1.8 楊-Mills Theory (See App. C.5 for verbose notes.)

◆1.8.1 Non-Abelian gauge theory

$$[T^a,T^b]=\mathrm{i}f^{ab}{}_cT^c,\qquad 0=f^D{}_{ab}f^E{}_{Dc}+f^D{}_{ca}f^E{}_{Db}+f^D{}_{bc}f^E{}_{Da},\qquad \mathrm{D}_\mu=\partial_\mu-\mathrm{i}gA_\mu$$
$$\operatorname{Tr}T^aT^b=\frac{1}{2}\delta^{ab},\qquad [\widetilde{T}^a]_i{}^j:=T^{\mathrm{ad}\ a}{}_i{}^j:=-\mathrm{i}f^{aij}\qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j:=\delta_i^j\partial_\mu+gf^{iaj}A^a_\mu.$$

$$\begin{aligned}F_{\mu\nu}&=\frac{\mathrm{i}}{g}\left[\mathrm{D}_\mu,\mathrm{D}_\nu\right] & \mathrm{D}_\mu\phi&=\partial_\mu\phi-\mathrm{i}gA^a_\mu(T^a_\phi\phi)\\ &=\partial_\mu A_\nu-\partial_\nu A_\mu+\frac{g}{\mathrm{i}}\left[A_\mu,A_\nu\right] & \mathrm{D}_\mu F_{\mu\nu}{}^a&=\partial_\mu F^a_{\mu\nu}+gf^{abc}A^b_\mu F^c_{\mu\nu},\\ &=\Big[\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\Big]T^a & \Big(\mathrm{D}_\mu F_{\nu\rho}&=\partial_\mu\lambda-\mathrm{i}g[A_\mu,F_{\nu\rho}]\Big).^{*1}\end{aligned}$$

$$\phi\mapsto V\phi:=\mathrm{e}^{\mathrm{i}g\theta}\phi\qquad A_\mu\mapsto V\left(A_\mu+\frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1}\qquad F_{\mu\nu}\mapsto VF_{\mu\nu}V^{-1}$$
$$\phi^a{}'\simeq\phi+\mathrm{i}g\theta^aT^a\phi\qquad A^a_\mu{}'\simeq A^a_\mu+\partial_\mu\theta^a+gf^{abc}A^b_\mu\theta^c\qquad F^a_{\mu\nu}{}'\simeq F^a_{\mu\nu}+gf^{abc}F^b_{\mu\nu}\theta^c$$

$$\epsilon^{\mu\nu\rho\sigma}\left[\mathrm{D}_\nu,[\mathrm{D}_\rho,\mathrm{D}_\sigma]\right]=\epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma}=0.$$

■Killing and Casimir Here we have two constants which **depend on representation** r.

$$\operatorname{Tr}(T^aT^b)=:C(r)\delta^{ab}\quad\text{(Killing form)},\qquad T^aT^a=:C_2(r)\cdot\mathbf{1}\quad\text{(quadratic Casimir operator)},\tag{1.46}$$

which satisfy

$$C(r)=\frac{d(r)}{d(\mathrm{ad})}C_2(r),\qquad T^aT^bT^a=\left[C_2(r)-\frac{1}{2}C_2(\mathrm{ad})\right]T^b,\tag{1.47}$$
$$f^{acd}f^{bcd}=C_2(\mathrm{ad})\delta^{ab},\qquad f^{abc}T^bT^c=\frac{\mathrm{i}}{2}C_2(\mathrm{ad})T^a.\tag{1.48}$$

For SU(N) For its fundamental representation N with definition $C(N):=\frac{1}{2}$, we have

$$C(N):=\frac{1}{2},\quad C_2(N)=\frac{N^2-1}{2N},\quad C(\mathrm{ad})=C_2(\mathrm{ad})=N;\quad (T^a)_{ij}(T^a)_{kl}=\frac{1}{2}\left(\delta_{il}\delta_{kj}-\frac{\delta_{ij}\delta_{kl}}{N}\right).$$

◆1.8.2 Abelian gauge theory

In Abelian gauge theory, V and fields are always commutative, and thus we have charge freedom (Q).

$$\mathrm{D}_\mu\phi=\left(\partial_\mu-\mathrm{i}gA_\mu Q\right)\phi\qquad \phi\mapsto\mathrm{e}^{igQ\theta}\phi\qquad F_{\mu\nu}=\frac{\mathrm{i}}{g}\left[\mathrm{D}_\mu,\mathrm{D}_\nu\right]=\partial_\mu A_\nu-\partial_\nu A_\mu$$
$$\mathrm{D}_\mu\lambda^a=\partial_\mu\lambda^a\qquad A_\mu\mapsto A_\mu+\partial_\mu\theta\qquad F_{\mu\nu}\mapsto F_{\mu\nu}$$

◆1.8.3 Lagrangian Block

$$\mathcal{L}\ni|\mathrm{D}_\mu\phi|^2-m^2|\phi|^2,\quad \overline{\psi}(\mathrm{i}\not{D}-m)\psi,\quad -\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu}\left(=-\frac{1}{2}\operatorname{Tr}F^{\mu\nu}F_{\mu\nu}\right),\quad \theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\tag{1.49}$$
$$-\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu}=-\frac{1}{2}\left[(\partial_\mu A^a_\nu)^2+A^a_\mu\partial^\mu\partial^\nu A^a_\nu\right]-gf^{abc}A^a_\mu A^b_\nu\partial^\mu A^{c\nu}-\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}\tag{1.50}$$

[*1] Note that we can use any representation T^a but must the same ones for $A^a_\mu T^a$ and λ^aT^a.

2 Standard Model

Any representations assumed to be *normalized Hermitian.* Note that the $SU(2)$ **2** representation is

$$T^a = \frac{1}{2}\sigma^a; \qquad \left[T^a, T^b\right] = \mathrm{i}\epsilon^{abc}T^c; \qquad T^{\pm} := T^1 \pm \mathrm{i}T^2. \tag{2.1}$$

We use the following abridged notations:

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.2}$$

2.1 Symmetries and Fields

	$\mathrm{SU}(3)_{\mathrm{strong}}$	$\mathrm{SU}(2)_{\mathrm{weak}}$	$\mathrm{U}(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_{\mathrm{L}}Q_i$: Left-handed quarks	**3**	**2**	1/6
$P_{\mathrm{L}}U_i$: Right-handed up-type quarks	**3**	**1**	2/3
$P_{\mathrm{R}}D_i$: Right-handed down-type quarks	**3**	**1**	−1/3
$P_{\mathrm{R}}L_i$: Left-handed leptons	**1**	**2**	−1/2
$P_{\mathrm{R}}E_i$: Right-handed leptons	**1**	**1**	−1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	**1**	**2**	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	**8**	**1**	0
W : Weak bosons	**1**	**3**	0
B : B boson	**1**	**1**	0

■Full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{湯川}}$

$$\text{where} \quad \mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right) H\right|^2 - V(H), \tag{2.4}$$

$$\begin{aligned}
\mathcal{L}_{\text{matter}} = {} & \overline{Q}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{L}} Q_i \\
& + \overline{U}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \frac{2}{3}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}} U_i \\
& + \overline{D}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu + \frac{1}{3}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}} D_i \\
& + \overline{L}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_2 W_\mu + \frac{1}{2}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{L}} L_i \\
& + \overline{E}_i \mathrm{i}\gamma^\mu \left(\partial_\mu + \mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}} E_i,
\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{\text{湯川}} = \overline{U}_i (y_u)_{ij} H P_{\mathrm{L}} Q_j - \overline{D}_i (y_d)_{ij} H^\dagger P_{\mathrm{L}} Q_j - \overline{E}_i (y_e)_{ij} H^\dagger P_{\mathrm{L}} L_j + \text{H.c.} \tag{2.6}$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix C.4.

■Gauge Kinetic Terms

the gauge kinetic terms can be expanded as

$$
\begin{aligned}
\mathcal{L}_{\text{gauge}} = &-\frac{1}{4}(\partial B)(\partial B)\\
&-\frac{1}{4}(\partial W^a)(\partial W^a) - g_2\epsilon^{abc}(\partial_\mu W^a_\nu)W^{\mu b}W^{\nu c} - \frac{{g_2}^2}{4}\left(\epsilon^{eab}W^a_\mu W^b_\nu\right)\left(\epsilon^{ecd}W^{c\mu}W^{d\nu}\right)\\
&-\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G^a_\nu)G^{\mu b}G^{\nu c} - \frac{{g_3}^2}{4}\left(f^{eab}G^a_\mu G^b_\nu\right)\left(f^{ecd}G^{c\mu}G^{d\nu}\right).
\end{aligned}
\tag{2.7}
$$

2.2 Higgs Mechanism

■Higgs Potential

The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2(H^\dagger H) + \lambda\left(H^\dagger H\right)^2. \tag{2.8}$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + \mathrm{i}\phi_2\\ v + (h + \mathrm{i}\phi_3)\end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.9}$$

Here h is the “Higgs boson,” and ϕ_i are 南部–Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2 h^2, \tag{2.10}$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$
\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right)H\right|^2 && (2.11)\\
&= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\Big[{g_2}^2{W_1}^2 + {g_2}^2{W_2}^2 + (g_1 B - g_2 W_3)^2\Big]. && (2.12)
\end{aligned}
$$

Redefining the gauge fields (with concerning the norms) as

$$W^\pm_\mu := \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}W^2_\mu), \qquad \begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix} := \begin{pmatrix}\cos\theta_{\text{w}} & -\sin\theta_{\text{w}}\\ \sin\theta_{\text{w}} & \cos\theta_{\text{w}}\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}, \tag{2.13}$$

where

$$
\begin{aligned}
&\tan\theta_{\text{w}} := \frac{g_1}{g_2}, \qquad e := -\frac{g_1 g_2}{\sqrt{{g_1}^2 + {g_2}^2}}; \qquad g_Z := \sqrt{{g_1}^2 + {g_2}^2}; && (2.14)\\
&g_1 = \frac{|e|}{\cos\theta_{\text{w}}} = g_Z\sin\theta_{\text{w}}, \qquad g_2 = \frac{|e|}{\sin\theta_{\text{w}}} = g_Z\cos\theta_{\text{w}}. && (2.15)
\end{aligned}
$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4}\Big[{g_2}^2 W^{+\mu}W^-_\mu + \frac{{g_Z}^2}{2}Z^\mu Z_\mu\Big]. \tag{2.16}$$

Here we have omitted the 南部–Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan\theta_{\mathrm{w}} Z_\mu, \tag{2.17}$$
$$g_2 W_\mu = \frac{g_2}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right) + \left(\frac{|e|}{\tan\theta_{\mathrm{w}}} Z_\mu + |e| A_\mu\right) T^3, \tag{2.18}$$

$$Z_\mu^0 := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_2 W_\mu^3 - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_1 W_\mu^3 + g_2 B_\mu) \tag{2.19}$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \tag{2.20}$$

■Gauge Term The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W^a_{\mu\nu} = {} & (\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ & - 4\mathrm{i}g\left[(\partial W^3)^{\mu\nu} W_\mu^+ W_\nu^- + (\partial W^+)^{\mu\nu} W_\mu^- W_\nu^3 + (\partial W^-)^{\mu\nu} W_\mu^3 W_\nu^+\right] \\ & - 2g^2(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left(W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- - 2W_\mu^3 W_\nu^3 W_\rho^+ W_\sigma^-\right), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_{\text{gauge}} := {} & -\frac{1}{4}\left[G^{a\mu\nu} G^a_{\mu\nu} + (\partial Z)^{\mu\nu}(\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu}(\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu}(\partial W^-)_{\mu\nu}\right] \\ & + \frac{\mathrm{i}|e|}{\tan\theta_{\mathrm{w}}}\Big[(\partial W^+)^{\mu\nu} W_\mu^- Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W_\nu^+ + (\partial Z)^{\mu\nu} W_\mu^+ W_\nu^-\Big] \\ & + \mathrm{i}|e|\Big[(\partial W^+)^{\mu\nu} W_\mu^- A_\nu + (\partial W^-)^{\mu\nu} A_\mu W_\nu^+ + (\partial A)^{\mu\nu} W_\mu^+ W_\nu^-\Big] \\ & + (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left[\frac{|e|^2}{2\sin^2\theta_{\mathrm{w}}} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- + \frac{|e|^2}{\tan^2\theta_{\mathrm{w}}} W_\mu^+ Z_\nu W_\rho^- Z_\sigma\right. \\ & \qquad \left. + \frac{|e|^2}{\tan\theta_{\mathrm{w}}}\left(W_\mu^+ Z_\nu W_\rho^- A_\sigma + W_\mu^+ A_\nu W_\rho^- Z_\sigma\right) + |e|^2 W_\mu^+ A_\nu W_\rho^- A_\sigma\right]. \end{aligned} \tag{2.21}$$

■湯川 Term

$$\begin{aligned} \mathcal{L}_{湯川} = {} & \overline{U} y_u H P_{\mathrm{L}} Q - \overline{D} y_d H^\dagger P_{\mathrm{L}} Q - \overline{E} y_e H^\dagger P_{\mathrm{L}} L + \text{H.c.} \\ = {} & \overline{U} y_u \epsilon^{\alpha\beta} H^\alpha P_{\mathrm{L}} Q^\beta - \overline{D} y_d {H^\dagger}^\alpha P_{\mathrm{L}} Q^\alpha - \overline{E} y_e {H^\dagger}^\alpha P_{\mathrm{L}} L^\alpha + \text{H.c.} \\ = {} & -\frac{v + h}{\sqrt{2}}\left(\overline{U} y_u P_{\mathrm{L}} Q^1 + \overline{D} y_d P_{\mathrm{L}} Q^2 + \overline{E} y_e P_{\mathrm{L}} L^2\right) + \text{H.c.} \end{aligned} \tag{2.22}$$

2.3 Full Lagrangian After Higgs Mechanism

Now we have the following Lagrangian (with omitting P_{L} etc.):

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} + {m_W}^2 W^+W^- + \frac{{m_Z}^2}{2} Z^2 \\
\text{【Higgs】}\quad & + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}{m_h}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
& + \frac{v{g_2}^2}{4} W^+W^- h + \frac{v({g_1}^2+{g_2}^2)}{8} Z^2 h \\
& + \frac{{g_2}^2}{4} W^+W^- h^2 + \frac{{g_1}^2+{g_2}^2}{8} Z^2 h^2 \\
& - \left(\frac{1}{\sqrt{2}} h\bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} h\bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} h\bar{E} y_e L^2 + \text{H.c.} \right) \\
\text{【SU(3)】}\quad & + \bar{Q}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) Q + \bar{U}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) U + \bar{D}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) D + \bar{L}\left(\mathrm{i}\partial\!\!\!/\right) L + \bar{E}\left(\mathrm{i}\partial\!\!\!/\right) E \\
\text{【}W\text{】}\quad & + \bar{Q}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+ T^+ + W\!\!\!/^- T^-\right) Q + \bar{L}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+ T^+ + W\!\!\!/^- T^-\right) L \\
\text{【}A\&Z^0\text{】}\quad & + \bar{Q}\left[\left(T^3 + \frac{1}{6}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right) Z\!\!\!/^0\right] Q \\
& + \bar{U}\left(\frac{2}{3}|e|A\!\!\!/ - \frac{2|e|s}{3c} Z\!\!\!/\right) U \\
& + \bar{D}\left(-\frac{1}{3}|e|A\!\!\!/ + \frac{|e|s}{3c} Z\!\!\!/\right) D \\
& + \bar{L}\left[\left(T^3 - \frac{1}{2}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right) Z\!\!\!/^0\right] L \\
& + \bar{E}\left(-|e|A\!\!\!/ + \frac{|e|s}{c} Z\!\!\!/\right) E \\
\text{【湯川項】}\quad & - \left(\frac{1}{\sqrt{2}} v\bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} v\bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} v\bar{E} y_e L^2 + \text{H.c.} \right)
\end{aligned}
\tag{2.23}
$$

2.4 Mass Eigenstates

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := vy_\bullet/\sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \qquad (m_i \geq 0). \tag{2.24}
$$

Using this Ψ and Φ, we can rotate the basis as

$$
Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \qquad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \tag{2.25}
$$

and now we have the 湯川 terms in mass eigenstates as

$$
\mathcal{L}_{\text{湯川}} = -\left(1 + \frac{1}{v}h\right)\left[(m_u)_i \overline{U}_i P_{\mathrm{L}} Q^1_i + (m_d)_i \overline{D}_i P_{\mathrm{L}} Q^2_i + (m_e)_i \overline{E}_i P_{\mathrm{L}} L^2_i + \text{H.c.}\right]. \tag{2.26}
$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark–W interactions do change drastically, as

$$\mathcal{L} \supset \overline{Q}\mathrm{i}\gamma^\mu \left(-\mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) P_\mathrm{L} Q \tag{2.27}$$

$$= \overline{Q}\frac{g_2}{\sqrt{2}}\left(\not{W}^+ T^+ + \not{W}^- T^-\right) P_\mathrm{L} Q \quad + \quad \text{(interaction terms with } Z \text{ and } A\text{)} \tag{2.28}$$

$$\mapsto \frac{g_2}{\sqrt{2}}\begin{pmatrix}\overline{Q}^1\Psi_u & \overline{Q}^2\Psi_d\end{pmatrix}\begin{pmatrix}0 & \not{W}^+ \\ \not{W}^- & 0\end{pmatrix} P_\mathrm{L}\begin{pmatrix}\Psi_u^\dagger Q^1 \\ \Psi_d^\dagger Q^2\end{pmatrix} + (\quad\cdots\quad) \tag{2.29}$$

$$= \frac{g_2}{\sqrt{2}}\left[\overline{Q}^2\not{W}^- X P_\mathrm{L} Q^1 + \overline{Q}^1\not{W}^+ X^\dagger P_\mathrm{L} Q^2\right] + (\quad\cdots\quad), \tag{2.30}$$

where $X := \Psi_d\Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP-symmetry.

> In our notation, CP-transformation of a spinor is described as
>
> $$\mathscr{C}\mathscr{P}\left(\psi\right) = -\mathrm{i}\eta^*(\overline{\psi}\gamma^2)^\mathrm{T}, \quad \mathscr{C}\mathscr{P}\left(\overline{\psi}\right) = \mathrm{i}\eta(\gamma^2\psi)^\mathrm{T}, \tag{2.31}$$
>
> where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,
>
> $$\begin{aligned}\mathscr{C}\mathscr{P}\left(\overline{Q}^2\not{W}^- X P_\mathrm{L} Q^1\right) &= (\gamma^2 Q^2)^\mathrm{T}\mathscr{P}(-\not{W}^+) X P_\mathrm{L}(\overline{Q}^1\gamma^2)^\mathrm{T} \\ &= -W_\mu^{+P}(\gamma^2 Q^2)^\mathrm{T}(\overline{Q}^1 X^\mathrm{T}\gamma^2 P_\mathrm{L}\gamma^{\mu\mathrm{T}})^\mathrm{T} \\ &= (\overline{Q}^1\not{W}^+ X^\mathrm{T} P_\mathrm{L} Q^2).\end{aligned} \tag{2.32}$$
>
> Therefore, we can see that the CP-symmetry is maintained if and only if $X^\mathrm{T} = X^\dagger$, that is, if and only if X is a real matrix.

以上より，標準模型の Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} \\
\text{【質量項】} \quad & + \; {m_W}^2 W^+ W^- \; + \; \frac{{m_Z}^2}{2} Z^2 \\
& - \; (\bar{U} M_u P_{\mathrm{L}} Q^1 + \bar{D} M_d P_{\mathrm{L}} Q^2 + \bar{E} M_e P_{\mathrm{L}} L^2 + \text{H.c.}) \\
\text{【Higgs Field】} \quad & + \; \frac{1}{2}(\partial_\mu h)^2 \; - \frac{1}{2}{m_h}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
\text{【Higgs との結合】} \quad & + \; \frac{v {g_2}^2}{4} W^+ W^- h \; + \; \frac{v({g_1}^2 + {g_2}^2)}{8} Z^2 h \\
& + \; \frac{{g_2}^2}{4} W^+ W^- h^2 \; + \; \frac{{g_1}^2 + {g_2}^2}{8} Z^2 h^2 \\
& - \; \left(\frac{1}{v} \bar{U} M_u P_{\mathrm{L}} Q^1 h + \frac{1}{v} \bar{D} M_d P_{\mathrm{L}} Q^2 h + \frac{1}{v} \bar{E} M_e P_{\mathrm{L}} L^2 h + \text{H.c.} \right) \\
\text{【SU(3) および微分項】} \quad & + \; \bar{Q} \left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{L}} Q \; + \; \bar{U} \left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{R}} U \; + \; \bar{D} \left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{R}} D \\
& + \; \bar{L} \left(\mathrm{i}\partial\!\!\!/ \right) P_{\mathrm{L}} L \; + \; \bar{E} \left(\mathrm{i}\partial\!\!\!/ \right) P_{\mathrm{R}} E \\
\text{【}W\text{ boson】} \quad & + \; \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 W\!\!\!/^- X P_{\mathrm{L}} Q^1 + \bar{Q}^1 W\!\!\!/^+ X^\dagger P_{\mathrm{L}} Q^2 \right] \qquad \text{【}\leftarrow CP\text{ and flavor violating!】} \\
& + \; \bar{L} \frac{g_2}{\sqrt{2}} \left(W\!\!\!/^+ T^+ + W\!\!\!/^- T^- \right) P_{\mathrm{L}} L \\
\text{【}A\&Z^0\text{ boson】} \quad & + \; \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| A\!\!\!/ + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) Z\!\!\!/^0 \right] P_{\mathrm{L}} Q \\
& + \; \bar{U} \left(\frac{2}{3} |e| A\!\!\!/ - \frac{2|e|s}{3c} Z\!\!\!/ \right) P_{\mathrm{R}} U \\
& + \; \bar{D} \left(-\frac{1}{3} |e| A\!\!\!/ + \frac{|e|s}{3c} Z\!\!\!/ \right) P_{\mathrm{R}} D \\
& + \; \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| A\!\!\!/ + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) Z\!\!\!/^0 \right] P_{\mathrm{L}} L \\
& + \; \bar{E} \left(-|e| A\!\!\!/ + \frac{|e|s}{c} Z\!\!\!/ \right) P_{\mathrm{R}} E
\end{aligned} \tag{2.33}
$$

となる。

2.5 Chiral Notation

以上の Lagrangian を chiral 表示で表すと，まず最初は

$$
\begin{aligned}
\mathcal{L} =&(\text{Higgs terms}) + (\text{Gauge fields strength}) \\
&+ Q_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \mathrm{i}g_2 W_{\mu} - \frac{1}{6}\mathrm{i}g_1 B_{\mu}\right) Q_{\mathrm{L}} \\
&+ U_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \frac{2}{3}\mathrm{i}g_1 B_{\mu}\right) U_{\mathrm{R}} \\
&+ D_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} + \frac{1}{3}\mathrm{i}g_1 B_{\mu}\right) D_{\mathrm{R}} \\
&+ L_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu} + \frac{1}{2}\mathrm{i}g_1 B_{\mu}\right) L_{\mathrm{L}} \\
&+ E_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} + \mathrm{i}g_1 B_{\mu}\right) E_{\mathrm{R}} \\
&- \left(U_{\mathrm{R}}^{\dagger} y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger} y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger} y_e H^{\dagger} L_{\mathrm{L}} + \text{H.c.}\right) \\
=&(\text{Higgs terms}) + (\text{Gauge fields strength}) \\
&+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger} \\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right) \\
&+ g_2\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}Q_{\mathrm{L}} + L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}L_{\mathrm{L}}\right) \\
&+ g_1\left(\frac{1}{6}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}Q_{\mathrm{L}} + \frac{2}{3}U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}U_{\mathrm{R}} - \frac{1}{3}D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}D_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}L_{\mathrm{L}} - E_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}E_{\mathrm{R}}\right) \\
&- \left(U_{\mathrm{R}}^{\dagger} y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger} y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger} y_e H^{\dagger} L_{\mathrm{L}} + \text{H.c.}\right)
\end{aligned}
\tag{2.34}
$$

であり，そして最終的には

$$
\begin{aligned}
\mathcal{L} =&(\text{Gauge bosons and Higgs}) \\
&+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger} \\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right) \\
&- m_u(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}}) - (\text{quarks}) - m_e(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}}) - (\text{leptons}) \\
&- \frac{m_u}{v}(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}})h - (\text{quarks}) - \frac{m_e}{v}(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}})h - (\text{leptons}) \\
&+ \frac{g_2}{\sqrt{2}}\left[\begin{pmatrix} d_{\mathrm{L}}^{\dagger} & s_{\mathrm{L}}^{\dagger} & b_{\mathrm{L}}^{\dagger}\end{pmatrix}\bar{\sigma}^{\mu}W_{\mu}^{-}X\begin{pmatrix} u_{\mathrm{L}} \\ c_{\mathrm{L}} \\ t_{\mathrm{L}}\end{pmatrix} + \begin{pmatrix} u_{\mathrm{L}}^{\dagger} & c_{\mathrm{L}}^{\dagger} & t_{\mathrm{L}}^{\dagger}\end{pmatrix}\bar{\sigma}^{\mu}W_{\mu}^{+}X^{\dagger}\begin{pmatrix} d_{\mathrm{L}} \\ s_{\mathrm{L}} \\ b_{\mathrm{L}}\end{pmatrix}\right] \\
&+ \frac{g_2}{\sqrt{2}}\left[\nu_e^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{+}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{-}\nu_e\right] \\
&+ |e|\left[\frac{2}{3}u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}u_{\mathrm{L}} - \frac{1}{3}d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}d_{\mathrm{L}} + \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}u_{\mathrm{R}} - \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}d_{\mathrm{R}} + (\text{quarks})\right. \\
&\qquad\qquad \left. -e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}e_{\mathrm{L}} - e_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}e_{\mathrm{R}} + (\text{leptons})\right] \\
&+ \frac{|e|s}{c}\left[\left(\frac{c^2}{2s^2} - \frac{1}{6}\right)u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}u_{\mathrm{L}} - \left(\frac{c^2}{2s^2} + \frac{1}{6}\right)d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}d_{\mathrm{L}} - \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}u_{\mathrm{R}} + \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}d_{\mathrm{R}}\right. \\
&\qquad\qquad \left. + \left(\frac{c^2}{2s^2} + \frac{1}{2}\right)\nu_e^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2}\right)e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}e_{\mathrm{L}} + e_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}e_{\mathrm{R}} + (\text{others})\right]
\end{aligned}
\tag{2.35}
$$

となる。