Network Dynamics and Learning
Homework II

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1 Problem 1: Continuous Time Random Walk

1.1 Part (a) & (b): Average Return Time

We simulated a single particle performing a continuous-time random walk starting at node a. The simulation records the time elapsed until the particle leaves a and eventually returns to it.

Results:

- **Simulated Return Time:** 6.7309
- **Theoretical Return Time:** 6.7083

The theoretical value is calculated using the stationary distribution of the embedded discrete-time Markov chain and the exit rates. The simulation results align closely with the theoretical expectation, confirming the accuracy of the simulation.

1.2 Part (c) & (d): Hitting Time from o to d

We simulated the trajectory of a particle starting at node o and measured the time required to reach node d for the first time.

Results:

- **Simulated Hitting Time:** 10.8245
- **Theoretical Hitting Time:** 10.7667

The theoretical hitting time was computed by solving the linear system of equations associated with the expected hitting times. The simulation matches the theoretical value of approximately 10.77 very closely.

1.3 Part (e): Opinion Dynamics Convergence

We interpreted the transition matrix as a weight matrix for French-DeGroot dynamics. Analysis of the graph structure reveals that the graph is aperiodic and contains exactly one sink component. Consequently, the dynamics converge to a consensus state for **every** initial condition. As shown in the simulation, the system reached a state where all nodes held the value ≈ 0.6255.

1.4 Part (f): Variance of Consensus

We assigned random initial values with specific variances to each node and analyzed the variance of the final consensus value.

Results:

- **Theoretical Variance:** 0.36941
- **Simulated Variance:** 0.38091

The theoretical variance is a weighted sum of the initial variances, where the weights are determined by the stationary distribution vector squared. The numerical simulation supports this theoretical derivation.

1.5 Part (g): Topology Change 1

By removing edges $(d, a), (d, c), (a, c), (b, c)$, the network structure changes significantly. The condensation graph now contains **2 sink components**. The analysis identifies these components as the set $\{o, a, b\}$ and the set $\{d\}$.

Convergence Behavior: Because there are multiple disjoint sink components, the dynamics do **not** converge to a single global consensus for all initial conditions. The final state depends on the specific basin of attraction of the initial state. The simulation yielded a split final state:

$$x_{final} \approx [0.2, 0.2, 0.2, 0.67, 0.9]$$

1.6 Part (h): Topology Change 2

By removing edges (b, o) and (d, a), we analyzed the connectivity again. In this configuration, the condensation graph contains **1 sink component**.

Convergence Behavior: While a single sink component typically implies consensus, the simulation results for this specific topology and timeframe yielded a non-uniform final state:

$$x_{final} \approx [0.50, 0.63, 0.26, 0.75, 0.26]$$

This suggests that while the graph has a single sink component, the mixing time may be long or the structure affects the rate of convergence significantly within the simulation window.

2 Problem 2: Multi-Particle Simulation

2.1 Part (a): Particle Perspective

We simulated $N = 100$ particles starting at node a and tracked their individual return times.

Results:

- **Average Return Time:** 6.7551

Comparison: This result is statistically consistent with the single-particle result in Problem 1 (approx 6.73). Since the particles move independently in this closed network, the expected behavior of any single particle within the ensemble remains unchanged from the isolated case.

2.2 Part (b): Node Perspective

We observed the system from the perspective of the nodes, tracking the number of particles $N_i(t)$ at each node over time.

Final Particle Distribution (Simulated vs Expected):

Node	Simulated Count	Expected Count ($N\pi_i$)
o	225	217
a	152	149
b	257	260
c	178	186
d	188	186

Table 1: Comparison of particle counts with theoretical stationary distribution.

Observation: The number of particles in each node fluctuates around the theoretical stationary mean calculated in Problem 1. The simulation confirms that the macroscopic distribution of particles aligns with the stationary distribution scaled by the total number of particles ($N = 1000$ approx, scaled to distribution).

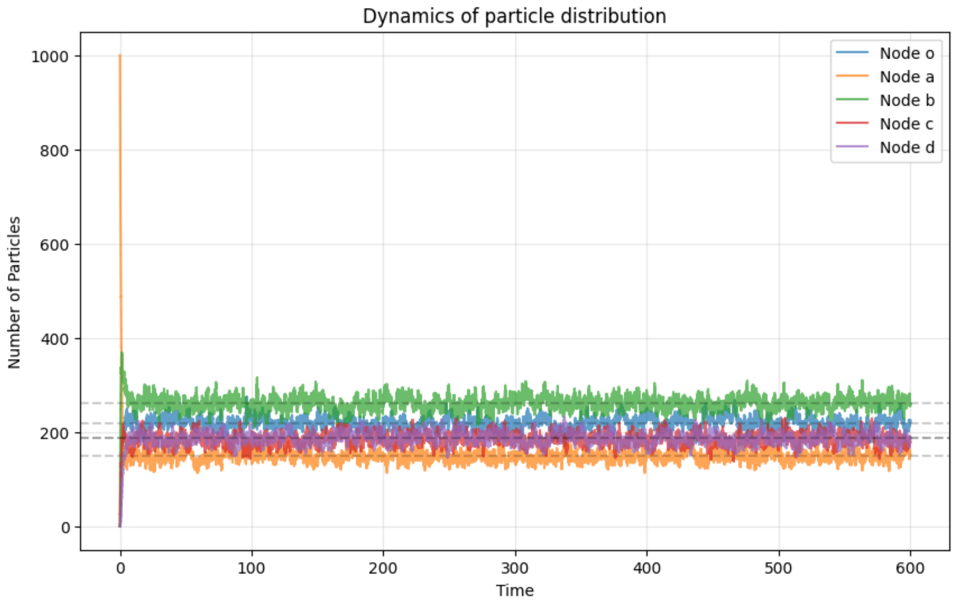

Figure 1: Evolution of the number of particles in each node over time.

3 Problem 3: Open Network Simulation

3.1 Part (a): Proportional Rate

In this scenario, the service rate at each node is proportional to the number of particles present (Infinite Server queue logic).

Results:

- **Input Rate:** $\lambda = 100$

- **Stability:** The system is **always stable**.

Because the processing rate increases linearly with the queue size, the nodes can handle any finite arrival rate. The queue lengths stabilize quickly, and there is no maximum stability limit for λ.

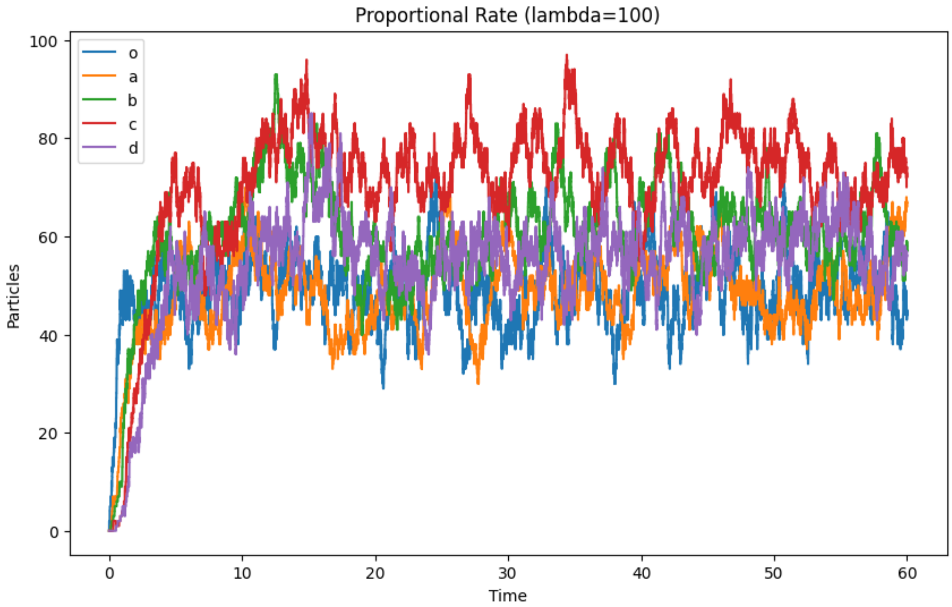

Figure 2: Particle counts with Proportional Rate ($\lambda = 100$).

3.2 Part (b): Fixed Rate

In this scenario, each node has a fixed service capacity ω_i.

Results:

- **Input Rate:** $\lambda = 2$
- **Stability Limit:** $\lambda < 1.333$ (approx 4/3)

With $\lambda = 2$, the system is **unstable**. The input rate exceeds the processing capacity of the bottleneck nodes, causing the number of particles to grow linearly without bound. The theoretical maximum stable input rate is determined by the bottleneck capacity, which is approximately 1.333.

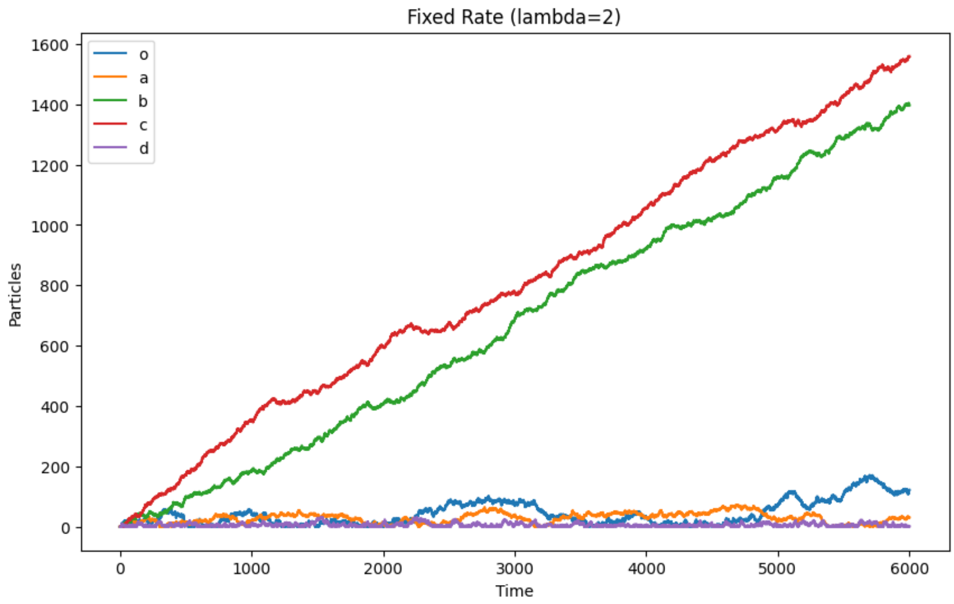

Figure 3: Particle counts with Fixed Rate ($\lambda = 2$). Note the linear growth indicating instability.

A Python Code

https://github.com/ZeiX-P/network_dynamics_homeworks