1 General Definitions and Tools

1.1 Notations and Conventions

◆1.1.1 Metric etc.

Minkowski Metric : $\eta^{\mu\nu} := \mathrm{diag}(+,-,-,-);\quad \epsilon^{0123}_{0123} := \pm 1$

Coordinates : $x^\mu := (t,x,y,z);\quad$ therefore $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$.

Gamma Matrices : $\{\gamma^\mu,\gamma^\nu\} := 2\eta^{\mu\nu};\quad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \dfrac{-\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

: therefore $\{\gamma^\mu,\gamma_5\} = 0, (\gamma_5)^2 = 1$.

Gamma Combinations : $1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5;\quad \sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu] = 0 \big/ \mathrm{i}\gamma^\mu\gamma^\nu$

Spinor ϵ and σ matrices : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$

: $(\sigma^\mu)_{\alpha\dot{\beta}} := (1,\boldsymbol{\sigma})_{\alpha\dot{\beta}},\quad (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (1,-\boldsymbol{\sigma})^{\dot{\alpha}\beta}$.

Pauli Matrices : $\sigma_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix},$

: $\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \begin{pmatrix}0&0\\1&0\end{pmatrix},$

: $\sigma^\mu := (1,\boldsymbol{\sigma}),\quad \bar{\sigma}^\mu := (1,-\boldsymbol{\sigma})$.

Fourier Transformation : $\widetilde{f}(k) := \displaystyle\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} f(x);\qquad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}kx}\widetilde{f}(k)$.

◆1.1.2 Fields

Scalar : $(\partial^2 + m^2)\phi = 0;\quad \phi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}px}\right]$

Dirac : $(i\partial\!\!\!/ - m)\psi = 0;\quad \psi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left[a^s_{\boldsymbol{p}}u^s(p)\mathrm{e}^{-\mathrm{i}px} + b^{s\dagger}_{\boldsymbol{p}}v^s(p)\mathrm{e}^{\mathrm{i}px}\right]$

Vector : $\partial^2 A^\mu = 0;\quad A^\mu(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{r=0..3}\left[a^r_{\boldsymbol{p}}\epsilon^r(p)\mathrm{e}^{-\mathrm{i}px} + a^{r\dagger}_{\boldsymbol{p}}\epsilon^{r*}(p)\mathrm{e}^{\mathrm{i}px}\right]$

TODO: 南部–Goldstone; Gravitino

◆1.1.3 Electromagnetism

Electromagnetic Fields : $A^\mu = (\phi, \boldsymbol{A})$ 【We can invert the signs, but cannot lower the index.】

Maxwell Equations : $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu;\qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0,\quad \partial_\mu F^{\mu\nu} = ej^\nu$

Our Old Language : $\nabla\cdot\boldsymbol{B} = 0,\ \nabla\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = 0;\quad \nabla\cdot\boldsymbol{E} = ej^0,\ (\nabla\times\boldsymbol{B})_i - \frac{\partial}{\partial t}E_i = ej^i$.

$$F_{\mu\nu} = \begin{pmatrix} 0 & & \boldsymbol{E} & \\ & 0 & -B_3 & B_2 \\ -\boldsymbol{E} & B_3 & 0 & -B_1 \\ & -B_2 & B_1 & 0 \end{pmatrix};\quad F_{\mu\nu}F^{\mu\nu} = -2\left(\|\boldsymbol{E}\|^2 - \|\boldsymbol{B}\|^2\right)$$

1.2 SPINOR FIELDS

Spinor : ξ_α, $\xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta$; Lorentz tr. : $\xi_\alpha \mapsto \Lambda_\alpha{}^\beta \xi_\beta$, $\xi^\alpha \mapsto \xi^\beta \Lambda^{-1}{}_\beta{}^\alpha$;

: $\bar\eta^{\dot\alpha} := (\eta^\alpha)^*$ $\bar\eta_{\dot\alpha} := (\eta_\alpha)^*$: $\bar\eta^{\dot\alpha} \mapsto \Lambda^{\dagger-1\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta}$, $\bar\eta_{\dot\alpha} \mapsto \bar\eta_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}$.

Kinetic term : $\mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi \quad (= \mathrm{i}\,\eta\sigma^\mu\partial_\mu\bar\eta\,)$

Mass term : [Majorana] $-\frac{1}{2}(m_{\mathrm{M}}\xi\xi + m_{\mathrm{M}}^*\bar\xi\bar\xi)$ [Dirac] $-(m_{\mathrm{D}}\xi\eta + m_{\mathrm{D}}^*\bar\xi\bar\eta)$

Dirac fermion : $\mathcal{L}_{\mathrm{Dirac}} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi + \mathrm{i}\,\eta\sigma^\mu\partial_\mu\bar\eta - m(\xi\eta + \bar\xi\bar\eta) = \overline{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$

Majorana fermion : $\mathcal{L}_{\mathrm{Majorana}} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi - \frac{m}{2}(\xi\xi + \bar\xi\bar\xi) = \frac{1}{2}\overline{\psi}_{\mathrm{M}}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi_{\mathrm{M}}$

Charge conjugate : $\psi^{\mathrm{C}} := C(\overline{\psi})^{\mathrm{T}}$ $[(\psi_{\mathrm{M}})^{\mathrm{C}} = \psi_{\mathrm{M}}]$

◆1.2.1 Chiral Notation (Peskin)

Gamma Matrices : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; $P_{\mathrm{L}}^{\mathrm{R}} = \dfrac{1 \pm \gamma_5}{2}$.

Fields : $\psi = \begin{pmatrix} \xi_\alpha \\ \bar\eta^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}$; $\overline{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\alpha & \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm{R}}^\dagger & \psi_{\mathrm{L}}^\dagger \end{pmatrix}$; $\psi_{\mathrm{M}} = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$.

: $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar\sigma}\xi^s \end{pmatrix}$; $v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}$

: $\Big[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\Big]$.

Charge conj. : $C := -\mathrm{i}\gamma^2\gamma^0 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$ $\begin{cases} -C = C^{-1} = C^\dagger = C^{\mathrm{T}} \\ C = C^* \end{cases}$, $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$.

: $\psi^{\mathrm{C}} = C(\overline{\psi})^{\mathrm{T}} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} \eta_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$, $\overline{\psi}^{\mathrm{C}} = \psi^{\mathrm{T}}C = \mathrm{i}\overline{\psi}^*\gamma^2$

Weyl Equations : $\mathrm{i}\bar\sigma\cdot\partial\psi_{\mathrm{L}} = m\psi_{\mathrm{R}}$; $\mathrm{i}\sigma\cdot\partial\psi_{\mathrm{R}} = m\psi_{\mathrm{L}}$

z-boost limit : Halt: $u^s = \sqrt{m}\left(\begin{smallmatrix} \xi^s \\ \xi^s \end{smallmatrix}\right)$, $v^s = \sqrt{m}\left(\begin{smallmatrix} \eta^s \\ -\eta^s \end{smallmatrix}\right)$;

: Slow: $\sqrt{p\cdot\sigma} \simeq \sqrt{m}(1 - \boldsymbol{v}\cdot\boldsymbol{\sigma}/2)$, $\sqrt{p\cdot\bar\sigma} \simeq \sqrt{m}(1 + \boldsymbol{v}\cdot\boldsymbol{\sigma}/2)$;

: Extreme: $u^s = \sqrt{2E}\begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\xi^s \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\xi^s \end{pmatrix}$, $v^s = \sqrt{2E}\begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\eta^s \\ -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\eta^s \end{pmatrix}$.

◆1.2.2 Dirac Notation

Gamma Matrices : $\hat\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat\gamma^i = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$, $\hat\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\hat{P}_{\mathrm{L}}^{\mathrm{R}} = \dfrac{1 \pm \gamma_5}{2}$.

: $\hat\sigma^{0i} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$, $\hat\sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.

Fields : $\hat\psi = \begin{pmatrix} \psi_{\mathrm{A}} \\ \psi_{\mathrm{B}} \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\psi = \dfrac{1}{\sqrt{2}}\begin{pmatrix} \psi_{\mathrm{L}} + \psi_{\mathrm{R}} \\ -\psi_{\mathrm{L}} + \psi_{\mathrm{R}} \end{pmatrix}$; $\hat\psi_{\mathrm{M}} = \begin{pmatrix} \psi_{\mathrm{A}} \\ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\psi_{\mathrm{A}}^* \end{pmatrix}$.

: $\hat{\overline{\psi}} = \hat\psi^\dagger\gamma^0 = \begin{pmatrix} \psi_{\mathrm{A}}^\dagger & -\psi_{\mathrm{B}}^\dagger \end{pmatrix}$

: $\hat{u}^s(p) = \begin{pmatrix} \sqrt{p^0 + m}\,\xi^s \\ \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\xi^s \end{pmatrix}$; $\hat{v}^s(p) = \begin{pmatrix} -\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\eta^s \\ -\sqrt{p^0 + m}\,\eta^s \end{pmatrix}$

: $\Big[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\Big]$

Charge conj. : $\hat{C} = -\mathrm{i}\hat{\gamma}^2\hat{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ with $C = -C^{-1} = -C^\dagger$, $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$.

z-boost limit : Halt: $\hat{u}^s = \sqrt{2m}\begin{pmatrix} \xi^s \\ 0 \end{pmatrix}$, $\hat{v}^s = -\sqrt{2m}\begin{pmatrix} 0 \\ \eta^s \end{pmatrix}$;

: Slow: $\sqrt{p^0+m} \simeq \sqrt{2m}(1+\frac{v^2}{8})$, $\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}} \simeq \sqrt{\frac{m}{2}}(\boldsymbol{v}\cdot\boldsymbol{\sigma})$;

: Extreme: $\hat{u}^s = \sqrt{E}\begin{pmatrix} \xi^s \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\xi^s \end{pmatrix}$, $\hat{v}^s = -\sqrt{E}\begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\eta^s \\ \eta^s \end{pmatrix}$

◆1.2.3 CPT transformations

【 Note that these expressions are valid under the above frameworks. 】

In the following, CP means "P, then C" **in algebraic sense**. Be careful to the order.

$$\psi(t,\boldsymbol{x}) \xrightarrow{P} \eta_P\gamma^0\psi(t,-\boldsymbol{x}) \qquad \overline{\psi} \xrightarrow{P} \eta_P^*\overline{\psi}\gamma^0$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{T} \eta_T C\gamma_5\psi(-t,\boldsymbol{x}) \qquad \overline{\psi} \xrightarrow{T} -\eta_T^*\overline{\psi}C\gamma_5$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{C} \eta_C C\overline{\psi}^{\mathrm{T}}(t,\boldsymbol{x}) = C\gamma^0\psi^* \qquad \overline{\psi} \xrightarrow{C} \eta_C^*\overline{\psi}^*\gamma^0 C = -\eta_C^*(C\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{CP} \eta_{CP}(\overline{\psi}\gamma^0 C)^{\mathrm{T}} \qquad \overline{\psi} \xrightarrow{CP} \eta_{CP}^*(C\gamma^0\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{CPT} (\overline{\psi}\gamma^0\gamma_5)^{\mathrm{T}} \qquad \overline{\psi} \xrightarrow{CPT} (\gamma^0\gamma_5\psi)^{\mathrm{T}}$$

Note that T-transformation is anti-unitary, and $\eta_{CPT}=1$. Especially, photon is $(P,T,C)=(-,+,-)$.

	ϕ	A^μ	$\bar\psi\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\sigma^{\mu\nu}\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$	$\mathrm{i}\bar\psi\gamma_5\psi$	∂_μ
P	ϕ	$-{+}{+}{+}A^\mu$	$+$	$+{-}{-}{-}$	$(+{-}{-}{-})(+{-}{-}{-})$	$-{+}{+}{+}$	$-$	$+{-}{-}{-}$
T	ϕ	$+{-}{-}{-}A^\mu$	$+$	$+{-}{-}{-}$	$-(+{-}{-}{-})(+{-}{-}{-})$	$+{-}{-}{-}$	$-$	$-{+}{+}{+}$
C	ϕ^*	$+A^{\mu*}$	$+$	$-$	$-$	$+$	$+$	$+$
CPT	ϕ^*	$-A^{\mu*}$	$+$	$-$	$+$	$-$	$+$	$-$

◆1.2.4 Noether current

Infinitesimal transformation : $\phi(x) \mapsto \phi'(x) := \phi(x) + \alpha\Delta\phi(x)$

Correspondent transformation : $\alpha\Delta\mathcal{L} = \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\cancel{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]}\Delta\phi$

: So, defining $\alpha\partial_\mu\mathcal{J}^\mu(x) := \mathcal{L}'(x) - \mathcal{L}(x)$,

Noether current : $j^\mu(x) := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu$; $\partial_\mu j^\mu(x) = 0$

Noether charge : $Q := \int j^0\mathrm{d}^3x$

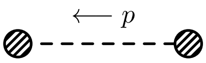
Energy-momentum tensor : $T^\mu{}_\nu = \partial_\mu\mathcal{L}(\partial_\mu\phi)\partial_\nu\phi - \mathcal{L}\delta^\mu_\nu$; $\mathcal{H} = T^{00}$, $\mathcal{P}^i = T^{0i}$.

: $T^\mu{}_\nu$ is the variation along μ in respect to the modification a^ν.

1.3 Feynman Rules

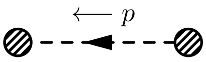
■Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

$\phi\ \phi =$ 

$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

$$\mathcal{L} \supset |\partial_\mu\phi|^2 - m^2|\phi|^2$$

$\phi^*\ \phi =$ 

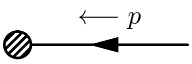
$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

(External lines equal to 1 in both cases.)

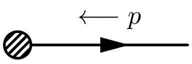
■Dirac Fermion

$$\mathcal{L} \supset \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi$$
$$= \mathrm{i}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \bar{\xi}\bar{\chi})$$

Initial state

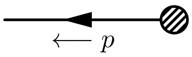
$\psi\ |\boldsymbol{p}, s\rangle =$ 

$= u^s(p)$

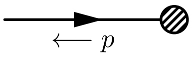
$\overline{\psi}\ |\boldsymbol{p}, s\rangle =$ 

$= \overline{v}^s(p)$

Final state

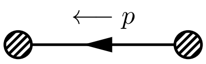
$\langle\boldsymbol{p}, s|\ \overline{\psi} =$ 

$= \overline{u}^s(p)$

$\langle\boldsymbol{p}, s|\ \psi =$ 

$= v^s(p)$

Propagator

$\psi\ \overline{\psi} =$ 

$= \dfrac{\mathrm{i}(p\!\!\!/ + m)}{p^2 - m^2 + \mathrm{i}\epsilon}$

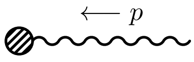
■Majorana Fermion

$$\mathcal{L} \supset \tfrac{1}{2}\overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi$$
$$= \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda - \tfrac{m}{2}(\lambda\lambda + \bar{\lambda}\bar{\lambda})$$

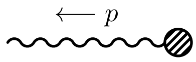
Initial state

■Abelian Gauge Theory (Photon)

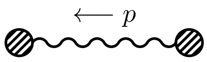
$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}D\!\!\!\!/\,\psi + |\mathrm{D}_\mu\phi|^2$$
$$(\mathrm{D}_\mu = \partial_\mu - \mathrm{i}QA_\mu)$$

$A_\mu\ |\boldsymbol{p};あ\rangle =$ 

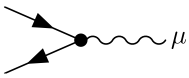
$= \epsilon^{あ}_\mu(p)$

$\langle\boldsymbol{p};あ|\ A_\mu =$ 

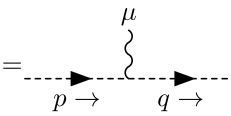
$= \epsilon^{あ*}_\mu(p)$

$A_\mu\ A_\nu =$ 

$= \dfrac{-\mathrm{i}\eta_{\mu\nu}}{p^2 + \mathrm{i}\epsilon}$

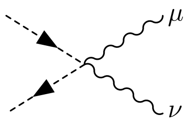
$Q\overline{\psi}A\!\!\!/\psi =$ 

$= \mathrm{i}Q\gamma^\mu$

$\mathrm{i}QA^\mu\phi^*\partial_\mu\phi$
$+\mathrm{H.c.}$ $=$ 

$= \mathrm{i}Q(p^\mu + q^\mu)$

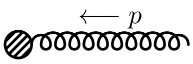
(Momentum must be taken along the arrow)

$Q^2A^2|\phi|^2 =$ 

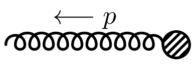
$= 2\mathrm{i}Q^2\eta^{\mu\nu}$

■Non-Abelian Gauge Theory (Gluon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}D\!\!\!\!/\,\psi + |\mathrm{D}_\mu\phi|^2$$
$$(\mathrm{D}_\mu = \partial_\mu - \mathrm{i}gA_\mu)$$

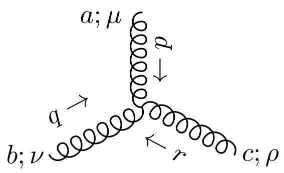
$A^b_\mu\ |\boldsymbol{p};あ, a\rangle =$ 

$= \epsilon^{あ}_\mu(p)\delta^{ab}$

$\langle\boldsymbol{p};あ, a|\ A^b_\mu =$ 

$= \epsilon^{あ*}_\mu(p)\delta^{ab}$

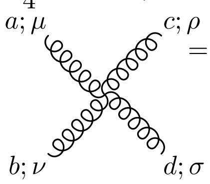
$-gf^{abc}A^{\mu a}A^{\nu b}(\partial_\mu A^c_\nu) =$



$$= gf^{abc}\big[\eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-r)^\mu + \eta^{\rho\mu}(r-p)^\nu\big]$$

(Momentum are in incoming directions)

$-\dfrac{1}{4}g^2(f^{abe}A^a_\mu A^b_\nu)(f^{cde}A^c_\rho A^d_\sigma) =$



$$= -\mathrm{i}g^2\big[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})\big]$$

TODO: vertex は lagrangian の $(n!)$i 倍)

1.4 Field Calculation Techniques

◆1.4.1 Dirac Field Techniques

Dirac Equations	:	$(\not{p}-m)u^s(p)=0;\quad (\not{p}+m)v^s(p)=0$
	:	$\bar{u}^s(p)(\not{p}-m)=0;\quad \bar{v}^s(p)(\not{p}+m)=0$
Dirac Components	:	$u^{r\dagger}(p)u^s(p)=2E_{\boldsymbol{p}}\delta^{rs};\quad v^{r\dagger}(p)v^s(p)=2E_{\boldsymbol{p}}\delta^{rs}$
	:	$\bar{u}^r(p)u^s(p)=2m\delta^{rs};\quad \bar{v}^r(p)v^s(p)=-2m\delta^{rs};\quad \bar{u}^r(p)v^s(p)=\bar{v}^r(p)u^s(p)=0$
Spin Sums	:	$\sum_{\text{spin}}u^s(p)\bar{u}^s(p)=\not{p}+m;\quad \sum_{\text{spin}}v^s(p)\bar{v}^s(p)=\not{p}-m$
Charge Conj.	:	$-C=C^{-1}=C^\dagger=C^{\mathrm{T}},\quad C^{-1}\gamma^\mu C=-C\gamma^\mu C=-\gamma^{\mu\mathrm{T}},\quad C^{-1}\gamma^0C=-\gamma^0$
	:	$C=C^*\quad,\quad \psi^{\mathrm{C}}=C(\overline{\psi})^{\mathrm{T}},\quad \overline{\psi}^{\mathrm{C}}=\psi^{\mathrm{T}}C$
u & v	:	$u^*=-\mathrm{i}\gamma^2v,\quad v^{\mathrm{T}}=-\mathrm{i}u^\dagger\gamma^2=\overline{u}C^{-1},\quad v=C\overline{u}^{\mathrm{T}};\quad \overline{u}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}v_{\mathrm{A}}$
	:	$v^*=-\mathrm{i}\gamma^2u,\quad u^{\mathrm{T}}=-\mathrm{i}v^\dagger\gamma^2=\overline{v}C^{-1},\quad u=C\overline{v}^{\mathrm{T}};\quad \overline{v}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}u_{\mathrm{A}}$

◆1.4.2 Polarization Sum

Single photon case $\quad M=\epsilon^*_\mu(k)M^\mu$

When Ward identity $k_\mu M^\mu=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu*}=\eta_{\mu\nu}M^\mu M^{\nu*}. \tag{1.1}$$

Double photons case $\quad M=\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu}=k'_\nu M^{\mu\nu}=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\rho(k)\epsilon'^*_\nu(k')\epsilon'_\sigma(k')M^{\mu\nu}M^{\rho\sigma*}=\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \tag{1.2}$$

【 See Sec. C.3 for verbose information. 】

◆1.4.3 Fierz transformations

For Dirac spinors a,b,c,d,

$$\begin{aligned}
S(a,b;c,d)&:=(\bar{a}b)(\bar{c}d);\\
V(a,b;c,d)&:=(\bar{a}\gamma^\mu b)(\bar{c}\gamma_\mu d);\\
T(a,b;c,d)&:=\frac{1}{2}(\bar{a}\sigma^{\mu\nu}b)(\bar{c}\sigma_{\mu\nu}d);\\
A(a,b;c,d)&:=(\bar{a}\gamma^\mu\gamma_5 b)(\bar{c}\gamma_\mu\gamma_5 d);\\
P(a,b;c,d)&:=(\bar{a}\gamma_5 b)(\bar{c}\gamma_5 d);
\end{aligned}
\qquad
\begin{pmatrix}S(a,b;c,d)\\V(a,b;c,d)\\T(a,b;c,d)\\A(a,b;c,d)\\P(a,b;c,d)\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&-1&-1\\4&-2&0&-2&4\\6&0&-2&0&-6\\-4&-2&0&-2&-4\\-1&1&-1&-1&1\end{pmatrix}\begin{pmatrix}S(a,d;c,b)\\V(a,d;c,b)\\T(a,d;c,b)\\A(a,d;c,b)\\P(a,d;c,b)\end{pmatrix}$$

Also defining $V_{\mathrm{LR}}(a,b;c,d):=(\bar{a}\gamma^\mu P_{\mathrm{L}}b)(\bar{c}\gamma_\mu P_{\mathrm{R}}d)$ and so on,

$$V_{\mathrm{LL}}(a,b;c,d)=-V_{\mathrm{LL}}(a,d;c,b)\qquad S_{\mathrm{RL}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{LR}}(a,d;b,c)-A_{\mathrm{LR}}(a,d;b,c)\right] \tag{1.3}$$

$$V_{\mathrm{RR}}(a,b;c,d)=-V_{\mathrm{RR}}(a,d;c,b)\qquad S_{\mathrm{LR}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{RL}}(a,d;b,c)-A_{\mathrm{RL}}(a,d;b,c)\right] \tag{1.4}$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta};\qquad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{1.5}$$

◆1.4.4 Gordon identity

For $P := p' + p$ and $q := p' - p$,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \qquad \bar{u}(p')\gamma^\mu v(p) = \bar{u}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \tag{1.6}$$

$$\bar{v}(p')\gamma^\mu v(p) = -\bar{v}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]v(p) \qquad \bar{v}(p')\gamma^\mu u(p) = -\bar{v}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \tag{1.7}$$

◆1.4.5 Color Sum

(Here T^a is **3** of SU(3). For other representations or gauge groups, see Sec. 1.9.)

$$\mathrm{Tr}(T^aT^b) := \frac{1}{2}\delta^{ab} \qquad \text{(That is, } T^a\text{'s are } \tfrac{1}{2}\times\text{Gell-Mann matrices.)} \tag{1.8}$$

$$\sum_a T^aT^a = \frac{4}{3}\cdot\mathbf{1}, \qquad \sum_{c,d} f^{acd}f^{bcd} = 3\delta^{ab} \qquad \sum_a T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} - \frac{1}{6}\delta_{ij}\delta_{kl} \tag{1.9}$$

$$\sum_a T^aT^bT^a = -\frac{1}{6}T^b \qquad \sum_{b,c} f^{abc}T^bT^c = \frac{3\mathrm{i}}{2}T^a \qquad f^{Dab}f^{EDc} + f^{Dca}f^{EDb} + f^{Dbc}f^{EDa} = 0 \tag{1.10}$$

1.5 Miscellaneous Techniques

$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2$

$\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$

$\sqrt{p_\mu\sigma^\mu} = \dfrac{p_\mu\sigma^\mu + m}{\sqrt{2(m+p^0)}}$

$\sigma^i\sigma^j = \delta_{ij}\sigma^0 + \mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^\mu\sigma^\nu = \mathrm{i}\epsilon^{0\mu\nu\rho}\sigma^\rho + \delta^\mu_0\sigma^\nu + \delta^\nu_0\sigma^\mu - \eta^{\mu\nu}\sigma^0$

$[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^i, \sigma^j = 2\delta_{ij}$

TODO: TODO:

- Majorana Ferminos
- Feynman Rules(A.1)

1.6 Dirac's Gamma Algebras

◆1.6.1 Traces

$$\mathrm{Tr}(\text{any odd \# of } \gamma\text{'s}) = 0 \tag{1.11}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \tag{1.12}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \tag{1.13}$$
$$\mathrm{Tr}(\gamma_5 \text{ and any odd \# of } \gamma\text{'s}) = 0 \tag{1.14}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0 \tag{1.15}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{1.16}$$

Generally, for some γ-matrices $A, B, C, \dots,$

$$\begin{aligned}\mathrm{Tr}(ABCDEF\cdots) &= \eta^{AB}\,\mathrm{Tr}(CDEF\cdots) - \eta^{AC}\,\mathrm{Tr}(BDEF\cdots)\\ &\quad + \eta^{AD}\,\mathrm{Tr}(BCEF\cdots) - \eta^{AE}\,\mathrm{Tr}(BCDF\cdots) + \cdots,\end{aligned} \tag{1.17}$$

$$\mathrm{Tr}(ABCDEF\cdots\gamma_5) = \textbf{Not Established.} \tag{1.18}$$

To prove the second equation, we use following technique:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{Tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu); \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots\gamma_5) = \mathrm{Tr}(\gamma_5\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu). \tag{1.19}$$

◆1.6.2 Contractions

$$\gamma^\mu\gamma_\mu = 4 \tag{1.20}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{1.21}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} \tag{1.22}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{1.23}$$

Generally, for some γ-matrices $A, B, C, \dots,$

$$\text{ODD \#}: \quad \gamma^\mu ABC\cdots\gamma_\mu = -2(\cdots CBA), \tag{1.24}$$
$$\text{EVEN \#}: \quad \gamma^\mu ABC\cdots\gamma_\mu = \mathrm{Tr}(ABC\cdots) - \mathrm{Tr}(ABC\cdots\gamma_5)\cdot\gamma_5. \tag{1.25}$$

Contractions in d-dimension

$$\gamma^\mu\gamma_\mu = d \tag{1.26}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu \tag{1.27}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho \tag{1.28}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{1.29}$$

Contractions of ϵ's

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu; \quad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \tag{1.30}$$
$$\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\alpha'\beta'\gamma'} = -\left(\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\alpha_{\beta'}\delta^\beta_{\gamma'}\delta^\gamma_{\alpha'} + \delta^\alpha_{\gamma'}\delta^\beta_{\alpha'}\delta^\gamma_{\beta'} - \delta^\alpha_{\alpha'}\delta^\beta_{\gamma'}\delta^\gamma_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\delta^\gamma_{\gamma'} - \delta^\alpha_{\gamma'}\delta^\beta_{\beta'}\delta^\gamma_{\alpha'}\right) \tag{1.31}$$

1.7 Loop Integrals and Dimensional Regularization

◆1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots x_n\, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{1.32}$$

$$\frac{1}{A_1 A_2} = \int_0^1 \mathrm{d}x \frac{1}{[xA_1 + (1-x)A_2]^2} \tag{1.33}$$

◆1.7.2 d-dimensional integrals in Minkowski space

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{1.34}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.35}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.36}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{1.37}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{4} \tag{1.38}$$

Here we can use following expansions: $(\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - (d-4)\frac{\log\Delta}{2} + \mathrm{O}\left((d-4)^2\right) \quad \text{around } d=4, \tag{1.39}$$

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathrm{O}(x) \quad \text{around } x=0, \tag{1.40}$$

$$\Gamma(x) = \frac{(-1)^n}{n!}\left[\frac{1}{x+n} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + \mathrm{O}(x+n)\right] \quad \text{around } x=-n. \tag{1.41}$$

and we get following expansion:

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left[\left(\frac{2}{4-d} - \gamma + \log 4\pi\right) - \log\Delta + \mathrm{O}(4-d)\right]. \tag{1.42}$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S-matrix elements. To compute the S-matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \to \left(\frac{1}{\Delta - \mathrm{i}\epsilon}\right)^{n-\frac{d}{2}}. \tag{1.43}$$

1.8 Cross Sections and Decay Rates

■General expression (The mass dimension of $\mathcal{M}$ is $2 - N_\mathrm{f}$ for $\mathrm{d}\sigma$ and $3 - N_\mathrm{f}$ for $\mathrm{d}\Gamma$.)

$$\mathrm{d}\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(p_A, p_B \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \{p_f\}\right) \tag{1.44}$$

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(m_A \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(m_A - \{p_f\}\right) \quad \text{(in A-rest frame.)} \tag{1.45}$$

■2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\delta^{(4)}\left(E_\mathrm{cm} - (p_1 + p_2)\right) \qquad \text{(in center-of-mass frame)} \tag{1.46}$$

$$= \int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}\frac{2\left\|\boldsymbol{p_1}\right\|}{E_\mathrm{cm}} \tag{1.47}$$

$$= \frac{1}{8\pi}\sqrt{1 - \frac{2({m_1}^2 + {m_2}^2)}{E_\mathrm{cm}^2} + \frac{({m_1}^2 - {m_2}^2)^2}{E_\mathrm{cm}^4}} \quad \longrightarrow [m_2 = 0]\ \frac{1}{8\pi}\left(1 - \frac{{m_1}^2}{{E_\mathrm{cm}}^2}\right) \tag{1.48}$$

■Kinematics of Decay

$$K \to p_1 + p_2 \qquad \text{or} \qquad \begin{pmatrix} M \\ \boldsymbol{0} \end{pmatrix} \to \begin{pmatrix} \sqrt{p^2 + {m_1}^2} \\ \boldsymbol{p} \end{pmatrix} + \begin{pmatrix} \sqrt{p^2 + {m_2}^2} \\ -\boldsymbol{p} \end{pmatrix}; \tag{1.49}$$

$$\left\|\boldsymbol{p}\right\|^2 = \frac{1}{4}\left[M^2 - 2\left({m_1}^2 + {m_2}^2\right) + \frac{\left({m_1}^2 - {m_2}^2\right)^2}{M^2}\right] \approx \left(\frac{M^2 - {m_1}^2}{2M}\right)^2$$

$$E_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2M}, \qquad E_2 = \frac{M^2 + {m_2}^2 - {m_1}^2}{2M};$$

$$K \cdot p_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2}, \qquad p_1 \cdot p_2 = \frac{M^2 - \left({m_1}^2 + {m_2}^2\right)}{2}.$$

■Mandelstam Variables

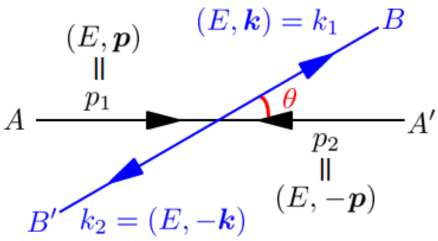
For $p_1 + p_2 \to k_1 + k_2$ collision,

$$s = (p_1 + p_2)^2 = (k_1 + k_2)^2,$$
$$t = (p_1 - k_1)^2 = (p_2 - k_2)^2,$$
$$u = (p_1 - k_2)^2 = (p_2 - k_1)^2,$$

and

$$s + t + u = p_1^2 + p_2^2 + k_1^2 + k_2^2 = \sum m^2.$$

■Kinematics of Collision (Same Mass)



$$\left\|\boldsymbol{p}\right\|^2 = E^2 - m_A^2 \qquad \boldsymbol{p} \cdot \boldsymbol{k} = \left\|\boldsymbol{p}\right\|\left\|\boldsymbol{k}\right\|\cos\theta$$
$$\left\|\boldsymbol{k}\right\|^2 = E^2 - m_B^2$$

$$p_1 \cdot p_2 = s/2 - m_A^2 \quad p_1 \cdot k_1 = p_2 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - t);$$
$$k_1 \cdot k_2 = s/2 - m_B^2 \quad p_1 \cdot k_2 = p_1 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - u);$$

$$s = 4E^2,$$

$$(p_1 - p_2)^2 = -4(E^2 - m_A^2) \qquad t = -(2E^2 - m_A^2 - m_B^2) + 2\boldsymbol{p}\cdot\boldsymbol{k}$$
$$(k_1 - k_2)^2 = -4(E^2 - m_B^2) \qquad u = -(2E^2 - m_A^2 - m_B^2) - 2\boldsymbol{p}\cdot\boldsymbol{k}$$

1.9 楊-Mills Theory (See App. C.5 for verbose notes.)

◆1.9.1 Non-Abelian gauge theory

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_c T^c, \qquad 0 = f^D{}_{ab} f^E{}_{Dc} + f^D{}_{ca} f^E{}_{Db} + f^D{}_{bc} f^E{}_{Da}, \qquad \mathrm{D}_\mu = \partial_\mu - \mathrm{i}gA_\mu$$

$$\operatorname{Tr} T^a T^b = \frac{1}{2}\delta^{ab}, \qquad [\widetilde{T}^a]_i{}^j := T^{\mathrm{ad}\ a}{}_i{}^j := -\mathrm{i}f^{aij} \qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j := \delta_i^j \partial_\mu + g f^{iaj} A^a_\mu.$$

$$\begin{aligned} F_{\mu\nu} &= \frac{\mathrm{i}}{g}\,[\mathrm{D}_\mu, \mathrm{D}_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{\mathrm{i}}\,[A_\mu, A_\nu] \\ &= \Big[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\Big] T^a \end{aligned}$$

$$\begin{aligned} &\mathrm{D}_\mu\phi = \partial_\mu\phi - \mathrm{i}gA^a_\mu (T^a_\phi \phi) \\ &\mathrm{D}_\mu F_{\mu\nu}{}^a = \partial_\mu F^a_{\mu\nu} + g f^{abc} A^b_\mu F^c_{\mu\nu}, \\ &\Big(\mathrm{D}_\mu F_{\nu\rho} = \partial_\mu \lambda - \mathrm{i}g[A_\mu, F_{\nu\rho}]\Big).^{*1} \end{aligned}$$

$$\phi \mapsto V\phi := \mathrm{e}^{\mathrm{i}g\theta}\phi \qquad A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1} \qquad F_{\mu\nu} \mapsto V F_{\mu\nu} V^{-1}$$

$$\phi^{a\prime} \simeq \phi + \mathrm{i}g\theta^a T^a \phi \qquad A^a_\mu{}' \simeq A^a_\mu + \partial_\mu\theta^a + g f^{abc} A^b_\mu \theta^c \qquad F^a_{\mu\nu}{}' \simeq F^a_{\mu\nu} + g f^{abc} F^b_{\mu\nu}\theta^c$$

$$\epsilon^{\mu\nu\rho\sigma}\,[\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\sigma]] = \epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma} = 0.$$

■Killing and Casimir Here we have two constants which **depend on representation** r.

$$\operatorname{Tr}(T^a T^b) =: C(r)\delta^{ab} \quad \text{(Killing form)}, \qquad T^a T^a =: C_2(r)\cdot \mathbf{1} \quad \text{(quadratic Casimir operator)}, \tag{1.50}$$

which satisfy

$$C(r) = \frac{d(r)}{d(\mathrm{ad})} C_2(r), \qquad T^a T^b T^a = \left[C_2(r) - \frac{1}{2}C_2(\mathrm{ad})\right] T^b, \tag{1.51}$$

$$f^{acd} f^{bcd} = C_2(\mathrm{ad})\delta^{ab}, \qquad f^{abc} T^b T^c = \frac{\mathrm{i}}{2} C_2(\mathrm{ad}) T^a. \tag{1.52}$$

<u>For SU(N)</u> For its fundamental representation N with definition $C(N) := \frac{1}{2}$, we have

$$C(N) := \frac{1}{2}, \quad C_2(N) = \frac{N^2-1}{2N}, \quad C(\mathrm{ad}) = C_2(\mathrm{ad}) = N; \quad (T^a)_{ij}(T^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{\delta_{ij}\delta_{kl}}{N}\right).$$

◆1.9.2 Abelian gauge theory

In Abelian gauge theory, V and fields are always commutative, and thus we have charge freedom (Q).

$$\mathrm{D}_\mu\phi = (\partial_\mu - \mathrm{i}gA_\mu Q)\,\phi \qquad \phi \mapsto \mathrm{e}^{igQ\theta}\phi \qquad F_{\mu\nu} = \frac{\mathrm{i}}{g}\,[\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathrm{D}_\mu\lambda^a = \partial_\mu\lambda^a \qquad A_\mu \mapsto A_\mu + \partial_\mu\theta \qquad F_{\mu\nu} \mapsto F_{\mu\nu}$$

◆1.9.3 Lagrangian Block

$$\mathcal{L} \ni |\mathrm{D}_\mu\phi|^2 - m^2|\phi|^2, \quad \overline{\psi}(\mathrm{i}\not{\mathrm{D}} - m)\psi, \quad -\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu}\ \Big(= -\frac{1}{2}\operatorname{Tr} F^{\mu\nu}F_{\mu\nu}\Big), \quad \theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{1.53}$$

$$-\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} = -\frac{1}{2}\left[(\partial_\mu A^a_\nu)^2 + A^a_\mu \partial^\mu\partial^\nu A^a_\nu\right] - g f^{abc} A^a_\mu A^b_\nu \partial^\mu A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \tag{1.54}$$

---

*1 Note that we can use any representation T^a but must the same ones for $A^a_\mu T^a$ and $\lambda^a T^a$.

2 Standard Model

Any representations assumed to be *normalized Hermitian.* Note that the SU(2) **2** representation is

$$T^a = \frac{1}{2}\sigma^a; \qquad \left[T^a, T^b\right] = \mathrm{i}\epsilon^{abc}T^c; \qquad T^{\pm} := T^1 \pm \mathrm{i}T^2. \tag{2.1}$$

We use the following abridged notations:

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad\qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.2}$$

2.1 Symmetries and Fields

	$\mathrm{SU}(3)_{\mathrm{strong}}$	$\mathrm{SU}(2)_{\mathrm{weak}}$	$\mathrm{U}(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_{\mathrm{L}}Q_i$: Left-handed quarks	**3**	**2**	1/6
$P_{\mathrm{L}}U_i$: Right-handed up-type quarks	**3**	**1**	2/3
$P_{\mathrm{R}}D_i$: Right-handed down-type quarks	**3**	**1**	−1/3
$P_{\mathrm{R}}L_i$: Left-handed leptons	**1**	**2**	−1/2
$P_{\mathrm{R}}E_i$: Right-handed leptons	**1**	**1**	−1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	**1**	**2**	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	**8**	**1**	0
W : Weak bosons	**1**	**3**	0
B : B boson	**1**	**1**	0

■Full Lagrangian $\mathcal{L} = \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{Higgs}} + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{湯川}$

where
$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{\mathrm{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right)H\right|^2 - V(H), \tag{2.4}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{matter}} = {} & \overline{Q}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{L}}Q_i \\ & + \overline{U}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \frac{2}{3}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}}U_i \\ & + \overline{D}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu + \frac{1}{3}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}}D_i \\ & + \overline{L}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_2 W_\mu + \frac{1}{2}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{L}}L_i \\ & + \overline{E}_i \mathrm{i}\gamma^\mu \left(\partial_\mu + \mathrm{i}g_1 B_\mu\right) P_{\mathrm{R}}E_i,\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{湯川} = \overline{U}_i (y_u)_{ij} H P_{\mathrm{L}} Q_j - \overline{D}_i (y_d)_{ij} H^\dagger P_{\mathrm{L}} Q_j - \overline{E}_i (y_e)_{ij} H^\dagger P_{\mathrm{L}} L_j + \mathrm{H.c.} \tag{2.6}$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix C.4.

■Gauge Kinetic Terms the gauge kinetic terms can be expanded as

$$\begin{aligned}\mathcal{L}_{\text{gauge}} =& -\frac{1}{4}(\partial B)(\partial B)\\ &-\frac{1}{4}(\partial W^a)(\partial W^a) - g_2\epsilon^{abc}(\partial_\mu W^a_\nu)W^{\mu b}W^{\nu c} - \frac{{g_2}^2}{4}\left(\epsilon^{eab}W^a_\mu W^b_\nu\right)\left(\epsilon^{ecd}W^{c\mu}W^{d\nu}\right)\\ &-\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G^a_\nu)G^{\mu b}G^{\nu c} - \frac{{g_3}^2}{4}\left(f^{eab}G^a_\mu G^b_\nu\right)\left(f^{ecd}G^{c\mu}G^{d\nu}\right).\end{aligned} \tag{2.7}$$

2.2 Higgs Mechanism

■Higgs Potential The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2(H^\dagger H) + \lambda\left(H^\dagger H\right)^2. \tag{2.8}$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + \mathrm{i}\phi_2\\ v + (h + \mathrm{i}\phi_3)\end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.9}$$

Here h is the "Higgs boson," and ϕ_i are 南部–Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2 h^2, \tag{2.10}$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right)H\right|^2 \tag{2.11}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\Big[{g_2}^2{W_1}^2 + {g_2}^2{W_2}^2 + (g_1 B - g_2 W_3)^2\Big]. \tag{2.12}$$

Redefining the gauge fields (with concerning the norms) as

$$W^\pm_\mu := \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}W^2_\mu), \qquad \begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix} := \begin{pmatrix}\cos\theta_{\rm w} & -\sin\theta_{\rm w}\\ \sin\theta_{\rm w} & \cos\theta_{\rm w}\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}, \tag{2.13}$$

where

$$\tan\theta_{\rm w} := \frac{g_1}{g_2}, \qquad e := -\frac{g_1 g_2}{\sqrt{{g_1}^2 + {g_2}^2}}; \qquad g_Z := \sqrt{{g_1}^2 + {g_2}^2}; \tag{2.14}$$

$$g_1 = \frac{|e|}{\cos\theta_{\rm w}} = g_Z\sin\theta_{\rm w}, \qquad g_2 = \frac{|e|}{\sin\theta_{\rm w}} = g_Z\cos\theta_{\rm w}. \tag{2.15}$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4}\Big[{g_2}^2 W^{+\mu}W^-_\mu + \frac{{g_Z}^2}{2}Z^\mu Z_\mu\Big]. \tag{2.16}$$

Here we have omitted the 南部–Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan\theta_{\rm w} Z_\mu, \tag{2.17}$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) + \left(\frac{|e|}{\tan\theta_{\rm w}} Z_\mu + |e| A_\mu\right) T^3, \tag{2.18}$$

$$Z^0_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_2 W^3_\mu - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_1 W^3_\mu + g_2 B_\mu) \tag{2.19}$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \tag{2.20}$$

■Gauge Term The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W^a_{\mu\nu} = {} & (\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ & - 4{\rm i}g\left[(\partial W^3)^{\mu\nu} W^+_\mu W^-_\nu + (\partial W^+)^{\mu\nu} W^-_\mu W^3_\nu + (\partial W^-)^{\mu\nu} W^3_\mu W^+_\nu\right] \\ & - 2g^2(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left(W^+_\mu W^+_\nu W^-_\rho W^-_\sigma - 2W^3_\mu W^3_\nu W^+_\rho W^-_\sigma\right), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_{\rm gauge} := {} & -\frac{1}{4}\left[G^{a\mu\nu} G^a_{\mu\nu} + (\partial Z)^{\mu\nu}(\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu}(\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu}(\partial W^-)_{\mu\nu}\right] \\ & + \frac{{\rm i}|e|}{\tan\theta_{\rm w}}\Big[(\partial W^+)^{\mu\nu} W^-_\mu Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W^+_\nu + (\partial Z)^{\mu\nu} W^+_\mu W^-_\nu\Big] \\ & + {\rm i}|e|\Big[(\partial W^+)^{\mu\nu} W^-_\mu A_\nu + (\partial W^-)^{\mu\nu} A_\mu W^+_\nu + (\partial A)^{\mu\nu} W^+_\mu W^-_\nu\Big] \\ & + (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\Bigg[\frac{|e|^2}{2\sin^2\theta_{\rm w}} W^+_\mu W^+_\nu W^-_\rho W^-_\sigma + \frac{|e|^2}{\tan^2\theta_{\rm w}} W^+_\mu Z_\nu W^-_\rho Z_\sigma \\ & \qquad + \frac{|e|^2}{\tan\theta_{\rm w}}\left(W^+_\mu Z_\nu W^-_\rho A_\sigma + W^+_\mu A_\nu W^-_\rho Z_\sigma\right) + |e|^2 W^+_\mu A_\nu W^-_\rho A_\sigma\Bigg]. \end{aligned} \tag{2.21}$$

■湯川 Term

$$\begin{aligned} \mathcal{L}_{湯川} = {} & \overline{U} y_u H P_{\rm L} Q - \overline{D} y_d H^\dagger P_{\rm L} Q - \overline{E} y_e H^\dagger P_{\rm L} L + \text{H.c.} \\ = {} & \overline{U} y_u \epsilon^{\alpha\beta} H^\alpha P_{\rm L} Q^\beta - \overline{D} y_d {H^\dagger}^\alpha P_{\rm L} Q^\alpha - \overline{E} y_e {H^\dagger}^\alpha P_{\rm L} L^\alpha + \text{H.c.} \\ = {} & -\frac{v+h}{\sqrt{2}}\left(\overline{U} y_u P_{\rm L} Q^1 + \overline{D} y_d P_{\rm L} Q^2 + \overline{E} y_e P_{\rm L} L^2\right) + \text{H.c.} \end{aligned} \tag{2.22}$$

2.3 Full Lagrangian After Higgs Mechanism

Now we have the following Lagrangian (with omitting P_{L} etc.):

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} + {m_W}^2 W^+W^- + \frac{{m_Z}^2}{2}Z^2 \\
\text{【Higgs】}\quad & + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}{m_h}^2h^2 - \sqrt{\frac{\lambda}{2}}m_h h^3 - \frac{1}{4}\lambda h^4 \\
& + \frac{v{g_2}^2}{4}W^+W^-h + \frac{v({g_1}^2+{g_2}^2)}{8}Z^2h \\
& + \frac{{g_2}^2}{4}W^+W^-h^2 + \frac{{g_1}^2+{g_2}^2}{8}Z^2h^2 \\
& - \left(\frac{1}{\sqrt{2}}h\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}h\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}h\bar{E}y_eL^2 + \text{H.c.}\right) \\
\text{【SU(3)】}\quad & + \bar{Q}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)Q + \bar{U}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)U + \bar{D}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)D + \bar{L}\left(\mathrm{i}\partial\!\!\!/\right)L + \bar{E}\left(\mathrm{i}\partial\!\!\!/\right)E \\
\text{【}W\text{】}\quad & + \bar{Q}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+T^+ + W\!\!\!/^-T^-\right)Q + \bar{L}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+T^+ + W\!\!\!/^-T^-\right)L \\
\text{【}A\&Z^0\text{】}\quad & + \bar{Q}\left[\left(T^3+\frac{1}{6}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right)Z\!\!\!/^0\right]Q \\
& + \bar{U}\left(\frac{2}{3}|e|A\!\!\!/ - \frac{2|e|s}{3c}Z\!\!\!/\right)U \\
& + \bar{D}\left(-\frac{1}{3}|e|A\!\!\!/ + \frac{|e|s}{3c}Z\!\!\!/\right)D \\
& + \bar{L}\left[\left(T^3-\frac{1}{2}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right)Z\!\!\!/^0\right]L \\
& + \bar{E}\left(-|e|A\!\!\!/ + \frac{|e|s}{c}Z\!\!\!/\right)E \\
\text{【湯川項】}\quad & - \left(\frac{1}{\sqrt{2}}v\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}v\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}v\bar{E}y_eL^2 + \text{H.c.}\right)
\end{aligned}
\tag{2.23}
$$

2.4 Mass Eigenstates

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := vy_\bullet/\sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \qquad (m_i \geq 0). \tag{2.24}
$$

Using this Ψ and Φ, we can rotate the basis as

$$
Q^1 \mapsto \Psi_u^\dagger Q^1,\quad Q^2 \mapsto \Psi_d^\dagger Q^2,\quad L \mapsto \Psi_e^\dagger L,\qquad U \mapsto \Phi_u^\dagger U,\quad D \mapsto \Phi_d^\dagger D,\quad E \mapsto \Phi_e^\dagger E, \tag{2.25}
$$

and now we have the 湯川 terms in mass eigenstates as

$$
\mathcal{L}_{\text{湯川}} = -\left(1+\frac{1}{v}h\right)\left[(m_u)_i\overline{U}_iP_{\mathrm{L}}Q_i^1 + (m_d)_i\overline{D}_iP_{\mathrm{L}}Q_i^2 + (m_e)_i\overline{E}_iP_{\mathrm{L}}L_i^2 + \text{H.c.}\right]. \tag{2.26}
$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark–W interactions do change drastically, as

$$\begin{aligned}
\mathcal{L} &\supset \overline{Q}\mathrm{i}\gamma^\mu \left(-\mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) P_\mathrm{L} Q && (2.27)\\
&= \overline{Q}\frac{g_2}{\sqrt{2}}\left(\not{W}^+ T^+ + \not{W}^- T^-\right) P_\mathrm{L} Q \quad + \quad \text{(interaction terms with } Z \text{ and } A) && (2.28)\\
&\mapsto \frac{g_2}{\sqrt{2}}\begin{pmatrix}\overline{Q}^1\Psi_u & \overline{Q}^2\Psi_d\end{pmatrix}\begin{pmatrix}0 & \not{W}^+\\ \not{W}^- & 0\end{pmatrix} P_\mathrm{L}\begin{pmatrix}\Psi_u^\dagger Q^1\\ \Psi_d^\dagger Q^2\end{pmatrix} + (\quad\cdots\quad) && (2.29)\\
&= \frac{g_2}{\sqrt{2}}\left[\overline{Q}^2\not{W}^- X P_\mathrm{L} Q^1 + \overline{Q}^1\not{W}^+ X^\dagger P_\mathrm{L} Q^2\right] + (\quad\cdots\quad), && (2.30)
\end{aligned}$$

where $X := \Psi_d\Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP-symmetry.

> In our notation, CP-transformation of a spinor is described as
>
> $$\mathscr{C}\mathscr{P}\left(\psi\right) = -\mathrm{i}\eta^*(\overline{\psi}\gamma^2)^\mathrm{T}, \quad \mathscr{C}\mathscr{P}\left(\overline{\psi}\right) = \mathrm{i}\eta(\gamma^2\psi)^\mathrm{T}, \tag{2.31}$$
>
> where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,
>
> $$\begin{aligned}\mathscr{C}\mathscr{P}\left(\overline{Q}^2\not{W}^- X P_\mathrm{L} Q^1\right) &= (\gamma^2 Q^2)^\mathrm{T}\mathscr{P}(-\not{W}^+) X P_\mathrm{L}(\overline{Q}^1\gamma^2)^\mathrm{T}\\ &= -W_\mu^{+\,P}(\gamma^2 Q^2)^\mathrm{T}(\overline{Q}^1 X^\mathrm{T}\gamma^2 P_\mathrm{L}\gamma^{\mu\,\mathrm{T}})^\mathrm{T}\\ &= (\overline{Q}^1\not{W}^+ X^\mathrm{T} P_\mathrm{L} Q^2).\end{aligned}\tag{2.32}$$
>
> Therefore, we can see that the CP-symmetry is maintained if and only if $X^\mathrm{T} = X^\dagger$, that is, if and only if X is a real matrix.

以上より，標準模型の Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} \\
\text{【質量項】} \quad & + \quad m_W{}^2 W^+ W^- \quad + \quad \frac{m_Z{}^2}{2} Z^2 \\
& - \quad \left(\bar{U} M_u P_{\rm L} Q^1 \; + \; \bar{D} M_d P_{\rm L} Q^2 \; + \; \bar{E} M_e P_{\rm L} L^2 \; + \; \text{H.c.} \right) \\
\text{【 Higgs Field 】} \quad & + \quad \frac{1}{2}(\partial_\mu h)^2 \quad - \frac{1}{2} m_h{}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
\text{【 Higgs との結合】} \quad & + \quad \frac{v g_2{}^2}{4} W^+ W^- h \quad + \quad \frac{v(g_1{}^2 + g_2{}^2)}{8} Z^2 h \\
& + \quad \frac{g_2{}^2}{4} W^+ W^- h^2 \quad + \quad \frac{g_1{}^2 + g_2{}^2}{8} Z^2 h^2 \\
& - \quad \left(\frac{1}{v} \bar{U} M_u P_{\rm L} Q^1 h \, + \, \frac{1}{v} \bar{D} M_d P_{\rm L} Q^2 h \, + \, \frac{1}{v} \bar{E} M_e P_{\rm L} L^2 h \; + \; \text{H.c.} \right) \\
\text{【 SU(3) および微分項】} \quad & + \quad \bar{Q} \left(\mathrm{i}\not\partial + g_3 \not{G} \right) P_{\rm L} Q \quad + \quad \bar{U} \left(\mathrm{i}\not\partial + g_3 \not{G} \right) P_{\rm R} U \quad + \quad \bar{D} \left(\mathrm{i}\not\partial + g_3 \not{G} \right) P_{\rm R} D \\
& + \quad \bar{L} \left(\mathrm{i}\not\partial \right) P_{\rm L} L \quad + \quad \bar{E} \left(\mathrm{i}\not\partial \right) P_{\rm R} E \\
\text{【 } W \text{ boson 】} \quad & + \quad \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \not{W}^- X P_{\rm L} Q^1 + \bar{Q}^1 \not{W}^+ X^\dagger P_{\rm L} Q^2 \right] \qquad \text{【} \leftarrow CP \text{ and flavor violating!】} \\
& + \quad \bar{L} \frac{g_2}{\sqrt{2}} \left(\not{W}^+ T^+ + \not{W}^- T^- \right) P_{\rm L} L \\
\text{【 } A\&Z^0 \text{ boson 】} \quad & + \quad \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| \not{A} + \left(\frac{|e| c}{s} T^3 - \frac{|e| s}{6c} \right) \not{Z}^0 \right] P_{\rm L} Q \\
& + \quad \bar{U} \left(\frac{2}{3} |e| \not{A} - \frac{2|e| s}{3c} \not{Z} \right) P_{\rm R} U \\
& + \quad \bar{D} \left(-\frac{1}{3} |e| \not{A} + \frac{|e| s}{3c} \not{Z} \right) P_{\rm R} D \\
& + \quad \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| \not{A} + \left(\frac{|e| c}{s} T^3 + \frac{|e| s}{2c} \right) \not{Z}^0 \right] P_{\rm L} L \\
& + \quad \bar{E} \left(-|e| \not{A} + \frac{|e| s}{c} \not{Z} \right) P_{\rm R} E
\end{aligned}
\tag{2.33}
$$

となる。

2.5 Chiral Notation

In the chiral expression, the Lagrangian is written as

$$
\begin{aligned}
\mathcal{L} =&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ Q_\mathrm{L}^\dagger \mathrm{i}\bar\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) Q_\mathrm{L}\\
&+ U_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \frac{2}{3}\mathrm{i}g_1 B_\mu\right) U_\mathrm{R}\\
&+ D_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu + \frac{1}{3}\mathrm{i}g_1 B_\mu\right) D_\mathrm{R}\\
&+ L_\mathrm{L}^\dagger \mathrm{i}\bar\sigma^\mu \left(\partial_\mu - \mathrm{i}g_2 W_\mu + \frac{1}{2}\mathrm{i}g_1 B_\mu\right) L_\mathrm{L}\\
&+ E_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu + \mathrm{i}g_1 B_\mu\right) E_\mathrm{R}\\
&- \left(U_\mathrm{R}^\dagger y_u H Q_\mathrm{L} + D_\mathrm{R}^\dagger y_d H^\dagger Q_\mathrm{L} + E_\mathrm{R}^\dagger y_e H^\dagger L_\mathrm{L} + \text{H.c.}\right)\\
=&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ \mathrm{i}Q_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu Q_\mathrm{L} + \mathrm{i}U_\mathrm{R}\bar\sigma^\mu\partial_\mu U_\mathrm{R}^\dagger + \mathrm{i}D_\mathrm{R}\bar\sigma^\mu\partial_\mu D_\mathrm{R}^\dagger + \mathrm{i}L_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu L_\mathrm{L} + \mathrm{i}E_\mathrm{R}\bar\sigma^\mu\partial_\mu E_\mathrm{R}^\dagger\\
&+ g_3\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu G_\mu Q_\mathrm{L} + U_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu U_\mathrm{R} + D_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu D_\mathrm{R}\right)\\
&+ g_2\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu Q_\mathrm{L} + L_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu L_\mathrm{L}\right)\\
&+ g_1\left(\frac{1}{6}Q_\mathrm{L}^\dagger\bar\sigma^\mu B_\mu Q_\mathrm{L} + \frac{2}{3}U_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu U_\mathrm{R} - \frac{1}{3}D_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu D_\mathrm{R} - \frac{1}{2}L_\mathrm{L}^\dagger\bar\sigma^\mu B_\mu L_\mathrm{L} - E_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu E_\mathrm{R}\right)\\
&- \left(U_\mathrm{R}^\dagger y_u H Q_\mathrm{L} + D_\mathrm{R}^\dagger y_d H^\dagger Q_\mathrm{L} + E_\mathrm{R}^\dagger y_e H^\dagger L_\mathrm{L} + \text{H.c.}\right),
\end{aligned}
\tag{2.34}
$$

and finally we obtain

$$
\begin{aligned}
\mathcal{L} =&(\text{Gauge bosons and Higgs})\\
&+ \mathrm{i}Q_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu Q_\mathrm{L} + \mathrm{i}U_\mathrm{R}\bar\sigma^\mu\partial_\mu U_\mathrm{R}^\dagger + \mathrm{i}D_\mathrm{R}\bar\sigma^\mu\partial_\mu D_\mathrm{R}^\dagger + \mathrm{i}L_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu L_\mathrm{L} + \mathrm{i}E_\mathrm{R}\bar\sigma^\mu\partial_\mu E_\mathrm{R}^\dagger\\
&+ g_3\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu G_\mu Q_\mathrm{L} + U_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu U_\mathrm{R} + D_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu D_\mathrm{R}\right)\\
&- m_u(u_\mathrm{R}^\dagger u_\mathrm{L} + u_\mathrm{L}^\dagger u_\mathrm{R}) - (\text{quarks}) - m_e(e_\mathrm{R}^\dagger e_\mathrm{L} + e_\mathrm{L}^\dagger e_\mathrm{R}) - (\text{leptons})\\
&- \frac{m_u}{v}(u_\mathrm{R}^\dagger u_\mathrm{L} + u_\mathrm{L}^\dagger u_\mathrm{R})h - (\text{quarks}) - \frac{m_e}{v}(e_\mathrm{R}^\dagger e_\mathrm{L} + e_\mathrm{L}^\dagger e_\mathrm{R})h - (\text{leptons})\\
&+ \frac{g_2}{\sqrt{2}}\left[\begin{pmatrix} d_\mathrm{L}^\dagger & s_\mathrm{L}^\dagger & b_\mathrm{L}^\dagger\end{pmatrix}\bar\sigma^\mu W_\mu^- X\begin{pmatrix} u_\mathrm{L}\\ c_\mathrm{L}\\ t_\mathrm{L}\end{pmatrix} + \begin{pmatrix} u_\mathrm{L}^\dagger & c_\mathrm{L}^\dagger & t_\mathrm{L}^\dagger\end{pmatrix}\bar\sigma^\mu W_\mu^+ X^\dagger\begin{pmatrix} d_\mathrm{L}\\ s_\mathrm{L}\\ b_\mathrm{L}\end{pmatrix}\right]\\
&+ \frac{g_2}{\sqrt{2}}\left[\nu_e^\dagger\bar\sigma^\mu W_\mu^+ e_\mathrm{L} + e_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu^-\nu_e\right]\\
&+ |e|\left[\frac{2}{3}u_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu u_\mathrm{L} - \frac{1}{3}d_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu d_\mathrm{L} + \frac{2}{3}u_\mathrm{R}^\dagger\sigma^\mu A_\mu u_\mathrm{R} - \frac{1}{3}d_\mathrm{R}^\dagger\sigma^\mu A_\mu d_\mathrm{R} + (\text{quarks})\right.\\
&\qquad\left. - e_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu e_\mathrm{L} - e_\mathrm{R}^\dagger\sigma^\mu A_\mu e_\mathrm{R} + (\text{leptons})\right]\\
&+ \frac{|e|s}{c}\left[\left(\frac{c^2}{2s^2} - \frac{1}{6}\right)u_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu u_\mathrm{L} - \left(\frac{c^2}{2s^2} + \frac{1}{6}\right)d_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu d_\mathrm{L} - \frac{2}{3}u_\mathrm{R}^\dagger\sigma^\mu Z_\mu u_\mathrm{R} + \frac{1}{3}d_\mathrm{R}^\dagger\sigma^\mu Z_\mu d_\mathrm{R}\right.\\
&\qquad\left. + \left(\frac{c^2}{2s^2} + \frac{1}{2}\right)\nu_e^\dagger\bar\sigma^\mu Z_\mu\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2}\right)e_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu e_\mathrm{L} + e_\mathrm{R}^\dagger\sigma^\mu Z_\mu e_\mathrm{R} + (\text{others})\right].
\end{aligned}
\tag{2.35}
$$

2.6 Values of SM Parameters (Extracted from PDG 2010 / 2012)

◆2.6.1 Experimental Values

Theoretical Parameters 【 These values are all in $\overline{\text{MS}}$ scheme. 】

$$\alpha_{\text{EM}}^{-1}(0) = 137.035999074(44) \qquad G_{\text{F}} = \frac{{g_2}^2}{4\sqrt{2}{m_W}^2} = \frac{1}{\sqrt{2}v^2} = 1.1663787(6) \times 10^{-5}\,\text{GeV}^{-2}$$

$$\alpha_{\text{EM}}^{-1}(m_Z) = 127.944(14) \qquad m_W(m_W) = 80.385(15)\,\text{GeV} \qquad \Gamma_W \approx 2.085(42)\,\text{GeV}$$

$$\alpha_{\text{EM}}^{-1}(m_\tau) = 133.471(14) \qquad m_Z(m_Z) = 91.1876(21)\,\text{GeV} \qquad \Gamma_Z \approx 2.4952(23)\,\text{GeV}$$

$$\alpha_{\text{s}}(m_Z) = 0.1184(7) \qquad \sin^2\theta_{\text{W}}(m_Z) = 0.23116(12) \qquad \sin^2\theta_{\text{eff}} = 0.23146(12)$$

Masses and Lifetimes 【 t pole mass is the “MC mass”. Quark $\overline{\text{MS}}$ mass at 1 GeV can be obtained by ×1.35. 】

$e : 0.510998928(11)\,\text{MeV}$ $\qquad \mu : 105.6583715(35)\,\text{MeV}$ $\qquad \tau : 1.77682(16)\,\text{GeV}$

$[\overline{\text{MS}}\ (2\,\text{GeV})]$	$u : 2.3^{+0.7}_{-0.5}\,\text{MeV}$		
	$d : 4.8^{+0.7}_{-0.3}\,\text{MeV}$		
	$s : 95 \pm 5\,\text{MeV}$		
$[\overline{\text{MS}}(m)]$	$c : 1.275(25)\,\text{GeV}$	$[\text{pole}]$	$c : 1.67(7)\,\text{GeV}$
	$b : 4.18(3)\,\text{GeV}$		$b : 4.78(6)\,\text{GeV}$
	$t : 160^{+5}_{-4}\,\text{GeV}$		$t : 173.5(6)(8)\,\text{GeV}$

$\pi^\pm : 139.57018(35)\,\text{MeV}$ $\qquad K^\pm : 493.677(16)\,\text{MeV}$ $\qquad p : 938.272046(21)\,\text{MeV}$

$\pi^0 : 134.9766(6)\,\text{MeV}$ $\qquad K^0 : 497.614(24)\,\text{MeV}$ $\qquad n : 939.565379(21)\,\text{MeV}$

$\mu : 2.1969811(22)\,\mu\text{s}\ (659\,\text{m})$ $\qquad \pi^\pm : 2.6033(5) \times 10^{-8}\,\text{s}$ $\qquad K^\pm : 1.2380(21) \times 10^{-8}\,\text{s}\ (3.7\,\text{m})$

$\tau : 2.906(10) \times 10^{-13}\,\text{s}\ (87\,\mu\text{m})$ $\qquad \pi^0 : 8.52(18) \times 10^{-17}\,\text{s}$ $\qquad K_{\text{S}}^0 : 8.9564(33) \times 10^{-11}\,\text{s}\ (2.68\,\text{cm})$

$K_{\text{L}}^0 : 5.116(21) \times 10^{-8}\,\text{s}\ (15.3\,\text{m})$

Other Important Values

$a_e = 11596521.8076(27) \times 10^{-10}$ $\qquad d_e^{\text{EDM}} < 10.5 \times 10^{-28}\,e\,\text{cm}$ $\qquad \text{Br}(\tau \to e) = 17.83(4)\%$

$a_\mu = 11659209(6) \times 10^{-10}$ $\qquad d_\mu^{\text{EDM}} = -1(9) \times 10^{-20}\,e\,\text{cm}$ $\qquad \text{Br}(\tau \to \mu) = 17.41(4)\%$

$\sin^2 2\theta_{12} = 0.857(24)$ $\qquad \text{Br}(\tau \to \text{had}) \sim 64.8\%$

$\sin^2 2\theta_{23} > 0.95$ $\qquad \Delta m_{\nu 21}^2 = 7.50(20) \times 10^{-5}\,\text{eV}^2$

$\sin^2 2\theta_{13} = 0.098(13)$ $\qquad \left|\Delta m_{\nu 32}^2\right| = 0.00232(^{12}_{08})\,\text{eV}^2$

CKM matrix

$$V_{\text{CKM}} = \begin{pmatrix} 0.97425(22) & 0.2252(9) & 0.0084(6) \\ 0.230(11) & 1.006(23) & 0.0429(26) \\ 0.00415(49) & 0.0409(11) & 0.89(7) \end{pmatrix} \approx \begin{pmatrix} 0.97427(15) & 0.22534(65) & 0.00351(^{15}_{14}) \\ 0.22520(65) & 0.97344(16) & 0.0412(^{11}_{05}) \\ 0.00867(^{29}_{31}) & 0.0404(^{11}_{05}) & 0.999146(^{21}_{46}) \end{pmatrix}$$

$$\lambda = 0.22535(65), \quad A = 0.811^{+0.022}_{-0.012}, \quad \bar\rho = 0.131^{+0.026}_{-0.013}, \quad \bar\eta = 0.345^{+0.013}_{-0.014}; \quad J = (2.96^{+20}_{-16}) \times 10^{-5}$$

◆2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$|e| \sim 0.313, \qquad g_1 \sim 0.357, \qquad g_2 \sim 0.652, \qquad g_Z \sim 0.743; \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246\,\mathrm{GeV} \tag{2.36}$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.997 \end{pmatrix} \quad y_d \sim \begin{pmatrix} 3\times 10^{-5} & 0 & 0 \\ 0 & 0.0005 & 0 \\ 0 & 0 & 0.02 \end{pmatrix} \quad y_e \sim \begin{pmatrix} 3\times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix} \tag{2.37}$$

Also, for $m_h \sim 125\,\mathrm{GeV}$, we can estimate the Higgs potential as $\mu \sim 88\,\mathrm{GeV}$ and $\lambda \sim 0.13$.

3 Supersymmetry for $\eta = \mathrm{diag}(+,-,-,-)$

3.1 Spinor Convention

(See App. C.1.1 for a verbose explanation.)

ϵ tensor : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)

Sum Rule : ${}^{\alpha}{}_{\alpha}$ and ${}_{\dot\alpha}{}^{\dot\alpha}$, except for $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$, $\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta$, $\xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\xi^{\dot\beta}$, $\xi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\xi_{\dot\beta}$.

Lorentz 変換 : $\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$, $\bar\psi'_{\dot\alpha} = \bar\psi_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}$, $\psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha$, $\bar\psi'^{\dot\alpha} = (\Lambda^{-1})^{\dagger\dot\alpha}{}_{\dot\beta}\bar\psi'^{\dot\beta}$.

σ matrices : $(\sigma^\mu)_{\alpha\dot\beta} := (1,\boldsymbol{\sigma})_{\alpha\dot\beta}$, $(\bar\sigma^\mu)^{\dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1,-\boldsymbol{\sigma})^{\dot\alpha\beta}$.

3.2 Spinor Calculation Cheatsheet

$$\eta = (+,-,-,-), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1; \qquad \textbf{Left Differential};$$
$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot1\dot2} = \epsilon_{\dot2\dot1} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar\xi^{\dot\alpha} := \epsilon^{\dot\alpha\dot\beta}\bar\xi_{\dot\beta}, \quad \bar\xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta}$$
$$\bar\sigma^{\mu\dot\alpha\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} \qquad \sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\beta\beta}, \qquad \sigma^\mu := (1,\boldsymbol{\sigma}), \quad \bar\sigma^\mu := (1,-\boldsymbol{\sigma})$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta := \tfrac14\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} := \tfrac14\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)^{\dot\alpha}{}_{\dot\beta} = (\sigma^{\nu\mu})^{\dagger\dot\alpha}{}_{\dot\beta}.$$

$$\begin{array}{llll}
\theta^\alpha\theta^\beta = -\frac12\epsilon^{\alpha\beta}\theta\theta & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac12\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & (\theta\phi)(\theta\psi) = -\frac12(\psi\phi)(\theta\theta) & (\theta\sigma^\nu\bar\theta)\theta^\alpha = \frac12\theta\theta(\bar\theta\bar\sigma^\nu)^\alpha \\
\theta_\alpha\theta_\beta = \frac12\epsilon_{\alpha\beta}\theta\theta & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac12\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta & (\bar\theta\bar\phi)(\bar\theta\bar\psi) = -\frac12(\bar\psi\bar\phi)(\bar\theta\bar\theta) & (\theta\sigma^\nu\bar\theta)\bar\theta_{\dot\alpha} = -\frac12\bar\theta\bar\theta(\theta\sigma^\mu)_{\dot\alpha} \\
\theta^\alpha\theta_\beta = \frac12\delta^\alpha_\beta\theta\theta & \bar\theta_{\dot\alpha}\bar\theta^{\dot\beta} = \frac12\delta^{\dot\beta}_{\dot\alpha}\bar\theta\bar\theta & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac12\theta\theta\bar\theta\bar\theta\eta^{\mu\nu} & (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) = \frac12(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\bar\theta\bar\theta \\
\theta\sigma^\mu\bar\sigma^\nu\theta = \eta^{\mu\nu}\theta\theta & \bar\theta\bar\sigma^\mu\sigma^\nu\bar\theta = \eta^{\mu\nu}\bar\theta\bar\theta & &
\end{array}$$

$$\begin{array}{ll}
\sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar\sigma^\rho\sigma^\nu + \sigma^\nu\bar\sigma^\rho\sigma^\mu = 2\left(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho\right) \\
\bar\sigma^\mu\sigma^\nu = \eta^{\mu\nu} + 2\bar\sigma^{\mu\nu} & \bar\sigma^\mu\sigma^\rho\bar\sigma^\nu + \bar\sigma^\nu\sigma^\rho\bar\sigma^\mu = 2\left(\eta^{\mu\rho}\bar\sigma^\nu + \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\nu}\bar\sigma^\rho\right) \\
\sigma^{\mu\nu} = -\sigma^{\nu\mu} & \sigma^\mu\bar\sigma^\nu\sigma^\rho - \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\
\bar\sigma^{\mu\nu} = -\bar\sigma^{\nu\mu} & \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma \\
\mathrm{Tr}\,\bar\sigma^\mu\sigma^\nu = \mathrm{Tr}\,\sigma^\mu\bar\sigma^\nu = 2\eta^{\mu\nu} & \mathrm{Tr}\,\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\frac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\mathrm{Tr}\,\sigma^{\mu\nu} = \mathrm{Tr}\,\bar\sigma^{\mu\nu} = 0 & \mathrm{Tr}\,\bar\sigma^{\mu\nu}\bar\sigma^{\rho\sigma} = -\frac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\dot\beta\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} - \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + (\epsilon\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}\right] \\
\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} + \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = \eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} - 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar\sigma^{\sigma\nu})_{\dot\alpha\dot\beta}
\end{array}$$

$$\begin{array}{lll}
\bar\sigma^{\mu\dot\alpha\alpha}\bar\sigma^{\dot\beta\beta}_\mu = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} & \epsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\mathrm{i}\sigma^{\mu\nu} \\
\sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} = -2\mathrm{i}\bar\sigma^{\mu\nu}
\end{array}$$

$$\begin{array}{ll}
\bar\xi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\xi = (\bar\chi\bar\sigma^\mu\xi)^* = -(\xi\sigma^\mu\bar\chi)^* & (\psi\phi)\chi_\alpha = -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\
\xi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\xi = (\bar\chi\bar\sigma^\nu\sigma^\mu\bar\xi)^* = (\bar\xi\bar\sigma^\mu\sigma^\nu\bar\chi)^* & (\psi\phi)\bar\chi_{\dot\alpha} = \frac12(\phi\sigma^\mu\bar\chi)(\psi\sigma_\mu)_{\dot\alpha}
\end{array}$$

$$\begin{array}{lll}
\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\beta} = -\dfrac{\partial}{\partial\theta_\alpha} & \dfrac{\partial}{\partial\theta^\alpha}\theta\theta = 2\theta_\alpha & \epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta^\beta}\theta\theta = \dfrac{\partial}{\partial\theta_\alpha}\dfrac{\partial}{\partial\theta^\alpha}\theta\theta = 4 \\
\epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\beta} = -\dfrac{\partial}{\partial\theta^\alpha} & \dfrac{\partial}{\partial\theta_\alpha}\theta\theta = -2\theta^\alpha & \epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\alpha}\dfrac{\partial}{\partial\theta_\beta}\theta\theta = \dfrac{\partial}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta_\alpha}\theta\theta = -4 \\
\epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -2\bar\theta_{\dot\alpha} & \epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}}\bar\theta\bar\theta = \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 4 \\
\epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 2\bar\theta^{\dot\alpha} & \epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}}\bar\theta\bar\theta = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -4
\end{array}$$

3.3 General Relations

(Note: $P_\mu = \mathrm{i}\partial_\mu$ in our convention.)

$$Q_\alpha := \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad D_\alpha := \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad y := x - \mathrm{i}\theta\sigma\bar\theta,$$

$$\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad y^\dagger = x + \mathrm{i}\theta\sigma\bar\theta$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{D_\alpha, \bar D_{\dot\alpha}\} = 2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad (\text{others}) = 0.$$

$$\begin{array}{lllll}
 & < x\text{-basis} > & < y\text{-basis} > & < y^\dagger\text{-basis} > & \\
D_\alpha & = \dfrac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu & = \dfrac{\partial}{\partial\theta^\alpha} - 2\mathrm{i}(\sigma^\mu\bar\theta)_\alpha\dfrac{\partial}{\partial y^\mu} & = \dfrac{\partial}{\partial\theta^\alpha} & (3.1) \\
\bar D_{\dot\alpha} & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2\mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\dfrac{\partial}{\partial(y^\dagger)^\mu} & (3.2) \\
D^\alpha & = -\dfrac{\partial}{\partial\theta_\alpha} + \mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\partial_\mu & = -\dfrac{\partial}{\partial\theta_\alpha} + 2\mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\dfrac{\partial}{\partial y^\mu} & = -\dfrac{\partial}{\partial\theta_\alpha} & (3.3) \\
\bar D^{\dot\alpha} & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} - \mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} - 2\mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\dfrac{\partial}{\partial(y^\dagger)^\mu} & (3.4)
\end{array}$$

$$\phi(y) = \phi(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac14\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y^\dagger) - 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \tag{3.5}$$

$$\phi(y^\dagger) = \phi(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac14\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y) + 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y) \tag{3.6}$$

$$\phi(x) = \phi(y) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \frac14\theta\theta\bar\theta\bar\theta\partial^2\phi(y) = \phi(y^\dagger) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \frac14\theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \tag{3.7}$$

3.4 Chiral Superfields : $\bar{D}_{\dot\alpha}\Phi = 0$

Explicit Expression

$$\begin{aligned}
\Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) && (3.8)\\
&= \phi(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) && (3.9)\\
&= \phi(y^\dagger) - 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) + \sqrt{2}\theta\psi(y^\dagger) - \sqrt{2}\mathrm{i}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(y^\dagger) + \theta\theta F(y^\dagger) && (3.10)\\
\Phi^\dagger &= \phi^*(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta\bar\theta F^*(y^\dagger) && (3.11)\\
&= \phi^*(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \bar\theta\bar\theta F^*(x) && (3.12)\\
&= \phi^*(y) + 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi^*(y) + \sqrt{2}\bar\theta\bar\psi(y) - \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(y) + \bar\theta\bar\theta F^*(y) && (3.13)
\end{aligned}$$

Product of Chiral Superfields

$$\begin{aligned}
\Phi_i^\dagger\Phi_j(x,\theta,\bar\theta) = {}& \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j\\
&- \mathrm{i}\theta\sigma^\mu\bar\theta\left(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j\right) + 2\bar\theta\bar\psi_i\theta\psi_j\\
&+ \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\left(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j\right)\sigma^\mu\bar\theta + \sqrt{2}\theta\theta\bar\theta\bar\psi_iF_j\\
&- \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\left(\partial_\mu\bar\psi_i\phi_j - \bar\psi_i\partial_\mu\phi_j\right) + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j && (3.14)\\
&+ \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \frac{1}{4}\phi_i^*\partial^2\phi_j - \frac{1}{4}\partial^2\phi_i^*\phi_j + \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j - \frac{\mathrm{i}}{2}\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_j + \frac{\mathrm{i}}{2}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]\\
\rightsquigarrow{}& \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j\\
&- 2\mathrm{i}(\theta\sigma^\mu\bar\theta)(\phi_i^*\partial_\mu\phi_j) + \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar\theta\bar\sigma^\mu\psi_j + \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j && (3.15)\\
&+ 2\bar\theta\bar\psi_i\theta\psi_j + \sqrt{2}\theta\theta\bar\theta\bar\psi_iF_j + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j\\
&+ \theta\theta\bar\theta\bar\theta\left[F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned}$$

$$\begin{aligned}
\Phi_i\Phi_j(\text{in } y\text{-basis}) &= \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right] && (3.16)\\
\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right]\\
&\quad + \theta\theta\left[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right] && (3.17)
\end{aligned}$$

(Products of chiral superfields are still chiral superfields.)

$$\mathrm{e}^{\mathrm{i}k\Phi} = \mathrm{e}^{\mathrm{i}k\phi(y)}\left[1 + \mathrm{i}k\left(\sqrt{2}\theta\psi(y) + \theta\theta F(y)\right) + \frac{k^2}{2}\theta\theta\psi(y)\psi(y)\right] \tag{3.18}$$

Lagrangian Blocks

$$\mathcal{L}_{\text{kin.}} = \Phi_i^\dagger\Phi_j\Big|_{\theta\theta\bar\theta\bar\theta} \rightsquigarrow F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j \tag{3.19}$$

$$\begin{aligned}
\mathcal{L}_{\text{super}} &= W\Big|_{\theta\theta} + W^*\Big|_{\bar\theta\bar\theta} = \int \mathrm{d}^2\theta\left[\lambda_i\Phi_i + m_{ij}\Phi_i\Phi_j + y_{ijk}\Phi_i\Phi_j\Phi_k\right] + \text{H.c.}\\
&= \lambda_iF_i + m_{ij}\left(\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right) + y_{ijk}\Big[(F_i\phi_j\phi_k - \psi_i\psi_j\phi_k) + (jki \text{ and } kij \text{ terms})\Big] && (3.20)
\end{aligned}$$

3.5 Vector Superfields and Gauge Theory : $V = V^\dagger$

◆3.5.1 Abelian Case — Field Construction

Explicit Expression

$$\begin{aligned}
V &= C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar\theta\bar\chi(x) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(x)+\mathrm{i}N(x)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(x)-\mathrm{i}N(x)\right] - \theta\sigma^\mu\bar\theta A_\mu(x) \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(x) + \frac{1}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right] + \bar\theta\bar\theta\theta\left[\lambda(x) - \frac{1}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) - \frac{1}{2}\partial^2 C(x)\right] \qquad (3.21)\\
&= C(y) + \mathrm{i}\theta\chi(y) - \mathrm{i}\bar\theta\bar\chi(y) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y)+\mathrm{i}N(y)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y)-\mathrm{i}N(y)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y) - \mathrm{i}\partial_\mu C(y)\right] \\
&\quad + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\left[\lambda(y) - \sigma^\mu\partial_\mu\bar\chi(y)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y) - \partial^2 C(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] \qquad (3.22)\\
&= C(y^\dagger) + \mathrm{i}\theta\chi(y^\dagger) - \mathrm{i}\bar\theta\bar\chi(y^\dagger) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y^\dagger)+\mathrm{i}N(y^\dagger)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y^\dagger)-\mathrm{i}N(y^\dagger)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y^\dagger) + \mathrm{i}\partial_\mu C(y^\dagger)\right] \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(y^\dagger) + \bar\sigma^\mu\partial_\mu\chi(y^\dagger)\right] + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) - \partial^2 C(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] \qquad (3.23)
\end{aligned}$$

Supersymmetric Gauge Transformation : $V \to V + \Phi + \Phi^\dagger$

$$\begin{aligned}
C &\mapsto C + (\phi + \phi^*) & A_\mu &\mapsto A_\mu + \mathrm{i}\partial_\mu(\phi - \phi^*) \\
\chi &\mapsto \chi - \mathrm{i}\sqrt{2}\psi & \lambda &\mapsto \lambda \\
M + \mathrm{i}N &\mapsto M + \mathrm{i}N - 2\mathrm{i}F & D &\mapsto D
\end{aligned} \qquad (3.24)$$

Wess-Zumino Gauge $\quad C = \chi = M = N = 0$

Fixing this gauge breaks SUSY, but still allows the usual gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu\alpha, \quad \lambda \mapsto \lambda, \quad D \mapsto D. \qquad (3.25)$$

$$\begin{aligned}
V &= -\theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x) & V^2 &= \frac{1}{2}\theta\theta\bar\theta\bar\theta A_\mu A^\mu \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y) + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] & V^3 &= 0 \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y^\dagger) + \theta\theta\bar\theta\bar\lambda(y^\dagger) + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] \\
\mathrm{e}^{kV} &= 1 - k\theta\sigma^\mu\bar\theta A_\mu(x) + k\theta\theta\bar\theta\bar\lambda + k\bar\theta\bar\theta\theta\lambda + \theta\theta\bar\theta\bar\theta\left[\frac{k}{2}D + \frac{k^2}{4}A_\mu A^\mu\right]
\end{aligned}$$

Field Strength

$$W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V, \quad \bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha}V; \qquad W_\alpha \mapsto W_\alpha \quad \text{(gauge invariant)} \qquad (3.26)$$

$$\bar D_{\dot\beta}W_\alpha = D_\beta \bar W_{\dot\alpha} = 0; \qquad D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} \qquad (3.27)$$

$$W_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \mathrm{i}\theta\theta\left[\sigma^\mu\partial_\mu\bar\lambda(y)\right]_\alpha \qquad (3.28)$$

$$\bar W_{\dot\alpha} = \bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha}D(y^\dagger) + \mathrm{i}(\bar\theta\bar\sigma^{\mu\nu})_{\dot\alpha}F_{\mu\nu}(y^\dagger) - \mathrm{i}\bar\theta\bar\theta\left[\partial_\mu\lambda(y^\dagger)\sigma^\mu\right]_{\dot\alpha} \qquad (3.29)$$

$$W^\alpha W_\alpha\big|_{\theta\theta} = -\frac{1}{4}\bar D\bar D W^\alpha D_\alpha V \rightsquigarrow -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + 2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 \qquad (3.30)$$

Lagrangian Blocks

($\mathcal{L}_{\rm inv}$ is SUSY- and gauge-invariant, while $\mathcal{L}_{\rm mass}$ is not gauge-invariant.)

$$\begin{aligned}
\mathcal{L}_{\rm inv} &= \frac{\tau}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \frac{\tau^*}{4}\bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big|_{\bar\theta\bar\theta} \qquad \left(\text{with}\quad \tau := 1 + \frac{\mathrm{i}\theta}{8\pi^2}\right)\\
&\rightsquigarrow -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2 - \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \qquad (3.31)\\
\mathcal{L}_{\rm mass} &= m^2V^2\Big|_{\theta\theta\bar\theta\bar\theta}\\
&= m^2\left(\frac{1}{2}A^\mu A_\mu + \mathrm{i}\bar\chi\bar\sigma^\mu\partial_\mu\chi - (\lambda\chi + \bar\lambda\bar\chi) + \frac{1}{2}(M^2+N^2) + CD + \frac{1}{2}\partial_\mu C\partial^\mu C\right)\\
&= \frac{1}{2}\partial_\mu C'\partial^\mu C' + \mathrm{i}\bar\chi'\bar\sigma^\mu\partial_\mu\chi' + \frac{m^2}{2}A^\mu A_\mu - m(\lambda\chi' + \bar\lambda\bar\chi') + mC'D + \frac{m^2}{2}(M^2+N^2) \qquad (3.32)
\end{aligned}$$

◆3.5.2 Abelian Case — Gauge Theory

Here we turn on the coupling constant g. When $\Lambda = \mathrm{i}\lambda(y) + \sqrt{2}\theta\xi(y) + \theta\theta K(y)$,

$$\mathcal{L} \ni \Phi^\dagger \mathrm{e}^{2gqV}\Phi; \qquad \Phi \mapsto \mathrm{e}^{\mathrm{i}qg\Lambda}\Phi,\quad \Phi^\dagger \mapsto \Phi^\dagger\mathrm{e}^{-\mathrm{i}qg\Lambda^\dagger}; \qquad 2V \mapsto 2V - \mathrm{i}(\Lambda - \Lambda^\dagger) \qquad (3.33)$$

$$\begin{aligned}
&\phi \mapsto \mathrm{e}^{-qg\lambda}\phi && C \mapsto C + \operatorname{Re}\lambda && M + \mathrm{i}N \mapsto M + \mathrm{i}N - K\\
&\psi \mapsto \mathrm{e}^{-qg\lambda}\left(\psi + \mathrm{i}qg\phi\cdot\xi\right) && \chi \mapsto \chi - \tfrac{1}{\sqrt2}\xi && A_\mu \mapsto A_\mu - \partial_\mu(\operatorname{Im}\lambda)\\
&F \mapsto \mathrm{e}^{-qg\lambda}\left(F + \mathrm{i}qg\phi K - \mathrm{i}qg\xi\psi + \frac{(qg)^2}{2}\xi\xi\phi\right) && \lambda \mapsto \lambda && D \mapsto D
\end{aligned}$$

Lagrangian block

(Very similar to the gauge transformations in Sec. 1.9.2.)

$$\begin{aligned}
\mathcal{L}_{\rm gauge} &= \frac{1}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \text{H.c.} && \rightsquigarrow -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2\\
\mathcal{L}_{\rm chiral} &= \Phi^\dagger\mathrm{e}^{2gqV}\Phi\Big|_{\theta\theta\bar\theta\bar\theta} && \rightsquigarrow F^*F + \mathrm{D}_\mu\phi^*\mathrm{D}^\mu\phi + \mathrm{i}\bar\psi\bar\sigma^\mu\mathrm{D}_\mu\psi + qgD\phi^*\phi - \sqrt{2}gq\left(\phi^*\lambda\psi + \phi\bar\lambda\bar\psi\right)\\
\mathcal{L}_{\cancel{CP}} &= \frac{\mathrm{i}\theta}{32\pi^2}W^\alpha W_\alpha\Big|_{\theta\theta} + \text{H.c.} && \rightsquigarrow -\frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\
& && \text{with}\quad \mathrm{D}_\mu[\phi,\psi] = (\partial_\mu - \mathrm{i}gqA_\mu)[\phi,\psi], \qquad \mathrm{D}_\mu\lambda = \partial_\mu\lambda,
\end{aligned}$$

◆3.5.3 Non-Abelian Case

T^a : generators (Hermitian);
$\operatorname{Tr}T^aT^b = K\delta^{ab}$ $(K>0)$, $\qquad [T^a, T^b] = \mathrm{i}f^{abc}T^c$ (f is anti-symmetric)

Explicit Expression Same as the Abelian case.

Supersymmetric Gauge Transformation

$$\mathcal{L} \ni \Phi^\dagger\mathrm{e}^{2g\widetilde V}\Phi; \qquad \Phi \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda}\Phi,\quad \Phi^\dagger \mapsto \Phi^\dagger\mathrm{e}^{-\mathrm{i}g\widetilde\Lambda^\dagger}; \qquad \mathrm{e}^{2g\widetilde V} \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda^\dagger}\mathrm{e}^{2g\widetilde V}\mathrm{e}^{-\mathrm{i}g\widetilde\Lambda} \qquad (3.34)$$
$$\text{with } \widetilde V := V^aT^a,\quad \widetilde\Lambda := \Lambda^aT^a,\quad \widetilde\Lambda^\dagger := (\Lambda^a)^\dagger T^a$$

$$\begin{aligned}
2\widetilde V &\mapsto 2\widetilde V - \mathrm{i}(\widetilde\Lambda - \widetilde\Lambda^\dagger) - \frac{g}{2}\left([\widetilde\Lambda, \widetilde\Lambda^\dagger] + \mathrm{i}[2\widetilde V, \widetilde\Lambda + \widetilde\Lambda^\dagger]\right) + \cdots \qquad (3.35)\\
&= \left[2V^a - \mathrm{i}(\Lambda^a - \Lambda^{\dagger a}) + \frac{g}{2}\Big(-\mathrm{i}\Lambda^b\Lambda^{\dagger c} + 2V^b(\Lambda^c + \Lambda^{\dagger c})\Big)f^{abc} + \cdots\right]T^a \qquad (3.36)
\end{aligned}$$

We do not present the transformations of the components; note that λ and D are transformed in non-Abelian theories.

Wess-Zumino Gauge

$$V^a = -\theta\sigma^\mu\bar{\theta}A^a_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}^a(x) + \bar{\theta}\bar{\theta}\theta\lambda^a(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D^a(x) \tag{3.37}$$

$$\mathrm{e}^{kV^aT^a} = 1 + kV^aT^a + \frac{k^2}{4}\theta\theta\bar{\theta}\bar{\theta}A^a_\mu A^{b\mu}T^aT^b \tag{3.38}$$

Note that the lowest order term of the gauge transformation is independent of V, which guarantees that we can still take the Wess-Zumino gauge. The gauge transformation is restricted as $\mathrm{e}^{\mathrm{i}g\widetilde{\Lambda}}$, where $\Lambda^a = \xi^a(y) = \xi^a(x) - \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\xi^a(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\xi^a(x) : \xi \in \mathbb{R}$.

$$\begin{aligned} 2V^e \mapsto\ & 2\left(V^e - gf^{eab}\xi^aV^b + 6g^2f^{abc}f^{ade}V^b\xi^c\xi^d\right) \\ & + \theta\sigma^\mu\bar{\theta}\left(-2\partial_\mu\xi^e + gf^{eab}\xi^a\partial_\mu\xi^b + 4g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \end{aligned} \tag{3.39}$$

$$\begin{aligned} A^e_\mu \mapsto\ & A^e_\mu + gf^{eab}A^a_\mu\xi^b + 6g^2f^{abc}f^{ade}A^b_\mu\xi^c\xi^d \\ & + \left(\partial_\mu\xi^e - \frac{g}{2}f^{eab}\xi^a\partial_\mu\xi^b - 2g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \end{aligned} \tag{3.40}$$

$$\lambda^e \mapsto \lambda^e + gf^{eab}\lambda^a\xi^b + 6g^2f^{abc}f^{ade}\lambda^b\xi^c\xi^d + \cdots \tag{3.41}$$

$$D^e \mapsto D^e + gf^{eab}D^a\xi^b + 6g^2f^{abc}f^{ade}D^b\xi^c\xi^d + \cdots \tag{3.42}$$

Note that C, χ, M and N are kept invariant automatically, for now we are under Wess-Zumino gauge.

Field Strength [*2]

$$\widetilde{W}_\alpha = -\frac{1}{8g}\bar{D}\bar{D}\mathrm{e}^{-2g\widetilde{V}}D_\alpha\mathrm{e}^{2g\widetilde{V}} \qquad \bar{D}_{\dot{\beta}}W_\alpha = 0 \qquad W_\alpha \mapsto \mathrm{e}^{\mathrm{i}g\widetilde{\Lambda}}W_\alpha\mathrm{e}^{-\mathrm{i}g\Lambda^\dagger} \tag{3.43}$$

$$W^a_\alpha = \lambda^a_\alpha(y) + \theta_\alpha D^a(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) + \mathrm{i}\theta\theta(\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^a(y))_\alpha \tag{3.44}$$

$$\mathrm{Tr}\,\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} = \mathrm{Tr}\left[DD + \mathrm{i}\lambda\sigma^\mu\mathrm{D}_\mu\bar{\lambda} + \mathrm{i}\mathrm{D}_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \tag{3.45}$$

$$= K\left[D^aD^a + \mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^a + \mathrm{i}\mathrm{D}_\mu\bar{\lambda}^a\bar{\sigma}^\mu\lambda^a - \frac{1}{2}F^{\mu\nu\,a}F^a_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right] \tag{3.46}$$

Lagrangian Block

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4K}\mathrm{Tr}\,\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} + \text{H.c.} \qquad \rightsquigarrow -\frac{1}{4}F^{\mu\nu\,a}F^a_{\mu\nu} + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^a + \frac{1}{2}D^aD^a \tag{3.47}$$

$$\mathcal{L}_{\cancel{CP}} = \frac{\mathrm{i}}{32K\pi^2}\mathrm{Tr}\,\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} + \text{H.c.} \qquad \rightsquigarrow -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.48}$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = \Phi^\dagger_i[\mathrm{e}^{2gV^aT^a}]_{ij}\Phi_j\Big|_{\theta\theta\bar{\theta}\bar{\theta}} \qquad & \rightsquigarrow \mathrm{D}^\mu\phi^*_i\mathrm{D}_\mu\phi_i + \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathrm{D}_\mu\psi_i + F^*_iF_i \\ & \quad + gD^a(\phi^*T^a\phi) - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar{\psi}\bar{\lambda}T^a\phi\right) \end{aligned} \tag{3.49}$$

$$\begin{aligned} \mathrm{D}_\mu\phi_i &= \partial_\mu\phi - \mathrm{i}gA^a_\mu(T^a\phi)_i & F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu], \\ \mathrm{D}_\mu\phi^*_i &= \partial_\mu\phi + \mathrm{i}gA^a_\mu(\phi^*T^a)_i & \mathrm{D}_\mu\lambda^a &= \partial_\mu\lambda^a + gf^{abc}A^b_\mu\lambda^c, \\ \mathrm{D}_\mu\psi_i &= \partial_\mu\psi - \mathrm{i}gA^a_\mu(T^a\psi)_i & \mathrm{D}_\mu\bar{\lambda}^a &= \partial_\mu\bar{\lambda}^a + gf^{abc}A^b_\mu\bar{\lambda}^c, \end{aligned}$$

[*2] Note the signs. $\bar{D}\bar{D}\mathrm{e}^{2gV}D_\alpha\mathrm{e}^{-2gV}$ is not gauge invariant! Also the curvature tensor and the covariant derivative is well-known ones: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$, and $\mathrm{D}_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - \mathrm{i}g[A_\mu, \bar{\lambda}] = \partial_\mu\bar{\lambda} + gf^{abc}A^b_\mu\bar{\lambda}^cT^a$.

3.6 Minimal Supersymmetric Standard Model

◆3.6.1 Definitions

Gauge Group and Superfields[*3]

$$\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_{\mathrm{Y}} \quad (\times \mathrm{Z}_{2R} : R\text{-parity}); \tag{3.50}$$

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$1/3$	
L_i		$\mathbf{2}$	$-1/2$		1
$\bar{U}_i$	$\bar{\mathbf{3}}$		$-2/3$	$-1/3$	
$\bar{D}_i$	$\bar{\mathbf{3}}$		$1/3$	$-1/3$	
$\bar{E}_i$			1		-1
H_{u}		$\mathbf{2}$	$1/2$		
H_{d}		$\mathbf{2}$	$-1/2$		

Field	SU(3)	SU(2)	U(1)
g	$\mathbf{8}$		
W		$\mathbf{3}$	
B			

Superpotential and $\cancel{\text{SUSY}}$-terms

$$W_{\mathrm{RPC}} = \mu H_{\mathrm{u}} H_{\mathrm{d}} - y_{\mathrm{u}ij}\bar{U}_i H_{\mathrm{u}} Q_j + y_{\mathrm{d}ij}\bar{D}_i H_{\mathrm{d}} Q_j + y_{\mathrm{e}ij}\bar{E}_i H_{\mathrm{d}} L_j \tag{3.51}$$

$$W_{\mathrm{RPV}} = \mu_i H_{\mathrm{u}} L_i + \frac{1}{2}\lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2}\lambda''_{ijk}\bar{U}_i \bar{D}_j \bar{D}_k \tag{3.52}$$

$$\mathcal{L}_{\cancel{\mathrm{SUSY}}} = -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{w}\widetilde{w} + M_1\widetilde{b}\widetilde{b} + \text{H.c.}\right) - V_{\cancel{\mathrm{SUSY}}}; \tag{3.53}$$

$$\begin{aligned} V^{\mathrm{RPC}}_{\cancel{\mathrm{SUSY}}} = &\left(\widetilde{q}^* m_Q^2 \widetilde{q} + \widetilde{l}^* m_L^2 \widetilde{l} + \widetilde{u}_{\mathrm{R}} m_{\bar{U}}^2 \widetilde{u}_{\mathrm{R}}^* + \widetilde{d}_{\mathrm{R}} m_{\bar{D}}^2 \widetilde{d}_{\mathrm{R}}^* + \widetilde{e}_{\mathrm{R}} m_{\bar{E}}^2 \widetilde{e}_{\mathrm{R}}^* + m_{H_{\mathrm{u}}}^2 |H_{\mathrm{u}}|^2 + m_{H_{\mathrm{d}}}^2 |H_{\mathrm{d}}|^2\right) \\ &+ \left(-\widetilde{u}_{\mathrm{R}}^* H_{\mathrm{u}} A^u \widetilde{q} + \widetilde{d}_{\mathrm{R}}^* H_{\mathrm{d}} A^d \widetilde{q} + \widetilde{e}_{\mathrm{R}}^* H_{\mathrm{d}} A^e \widetilde{l} + b H_{\mathrm{u}} H_{\mathrm{d}} + \text{H.c.}\right) m \\ &+ \left(-\widetilde{u}_{\mathrm{R}}^* H_{\mathrm{d}}^* C^u \widetilde{q} + \widetilde{d}_{\mathrm{R}}^* H_{\mathrm{u}}^* C^d \widetilde{q} + \widetilde{e}_{\mathrm{R}}^* H_{\mathrm{u}}^* C^e \widetilde{l} + \text{H.c.}\right) \end{aligned} \tag{3.54}$$

$$V^{\mathrm{RPV}}_{\cancel{\mathrm{SUSY}}} = \left(b_i H_{\mathrm{u}} \widetilde{l}_i + \frac{1}{2}A_{ijk}\widetilde{l}_i \widetilde{l}_j \widetilde{e}_{\mathrm{R}k}^* + A'_{ijk}\widetilde{l}_i \widetilde{q}_j \widetilde{d}_{\mathrm{R}k}^* + \frac{1}{2}A''_{ijk}\widetilde{u}_{\mathrm{R}i}^* \widetilde{d}_{\mathrm{R}j}^* \widetilde{d}_{\mathrm{R}k}^* + M_{Li}^2 H_{\mathrm{d}}^* \widetilde{l}_i + \text{H.c.}\right) \tag{3.55}$$

$$+ \left(C^1_{ijk}\widetilde{l}_i^* \widetilde{q}_j \widetilde{u}_{\mathrm{R}k}^* + C^2_i H_{\mathrm{u}}^* H_{\mathrm{d}} \widetilde{e}_{\mathrm{R}i}^* + C^3_{ijk}\widetilde{d}_{\mathrm{R}i}\widetilde{u}_{\mathrm{R}j}^* \widetilde{e}_{\mathrm{R}k}^* + \frac{1}{2}C^4_{ijk}\widetilde{d}_{\mathrm{R}i}\widetilde{q}_j \widetilde{q}_k + \text{H.c.}\right), \tag{3.56}$$

where we define $\lambda_{ijk} = -\lambda_{jik}$, $\lambda''_{ijk} = -\lambda''_{ikj}$, and $C^4_{ijk} = C^4_{ikj}$

[*3] For left-handed fermions, the superfield will be written as, e.g., Q, and it contains a scalar component $\widetilde{q}$ and a chiral fermion Q_α. Their complex conjugates will be shown as $\widetilde{q}^*$ and $\bar{Q}_{\dot{\alpha}}$ as is done in the previous section.

For right-handed fermions such as $\bar{U}$, the superfield will be written as $\bar{U}$. Its scalar component is written as (or, in other words, equivalent to) $\widetilde{u}_{\mathrm{R}}^*$, and its fermionic one is U^{c}_α. *c is just a label;* does not mean charge or complex conjugation. Their complex conjugates are $\widetilde{u}_{\mathrm{R}}$ and $\bar{U}^{\mathrm{c}}_{\dot{\alpha}}$.

They form a Dirac fermion as $U = \begin{pmatrix} u_\alpha \\ \bar{U}^{\mathrm{c}\dot{\alpha}} \end{pmatrix} =: \begin{pmatrix} U_{\mathrm{L}} \\ U_{\mathrm{R}} \end{pmatrix}$; its *charge* conjugate is $U^{\mathrm{C}} = \begin{pmatrix} U^{\mathrm{C}}_\alpha \\ \bar{u}^{\dot{\alpha}} \end{pmatrix}$.

Majorana fermions are written as, e.g., $\widetilde{b} = \widetilde{b}^{\mathrm{C}} = \begin{pmatrix} \widetilde{b}_\alpha \\ \bar{\widetilde{b}}^{\dot{\alpha}} \end{pmatrix}$. Here $\bar{\widetilde{b}}_{\dot{\alpha}}$ is the complex conjugate of $\widetilde{b}_\alpha$.

scalars : $\widetilde{q}\,(\widetilde{u}_{\mathrm{L}}, \widetilde{d}_{\mathrm{L}}), \widetilde{u}_{\mathrm{R}}^*, \widetilde{d}_{\mathrm{R}}^*, \widetilde{l}\,(\widetilde{e}_{\mathrm{L}}, \widetilde{\nu}), \widetilde{e}_{\mathrm{R}}^*, H_{\mathrm{u}}\,(H_{\mathrm{u}}^+, H_{\mathrm{u}}^0), H_{\mathrm{d}}\,(H_{\mathrm{d}}^0, H_{\mathrm{d}}^-)$

$\widetilde{q}^*\,(\widetilde{u}_{\mathrm{L}}^*, \widetilde{d}_{\mathrm{L}}^*), \widetilde{u}_{\mathrm{R}}, \widetilde{d}_{\mathrm{R}}, \widetilde{l}^*\,(\widetilde{e}_{\mathrm{L}}^*, \widetilde{\nu}^*), \widetilde{e}_{\mathrm{R}}, H_{\mathrm{u}}^*, H_{\mathrm{d}}^*$

Diracs : $U\,(U_{\mathrm{L}}, U_{\mathrm{R}}), D\,(D_{\mathrm{L}}, D_{\mathrm{R}}), E\,(E_{\mathrm{L}}, E_{\mathrm{R}}), \nu$

Weyls : $Q\,(u, d), U^{\mathrm{c}}, D^{\mathrm{c}}, L\,(\nu, e), E^{\mathrm{c}}$

$\bar{Q}\,(\bar{u}, \bar{d}), \bar{U}^{\mathrm{c}}, \bar{D}^{\mathrm{c}}, \bar{L}\,(\bar{\nu}, \bar{e}), \bar{E}^{\mathrm{c}}$

$\widetilde{h}_{\mathrm{u}}\,(\widetilde{h}_{\mathrm{u}}^+, \widetilde{h}_{\mathrm{u}}^0), \widetilde{h}_{\mathrm{d}}\,(\widetilde{h}_{\mathrm{d}}^0, \widetilde{h}_{\mathrm{d}}^-), \widetilde{b}, \widetilde{w}, \widetilde{g}$

$\bar{\widetilde{h}}_{\mathrm{u}}\,(\bar{\widetilde{h}}_{\mathrm{u}}^+, \bar{\widetilde{h}}_{\mathrm{u}}^0), \bar{\widetilde{h}}_{\mathrm{d}}\,(\bar{\widetilde{h}}_{\mathrm{d}}^0, \bar{\widetilde{h}}_{\mathrm{d}}^-), \bar{\widetilde{b}}, \bar{\widetilde{w}}, \bar{\widetilde{g}}$

◆3.6.2 Lagrangian Build Block

$$\mathcal{L}_{\mathrm{K;\,CP}} = -\frac{1}{4}F^{\mu\nu\,a}F^a_{\mu\nu} + \mathrm{D}^\mu\phi_i^*\mathrm{D}_\mu\phi_i + \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathrm{D}_\mu\psi_i + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^a - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar{\psi}\bar{\lambda}T^a\phi\right) \tag{3.57}$$

$$\mathcal{L}^{\cancel{\mathrm{SUSY}}}_{\mathrm{gaugino}} = -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{w}\widetilde{w} + M_1\widetilde{b}\widetilde{b} + \mathrm{H.c.}\right) \tag{3.58}$$

$$\mathcal{L}_{\mathrm{scalar}} = -\left(\sum V^F + \sum V^D + \sum V_{\cancel{SUSY}}\right) \tag{3.59}$$

$$\mathcal{L}^{\mathrm{RPC}}_{\mathrm{S;\,fermi}} = -\left(\mu\widetilde{h}_\mathrm{u}\widetilde{h}_\mathrm{d} - y_{\mathrm{u}ij}U^\mathrm{c}_iH_\mathrm{u}Q_j + y_{\mathrm{d}ij}D^\mathrm{c}_iH_\mathrm{d}Q_j + y_{\mathrm{e}ij}E^\mathrm{c}_iH_\mathrm{d}L_j + \cdots + \mathrm{H.c.}\right) \tag{3.60}$$

$$\mathcal{L}^{\mathrm{RPV}}_{\mathrm{S;\,fermi}} = -\left(\mu_i\widetilde{h}_\mathrm{u}L_i + \frac{1}{2}\lambda_{ijk}L_iL_jE^\mathrm{c}_k + \lambda'{}_{ijk}L_iQ_jD^\mathrm{c}_k + \frac{1}{2}\lambda''{}_{ijk}U^\mathrm{c}_iD^\mathrm{c}_jD^\mathrm{c}_k + \cdots + \mathrm{H.c.}\right) \tag{3.61}$$

$$\mathcal{L}_{\mathrm{K;}\cancel{CP}} = -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.62}$$

The scalar potential is decomposed as

$$-F_{H_\mathrm{u}}{}^{a*} = \epsilon^{ab}\left(\mu H^b_\mathrm{d} - y_{\mathrm{u}ij}\bar{U}^x_iQ^{bx}_j + \mu_iL^b_i\right) \tag{3.63}$$

$$-F_{H_\mathrm{d}}{}^{a*} = \epsilon^{ab}\left(-\mu H^b_\mathrm{u} + y_{\mathrm{d}ij}\bar{D}^x_iQ^{bx}_j + y_{\mathrm{e}ij}\bar{E}_iL^b_j\right) \tag{3.64}$$

$$-F_{Q_i}{}^{ax*} = \epsilon^{ab}\left(y_{\mathrm{u}ji}H^b_\mathrm{u}\bar{U}^x_j - y_{\mathrm{d}ji}H^b_\mathrm{d}\bar{D}^x_j - \lambda'{}_{jik}L^b_i\bar{D}^x_j\right) \tag{3.65}$$

$$-F_{L_i}{}^{a*} = \epsilon^{ab}\left(-\mu_iH^b_\mathrm{u} - y_{\mathrm{e}ji}H^b_\mathrm{d}\bar{E}_j + \lambda_{ijk}L^b_j\bar{E}_k + \lambda'_{ijk}Q^{bx}_j\bar{D}^x_k\right) \tag{3.66}$$

$$-F_{\bar{U}i}{}^{x*} = \left(-\epsilon^{ab}y_{\mathrm{u}ij}H^a_\mathrm{u}Q^{bx}_j + \frac{1}{2}\epsilon^{xyz}\lambda''_{ijk}\bar{D}^y_j\bar{D}^z_k\right) \tag{3.67}$$

$$-F_{\bar{D}i}{}^{x*} = \left(\epsilon^{ab}y_{\mathrm{d}ij}H^a_\mathrm{d}Q^{bx}_j + \epsilon^{ab}\lambda'{}_{jki}L^a_jQ^{bx}_k + \epsilon^{yzx}\lambda''_{jki}\bar{U}^y_j\bar{D}^z_k\right) \tag{3.68}$$

$$-F_{\bar{E}i}{}^{*} = \left(\epsilon^{ab}y_{\mathrm{e}ij}H^a_\mathrm{d}L^b_j + \frac{1}{2}\epsilon^{ab}\lambda_{jki}L^a_jL^b_k\right) \tag{3.69}$$

and

$$D_g{}^\alpha = -g_3\sum_{i=1}^{3}\left[\sum_{a=1,2}Q^{ax*}_i(T^\alpha)_{xy}Q^{ay}_i - \bar{U}^{x*}_i(T^\alpha)_{xy}\bar{U}^y_i - \bar{D}^{x*}_i(T^\alpha)_{xy}\bar{D}^y_i\right] \tag{3.70}$$

$$D_W{}^\alpha = -g_2\left[\sum_{i=1}^{3}\sum_{x=1}^{3}Q^{ax*}_i(T^\alpha)_{ab}Q^{by}_i + \sum_{i=1}^{3}L^{a*}_i(T^\alpha)_{ab}L^b_i + H^{a*}_\mathrm{u}(T^\alpha)_{ab}H^b_\mathrm{u} + H^{a*}_\mathrm{d}(T^\alpha)_{ab}H^b_\mathrm{d}\right] \tag{3.71}$$

$$D_B = -g_1\left[\frac{1}{6}|Q^{ax}_i|^2 - \frac{1}{2}|L^a_i|^2 - \frac{2}{3}|\bar{U}^x_i|^2 + \frac{1}{3}|\bar{D}^x_i|^2 + |\bar{E}_i|^2 + \frac{1}{2}|H^a_\mathrm{u}|^2 - \frac{1}{2}|H^a_\mathrm{d}|^2\right]. \tag{3.72}$$

Here we use the superfield notation for simple appearance.

◆3.6.3 Scalar Potential (Verbose)

$$
\begin{aligned}
V^F_{H_\mathrm{u}} &= |\mu|^2|H_\mathrm{d}|^2 + \sum_a \Big(|\bar{U}y^uQ^a|^2 + |\mu_i L_i^a|^2\Big) \\
&\quad + \Big[\mu^*\mu_i H_\mathrm{d}^* L_i - \mu^* H_\mathrm{d}^*\bar{U}y^uQ - \mu_i^* L_i^*\bar{U}y^uQ + \mathrm{H.c.}\Big]
\end{aligned} \tag{3.73}
$$

$$
\begin{aligned}
V^F_{H_\mathrm{d}} &= |\mu|^2|H_\mathrm{u}|^2 + \sum_a \Big(|\bar{D}y^dQ^a|^2 + |\bar{E}y^eL^a|^2\Big) \\
&\quad + \Big[-\mu^* H_\mathrm{u}^*\bar{D}y^dQ - \mu^* H_\mathrm{u}^*\bar{E}y^eL + (\bar{D}y^dQ)^*(\bar{E}y^eL) + \mathrm{H.c.}\Big]
\end{aligned} \tag{3.74}
$$

$$
\begin{aligned}
V^F_{Q} &= |H_\mathrm{u}|^2|\bar{U}_i y^u_{ij}|^2 + |H_\mathrm{d}|^2|\bar{D}_i y^d_{ij}|^2 + \lambda'^*_{jik}\lambda'_{lim}L_j^*L_l\bar{D}_k^*\bar{D}_m \\
&\quad + \Big[-y^{u*}_{ji}y^d_{ki}H_\mathrm{u}^*H_\mathrm{d}\bar{U}_j^*\bar{D}_k - y^{u*}_{ji}\lambda'_{lim}H_\mathrm{u}^*L_l\bar{U}_j^*\bar{D}_m + y^{d*}_{ji}\lambda'_{lim}H_\mathrm{d}^*L_l\bar{D}_j^*\bar{D}_m + \mathrm{H.c.}\Big]
\end{aligned} \tag{3.75}
$$

$$
\begin{aligned}
V^F_{L} &= |\mu_i|^2|H_\mathrm{u}|^2 + |H_\mathrm{d}|^2(\bar{E}y^ey^{e\dagger}\bar{E}^*) + \lambda'_{ijk}\lambda'_{ilm}(Q_j^*\bar{D}_k^*)Q_l\bar{D}_m + \lambda^*_{ijk}\lambda_{ilm}L_j^*L_l\bar{E}_k^*\bar{E}_m \\
&\quad + \Big[\mu_i^* y^e_{ji}\bar{E}_jH_\mathrm{u}^*H_\mathrm{d} - \mu_i^*\lambda'_{ijk}H_\mathrm{u}^*Q_j\bar{D}_k - \mu_i^*\lambda_{ijk}H_\mathrm{u}^*L_j\bar{E}_k \\
&\qquad -y^{e*}_{ji}\lambda'_{ilm}\bar{E}_j^*H_\mathrm{d}^*Q_l\bar{D}_m - y^{e*}_{ji}\lambda_{ilm}\bar{E}_j^*H_\mathrm{d}^*L_l\bar{E}_m + \lambda'^*_{ijk}\lambda_{ilm}\bar{D}_k^*Q_j^*L_l\bar{E}_m + \mathrm{H.c.}\Big]
\end{aligned} \tag{3.76}
$$

$$
V^F_{\bar{U}} = y^{u*}_{ij}y^u_{ik}\epsilon^{ab}\epsilon^{cd}H_\mathrm{u}^{a*}H_\mathrm{u}^cQ_j^{b*}Q_k^d + \frac{1}{2}\lambda''^*_{ijk}\lambda''_{ilm}(\bar{D}_j^*\bar{D}_l)(\bar{D}_k^*\bar{D}_m) - \Big[y^{u*}_{il}\lambda''_{ijk}H_\mathrm{u}^*Q_l^*\bar{D}_j\bar{D}_k + \mathrm{H.c.}\Big] \tag{3.77}
$$

$$
\begin{aligned}
V^F_{\bar{D}} &= \epsilon^{ab}\epsilon^{cd}\left(y^d_{ij}H_\mathrm{d}^a + \lambda'_{kji}L_k^a\right)^*\left(y^d_{il}H_\mathrm{d}^c + \lambda'_{mli}L_m^c\right)Q_j^{b*}Q_l^d + \lambda''^*_{jki}\lambda''_{lmi}\left(\bar{U}_j^*\bar{U}_l\ \bar{D}_k^*\bar{D}_m - \bar{U}_j^*\bar{D}_m\ \bar{D}_k^*\bar{U}_l\right) \\
&\quad + \Big[\lambda''_{lmi}(y^{d*}_{ij}H_\mathrm{d}^* + \lambda'^*_{kji}L_k^*)Q_j^*\bar{U}_l\bar{D}_m + \mathrm{H.c.}\Big]
\end{aligned} \tag{3.78}
$$

$$
V^F_{\bar{E}} = \epsilon^{ab}\epsilon^{cd}\left(y^e_{ij}H_\mathrm{d}^a + \frac{1}{2}\lambda_{kji}L_k^a\right)^* L_j^{*b}\left(y^e_{il}H_\mathrm{d}^c + \frac{1}{2}\lambda_{mli}L_m^c\right)L_l^d \tag{3.79}
$$

$$
V^D_g = \frac{g_3^2}{2}\left\{\sum_{\alpha=1}^{8}\sum_{i=1}^{3}\left[\sum_{a=1,2}Q_i^{a*}(t^\alpha)Q_i^a - \bar{U}_i^*(t^\alpha)\bar{U}_i - \bar{D}_i^*(t^\alpha)\bar{D}_i\right]\right\}^2 \tag{3.80}
$$

$$
V^D_W = \frac{g_2^2}{2}\left[\sum_{i=1}^{3}\sum_{x=1}^{3}Q_i^{x*}(T^\alpha)Q_i^x + \sum_{i=1}^{3}L_i^*(T^\alpha)L_i + H_\mathrm{u}^*(T^\alpha)H_\mathrm{u} + H_\mathrm{d}^*(T^\alpha)H_\mathrm{d}\right]^2 \tag{3.81}
$$

$$
V^D_B = \frac{g_1^2}{2}\left[\sum_i\left(\frac{1}{6}|Q_i|^2 - \frac{1}{2}|L_i|^2 - \frac{2}{3}|\bar{U}_i|^2 + \frac{1}{3}|\bar{D}_i|^2 + |\bar{E}_i|^2\right) + \frac{1}{2}|H_\mathrm{u}|^2 - \frac{1}{2}|H_\mathrm{d}|^2\right]^2 \tag{3.82}
$$

$$
\begin{aligned}
V^\mathrm{RPC}_{\cancel{\mathrm{SUSY}}} &= \left(Q^*m_Q^2Q + L^*m_L^2L + \bar{U}^*m_{\bar{U}}^2\bar{U} + \bar{D}^*m_{\bar{D}}^2\bar{D} + \bar{E}^*m_{\bar{E}}^2\bar{E} + m_{H_\mathrm{u}}^2|H_\mathrm{u}|^2 + m_{H_\mathrm{d}}^2|H_\mathrm{d}|^2\right) \\
&\quad + \left(-\bar{U}H_\mathrm{u}A^uQ + \bar{D}H_\mathrm{d}A^dQ + \bar{E}H_\mathrm{d}A^eL + BH_\mathrm{u}H_\mathrm{d} + \mathrm{H.c.}\right) \\
&\quad + \left(-\bar{U}H_\mathrm{d}^*C^uQ + \bar{D}H_\mathrm{u}^*C^dQ + \bar{E}H_\mathrm{u}^*C^eL + \mathrm{H.c.}\right)
\end{aligned} \tag{3.83}
$$

$$
\begin{aligned}
V^\mathrm{RPV}_{\cancel{\mathrm{SUSY}}} &= \left(B_iH_\mathrm{u}L_i + \frac{1}{2}A_{ijk}L_iL_j\bar{E}_k + A'_{ijk}L_iQ_j\bar{D}_k + \frac{1}{2}A''_{ijk}\bar{U}_i\bar{D}_j\bar{D}_k + \mathrm{H.c.}\right) \\
&\quad + \left(C^1_{ijk}L_i^*Q_j\bar{U}_k + C^2_iH_\mathrm{u}^*H_\mathrm{d}\bar{E}_i + C^3_{ijk}\bar{D}_i^*\bar{U}_j\bar{E}_k + \frac{1}{2}C^4_{ijk}\bar{D}_i^*Q_jQ_k + \mathrm{H.c.}\right) \\
&\quad + \left(M^2_{Li}H_\mathrm{d}^*L_i + \mathrm{H.c.}\right)
\end{aligned} \tag{3.84}
$$

With R-parity

$$
\begin{aligned}
V_{\text{full}}^{\text{RPC}} = & \left(Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E}\right) \\
& + \left(|\mu|^2 + m_{H_{\text{u}}}^2\right) |H_{\text{u}}|^2 + \left(|\mu|^2 + m_{H_{\text{d}}}^2\right) |H_{\text{d}}|^2 + (B H_{\text{u}} H_{\text{d}} + \text{H.c.}) \\
& + \left[(-\mu^* y^u H_{\text{d}}^* - A^u H_{\text{u}} - C^u H_{\text{d}}^*)_{ij} \bar{U}_i Q_j + \text{H.c.}\right] \\
& + \left[\left(-\mu^* y^d H_{\text{u}}^* + A^d H_{\text{d}} + C^d H_{\text{u}}^*\right)_{ij} \bar{D}_i Q_j + \text{H.c.}\right] \\
& + \left[(-\mu^* y^e H_{\text{u}}^* + A^e H_{\text{d}} + C^e H_{\text{u}}^*)_{ij} \bar{E}_i L_j + \text{H.c.}\right] \\
& + |H_{\text{u}}|^2 |\bar{U} y^u|^2 + |H_{\text{d}}|^2 |\bar{D} y^d|^2 + |H_{\text{d}}|^2 |\bar{E} y^e|^2 + |H_{\text{u}}|^2 |y^u Q|^2 + |H_{\text{d}}|^2 |y^d Q|^2 + |H_{\text{d}}|^2 |y^e L|^2 \\
& + \sum_a \left(|\bar{U} y^u Q^a|^2 + |\bar{D} y^d Q^a|^2 + |\bar{E} y^e L^a|^2\right) \\
& + \left[(\bar{D} y^d Q)^* (\bar{E} y^e L) - y_{ji}^{u*} y_{ki}^d H_{\text{u}}^* H_{\text{d}} \bar{U}_j^* \bar{D}_k + \text{H.c.}\right] \\
& - \left[y_{ki}^{u*} y_{kj}^u (H_{\text{u}}^* Q_j)(Q_i^* H_{\text{u}}) + y_{ki}^{d*} y_{kj}^d (H_{\text{d}}^* Q_j)(Q_i^* H_{\text{d}}) + y_{ki}^{e*} y_{kj}^e (H_{\text{d}}^* L_j)(L_i^* H_{\text{d}})\right] \\
& + \frac{g_3^2}{2} \left\{\sum_{\alpha=1}^{8} \sum_{i=1}^{3} \left[\sum_{a=1,2} Q_i^{a*} (t^\alpha) Q_i^a - \bar{U}_i^* (t^\alpha) \bar{U}_i - \bar{D}_i^* (t^\alpha) \bar{D}_i\right]\right\}^2 \\
& + \frac{g_2^2}{2} \left[\sum_{i=1}^{3} \sum_{x=1}^{3} Q_i^{x*} (T^\alpha) Q_i^x + \sum_{i=1}^{3} L_i^* (T^\alpha) L_i + H_{\text{u}}^* (T^\alpha) H_{\text{u}} + H_{\text{d}}^* (T^\alpha) H_{\text{d}}\right]^2 \\
& + \frac{g_1^2}{2} \left[\sum_i \left(\frac{1}{6}|Q_i|^2 - \frac{1}{2}|L_i|^2 - \frac{2}{3}|\bar{U}_i|^2 + \frac{1}{3}|\bar{D}_i|^2 + |\bar{E}_i|^2\right) + \frac{1}{2}|H_{\text{u}}|^2 - \frac{1}{2}|H_{\text{d}}|^2\right]^2
\end{aligned} \tag{3.85}
$$

With Bilinear R-parity Violation

$$V_{H_{\text{u}}}^F \mathrel{+}= \sum_a |\mu_i L_i^a|^2 + \left[\mu^* \mu_i H_{\text{d}}^* L_i - \mu_i^* L_i^* \bar{U} y^u Q + \text{H.c.}\right] \tag{3.86}$$

$$V_L^F \mathrel{+}= |\mu_i|^2 |H_{\text{u}}|^2 + \left[\mu_i^* y_{ji}^e \bar{E}_j H_{\text{u}}^* H_{\text{d}} + \text{H.c.}\right] \tag{3.87}$$

$$V_{\cancel{\text{SUSY}}}^{\text{RPV}} = \left(B_i H_{\text{u}} L_i + M_{Li}^2 H_{\text{d}}^* L_i + \text{H.c.}\right) \tag{3.88}$$

With Trilinear leptonic R-parity Violation

$$V_Q^F \mathrel{+}= |\lambda'_{jik} L_j \bar{D}_k|^2 + \left[-y_{ji}^{u*} \lambda'_{lim} H_{\text{u}}^* L_l \bar{U}_j^* \bar{D}_m + y_{ji}^{d*} \lambda'_{lim} H_{\text{d}}^* L_l \bar{D}_j^* \bar{D}_m + \text{H.c.}\right] \tag{3.89}$$

$$
\begin{aligned}
V_L^F \mathrel{+}= & |\lambda'_{ijk} Q_j \bar{D}_k|^2 + |\lambda_{ijk} L_j \bar{E}_k|^2 \\
& + \left[-y_{mi}^{e*} \lambda'_{ijk} \bar{E}_m^* H_{\text{d}}^* Q_j \bar{D}_k - y_{mi}^{e*} \lambda_{ijk} \bar{E}_m^* H_{\text{d}}^* L_j \bar{E}_k + \lambda'^*_{ijk} \lambda_{ilm} \bar{D}_k^* Q_j^* L_l \bar{E}_m + \text{H.c.}\right]
\end{aligned} \tag{3.90}
$$

$$
\begin{aligned}
V_{\bar{D}}^F \mathrel{+}= & \lambda'^*_{kji} \lambda'_{mli} \left[(L_k^* L_m)(Q_j^* Q_l) - (L_k^* Q_l)(Q_j^* L_m)\right] \\
& + \left\{y_{ij}^{d\,*} \lambda'_{mli} \left[(H_{\text{d}}^* L_m)(Q_j Q_l) - (H_{\text{d}}^* Q_l)(Q_j^* L_m)\right] + \text{H.c.}\right\}
\end{aligned} \tag{3.91}
$$

$$V_{\bar{E}}^F = \frac{1}{2} \lambda_{kji}^* \lambda_{mli} (L_k^* L_m)(L_j^* L_l) + \lambda_{kji}^* y_{ilp}^e \left[(L_k^* H_{\text{d}})(L_j^* L_l) + \text{H.c.}\right] \tag{3.92}$$

◆3.6.4 Full Lagrangian (in Gauge eigenstates)

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{\mu\nu\,a}W^a_{\mu\nu} - \frac{1}{4}G^{\mu\nu\,a}G^a_{\mu\nu} \quad {}^{*4} \tag{3.93}$$

$$\begin{aligned}\mathcal{L}_{\text{gaugino}} &= -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{w}\widetilde{w} + M_1\widetilde{b}\widetilde{b} + \text{H.c.}\right)\\ &\quad + \mathrm{i}\bar{\widetilde{b}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{b}^a + \mathrm{i}\bar{\widetilde{w}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{w}^a + \mathrm{i}\bar{\widetilde{g}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{g}^a + \mathrm{i}g_2\epsilon^{abc}\bar{\widetilde{w}}^a\bar{\sigma}^\mu W^b_\mu\widetilde{w}^c + \mathrm{i}g_3 f^{abc}\bar{\widetilde{g}}^a\bar{\sigma}^\mu G^b_\mu\widetilde{g}^c\end{aligned} \tag{3.94}$$

$$\mathcal{L}_{\not{CP}} = -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma} - \frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}W^a_{\mu\nu}W^a_{\rho\sigma} - \frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} \tag{3.95}$$

$$\begin{aligned}\mathcal{L}_{\text{scalar}} &= \left[\left(\partial^\mu + \mathrm{i}g_3G^\mu + \mathrm{i}g_2W^\mu + \tfrac{1}{6}\mathrm{i}g_1B^\mu\right)\widetilde{q}_i^*\right]\left[\left(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \tfrac{1}{6}\mathrm{i}g_1B_\mu\right)\widetilde{q}_i\right]\\ &\quad + \left[\left(\partial^\mu + \mathrm{i}g_3G^\mu + \tfrac{2}{3}\mathrm{i}g_1B^\mu\right)\widetilde{u}_{\mathrm{R}i}^*\right]\left[\left(\partial_\mu - \mathrm{i}g_3G_\mu - \tfrac{2}{3}\mathrm{i}g_1B_\mu\right)\widetilde{u}_{\mathrm{R}i}\right]\\ &\quad + \left[\left(\partial^\mu + \mathrm{i}g_3G^\mu - \tfrac{1}{3}\mathrm{i}g_1B^\mu\right)\widetilde{d}_{\mathrm{R}i}^*\right]\left[\left(\partial_\mu - \mathrm{i}g_3G_\mu + \tfrac{1}{3}\mathrm{i}g_1B_\mu\right)\widetilde{d}_{\mathrm{R}i}\right]\\ &\quad + \left[\left(\partial^\mu + \mathrm{i}g_2W^\mu - \tfrac{1}{2}\mathrm{i}g_1B^\mu\right)\widetilde{l}_i^*\right]\left[\left(\partial_\mu - \mathrm{i}g_2W_\mu + \tfrac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{l}_i\right]\\ &\quad + \left[\left(\partial^\mu - \mathrm{i}g_1B^\mu\right)\widetilde{e}_{\mathrm{R}i}^*\right]\left[\left(\partial_\mu + \mathrm{i}g_1B_\mu\right)\widetilde{e}_{\mathrm{R}i}\right]\\ &\quad + \left[\left(\partial^\mu + \mathrm{i}g_2W^\mu + \tfrac{1}{2}\mathrm{i}g_1B^\mu\right)H_{\mathrm{u}}^*\right]\left[\left(\partial_\mu - \mathrm{i}g_2W_\mu - \tfrac{1}{2}\mathrm{i}g_1B_\mu\right)H_{\mathrm{u}}\right]\\ &\quad + \left[\left(\partial^\mu + \mathrm{i}g_2W^\mu - \tfrac{1}{2}\mathrm{i}g_1B^\mu\right)H_{\mathrm{d}}^*\right]\left[\left(\partial_\mu - \mathrm{i}g_2W_\mu + \tfrac{1}{2}\mathrm{i}g_1B_\mu\right)H_{\mathrm{d}}\right],\end{aligned} \tag{3.96}$$

$$\begin{aligned}\mathcal{L}_{\text{fermion}} &= \mathrm{i}\bar{Q}_i\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \frac{1}{6}\mathrm{i}g_1B_\mu\right)Q_i\\ &\quad + \mathrm{i}\bar{U}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3[-G_\mu^{\mathrm{T}}] + \frac{2}{3}\mathrm{i}g_1B_\mu\right)U_i^{\mathrm{c}} + \mathrm{i}\bar{D}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3[-G_\mu^{\mathrm{T}}] - \frac{1}{3}\mathrm{i}g_1B_\mu\right)D_i^{\mathrm{c}}\\ &\quad + \mathrm{i}\bar{L}_i\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)L_i + \mathrm{i}\bar{E}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_1B_\mu\right)E_i^{\mathrm{c}}\\ &\quad + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{u}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{u}} + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{d}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{d}},\end{aligned} \tag{3.97}$$

$$\begin{aligned}\mathcal{L}_{\text{SFG}} &= -\sqrt{2}g_3\Big[\left(\widetilde{q}_i^*\tau^aQ_i + \widetilde{u}_{\mathrm{R}i}[-\tau^{a\mathrm{T}}]U_i^{\mathrm{c}} + \widetilde{d}_{\mathrm{R}i}[-\tau^{a\mathrm{T}}]D_i^{\mathrm{c}}\right)\widetilde{g}^a\\ &\qquad\qquad + \bar{\widetilde{g}}^a\left(\bar{Q}_i\tau^a\widetilde{q}_i + \bar{U}_i^{\mathrm{c}}[-\tau^{a\mathrm{T}}]\widetilde{u}_{\mathrm{R}i}^* + \bar{D}_i^{\mathrm{c}}[-\tau^{a\mathrm{T}}]\widetilde{d}_{\mathrm{R}i}^*\right)\Big]\\ &\quad - \sqrt{2}g_2\Big[\left(\widetilde{q}_i^*T^aQ_i + \widetilde{l}_i^*T^aL_i + H_{\mathrm{u}}^*T^a\widetilde{h}_{\mathrm{u}} + H_{\mathrm{d}}^*T^a\widetilde{h}_{\mathrm{d}}\right)\widetilde{w}^a\\ &\qquad\qquad + \bar{\widetilde{w}}^a\left(\bar{Q}_iT^a\widetilde{q}_i + \bar{L}_iT^a\widetilde{l}_i + \bar{\widetilde{h}}_{\mathrm{u}}T^aH_{\mathrm{u}} + \bar{\widetilde{h}}_{\mathrm{d}}T^aH_{\mathrm{d}}\right)\Big]\\ &\quad - \sqrt{2}g_1\Big[\left(\tfrac{1}{6}\widetilde{q}_i^*Q_i - \tfrac{2}{3}\widetilde{u}_{\mathrm{R}i}U_i^{\mathrm{c}} + \tfrac{1}{3}\widetilde{d}_{\mathrm{R}i}D_i^{\mathrm{c}} - \tfrac{1}{2}\widetilde{l}_i^*L_i + \widetilde{e}_{\mathrm{R}i}E_i^{\mathrm{c}} + \tfrac{1}{2}H_{\mathrm{u}}^*\widetilde{h}_{\mathrm{u}} - \tfrac{1}{2}H_{\mathrm{d}}^*\widetilde{h}_{\mathrm{d}}\right)\widetilde{b}\\ &\qquad\qquad + \bar{\widetilde{b}}\left(\tfrac{1}{6}\bar{Q}_i\widetilde{q}_i - \tfrac{2}{3}\bar{U}_i^{\mathrm{c}}\widetilde{u}_{\mathrm{R}i}^* + \tfrac{1}{3}\bar{D}_i^{\mathrm{c}}\widetilde{d}_{\mathrm{R}i}^* - \tfrac{1}{2}\bar{L}_i\widetilde{l}_i + \bar{E}_i^{\mathrm{c}}\widetilde{e}_{\mathrm{R}i}^* + \tfrac{1}{2}\bar{\widetilde{h}}_{\mathrm{u}}H_{\mathrm{u}} - \tfrac{1}{2}\bar{\widetilde{h}}_{\mathrm{d}}H_{\mathrm{d}}\right)\Big]\end{aligned} \tag{3.98}$$

$$\begin{aligned}\mathcal{L}_{\text{super}}^{\text{RPC}} &= -\mu\widetilde{h}_{\mathrm{u}}\widetilde{h}_{\mathrm{d}} + y_{\mathrm{u}ij}U_i^{\mathrm{c}}H_{\mathrm{u}}Q_j - y_{\mathrm{d}ij}D_i^{\mathrm{c}}H_{\mathrm{d}}Q_j - y_{\mathrm{e}ij}E_i^{\mathrm{c}}H_{\mathrm{d}}L_j\\ &\quad + y_{\mathrm{u}ij}U_i^{\mathrm{c}}\widetilde{h}_{\mathrm{u}}\widetilde{q}_j + y_{\mathrm{u}ij}\widetilde{u}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{u}}Q_j - y_{\mathrm{d}ij}D_i^{\mathrm{c}}\widetilde{h}_{\mathrm{d}}\widetilde{q}_j - y_{\mathrm{d}ij}\widetilde{d}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{d}}Q_j\\ &\quad - y_{\mathrm{e}ij}E_i^{\mathrm{c}}\widetilde{h}_{\mathrm{d}}\widetilde{l}_j - y_{\mathrm{e}ij}\widetilde{e}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{d}}L_j + \text{H.c.}\end{aligned} \tag{3.99}$$

$$\mathcal{L}_{\text{pot.}}^{\text{RPC}} = -(3.85)\left[V_{\text{full}}^{\text{RPC}}\right] \tag{3.100}$$

[*4] Further decomposed results are shown in *Standard Model* section, Eqs. (2.7) and (2.21).

Fermion Composition

$$\mathcal{L}_{\text{gaugino}} = \frac{1}{2}\overline{\widetilde{b}}\left(\mathrm{i}\partial\!\!\!/ - M_1\right)\widetilde{b} + \frac{1}{2}\overline{\widetilde{w}}\left(\mathrm{i}D\!\!\!\!/ - M_2\right)\widetilde{w} + \frac{1}{2}\overline{\widetilde{g}}\left(\mathrm{i}D\!\!\!\!/ - M_3\right)\widetilde{g} \tag{3.101}$$

$$\begin{aligned}
\mathcal{L}_{\text{fermion}} &= (2.5)\Big[\mathcal{L}^{\text{SM}}_{\text{matter}}\Big] \\
&\qquad - \left(\mu\widetilde{h}_{\mathrm{u}}\widetilde{h}_{\mathrm{d}} + \text{H.c.}\right) + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{u}}\bar{\sigma}^{\mu}\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{u}} + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{d}}\bar{\sigma}^{\mu}\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{d}} \\
&= \mathcal{L}^{\text{SM}}_{\text{matter}} - \Big[\mu\left(\widetilde{h}^+_{\mathrm{u}}\widetilde{h}^-_{\mathrm{d}} - \widetilde{h}^0_{\mathrm{u}}\widetilde{h}^0_{\mathrm{d}}\right) + \text{H.c.}\Big] \\
&\qquad + \mathrm{i}\bar{\widetilde{h}}{}^+_{\mathrm{u}}\bar{\sigma}^{\mu}\left(\partial_\mu - \frac{1}{2}\mathrm{i}g_2W^3_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}^+_{\mathrm{u}} + \mathrm{i}\bar{\widetilde{h}}{}^-_{\mathrm{d}}\bar{\sigma}^{\mu}\left(\partial_\mu + \frac{1}{2}\mathrm{i}g_2W^3_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}^-_{\mathrm{d}} \\
&\qquad + \mathrm{i}\bar{\widetilde{h}}{}^0_{\mathrm{u}}\bar{\sigma}^{\mu}\left(\partial_\mu + \frac{1}{2}\mathrm{i}g_2W^3_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}^0_{\mathrm{u}} + \mathrm{i}\bar{\widetilde{h}}{}^0_{\mathrm{d}}\bar{\sigma}^{\mu}\left(\partial_\mu - \frac{1}{2}\mathrm{i}g_2W^3_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}^0_{\mathrm{d}} \\
&\qquad + \frac{g_2}{\sqrt{2}}\left(\bar{\widetilde{h}}{}^0_{\mathrm{u}}\bar{\sigma}^{\mu}W^-_\mu\widetilde{h}^+_{\mathrm{u}} + \bar{\widetilde{h}}{}^0_{\mathrm{d}}\bar{\sigma}^{\mu}W^+_\mu\widetilde{h}^-_{\mathrm{d}} + \bar{\widetilde{h}}{}^+_{\mathrm{u}}\bar{\sigma}^{\mu}W^+_\mu\widetilde{h}^0_{\mathrm{u}} + \bar{\widetilde{h}}{}^-_{\mathrm{d}}\bar{\sigma}^{\mu}W^-_\mu\widetilde{h}^0_{\mathrm{d}}\right)
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
\mathcal{L}_{\text{SFG}} = -\sqrt{2}g_3\Big[&\widetilde{u}^*_{\mathrm{L}i}\tau^a\left(\bar{\widetilde{g}}{}^aP_{\mathrm{L}}U_i\right) + \widetilde{d}^*_{\mathrm{L}i}\tau^a\left(\bar{\widetilde{g}}{}^aP_{\mathrm{L}}D_i\right) - \left(\overline{U}_iP_{\mathrm{L}}\widetilde{g}^a\right)\tau^a\widetilde{u}_{\mathrm{R}i} - \left(\overline{D}_iP_{\mathrm{L}}\widetilde{g}^a\right)\tau^a\widetilde{d}_{\mathrm{R}i} \\
&+ \left(\overline{U}_iP_{\mathrm{R}}\widetilde{g}^a\right)\tau^a\widetilde{u}_{\mathrm{L}i} + \left(\overline{D}_iP_{\mathrm{R}}\widetilde{g}^a\right)\tau^a\widetilde{d}_{\mathrm{L}i} - \widetilde{u}^*_{\mathrm{R}i}\tau^a\left(\bar{\widetilde{g}}{}^aP_{\mathrm{R}}U\right) - \widetilde{d}^*_{\mathrm{R}i}\tau^a\left(\bar{\widetilde{g}}{}^aP_{\mathrm{R}}D\right)\Big] \\
- g_2\Big[&\widetilde{w}^+\left(d_i\widetilde{u}_{\mathrm{L}i}^* + e_i\widetilde{\nu}_i^* + \widetilde{h}^0_{\mathrm{u}}H^{+*}_{\mathrm{u}} + \widetilde{h}^-_{\mathrm{d}}H^{0*}_{\mathrm{d}}\right) + \left(\widetilde{d}_{\mathrm{L}i}\bar{U}_i + \widetilde{e}_{\mathrm{L}i}\bar{\nu}_i + H^0_{\mathrm{u}}\bar{\widetilde{h}}{}^+_{\mathrm{u}} + H^-_{\mathrm{d}}\bar{\widetilde{h}}{}^0_{\mathrm{d}}\right)\bar{\widetilde{w}}{}^- \\
&+ \widetilde{w}^-\left(u_i\widetilde{d}_{\mathrm{L}i}^* + \nu_i\widetilde{e}_{\mathrm{L}i}^* + \widetilde{h}^+_{\mathrm{u}}H^{0*}_{\mathrm{u}} + \widetilde{h}^0_{\mathrm{d}}H^{-*}_{\mathrm{d}}\right) + \left(\widetilde{u}_{\mathrm{L}i}\bar{D}_i + \widetilde{\nu}_i\bar{E}_i + H^+_{\mathrm{u}}\bar{\widetilde{h}}{}^0_{\mathrm{u}} + H^0_{\mathrm{d}}\bar{\widetilde{h}}{}^-_{\mathrm{d}}\right)\bar{\widetilde{w}}{}^+\Big] \\
- \sqrt{2}g_2\Big[&\tfrac{1}{2}\left(\widetilde{u}_{\mathrm{L}i}^*u_i + \widetilde{\nu}_i^*\nu_i + H^{+*}_{\mathrm{u}}\widetilde{h}^+_{\mathrm{u}} + H^{0*}_{\mathrm{d}}\widetilde{h}^0_{\mathrm{d}} - \widetilde{d}_{\mathrm{L}i}^*d_i - \widetilde{e}_{\mathrm{L}i}^*e_i - H^{0*}_{\mathrm{u}}\widetilde{h}^0_{\mathrm{u}} - H^{-*}_{\mathrm{d}}\widetilde{h}^-_{\mathrm{d}}\right)\widetilde{w}^3 \\
&\qquad + \tfrac{1}{2}\bar{\widetilde{w}}{}^a\left(\bar{U}_i\widetilde{u}_{\mathrm{L}i} + \bar{\nu}_i\widetilde{\nu}_i + \bar{\widetilde{h}}{}^+_{\mathrm{u}}H^+_{\mathrm{u}} + \bar{\widetilde{h}}{}^0_{\mathrm{d}}H^0_{\mathrm{d}} - \bar{D}_i\widetilde{d}_{\mathrm{L}i} - \bar{E}_i\widetilde{e}_{\mathrm{L}i} - \bar{\widetilde{h}}{}^0_{\mathrm{u}}H^0_{\mathrm{u}} - \bar{\widetilde{h}}{}^-_{\mathrm{d}}H^-_{\mathrm{d}}\right)\Big] \\
- \sqrt{2}g_1\Big[&\tfrac{1}{6}\widetilde{q}_i^*\bar{\widetilde{b}}P_{\mathrm{L}}Q_i - \tfrac{2}{3}\widetilde{u}_{\mathrm{R}i}\overline{U}_iP_{\mathrm{L}}\widetilde{b} + \tfrac{1}{3}\widetilde{d}_{\mathrm{R}i}\overline{D}_iP_{\mathrm{L}}\widetilde{b} - \tfrac{1}{2}\widetilde{l}_i^*\bar{\widetilde{b}}P_{\mathrm{L}}L_i + \widetilde{e}_{\mathrm{R}i}\overline{E}_iP_{\mathrm{L}}\widetilde{b} \\
&\qquad + \tfrac{1}{6}\widetilde{q}_i\overline{Q}_iP_{\mathrm{R}}\widetilde{b} - \tfrac{2}{3}\widetilde{u}^*_{\mathrm{R}i}\bar{\widetilde{b}}P_{\mathrm{R}}U_i + \tfrac{1}{3}\widetilde{d}^*_{\mathrm{R}i}\bar{\widetilde{b}}P_{\mathrm{R}}D_i - \tfrac{1}{2}\widetilde{l}_i\overline{L}_iP_{\mathrm{R}}\widetilde{b} + \widetilde{e}^*_{\mathrm{R}i}\bar{\widetilde{b}}P_{\mathrm{R}}E_i \\
&\qquad + \tfrac{1}{2}H^*_{\mathrm{u}}\widetilde{h}_{\mathrm{u}}\widetilde{b} - \tfrac{1}{2}H^*_{\mathrm{d}}\widetilde{h}_{\mathrm{d}}\widetilde{b} + \tfrac{1}{2}H_{\mathrm{u}}\bar{\widetilde{h}}_{\mathrm{u}}\bar{\widetilde{b}} - \tfrac{1}{2}H_{\mathrm{d}}\bar{\widetilde{h}}_{\mathrm{d}}\bar{\widetilde{b}}\Big]
\end{aligned} \tag{3.103}$$

$$\begin{aligned}
\mathcal{L}_{\text{湯川}} = \Big(&-H^0_{\mathrm{u}}\overline{U}y_{\mathrm{u}}P_{\mathrm{L}}U - H^0_{\mathrm{d}}\overline{D}y_{\mathrm{d}}P_{\mathrm{L}}D - H^0_{\mathrm{d}}\overline{E}y_{\mathrm{e}}P_{\mathrm{L}}E \\
&\qquad + H^+_{\mathrm{u}}\overline{U}y_{\mathrm{u}}P_{\mathrm{L}}D + H^-_{\mathrm{d}}\overline{D}y_{\mathrm{d}}P_{\mathrm{L}}U + H^-_{\mathrm{d}}\overline{E}y_{\mathrm{e}}P_{\mathrm{L}}\nu + \text{H.c.}\Big) \\
&+ \widetilde{u}_{\mathrm{L}j}\left(y_{\mathrm{d}ij}D^{\mathrm{c}}_i\widetilde{h}^-_{\mathrm{d}} - y_{\mathrm{u}ij}U^{\mathrm{c}}_i\widetilde{h}^0_{\mathrm{u}}\right) + \widetilde{d}_{\mathrm{L}j}\left(y_{\mathrm{u}ij}U^{\mathrm{c}}_i\widetilde{h}^+_{\mathrm{u}} - y_{\mathrm{d}ij}D^{\mathrm{c}}_i\widetilde{h}^0_{\mathrm{d}}\right) + y_{\mathrm{e}ij}E^{\mathrm{c}}_i\widetilde{h}^-_{\mathrm{d}}\widetilde{\nu}_j - y_{\mathrm{e}ij}E^{\mathrm{c}}_i\widetilde{h}^0_{\mathrm{d}}\widetilde{e}_{\mathrm{L}j} \\
&+ y_{\mathrm{u}ij}\widetilde{u}^*_{\mathrm{R}i}\left(\widetilde{h}^+_{\mathrm{u}}d_j - \widetilde{h}^0_{\mathrm{u}}u_j\right) + y_{\mathrm{d}ij}\widetilde{d}^*_{\mathrm{R}i}\left(\widetilde{h}^-_{\mathrm{d}}u_j - \widetilde{h}^0_{\mathrm{d}}d_j\right) + y_{\mathrm{e}ij}\widetilde{e}^*_{\mathrm{R}i}\left(\widetilde{h}^-_{\mathrm{d}}\nu_j - \widetilde{h}^0_{\mathrm{d}}e_j\right) \\
&+ \widetilde{u}^*_{\mathrm{L}j}\left(y^*_{\mathrm{d}ij}\bar{D}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^-_{\mathrm{d}} - y^*_{\mathrm{u}ij}\bar{U}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^0_{\mathrm{u}}\right) + \widetilde{d}^*_{\mathrm{L}j}\left(y^*_{\mathrm{u}ij}\bar{U}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^+_{\mathrm{u}} - y^*_{\mathrm{d}ij}\bar{D}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^0_{\mathrm{d}}\right) + \widetilde{\nu}^*{}_jy^*_{\mathrm{e}}{}_{ij}\bar{E}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^-_{\mathrm{d}} - \widetilde{e}^*_{\mathrm{L}j}y^*_{\mathrm{e}}{}_{ij}\bar{E}^{\mathrm{c}}_i\bar{\widetilde{h}}{}^0_{\mathrm{d}} \\
&+ y^*_{\mathrm{u}ij}\widetilde{u}_{\mathrm{R}i}\left(\bar{\widetilde{h}}{}^+_{\mathrm{u}}\bar{d}_j - \bar{\widetilde{h}}{}^0_{\mathrm{u}}\bar{u}_j\right) + y^*_{\mathrm{d}ij}\widetilde{d}_{\mathrm{R}i}\left(\bar{\widetilde{h}}{}^-_{\mathrm{d}}\bar{u}_j - \bar{\widetilde{h}}{}^0_{\mathrm{d}}\bar{d}_j\right) + y^*_{\mathrm{e}}{}_{ij}\widetilde{e}_{\mathrm{R}i}\left(\bar{\widetilde{h}}{}^-_{\mathrm{d}}\bar{\nu}_j - \bar{\widetilde{h}}{}^0_{\mathrm{d}}\bar{e}_j\right)
\end{aligned} \tag{3.104}$$

and the rest part is $\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}^{\text{RPC}}_{\text{pot.}}$.

3.7 MINIMAL GMSB

$$W_{\text{mess}} = \lambda_i X \bar{\Phi}_i \Phi_i \rightsquigarrow \lambda_i (M_X + F_X \theta\theta) \Phi_i \Phi_i \quad : \quad M_i^{\text{F}} = \lambda_i M_X, \quad \left(M_i^{\text{S}}\right)^2 = \lambda_i^2 M_X^2 - \lambda_i F_X.$$

If F_i/M_i is universal and the messengers feel only a single X,[*5]

$$\Lambda := \frac{F_X}{M_X}, \quad M_{\text{mess}} := \lambda M_X, \qquad M_a(M_{\text{mess}}) \simeq \frac{\alpha_a}{4\pi}\Lambda_{\text{G}}, \qquad m^2(M_{\text{mess}}) \simeq 2\Lambda_{\text{S}}^2 \sum \frac{\alpha_a^2}{(4\pi)^2} C_a, \tag{3.106}$$

C_3 is 4/3 for colored particles, C_2 is 3/4 for **2**, and $C_1 = (5/3)Y^2$. Also the ~~SUSY~~ scales are

$$\Lambda_{\text{G}} := N_5 \frac{F}{M} = N_5 \Lambda, \qquad \Lambda_{\text{S}} := N_5 \frac{F^2}{M^2} = N_5 \Lambda^2; \qquad \text{where } N_5 \text{ is } (\#\mathbf{5}) + 3 \times (\#\mathbf{10}), \tag{3.107}$$

Note that $(\Lambda_{\text{G}}, \Lambda_{\text{S}})$ can be used as the defining parameters instead of (N_5, Λ).

The total SUSY-breaking F_{total} is different from that felt by messengers; $F_{\text{total}} =: F/k$ with $k < 1$.

$$m_{3/2} = \frac{F_{\text{total}}}{\sqrt{3} M_{\text{pl}}^{\text{R}}} = \frac{F/k}{\sqrt{3} M_{\text{pl}}^{\text{R}}} = \frac{M_{\text{mess}} \Lambda}{\lambda k \cdot \sqrt{3} M_{\text{pl}}^{\text{R}}}. \tag{3.108}$$

[*5] If non universal, Λ_{S} cannot be written in simple form, and

$$\Lambda_{\text{G}} = \sum n_i \frac{F_i}{M_i} \left(1 + \mathrm{O}\left(\frac{F_i^2}{M_i^4}\right)\right), \tag{3.105}$$

where n_i is twice of the Dynkin index; 1 for **5**, and 3 for **10**.

3.8 Gravitino and Goldstino

◆3.8.1 Supercurrent

$S^\mu_\alpha = (\sigma^\nu\bar\sigma^\mu\chi_i)_\alpha \mathrm{D}_\nu\phi^{*i} + \mathrm{i}(\sigma^\mu\chi^{\dagger i})_\alpha W^*_\phi - \frac{1}{2\sqrt2}(\sigma^\nu\bar\sigma^\rho\sigma^\mu\lambda^{\dagger a})_\alpha F^a_{\nu\rho} + \frac{\mathrm{i}}{\sqrt2}g_a(\phi^*T^a\phi)(\sigma^\mu\lambda^{\dagger a})_\alpha$

With a Majorana particle $X = \begin{pmatrix} X_\alpha \\ \bar X^{\dot\alpha}\end{pmatrix}$,

$$
\begin{aligned}
X^\alpha S^\mu_\alpha &= \mathrm{D}_\nu\phi^*(\overline{X}\gamma^\nu\gamma^\mu P_\mathrm{L}\chi) + \mathrm{i}(\overline{X}\gamma^\nu P_\mathrm{R}\chi)W^*_\phi - \frac{1}{4\sqrt2}F^a_{\nu\rho}\overline{X}[\gamma^\nu,\gamma^\rho]\gamma^\mu P_\mathrm{R}\lambda^a + \frac{\mathrm{i}g_a(\phi^*T^a\phi)}{\sqrt2}\overline{X}\gamma^\mu P_\mathrm{R}\lambda^a \\
&\overset{\text{h.c.}}{\rightsquigarrow} \mathrm{D}_\nu\phi(\overline{X}\gamma^\nu\gamma^\mu P_\mathrm{R}\chi) + \mathrm{i}(\overline{X}\gamma^\nu P_\mathrm{L}\chi)W_\phi - \frac{1}{4\sqrt2}F^a_{\nu\rho}\overline{X}[\gamma^\nu,\gamma^\rho]\gamma^\mu P_\mathrm{L}\lambda^a + \frac{\mathrm{i}g_a(\phi^*T^a\phi)}{2}\overline{X}\gamma^\mu P_\mathrm{L}\lambda^a
\end{aligned}
$$

where $W_\phi = \frac{\delta}{\delta\phi}W|_\mathrm{scalar}$, $W_\phi = \frac{\delta}{\delta\phi^*}W^*|_\mathrm{scalar}$.

◆3.8.2 Goldstino

$$
\mathcal{L}_\mathrm{goldstino} = i\widetilde{\mathcal{G}}^\dagger\bar\sigma^\mu\partial_\mu\widetilde{\mathcal{G}} - \frac{1}{F_\mathrm{total}}\left[\widetilde{\mathcal{G}}\partial_\mu S^\mu + \text{H.c.}\right], \tag{3.109}
$$

especially interaction terms contain

$$
-\frac{1}{F_\mathrm{total}}\left[\left(\partial_\nu\phi^*\cdot\overline{\chi}\gamma^\mu\gamma^\nu P_\mathrm{L}\partial_\mu\widetilde{\mathcal{G}} + \text{H.c.}\right) - \frac{1}{4\sqrt2}(\overline{\lambda}\gamma^\mu[\gamma^\nu,\gamma^\rho]\partial_\mu\widetilde{\mathcal{G}})F^a_{\nu\rho}\right]. \tag{3.110}
$$

If all the particles are on-shell, this can be reduced to

$$
\frac{1}{F_\mathrm{total}}\left[(m_\chi^2 - m_\phi^2)\phi^*\overline{\chi}P_\mathrm{L}\widetilde{\mathcal{G}} + \frac{1}{4\sqrt2}(m_\lambda\overline{\lambda}[\gamma^\nu,\gamma^\rho]P_\mathrm{R}\widetilde{\mathcal{G}})F^a_{\nu\rho} + \text{H.c.}\right]. \tag{3.111}
$$

<u>**TODO**</u>**:** 実はよくわかってない

◆3.8.3 Gravitino $\psi = \begin{pmatrix}\psi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix}$

Lagrangian: $\mathcal{L} = -\frac12\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \frac{m_{3/2}}{4}\overline{\psi}_\mu\left[\gamma^\mu,\gamma^\nu\right]\psi_\nu - \frac{1}{\sqrt2 M_\mathrm{pl}^\mathrm{R}}\left(\psi^\alpha_\mu S^\mu_\alpha + \text{H.c.}\right)$

EOM: $\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma + \frac{m_{3/2}}{2}\left[\gamma^\mu,\gamma^\nu\right]\psi_\nu = 0$ (Rarita–Schwinger eq.)

$\Longrightarrow m_{3/2}\left(\partial\!\!\!/\psi\!\!\!/ - \gamma^\nu\partial\!\!\!/\psi_\nu\right) = 0,\quad \left(\partial\!\!\!/\psi\!\!\!/ - \gamma^\nu\partial\!\!\!/\psi_\nu\right) - 3\mathrm{i}m_{3/2}\psi\!\!\!/ = 0.$

⟪if massive $\Longrightarrow \psi\!\!\!/ = 0,\quad \partial_\mu\psi^\mu = 0,\quad (\mathrm{i}\partial\!\!\!/ - m_{3/2})\psi_\mu = 0$ (Dirac eq.).⟫

Field: $\psi_\mu = \displaystyle\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol p}}}\sum_{s=\pm1/2,\pm3/2}\left[a^s_{\boldsymbol p}u^s_\mu(p)\mathrm{e}^{-\mathrm{i}px} + a^{s\dagger}_{\boldsymbol p}v^s_\mu(p)\mathrm{e}^{\mathrm{i}px}\right].$

Of course, u and v are as usual related as $v^s_\mu(p) = \left[u^s_\mu(p)\right]^\mathrm{C} = C\overline{u}^\mathrm{T}$.

Spin sum: $\Pi_{\mu\nu}(p) := \sum_s\psi^s_\mu(p)\overline{\psi}^s_\nu(p)$

$= -(p\!\!\!/ + m_{3/2})\left(\eta^{\mu\nu} - \frac{p_\mu p_\nu}{m_{3/2}^2}\right) - \frac13\left(\gamma^\mu + \frac{p^\mu}{m_{3/2}}\right)(p\!\!\!/ - m_{3/2})\left(\gamma^\nu + \frac{p^\nu}{m_{3/2}}\right)$

Obviously, $\gamma^\mu\Pi_{\mu\nu}(p) = p^\mu\Pi_{\mu\nu}(p) = (p\!\!\!/ - m_{3/2})\Pi_{\mu\nu}(p) = 0.$

Formulae: $k^\mu k^\nu\Pi_{\mu\nu}(p) = \frac23\left[\frac{(p\cdot k)^2}{m_{3/2}^2} - k^2\right](p\!\!\!/ + m_{3/2}),\quad \eta^{\mu\nu}\Pi_{\mu\nu}(p) = -2\left(p\!\!\!/ + m_{3/2}\right).$

4 Supergravity

4.1 Minimal SUGRA Lagrangian

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2} eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \bar{\sigma}_\nu \mathrm{D}_\rho \psi_\sigma \tag{4.1}$$

where

$$\mathrm{D}_\mu \psi_\nu := \partial_\mu \psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \qquad \left[\omega_\mu{}^{ab} : \text{“spin 接続”}\right] \tag{4.2}$$

$$e := \det e_a{}^\mu \tag{4.3}$$

$$M := 1/\sqrt{8\pi G} \quad \text{(Reduced Planck mass)} \tag{4.4}$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \tag{4.5}$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \tag{4.6}$$

4.2 General SUGRA Lagrangian

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \qquad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \qquad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \tag{4.7}$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
  - Real function of chiral multiplets.
  - In global SUSY, $\int \mathrm{d}^4\theta K$ yields kinetic terms of the chiral multiplet.
  - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int \mathrm{d}^4\theta\ \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \tag{4.8}$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
  - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
  - $(a), (b), \ldots$ are indices for adjoint representation of gauge group.
  - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}\mathrm{D}_\mu\phi^i\mathrm{D}^\mu\phi^{*j} - \frac{1}{2}eg^2D_{(a)}D^{(a)} \\
& + \mathrm{i}eg_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu\mathrm{D}_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\mathrm{D}_\rho\psi_\sigma \\
& - \frac{1}{4}ef^{\mathrm{R}}{}_{(ab)}F^{(a)}_{\mu\nu}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f^{\mathrm{I}}{}_{(ab)}f^{(a)}_{\mu\nu}f^{(b)}_{\rho\sigma} \\
& + \frac{\mathrm{i}}{2}e\left[\lambda_{(a)}\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^{(a)}\right] - \frac{1}{2}f^{\mathrm{I}}{}_{(ab)}\mathrm{D}_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_{i*}f^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F^{(b)}_{\mu\nu} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F^{(b)}_{\mu\nu} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{\mathrm{i}}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F^{(a)}_{\nu\rho} + \hat{F}^{(a)}_{\nu\rho}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f^{\mathrm{R}}{}_{(ab)} + f^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(bc)}\partial_{j^*}f^*_{(ad)}\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f^*_{(ab)}\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_{i^*}f^*_{(ac)}\partial_{j^*}f^*_{(bd)}\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f^*_{(cd)}\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{\mathrm{i}}{2}\sqrt{2}e\mathrm{e}^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}e\mathrm{e}^{K/2}\left[\mathrm{D}_iD_jW\chi^i\chi^j + \mathrm{D}_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}e\mathrm{e}^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f^*_{(ab)}\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned} \tag{4.9}
$$

付録 A　Mathematics

A.1　ALGEBRA

◆A.1.1　Algebraic Structure

Semigroup	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative.
Monoid	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit.
Group	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit, Inverse.
Module (加群/可換群)	:	For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit, Inverse.
Semimodule	:	For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit.
Ring (環)	:	$+$: Module, $\times$: Semigroup(Monoid), Distributive.
Semiring	:	$+$: Semimodule, $\times$: Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.
Field	:	$+$: Module, $\times$: Commutative Monoid, a^{-1} but 0, $1 \neq 0$, Distributive.

■Vector Space

Vector space on K	:	For $v \in (V, +)_{\text{module}}$ and $k \in K_{\text{field}}$,
(K-module)	:	$kv \in (V, +)$; Compatible, Distributive, $1v = v$.
Norm	:	$\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|kx\| = k\,\|x\|$, $\|x + y\| \leq \|x\| + \|y\|$
Inner product	:	$\langle x \rangle\, x \geq 0$, $\langle x \rangle\, x = 0 \Leftrightarrow x = 0$,$\langle x \rangle\, y = \langle y \rangle\, x$,
	:	$\langle x + y \rangle\, z = \langle x \rangle\, z + \langle y \rangle\, z$, $\langle kx \rangle\, y = k\,\langle x \rangle\, y$

■K-algebra　K-algebra $C(V)$ とは，vector 空間 V に，distributive な乗法を入れたもの：

$xy \in C(V)$;　$(xy)z = x(yz)$, $(x + y)z = xz + yz$, $x(y + z) = xy + xz$, $k(xy) = (kx)y = x(ky)$.

◆A.1.2　Lie Algebra

Lie Algebra :　For a Finite-dimensional K-module $(A, +)$ and $x, y, z \in (A, +)$, $a, b \in K$,

:　$[u, v] \in (A, +)$ (Lie product), and

:　Bilinear: $[ax + by, z] = a[x, z] + b[y, z]$, $[x, ay + bz] = a[x, y] + b[x, z]$,

:　Alternating: $[x, x] = 0$　$(\Longrightarrow [x, y] = -[y, x])$,

:　Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

則ち $[A, B] := AB - BA$ として閉じていれば Lie algebra となる。

◆A.1.3　Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.
For $C(V)$, **Clifford algebra** $(C(V), \theta)$ is defined as

- $C(V)$: K-algebra with 1　$(1x = x1 = x)$,
- $\theta : V \to C(V)$, homomorphism, $\theta(x)^2 = \langle x \rangle\, x1$,
- Any C': K-algebra with 1 and any homomorphism $\phi : V \to C'$ with $\phi(x)^2 = \langle x \rangle\, x1$, there's unique $\bar{\phi} : C \to C'$, homomorphism, $\bar{\phi}(1) = 1$.

> The Gamma matrices form Clifford Algebra:
>
> $V : \mathbb{R}^4$ with $\{G_0, G_1, G_2, G_3\}$,　$\langle G_0 \rangle\, G_0 = 1, \langle G_i \rangle\, G_i = -1$;　　$C(V) : M(n, \mathbb{C})$.

◆A.1.4 Multilinear Algebra

可換環 K 上の vector 空間 V とその双対空間 V^* について：

Tensor Algebra $T(V)$: 線形写像 $f : V \to T(V)$ を持つ K-algebra であり，別の K-algebra A への線型写像 $g : V \to A$ が与えられたときに可換な準同型 $h : T(V) \to A$ s.t. $h \circ f = g$ が一意に存在するもの。

Symmetric Algebra $S(V)$: 上の定義で，$C(V)$（および A）を可換 K-algebra としたもの。

Exterior Algebra $\wedge(V)$: 上の定義で，$g(\cdots vv \cdots) = 0$ を要請したもの。

■Tensor and Tensor Space

p 次反変 Tensor 積 : $T^p(V) := V^{\otimes p} = V \otimes V \otimes \cdots \otimes V$

q 次共変 Tensor 積 : $T_q(V) := (V^*)^{\otimes q} = V^* \otimes V^* \otimes \cdots \otimes V^*$

混合 Tensor 積 : $T^p_q(V) := T^p(V) \otimes T_q(V)$（ただし V と V^* の順序を変えたものは同型）

Tensor Space : $T(V) := \oplus T^p_q(V)$

Tensor space の代数 $(+, \otimes)$ は環を為しており，また更に：

Contraction : V^* は $V \to K$ なので，$T^p_q(V) \to T^{p-1}_{q-1}(V)$ が定義される。

内積 : $T_2(V) \ni g_{ij} : V \times V \to \mathbb{R}$ （通常は対称にする。正定値としてもよい。）

添字の上げ下げ : $T_2(V) \cong \mathrm{Hom}(V, V^*)$（同型）なので，内積から誘導される。

: 同型なので，$g^{ij} : \mathrm{Hom}(V^*, V)$ は逆写像になる。

■Grassmann Operator

台集合 V を Hilbert 空間であるとする。$V \ni v$ について

$$\|\psi v\|^2 = (\psi_{ab}v_b)^*(\psi_{ac}v_c) = v_b^* v_c (\psi_{ab})^* \psi_{ac} \geq 0 \qquad \therefore (\psi_{ab})^*\psi_{ac} = -\left(\psi_{ab}(\psi_{ac})^*\right)^* \tag{A.1}$$

即ち反可換な作用素について $(ab)^\dagger = a^\dagger b^\dagger$, $(ab)^\mathrm{T} = -a^\mathrm{T} b^\mathrm{T}$, $(ab)^* = b^* a^*$ である。

TODO: ψ の正体がわからない……。。。

◆A.1.5 Lie Group and Lie Algebra

- 群 G が **Lie group** である … G が同時に C^∞ 多様体であり，積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が **COMPLEX Lie group** である … 積演算と逆元写像が共に正則写像である。

- Lie 群 G の単位元における接空間を，G の **Lie algebra** $\mathfrak{g}$ という。
  - $\mathfrak{g}$ は G の左不変な vector 場全体である。
  - $\mathfrak{g}$ は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると，
  - その Lie 代数の基底 B_i に対して **structure constant** c が $[B_i, B_j] = c^k_{ij} B_k$ として定義できる。

* * *

- Compact Lie 群は線型 Lie 群である。
- G として **Linear group** $\mathrm{GL}(n;\mathbb{R})$ を考えると，
  - その Lie 代数は n 次実正方行列全体となる。
  - Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は，$\mathrm{GL}(n;\mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では，Lie 群として $\mathrm{GL}(n;\mathbb{R})$ の部分群を考えることにし，Lie 代数の元を行列により表現する。

◆A.1.6 Matrix Representation

- Lie 群 G の Lie 代数の基底の組を，G の **generators** と言う。
- $\mathrm{GL}(n;\mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し，以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1 + \cdots + x_m T_m} \qquad (x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1} \cdots \mathrm{e}^{x_m T_m} \tag{A.2}$$

- Lie 群 G が compact である $\cdots$
  1. 多様体 G が compact である。 **TODO:** これは何故同値なのか？
  2. G の生成子 $\{T_i\}$ を，$\mathrm{Tr}(T_i T_j) = k\,\delta_{ij}$ かつ $k > 0$ となるように取り替えることができる。
  【この基底の下では構造定数が完全反対称になる。】
- Compact 群 G は，**unitary representation** を持つ。
  故に，単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i \in \mathbb{R}$ により，G の元を

$$\mathrm{e}^{\mathrm{i}x^i T^i} \tag{A.3}$$

  と表すことが出来る。

◆A.1.7 結論

Compact Lie 群の元のうち，単位元近傍にあるもの V は，

Hermitian Representation

$$V = \exp(\mathrm{i}x^i T^i) \qquad \text{where} \quad T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R},$$
$$[T^i, T^j] = \mathrm{i} f^{ijk} T^k, \quad \mathrm{Tr}(T^i T^j) = \lambda\,\delta^{ij} > 0; \qquad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \qquad \text{where} \quad R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R},$$
$$[R^i, R^j] = -f^{ijk} R^k, \quad \mathrm{Tr}(R^i R^j) = -\lambda\,\delta^{ij} < 0; \qquad f \in \mathbb{R}$$

と表すことが出来る。

付録 B　Statistics

Histogram の階級数についての Sturges の公式 $k \approx 1 + \log_2 n$ （n:観測値の数）

■分布の代表値

$$\text{算術平均 } \overline{x} := \frac{1}{n}\sum x_i, \quad \text{幾何平均 } x_G := \left(\prod x_i\right)^{1/n}, \quad \text{調和平均 } x_H := n\left(\sum \frac{1}{x_i}\right)^{-1}; \tag{B.1}$$

中央値，n 分位点，最頻値，mid-range，…… (B.2)

■分布の散らばり

$$\text{平均偏差 } d := \frac{1}{n}\sum |x_i - \overline{x}|, \quad \text{標準偏差（分散）} S^2 := \frac{1}{n}\sum (x_i - \overline{x})^2, \quad \text{変動係数 } C.V. := S_x/\overline{x}; \tag{B.3}$$

$$\text{平均差 } M.D. := \frac{1}{n^2}\sum_i \sum_j |x_i - x_j|, \quad \text{Gini 係数 } G.I. := \frac{M.D.}{2\overline{x}} = \frac{1}{2n^2\overline{x}}\sum_i \sum_j |x_i - x_j|; \tag{B.4}$$

$$\text{Entropy } H = -\sum p_i \log p_i \quad (p\text{:相対頻度}) \quad \cdots \text{ 一ヶ所集中} = 0 \leq H \leq 1 = \text{等確率} \tag{B.5}$$

範囲，四分位偏差，…… (B.6)

■相関を表す量

$$\text{共分散 } C_{xy} := \frac{1}{n}\sum (x_i - \overline{x})(y_i - \overline{y}) \qquad \text{相関係数 } r_{xy} := \frac{C_{xy}}{S_x S_y} \quad -1 \leq r_{xy} \leq 1, \text{線型不変} \tag{B.7}$$

$$\text{偏相関関数 } r_{12;3} := \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - {r_{13}}^2}\sqrt{1 - {r_{23}}^2}} \qquad \text{系列相関関数 } r_h := \frac{1}{S_x}\sum_{i=1}^{n-h} \frac{(x_i - \overline{x})(x_i + h - \overline{x})}{n - h} \tag{B.8}$$

順位相関関数 … 順位の組 $\{R_i\}$，$\{R'_i\}$ の間の相関

$$\text{Spearman}: \ r_{\mathrm{S}} := 1 - \frac{6}{n(n^2 - 1)}\sum (R_i - R'_i)^2 \quad \text{（通常の相関関数）} \tag{B.9}$$

$$\text{Kendall}: \ r_{\mathrm{K}} := \frac{\sum G_{ij}}{n(n-1)/2} \qquad \text{where} \quad G_{ij} := (i, j) \text{ に対して同順なら } +1\text{，逆順なら } -1 \tag{B.10}$$

■条件付き確率

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{B.11}$$

Bayes の定理: 事象 $\{H_i\}$ が互いに排反かつ全体を尽くしているとき，

$$P(A) = \sum P(A \cap H_i) \quad \text{によって} \quad P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\sum_k P(H_k)P(A|H_k)}. \tag{B.12}$$

相関係数の分布 （ρ:母集団の（真の）相関係数）

$$f(r) = \frac{\left(1 - \rho^2\right)^{(n-1)/2}\left(1 - r^2\right)^{(n-4)/2}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\sum_{i=0}^{\infty} \frac{(2\rho r)^i}{i!}\left[\Gamma\left(\frac{n-1+i}{2}\right)\right]^2 \tag{B.13}$$

■確率分布に対する Moment

$$\text{Moment } \mu_r := \langle X^r \rangle\,; \quad \mu'_r := \langle (X-\mu)^r \rangle \qquad \text{標準化 moment } \alpha_r := \langle (X-\mu)^r \rangle / \sigma^r \tag{B.14}$$

$$\text{Moment 母関数 } M_X(t) := \langle \exp(tX) \rangle \Longrightarrow \mu_r = \frac{\mathrm{d}^n}{\mathrm{d}t^n} M_X(t) \tag{B.15}$$

$$\text{期待値 } \mu := \mu_1; \qquad \text{計算の便法:}\ \mu'_2 = \left\langle X^2 \right\rangle - \mu^2 \tag{B.16}$$

$$\text{分散 } \sigma^2 := \mu'_2; \quad \text{標準偏差 } \sigma := \sqrt{\sigma^2}; \qquad \mu'_3 = \left\langle X^3 \right\rangle - 3\mu\mu_2 + 2\mu^3 \tag{B.17}$$

$$\text{歪度 } C_{\text{skew}} := \alpha_3; \quad \text{尖度 } C_{\text{kurt}} := \alpha_4 - 3 \qquad \mu'_4 = \left\langle X^4 \right\rangle - 4\mu\mu_3 + 6\mu^2\mu_2 - 3\mu^4 \tag{B.18}$$

■Chebyshev の不等式　いかなる確率変数に対しても，$P\Big(|X-\mu| \geq k\sigma\Big) \leq \dfrac{1}{k^2}$.

■Stirling の公式

$$\log n! = \left(n + \frac{1}{2}\right)\log n - n + \frac{\log 2\pi}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + \mathrm{O}(n^{-11}) \tag{B.19}$$

B.1 離散型確率分布

■超幾何分布　(A,B) が $(M, N-M)$ 個あるとき，n 個取り出して $(k, n-k)$ 個である確率。非復元捕獲。

$$P_k = \frac{{}_M\mathrm{C}_k\ {}_{N-M}\mathrm{C}_{n-k}}{{}_N\mathrm{C}_n} \qquad E = np, \quad V = np(1-p)\frac{N-n}{N-1} \qquad (p := M/N) \tag{B.20}$$

■二項分布　確率 p で起きる事象が，n 回のうち k 回起こる確率。復元捕獲，Bernoulli 試行。

$$P_k = {}_n\mathrm{C}_k \cdot p^k (1-p)^{n-k} \qquad E = np, \quad V = np(1-p) \qquad (n = 1 : \text{Bernoulli 分布}) \tag{B.21}$$

■Poisson 分布　二項分布において $np = \lambda$ 一定で $n \to \infty,\ p \to 0$ として，$P_k = \dfrac{\mathrm{e}^{-\lambda}\lambda^k}{k!}$,　$E = V = \lambda$.

■幾何分布　確率 p の事象が起こるまでの失敗回数 k の分布。

$$P_k = p(1-p)^k \qquad E = \frac{1-p}{p}, \quad V = \frac{1-p}{p^2}. \tag{B.22}$$

■負の二項分布 (Pascal 分布)　確率 p の事象が n 回起こるまでの失敗回数 k の分布。(試行は $n+k$ 回)

$$P_k = {}_{n+k-1}\mathrm{C}_k\ p^n (1-p)^k \qquad E = \frac{k(1-p)}{p}, \quad V = \frac{k(1-p)}{p^2}. \tag{B.23}$$

■一様分布

$$P_k = \frac{1}{N}, \qquad E = \frac{N+1}{2}, \quad V = \frac{N^2-1}{12}. \tag{B.24}$$

B.2　連続型確率分布

■**正規分布**

$$\mathrm{N}[\mu,\sigma]=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right];\qquad E=\mu,\quad V=\sigma^2. \tag{B.25}$$

■**指数分布**

$$\mathrm{Ex}[\lambda]=\mathrm{Ga}[\lambda,1]=\lambda\mathrm{e}^{-\lambda x}\quad(x\geq 0);\qquad E=\frac{1}{\lambda},\quad V=\frac{1}{\lambda^2}. \tag{B.26}$$

■**Gamma 分布**

$$\mathrm{Ga}[\lambda,\alpha]=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\lambda x}\quad(x\geq 0);\qquad E=\frac{\alpha}{\lambda},\quad V=\frac{\alpha}{\lambda^2}. \tag{B.27}$$

■**χ^2 分布**

$$\chi^2[n]=\mathrm{Ga}[1/2,n/2]=\frac{1}{\Gamma(n/2)}\sqrt{\frac{x^{n-2}\mathrm{e}^{-x}}{2^n}}\quad(x\geq 0);\qquad E=n,\quad V=2n. \tag{B.28}$$

■**Beta 分布**

$$\mathrm{Be}[\alpha,\beta]=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\quad(0<x<1),\qquad E=\frac{\alpha}{\alpha+\beta},\quad V=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \tag{B.29}$$

$$\text{where}\quad B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{B.30}$$

■**Cauchy 分布**

$$f[\alpha,\lambda]=\frac{\alpha}{\pi}\left[(x-\lambda)^2+\alpha^2\right];\qquad E \text{ と } V \text{ は定義されない。} \tag{B.31}$$

■**対数正規分布**　所得の分布。$\log x$ が正規分布 $N[\mu,\sigma]$ に従うとき，

$$f[\mu,\sigma]=\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}\exp\frac{-(\log x-\mu)^2}{2\sigma^2};\qquad E=\mathrm{e}^{\mu+\sigma^2/2},\quad V=\mathrm{e}^{2\mu+2\sigma^2}-\mathrm{e}^{2\mu+\sigma^2}. \tag{B.32}$$

■**Pareto 分布**　高額所得の分布。E と V はそれぞれ $a>1$, $a>2$ でのみ定義される。

$$f_{[a,x_0]}=\frac{a}{x_0}\left(\frac{x_0}{x}\right)^{a+1}\quad(x\geq x_0;\quad a>0);\qquad E=\frac{ax_0}{a-1},\qquad V=\frac{ax_0^2}{a-2}-\left(\frac{ax_0}{a-1}\right)^2. \tag{B.33}$$

■**Weibull 分布**　$a>0$, $b>0$ とする。

$$f[a,b]=\frac{b}{a^b}x^{b-1}\exp\left[-\left(\frac{x}{a}\right)^b\right]\quad(x\geq 0);\qquad E=a\Gamma\left(1+\frac{1}{b}\right),\quad V=a^2\left[\Gamma\Big(1+\frac{2}{b}\Big)-\left[\Gamma\Big(1+\frac{1}{b}\Big)\right]^2\right] \tag{B.34}$$

付録 C　Verbose Notes

C.1　Spinor Fields

◆C.1.1　Lorentz group and Lorentz algebra

Metric	:	$\eta = \mathrm{diag}(+1,-1,-1,-1), \qquad \eta = \mathrm{diag}(-1,+1,+1,+1).$
Lorentz transf. in $\mathbb{R}^{1,3}$	:	Linear transf. $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ which conserve x^2.
	:	$\Longrightarrow \eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma$. and form a group L.
	:	$(\Longrightarrow \quad (\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^\alpha{}_\beta =: \Lambda_\nu{}^\mu \quad \Longrightarrow \quad \Lambda^\mu{}_\nu\Lambda_\mu{}^\rho = \delta^\rho_\nu)$
Disconnected parts of L	:	$L_0 := \{\det\Lambda = +1 \wedge {\Lambda_0}^0 > 0\} \quad L_P := \{\det\Lambda = -1 \wedge {\Lambda_0}^0 > 0\}$
	:	$L_T := \{\det\Lambda = +1 \wedge {\Lambda_0}^0 < 0\} \quad L_{PT} := \{\det\Lambda = -1 \wedge {\Lambda_0}^0 < 0\}$
	:	(L_0 is identical with $(\mathrm{SO}(1,3), \mathrm{SO}(3,1))$.)
Infinitesimal one in L_0	:	$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ (for $\eta = \eta\Lambda\Lambda$)

微小変換は $\epsilon^\mu{}_\nu = \begin{pmatrix} 0 & \beta_x & \beta_y & \beta_z \\ \beta_x & 0 & -\theta_z & \theta_y \\ \beta_y & \theta_z & 0 & -\theta_x \\ \beta_y & -\theta_y & \theta_x & 0 \end{pmatrix}$ の形となっているので，回転生成子 $\boldsymbol{J}$ と加速生成子 $\boldsymbol{K}$ は

$$\Lambda = \exp\left[\kappa\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] \quad\Longrightarrow\quad J_x = -\kappa\mathrm{i}\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}, \quad K_x = -\kappa\mathrm{i}\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \tag{C.1}$$

の形である。ここで $\kappa = \pm1$ は notation である。

一方，微小変換から生成子を $\epsilon^\mu{}_\nu =: \mp\frac{\mathrm{i}}{2}\epsilon^{\rho\sigma}(J_{\rho\sigma})^\mu{}_\nu$ と定義すると，計量によらずに $\epsilon_{\mu\nu}$ は反対称となり，

$$\begin{aligned} \boldsymbol{\theta} &= (+,-)(\epsilon^{23},\epsilon^{31},\epsilon^{12}), & \boldsymbol{\beta} &= (+,-)(\epsilon^{10},\epsilon^{20},\epsilon^{30}). \\ &= (+,-)(\epsilon_{23},\epsilon_{31},\epsilon_{12}) & &= (-,+)(\epsilon_{10},\epsilon_{20},\epsilon_{30}) \end{aligned}$$

で，従って $J^{\rho\sigma}$ も反対称。$(J_{\rho\sigma})^\mu{}_\nu = \pm\mathrm{i}\left(\delta^\mu_\rho\eta_{\sigma\nu} - \delta^\mu_\sigma\eta_{\rho\nu}\right)$ となり，交換関係が得られ，これが閉じているので Lie 代数であることもわかる：

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mp\mathrm{i}(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}). \tag{C.2}$$

生成子の具体形は計量に依存し，$J_{10}{}^\mu{}_\nu = (\pm\mathrm{i}, \mp\mathrm{i})\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}_{\mu\nu}$, $J_{23}{}^\mu{}_\nu = (\pm\mathrm{i}, \mp\mathrm{i})\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}_{\mu\nu}$ となる。よって，ここでの複号の取り方と κ および計量の定義によって，$\boldsymbol{J}\cdot\boldsymbol{K}$ と $J_{\rho\sigma}$ の対応が定まることになる。

＊　＊　＊

κ と複号について $(-,上)$,$(+,下)$,$(-,下)$,$(+,上)$ を取れば

$$\boldsymbol{J} = (J_{23}, J_{31}, J_{12}), \quad \boldsymbol{K} = (J_{10}, J_{20}, J_{30}); \tag{C.3}$$

となる：

$$\Lambda = \exp\epsilon = \exp\left[\kappa\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] = \exp\left[\mp\mathrm{i}(\epsilon^{\rho\sigma}J_{\rho\sigma})/2\right]. \tag{C.4}$$

◆C.1.2 Lorentz group and $\mathrm{SL}(2,\mathbb{C})$

次に，連結 Lie 群 L_0 が，連結 Lie 群 $\mathrm{SL}(2,\mathbb{C})/\mathrm{Z}_2$ と同型であることを見る：

$$\mathfrak{sl}(2,\mathbb{C}) :=\{a\in\mathfrak{gl}(2,\mathbb{C})\mid \mathrm{Tr}(a)=0\},\qquad \mathrm{SL}(2,\mathbb{C}):=\{g\in\mathrm{GL}(2,\mathbb{C})|\ \det(g)=1\}. \tag{C.5}$$

まず,σ^μ を（極めて一般的に） $\sigma^\mu := (\alpha 1, \beta\boldsymbol{\sigma})$ と定義する ($\alpha=\beta=\pm1$)。$x^2=(+-)\det(x_\mu\sigma^\mu)$ なので

$$f^g:(x_\mu\sigma^\mu)\mapsto g(x_\mu\sigma^\mu)g^\dagger;\quad g\in\mathrm{SL}(2,\mathbb{C}) \tag{C.6}$$

は x^2 を保存する。よって Lorentz 変換であり，生成子を比べることで局所同型だとわかる：

> $\mathrm{SL}(2,\mathbb{C})\ni g=\exp(-\mathrm{i}a)$ として $x_\mu(g\sigma^\mu g^\dagger)={\Lambda_\mu}^\nu x_\nu\sigma^\mu$ を微小展開すると
>
> $$\Lambda^\mu{}_\nu\sigma^\nu = g^{-1}\sigma^\mu(g^{-1})^\dagger\quad\Longrightarrow\quad \epsilon^\mu{}_\nu\sigma^\nu=\mathrm{i}(a\sigma^\mu-\sigma^\mu a^\dagger) \tag{C.7}$$
>
> であり，ここから g がわかる：
>
> $$g=\exp\left(-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}-\frac{\alpha\beta}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right). \tag{C.8}$$

このことを別の観点から見る。Lorentz 群の生成子の交換関係を見ると，（正しく複号を取った場合）

$$[J_i,J_j]=\mathrm{i}\epsilon_{ijk}J_k,\qquad [J_i,K_j]=\mathrm{i}\epsilon_{ijk}K_k,\qquad [K_i,K_j]=-\mathrm{i}\epsilon_{ijk}J_k \tag{C.9}$$

となるので，

$$\boldsymbol{A}:=\frac{1}{2}(\boldsymbol{J}+\mathrm{i}\boldsymbol{K}),\qquad \boldsymbol{B}:=\frac{1}{2}(\boldsymbol{J}-\mathrm{i}\boldsymbol{K}). \tag{C.10}$$

と定義すると

$$[A_i,A_j]=\mathrm{i}\epsilon_{ijk}A_k,\qquad [B_i,B_j]=\mathrm{i}\epsilon_{ijk}B_k,\qquad [A_i,B_j]=0, \tag{C.11}$$

となり，Lorentz 群が $\mathrm{SU}(2)\times\mathrm{SU}(2)$ に分解できる。

C.2 Weyl Spinor

$\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ に対して $(1/2,0)$ 表現を為すものを左巻き spinor ξ，$(0,1/2)$ 表現を為すものを右巻き spinor $\bar\xi$ と定義する。

$$\xi\mapsto\left(1-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\xi\qquad\qquad \bar\xi\mapsto\left(1-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\bar\xi. \tag{C.12}$$

$\alpha\beta=1$ とすると g は左巻き spinor の変換子となる。記号を $\xi_\alpha\mapsto {g_\alpha}^\beta\xi_\beta$, $\bar\xi^{\dot\alpha}\mapsto (g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}$ と定義する。

次に，$\xi^\alpha\chi_\alpha$ および $\bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha}$ が scalar となるようにしたい。$E=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ としておくと，$-Eg^{\mathrm{T}}E=g^{-1}$ より

$$(\xi')^\alpha=\xi^\beta(g^{-1})_\beta{}^\alpha=-\xi^\beta(E_{\beta A}{g_B}^A E_{B\alpha})\qquad\therefore(-E_{\gamma\alpha})(\xi')^\alpha=-{g_\gamma}^\beta(E_{\beta\delta}\xi^\delta). \tag{C.13}$$

よって，$\epsilon^{12}=\epsilon_{21}=1$ として $\xi^\alpha:=\epsilon^{\alpha\beta}\xi_\beta$，$\xi_\alpha=\epsilon_{\alpha\beta}\xi^\beta$ とすれば良い。

同様に，$\bar\xi'_{\dot\alpha}=-E(g^\dagger)^{-1}E\bar\xi_{\dot\beta}$ から $(E^{\dot\alpha\dot\beta}\bar\xi_{\dot\beta})'=(g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}(E^{\dot\beta\dot\gamma}\bar\xi_{\dot\gamma})$ となる。$\epsilon_{12}=\epsilon_{\dot1\dot2}$ として $\bar\xi_{\dot\alpha}:=\epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\alpha}$ とするのが一般的である。また，このことから $(\xi_a)^*=\bar\xi_{\dot\alpha}$（或いは $\xi^\dagger=\bar\xi$）が分かる。

$x_\mu\sigma^\mu\mapsto x_\mu(g\sigma^\mu g^\dagger)$ であるので，$x_\mu(\xi^\alpha\sigma^\mu\bar\chi^{\dot\alpha})$ は scalar である。よって，$\sigma^\mu_{\alpha\dot\alpha}$ のように書ける。更にここで $x_\mu(\bar\xi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\chi_\alpha)$ も scalar となるように $\bar\sigma$ を定めよう。

$$\bar\xi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\chi_\alpha=-\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\chi^\beta\bar\sigma^{\mu\dot\alpha\alpha}\bar\xi^{\dot\beta}\qquad\therefore\sigma^\mu_{\beta\dot\beta}\propto\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\alpha\alpha} \tag{C.14}$$

であり，あとは convention である。

$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^{\alpha} := \epsilon^{\alpha\beta}\xi_{\beta}, \quad \xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta}, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\xi_{\alpha} \mapsto g_{\alpha}{}^{\beta}\xi_{\beta}, \qquad \xi^{\alpha} \mapsto \xi^{\beta}(g^{-1})_{\beta}{}^{\alpha}, \qquad \bar{\xi}_{\dot{\alpha}} \mapsto \bar{\xi}_{\dot{\beta}}(g^{\dagger})^{\dot{\beta}}{}_{\dot{\alpha}} \qquad \bar{\xi}^{\dot{\alpha}} \mapsto (g^{\dagger})^{-1\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \quad \sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta} \quad \therefore \sigma^{\mu} := (\alpha 1, \beta\boldsymbol{\sigma}), \quad \bar{\sigma}^{\mu} := (\alpha 1, -\beta\boldsymbol{\sigma})$$

C.3 Polarization Sum

Firstly we focus on the single photon case $M = \epsilon^{*}_{\mu}(k)M^{\mu}$. In this case, the replacement

$$\sum_{\text{pol.}} \epsilon_{\mu}\epsilon'_{\nu} \to \eta_{\mu\nu} \tag{C.15}$$

is valid. Let us prove this validity. First we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^{*}_{\mu}(k)\epsilon_{\nu}(k)M^{\mu}M^{\nu *} = |M^1|^2 + |M^2|^2, \tag{C.16}$$

while

$$\eta_{\mu\nu}M^{\mu}M^{\nu *} = |M^1|^2 + |M^2|^2 \tag{C.17}$$

for Ward identity $k_{\mu}M^{\mu} = 0$. Now we can see the validity easily.

Next we think about the double photons case [*6] $M = \epsilon^{*}_{\mu}(k)\epsilon'^{*}_{\nu}(k')M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \qquad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \tag{C.18}$$

$$k' = (E, 0, 0, -E) \qquad \epsilon' = (0, \cos\theta, \sin\theta, 0) \oplus (0, -\sin\theta, \cos\theta, 0). \tag{C.19}$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^{*}_{\mu}(k)\epsilon_{\nu}(k)\epsilon^{*}_{\rho}(k')\epsilon_{\sigma}(k')M^{\mu\rho}M^{\nu\sigma *} \tag{C.20}$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2 \tag{C.21}$$

$$\stackrel{?}{=} \eta^{\mu\nu}\eta^{\rho\sigma}M^{\mu\rho}M^{\nu\sigma *}. \tag{C.22}$$

Badly, our Ward identities

$$k_{\mu}\epsilon'^{*}_{\nu}(k')M^{\mu\nu} = \epsilon^{*}_{\mu}(k)k'_{\nu}M^{\mu\nu} = 0 \tag{C.23}$$

do not help us; now we want to be given

$$k_{\mu}M^{\mu\nu} = k'_{\nu}M^{\mu\nu} = 0, \tag{C.24}$$

to recover validity of the replacement.

The difference between (C.23) and (C.24) is that the former considers (and tries to sum up) all polarizations but the latter does only physical ones. Actually, as long as we are summing up all polarizations, the replacement is still valid; these two conditions are equivalent because of cancellation of unphysical polarizations. However, once we restrict the polarizations (for example with using a relation $\epsilon \cdot k = 0$), we can no more use (C.24) and thus the replacement becomes invalid.

[*6] This part is derived from 濱口幸一's notebook.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our latter conditions (C.24), and define $\widetilde{M}^{\mu\nu}$ and $\widetilde{M}$ as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \tag{C.25}$$

$$\widetilde{M} := \epsilon^*_\mu(k)\epsilon'^*_\nu(k')\widetilde{M}^{\mu\nu}. \tag{C.26}$$

Here $\widetilde{M}^{\mu\nu} \neq M^{\mu\nu}$ but $\widetilde{M} = M$; thus $\widetilde{M}$ satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$). However, we cannot utilize the replacement for $\widetilde{M}$, while it is valid for M. If you did the replacement, a wrong result would come out, like

$$\eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \eta_{\mu\rho}\eta_{\nu\sigma}\left(M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu\right)\left(M^{\rho\sigma *} + k^\rho p^{\sigma *} + p'^{\rho *} k'^\sigma\right) \tag{C.27}$$

$$= \sum_{\text{pol.}} |M|^2 + \left[(k\cdot p'^*)(k'\cdot p) + \text{H.c.}\right]. \tag{C.28}$$

After all, we have obtained following expression:

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} |\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}|^2 = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *}$$
$$= \sum_{\text{pol.}} |\widetilde{M}|^2 = \sum_{\text{pol.}} |\epsilon^*_\mu(k)\epsilon'^*_\nu(k')\widetilde{M}^{\mu\nu}|^2 \neq \eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \sum_{\text{pol.}} |\widetilde{M}|^2 + \left[(k\cdot p'^*)(k'\cdot p) + \text{H.c.}\right]. \tag{C.29}$$

To check the Ward identity always helps us!

C.4 Phantom Terms in the Gauge Theory

You may think we forget to introduce $\overline{\psi}\gamma_5\psi,\ \overline{\psi}\gamma_5\not{D}\psi,\ \epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma},\ \epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma}$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_{L} and P_{R},
- the last term is equivalent to the third term as
$$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\frac{1}{2}[D_\mu, D_\nu]F^a_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}.$$

Therefore, we have to discuss only the ϵFF terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma &= \epsilon^{\mu\nu\rho\sigma}\left(-f^{acd}f^{abe} - f^{adb}f^{ace}\right)A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\ &= -2\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\ &= 0,\end{aligned} \tag{C.30}$$

$$\begin{aligned}\therefore\ \epsilon^{\nu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^a_\nu \partial_\rho A^a_\sigma + 4g\epsilon^{\mu\nu\rho\sigma}f^{abc}A^a_\mu A^b_\nu \partial_\rho A^c_\sigma \\ &= 2\partial_\mu G^\mu,\end{aligned} \tag{C.31}$$

where G^μ is the Chern–Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma + \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) = \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right). \tag{C.32}$$

See Appendix C.7 for the instanton effect.

C.5 楊-Mills Theory

◆C.5.1 General Gauge Theory

For any Lie group G, we can consider "gauge transformation" $\phi(x) \mapsto V(x)\phi(x)$, where $V : \mathbb{R}^{1,3} \to G$. Also we can define a "connection field" $A_\mu(x)$ as:

$$\phi_\parallel(x+\mathrm{d}x) := \phi(x) + \mathrm{i}gA_\mu(x)\phi(x)\mathrm{d}x^\mu \qquad \text{s.t.} \quad \phi_\parallel(x+\mathrm{d}x) \mapsto V(x+\mathrm{d}x)\phi_\parallel(x+\mathrm{d}x). \tag{C.33}$$

Then the covariant derivative D_μ can be defined as

$$\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu := \Delta_{\mathrm{d}x}\phi(x) := \phi(x+\mathrm{d}x) - \phi_\parallel(x+\mathrm{d}x) \qquad \therefore \mathrm{D}_\mu := \partial_\mu - \mathrm{i}gA_\mu. \tag{C.34}$$

Note that $\Delta_{\mathrm{d}x}\phi(x) \mapsto V(x+\mathrm{d}x)\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu$, which means $\mathrm{D}_\mu\phi(x) \mapsto V(x)\mathrm{D}_\mu\phi(x)$. Now we can see

$$\phi \mapsto V\phi, \qquad A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1}, \qquad \mathrm{D}_\mu \mapsto V\mathrm{D}_\mu V^{-1}. \tag{C.35}$$

We can do another discussion: we can define D_μ as a kind of derivative which satisfies (C.35).

Next we introduce the curvature tensor, or "field strength" as

$$\Delta\phi(x) := \phi_\parallel^{xy}(x+\mathrm{d}x+\mathrm{d}y) - \phi_\parallel^{yx}(x+\mathrm{d}x+\mathrm{d}y) = [\mathrm{D}_\mu, \mathrm{D}_\nu]\,\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu =: -\mathrm{i}gF_{\mu\nu}\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu; \tag{C.36}$$

$$F_{\mu\nu}(x) := \frac{\mathrm{i}}{g}[\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}g\,[A_\mu(x), A_\nu(x)]\,. \tag{C.37}$$

$\Delta\phi(x)$ is transformed in terms of $V(x+\mathrm{d}x+\mathrm{d}y) \simeq V(x)$, thus $F_{\mu\nu}(x) \mapsto V(x)F_{\mu\nu}(x)V^{-1}(x)$.

◆C.5.2 Compact Gauge Theory

Generators If the gauge group G is **compact**, it has a finite-dimensional unitary representation. The generators T_a can be taken to be Hermitian, and $V(x) = \exp[\mathrm{i}g\theta^a(x)T^a]$ for $\theta^a(x) \in \mathbb{R}$;

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_cT^c \quad (f \in \mathbb{R}) \qquad 0 = f^D{}_{ab}f^E{}_{Dc} + f^D{}_{ca}f^E{}_{Db} + f^D{}_{bc}f^E{}_{Da} \tag{C.38}$$

For the sake of the compactness Killing form is positive-definite, where we can normalize the generators as $\operatorname{Tr} T^aT^b = \frac{1}{2}\delta^{ab}$, and the structure constant f^{abc} would be totally antisymmetric.

Adjoint Representations

$$[\widetilde{T}^a]_i{}^j := -\mathrm{i}f^{aij}; \qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j := \delta_i^j\partial_\mu + gf^{iaj}A_\mu^a. \tag{C.39}$$

Field Expansion In this *normalized Hermitian* basis, the relations would be[*7]

$$\begin{aligned} \phi' &= \mathrm{e}^{\mathrm{i}gT^a\theta^a}\phi; & F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\ A^a_\mu{}' &\simeq A^a_\mu + \partial_\mu\theta^a + gf^{abc}A^b_\mu\theta^c & F^a_{\mu\nu}{}' &= [\mathrm{e}^{\mathrm{i}g\theta^c\widetilde{T}^c}]^{ab}F^b_{\mu\nu} \\ &= A^a_\mu + (\widetilde{\mathrm{D}}_\mu\theta)^a, & &\simeq F^a_{\mu\nu} + gf^{abc}F^b_{\mu\nu}\theta^c. \end{aligned}$$

Covariant Derivative For a field λ^a under the adjoint representation,

$$(\mathrm{D}_\mu\lambda)^a = \partial_\mu\lambda^a + gf^{abc}A^b_\mu\lambda^c \qquad \text{or} \quad \mathrm{D}_\mu\lambda^aT^a = \partial_\mu\lambda^aT^a - \mathrm{i}g[A^b_\mu T^b, \lambda^aT^a] \text{ [*8]} \tag{C.40}$$

Bianchi Equation

$$\epsilon^{\mu\nu\rho\sigma}\,[\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\sigma]] = \epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma} = 0. \tag{C.41}$$

[*7] We can expand A_μ in T^a-basis, because it is induced by the gauge transformation.

[*8] Note that we can use any representation T^a but must the same ones for $A^a_\mu T^a$ and λ^aT^a.

C.6 Spinor

■$\eta^{\mu\nu} = (-,+,+,+)$ case

Grassmann Number	:	$(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$
	:	$\Longrightarrow$ for $a, b \in \mathbb{G}^{\mathbb{R}}$, $ab \in \mathrm{i}\mathbb{G}^{\mathbb{R}}$
γ matrix	:	$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$
	:	$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ etc...
	:	$(\mathrm{i}\gamma^0)^\dagger := \mathrm{i}\gamma^0, \quad {\gamma^i}^\dagger := \gamma^i$
Dirac Conjugate	:	$\bar{\psi} = \mathrm{i}\psi^\dagger\gamma^0$

TODO: SPINOR

C.7 Instanton

TODO: INSTANTON

付録 D　Supersymmetry in the text by Wess & Bagger

D.1　Spinor Convention

ϵ tensor　:　$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$　　(definition)

Sum Rule　:　${}^{\alpha}{}_{\alpha}$ and ${}_{\dot{\alpha}}{}^{\dot{\alpha}}$, except for　$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$,　$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta$,　$\xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}$,　$\xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}$.

Lorentz 変換 :　$\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$,　$\bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dagger\dot{\beta}}{}_{\dot{\alpha}}$,　$\psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha$,　$\bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dagger\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}'^{\dot{\beta}}$.

σ matrices　:　$(\sigma^\mu)_{\alpha\dot{\beta}} := (-1, \boldsymbol{\sigma})_{\alpha\dot{\beta}}$,　$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (-1, -\boldsymbol{\sigma})^{\dot{\alpha}\beta}$.

(See App. C.1.1 for a verbose explanation.)

D.2　Spinor Calculation Cheatsheet

$$\eta = (-,+,+,+), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1$$

$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \qquad \sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}, \qquad \sigma^\mu := (-1, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu := (-1, -\boldsymbol{\sigma})$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \tfrac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha{}^\beta, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \tfrac{1}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}} = (\sigma^{\nu\mu})^{\dagger\dot{\alpha}}{}_{\dot{\beta}}.$$

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \qquad \theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta \qquad (\theta\phi)(\theta\psi) = -\tfrac{1}{2}(\psi\phi)(\theta\theta) \qquad (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = -\tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}$$

$$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\tfrac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta}) \qquad (\sigma^\mu\bar{\theta})_\alpha(\theta\sigma^\nu\bar{\theta}) = \tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha\bar{\theta}\bar{\theta}$$

$$\theta\sigma^\mu\bar{\sigma}^\nu\theta = -\eta^{\mu\nu}\theta\theta \qquad \bar{\theta}\bar{\sigma}^\mu\sigma^\nu\bar{\theta} = -\eta^{\mu\nu}\bar{\theta}\bar{\theta} \qquad (\theta\sigma^\mu)_{\dot{\alpha}}(\theta\sigma^\nu\bar{\theta}) = \tfrac{1}{2}(\bar{\theta}\bar{\sigma}^\nu\sigma^\mu)_{\dot{\alpha}}\theta\theta$$

$$\sigma^\mu\bar{\sigma}^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu} \qquad \sigma^\mu\bar{\sigma}^\rho\sigma^\nu + \sigma^\nu\bar{\sigma}^\rho\sigma^\mu = -2\left(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho\right)$$

$$\bar{\sigma}^\mu\sigma^\nu = -\eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu} \qquad \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu = -2\left(\eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\nu}\bar{\sigma}^\rho\right)$$

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu} \qquad \sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma$$

$$\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu} \qquad \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma$$

$$\operatorname{Tr}\bar{\sigma}^\mu\sigma^\nu = \operatorname{Tr}\sigma^\mu\bar{\sigma}^\nu = -2\eta^{\mu\nu} \qquad \operatorname{Tr}\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\tfrac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$$

$$\operatorname{Tr}\sigma^{\mu\nu} = \operatorname{Tr}\bar{\sigma}^{\mu\nu} = 0 \qquad \operatorname{Tr}\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} = -\tfrac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \qquad \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} - \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right]$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \qquad \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} + \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} = -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_\mu = -2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \qquad \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} \qquad \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2\mathrm{i}\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} \qquad \epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \qquad \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = 2\mathrm{i}\bar{\sigma}^{\mu\nu}$$

$$\bar{\xi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\xi} = (\bar{\chi}\bar{\sigma}^\mu\xi)^* = -(\xi\sigma^\mu\bar{\chi})^* \qquad (\psi\phi)\chi_\alpha = -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha$$

$$\xi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\xi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\xi})^* = (\bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^* \qquad (\psi\phi)\bar{\chi}_{\dot{\alpha}} = -\tfrac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot{\alpha}}$$

In the following equations, we chose left-differential notation.

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha} \qquad \frac{\partial}{\partial\theta^\alpha}\theta\theta = 2\theta_\alpha \qquad \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta = 4$$

$$\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} = -\frac{\partial}{\partial\theta^\alpha} \qquad \frac{\partial}{\partial\theta_\alpha}\theta\theta = -2\theta^\alpha \qquad \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta = -4$$

$$\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} = -2\bar{\theta}_{\dot{\alpha}} \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} = 4$$

$$\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} = 2\bar{\theta}^{\dot{\alpha}} \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} = -4$$

D.3 Chiral Superfields : $\bar{D}_{\dot\alpha}\Phi = 0$

■Explicit Expression

$$\begin{aligned}\Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) && \text{(D.1)}\\ &= \phi(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) + \sqrt{2}\theta\psi(x) + \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) && \text{(D.2)}\\ \Phi^\dagger &= \phi^*(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \bar\theta\bar\theta F^*(x) && \text{(D.3)}\end{aligned}$$

■Product of Chiral Superfields

$$\begin{aligned}\Phi_i^\dagger\Phi_j(x,\theta,\bar\theta) \rightsquigarrow{}& \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j\\ &+ 2\mathrm{i}(\theta\sigma^\mu\bar\theta)(\phi_i^*\partial_\mu\phi_j) - \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar\theta\bar\sigma^\mu\psi_j - \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j\\ &+ 2\bar\theta\bar\psi_i\theta\psi_j + \sqrt{2}\theta\theta\bar\theta\bar\psi_i F_j + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j\\ &+ \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \partial^\mu\phi_i^*\partial_\mu\phi_j - \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right] && \text{(D.4)}\end{aligned}$$

$$\begin{aligned}\Phi_i\Phi_j(\text{in } y\text{-basis}) &= \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right] && \text{(D.5)}\\ \Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right]\\ &\quad + \theta\theta\left[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right] && \text{(D.6)}\end{aligned}$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

付録 E　Cherry on the Cake

■Conversion of Units

$$
\begin{aligned}
1\,\mathrm{GeV} &= \frac{1}{6.5821\times 10^{-25}\,\mathrm{s}} = \frac{1}{2.086\times 10^{-32}\,\mathrm{yr}} = \frac{1}{0.19733\,\mathrm{fm}} = 1.1605\times 10^{13}\,\mathrm{K} = 1.7827\times 10^{-24}\,\mathrm{g} \\
&= \frac{1.519268\times 10^{24}}{1\,\mathrm{s}} = \frac{4.79\times 10^{31}}{1\,\mathrm{yr}} = \frac{5.0677}{1\,\mathrm{fm}}
\end{aligned}
\tag{E.1}
$$

$$
1\,\mathrm{K} = 8.6173\times 10^{-5}\,\mathrm{eV} = \frac{1}{8.0591\times 10^{-21}\,\mathrm{s}} = \frac{1.2408\times 10^{20}}{1\,\mathrm{s}} \tag{E.2}
$$

$$
1\,\mathrm{GeV}^2 = \frac{1}{3.8938\times 10^{-4}\,\mathrm{barn}} = \frac{2568.2}{1\,\mathrm{barn}} \qquad (1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2) \tag{E.3}
$$

$$
\begin{aligned}
&1\,\text{tropical yr} = 3.1557\times 10^{7}\,\mathrm{s}, \quad 1\,\text{sidereal yr} = 3.1558\times 10^{7}\,\mathrm{s}; \\
&1\,\mathrm{s} = 3.1689\times 10^{-8}\,\text{tr-yr} = 3.1688\times 10^{-8}\,\text{sr-yr}.
\end{aligned}
\tag{E.4}
$$

■Physical Constants

$$
G_{\mathrm{F}} = \frac{1}{\sqrt{2}v^2} = 1.16637(1)\times 10^{-5}\,\mathrm{GeV}^{-2}, \qquad G_{\mathrm{N}} = 6.70881(67)\times 10^{-39}\,\mathrm{GeV}^{-2} \tag{E.5}
$$

$$
\sqrt{G_{\mathrm{N}}} = 1.61624(8)\times 10^{-35}\,\mathrm{m} = \frac{1}{1.22089(6)\times 10^{19}\,\mathrm{GeV}} = \frac{1}{2.17644(11)\times 10^{-8}\,\mathrm{kg}} \tag{E.6}
$$

$$
\sqrt{8\pi G_{\mathrm{N}}} = 8.1026(4)\times 10^{-35}\,\mathrm{m} = \frac{1}{2.4353(1)\times 10^{18}\,\mathrm{GeV}} = \frac{1}{4.3413(2)\times 10^{-9}\,\mathrm{kg}} \tag{E.7}
$$

■Component of Spinor in Weyl Representation

$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta,\ \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta},\ \chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta};\quad \epsilon^{12} = \epsilon^{\dot1\dot2} = 1,\ \epsilon_{12} = \epsilon_{\dot1\dot2} = -1.)$

$$
\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar\chi^{\dot1} \\ \bar\chi^{\dot2} \end{pmatrix} \quad \xrightarrow{C} \quad \psi^{\mathrm{c}} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} -(\bar\chi^{\dot2})^* \\ (\bar\chi^{\dot1})^* \\ (\xi_2)^* \\ -(\xi_1)^* \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \bar\xi_{\dot2} \\ -\bar\xi_{\dot1} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix} \tag{E.8}
$$

$$
\overline{\psi} = \begin{pmatrix} \chi^\alpha & \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \chi^1 & \chi^2 & \bar\xi_{\dot1} & \bar\xi_{\dot2} \end{pmatrix} \quad \xrightarrow{C} \quad \overline{\psi^{\mathrm{c}}} = \mathrm{i}\psi^{\mathrm{T}}\gamma^0\gamma^2 = \begin{pmatrix} \xi_2 & -\xi_1 & -\bar\chi^{\dot2} & \bar\chi^{\dot1} \end{pmatrix} = \begin{pmatrix} \xi^\alpha & \bar\chi_{\dot\alpha} \end{pmatrix} \tag{E.9}
$$

$$
A^\alpha B_\alpha = \overline{\psi}_{A^{\mathrm{c}}} P_{\mathrm{L}} \psi_B = \overline{\psi}_{B^{\mathrm{c}}} P_{\mathrm{L}} \psi_A \qquad \bar A_{\dot\alpha} \bar B^{\dot\alpha} = \overline{\psi}_A P_{\mathrm{R}} \psi_{B^{\mathrm{c}}} = \overline{\psi}_B P_{\mathrm{R}} \psi_{A^{\mathrm{c}}} \tag{E.10}
$$

$$
\begin{aligned}
\overline{\psi} \not{A} \chi &= \overline{\psi}_{\mathrm{L}} \not{A} P_{\mathrm{L}} \chi_{\mathrm{L}} + \overline{\psi}_{\mathrm{R}} \not{A} P_{\mathrm{R}} \chi_{\mathrm{R}} \\
&= \overline{\psi}_{\mathrm{L}} \not{A} P_{\mathrm{L}} \chi_{\mathrm{L}} - \overline{\chi}{}^{\mathrm{c}}_{\mathrm{R}} \not{A} P_{\mathrm{L}} \psi^{\mathrm{c}}_{\mathrm{R}}
\end{aligned}
\tag{E.11}
$$