{t}, t)\right] = \delta^3(\vec{x} - \vec{y}) \tag{2.5}$$

, generalizing the particle bracket

$$[q, p] = 1 \tag{2.6}$$

.This completes the transition from particle mechanics to continuum field theory, laying the groundwork for quantization.

Now, we talk about the two evidence for QFT motivating quantization: UV catastrophe and spontaneous emission. A historical puzzle was the ultraviolet catastrophe in blackbody radiation. Classical electromagnetism, together with the equipartition theorem, predicted that a cavity at temperature T should have an energy density $u(\nu) \sim \nu^2 T$ per frequency interval (the Rayleigh–Jeans law). This expression diverges as $\nu \to \infty$, in stark contradiction with expeerimental spectra, which show a peak and then rapid decay at high frequency. Max Planck resolved the discrepancy by postulating that the energy of electromagnetic modes is quantized in units of $E = h\nu = \hbar\omega$. This leads to the Planck distribution

$$u(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} \tag{2.7}$$

which agrees with observation. Planck's solution implied that electromagnetic waves behave as if they are composed of quanta (photons) of energy $h\nu$, revealing that the classical description of fields must break down at small scales.

to be continued, the blackbody radiation picture

Another phenomenon is spontaneous emission by excited atoms. In the old semi-classical picture (quantized atoms in a classical field), a stationary bound electron would not radiate, so a truly isolated excited atom would remain excited forever. However, experiments show that excited atoms decay spontaneously, emitting photons even without external perturbations. A purely classical electromagnetic field cannot explain this effect; it is naturally explained when the field is quantized. In quantum electrodynamics, vacuum fluctuations allow an excited atom to emit a photon via the field's creation operator. Thus spontaneous emission provides further evidence that the electromagnetic field has quantized excitations (photons).

Together, these examples indicate that classical field theory is inadequate at microscopic scales. Quantizing the field (introducing photons and similar quanta) is necessary to account for observed phenomena, paving the way for a quantum theory of fields.

QFT is as a merger of special relativity and quantum mechanics. When attempting to combine quantum mechanics with special relativity, one encounters problems that point toward quantum field theory. A key issue is particle creation and annihilation. In relativistic processes, kinetic energy can convert into mass, producing new particles (as seen in particle accelerators). A fixed-particle-number Schrödinger equation cannot accommodate changing particle number. Even the relativistic single-particle equations hint at this need: the Klein–Gordon and Dirac equations have solutions with arbitrarily high (including negative) energy. Dirac reinterpreted the negative-energy solutions as antiparticles, suggesting that particles and antiparticles must be treated on equal footing. Thus a full description must allow creation and annihilation of particle-antiparticle pairs, which is naturally accomplished by quantum fields.

Another essential requirement is locality (causality). In a relativistic theory, no influence can propagate faster than light. In QFT this is enforced by requiring that field operators

commute or anticommute at spacelike separations. Specifically, one imposes

$$[\phi(\vec{x},t),\phi(\vec{y},t)]=0, \quad \text{if } (x-y)^2<0 \tag{2.8}$$

so that measurements at spacelike-separated points do not affect each other. Consistency with both Lorentz invariance and particle statistics leads to the spin-statistics theorem: integer-spin (bosonic) fields satisfy commutation relations and can occupy the same state, while half-integer-spin (fermionic) fields satisfy anticommutation relations and obey the Pauli exclusion principle. In summary, requiring both relativity and quantum principles forces us to treat fields as the fundamental objects; particles then emerge as quanta of these fields. This is the essence of quantum field theory.

To establish the QFT, we should have to get the idea called cannonical quantization principles. Canonical quantization promotes the classical fields to quantum operators, analogous to quantizing the harmonic oscillator. Starting from a classical Lagrangian density, one defines the conjugate momentum field

$$\pi(\vec{x},t)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \tag{2.9}$$

For the free scalar Lagrangaian $\mathcal{L}=\frac{1}{2}\left[(\partial_0\phi)^2-(\nabla\phi)^2-m^2\phi^2\right]$, one finds $\pi=\dot{\phi}$. In the quantum theory, ϕ and π become operators with equal-time commutation relations,

$$\begin{aligned}&[\phi(t,\vec{x}),\pi(t,\vec{y})]=i\delta^3(\vec{x}-\vec{y})\\&[\phi(t,\vec{x}),\phi(t,\vec{y})]=[\pi(t,\vec{x}),\pi(t,\vec{y})]=0\end{aligned} \tag{2.10}$$

where we have set $\hbar=1$. These generalize the quantum mechanical commutator $[\hat{q},\hat{p}]=i$. The Hamiltonian operator is obtained by substituting the quantum fields into $H=\int d^3x\mathcal{H}$, where for the free scalar $\mathcal{H}=\frac{1}{2}\left(\pi^2+(\nabla\phi)^2+m^2\phi^2\right)$.

To solve the free theory, one expands the field in Fourier modes. For a real scalar field,

$$\phi(\vec{x},t)=\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_p}}\left(a_pe^{-ipx}+a_p^\dagger e^{ipx}\right) \tag{2.11}$$

with $\omega_p=\sqrt{\vec{p}^2+m^2}$ and the opertaors a_p and $a_p^\dagger$ satisfy

$$\left[a_{\vec{p}},a_{\vec{p}'}^\dagger\right]=(2\pi)^3\delta^3(\vec{p}-\vec{p}') \tag{2.12}$$

with other commutators vanishing. Substituting this expansion into the Hamiltonian yields

$$H=\int d^3p\omega_pa_p^\dagger a_p+(\text{infinite constant}) \tag{2.13}$$

The infinite constant is the vacuum zero-point energy, which can be subtracted by normal ordering or renormalization. The key result is that $a_{\vec{p}'}^\dagger$ creates a particle of momentum $\vec{p}$ and $a_{\vec{p}}$ annihilates one. Each field mode acts like an independent quantum harmonic oscillator. In the case of fermionic fields, one imposes anticommutation relations instead, which automatically enforce the Pauli exclusion principle.

We have seen that the integrals and another something which these imply the second quantization and Fock space. The quantized fields act on a Fock space of many-particle states. If $\mathcal{H}_n$ denotes the n-particle Hilbert space, the Fock space is

$$\mathcal{F} = \oplus_n \mathcal{H}_n \tag{2.14}$$

where the $\mathcal{H}_0$ is the vacuum sector (one-dimensional, spanned by the vacuum state $|0\rangle$). Creation and annihilation operators build up this space. For example,

$$a^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle \tag{2.15}$$

and

$$a_{\vec{p}}\vec{q} = (2\pi)^3\delta^3(\vec{p}-\vec{q})\,|0\rangle \tag{2.16}$$

so $a^\dagger_{\vec{p}}$ creates a one-particle state with momentum $\vec{p}$ and the $a_{\vec{p}}$ annihilates it. A general state in Fock space is a superposition of components with $0, 1, 2, \cdots$ particles. The number operator $N_{\vec{p}} = a^\dagger_{\vec{p}} a_{\vec{p}}$ counts the number of quanta with momentum $\vec{p}$.

This formalism naturally incorporates particle creation and annihilation. For instance, interacting fields can transform a one-particle state into a two-particle state, and so on. The commutation relations of the creation/annihilation operators enforce the correct statistics: bosonic operators satisfy $\left[a, a^\dagger\right] = 1$ (Bose-Einstein statistics), whereas fermionic operators $b, b^\dagger$ satisfy $\left\{b, b^\dagger\right\} = 1$ (Fermi–Dirac statistics), which implies the Pauli exclusion principle. Thus, in Fock space the quantum field automatically handles systems with any number of identical particles.

If all the fields in the universe were uniform, then our research would obviously have no meaning at all. By making the universe non-uniform, or by disturbing the fields in the universe, we can obtain local matter or equations. Or say, we talk about the perturbation theory and the importance of inhomogeneity.

Exact solutions of interacting quantum field theories are rarely available, so one typically uses perturbation theory. One splits the Lagrangian into a solvable free part and a perturbation, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where $\mathcal{L}_I$ contains interactions terms (for example, $\lambda\phi^4/4!$ or $g\bar{\psi}\gamma^\mu A_\mu\psi$). The free Lagrangian $\mathcal{L}_0$ yields linear (homogeneous) equations of motion whose solutions are known, while the interaction $\mathcal{L}_I$ introduces nonlinear or inhomogeneous terms.

For instance, consider a scalar field equation with a source

$$\left(\Box + m^2\right)\phi(x) = J(x) \tag{2.17}$$

This inhomogeneous Klein–Gordon equation can be formally solved by using a Green's function (propagator). Its solution is

$$\phi(x) = \phi_0(x) + \int d^4y D_F(x-y)J(y) + \cdots \tag{2.18}$$

where $\phi_0(x)$ is the homogeneous solution and $D_F(x-y)$ is the Feynman propagator of the free theory. In a quantum interacting theory, one can similarly treat the interaction terms

as effective sources. One formally expands the interacting field in terms of the free field plus integrals over interaction vertices.

This viewpoint leads to a perturbative expansion for correlation functions and amplitudes. Each order in the expansion involves integrals over spacetime of products of propagators connecting the interaction points. Graphically, these terms are represented by Feynman diagrams: each internal line represents a propagator, and each vertex represents an interaction from $\mathcal{L}_I$. In the interaction picture, this systematic expansion is given by the Dyson series for the time-evolution operator. In summary, treating $\mathcal{L}_I$ as an inhomogeneous source term allows one to compute amplitudes order by order via Feynman diagrams, which is practical when the coupling is small.

But, how we describe an interaction? That's the topic: S-matrix and interaction formalism. The scattering matrix (S-matrix) formalism captures how in-state particles evolve into out-state particles through interactions. In a scattering experiment, one starts with an initial free-particle state $|i\rangle$ in the distant past ($t \to -\infty$) and observes a final free-particle state $|f\rangle$ in the distant future ($t \to +\infty$). The S-matrix element $\langle f| S |i\rangle$ is the amplitude for the initial state to evolve into the final state. Unitarity of S ensures conservation of total probability.

In the interaction picture, the time-evolution operator can be expressed in terms of the interaction Hamiltonian H_I. One obtains the formal result

$$S = Te^{(-i\int_{-\infty}^{+\infty} dtH_I(t))} \tag{2.19}$$

where T denotes time-ordering. Expanding the exponential yields the Dyson series:

$$S = 1 + (-i)\int d^4x\mathcal{H}_I(x) + \frac{(-i)^2}{2!}\int d^4xd^4yT\left\{\mathcal{H}_I(x)\mathcal{H}_I(y)\right\} \tag{2.20}$$

with $\mathcal{H}_I(x)$ the interaction Hamiltonian density (*the details seen on the next chapters, please*) . Each term involves time-ordered products of field operators at different spacetime points. Time-ordering ensures that operators are ordered by their time arguments, preserving causality also you it vanishes the commutation relations in the brackets which is useful to the theoretical physics to deal with some formalism things.

To compute physical amplitudes, one uses the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula, which relates S-matrix elements to time-ordered correlation functions of fields. Practically, one computes the vacuum expectation value of a time-ordered product $\langle 0| T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|0\rangle$ and then "amputates" the external propagators corresponding to incoming and outgoing particles to obtain $\langle f| S |i\rangle$. Conceptually, the S-matrix formalism emphasizes that interactions occur at localized spacetime points (vertices) determined by $\mathcal{H}_I$.

An useful tool or visualization is the genius Feynman diagrams. And we know that our commonly used physical space is the position space and the momentum space. So we have a short view of Feynman rules in position space and momentum space.

Feynman diagrams provide a graphical shorthand for perturbative expansions. Each diagram corresponds to a mathematical expression determined by Feynman rules, which depend

on the theory's interaction Lagrangian. In momentum space, the general rules include (*the details seen on the later chapters, please*):

- Assign to each internal line carrying momentum p a propagator factor $\frac{i}{p^2-m^2+i\varepsilon}$ (for a scalar of mass m; other fields have analogous propagators with appropriate numerators).
- At each interaction vertex, include the factor from the interaction Lagrangian (for example, $-i\lambda$ for a $\lambda\phi^4$ vertex) and enforce conservation of four-momentum. Momentum conservation is implemented by a delta-function at each vertex, typically giving an overall factor $(2\pi)^4\delta^4(\sum p_{\text{in}} - \sum p_{\text{out}})$.
- Integrate over each undetermined loop momentum k with $\int \frac{d^4k}{(2\pi)^4}$
- Include any symmetry or combinatorial factors for identical particles or identical vertices.
- Attach to each external line a factor corresponding to the particle's wavefunction (often taken as 1 for on-shell external scalar lines).

For example, in a real scalar ϕ^4 theory each four-point vertex contributes $-i\lambda$ and each internal line contributes $i/(p^2 - m^2 + i\varepsilon)$. One then integrates over loop momenta and imposes the momentum-conserving delta functions. External scalar lines each contribute a factor of 1 (for the normalized external wavefunction).

In position space, one works with the coordinate-space propagator. The Feynman propagator for a scalar field is

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{ip(x-y)} \tag{2.21}$$

The Feynman rules in x-space assign a factor $D_F(x-y)$ to each internal line connecting spacetime points x and y. At each interaction vertex one integrates over the vertex position, $\int d^4x(-i\lambda)$ for a ϕ^4 vertex. External lines correspond to field insertions at the external points. Working in position space can be useful for time-dependent or non-perturbative analyses, while momentum-space rules are standard for scattering calculations.

In summary, Feynman rules translate each diagram's lines and vertices into an integral expression. Propagators encode particle propagation between interaction points, vertex factors encode the interaction strengths from the Lagrangian, and momentum-conserving δ-functions enforce translation symmetry. Evaluating these integrals yields the scattering amplitude, from which physical probabilities are derived.

This chapter is based on my QFT-slef-learning report in Yunnan University. In fact, the content of this chapter is adapted from the lecture notes of my group meeting report. It offers a global perspective, and of course, mainly the field theory part (the Part I of the textbook).

Chapter 3

Classical Field Theory

As we had illustrated formerly that the quantum field theory is just quantum mechanics with as infinite number of oscillators. We already saw that it can do some remarkable things, such as explain spontaneous emission without the statistical theory. (But it also seems to lead to aburdities, such as an infinite shift in the energy levels of hydrogen atom). Remember that quantum field theory is not abusrd but extremely predictive.That means you should be vary careful about how we do calculations that the theory gets right without infinities. These are called the *tree-level processes*, which means they are leading order in an expansion in $\hbar$. Since takeing $\hbar \to 0$ gives the classical limit, tree-level calculations are closely related to calculations in classical field theory, which is the subject of this chapter.

A classical field theory is just a mechanical system with a continuous set of degrees of freedom. Differently, we often use the density of Hamiltonians or Lagrangians in place of Hamiltonian or Lagrangians: (notice that we often omit the word *density*, so that whenever we refer to Hamiltonians and Lagrangians we refer to their density)

$$H = \int d^3x\mathcal{H}, \quad L = \int d^3x\mathcal{L} \tag{3.1}$$

and the dynamics for a Lagrangian system are determined by *the principle of least action*. The action is the integral over the time:

$$S = \int dtL = \int d^4x\mathcal{L} \tag{3.2}$$

Formally, the Hamiltonian is a functional of fields and their conjugate momenta $\mathcal{H} = \mathcal{H}[\phi, \pi]$ while the Lagrangian is the *Legendre transform of the Hamiltonian* defined as

$$\mathcal{L}[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}]\dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]] \tag{3.3}$$

where items satisfies

$$\dot{\phi} = \partial_t\phi$$

and $\pi[\phi, \dot{\phi}]$ is defined by

$$\frac{\partial\mathcal{H}[\phi, \pi]}{\partial\pi} = \dot{\phi}$$

The inverse transform is

$$\mathcal{H}[\phi,\pi]=\pi\dot{\phi}[\phi,\pi]-\mathcal{L}[\phi,\dot{\phi}[\phi,\pi]] \tag{3.4}$$

where the item $\dot{\phi}[\phi,\pi]$ is defined by

$$\frac{\partial\mathcal{L}[\phi,\dot{\phi}]}{\partial\dot{\phi}}=\pi$$

What a coup is that you can regard the ϕ as the analog of q and regard the π as the analog of p which represents that the thought of field is the field version of the so-called analytical mechanics.

Also, we can derive the Hamilontian with the sum of the kinetic and potential energies of a system:

$$\mathcal{H}=\mathcal{K}+\mathcal{V} \tag{3.5}$$

while the Lagrangian was shown as:

$$\mathcal{L}=\mathcal{K}-\mathcal{V} \tag{3.6}$$

The Hamiltonian corresponds to a conserved quantity that is the total energy of a system while the Lagranian does not. The question is that, Hamiltonians, however, is that they are not Lorentz invariant. Although the Hamiltonian picks out the energy or pricisely the energy eigenvalue, which is not a Lorentz scalar. On the contrary, it is the 0 component of a Lorentz vector $P_\mu=\left(H,\vec{p}\right)$ (4-momentum vector). The Hamiltonian density is the 00 component of a Lorentz tensor, the energy-momentum tensor $\mathcal{T}_{\mu\nu}$ which we will see it is the Noether current given by symmery under global space-time translations.

What's more? The Matthew D. Schwartz taught us a fact or a convention. Hamiltonians are great for non-relativistic sysytems (such as we have already learnt in quantum mechanics) , but for relativistic systems we will almost exclusively use Lagrangians.

And there are many terms we should understand clearly. We do not usually talk about kinetic and potential energy in quantum field theory. Instead we talk about *kenetic terms* and then about *interactions*, for reasons that will become clear after we have done a few calculations (shut up and do calclations!). *Kinetic terms* are *bilinear*, meaning they have exactly two fields as we saw the square item in Newton's mechanics. So kinetic terms may look like:

$$\mathcal{L}_K\supset\frac{1}{2}\phi\Box\phi,\quad\bar{\psi}\partial\!\!\!/\psi,\quad\frac{1}{4}F_{\mu\nu}^2,\quad\frac{1}{2}m^2\phi^2,\quad\frac{1}{2}\phi_1\Box\phi_2$$

Anything with just two fields of the same or different type can be called a kinetic term. The kinetic terms tell you about the free (non-interacting) behavior. Fields with kinetic terms are said to be *dynamical* or *propagating*. More precisely, a field should have time derivatives in its kinetic term to be dynamical. It is also sometimes useful to think of a *mass term*, such as $m^2\phi^2$, as an interaction rather than a kinetic term. *Interactions* have three or more fields like $\mathcal{L}\supset\lambda\phi^3$, $g\bar{\psi}A\!\!\!/\psi$,$g\partial_\mu\phi A_\mu\phi^\star$,$g^2A_\mu^2A_\nu^2$,$1/M_{PI}\partial_\mu h_{\mu\nu}\partial_\nu h_{\alpha\beta}h_{\alpha\beta}$,$\cdots$ Since the interactions are everything but the kinetic terms, we also sometimes write $\mathcal{L}=-\mathcal{V}=-\mathcal{H}_{int}$. It is helpful if the coefficients of the interaction terms are small in some sense, so that the fields are *weakly interacting* and we can do *perturbation theory*.

From now, we focus on the principal of least action and consider the varing of $\phi \to \phi + \delta\phi$, and the Lagranian is the functional of a field and its first derivatives[1], then we can do functional computation as following:

$$\begin{aligned}
\delta S &= \int d^4x[\mathcal{L} + \delta\mathcal{L} - \mathcal{L}] \\
&= \int d^4x\big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\big] \\
&= \int d^4x\big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)\big] \\
&= \int d^4x\big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \partial_\mu\big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\big) - \big(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\big)\delta\phi\big] \\
&= \int d^4x\{\big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\big]\delta\phi + \partial_\mu\big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\big]\} \\
&= \{\int d^4x\big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\big]\delta\phi\} + \big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\big]\Big|_{\text{boundaries}} \\
&= \int d^4x\big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\big]\delta\phi
\end{aligned} \tag{3.7}$$

Above the computation is the variation of the action[2], and we should note that the principle of the least action is beyond the classical mechanics as the equations of motion determined by this principle in classical field theory. Namely it should be insensitive to small variations of those field $\delta S/\delta\phi = 0$, and if this holds for all variations, it gives $\delta S = 0$, thus:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0} \tag{3.8}$$

These are the *Euler-Lagrange equtions* while they give the *equations of motion* following from a Lagrangian.

Let's check the principle from two examples. Given an action as the following:

$$S = \int d^4x\mathcal{L} = \int d^4x\big[\frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) - \mathcal{V}[\phi]\big]$$

and we put the Lagrangian into the E-L equtions and we can easily get the motion eqution:

$$-\mathcal{V}'[\phi] - \partial_\mu(\partial_\mu\phi) = 0$$

where we often note the $\Box = \partial_\mu^2$ as the *d'Alembertian*.

What an important one is the Lagrangian as the form as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m\phi^2 \tag{3.9}$$

[1]The form $\mathcal{L}[\phi, \partial_\mu\phi]$ is the form that "classical" Lagrangians had. If only first derivatives are involved, boundary conditions can be specified by initial positions and velosities only in accordance with Newton's idea. And actually, the complex couping fields items in kinetic or interactions must occur due to the quantum effects in all but the simplest *renormalizable* field theories such as geneic in *effective field theories*.

[2]While computing, we use the method of integration by parts and we exchange the variation and derivative which we often do in physics.

When we put it in the E-L eqution, it will give an important equation as :

$$\boxed{\big(\Box + m^2\big)\phi = 0} \tag{3.10}$$

The equation (3.10) is known as the *Klein-Gordon equation* which describes the equation of motion for a *free scalar field*.

When we meet a physical quantity, a natural question is that will the quantity give some symmetry or what kind of symmetry is inside? Now, we will show the topic of syemmery *Noether's theorem*.

We must clarify the concept that the symmetries we are referring to are not the "folded" symmetries obtained by folding a piece of paper , like the symmetries learned by schoolchildren[3].The so-called symmetry in physics is actually a kind of invariance, under a certain transformation, the physical quantity we study does not change, then it is said that this is a certain symmetry.

As for our physical quantity, the Lagranian, it may be invariant under some special type of variation $\phi \to \phi + \delta\phi$. For example, given a Lagrangian for a complex field ϕ as:

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 \tag{3.11}$$

it is the square, so when we plug the complex phrase in the state ϕ it will be invariant. Under transformation $\phi \to e^{-i\alpha}\phi$ for any $\alpha \in \mathbb{R}$, the Lagrangian is invariant. We claim that this transformation is a *symmetry* of the Lagrangian. Notice that the Lagrangian has two independent real degrees of freedom in such a complex field ϕ, actually you can take the $\phi = \phi_1 + i\phi_2$ then the Lagrangian is determined by two fields ϕ_1, ϕ_2.

To fit our symmetry, the parameter α is continuous while we can deal the degrees of freedom as the kind of special discrete symmetry, namely the conjugate ϕ and $\phi^\star$. Consider that our example is as the form as the scalar free field which should fit the Klein-Gordon eqution mentioned above.

We use a *continuous* parameter to describe the transformation and the symmetry which exactly mean the Lagrangian $\mathcal{L}$ is invariant under the transformation by parameter α saying that the functional derivative of $\mathcal{L}$ with respect to parameter α is 0, thus:

$$\frac{\delta \mathcal{L}}{\delta \alpha} = 0 \tag{3.12}$$

We have already compute the δL in the (3.7), then we can quickly write the $\delta\mathcal{L}/\delta\alpha$:

$$\frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \Big\{ \Big[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \Big] \frac{\delta \phi_n}{\delta \alpha} + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \Big] \Big\} = 0 \tag{3.13}$$

where ϕ_n are the set of fields the Lagrangian depends on[4].

[3]Although folding a piece of paper does have a spatial symmetry

[4]Or you can regard ϕ_n as the ϕ and $\phi^\star$ in our example mentioned above

When the equations of motion are satisfied[5], the(3.13) will be written as:

$$\sum_n \partial_\mu \big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}\big] = 0 \tag{3.14}$$

implies the important *conservative equation*:

$$\boxed{\partial_\mu J_\mu = 0} \tag{3.15}$$

where

$$\boxed{J_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}} \tag{3.16}$$

is known as a *Noether Current*.

The Noether current can be used to describe the symmetry. For example, we can compute the Noether current from the example Lagrangian above $\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2$. We note that $\phi_1 = \phi$, $\phi_2 = \phi^\star$, and their variant derivative $\delta\phi/\delta\alpha = -i\phi$, $\delta\phi^\star/\delta\alpha = i\phi^\star$, so that

$$J_\mu = i\big(\phi^\star \partial_\mu \phi - \phi \partial_\mu \phi^\star\big) \tag{3.17}$$

and note that the symmetry is continuous so that we can take small variantions. As this is the scalar free field, we can check that

$$\partial_\mu J_\mu = i\big(\phi^\star \Box \phi - \phi \Box \phi^\star\big) = im^2\big(\phi\phi^\star - \phi^\star\phi\big) = 0 \tag{3.18}$$

which vanishes when they are satisfied with the equations of motion[6].

In the previous paragraph,we should note that the equation of motion is satisfied, and in this case vector field J_μ that satisfies $\partial_\mu J_\mu = 0$ is called a *conserved current*. It is called *conserved* beacuse it is actually the *charge continuity equation*. Here, the total charge Q, defined as

$$Q = \int d^3x J_0 \tag{3.19}$$

satisfies

$$\partial_t Q = \int d^3x \partial_t J_0 = \int d^3x \nabla \cdot \vec{J} = 0 \tag{3.20}$$

The equation above is similar to the "classical" charge conserved, like the equation $\partial\rho/\partial t + \nabla \cdot (\rho\vec{v}) = 0$ in the fluid mechanics[7] which called the Euler equation[8]. Also, we have assumed $\vec{J}$ vanishes at the spaital boundary which is reasonable in physics, since, by assumption, nothing is leaving our experiment. That says, the total charge does not change with time, and is conserved.

So now, we can formally explain what *Noether's theorem* is:

[5]In contrast to (3.7), the (3.13) holds all ϕ_n because it is derived by continuous symmetry's computation of functional derivative so the field configurations ϕ_n can not fit in the equation of motion corresponds to a symmetry

[6]We used the equations $\Box\phi = -m^2\phi$ and $\Box\phi^\star = -m^2\phi^\star$.

[7]To fit with the total charge, you can rewrite the equation through the Gauss's Law

[8]You may notice that the $\partial_\mu J_\mu$ shows that $\partial_t J_0 - \nabla \cdot \vec{J}$ has a minus sign difference. It doesn't matter because of the difference of definition from the J_0 or the total charge Q

If a Lagrangian has a continuous symmetry then there exists a current associated with that symmetry that is conserved when the equations of motion are satisfied.

Notice that the symmetry must be continuous otherwise we cannot induce the Noether's theorem because the small variantions $\delta\mathcal{L}/\delta\alpha$ has no meaning which also can be expressed that Noether's theorem does not apply to discrete symmetries[9]. And now we can summarize a few points[10]:

1. The symmetry must be continuous, otherwise $\delta\alpha$ has no meaning.

2. The current is conserved on-shell, that is, when the equations of motion are satisfied.

3. It works for global symmetries, parametrized by numbers α, not only for local (gauge) symmetries parametrized by functions $\alpha(x)$.

We have expounded the symmetry especially the Noerther's theorem. We deal with the Noether current J_μ. But now we deal with an important case which is the global symmetry to the action not the Lagrangian[11]. That's the global space-time translations and the symmetry gives a conserved current, the *energy-momentum tensor* $\mathcal{T}_{\mu\nu}$.

We take a translation from the whole space-time manifold, and we say that physics at the point x should be same as the point y[12]. We note the translation as an infinitesimal 4-vector ξ^μ, then for the scalar field, we have:

$$\phi(x) \to \phi(x+\xi) \tag{3.21}$$

Using the field expansion, we can expand the equation as

$$\phi(x+\xi) = \phi(x) + \sum_\nu \frac{\partial\phi(x)}{\partial x^\nu} + \cdots = \phi(x) + \xi^\nu\partial_\nu\phi(x) + \cdots \tag{3.22}$$

That implies that the infinitesimal translation means:

$$\frac{\delta\phi}{\delta\xi^\nu} = \partial_\nu\phi \tag{3.23}$$

[9]Such as the symmetry under $\phi \to -\phi$ of $\mathcal{L} = 1/2\phi\Box\phi - m^2\phi^2 - \lambda\phi^4$ with ϕ real even though it is exactly variant under the transformation.

[10]There is no Noether's theorem in general relativity, but we can derive the global conservation in GR instead of the local conservation.

[11]Notice that Noether's theorem fits to the continuous symmetry to the Lagrangaians, but we deal with the case including the time part's translations so this is the kind of symmetry to action, naturally the Noether's theorem is not working.

[12]Notice that this is a translation of spacetime as a whole and not a change in the coordinate system.

samely as the Lagrangian:

$$\frac{\delta \mathcal{L}}{\delta \xi^{\nu}} = \partial_{\nu} \mathcal{L} \tag{3.24}$$

Since this is a total derivative, then the variantion of action S :

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \xi^{\nu} \partial_{\nu} \mathcal{L} = 0 \end{aligned} \tag{3.25}$$

which is why we sometimes say this is a symmetry of the action, not the Lagrangian.

And what's the conserved current? As we have computed in (3.13), and use the equations of motion, then the variantion of the Lagrangian form a translation ξ^{ν} can be written as:

$$\frac{\partial \mathcal{L}}{\partial \xi^{\nu}} = \partial_{\mu} \big(\sum_{n} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_n)} \frac{\delta \phi_n}{\delta \xi^{\nu}} \big) \tag{3.26}$$

using the (3.23), we have an equation:

$$\partial_{\mu} \big(\sum_{n} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_n)} \partial_{\nu} \phi_n \big) = \partial_{\nu} \mathcal{L} \tag{3.27}$$

namely:

$$\partial_{\mu} \big(\sum_{n} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_n)} \partial_{\nu} \phi_n - g_{\mu\nu} \mathcal{L} = 0 \big) \tag{3.28}$$

which is the conserved equation $\partial_{\mu} \mathcal{T}_{\mu\nu} = 0$ producing four Noether currents, and the $\mathcal{T}_{\mu\nu}$ is:

$$\boxed{\mathcal{T}_{\mu\nu} = \sum_{n} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_n)} \partial_{\nu} \phi_n - g_{\mu\nu} \mathcal{L}} \tag{3.29}$$

The four Noether currents correspond to four conserved quantities, the energy and the momentum. $\mathcal{T}_{\mu\nu}$ is called the *energy-momentum tensor*. We see the 00 component of the energy-momentum tensor:

$$\varepsilon = \mathcal{T}_{00} = \sum_{n} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \dot{\phi}_n - \mathcal{L} \tag{3.30}$$

which has the same form as the Legendre transform of Lagrangian for the $\partial \mathcal{L} / \partial \dot{\phi}_n = \pi_n$. Then the 00 component of energy-momentum tensor is the energy density. By the way, the definition of energy-momentum tensor in (3.29) is called the *canonical energy-momentum tensor* for the difference definition from the general relativity[13].

There's an another thing we should clarify. The concept of the current is abstract current which we use it to describe the continuous symmetry. And with that comes the concept of the abstract charge. Both the conserved vector J_{μ} associated with a global symmetry and the energy-momentum tensor $\mathcal{L}_{\mu\nu}$ are types of currents. The currents are used in many ways in quantum field theory, for example:

[13]There is another way to derive the energy-momentum tensor in general relativity and Lagrangian form in GR. See textbook *Introduction to differential geometry and general relativity* written by Canbin Liang and Bin Zhou

1. Currents can be Noether currents associated with a symmetry.

2. Currents can refer to *external* currents[14].

3. Currents can be used as sources for fields appearing in the Lagrangians.

4. Currents can be place-holders for certain terms in a Lagrangian.

Now, we gonna to do some calculations, the Coulomb's law and Coulomb potential, using the classical field theory we developed above, and we will induce the method of Green's function and the Feynman diagrams (rules) in these examples. We start with an external current:

$$J_\mu(x) = \begin{cases} J_0(x) = \rho(x) = e\delta^3(x) \\ J_i(x) = 0 \end{cases} \tag{3.31}$$

whose Lagrangian[15] is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - A_\mu J_\mu \tag{3.32}$$

and the E-L equations[16] imply the :

$$\partial_\mu F_{\mu\nu} = J_\nu \tag{3.33}$$

which is the Maxwell equations and choose the Lorenz gauge, $\partial_\mu A_\mu = 0$, then:

$$\Box A_\nu(x) = J_\nu(x) \tag{3.34}$$

and the formal solution :

$$A_\nu(x) = \frac{1}{\Box} J_\nu(x) \tag{3.35}$$

The $1/\Box$ just means the inverse of the $\Box$. It says that the A_ν field is determined by the source J_ν after it propagates with the *propagator*. It's an insightful way to understand.

$$\Pi_A = \frac{1}{\Box} \tag{3.36}$$

Expect the electromagenetic waves' solutions, the point charge at the origin will gives:

$$\begin{aligned} A_i &= 0 \\ A_0(x) &= \frac{e}{\Box}\delta^3(x) \end{aligned} \tag{3.37}$$

To solve the equations, we develop the tools of fourier transform first. We have the general transform:

$$\delta^3(\vec{x}) = \int \frac{d^3k}{(2\pi)^2} e^{i\vec{k}\vec{x}} \tag{3.38}$$

[14]These are given background configurations, such as electrons flowing through a wire

[15]Please look up the textbook of electrodynamics and write the Lagrangian of electromagnetic field

[16]Using the symmetry of $F_{\mu\nu}$

Since the Laplacian is $\Delta = \partial^2$ for space parts, therefore:

$$\begin{aligned}\Delta\delta^3(\vec{x}) &= \Delta \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \\ &= (\partial_x^2 + \partial_y^2 + \partial_z^2) \int \frac{d^3k}{(2\pi)^3} e^{ik_x x} e^{ik_y y} e^{ik_z z} \\ &= i^2(k_x^2 + k_y^2 + k_z^2) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \\ &= (i\vec{k})^2 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}\end{aligned}$$

which we can extent the relations as following:

$$\begin{aligned}\Delta^2\delta^3(\vec{x}) &= (i\vec{k})^{2n} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \\ &= -(\vec{k})^2 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}\end{aligned} \tag{3.39}$$

or we can rewrite the equation as the fourier transform equation, that means:

$$\left[\Delta^2\delta^3(\vec{k})\right]^{-1} = -(\vec{k})^2 \tag{3.40}$$

Now, let's check the d'Alembertian for a lorentz-invariant quantities:

$$\delta^4(x) = \int \frac{d^4k}{(2\pi)} e^{ikx} \tag{3.41}$$

where the item kx means the summarizition $k_0x_0 - \vec{k}\cdot\vec{x}$. Therefore,

$$\Box\delta^4(x) = -\int \frac{d^4k}{(2\pi)} e^{ikx} k^2 e^{ikx} \tag{3.42}$$

thus, the extension:

$$\Box^n\delta^4(x) = \int \frac{d^4k}{(2\pi)^4} (-k^2)^n e^{ikx} \tag{3.43}$$

The fourier's law tells us that every functions can be expanded by fourier transform:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{f}(k) e^{ikx} \tag{3.44}$$

then

$$\Box^n f(x) = \int \frac{d^4k}{(2\pi)^4} \Box^n \widetilde{f} e^{ikx} = \int \frac{d^4k}{(2\pi)^4} (-k^2)^n \widetilde{f}(k) e^{ikx} \tag{3.45}$$

or namely

$$\widetilde{\left[\Box^n f\right]}(k) = (-k^2)^n \widetilde{f}(k) \tag{3.46}$$

.

Then we develop the important tools for Δ and $\Box$ which for a field theorist, they mean[17]:

$$\Delta \rightleftharpoons -\vec{k}^2, \quad \Box \rightleftharpoons -k^2 \tag{3.47}$$

are the linear and it will simplfy our computation.

So, let's back to our example, the equation:

$$A_0(x) = \frac{e}{\Box}\delta^3(\vec{x}) = -\frac{e}{\Delta}\delta^3(\vec{x}) \tag{3.48}$$

use our tools, then we will have the integral in k-space[18]:

$$\begin{aligned} A_0(x) &= -\frac{e}{\Delta}\delta^3(\vec{x}) \\ &= \frac{e}{(2\pi)^3}\int \frac{d^3k}{\vec{k}^2}e^{i\vec{k}\vec{x}} \\ &= \frac{e}{(2\pi)^3}\int d^3k\frac{e^{i\vec{k}\vec{x}}}{k^2} \\ &= \frac{e}{(2\pi)^3}\iiint k^2\sin\theta d\theta\varphi dk\frac{e^{ikr\cos\theta}}{k^2} \\ &= \frac{e}{(2\pi)^3}\int_0^{\infty} k^2dk\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\varphi\frac{e^{ikr\cos\theta}}{k^2} \\ &= \frac{e}{8\pi^2}\frac{1}{ir}\int_{-\infty}^{+\infty} dk\frac{e^{ikr}-e^{-ikr}}{k} \end{aligned} \tag{3.49}$$

and we use the method of mathematics in physics, the theory of complex variables functions,

$$\int_{-\infty}^{+\infty} dk\frac{e^{ikr}-e^{-ikr}}{k} = \lim_{\delta\to 0}\big[\int_{-\infty}^{+\infty} dk\frac{e^{ikr}-e^{-ikr}}{k+i\delta}\big] = \lim_{\delta\to 0} 2\pi i e^{-\delta r} = 2\pi i \tag{3.50}$$

therefore the equation

$$A_0(x) = \frac{e}{4\pi}\frac{1}{r} \tag{3.51}$$

shows the Coulomb potential.

From the important example above, we can summarize a kind of equations whose form follow the form:

$$\text{Field} = \frac{1}{\Box}\text{Current} \tag{3.52}$$

We add some items in the Lagrangian $\mathcal{L}$. We consider the Lagrangian of a charged object radiating the A filed,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \phi^\star\Box\phi - ieA_\mu\big(\phi^\star\partial_\mu\phi - \phi\partial_\mu\phi^\star\big) \tag{3.53}$$

We can get the current in the space-time is $J_\mu = ie\big(\phi^\star\partial_\mu\phi - \phi\partial_\mu\phi^\star\big)$ and we can get the same eqution in (3.52) in Lorenz gauge.

[17] In electrodynamics, we used to do the same with derivative operators, for example, by treating the derivative operator directly as $\nabla = i\vec{k}$ and the time deribative operator as $\partial_t = -i\omega$ for the plane wave solution or low order spherical wave solution.

[18] use the correct integral on the spherical surface and use the general integrand.Just one step by step

Using the propagators is a very useful way to solve these types of eqution, and quite general. And we gonna to introduce an example about the graviton and we shall use the example's perturbation solution to induce the Green's function method.

Given the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{3}\lambda h^3 + Jh \tag{3.54}$$

where the h represents the gravitational potential. As the E-L equation (3.8), we get the equation of motion:

$$\Box h - \lambda h^2 - J = 0 \tag{3.55}$$

Solve the perturbatively in λ, for the $\lambda = 0$, the zero solution is

$$h_0 = \frac{1}{\Box}J \tag{3.56}$$

. Then we plug in

$$h = h_0 + h_1 \tag{3.57}$$

with $h_1 = \mathcal{O}(\lambda^1)$.Then

$$\Box(h_0 + h_1) - \lambda(h_0 + h_1)^2 - J = 0 \tag{3.58}$$

implies

$$\Box h_1 = \lambda h_0^2 + \mathcal{O}(\lambda^2) \tag{3.59}$$

so that:

$$h_1 = \lambda\frac{1}{\Box}(h_0 h_0) = \lambda\frac{1}{\Box}\big[\big(\frac{1}{\Box}J\big)\big(\frac{1}{\Box}J\big)\big] \tag{3.60}$$

So the solution to order λ in perturation theory is

$$h = \frac{1}{\Box}J + \lambda\frac{1}{\Box}\big[\big(\frac{1}{\Box}J\big)\big(\frac{1}{\Box}J\big)\big] + \mathcal{O}(\lambda^2) \tag{3.61}$$

which can keep iterating.

This is the *Green's function method*, and the

$$\Pi = \frac{1}{\Box} \tag{3.62}$$

is called *2-point Green's function* or *propagator*. As its name illustrates, the propagator is a kind of tools to decribe the propagating having nothing to do with the source exactly determined by the kinetic terms for a field which is insightful to physical pictures.

About this expansion, we can define the propagator as the solution to[19] :

$$\Box_x \Pi(x, y) = -\delta^4(x - y) \tag{3.63}$$

Drop out the boundaries in infinite space, we can get

$$\Pi(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}\frac{1}{k^2} \tag{3.64}$$

[19]See the textbook *Methods of mathematics in physics*

The 2-point propagator is symmetry,

$$\Pi(x,y) = \Pi(y,x) \tag{3.65}$$

Using the $\Box_y \Pi(x,y) = -\delta^4(x,y)$, we can write a field as

$$\begin{aligned} h(x) &= \int d^4y \delta^4(x-y) h(y) \\ &= -\int d^4y \big[\Box_y \Pi(x,y)\big] h(y) \\ &= \int d^4y \Pi(x,y) \Box_y h(y) \quad \text{integrated by parts} \end{aligned} \tag{3.66}$$

therefore,

$$h_0(x) = \int d^4y \Pi(x,y) J(y) \tag{3.67}$$

The equation $\Box h_1 = \lambda h_0$ can be rewritten as a new form easy to compute:

$$\begin{aligned} \Box_\omega h_1(\omega) &= \lambda h_0^2(\omega) \\ &= \lambda h_0(\omega) h_0(\omega) \\ &= \lambda \int d^4y \Pi(\omega,y) J(y) \int d^4z \Pi(\omega,z) J(z) \end{aligned} \tag{3.68}$$

So the final soluton in order of λ is :

$$h(x) = \int d^4y \Pi(x,y) J(y) + \lambda \int d^4y \Pi(\omega,y) J(y) \int d^4z \Pi(\omega,z) J(z) + \mathcal{O}(\lambda^2) \tag{3.69}$$

There is a Feynman diagram to show this process. But we keep this till the later chapter about formal Feynman diagrams and rules. Here, the Feynman rules for this classical field theory examphy are : 1.Draw a point x and a line from x to a new point x_i. 2.Either truncate a line at a source J or let the line branch into two lines adding a new point and a factor of λ. 3.Repeat previous step. 4.The final value for $h(x)$ is given by graphs up to some order in λ with the ends capped by currents $J(x_i)$, the lines replaced by propagators $\Pi(x_i, x_j)$, and all internal points integrated over.

All the above is the content of this chapter, in this chapter, we re-interpret some important contents of the classical field theory which will be helpful to study the quantum field theory later.

Chapter 4

Old-Fashioned Perturbation Theory (OFPT)

In this chapter, we briefly introduce the old-fashioned perturbation theory (OFPT) and then do a little calculation using the theory. At the same time, we recall a bit of confusion that Oppenheimer obtained using OFPT, namely the Lamb shift, which will show us that it is a difficult and important thing to properly handle and clarify all infinite divergences.

One important thing is that there may have some infinities and off-shell cases. And we gonna to try to explain where off-shellness comes from, why you do not need it, but why you want it anyway.

Let's start with the Schwinger's work in OFPT. Just as in quantum mechanics we have leant before, perturbation theory in quantum field theory works by spliting the Hamiltonian up into two parts:

$$H = H_0 + V \tag{4.1}$$

where the H_0 is what we have completely understand and the V can be regarded as small parts[1]. We know the state of a system at early times and would like to know the state at late times. For system described by H_0, we note the complete set as $\{|\phi\rangle\}$, then the eigenequations are:

$$H_0 |\phi\rangle = E |\phi\rangle \tag{4.2}$$

If the energies E are continuous, we shall be able to find an eigenstate $|\psi\rangle$ of the full Hamiltonian with the same energy eigenvalue:

$$H_0 |\psi\rangle = E |\psi\rangle \tag{4.3}$$

Both with the two eigenequations above, we can formally write the linear combination with two states[2]:

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} V |\psi\rangle \tag{4.4}$$

[1]The difference from quantum mechanics is that in quantum field theory the states often have a continuous range of energies.

[2]We pick out the state $|\phi\rangle$ that is orthogonal to all other vectors and here we omit the degenerate perturbation problems. Just use the linear algebra.

which is called the *Lippmann-Schwinger equation*[3].

Notice that, the alegbra part of $\frac{1}{E-H_0}$ is an inverse operator called *Lippmann-Schwinger kernel* which is a kind of *Green's function*:

$$\Pi_{LS} = \frac{1}{E-H_0} \tag{4.5}$$

We define the *transfer matrix* T, satisfying the linear transformation:

$$V|\psi\rangle = T|\phi\rangle \tag{4.6}$$

thus product operator V:

$$T|\phi\rangle = V|\phi\rangle + V\frac{1}{E-H_0}T|\phi\rangle$$

therefore

$$T = V + V\frac{1}{E-H_0}T \tag{4.7}$$

namely

$$T = V + V\Pi_{LS}T \tag{4.8}$$

We expand the equation in order by V:

$$T = V + V\Pi_{LS}V + V\Pi_{LS}V\Pi_{LS}V + \cdots \tag{4.9}$$

In Heisenberg's picture, we write the martix elements of transfer matrix T_{fi} of the initial state mapping to final state[4]:

$$\begin{aligned}\langle\phi_f|T|\phi_i\rangle &= \langle\phi_f|V|\phi_i\rangle + \langle\phi_f|V\Pi_{LS}V|\phi_i\rangle + \cdots \\ &= \langle\phi_f|V|\phi_i\rangle + \langle\phi_f|V\frac{1}{E-H_0}|\phi_j\rangle\langle\phi_j|V|\phi_i\rangle + \cdots\end{aligned} \tag{4.10}$$

or the matrix product form:

$$T_{fi} = V_{fi} + V_{fi}\Pi_{LS}(j)V_{ji} + V_{fj}\Pi_{LS}(j)V_{jk}\Pi_{LS}(k)V_{ki} + \cdots \tag{4.11}$$

where $\Pi_{LS}(k) = \frac{1}{E-E_k}$ and $E = E_i = E_f$ is the energy of the initial and final state we are interested in.

The expansion is the so-called *old-fashioned perturbation theory (OFPT).*

So how we understand the perturbation's physical picture? Actually, in each term appeared in the equation, creates an intermediate state $|\phi_j\rangle$ which propagates with the propagator $\Pi_{LS}(j)$ until it hits another potential, where it creates a new field $|\phi_k\rangle$ which then propagates and so on, until they hit the final potential factor, which transitions it to the final state. The nice diagrammatic way of drawing this series is the Feynman graphs.

[3]It is useful in scattering theory. In scattering calculations the V acts at intermediate times to induce transitions among states $|\phi\rangle$ that are assumed to be free (non-interacting) at early and late times. It says the full wavefunction $|\psi\rangle$ is given by the free wavefunction $|\phi\rangle$ plus a scattering term.

[4]insert the complete set and use the Einstein's summation convention

Schwartz introduced the Feynman ruled for OFPT and developed it by an important example, an electron scattering off another electron and the Coulomb's law revisited, in his textbook. We don't note this parts rather recalling the Oppenheimer's confusion. We will recall the scartting example till we discuss the Feynman rules in later chapter.

So now, we revisit the second-quantized Hamiltonian from the photon field:

$$H = \int \frac{d^3k}{(2\pi)^3}\omega_k\big(a_k^\dagger a_k + \frac{1}{2}\big) \tag{4.12}$$

with $\omega_k = |\vec{k}|$.

The problem is that, if we compute the energy of vacuum[5], the Hamilontian gives:

$$E_0 = \langle 0|\, H\, |0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\vec{k}| = \infty \tag{4.13}$$

which we have used the communicate of the creation and annihilation operator.

That's a bad answer? Fortunately, there is an easy way out of this paradoxical infinity: How do you measure the energy of vacuum? Yes, we cannot. Only energy differences are measurable, and in these differences the *zero-point energy*, the energy of the ground state, drops out. This is the basic idea behind renormalization–infinities can appear in intermediate calculations but they must drop out of physical observables[6] . Because this kind of infinity, the Oppenheimer used to concluded that QED was wrong[7].

Schwartz used the example of *linear Stark effect* and *Lamb shift* to illustrate the confusions. But we also recall this part in the later chapter which we will develop the formal Feynman diagrams and rules. Let's cut to the chase.What went wrong? In the Stark effect calculation we only had to sum over excited electron states, through $\sum_{m>0}|\psi_m\rangle\langle\psi_m|$, which was finite. For the Lamb shift calculation, the sum was also over photon states, which was divergent. It diverged because the phase space for photons, d^3k, is larger than the suppression, $1/|\vec{k}|$, due to the energies of the intermediate excited states. In terms of Feynman diagrams, the difference is that in the latter case we do not consider interactions with a fixed external field, but integrate over dynamical fields, corresponding to intermediate state photons. Since the photons relevant to the $\langle\psi_0|\, H_{int}\, |\psi_m; 1_k\rangle$ matrix element are the same as the photons relevant to the second, $\langle\psi_m; 1_k|\, H_{int}\, |\psi_0\rangle$ matrix element, the photon lines represent the same state and should be represented by a single line. Thus the diagram contracts, and the Stark effect diagram becomes a loop diagram for the Lamb shift. These pictures are just shorthand for the perturbation expansion. The loop means that there is an unknown momentum, $\vec{k}$, over which we have to integrate. Momentum must be conserved, but it can split between the atom and the photon in an infinite number of ways.

[5]Even though the energy of vacuum is ∞, the energy density is limited in local, and the question is that how it affect the curve.

[6]The zero-point energy we have met in cosmology, the Casimir effect.

[7]In fact, the result is not infinite but a finite calculable quantity known as the Lamb shift, which agrees perfectly with data. However, it is instructive to understand Oppenheimer's argument.

There was actually nothing wrong with Oppenheimer's calculation. He did get the answer that OFPT predicts. What he missed was that there are other infinities that eventually cancel this infinity (for example, the electron mass is infinite too, so in fact his conclusion was on the right track). This discussion was really just meant as a preview to demonstrate the complexities we will be up against. To sort out all these infinities, it will be really helpful, but not strictly necessary, to have a formalism that keeps the symmetries, in particular Lorentz invariance, manifest along the way. Although Schwinger was able to tame the infinities using OFPT, his techniques were not for everyone. In his own words, "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses" .

Chapter 5

Cross Sections and Decay Rates

In this chapter, the author provided a useful tool for us to understand the high-energy physics experiments which said that:

> The twentieth century witnessed the invention and development of collider physics as an efficient way to determine which particles exist in nature, their properties, and how they interact. In early experiments, such as Rutherford's discovery of the nucleus in 1911 using α-particles or Anderson's discovery of the positron in 1932 from cosmic rays, the colliding particles came from nature. *Colliders provide a great way to study fundamental interactions because they begin with initial states of essentially fixed momenta, i.e. plane waves, and end up with final states, which also have fixed momenta.*

The experimentally measurable quantities that are based on QM are differential probabilities. These probabilities are given by the modulus squared on inner products of states. We can write such inner products as $\langle f; t_f | i; t_i \rangle$, where $|i; t_i\rangle$ is the initial state we start with at time t_i and $\langle f; t_f|$ is the final state we are interested in at some later time t_f. Since quantum field theory is just quantum mechanics with lots of fields, the experimental quantities we will be able to predict are also of the form $|\langle f; t_f | i; t_i \rangle|^2$. One thing should be cleared that the notation of $\langle f; t_f | i; t_i \rangle$ refers to the Schrodinger picture representation, where the states evolve in time. In Heisenberg picture, which will be the default picture for quantum field theory, we leave the states alone and put all the time evolution into an operator. In the special case where we evolve momentum eigenstates from $t = -\infty$ to $t = +\infty$, relevant for collider physics applications, we give the time-evolution operator a special name: **the scattering or S-matrix**. The **S-matrix** is defined as

$$\langle f | S | i \rangle_{\text{Heisenberg}} = \langle f; \infty | i; -\infty \rangle_{\text{Schrodinger}} \tag{5.1}$$

The S-matrix is defined assuming that all of the things that change the state (the interactions) happen in a finite time interval, so that at asymptotic times, $t = \pm\infty$, the states are free of interactions. Free states at $t = \pm\infty$ are known as *asymptotic states*.

But in this chapter, we relate S-matrix elements to scattering cross sections, which are directly measured in collider experiments.

The cross section is a natural quantity to measure experimentally. The cross-sectional area, $\sigma = \pi r^2$, of the nucleus is given by

$$\sigma = \frac{\text{number of particles scattered}}{\text{time} \times \text{number density in beam} \times \text{velocity of beam}} = \frac{1}{T}\frac{1}{\Phi}N \tag{5.2}$$

where the T is the time of experiment and Φ is the incoming flux (exactly the number density $\times$ velocity of beam) In additionally, the number of scattering, N, is determined completely by the *short-distance* interactions among the particles.

The *differential cross section*, $\frac{d\sigma}{d\Omega}$, which gives the number of scattered particles in a certain solid angle $d\Omega$. Classically, this gives us information about the shape of the object or form of the potential off of which the α-particles are scattered[1].

In QM, taking the derivative of both, that says $d\sigma = \frac{1}{T}\frac{1}{\Phi}dN$, while the derivative has the natural meaning in QM, probabilities $P = N/N_{inc}$, where the N is the number of particles scattering into a given area and N_{inc} is the number of incident particles. So the quantum mechanical cross section is then naturally:

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP \tag{5.3}$$

where Φ is the flux. But we should notice that the equation is normalized as if the beam has just one particle, and P is now the quantum mechanical probability of scattering. The differential quantities $d\sigma$ and dP are differential in kinematical variables, such as the angles and energies of the final state particles. The differential number of scattering events measures in a collider experiment is;

$$dN = L \times d\sigma \tag{5.4}$$

where L is the *luminosity*, which is defined by this equation.

One more thing, we should to prove the item

$$\int \frac{d^3p}{2\omega_p} \tag{5.5}$$

is Lorentz invariant which is the measure in $\mathbb{M}^4$ and we have to prove some characters in δ function. At first, there appears a strange term of $1/2\omega_p$. How it apears? That should be a physical problem not a mathematical structure problem. So, the simplest Lorentz invariant in $\mathbb{M}^4$ (momentum phase space) is

$$\int d^4p \tag{5.6}$$

but we can restrict this from physics so we can change the form.

In consideration of limit of **on-shell condition**, we can add the δ function to restrict it. Notice that

$$\delta^4(p^\mu p_\mu - m^2) = \delta^4(p^0 p_0 - |\vec{p}|^2 - m^2) = \delta^4(p^0 p_0 - \omega_p^2) \tag{5.7}$$

[1]This supports a kind of the "microscope" of the shape or potential

where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$ or $E_p = \sqrt{|\vec{p}|^2 + m^2}$.

Also we have to restrict the energy. Because the $\omega_p = E_p = \sqrt{|\vec{p}|^2 + m^2}$, so we have to add the condition by $\theta(p^0)$ function to ensure the **positive solution**. Thus, the Lorentz invariant form is

$$\int d^4p\delta^4(p^0p_0 - \omega_p^2)\theta(p^0) \tag{5.8}$$

Now, wo introduce the character of δ function,

$$\delta\left(g(x)\right) = \sum_i \frac{\delta(x - x_i)}{g'(x_i)} \quad \text{where zero point } g(x_i) = 0 \tag{5.9}$$

Therefore, the zero points of $p^0p_0 - \omega_p^2$ are $p_1^0 = \omega_p$ and $p_2^0 = -\omega_p$, so the $\delta(p^0p_0 - \omega_p^2)$ can be written as

$$\delta^4(p^0p_0 - \omega_p^2) = \left[\frac{\delta^4(p_0 - \omega_p)}{2\omega_p} - \frac{\delta^4(p_0 + \omega_p)}{2\omega_p}\right] \tag{5.10}$$

where the second item implies the negative energy solution.

So, the computation :

$$\begin{aligned}
\int d^4p\delta^4(p^0p_0 - \omega_p^2)\theta(p_0) &= \int d^4p \left[\frac{\delta^4(p_0 - \omega_p)}{2\omega_p} - \frac{\delta^4(p_0 + \omega_p)}{2\omega_p}\right]\theta(p_0) \\
&= \int dp_0 \int d^3p \left[\frac{\delta^4(p_0 - \omega_p)}{2\omega_p} - \frac{\delta^4(p_0 + \omega_p)}{2\omega_p}\right]\theta(p_0) \\
&= \int dp_0 \int d^3p \frac{\delta(p_0 - \omega_p)}{2\omega_p}\theta(p_0) \\
&= \int \frac{d^3p}{2\omega_p}
\end{aligned} \tag{5.11}$$

Therefore,

$$\int \frac{d^3p}{2\omega_p} \tag{5.12}$$

is Lorentz invariant.

Now, let's turn back to our main topic.

to be supplemented

Chapter 6

The S-Matrix and Time-Ordered Products

In this chapter here, our main topic is the LSZ (Lehmann-Symanzik-Zimmermann) reduction formula. As discussed in Chapter 5, scattering experiments have been an efficient way to determine the particles that exist and how they interact. As we know, all of the interesting interacting physics is encoded in how often given initial states produce given final states, that is, in the *S-matrix*. We assume that the process happened in finite time which fitting to the collider scattering experiments. This also means that if there were always interactions, it would not be possible to set up our initial states at $t = -\infty$ or find the desired final states at $t = +\infty$. Without interactions at asymptotic times, the states we scatter can be defined as on=shell one-particle states of given momenta, known as **asymptotic states**. We gonna to derive an expression for the S-matrix using only that the system is free at asymptotic times. And after that, we will derive and use the Feynman rules which is a powerful weapon to the interacting theory.

Before the driving, we give the LSZ reduction formula at first. The LSZ relates the S-matrix elements $\langle f|\, S\, |i\rangle$ for n asymptotic momentum eigenstates to an expression involving the quantum fields $\phi(x)$:

$$\boxed{\langle f|\, S\, |i\rangle = \left[i\int d^4x_1 e^{-ip_1x_1}(\Box + m^2)\right]\cdots\left[i\int d^4x_n e^{ip_nx_n}(\Box + m^2)\right] \times \langle\Omega|\, T\{\phi(x_1)\phi(x_2)\phi(x_3)\cdots\phi(x_n)\}\, |\Omega\rangle} \tag{6.1}$$

to be supplemented.

the detailed content is on my handwritten notebook.

Chapter 7

Feynman Rules

We have already seen that the scattering cross sections are naturally expressed in terms of time-ordered products of fields. The S-matrix has the form

$$\langle f|\, S\, |i\rangle \sim \langle \Omega|\, T\{\phi(x_1)\cdots\phi(x_n)\}\, |\Omega\rangle \tag{7.1}$$

where $|\Omega\rangle$ is the ground state or vacuum in the interacting theory. In this expression the fields $\phi(x)$ are not free but are the full interacting quantum fields. We also know that the time-ordered product of two fields is given by the Feynman propagator in the free theory:

$$D_F(x,y) \equiv \langle 0|\, T\{\phi_0(x)\phi_0(y)\}\, |0\rangle = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x-y)} \tag{7.2}$$

where $|0\rangle$ is the ground state in the free theory.

There are position-space Feynman rules, for calculating time-ordered products, and also momentum-space Feynman rules, for calculating S-matrix elements. The momentum-space Feynman rules are by far the more important- they provide an extremely effcient way to set up calculations of physical results in quantum field theory. The momentum-space Feynman rules are the main result of this Part I.

At first, we will derive the Feynman rules from the Lagrangian formulation of time evolution and quantization which is the quickese way to connect Feynman diagrams to classical field theory. Then, we will derive it based on expansion of the full interacting Hamiltonian around the free Hamiltonian.

In chapter 2, we got the free field theory which implies two top priority formula:

$$[\phi(\vec{x},t),\phi(\vec{x}',t)] = 0 \tag{7.3}$$

$$[\phi(\vec{x},t),\partial_t\phi(\vec{x}',t)] = i\hbar\delta^3(\vec{x}-\vec{x}') \tag{7.4}$$

which called the equal-time commutation relation.

However, we used to set that $\hbar = c = 1$ in previous chapters. The $\hbar$ here in (7.4) just imply the different from classical limit. And we used to derive the free field eqution $(\Box + m^2)\phi = 0$ by the Euler-Lagrange equation from field theory. While in an interacting theory, we must

generalize these eqution to specify how the dynamics is determined. Then we the quantizied Hesenberg motion eqution from quantum mechanics

$$i\frac{\partial}{\partial t}\phi(x) = [\phi, H] \tag{7.5}$$

which we regard it as one natural approach for an interacting quantum field theory lead to the Hamiltoian derivation of the Feynman rules in the later content.

Our first topic will be developed by the Lagrange approach where the Hamilton's equations are replaced by the Euler-Lagrange equations (we will simpfy the note by "E-L equations") derived from a Lagrangian $\mathcal{L}$ and the generalization of $(\Box + m^2)\phi = 0$. We also assume (7.3) and (7.4) are still satisfied[1]. This is a natural assumption, since the Hilbert space for the interacting theory is the same as the free theory. I think this is very physical which we can understand from physics. The equal-time communication relation of (7.3) implies that the **causality** is satisfied in quantum field theory. Because at the same time but at different points in space, all operators (or say physical quantity), in particular fields, should be simultaneously observable and commute otherwise our causality will broke up.

Now, let us talk about the (7.4). This formula can be seen as QFT version of the basic communtation relation in quantum mechanics of $[\hat{x}, \hat{p}] = i\hbar$. We can understand the $\phi(x)$ is the QFT version of x since we can know that $\phi_0(x)\left|0\right\rangle \sim \left|x\right\rangle$ and the $\hat{p}$ is also. In another word, the (7.4) is the equivalent of the canonical commutation relation from the quantum mechanics $[\hat{x}, \hat{p}] = i\hbar$. It indicates that a quantity and its time derivative are not simultaneously observable, the hallmark of the **uncertainty principle**.

We have known how to calculate $\left\langle 0\right| T\{\phi(x)\phi(x')\}\left|0\right\rangle$ in the free theory. Now we calculate this commutator in an interacting theory. Firstly, we prove the formula below:

$$\left(\Box + m^2\right)\left\langle \Omega\right| T\{\phi(x)\phi(x')\}\left|\Omega\right\rangle = \left\langle \Omega\right| T\{\left(\Box + m^2\right)\phi(x)\phi(x')\}\left|\Omega\right\rangle - i\hbar\delta^4(x - x')$$

where the $\left|\Omega\right\rangle$ is the vacuum in the interacting theory. The $\delta^4(x - x')$ on right side is critically important. It signifies the difference between the classical and quantum theories in a way that will be clear shortly. Notice that:

$$\begin{aligned}
\partial_t \left\langle \Omega\right| T\{\phi(x)\phi(x')\}\left|\Omega\right\rangle &= \partial_t \left[\left\langle \Omega\right| \phi(x)\phi(x')\left|\Omega\right\rangle \theta(t - t') + \left\langle \Omega\right| \phi(x')\phi(x)\left|\Omega\right\rangle \theta(t' - t)\right] \\
&= \left\langle \Omega\right| \partial_t\phi(x)\phi(x')\left|\Omega\right\rangle \theta(t - t') + \left\langle \Omega\right| \phi(x')\partial_t\phi(x)\left|\Omega\right\rangle \theta(t' - t) \\
&+ \left\langle \Omega\right| \phi(x)\phi(x')\left|\Omega\right\rangle \partial_t\theta(t - t') + \left\langle \Omega\right| \phi(x')\phi(x)\left|\Omega\right\rangle \partial_t\theta(t' - t) \\
&= \left\langle \Omega\right| T\{\partial_t\phi(x)\phi(x')\}\left|\Omega\right\rangle + \left\langle \Omega\right| \phi(x)\phi(x')\left|\Omega\right\rangle \delta(t - t') - \left\langle \Omega\right| \phi(x')\phi(x)\left|\Omega\right\rangle \delta(t - t') \\
&= \left\langle \Omega\right| T\{\partial_t\phi(x)\phi(x')\}\left|\Omega\right\rangle + \left\langle \Omega\right| \left[\phi(x)\phi(x') - \phi(x')\phi(x)\right]\left|\Omega\right\rangle \delta(t - t') \\
&= \left\langle \Omega\right| T\{\partial_t\phi(x)\phi(x')\}\left|\Omega\right\rangle + \left\langle \Omega\right| \left[\phi(x), \phi(x')\right]\delta(t - t') \\
&= \left\langle \Omega\right| T\{\partial_t\phi(x)\phi(x')\}\left|\Omega\right\rangle + 0 \\
&= \left\langle \Omega\right| T\{\partial_t\phi(x)\phi(x')\}\left|\Omega\right\rangle
\end{aligned} \tag{7.6}$$

[1] We have do proof of this through GP. Zhang's GFT course

and we used the $\partial_x\theta(x) = \delta(x)$ and the $\delta(t-t')$ forces the $t = t'$ therefore there exists the equal time and we use the equal time commutator $[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$ leading the vanish of the second term from above. Simply,

$$\partial_t \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle = \langle\Omega| T\{\partial_t\phi(x)\phi(x')\} |\Omega\rangle \tag{7.7}$$

Take the second derivative:

$$\begin{aligned}
\partial_t^2 \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle &= \partial_t \langle\Omega| T\{\partial_t\phi(x)\phi(x')\} |\Omega\rangle \\
&= \partial_t \left[\langle\Omega| \partial_t\phi(x)\phi(x') |\Omega\rangle \theta(t-t') + \langle\Omega| \phi(x')\partial_t\phi(x) |\Omega\rangle \theta(t'-t)\right] \\
&= \langle\Omega| T\{\partial_t^2\phi(x)\phi(x')\} |\Omega\rangle + \langle\Omega| \partial_t\phi(x)\phi(x') - \phi(x')\partial_t\phi(x) |\Omega\rangle \delta(t-t') \\
&= \langle\Omega| T\{\partial_t^2\phi(x)\phi(x')\} |\Omega\rangle + \langle\Omega| [\partial_t\phi(x), \phi(x')] |\Omega\rangle \delta(t-t') \\
&= \langle\Omega| T\{\partial_t^2\phi(x)\phi(x')\} |\Omega\rangle - i\hbar\delta^3(\vec{x}-\vec{x}')\delta(t-t') \\
&= \langle\Omega| T\{\partial_t^2\phi(x)\phi(x')\} |\Omega\rangle - i\hbar\delta^4(x-x')
\end{aligned} \tag{7.8}$$

where we have used $\partial_t\phi(x') = 0$ and the $\delta(t-t')$ forces the equal time again and we can use the equal-time commutator $[\phi(x), \partial_t\phi(x')] = i\hbar\delta^3(\vec{x}-\vec{x}')$ and the normalization of $\langle\Omega|\Omega\rangle = 1$.

Here, we just get the relation of $\partial^2/\partial t^2$ but our goal is $\Box + m^2 = \partial_t^2 - \partial_x^2 + m^2$. The m^2 can be seen as a constant, so we can insert the m^2 in anywhere. Then we have to deal with the second derivative of space ∂_x^2 or ∇^2. Same as the above, we have

$$\begin{aligned}
\partial_x \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle &= \partial_x \left[\langle\Omega| \phi(x)\phi(x') |\Omega\rangle \theta(t-t') + \langle\Omega| \phi(x')\phi(x) |\Omega\rangle \theta(t'-t)\right] \\
&= \langle\Omega| \partial_x^2\phi(x)\phi(x') |\Omega\rangle \theta(t-t') + \langle\Omega| \phi(x')\partial_x^2\phi(x) |\Omega\rangle \delta(t'-t) \\
&= \langle\Omega| T\{\partial_x^2\phi(x)\phi(x')\} |\Omega\rangle
\end{aligned} \tag{7.9}$$

where we have used the formula $\partial_x\phi(x') = 0$ and $\partial_x\theta(t-t') = 0$.

Above all, we get that

$$\left(\Box + m^2\right) \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle = \langle\Omega| T\{\left(\Box + m^2\right)\phi(x)\phi(x')\} |\Omega\rangle - i\hbar\delta^4(x-x') \tag{7.10}$$

For example, in the free theory, $\langle 0| T\{\phi_0(x)\phi_0(y)\} |0\rangle = \hbar D_F(x,y)$ and $\left(\Box + m^2\right)\phi_0(X) = 0$, then (7.10) implies

$$\left(\Box_x + m^2\right) D_F(x,y) = -\delta^4(x-y) \tag{7.11}$$

To simplefy the equation expression, we note $\langle\cdots\rangle = \langle\Omega| T\{\cdots\} |\Omega\rangle$ for time-ordered correlation functions in the interacting theory, (7.10) can be written as

$$\left(\Box + m^2\right) \langle\phi(x)\phi(x')\rangle = \langle\left(\Box + m^2\right)\phi(x)\phi(x')\rangle - i\hbar\delta^4(x-x') \tag{7.12}$$

Let's insert more fields which can be expressed as a normal form

$$\Box_x \langle\phi(x)\phi(x_1)\cdots\phi(x_n)\rangle = \langle\Box_x\phi(x)\phi(x_1)\cdots\phi(x_n)\rangle - i\hbar\sum_j \delta^4(x-x_j) \langle\phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)\rangle$$

It is not difficult to imagine that the above formula should be summarized from some law. So we calculate the 3 fields $\Box_x \langle\phi(x)\phi(x_1)\phi(x_2)\rangle$ which the M.D. Schwartz recommend we

calculating it. Then, we can summarize the law from the 3 fields and our orginal 2 field formula. Similar to the 2 field formula, the $\Box_x = \partial_t^2 - \partial_x^2$, is essential to our calculating especially the time second derivative which will calculate on the $\theta(t)$ part while the ∂_x does not.

For the time-order product of 3 fields, they will appear the permutation problem. So we simply note the time first derivative

$$\begin{aligned}\partial_t \langle\phi(x)\phi(x_1)\phi(x_2)\rangle &= \partial_t \langle\Omega| T\{\phi(x)\phi(x_1)\phi_(x_2)\} |\Omega\rangle \\ &= \sum_{\text{Permutation}}^{A_3^3=6} \partial_t \left[\langle\Omega| \phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \theta(t-t_1)\theta(t_1-t_2)\right]\end{aligned} \tag{7.13}$$

Now, foucs on the single kind "$\phi(x)\phi(x_1)\phi(x_2)$":

$$\begin{aligned}&\partial_t \left[\langle\Omega| \phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \theta(t-t_1)\theta(t_1-t_2)\right] \\ &= \langle\Omega| \partial_t\phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \theta(t-t_1)\theta(t_1-t_2) + \langle\Omega| \phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \delta(t-t_1)\theta(t_1-t_2)\end{aligned} \tag{7.14}$$

which we need to emphasis that the derivative of t or x only take effect on the quantity involve the t and x. Any permutations and combinations both have the similar form as above:

$$\langle\Omega| \partial_t\phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \theta(t-t_1)\theta(t_1-t_2) \tag{P.1}$$

$$\langle\Omega| \partial_t\phi(x)\phi(x_2)\phi(x_1) |\Omega\rangle \theta(t-t_2)\theta(t_2-t_1) \tag{P.2}$$

$$\langle\Omega| \phi(x_1)\partial_t\phi(x)\phi(x_2) |\Omega\rangle \theta(t_1-t)\theta(t-t_2) \tag{P.3}$$

$$\langle\Omega| \phi(x_1)\phi(x_2)\partial_t\phi(x) |\Omega\rangle \theta(t_1-t_2)\theta(t_2-t) \tag{P.4}$$

$$\langle\Omega| \phi(x_2)\partial_t\phi(x)\phi(x_1) |\Omega\rangle \theta(t_2-t)\theta(t-t_1) \tag{P.5}$$

$$\langle\Omega| \phi(x_2)\phi(x_1)\partial_t\phi(x) |\Omega\rangle \theta(t_2-t_1)\theta(t_1-t) \tag{P.6}$$

Notice that, the derivative of x in this formula gives $\partial_x\phi(x_j) = 0$ and $\partial_x\theta(t) = 0$. This means that the derivative of x can be inserted in angwhere in the formula. And the ∂_x^2 is so. We can directly insert the ∂_x^2 in it while ∂_t^2 does not. Then we can say: the d 'Alembert operator can be written as the follow:

$$\Box_x \langle\phi(x)\phi(x_1)\phi(x_2)\rangle = \langle\Box_x\phi(x)\phi(x_1)\phi(x_2)\rangle + \text{Remainder}\{t\} \tag{7.15}$$

which the Remainder$\{t\}$ implies that the remainder is caused by time derivative.

Since the $\Box_x$ in the second derivative especially the ∂_t^2, we can decompose the reaminder as

$$\text{Remainder}\{t\} = \partial_t \,(\text{first remainder}) + (\text{second remainder}) \tag{7.16}$$

from the time first derivative and the second derivative.

Let's check the time first derivative. We can easily write the first remainder by taking the time derivative of (P.1)$\sim$(P.6). Then, as for the ∂_t, we have

$$\langle\Omega| \phi(x)\phi(x_1)\phi(x_2) |\Omega\rangle \delta(t-t_1)\theta(t_1-t_2) \tag{F.1}$$

$$\langle\Omega|\,\phi(x)\phi(x_2)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_2-t_1) \tag{F.2}$$

$$\langle\Omega|\,\phi(x_1)\phi(x)\phi(x_2)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_1-t) \tag{F.3}$$

$$\langle\Omega|\,\phi(x_2)\phi(x)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_1)\theta(t_2-t) \tag{F.4}$$

$$-\langle\Omega|\,\phi(x_1)\phi(x_2)\phi(x)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_1-t_2) \tag{F.5}$$

$$-\langle\Omega|\,\phi(x_2)\phi(x_1)\phi(x)\,|\Omega\rangle\,\delta(t-t_1)\theta(t_2-t_1) \tag{F.6}$$

$$-\langle\Omega|\,\phi(x_1)\phi(x)\phi(x_2)\,|\Omega\rangle\,\delta(t-t_1)\theta(t-t_2) \tag{F.7}$$

$$-\langle\Omega|\,\phi(x_2)\phi(x)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_2)\theta(t-t_1) \tag{F.8}$$

which we have used the $\partial_x\theta(x)=\delta(x)$ and we order all t be front in the δ function legally.

Samely, the remainder caused by ∂_t^2 can be listed as followings (just replace the $\phi(x)$ to $\partial_t\phi(x)$):

$$\langle\Omega|\,\partial_t\phi(x)\phi(x_1)\phi(x_2)\,|\Omega\rangle\,\delta(t-t_1)\theta(t_1-t_2) \tag{S.1}$$

$$\langle\Omega|\,\partial_t\phi(x)\phi(x_2)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_2-t_1) \tag{S.2}$$

$$\langle\Omega|\,\phi(x_1)\partial_t\phi(x)\phi(x_2)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_1-t) \tag{S.3}$$

$$\langle\Omega|\,\phi(x_2)\partial_t\phi(x)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_1)\theta(t_2-t) \tag{S.4}$$

$$-\langle\Omega|\,\phi(x_1)\phi(x_2)\partial_t\phi(x)\,|\Omega\rangle\,\delta(t-t_2)\theta(t_1-t_2) \tag{S.5}$$

$$-\langle\Omega|\,\phi(x_2)\phi(x_1)\partial_t\phi(x)\,|\Omega\rangle\,\delta(t-t_1)\theta(t_2-t_1) \tag{S.6}$$

$$-\langle\Omega|\,\phi(x_1)\partial_t\phi(x)\phi(x_2)\,|\Omega\rangle\,\delta(t-t_1)\theta(t-t_2) \tag{S.7}$$

$$-\langle\Omega|\,\phi(x_2)\partial_t\phi(x)\phi(x_1)\,|\Omega\rangle\,\delta(t-t_2)\theta(t-t_1) \tag{S.8}$$

Now, recall the method we used in the 2 fields problem. We constructed the equal-time commutator to simplify the formula and remove the awful terms. Now, we have eight equations from (F.1) to (F.8) and eight equations from (S.1) to (S.8). We try to constuct the commutator. To make this easier, we omit the $|\Omega\rangle$ and $\langle\Omega|$ symbols and use two three-dimensional vector symbols to represent that they are equal in time like that $\phi(\vec{x})$ and $\phi(\vec{x}_1)$ are equal time.

We observe the (F.1)$\sim$(F.8) and we can find that the $\delta(t-t_1)$ enforce the equal time $t=t_1$. Then we can combine the (F.1), (F.4), (F.6) and (F.7) and give

$$\begin{aligned}&\phi(\vec{x})\phi(\vec{x}_1)\phi(x_2)\theta(t-t_2)-\phi(\vec{x}_1)\phi(\vec{x})\phi(x_2)\theta(t-t_2)+\phi(x_2)\phi(\vec{x})\phi(\vec{x}_1)\theta(t_2-t)-\phi(x_2)\phi(\vec{x}_1)\phi(\vec{x})\theta(t_2-t)\\&=\left[\phi(\vec{x}),\phi(\vec{x}_1)\right]\phi(x_2)\theta(t-t_2)+\phi(x_2)\left[\phi(\vec{x}),\phi(\vec{x}_1)\right]\theta(t_2-t)\end{aligned} \tag{7.17}$$

equally, the $\delta(t-t_2)$ enforces the equal time $t=t_2$, and we have

$$\begin{aligned}&\phi(\vec{x})\phi(\vec{x}_2)\phi(x_1)\theta(t-t_1)-\phi(\vec{x}_2)\phi(\vec{x})\phi(x_1)\theta(t-t_1)+\phi(x_1)\phi(\vec{x})\phi(\vec{x}_2)\theta(t_1-t)-\phi(x_1)\phi(\vec{x}_2)\phi(\vec{x})\theta(t_1-t)\\&=\left[\phi(\vec{x}),\phi(\vec{x}_2)\right]\phi(x_1)\theta(t-t_1)+\phi(x_1)\left[\phi(\vec{x}),\phi(\vec{x}_2)\right]\theta(t_1-t)\end{aligned} \tag{7.18}$$

while the equal-time commutators

$$\begin{aligned}[\phi(\vec{x}),\phi(\vec{x}_1)]=[\phi(\vec{x},t),\phi(\vec{x}_1,t)]=0\\ [\phi(\vec{x}),\phi(\vec{x}_2)]=[\phi(\vec{x},t),\phi(\vec{x}_2,t)]=0\end{aligned} \tag{7.19}$$

imply

$$\text{first remainder}=0 \tag{7.20}$$

namely

$$\partial_t(\text{first remainder})=0 \tag{7.21}$$

This proves that our mind of constucting the commutator is success. Then we now deal with the second remainder. Equally, we just replace the $\phi(x)$ to $\partial_t\phi(x)$ as we already seen from (S.1) to (S.8) and replace the commutator relation to $[\phi(\vec{x},t),\partial_t(\vec{x}',t)]=i\hbar\delta^3(\vec{x}-\vec{x}')$. We give back the $\delta(t-t_1)$ and $\delta(t-t_2)$ and re-write the (7.17) and (7.18). Then, we get

$$\begin{aligned}&[\phi(\vec{x}),\phi(\vec{x}_1)]\,\phi(x_2)\delta(t-t_1)\theta(t-t_2)+\phi(x_2)\,[\phi(\vec{x}),\phi(\vec{x}_1)]\,\delta(t-t_1)\theta(t_2-t)\\ =&-i\hbar\delta^3(\vec{x}-\vec{x}_1)\delta(t-t_1)\phi(x_2)\theta(t-t_2)-i\hbar\delta^3(\vec{x}-\vec{x}_1)\phi(x_2)\theta(t_2-t)\\ =&-i\hbar\delta^4(x-x_1)\,[\phi(x_2)\theta(t-t_2)+\phi(x_2)\theta(t_2-t)]\end{aligned} \tag{7.22}$$

and

$$\begin{aligned}&[\phi(\vec{x}),\phi(\vec{x}_2)]\,\phi(x_1)\delta(t-t_2)\theta(t-t_1)+\phi(x_1)\,[\phi(\vec{x}),\phi(\vec{x}_2)]\,\delta(t-t_2)\theta(t_1-t)\\ =&-i\hbar\delta^4(x-x_2)\,[\phi(x_1)\theta(t-t_1)+\phi(x_1)\theta(t_1-t)]\end{aligned} \tag{7.23}$$

They show us that the second remainder have such a construction

$$\text{Remainder}\{t\}=\text{second remainder}=-i\hbar\sum_j\delta^4(x-x_j)\big[\text{time ordered product, no }\phi(x)\text{ and no }\phi(x_j)\big] \tag{7.24}$$

However, we just solve the 3 fields' $\Box_x\langle\phi(x)\phi(x_1)\phi(x_2)\rangle$. The time ordered product just implies one field's time order. But, let's recall the 2 field formula of (7.10) then we can summarize the n-fields' formula from the law of consturction of communtators:

$$\Box_x\langle\phi(x)\phi(x_1)\cdots\phi(x_n)\rangle=\langle\Box_x\phi(x)\phi(x_1)\cdots\phi(x_n)\rangle-i\hbar\sum_j\delta^4(x-x_j)\langle\phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)\rangle \tag{7.25}$$

which is useful!

We assunme that the quantum field satisfies the same equations of motion as the classical field we developed in Chapter 3. In particular, if our Lagrangian has the form

$$\mathcal{L}=-\frac{1}{2}\phi(\Box+m^2)\phi+\mathcal{L}_{\text{int}}[\phi] \tag{7.26}$$

then the motion equation developed from E-L equation is

$$\left(\Box+m^2\right)\phi-\mathcal{L}'_{\text{int}}[\phi]=0 \tag{7.27}$$

where $\mathcal{L}_{\text{int}}[\phi] = d\mathcal{L}_{\text{int}}[\phi]/d\phi$.

Then, we can get the following with (7.25):

$$\boxed{\left(\Box_x + m^2\right)\langle\phi_x\phi_1\cdots\phi_n\rangle = \langle\mathcal{L}'_{\text{int}}[\phi_x]\phi_1\cdots\phi_n\rangle - i\hbar\sum_j \delta^4(x-x_j)\langle\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_n\rangle} \tag{7.28}$$

where $\phi_x = \phi(x)$ and $\phi_j = \phi(x_j)$.

The (7.28) is called **Schwinger-Dyson equations**.

> *The Schwinger–Dyson equations encode the difference between the classical and quantum theories. Note that their derivation did not require any specification of the dynamics of the theory, only that the canonical commutation relations in* (7.19) *are satisfied. In particular, in a classical theory,* $[\phi(x',t), \partial_t\phi(x,t)] = 0$ *and therefore classical timeordered correlation functions would satisfy a similar equation but without the* $\delta^4(x-x_j)$ *terms (i.e.* $\hbar = 0$*). That is, in a classical theory, correlation functions satisfy the same differential equations as the fields within the correlation functions. In a quantum theory, that is true only up to* δ*-functions, which in this context are also called contact interactions. These contact interactions allow virtual particles to be created and destroyed, which permits closed loops to form in the Feynman diagrammatic expansion, as we will now see.*

Now, we will talk about the *position-space Feynman rules*. The Schweinger-Dyson equation specify a completely non-perturbative relationship among correlation functions in the fully theory. We have known that the m^2 can be inserted in anywhere since this is a constant scalar, so we focus on the $\Box$ term and we set the $\hbar = 1$ to reveal the quantum world. We know that

$$\Box_x D_{x1} = -i\delta_{x1} \tag{7.29}$$

for which we note that $\delta_{xi} = \delta^4(x - x_i)$ and $D_{ij} = D_{ji} = D_F(x,y)$. And maybe you can realize that we can represent something via the free theory generalizations. In the free theorym, as an example, we can write the 2-point function as

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= \int d^4x\delta_{1x}\langle\phi_x\phi_2\rangle \\ &= i\int d^4x(\Box_x D_{x1})\langle\phi_x\phi_2\rangle\end{aligned} \tag{7.30}$$

To fit with the Schwinger-Dyson equation, we have to use the skills of intergrating by part. Same as we do in the past, we drop out all the boundary items and the will appear the -1 when we exchange the derivatived object. And notice that the $\Box$ is the second derivative operator, so there will appear -1 twice namely nothing just exchanging the object.

Thus, we have

$$\langle\phi_1\phi_2\rangle = i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\rangle \tag{7.31}$$

With the Schwinger-Dyson equation, the equation above can be written as

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\rangle \\ &= i\int d^4x D_{x1}(-i)\delta_{x2} \\ &= \int d^4x D_{x1}\delta_{x2} \\ &= D_{12}\end{aligned} \tag{7.32}$$

Similarly, the 4-points function can be written directly as

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= \int d^4x\delta_{x1}\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= i\int d^4x(\Box_x D_{x1})\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= i\int d^4x D_{x1}\big(-i\delta_{x2}\langle\phi_3\phi_4\rangle - i\delta_{x3}\langle\phi_2\phi_4\rangle - i\delta_{x4}\langle\phi_2\phi_3\rangle\big) \\ &= \int d^4x D_{x1}\big(\delta_{x2}D_{34}+\delta_{x3}D_{24}+\delta_{x4}D_{23}\big) \\ &= D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23}\end{aligned} \tag{7.33}$$

and the corresponded diagrams show as below

to be supplemented, the Feynman diagrams

Now, it's time to consider the interactions. For example, we consider the Lagrangian as below

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \frac{g}{3!}\phi^3 \tag{7.34}$$

which tells us the $\mathcal{L}'_{\text{int}}$ from the S-D equation that

$$\mathcal{L}'_{\text{int}} = \frac{g}{2}\phi^2 \tag{7.35}$$

Same as the above, we can get

$$\langle\phi_1\phi_2\rangle = i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\rangle \tag{7.36}$$

and

$$\begin{aligned}\Box_x\langle\phi_x\phi_2\rangle &= \langle\mathcal{L}'_{\text{int}}\phi_2\rangle - i\delta_{x2} \\ &= \frac{g}{2}\langle\phi_x^2\phi_2\rangle - i\delta_{x2}\end{aligned} \tag{7.37}$$

which both imply that

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4x D_{x1}\big(\frac{g}{2}\langle\phi_x^2\phi_2\rangle - i\delta_{x2}\big)\\ &= D_{12} + i\frac{g}{2}\int d^4x D_{x1}\langle\phi_x^2\phi_2\rangle\\ &= D_{12} + i\frac{g}{2}\int d^4x D_{x1}\int d^4y\delta_{2y}\langle\phi_x^2\phi_y\rangle\\ &= D_{12} + i\frac{g}{2}\int d^4x d^4y D_{x1} i(\Box_y D_{2y})\langle\phi_x^2\phi_y\rangle\\ &= D_{12} - \frac{g}{2}\int d^4x d^4y D_{x1}D_{2y}\Box_y\langle\phi_x^2\phi_y\rangle\end{aligned} \tag{7.38}$$

where the $\langle\phi_x^2\phi_y\rangle$ can be seen as $\langle\phi_x\phi_x\phi_y\rangle$.

Thus, we can move on

$$\begin{aligned}\Box_y\langle\phi_x^2\phi_y\rangle &= \Box_y\langle\phi_y\phi_x\phi_x\rangle\\ &= \frac{g}{2}\langle\phi_x^2\phi_y^2\rangle - 2i\delta_{xy}\langle\phi_x\rangle\end{aligned} \tag{7.39}$$

and we can get the new form of $\langle\phi_1\phi_2\rangle$

$$\langle\phi_1\phi_2\rangle = D_{12} + ig\int d^4x D_{1x}D_{2x}\langle\phi_x\rangle - \frac{g^2}{4}\int d^4x d^4y D_{1x}D_{2y}\langle\phi_x^2\phi_y^2\rangle \tag{7.40}$$

We have dealed with the 4-points functions. The $\langle\phi_x^2\phi_y^2\rangle$ can be rewritten as

$$\begin{aligned}\langle\phi_x^2\phi_y^2\rangle &= \langle\phi_x\phi_x\phi_y\phi_y\rangle\\ &= 2D_{xy}^2 + D_{xx}D_{yy}\end{aligned} \tag{7.41}$$

then the g^2 term can be written as

$$-\frac{g^2}{4}\int d^4x d^4y D_{1x}D_{2y}(2D_{xy}^2 + D_{xx}D_{yy}) \tag{7.42}$$

Now, let's handle the g term, where the main goal is the $\langle\phi_x\rangle$. We use the same skill. Then,

$$\begin{aligned}\langle\phi_x\rangle = \int d^4y\delta_{xy} &= \int d^4y i\Box_y D_{xy}\langle\phi_y\rangle\\ &= i\int d^4y D_{xy}\Box_y\langle\phi_y\rangle\\ &= i\int d^4y D_{xy}\langle\frac{g}{2}\phi_y^2\rangle\\ &= i\frac{g}{2}\int d^4y D_{xy}\langle\phi_y\phi_y\rangle\end{aligned} \tag{7.43}$$

and we just use the (7.40) again. But if we are only interested in the g^2 term, then we just keep the D_{yy} term to fit with the order of polynomial. Namely, we have

$$\langle\phi_y\phi_y\rangle = i\frac{g}{2}\int d^4y D_{xy}D_{yy} \tag{7.44}$$

and insert it back to (7.40). Thus, we have the final form

$$\langle\phi_1\phi_2\rangle = D_{12} - g^2\int d^4x d^4y\big(\frac{1}{2}D_{1x}D_{xy}^2D_{y2} + \frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2} + \frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}\big) \tag{7.45}$$

to be supplemented, the Feynman diagrams

Now I need to quote Schwartz to illustrate the problem with these Feynman diagrams:*From these examples, and looking at the pictures, it is easy to infer the way the perturbative expansion will work for higher-order terms or more general interactions.*

1. *Start with (external) points x_i for each position at which fields in the correlation function are evaluated. Draw a line from each point.*

2. *A line can then either contract to an existing line, giving a Feynman propagator connecting the endpoints of the two lines, or it can split, due to an interaction. A split gives a new (internal) vertex proportional to the coefficient of $\mathcal{L}'_{int}[\phi]$* times i and new lines corresponding to the fields in $\mathcal{L}'_{\text{int}}[\phi]$

3. *At a given order in the perturbative couplings, the result is the sum of all diagrams with all the lines contracted, integrated over the positions of internal vertices.*

4. *Drop all the $n!$ factors in the coefficient of the interaction, but then divide by the geometrical symmetry factor for each diagram.*

These are the **Positon-Space Feynman Rules**. Actually, the first three pieces are very pictorial which can help us to quickly understand the propagtors in a view way. The last piece help us to determain the numerical factor.

To determine the numerical factor, it is conventional to write interactions normalized by the number of permutations of identical fields, for example the form of below

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4!}\phi^4, \quad \frac{g}{3!}\phi^3, \quad \frac{\kappa}{5!3!2!}\phi_1^5\phi_2^3\phi_3^2, \quad \cdots \tag{7.46}$$

When the derivative is taken to turn the interaction into a vertex, the prefactor becomes $\frac{1}{(n-1)!}$. The $(n-1)!$ term is then canceled by the number of permutations of the lines coming out of the vertex, not including the line coming in. And in some cases, such as the real scalar fields, some of the permutations give the same amplitude. (Read the content of this part first, and I will come back to supplement the specific understanding after I calculate the Feynman diagram. Also the Symmetry Diagrams)

Now, we reproduce the position-space Feynman rules via the time-dependent perturbation theory under the Hamilton picture. A natural assumption is that the motion equation is replaced by Heisenberg equations of motion

$$i\frac{\partial}{\partial t}\phi(x) = [\phi, H] \tag{7.47}$$

and the formal solution of this equation is

$$\phi(t, \vec{x}) = S^\dagger(t, t_0)\phi(\vec{x})S(t, t_0) \tag{7.48}$$

where the $S(t, t_0)$ is the time-evolution operator (the S-matrix) satisfies

$$i\frac{\partial}{\partial t}S(t, t_0) = H(t_0)S(t, t_0) \tag{7.49}$$

These are the dynamical equations in the Heisenberg picture where all the time dependence is in operators. States including the vacuum state $|\Omega\rangle$ in the Heisenberg picture are, by definition, time independent. As mentioned in Chapter 2, the Hamiltonian can either be defined at any given time as a functional of the fields $\phi(\vec{x})$ and $\pi(\vec{x})$ or equivalently as a functional of the creation and annihilation operators $a_p^\dagger$ and a_p. *We will not need an explicit form of the Hamiltonian for this derivation so we just assume it is some time-dependent operator* $H(t)$.

For the perturbation theory, we decompose the Hamiltonian by two parts—the exactly sovled part H_0 and the small perturbation part $V(t)$, namely

$$H(t) = H_0 + V(t) \tag{7.50}$$

As for an example, we deal with the ϕ^3 field to illstrate the content. The $V(t)$, can be written as

$$V(t) = \int d^3x \frac{g}{3!}\phi^3(t, \vec{x}) \tag{7.51}$$

What we need to notice is that the operators $\phi(t, \vec{x})$, H, H_0 and $V(t)$ are all based on the Heisenberg picture[2] But now, we are going to review the **Interaction Picture**[3].

In the gauge field theory lectures from Guangpeng Zhang, it said that it is convention to calculate the elements of S-matrix by the definition of scattering amplitude in the Interaction Picture. To clearly the time-evolution of the state in Schrodinger picture

$$i\frac{\partial}{\partial t}|\psi, t\rangle_S = H_S |\psi, t\rangle_S \tag{7.52}$$

where the H_S is the total Hamiltonian of the sysytem and conserved. Also, we have directly to know that it is Hermitian, namely $H = H^\dagger$. And in the picture of Schrodinger, the state is time-independent, while the quantity of physics be not. To transfer to the Heisenberg Picture, we take the unitary transformation:

$$|\psi\rangle_H \equiv e^{iH_S t}|\psi, t\rangle_S, \quad O_H(t) \equiv e^{iH_S t}O_S e^{-iH_S t} \tag{7.53}$$

which we can easily to prove that

$$i\frac{\partial}{\partial t}|\psi\rangle_H = 0 \quad i\frac{\partial}{\partial t}O_H(t) = [O_H(t), H_S] \tag{7.54}$$

Even though that the general operators is time dependent in Heisenberg picture, the H itself is time-independent as in the Schrodinger picture, namely noted as $H_S = H_H \equiv H$.

Now, we are back to our note. As in the (7.48), we can have thar the Heisenberg picture fields are related to the free fields by

$$\begin{aligned}\phi(t, \vec{x}) &= S^\dagger(t, t_0)e^{-iH_0(t-t_0)}\phi_0(t, \vec{x})e^{iH_0(t-t_0)}S(t, t_0) \\ &= U^\dagger(t, t_0)\phi_0(t, \vec{x})U(t, t_0)\end{aligned} \tag{7.55}$$

[2]Actually, we used to say this point in fomer.

[3]Refer the lectures by Guangpeng Zhang in the class of gauge field theory

The operator $U(t,t_0) \equiv e^{iH_0(t-t_0)}S(t,t_0)$ therefore relates the full Heisenberg picture fields to the free fields at the *same time t. The evolution begins from the time* t_0 *where the fields in two pictures (and the Schrodinger picture) are equal.*Also, we can compute the time derivative of the $U(t,t_0)$:

$$\begin{aligned} i\partial_t U(t,t_0) &= i\partial_t \left[e^{iH_0(t-t_0)}S(t,t_0)\right] \\ &= -e^{iH_0(t-t_0)}H_0 S(t,t_0) + e^{iH_0(t-t_0)}H(t_0)S(t,t_0) \\ &= e^{iH_0(t-t_0)}\left[-H_0 + H(t_0)\right]S(t,t_0) \\ &= e^{iH_0(t-t_0)}\left[-H_0 + H(t_0)\right]e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}S(t,t_0) \\ &= e^{iH_0(t-t_0)}\left[V_I(t_0)\right]e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}S(t,t_0) \\ &= V_I(t)U(t,t_0) \end{aligned} \tag{7.56}$$

where $V_I(t) \equiv e^{iH_0(t-t_0)}V(t)e^{iH_0(t-t_0)}$is the original Heisenberg picture potential $V(t)$ from (7.50), now expressed in the interaction picture.

So, now, we can find that there is an common solution to above equation which would be $U(t,t_0) = \exp(-i\int_{t_0}^{t} V_I(t')dt')$. But $V_I(t_1)$ does not necessarily commute with $V_I(t_2)$, so this is not the right answer . It turns out that the right answer is very similar[4]:

$$U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} dt' V_I(t')}\right\} \tag{7.57}$$

where the $T\{\cdots\}$ represents the time-ordering product operator we have meet in LSZ formula. But, you can see that it is not easy to understand for the reason why there appear a time-order product.

That has been emphasised above for that the potential may be not commute from different time but the time-order can sovle this problem for is takes all the permutation and combination. In this way, we can regard the time-ordering operator $T\{\cdots\}$ as a tool to make everything inside commute effectively[5]:

$$T\{A\cdots B\cdots\} = T\{B\cdots A\cdots\} \tag{7.58}$$

We can easierly prove the solution with time-ordering product satisfis the (7.56) since it has the right boundary conditons, namely $U(t,t) = 1$ which is also an exercise in GFT homework.

Time ordering product as such exponetial in (7.57) can be expanded is an obvious way as the following

$$\boxed{U(t,t_0) = 1 - i\int_{t_0}^{t} dt' V_I(t') - \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\{V_I(t')V_I(t'')\} + \cdots} \tag{7.59}$$

This is the **Dyson series**.

[4]This part can be proven via GP.Zhang's lectures and some other books.

[5]My personal view.

Dyson defined the time-ordered product and this series in his classic paper [Dyson, 1949]. In that paper he showed the equivalence of oldfashioned perturbation theory or, more exactly, the interaction picture method developed by Schwinger and Tomonaga based on time-dependent perturbation theory, and Feynman's method, involving space-time diagrams, which we are about to get to.

And now we take the same operations as the lectures of gague field theory. We remove the subscript on V for simplicity, and the formal equati will be rewritten as

$$i\partial_t U(t,t_0) = V(t)U(t,t_0) \tag{7.60}$$

We product the $-i$ and dt to integrate and we get

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt' V(t')U(t',t_0) \tag{7.61}$$

where the 1 is $U(t_0,t_0)$ which is the appropriate integration constant $U(t_0,t_0)=1$. Then we can take the iteration of $U(t,t_0)$

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt' V(t') + \cdots \tag{7.62}$$

and the second order

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} dt' V(t') \left[1 - i\int_{t_0}^{t'} dt'' V(t'') + \cdots\right] \\ &= 1 - i\int_{t_0}^{t} dt' V(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V(t')V(t'') + \cdots \end{aligned} \tag{7.63}$$

Notice that, there hide an assumption that implies the time order $t_0 < t'' < t' < t$. Recall the time-order product, we vanish all the possible permutations just by a permutations factor. Thus we consider $t' > t''$ and $t'' > t$ two possible terms by 2! which we rewrite

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V(t')V(t'') = \frac{1}{2!}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\left\{V(t')V(t'')\right\} \tag{7.64}$$

This form is well for that all the integrals have the same upper limit t.

Same as it, we can rewrite all the order as its expansion form

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt' V(t') + \frac{(-i)^2}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\left\{V(t')V(t'')\right\} + \cdots \tag{7.65}$$

namely the time-order product form (return the subscript)

$$U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} dt' V_I(t')}\right\} \tag{7.66}$$

We can compute the procudt of the series and prove that the U have the following relations

$$\begin{aligned} &U_{21}U_{12} = 1 \\ &U_{21}^{\dagger} = U_{21}^{-1} = U_{12} \\ &U_{32}U_{21} = U_{31} \end{aligned} \tag{7.67}$$

Finally, our defining relation, (7.55)

$$\phi(\vec{x},t) = U^\dagger(t,t_0)\phi_0(\vec{x},t)U(t,t_0) \tag{7.68}$$

and an example to show the convention

$$\phi(x_1) = \phi(\vec{x}_1,t_1) = U_{10}^\dagger\phi_0(\vec{x}_1,t_1)U_{10} = U_{01}\phi_0(x_1)U_{10} \tag{7.69}$$

Now, we will talk about the vacuum, the vacuum matrix elements. The M.D. Schwartz said in his textbook:

> *In deriving LSZ we used that the vacuum state $|\Omega\rangle$ was annihilated by the operators $a_p(t)$ in the interacting theory at a time $t = -\infty$. To relate this to a state for which we know how the free-field creation and annihilation operators act, we need to evolve it to the reference time t_0 where the free and interacting pictures are taken equal. This is straightforward: states evolve (in the Schrodinger picture) with $S(t,t_0)$, and thus $S(t,t_0)|\Omega\rangle$ is annihilated by $a_p(t_0)$ at $t = -\infty$. Equivalently (in the Heisenberg picture) the operator $a_p(t) = S^\dagger(t,t_0)a_p(t_0)S(t,t_0)$ annihilates $|\Omega\rangle$ at $t = -\infty$.*

In my personal view, we gonna to say that the vacuum are different from free theory and interacting theory. But we do can deal with it and find the relation.

In the free theory, there is a state $|0\rangle$, which is annihilated by the a_p. Since the a_p evolve with a simple phase rotation, the same state $|0\rangle$ is annihilated by the (free theory) a_p at any time. More precisely, even if we do note assume $|0\rangle$ has zero energy, then $a_p(t_0)e^{-iH_0(t-t_0)}|0\rangle = 0$ at $t = -\infty$. Since at the time t_0 the free and interacting theory creation and annihilation operators are equal, the a_p in both theories annihilate $e^{-iH_0(t-t_0)}|0\rangle$ and $S(t,t_0)|\Omega\rangle$. Thus, the two states must be proportional. Therefore

$$|\Omega\rangle = \mathcal{N}_i \lim_{t\to-\infty} S^\dagger(t,t_0)e^{-iH_0(t-t_0)}|0\rangle = \mathcal{N}_i U_{0-\infty}|0\rangle \tag{7.70}$$

for some number $\mathcal{N}_i$. Also, similarly, $\langle\Omega| = \mathcal{N}_f\langle 0|U_{\infty 0}$ for some number $\mathcal{N}_f$.

Let's re-check the correlation function in LSZ formula. To see clearly, we assume that $t_1 > t_2 > \cdots > t_n$, then

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle &= \langle\Omega|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle \\ &= \mathcal{N}_f\langle 0|U_{\infty 0}\phi(x_1)\phi(x_2)\cdots\phi(x_n)U_{0-\infty}|0\rangle\mathcal{N}_i \\ &= \mathcal{N}_i\mathcal{N}_f\langle 0|U_{\infty 0}U_{01}\phi_0(x_1)U_{10}U_{02}\phi_0(x_2)U_{20}\cdots U_{0n}U_{n0}\phi_0(x_n)U_{n0}U_{0-\infty}|0\rangle\end{aligned} \tag{7.71}$$

In the equation (7.58), we know that, the time-ordering operator can ignore the commutation. Or say, as long as there are a T operator, we can freely arrange the quantities. Thus,(actually and anyway, they are time-ordered by itself)

$$\begin{aligned}&\langle 0|T\{U_{\infty 0}U_{01}\phi_0(x_1)U_{10}U_{02}\phi_0(x_2)U_{20}\cdots U_{0n}U_{n0}\phi_0(x_n)U_{n0}U_{0-\infty}\}|0\rangle \\ =&\langle 0|\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)U_{\infty 0}U_{01}U_{10}\cdots U_{0-\infty}\}|0\rangle \\ =&\langle 0|T\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)U_{\infty,-\infty}\}|0\rangle\end{aligned} \tag{7.72}$$

The normalization should set so that $\langle\Omega|\Omega\rangle = 1$ just as $\langle 0|0\rangle = 1$ in the free theory. So we have

$$\mathcal{N}_i\mathcal{N}_f = \frac{1}{\langle 0|\, U_{\infty,\infty}\, |0\rangle} \tag{7.73}$$

and therefore we have

$$\langle\Omega|\, T\left\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\right\}|\Omega\rangle = \frac{\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)e^{-i\int_{t_0}^{t} dtV_I(t)}\right\}|0\rangle}{\langle 0|\, T\left\{e^{-i\int_{t_0}^{t} dtV_I(t)}\right\}|0\rangle} \tag{7.74}$$

Then, the only thing left to understand is what $V_I(t)$ is. We have defined the time t_0 as when the interacting fields are the same the free fields. As an example, a cubic interaction would be

$$V(t_0) = \int d^3x \frac{g}{3!}\phi^3(\vec{x},t) = \int d^3x \frac{g}{3!}\phi_0^3(\vec{x},t_0) = \int d^3x \frac{g}{3!} phi^3(\vec{x}) \tag{7.75}$$

The time dependence of the free fields is determined by free Hamilontian,

$$\phi_0(\vec{x},t) = e^{iH_0(t-t_0)}\phi_0(\vec{x})e^{-iH_0(t-t_0)} \tag{7.76}$$

and therefore

$$V_I = e^{iH_0(t-t_0)}\left[\int d^3x \frac{g}{3!}\phi_0^3(\vec{x})\right]e^{-iH_0(t-t_0)} = \int d^3x \frac{g}{3!}\phi_0^3(\vec{x},t) \tag{7.77}$$

So the interaction picture potential is expressed in terms of the free fields at all times. And now, we try to make it as the Lorentz-invariant form. Recall that the potential is related to the Lagrangian by $V_I = -\int d^3x \mathcal{L}_{\text{int}}[\phi_0]$, where $\mathcal{L}_{\text{int}}[\phi_0]$ is the interacting part of Lagrangaian density. Then the U relation can be written as

$$U_{\infty,-\infty} = T\left\{e^{-i\int_{-\infty}^{\infty} dtV_I(t)}\right\} = T\left\{e^{i\int_{-\infty}^{\infty} d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\} \tag{7.78}$$

and this form is the Lorentz-invariant form.

In summary, matrix elements of interacting fields in the interacting vacuum are given by

$$\langle\Omega|\, \phi(x_1)\phi(x_2)\cdots\phi(x_n)\,|\Omega\rangle = \frac{\langle 0|\, U_{\infty 1}\phi_0(x_1)U_{12}\phi_0(x_2)U_{23}\cdots\phi_0(x_n)U_{n,-\infty}\,|0\rangle}{\langle 0|\, U_{\infty,-\infty}\,|0\rangle} \tag{7.79}$$

where the $|\Omega\rangle$ is the ground state in the interacting theory and

$$U_{ij} = T\left\{e^{i\int_{t_j}^{t_i} d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\} \tag{7.80}$$

with $\mathcal{L}_{\text{int}}[\phi] = \mathcal{L}_{\text{int}}[\phi] - \mathcal{L}_0[\phi]$, where $\mathcal{L}_0[\phi]$ is the free Lagrangaian. The free Lagrangaian is defined as whatever goes into the free-field evolution, usually taken to be just kinetic terms.

For the special case of time-ordered products, sucha as what we need for S-matrix elements, this simplifies to

$$\boxed{\langle\Omega|\, T\left\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\right\}|0\rangle = \frac{\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\}|\Omega\rangle}{\langle 0|\, T\left\{e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\}|0\rangle}} \tag{7.81}$$

which is remarkably simple and manifestly Lorentz-invariant result.

And now, we gonna to talk about the relationship of (7.81) and position-space Feynman rules. And we take the example of our ϕ^3 field. Namely, the Lagrangaian is

$$\mathcal{L}_{\text{int}}[\phi] = \frac{g}{3!}\phi^3 \tag{7.82}$$

and we compute the $\langle\Omega| T\{\phi(x_1)\phi(x_2)\} |\Omega\rangle$.

The numerator of (7.81) can be expanded perturbatively in g as g, g^2:

$$\langle 0| T\left\{\phi_0(x_1)\phi_0(x_2)e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]} |0\rangle\right\} = \text{1-order} + \text{2-order} \tag{7.83}$$

notice the function exponetial:

$$e^{i\int d^4x\frac{g}{3!}\phi^3(x)} = 1 + i\int d^4x\frac{g}{3!}\phi^3 + \frac{1}{2}(\frac{ig}{3!})^2\int d^4x\phi^3(x)\int d^4y\phi^3(y) \tag{7.84}$$

Then the 0-order is:

$$\phi_0(x_1)\phi_0(x_2)$$

the 1-order being

$$ig\int d^4x\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)$$

and the 2-order is:

$$-\frac{1}{2}\frac{g^2}{(3i)^2}\phi_0(x_1)\phi_0(x_2)\int d^4xd^4y\phi_0^3(x)\phi_0^3(y)$$

thus

$$\begin{aligned}
&\langle 0| T\left\{\phi_0(x_1)\phi_0(x_2)e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\} |0\rangle \\
&= \langle 0| T\{\phi_0(x_1)\phi_0(x_2)\} |0\rangle + ig\int d^4x \langle 0| T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\right\} |0\rangle \\
&-\frac{1}{2}\frac{g^2}{(3!)^2}\int d^4xd^4y \langle 0| T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\right\} |0\rangle
\end{aligned} \tag{7.85}$$

which implies the 2-fields, 5-fields and 8-fields. We can compute by the former time-orderding product computation.

A similar expansion would result from any time-ordered product of interacting fields. Thus, we now only need to evaluate correlation functions of products of free fields. To do so, it is help to write $\phi_0(x) = \phi_+(x) + \phi_-(x)$, where

$$\phi_+(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p^\dagger e^{ipx}, \quad \phi_-(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p e^{-ipx} \tag{7.86}$$

with ϕ_+ containing only creation operators and ϕ_- only annihilation operators. Then products of ϕ_0 fields at different points become sums of products of ϕ_+ and ϕ_- fields at differents points.

We can compute the permutation and combination (actually they are the kind of the problems) and we can have the intuition of the so called Wick theorem. For example, we compute

the 8-fields product $\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\right\}|0\rangle$, its expansion

$$\begin{aligned}
&\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\right\}|0\rangle \\
=&\langle 0|\, T\left\{\left[\phi_+(x_1)+\phi_-(x_1)\right]\left[\phi_+(x_2)+\phi_-(x_2)\right]\left[\phi_+(x)+\phi_-(x)\right]^3\left[\phi_+(y)+\phi_-(y)\right]^3\right\}|0\rangle \\
=&\text{permutation and combination as} \\
=&\langle 0|\, T\left\{\phi_+(x_1)\phi_+(x_2)\phi_+^3(x)\phi_+^3(y)\right\}|0\rangle + 3\langle 0|\, T\left\{\phi_+(x_2)\phi_+(x_1)\phi_+^3(x)\phi_+^2(y)\phi_-(y)\right\}|0\rangle + \cdots
\end{aligned} \tag{7.87}$$

The last line indicates that the result is the sum of a set of products of ϕ_+ and ϕ_- operators evaluated at different points. In each element of this sum, a ϕ_+ would create a particle that, to give a non-zero result, must then be annihilated by some ϕ_- operator. The matrix element can only be non-zero if every particle that is created is destoryed, so every term must have four ϕ_+ operators and four ϕ_- operators. Each pairing of ϕ_+ with ϕ_- to get a Feynman propagators is called a ***contraction*** *(not to be confused with a Lorentz contraction). The result is then the sum of all possible contractions.*

Each contraction represents the creation and annihilation of a particle, with the creation happening earlier than the annihilation and gives a factor of the Feynman propagator:

$$\langle 0|\, T\left\{\phi_0(x)\phi_0(y)\right\}|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{ik(x-y)} \equiv D_F(x,y) \tag{7.88}$$

A time-ordered correlation function of free fields is given by a sum over all possible ways in which all of the fields in the product can be contracted with each other. This is a result known as **Wick's theorem**. And we will prove this theorem in the end of the chapter.

To see what is the contraction actually, we re-check the (7.85) to show how the Wick theorem works. The first term of the expansion is the $\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\right\}|0\rangle$, the term only containing one contraction, gives the propagator $D_F(x_1,x_2) = D_{12}$. The second term in (7.85) has an odd number of ϕ fields, and therefore cannot be completely contracted and must be vanish (*We will prove this key point and the Wick theorem in the end of this chapter.*) The third term in (7.85) invovles eight fields which is the even number , and there are multiple possible contractions (actually just a senior high school-level problems of permutation and combination):

$$\begin{aligned}
\langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\right\}|0\rangle &= \langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\right\}|0\rangle \\
&= 9D_{12}D_{xx}D_{xy}D_{yy} + 6D_{12}D_{xy}^3 \\
&+ 18D_{1x}D_{2x}D_{xy}D_{yy} + 9D_{1x}D_{2y}D_{xx}D_{yy} + 18D_{1x}D_{2y}D_{xy}^2 \\
&+ 18D_{1y}D_{2y}D_{xy}D_{xx} + 9D_{1y}D_{2x}D_{xx}D_{yy} + 18D_{1y}D_{2x}D_{xy}^2
\end{aligned} \tag{7.89}$$

As in (7.85), we have to integrate over x and y. Thus, many of these terms (those on the last

line) give the same contributions as other terms. We find, to order g^2,

$$\begin{aligned}\langle\Omega|\,T\left\{\phi(x_1)\phi(x_2)\right\}|\Omega\rangle = \frac{1}{\langle 0|\,T\left\{e^{i\int\mathcal{L}_{\rm int}}\right\}|0\rangle}\times\Big\{&D_{12}\\ &-g^2\int d^4x\int d^4y\big[\frac{1}{8}D_{12}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{12}D_{xy}^3+\frac{1}{2}D_{12}D_{2x}D_{xy}D_{yy}+\cdots\\ &+\frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2}+\frac{1}{2}D_{1x}D_{xy}^2D_{y2}+\cdots\big]\Big\}\end{aligned} \tag{7.90}$$

There is a trick which we can just draw the position Feynman diagrams and are the same as those coming from the Lagrangian approach former. Comparing to (7.38) we see that sum of terms is exactly the same, including combinatoric factors, with two exceptions: the $\langle 0|\,T\left\{e^{i\int\mathcal{L}_{\rm int}}\right\}|0\rangle$ factor and the first two terms on the second line. The two new terms correspond to diagrams

to be continued

There two differences precisely cancel.

To see the cancellation, note that the extra diagrams both include the **bubbles**.

> *That is, they have connected subgraphs not involving any external point. The bubbles are exactly what are in $\langle 0|\,T\left\{e^{i\int\mathcal{L}_{int}}\right\}|0\rangle$. To see this, note that Wick's theorem also applies to the denominator of* (7.81)*. Up to order g^2, Wick's theorem implies*

$$\langle 0|\,T\left\{e^{i\int d^4x\mathcal{L}_{\rm int}[\phi_0]}\right\}|0\rangle = \langle 0|0\rangle+\left(\frac{ig}{3!}\right)^2\frac{1}{2}\int d^4x\int d^4y\,\langle 0|\,T\left\{\phi_0^3(x)\phi_0^3(y)\right\}|0\rangle+\cdots \tag{7.91}$$

We have dropped the $\mathcal{O}(g)$ term since it involves an odd number of fields and therefore vanishes by Wick's theorem. Performing a similar expansion as above, we find

$$\langle 0|\,T\left\{e^{i\int d^4x\mathcal{L}_{\rm int}[\phi_0]}\right\}|0\rangle = 1+\left(\frac{ig}{3!}\right)^2\frac{1}{2}\int d^4x\int d^4\big[9D_{xx}D_{xy}D_{yy}+6D_{xy}^3\big]+\mathcal{O}(g^3) \tag{7.92}$$

These diagrams are the bubbles (*to be continued*).(7.90) including terms up to $\mathcal{O}(g^2)$ in the numerator and denominator, we find

$$\frac{\langle 0|\,T\left\{\phi_0(x_1)\phi_0(x_2)e^{i\int\mathcal{L}_{\rm int}}\right\}|0\rangle}{\langle 0|\,T\left\{e^{i\int\mathcal{L}_{\rm int}}\right\}|0\rangle} = \frac{D_{12}-g^2\int\left[\frac{1}{8}D_{12}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{12}D_{xy}^3+\cdots\right]}{1-g^2\int\left[\frac{1}{8}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{xy}^3\right]} \tag{7.93}$$

Since $\dfrac{1}{1+g^2x}=1-g^2x+\mathcal{O}(g^4)$, we can invert the denominator in perturbation theory to see that the bubbles exactly cancel.

> *More generally, the bubbles will cancel. Since the integrals in the expansion of the numerator corresponding to the bubbles never invovles any external point, they just factor out. The sum over all graphs, in the numerator, is then the sum over all graphs with no bubbles multiplying the sum over the bubbles.*

to be continued, the Feynman diagrams

The sum over bubbles is exactly $\langle 0|\, T\left\{e^{i\int \mathcal{L}_{\text{int}}}\right\}|0\rangle$. So,

$$\langle \Omega|\, T\left\{\phi(x_1)\phi(x_2)\right\}|\Omega\rangle = \langle 0|\, T\left\{\phi_0(x_1)\phi_0(x_2)e^{i\int \mathcal{L}_{\text{int}}}\right\}|0\rangle_{\text{no bubbles}} \tag{7.94}$$

where "no bubbles" means that every connected subgraphs involves an external point.

Now, we can talk about the Position-space Feynman rules which from the Hamiltonian and the Lagrangian approaches.

1. *There is a factor of* $\frac{1}{m!}$ *from the expansion of* $e^{i\mathcal{L}_{int}} = \sum \frac{1}{m!}(i\mathcal{L}_{int})^m$. *If we expand to order* m *there will be* m *identical vertices in the same diagram. We can also swap these vertices around, leaving the diagram looking the same. If we only include the diagram once in our final sum, the* $m!$ *from permuting the diagrams will cancel the* $\frac{1}{m!}$ *from the exponential. Neither of these factors were present in the Lagrangian approach, since internal vertices came out of the splitting of lines associated with external vertices, which was unambiguous, and there was no exponential to begin with.*

2. *If interactions are normalized as in* (7.46)*, then there will be a* $\frac{1}{j!}$ *for each interaction with* j *identical particles. This factor is canceled by the* $j!$ *ways of permuting the* j *identical lines coming out of the same internal vertex. In the Lagrangian approaches, one of the lines was already chosen so the factor was* $(j-1)!$*, with the missing* j *coming from using* $\mathcal{L}'_{int}[\phi]$ *instead of* $\mathcal{L}_{int}[\phi]$

Each point x_i in the original $n-$point function $\langle \Omega|\, T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|\Omega\rangle$ gets an external point and each interaction gives a new vertex whoes position is integrated over and whose coefficient is given by the coefficients in the Lagrangaian.

As long as the vertices are normalized with appropriate permutation factors, as in (7.46), the combinatoric factors will work out the same, as we saw in the example. In the Lagrangaian approach, we saw that coefficient of the diagram will be given by the coefficient of the interaction multiplied by the geometrical symmetry factor of the diagram. To see that this is also true for the Hamiltonian, we have to count the various combinatoric factors. The result is the same Feynman rules as were derived in the Lagrangian approach. In both cases, symmetry factors muse be added if there is some geometric symmetry (there rarely is in theories with complex fields , such as QED). In neither case do any of the diagrams include bubbles (subdiagrams that do not connect with any external vertex).

However, the position space is often "bigger" than momentum space. So the momentum-space Feynman rules is the most important result in this chapter. The position-space Feynman rules derived in either of the previous two sections give a recipe for computing time-ordered products in perturbation theory. Now we will see how those time-ordered products simplify when all the phase-space integrals over the propagators are performed to turn then into S-matrix elements. This will produce the momentum-space Feynman rules. (Why the momentum-space?

Becasuse it is far smaller than the position-space which we we deal with the multiple particles' numerous numbers of degrees of the freedom while the freedoms of momenta is less than its. Also, you can use the conservation of momenta to simplify the equations. Law of conservation is well important to the theoretical physicits.)

We consider the diagram (*to be continued, the Feynman diagrams*)

$$\mathcal{T}_1 = -\frac{g^2}{2}\int d^4x \int d^4y D_{1x} D_{xy}^2 D_{y2} \tag{7.95}$$

and same as the skill before (the m is always the constant and will do nothing to any we have to do with the equations) so for the simplicity we take the $m = 0$ and the D_{xy}, the Feynman propagator, will be written in the momentum-space as following:

$$D_{xy} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} e^{ip(x-y)} \tag{7.96}$$

then the (7.95) which have four expansions can be re-written as

$$\begin{aligned}\mathcal{T}_1 = &-\frac{g^2}{2}\int d^4x \int d^4y \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} \\ &\times e^{ip_1(x_1-x)} e^{ip_2(y-x_2)} e^{ip_3(x-y)} e^{ip_4(x-y)} \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{p_3^2 + i\varepsilon} \frac{i}{p_4^2 + i\varepsilon}\end{aligned} \tag{7.97}$$

Luckily, there are some non-couplings integrals so that we can produce the Fourier intergrals which leads to the $\delta -$ functions. We decompose the long exponetial functions to the x and y parts

$$e^{ip_1(x_1-x)} e^{ip_2(y-x_2)} e^{ip_3(x-y)} e^{ip_4(x-y)} = e^{ip_1x_1} e^{-ip_2x_2} e^{i(p_3+p_4-p_1)x} e^{i(p_2-p_3-p_4)y} \tag{7.98}$$

and we integral over the x and y we can get the $\delta-$functions:

$$\int d^4x \int d^4y e^{(ip_3+p_4-p_1)x} e^{i(p_2-p_3-p_4)y} \sim \delta^4(p_3 + p_4 - p_1)\delta^4(p_2 - p_3 - p_4) \tag{7.99}$$

We base on the p_3 respectively corresponding to momentum being conserved at the vertices labeled x and y in the Feynman diagram. We integrate over p_3 using the $\delta^4(p_3 + p_4 - p_1)$ which means that $p_3 = p_1 - p_4$ and integrate the $\delta^4(p_2 - p_3 - p_4)$ which transfers to the new $\delta-$function $\delta^4(p_1 - p_2)$. To the simplicity, we replace the p_4 as $p_4 = k$. Then, we gonna to have:

$$\begin{aligned}\mathcal{T}_1 = &-\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} e^{ip_1x_1} e^{-ip_2x_2} \\ &\times \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{(p_1 - k)^2 + i\varepsilon} \frac{i}{k^2 + i\varepsilon} (2\pi)^4 \delta^4(p_1 - p_2)\end{aligned} \tag{7.100}$$

Now, we use the LSZ formula (6.1) which we have induced in the chapter 5 to convert this to

a contribution to the S-matrix:

$$\begin{aligned}\langle f|\, S\,|i\rangle &= \left[i\int d^4x_1 e^{-ip_i x_1}\left(\Box + m^2\right)\right]\left[i\int d^4x_2 e^{ip_f x_2}\left(\Box + m^2\right)\right]\langle\Omega|\, T\left\{\phi_(x_1)\phi_(x_2)\right\}|\Omega\rangle \\ &= \left[i\int d^4x_1 e^{-ip_i x_1}\left(\Box\right)\right]\left[i\int d^4x_2 e^{ip_f x_2}\left(\Box\right)\right]\langle\Omega|\, T\left\{\phi_(x_1)\phi_(x_2)\right\}|\Omega\rangle \\ &= \left[-i\int d^4x_1 e^{-ip_i x_1}\left(p_i^2\right)\right]\left[i\int d^4x_2 e^{ip_f x_2}\left(p_f^2\right)\right]\langle\Omega|\, T\left\{\phi_(x_1)\phi_(x_2)\right\}|\Omega\rangle \\ &= -\int d^4x_1 e^{-ip_i x_1}(p_i^2)\int d^4x_2 e^{ip_f x_2}(p_f^2)\mathcal{T}_1 + \cdots\end{aligned} \tag{7.101}$$

and we used the conditions $m = o$ and we re-write the $\Box$ as in momentum operatort p^2 which we employ the operatorization of physical quantities in the context of quantum mechanics. From the (7.97), we know that, we can integrate the x_1 and x_2 for the terms $\mathcal{T}_1 \sim e^{ip_1 x_1}e^{-ip_2 x_2}$ as $e^{i(p_1-p_i)x_1}$ and $e^{i(p_f-p_2)x_2}$ transfering to $(2\pi)^4\delta^4(p_1 - p_i)$ and $(2\pi)^4\delta^4(p_2 - p_f)$.

When we do integrate, the terms will be 1 in such situations:

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{p_i^2}{p_i^2 + i\varepsilon} &= 1 \\ \lim_{\varepsilon\to 0}\frac{p_f^2}{p_f^2 + i\varepsilon} &= 1\end{aligned} \tag{7.102}$$

So, the $\langle f|\, S\,|i\rangle$ can be written as following simple form:

$$\langle f|\, S\,|i\rangle = -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p_i - k)^2 + i\varepsilon}\frac{i}{k^2 + i\varepsilon}\left(2\pi\right)^4\delta^4(p_i - p_f) + \cdots \tag{7.103}$$

The $\delta^4(p_i - p_f)$ term is the answer forces overall momentum conservation, and will always be present in any calculation. But we will always factor it out, as we did when we related differential scattering amplitudes to S-matrix elements. In chapter 5, we get the such result

$$S = 1 + (2\pi)^4\delta^4(\sum p_i)i\mathcal{M} \tag{7.104}$$

and then

$$\boxed{i\mathcal{M} = -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p_i - k)^2 + i\varepsilon}\frac{i}{k^2 + i\varepsilon} + \cdots} \tag{7.105}$$

We can summarize this procedure with the **Momentum-Space Feynman Rules**:

1. *Internal lines (those not connected to external points) get propagators* $\dfrac{i}{p^2 - m^2 + i\varepsilon}$.
2. *Vertices come from interactions in the Lagrangaian. They get factors of the coupling constant times i.*
3. *Lines connected to external points do not get propagators (their propagators are canceled by terms from the LSZ reduction formula).*
4. *Momentum is conserved at each vertex.*

5. *Integrate over undetermined 4-momenta.*

6. *Sum over all possible diagrams.*

The content of "Signs of momenta", "Disconnected graphs", "phi3 examples", "Mandelstam variables" and "Derivative couplings" of this chapter will be continued later

Part II

Quantum electrodynamics

Now we gonna to talk about the spin-1 and the gauge invariance.