

1 General Definitions and Tools

1.1 NOTATIONS AND CONVENTIONS

1.1.1 Metric etc.

$$\begin{aligned}
\text{Minkowski Metric} & : \eta^{\mu\nu} := \text{diag}(+, -, -, -); \quad \epsilon_{0123}^{0123} := \pm 1 \\
\text{Coordinates} & : x^\mu := (t, x, y, z); \quad \text{therefore } \partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right). \\
\text{Gamma Matrices} & : \{\gamma^\mu, \gamma^\nu\} := 2\eta^{\mu\nu}; \quad \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\
& : \text{therefore } \{\gamma^\mu, \gamma_5\} = 0, (\gamma_5)^2 = 1. \\
\\
\text{Gamma Combinations} & : 1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5; \quad \sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 0 / i\gamma^\mu\gamma^\nu \\
\text{Spinor } \epsilon \text{ and } \sigma \text{ matrices} & : \epsilon^{12} = \epsilon^{i2} = \epsilon_{21} = \epsilon_{2i} = 1 \\
& : (\sigma^\mu)_{\alpha\dot{\beta}} := (1, \boldsymbol{\sigma})_{\alpha\dot{\beta}}, \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (1, -\boldsymbol{\sigma})^{\dot{\alpha}\alpha}. \\
\\
\left| \begin{array}{l} \text{Pauli Matrices} : \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ : \sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ : \sigma^\mu := (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu := (1, -\boldsymbol{\sigma}). \end{array} \right.
\end{aligned}$$

$$\text{Fourier Transformation} : \tilde{f}(k) := \int d^4x e^{ikx} f(x); \quad f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k).$$

1.1.2 Fields

$$\begin{aligned}
\text{Scalar} : (\partial^2 + m^2)\phi &= 0; \quad \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx} \right] \\
\text{Dirac} : (i\partial\!\!\!/ - m)\psi &= 0; \quad \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} \left[a_{\mathbf{p}}^s u^s(p) e^{-ipx} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ipx} \right] \\
\text{Vector} : \partial^2 A^\mu &= 0; \quad A^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{r=0..3} \left[a_{\mathbf{p}}^r \epsilon^r(p) e^{-ipx} + a_{\mathbf{p}}^{r\dagger} \epsilon^{r*}(p) e^{ipx} \right]
\end{aligned}$$

TODO: 南部-Goldstone; Gravitino

1.1.3 Electromagnetism

$$\begin{aligned}
\text{Electromagnetic Fields} : A^\mu &= (\phi, \mathbf{A}) \quad [\text{We can invert the signs, but cannot lower the index.}] \\
\text{Maxwell Equations} : F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu; \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = ej^\nu \\
\text{Our Old Language} : \nabla \cdot \mathbf{B} &= 0, \quad \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0; \quad \nabla \cdot \mathbf{E} = ej^0, \quad (\nabla \times \mathbf{B})_i - \frac{\partial}{\partial t} E_i = ej^i. \\
& : F_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & \begin{matrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{matrix} \end{pmatrix}; \quad F_{\mu\nu} F^{\mu\nu} = -2 \left(\|\mathbf{E}\|^2 - \|\mathbf{B}\|^2 \right)
\end{aligned}$$

1.2 SPINOR FIELDS

$$\begin{aligned}
\text{Spinor} & : \xi_\alpha, \quad \xi^\alpha := \epsilon^{\alpha\beta} \xi_\beta; \quad \text{Lorentz tr.} : \xi_\alpha \mapsto \Lambda_\alpha{}^\beta \xi_\beta, \quad \xi^\alpha \mapsto \xi^\beta \Lambda^{-1}{}^\alpha{}_\beta; \\
& : \bar{\eta}^{\dot{\alpha}} := (\eta^\alpha)^* \quad \bar{\eta}_{\dot{\alpha}} := (\eta_\alpha)^* \quad : \bar{\eta}^{\dot{\alpha}} \mapsto \Lambda^{\dagger-1}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}}, \quad \bar{\eta}_{\dot{\alpha}} \mapsto \bar{\eta}_{\dot{\beta}} \Lambda^{\dagger}{}_{\dot{\alpha}}{}^{\dot{\beta}}. \\
\text{Kinetic term} & : i \bar{\xi} \bar{\sigma}^\mu \partial_\mu \xi \quad (= i \eta \sigma^\mu \partial_\mu \bar{\eta}) \\
\text{Mass term} & : [\text{Majorana}] -\frac{1}{2}(m_M \xi \xi + m_M^* \bar{\xi} \bar{\xi}) \quad [\text{Dirac}] -(m_D \xi \eta + m_D^* \bar{\xi} \bar{\eta}) \\
\text{Dirac fermion} & : \mathcal{L}_{\text{Dirac}} = i \bar{\xi} \bar{\sigma}^\mu \partial_\mu \xi + i \eta \sigma^\mu \partial_\mu \bar{\eta} - m(\xi \eta + \bar{\xi} \bar{\eta}) = \bar{\psi}(i \gamma^\mu \partial_\mu - m) \psi \\
\text{Majorana fermion} & : \mathcal{L}_{\text{Majorana}} = i \bar{\xi} \bar{\sigma}^\mu \partial_\mu \xi - \frac{m}{2}(\xi \xi + \bar{\xi} \bar{\xi}) = \frac{1}{2} \bar{\psi}_M (i \gamma^\mu \partial_\mu - m) \psi_M \\
\text{Charge conjugate} & : \psi^C := C(\bar{\psi})^T \quad [(\psi_M)^C = \psi_M]
\end{aligned}$$

1.2.1 Chiral Notation (Peskin)

$$\begin{aligned}
\text{Gamma Matrices} & : \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad P_L^R = \frac{1 \pm \gamma_5}{2}. \\
\text{Fields} & : \psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}; \quad \bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta^\alpha & \bar{\xi}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix}; \quad \psi_M = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}. \\
& : u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}; \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix} \\
& : [\eta^s = \xi^{-s} := -i \sigma^2 (\xi^s)^* = (\xi^2, -\xi^1)]. \\
\text{Charge conj.} & : C := -i \gamma^2 \gamma^0 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad \begin{cases} -C = C^{-1} = C^\dagger = C^T \\ C = C^* \end{cases}, \quad C^{-1} \gamma^\mu C = -\gamma^{\mu T}. \\
& : \psi^C = C(\bar{\psi})^T = -i \gamma^2 \psi^* = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\psi}^C = \psi^T C = i \bar{\psi}^* \gamma^2 \\
\text{Weyl Equations} & : i \bar{\sigma} \cdot \partial \psi_L = m \psi_R; \quad i \sigma \cdot \partial \psi_R = m \psi_L \\
z\text{-boost limit} & : \text{Halt: } u^s = \sqrt{m} \begin{pmatrix} \xi^s \\ -\eta^s \end{pmatrix}, \quad v^s = \sqrt{m} \begin{pmatrix} \eta^s \\ -\xi^s \end{pmatrix}; \\
& : \text{Slow: } \sqrt{p \cdot \sigma} \simeq \sqrt{m}(1 - \mathbf{v} \cdot \boldsymbol{\sigma}/2), \quad \sqrt{p \cdot \bar{\sigma}} \simeq \sqrt{m}(1 + \mathbf{v} \cdot \boldsymbol{\sigma}/2); \\
& : \text{Extreme: } u^s = \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi^s \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi^s \end{pmatrix}, \quad v^s = \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \eta^s \\ -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \eta^s \end{pmatrix}.
\end{aligned}$$

1.2.2 Dirac Notation

$$\begin{aligned}
\text{Gamma Matrices} & : \hat{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}^i = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{P}_L^R = \frac{1 \pm \gamma_5}{2}. \\
& : \hat{\sigma}^{0i} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \hat{\sigma}^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \\
\text{Fields} & : \hat{\psi} = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}; \quad \hat{\psi}_M = \begin{pmatrix} \psi_A \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi_A^* \end{pmatrix}. \\
& : \hat{\bar{\psi}} = \hat{\psi}^\dagger \gamma^0 = \begin{pmatrix} \psi_A^\dagger & -\psi_B^\dagger \end{pmatrix} \\
& : \hat{u}^s(p) = \begin{pmatrix} \sqrt{p^0 + m} \xi^s \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{p^0 + m}} \xi^s \end{pmatrix}; \quad \hat{v}^s(p) = \begin{pmatrix} -\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{p^0 + m}} \eta^s \\ -\sqrt{p^0 + m} \eta^s \end{pmatrix} \\
& : [\eta^s = \xi^{-s} := -i \sigma^2 (\xi^s)^* = (\xi^2, -\xi^1)]
\end{aligned}$$

Charge conj. : $\hat{C} = -i\hat{\gamma}^2\hat{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ with $C = -C^{-1} = -C^\dagger$, $C^{-1}\gamma^\mu C = -\gamma^{\mu\top}$.

z -boost limit : Halt: $\hat{u}^s = \sqrt{2m} \begin{pmatrix} \xi^s \\ 0 \end{pmatrix}$, $\hat{v}^s = -\sqrt{2m} \begin{pmatrix} 0 \\ \eta^s \end{pmatrix}$;
: Slow: $\sqrt{p^0 + m} \simeq \sqrt{2m}(1 + \frac{v^2}{8})$, $\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{p^0 + m}} \simeq \sqrt{\frac{m}{2}}(\mathbf{v} \cdot \boldsymbol{\sigma})$;
: Extreme: $\hat{u}^s = \sqrt{E} \begin{pmatrix} \xi^s \\ (\frac{1}{0} \frac{0}{-1}) \xi^s \end{pmatrix}$, $\hat{v}^s = -\sqrt{E} \begin{pmatrix} (\frac{1}{0} \frac{0}{-1}) \eta^s \\ \eta^s \end{pmatrix}$

1.2.3 CPT transformations

[Note that these expressions are valid under the above frameworks.]

In the following, CP means “ P , then C ” in algebraic sense. Be careful to the order.

$$\begin{array}{ll}
\psi(t, \mathbf{x}) \xrightarrow{P} \eta_P \gamma^0 \psi(t, -\mathbf{x}) & \bar{\psi} \xrightarrow{P} \eta_P^* \bar{\psi} \gamma^0 \\
\psi(t, \mathbf{x}) \xrightarrow{T} \eta_T C \gamma_5 \psi(-t, \mathbf{x}) & \bar{\psi} \xrightarrow{T} -\eta_T^* \bar{\psi} C \gamma_5 \\
\psi(t, \mathbf{x}) \xrightarrow{C} \eta_C C \bar{\psi}^\top(t, \mathbf{x}) = C \gamma^0 \psi^* & \bar{\psi} \xrightarrow{C} \eta_C^* \bar{\psi}^* \gamma^0 C = -\eta_C^* (C \psi)^\top \\
\psi(t, \mathbf{x}) \xrightarrow{CP} \eta_{CP} (\bar{\psi} \gamma^0 C)^\top & \bar{\psi} \xrightarrow{CP} \eta_{CP}^* (C \gamma^0 \psi)^\top \\
\psi(t, \mathbf{x}) \xrightarrow{CPT} (\bar{\psi} \gamma^0 \gamma_5)^\top & \bar{\psi} \xrightarrow{CPT} (\gamma^0 \gamma_5 \psi)^\top
\end{array}$$

Note that T -transformation is anti-unitary, and $\eta_{CPT} = 1$. Especially, photon is $(P, T, C) = (-, +, -)$.

	ϕ	A^μ	$\bar{\psi}\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$\bar{\psi}\gamma^\mu\gamma_5\psi$	$i\bar{\psi}\gamma_5\psi$	∂_μ
P	ϕ	$-++++A^\mu$	+	+----	$(+----)(+----)$	++++	-	+----
T	ϕ	$+----A^\mu$	+	+----	$-(+----)(+----)$	+----	-	-++++
C	ϕ^*	$+A^{\mu*}$	+	-	-	+	+	+
CPT	ϕ^*	$-A^{\mu*}$	+	-	+	-	+	-

1.2.4 Noether current

Infinitesimal transformation : $\phi(x) \mapsto \phi'(x) := \phi(x) + \alpha \Delta\phi(x)$

Correspondent transformation : $\alpha \Delta\mathcal{L} = \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta\phi$

: So, defining $\alpha \partial_\mu \mathcal{J}^\mu(x) := \mathcal{L}'(x) - \mathcal{L}(x)$,

Noether current : $j^\mu(x) := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{J}^\mu$; $\partial_\mu j^\mu(x) = 0$

Noether charge : $Q := \int j^0 d^3x$

Energy-momentum tensor : $T^\mu_\nu = \partial_\mu \mathcal{L}(\partial_\mu \phi) \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$; $\mathcal{H} = T^{00}$, $\mathcal{P}^i = T^{0i}$.

: T^μ_ν is the variation along μ in respect to the modification a^ν .

1.3 FEYNMAN RULES

■ Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2$$

$$\overbrace{\phi \phi} = \text{diagram} = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\mathcal{L} \supset |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$$\overbrace{\phi^* \phi} = \text{diagram} = \frac{i}{p^2 - m^2 + i\epsilon}$$

(External lines equal to 1 in both cases.)

■ Dirac Fermion

$$\mathcal{L} \supset \bar{\psi}(i\partial - m)\psi$$

$$= i\bar{\xi}\sigma^\mu\partial_\mu\xi + i\bar{\chi}\sigma^\mu\partial_\mu\chi - m(\xi\chi + \bar{\xi}\bar{\chi})$$

Initial state

$$\overbrace{\psi |p, s} = \text{diagram} = u^s(p)$$

$$\overbrace{\bar{\psi} |p, s} = \text{diagram} = \bar{v}^s(p)$$

Final state

$$\langle p, s | \bar{\psi} = \text{diagram} = \bar{u}^s(p)$$

$$\langle p, s | \psi = \text{diagram} = v^s(p)$$

Propagator

$$\overbrace{\psi \bar{\psi}} = \text{diagram} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

■ Majorana Fermion

$$\mathcal{L} \supset \frac{1}{2}\bar{\psi}(i\partial - m)\psi$$

$$= i\bar{\lambda}\sigma^\mu\partial_\mu\lambda - \frac{m}{2}(\lambda\lambda + \bar{\lambda}\bar{\lambda})$$

Initial state

■ Abelian Gauge Theory (Photon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi + |D_\mu\phi|^2$$

$$(D_\mu = \partial_\mu - iQ A_\mu)$$

$$\overbrace{A_\mu |p; \vec{a}} = \text{diagram} = \epsilon_\mu^{\vec{a}}(p)$$

$$\langle p; \vec{a} | A_\mu = \text{diagram} = \epsilon_\mu^{\vec{a}*}(p)$$

$$\overbrace{A_\mu A_\nu} = \text{diagram} = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$$

$$Q\bar{\psi}A\psi = \text{diagram} = iQ\gamma^\mu$$

$$iQA^\mu\phi^*\partial_\mu\phi + \text{H.c.} = \text{diagram} = iQ(p^\mu + q^\mu)$$

(Momentum must be taken along the arrow)

$$Q^2 A^2 |\phi|^2 = \text{diagram} = 2iQ^2\eta^{\mu\nu}$$

■ Non-Abelian Gauge Theory (Gluon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi + |D_\mu\phi|^2$$

$$(D_\mu = \partial_\mu - ig A_\mu)$$

$$\overbrace{A_\mu^b |p; \vec{a}, a} = \text{diagram} = \epsilon_\mu^{\vec{a}}(p)\delta^{ab}$$

$$\langle p; \vec{a}, a | A_\mu^b = \text{diagram} = \epsilon_\mu^{\vec{a}*}(p)\delta^{ab}$$

$$-gf^{abc}A^{\mu a}A^{\nu b}(\partial_\mu A_\nu^c) =$$

$$= gf^{abc}[\eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-r)^\mu + \eta^{\rho\mu}(r-p)^\nu]$$

(Momentum are in incoming directions)

$$-\frac{1}{4}g^2(f^{abe}A_\mu^aA_\nu^b)(f^{cde}A_\rho^cA_\sigma^d) =$$

$$= -ig^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})]$$

TODO: vertex は lagrangian の (n!) 倍

1.4 FIELD CALCULATION TECHNIQUES

1.4.1 Dirac Field Techniques

$$\begin{aligned}
\text{Dirac Equations} &: (\not{p} - m)u^s(p) = 0; \quad (\not{p} + m)v^s(p) = 0 \\
&: \bar{u}^s(p)(\not{p} - m) = 0; \quad \bar{v}^s(p)(\not{p} + m) = 0 \\
\text{Dirac Components} &: u^{r\dagger}(p)u^s(p) = 2E_p\delta^{rs}; \quad v^{r\dagger}(p)v^s(p) = 2E_p\delta^{rs} \\
&: \bar{u}^r(p)u^s(p) = 2m\delta^{rs}; \quad \bar{v}^r(p)v^s(p) = -2m\delta^{rs}; \quad \bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0 \\
\text{Spin Sums} &: \sum_{\text{spin}} u^s(p)\bar{u}^s(p) = \not{p} + m; \quad \sum_{\text{spin}} v^s(p)\bar{v}^s(p) = \not{p} - m \\
\text{Chage Conj.} &: -C = C^{-1} = C^\dagger = C^T, \quad C^{-1}\gamma^\mu C = -C\gamma^\mu C = -\gamma^{\mu T}, \quad C^{-1}\gamma^0 C = -\gamma^0 \\
&: C = C^*, \quad \psi^C = C(\bar{\psi})^T, \quad \bar{\psi}^C = \psi^T C \\
u \text{ \& } v &: u^* = -i\gamma^2 v, \quad v^T = -iu^\dagger \gamma^2 = \bar{u}C^{-1}, \quad v = C\bar{u}^T; \quad \bar{u}_A P_H u_B = -\bar{v}_B P_H v_A \\
&: v^* = -i\gamma^2 u, \quad u^T = -iv^\dagger \gamma^2 = \bar{v}C^{-1}, \quad u = C\bar{v}^T; \quad \bar{v}_A P_H u_B = -\bar{v}_B P_H u_A
\end{aligned}$$

1.4.2 Polarization Sum

Single photon case $M = \epsilon_\mu^*(k)M^\mu$

When Ward identity $k_\mu M^\mu = 0$ is valid,

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\nu(k)M^\mu M^{\nu*} = \eta_{\mu\nu}M^\mu M^{\nu*}. \quad (1.1)$$

Double photons case $M = \epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0$ is valid,

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\rho(k)\epsilon_\nu'^*(k')\epsilon_\sigma'(k')M^{\mu\nu}M^{\rho\sigma*} = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \quad (1.2)$$

[See Sec. C.3 for verbose information.]

1.4.3 Fierz transformations

For Dirac spinors a, b, c, d ,

$$\begin{aligned}
S(a, b; c, d) &:= (\bar{a}b)(\bar{c}d); \\
V(a, b; c, d) &:= (\bar{a}\gamma^\mu b)(\bar{c}\gamma_\mu d); \\
T(a, b; c, d) &:= \frac{1}{2}(\bar{a}\sigma^{\mu\nu}b)(\bar{c}\sigma_{\mu\nu}d); \\
A(a, b; c, d) &:= (\bar{a}\gamma^\mu\gamma_5 b)(\bar{c}\gamma_\mu\gamma_5 d); \\
P(a, b; c, d) &:= (\bar{a}\gamma_5 b)(\bar{c}\gamma_5 d);
\end{aligned}
\quad \begin{pmatrix} S(a, b; c, d) \\ V(a, b; c, d) \\ T(a, b; c, d) \\ A(a, b; c, d) \\ P(a, b; c, d) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 4 & -2 & 0 & -2 & 4 \\ 6 & 0 & -2 & 0 & -6 \\ -4 & -2 & 0 & -2 & -4 \\ -1 & 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} S(a, d; c, b) \\ V(a, d; c, b) \\ T(a, d; c, b) \\ A(a, d; c, b) \\ P(a, d; c, b) \end{pmatrix}$$

Also defining $V_{\text{LR}}(a, b; c, d) := (\bar{a}\gamma^\mu P_L b)(\bar{c}\gamma_\mu P_R d)$ and so on,

$$V_{\text{LL}}(a, b; c, d) = -V_{\text{LL}}(a, d; c, b) \quad V_{\text{RL}}(a, b; c, d) = \frac{1}{4} [V_{\text{LR}}(a, d; b, c) - A_{\text{LR}}(a, d; b, c)] \quad (1.3)$$

$$V_{\text{RR}}(a, b; c, d) = -V_{\text{RR}}(a, d; c, b) \quad V_{\text{LR}}(a, b; c, d) = \frac{1}{4} [V_{\text{RL}}(a, d; b, c) - A_{\text{RL}}(a, d; b, c)] \quad (1.4)$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}; \quad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \quad (1.5)$$

1.4.4 Gordon identity

For $P := p' + p$ and $q := p' - p$,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left[\frac{P^\mu + i\sigma^{\mu\nu}q_\nu}{2m} \right] u(p) \quad \bar{u}(p')\gamma^\mu v(p) = \bar{u}(p') \left[\frac{q^\mu + i\sigma^{\mu\nu}P_\nu}{2m} \right] v(p) \quad (1.6)$$

$$\bar{v}(p')\gamma^\mu v(p) = -\bar{v}(p') \left[\frac{P^\mu + i\sigma^{\mu\nu}q_\nu}{2m} \right] v(p) \quad \bar{v}(p')\gamma^\mu u(p) = -\bar{v}(p') \left[\frac{q^\mu + i\sigma^{\mu\nu}P_\nu}{2m} \right] u(p) \quad (1.7)$$

1.4.5 Color Sum

(Here T^a is **3** of SU(3). For other representations or gauge groups, see Sec. 1.9.)

$$\text{Tr}(T^a T^b) := \frac{1}{2}\delta^{ab} \quad (\text{That is, } T^a\text{'s are } \frac{1}{2} \times \text{Gell-Mann matrices.}) \quad (1.8)$$

$$\sum_a T^a T^a = \frac{4}{3} \cdot \mathbf{1}, \quad \sum_{c,d} f^{acd} f^{bcd} = 3\delta^{ab} \quad \sum_a T_{ij}^a T_{kl}^a = \frac{1}{2}\delta_{il}\delta_{kj} - \frac{1}{6}\delta_{ij}\delta_{kl} \quad (1.9)$$

$$\sum_a T^a T^b T^a = -\frac{1}{6}T^b \quad \sum_{b,c} f^{abc} T^b T^c = \frac{3i}{2}T^a \quad f^{Dab} f^{EDc} + f^{Dca} f^{EDb} + f^{Dbc} f^{EDa} = 0 \quad (1.10)$$

1.5 MISCELLANEOUS TECHNIQUES

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2$$

$$\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$$

$$\sqrt{p_\mu \sigma^\mu} = \frac{p_\mu \sigma^\mu + m}{\sqrt{2(m + p^0)}}$$

$$\sigma^i \sigma^j = \delta_{ij} \sigma^0 + i\epsilon_{ijk} \sigma^k$$

$$\sigma^\mu \sigma^\nu = i\epsilon^{0\mu\nu\rho} \sigma^\rho + \delta_0^\mu \sigma^\nu + \delta_0^\nu \sigma^\mu - \eta^{\mu\nu} \sigma^0$$

$$[\sigma^i, \sigma^j] = 2i\epsilon_{ijk} \sigma^k$$

$$\sigma^i, \sigma^j = 2\delta_{ij}$$

TODO: TODO:

- Majorana Fermions
- Feynman Rules(A.1)

1.6 DIRAC'S GAMMA ALGEBRAS

1.6.1 Traces

$$\text{Tr}(\text{any odd \# of } \gamma\text{'s}) = 0 \quad (1.11)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} \quad (1.12)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}) \quad (1.13)$$

$$\text{Tr}(\gamma_5 \text{ and any odd \# of } \gamma\text{'s}) = 0 \quad (1.14)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma_5) = 0 \quad (1.15)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma} \quad (1.16)$$

Generally, for some γ -matrices A, B, C, \dots ,

$$\begin{aligned} \text{Tr}(ABCDEF \dots) &= \eta^{AB} \text{Tr}(CDEF \dots) - \eta^{AC} \text{Tr}(BDEF \dots) \\ &+ \eta^{AD} \text{Tr}(BCEF \dots) - \eta^{AE} \text{Tr}(BCDF \dots) + \dots, \end{aligned} \quad (1.17)$$

$$\text{Tr}(ABCDEF \dots \gamma_5) = \text{Not Established}. \quad (1.18)$$

To prove the second equation, we use following technique:

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots) = \text{Tr}(\dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu); \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots \gamma_5) = \text{Tr}(\gamma_5 \dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu). \quad (1.19)$$

1.6.2 Contractions

$$\gamma^\mu \gamma_\mu = 4 \quad (1.20)$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \quad (1.21)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} \quad (1.22)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \quad (1.23)$$

Generally, for some γ -matrices A, B, C, \dots ,

$$\text{ODD \#} : \quad \gamma^\mu ABC \dots \gamma_\mu = -2(\dots CBA), \quad (1.24)$$

$$\text{EVEN \#} : \quad \gamma^\mu ABC \dots \gamma_\mu = \text{Tr}(ABC \dots) - \text{Tr}(ABC \dots \gamma_5) \cdot \gamma_5. \quad (1.25)$$

Contractions in d -dimension

$$\gamma^\mu \gamma_\mu = d \quad (1.26)$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu \quad (1.27)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu \gamma^\rho \quad (1.28)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma \quad (1.29)$$

Contractions of ϵ 's

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu} \epsilon_{\alpha\beta\gamma\nu} = -6\delta_\nu^\mu; \quad \epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} = -2(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu) \quad (1.30)$$

$$\epsilon^{\mu\alpha\beta\gamma} \epsilon_{\mu\alpha'\beta'\gamma'} = - \left(\delta_{\alpha'}^\alpha \delta_{\beta'}^\beta \delta_{\gamma'}^\gamma + \delta_{\beta'}^\alpha \delta_{\gamma'}^\beta \delta_{\alpha'}^\gamma + \delta_{\gamma'}^\alpha \delta_{\alpha'}^\beta \delta_{\beta'}^\gamma - \delta_{\alpha'}^\alpha \delta_{\gamma'}^\beta \delta_{\beta'}^\gamma - \delta_{\beta'}^\alpha \delta_{\alpha'}^\beta \delta_{\gamma'}^\gamma - \delta_{\gamma'}^\alpha \delta_{\beta'}^\beta \delta_{\alpha'}^\gamma \right) \quad (1.31)$$

1.7 LOOP INTEGRALS AND DIMENSIONAL REGULARIZATION

1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots x_n \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \quad (1.32)$$

$$\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{[xA_1 + (1-x)A_2]^2} \quad (1.33)$$

1.7.2 d -dimensional integrals in Minkowski space

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n i \Gamma(n - \frac{d}{2})}{(4\pi)^{d/2} \Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \quad (1.34)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (1.35)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (1.36)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \quad (1.37)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \frac{\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}}{4} \quad (1.38)$$

Here we can use following expansions: $(\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = 1 - (d-4) \frac{\log \Delta}{2} + O((d-4)^2) \quad \text{around } d = 4, \quad (1.39)$$

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x) \quad \text{around } x = 0, \quad (1.40)$$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left[\frac{1}{x+n} - \gamma + \sum_{k=1}^n \frac{1}{k} + O(x+n) \right] \quad \text{around } x = -n. \quad (1.41)$$

and we get following expansion:

$$\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = \frac{1}{(4\pi)^2} \left[\left(\frac{2}{4-d} - \gamma + \log 4\pi \right) - \log \Delta + O(4-d) \right]. \quad (1.42)$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S -matrix elements. To compute the S -matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \rightarrow \left(\frac{1}{\Delta - i\epsilon}\right)^{n - \frac{d}{2}}. \quad (1.43)$$

1.8 CROSS SECTIONS AND DECAY RATES

■ **General expression** (The mass dimension of \mathcal{M} is $2 - N_f$ for $d\sigma$ and $3 - N_f$ for $d\Gamma$.)

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left[\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(p_A, p_B \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \{p_f\}) \quad (1.44)$$

$$d\Gamma = \frac{1}{2m_A} \left[\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(m_A \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(m_A - \{p_f\}) \quad (\text{in } A\text{-rest frame.}) \quad (1.45)$$

■ **2-body phase space in center-of-mass frame**

$$\int \Pi_2 := \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} (2\pi)^4 \delta^{(4)}(E_{\text{cm}} - (p_1 + p_2)) \quad (\text{in center-of-mass frame}) \quad (1.46)$$

$$= \int \frac{d\Omega}{4\pi} \frac{1}{8\pi} \frac{2 \|\mathbf{p}_1\|}{E_{\text{cm}}} \quad (1.47)$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{E_{\text{cm}}^2} + \frac{(m_1^2 - m_2^2)^2}{E_{\text{cm}}^4}} \xrightarrow{m_2=0} \frac{1}{8\pi} \left(1 - \frac{m_1^2}{E_{\text{cm}}^2} \right) \quad (1.48)$$

■ **Kinematics of Decay**

$$K \rightarrow p_1 + p_2 \quad \text{or} \quad \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{p^2 + m_1^2} \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} \sqrt{p^2 + m_2^2} \\ -\mathbf{p} \end{pmatrix}; \quad (1.49)$$

$$\|\mathbf{p}\|^2 = \frac{1}{4} \left[M^2 - 2(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{M^2} \right] \approx \left(\frac{M^2 - m_1^2}{2M} \right)^2$$

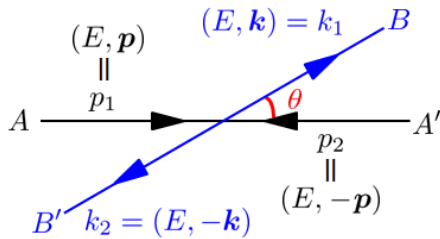
$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M};$$

$$K \cdot p_1 = \frac{M^2 + m_1^2 - m_2^2}{2}, \quad p_1 \cdot p_2 = \frac{M^2 - (m_1^2 + m_2^2)}{2}.$$

■ **Mandelstam Variables**

For $p_1 + p_2 \rightarrow k_1 + k_2$ collision, $s = (p_1 + p_2)^2 = (k_1 + k_2)^2$,
 $t = (p_1 - k_1)^2 = (p_2 - k_2)^2$,
 $u = (p_1 - k_2)^2 = (p_2 - k_1)^2$,
and $s + t + u = p_1^2 + p_2^2 + k_1^2 + k_2^2 = \sum m^2$.

■ **Kinematics of Collision (Same Mass)**



$$\|\mathbf{p}\|^2 = E^2 - m_A^2 \quad \mathbf{p} \cdot \mathbf{k} = \|\mathbf{p}\| \|\mathbf{k}\| \cos \theta$$

$$\|\mathbf{k}\|^2 = E^2 - m_B^2$$

$$p_1 \cdot p_2 = s/2 - m_A^2 \quad p_1 \cdot k_1 = p_2 \cdot k_2 = \frac{1}{2}(m_A^2 + m_B^2 - t);$$

$$k_1 \cdot k_2 = s/2 - m_B^2 \quad p_1 \cdot k_2 = p_1 \cdot k_2 = \frac{1}{2}(m_A^2 + m_B^2 - u);$$

$$s = 4E^2,$$

$$(p_1 - p_2)^2 = -4(E^2 - m_A^2)$$

$$(k_1 - k_2)^2 = -4(E^2 - m_B^2)$$

$$t = -(2E^2 - m_A^2 - m_B^2) + 2\mathbf{p} \cdot \mathbf{k}$$

$$u = -(2E^2 - m_A^2 - m_B^2) - 2\mathbf{p} \cdot \mathbf{k}$$

1.9 楊-MILLS THEORY

(See App. C.5 for verbose notes.)

1.9.1 Non-Abelian gauge theory

$$[T^a, T^b] = if^{ab}_c T^c, \quad 0 = f^D_{ab} f^E_{Dc} + f^D_{ca} f^E_{Db} + f^D_{bc} f^E_{Da}, \quad D_\mu = \partial_\mu - igA_\mu$$

$$\text{Tr } T^a T^b = \frac{1}{2} \delta^{ab}, \quad [\tilde{T}^a]_i^j := T^{\text{ad}}_a{}^j{}_i := -if^{aj}_i \quad [\tilde{D}_\mu]_i^j := \delta_i^j \partial_\mu + gf^{iaj} A_\mu^a.$$

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] \quad D_\mu \phi = \partial_\mu \phi - igA_\mu^a (T_\phi^a \phi)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{i} [A_\mu, A_\nu] \quad D_\mu F_{\mu\nu}^a = \partial_\mu F_{\mu\nu}^a + gf^{abc} A_\mu^b F_{\mu\nu}^c,$$

$$= \left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \right] T^a \quad \left(D_\mu F_{\nu\rho} = \partial_\mu \lambda - ig[A_\mu, F_{\nu\rho}] \right)^{*1}$$

$$\phi \mapsto V\phi := e^{ig\theta} \phi \quad A_\mu \mapsto V \left(A_\mu + \frac{i}{g} \partial_\mu \right) V^{-1} \quad F_{\mu\nu} \mapsto VF_{\mu\nu}V^{-1}$$

$$\phi^{a'} \simeq \phi + ig\theta^a T^a \phi \quad A_\mu^{a'} \simeq A_\mu^a + \partial_\mu \theta^a + gf^{abc} A_\mu^b \theta^c \quad F_{\mu\nu}^{a'} \simeq F_{\mu\nu}^a + gf^{abc} F_{\mu\nu}^b \theta^c$$

$$\epsilon^{\mu\nu\rho\sigma} [D_\nu, [D_\rho, D_\sigma]] = \epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0.$$

■ Killing and Casimir Here we have two constants which **depend on representation** r .

$$\text{Tr}(T^a T^b) =: C(r) \delta^{ab} \quad (\text{Killing form}), \quad T^a T^a =: C_2(r) \cdot \mathbf{1} \quad (\text{quadratic Casimir operator}), \quad (1.50)$$

which satisfy

$$C(r) = \frac{d(r)}{d(\text{ad})} C_2(r), \quad T^a T^b T^a = \left[C_2(r) - \frac{1}{2} C_2(\text{ad}) \right] T^b, \quad (1.51)$$

$$f^{acd} f^{bcd} = C_2(\text{ad}) \delta^{ab}, \quad f^{abc} T^b T^c = \frac{i}{2} C_2(\text{ad}) T^a. \quad (1.52)$$

For SU(N) For its fundamental representation N with definition $C(N) := \frac{1}{2}$, we have

$$C(N) := \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}, \quad C(\text{ad}) = C_2(\text{ad}) = N; \quad (T^a)_{ij} (T^a)_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{\delta_{ij} \delta_{kl}}{N} \right).$$

1.9.2 Abelian gauge theory

In Abelian gauge theory, V and fields are always commutative, and thus we have charge freedom (Q).

$$D_\mu \phi = (\partial_\mu - igA_\mu Q) \phi \quad \phi \mapsto e^{igQ\theta} \phi \quad F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu \lambda^a = \partial_\mu \lambda^a \quad A_\mu \mapsto A_\mu + \partial_\mu \theta \quad F_{\mu\nu} \mapsto F_{\mu\nu}$$

1.9.3 Lagrangian Block

$$\mathcal{L} \ni |D_\mu \phi|^2 - m^2 |\phi|^2, \quad \bar{\psi} (i\not{D} - m) \psi, \quad -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \left(= -\frac{1}{2} \text{Tr } F^{\mu\nu} F_{\mu\nu} \right), \quad \theta \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \quad (1.53)$$

$$-\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a = -\frac{1}{2} [(\partial_\mu A_\nu^a)^2 + A_\mu^a \partial^\mu \partial^\nu A_\nu^a] - gf^{abc} A_\mu^a A_\nu^b \partial^\mu A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu} \quad (1.54)$$

*1 Note that we can use any representation T^a but must the same ones for $A_\mu^a T^a$ and $\lambda^a T^a$.

2 Standard Model

Any representations assumed to be *normalized Hermitian*. Note that the $SU(2)$ **2** representation is

$$T^a = \frac{1}{2}\sigma^a; \quad [T^a, T^b] = i\epsilon^{abc}T^c; \quad T^\pm := T^1 \pm iT^2. \quad (2.1)$$

We use the following abridged notations:

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \quad (2.2)$$

2.1 SYMMETRIES AND FIELDS

	$SU(3)_{\text{strong}}$	$SU(2)_{\text{weak}}$	$U(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_L Q_i$: Left-handed quarks	3	2	1/6
$P_L U_i$: Right-handed up-type quarks	3	1	2/3
$P_R D_i$: Right-handed down-type quarks	3	1	-1/3
$P_R L_i$: Left-handed leptons	1	2	-1/2
$P_R E_i$: Right-handed leptons	1	1	-1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	1	2	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	8	1	0
W : Weak bosons	1	3	0
B : B boson	1	1	0

■ Full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{湯川}}$

$$\text{where } \mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W_{\mu\nu}^a - \frac{1}{4}G^{a\mu\nu}G_{\mu\nu}^a \quad (2.3)$$

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_\mu - ig_2 W_\mu - \frac{1}{2}ig_1 B_\mu \right) H \right|^2 - V(H), \quad (2.4)$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = & \bar{Q}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu - ig_2 W_\mu - \frac{1}{6}ig_1 B_\mu \right) P_L Q_i \\ & + \bar{U}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu - \frac{2}{3}ig_1 B_\mu \right) P_R U_i \\ & + \bar{D}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu + \frac{1}{3}ig_1 B_\mu \right) P_R D_i \\ & + \bar{L}_i i\gamma^\mu \left(\partial_\mu - ig_2 W_\mu + \frac{1}{2}ig_1 B_\mu \right) P_L L_i \\ & + \bar{E}_i i\gamma^\mu (\partial_\mu + ig_1 B_\mu) P_R E_i, \end{aligned} \quad (2.5)$$

$$\mathcal{L}_{\text{湯川}} = \bar{U}_i (y_u)_{ij} H P_L Q_j - \bar{D}_i (y_d)_{ij} H^\dagger P_L Q_j - \bar{E}_i (y_e)_{ij} H^\dagger P_L L_j + \text{H.c.} \quad (2.6)$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix C.4.

■ Gauge Kinetic Terms

the gauge kinetic terms can be expanded as

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = & -\frac{1}{4}(\partial B)(\partial B) \\ & -\frac{1}{4}(\partial W^a)(\partial W^a) - g_2 \epsilon^{abc}(\partial_\mu W_\nu^a)W^{\mu b}W^{\nu c} - \frac{g_2^2}{4}(\epsilon^{eab}W_\mu^a W_\nu^b)(\epsilon^{ecd}W^{c\mu}W^{d\nu}) \\ & -\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G_\nu^a)G^{\mu b}G^{\nu c} - \frac{g_3^2}{4}(f^{eab}G_\mu^a G_\nu^b)(f^{ecd}G^{c\mu}G^{d\nu}).\end{aligned}\quad (2.7)$$

2.2 HIGGS MECHANISM

■ Higgs Potential

The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2(H^\dagger H) + \lambda(H^\dagger H)^2. \quad (2.8)$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ v + (h + i\phi_3) \end{pmatrix}, \quad \text{where } v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (2.9)$$

Here h is the ‘‘Higgs boson,’’ and ϕ_i are 南部–Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2h^2, \quad (2.10)$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_\mu - ig_2 W_\mu - \frac{1}{2}ig_1 B_\mu \right) H \right|^2 \quad (2.11)$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8} \left[g_2^2 W_1^2 + g_2^2 W_2^2 + (g_1 B - g_2 W_3)^2 \right]. \quad (2.12)$$

Redefining the gauge fields (with concerning the norms) as

$$W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} := \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \quad (2.13)$$

where

$$\tan \theta_w := \frac{g_1}{g_2}, \quad e := -\frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}; \quad g_Z := \sqrt{g_1^2 + g_2^2}; \quad (2.14)$$

$$g_1 = \frac{|e|}{\cos \theta_w} = g_Z \sin \theta_w, \quad g_2 = \frac{|e|}{\sin \theta_w} = g_Z \cos \theta_w. \quad (2.15)$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4} \left[g_2^2 W^{+\mu} W_\mu^- + \frac{g_Z^2}{2} Z^\mu Z_\mu \right]. \quad (2.16)$$

Here we have omitted the 南部–Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan \theta_w Z_\mu, \quad (2.17)$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) + \left(\frac{|e|}{\tan \theta_w} Z_\mu + |e| A_\mu \right) T^3, \quad (2.18)$$

$$Z_\mu^0 := \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_2 W_\mu^3 - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_1 W_\mu^3 + g_2 B_\mu) \quad (2.19)$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \quad (2.20)$$

■Gauge Term The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W_{\mu\nu}^a &= (\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ &\quad - 4ig [(\partial W^3)^{\mu\nu} W_\mu^+ W_\nu^- + (\partial W^+)^{\mu\nu} W_\mu^- W_\nu^3 + (\partial W^-)^{\mu\nu} W_\mu^3 W_\nu^+] \\ &\quad - 2g^2 (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) (W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- - 2W_\mu^3 W_\nu^3 W_\rho^+ W_\sigma^-), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &:= -\frac{1}{4} [G^{a\mu\nu} G_{\mu\nu}^a + (\partial Z)^{\mu\nu} (\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu} (\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu} (\partial W^-)_{\mu\nu}] \\ &\quad + \frac{i|e|}{\tan \theta_w} [(\partial W^+)^{\mu\nu} W_\mu^- Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W_\nu^+ + (\partial Z)^{\mu\nu} W_\mu^+ W_\nu^-] \\ &\quad + i|e| [(\partial W^+)^{\mu\nu} W_\mu^- A_\nu + (\partial W^-)^{\mu\nu} A_\mu W_\nu^+ + (\partial A)^{\mu\nu} W_\mu^+ W_\nu^-] \\ &\quad + (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) \left[\frac{|e|^2}{2 \sin^2 \theta_w} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- + \frac{|e|^2}{\tan^2 \theta_w} W_\mu^+ Z_\nu W_\rho^- Z_\sigma \right. \\ &\quad \left. + \frac{|e|^2}{\tan \theta_w} (W_\mu^+ Z_\nu W_\rho^- A_\sigma + W_\mu^+ A_\nu W_\rho^- Z_\sigma) + |e|^2 W_\mu^+ A_\nu W_\rho^- A_\sigma \right]. \end{aligned} \quad (2.21)$$

■湯川 Term

$$\begin{aligned} \mathcal{L}_{\text{湯川}} &= \bar{U} y_u H P_L Q - \bar{D} y_d H^\dagger P_L Q - \bar{E} y_e H^\dagger P_L L + \text{H.c.} \\ &= \bar{U} y_u \epsilon^{\alpha\beta} H^\alpha P_L Q^\beta - \bar{D} y_d H^{\dagger\alpha} P_L Q^\alpha - \bar{E} y_e H^{\dagger\alpha} P_L L^\alpha + \text{H.c.} \\ &= -\frac{v+h}{\sqrt{2}} (\bar{U} y_u P_L Q^1 + \bar{D} y_d P_L Q^2 + \bar{E} y_e P_L L^2) + \text{H.c.} \end{aligned} \quad (2.22)$$

2.3 FULL LAGRANGIAN AFTER HIGGS MECHANISM

Now we have the following Lagrangian (with omitting P_L etc.):

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{gauge}} + m_W^2 W^+ W^- + \frac{m_Z^2}{2} Z^2 \\
\text{[Higgs]} & + \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} m_h^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4} \lambda h^4 \\
& + \frac{v g_2^2}{4} W^+ W^- h + \frac{v (g_1^2 + g_2^2)}{8} Z^2 h \\
& + \frac{g_2^2}{4} W^+ W^- h^2 + \frac{g_1^2 + g_2^2}{8} Z^2 h^2 \\
& - \left(\frac{1}{\sqrt{2}} h \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} h \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} h \bar{E} y_e L^2 + \text{H.c.} \right) \\
\text{[SU(3)]} & + \bar{Q} (i \not{\partial} + g_3 \not{G}) Q + \bar{U} (i \not{\partial} + g_3 \not{G}) U + \bar{D} (i \not{\partial} + g_3 \not{G}) D + \bar{L} (i \not{\partial}) L + \bar{E} (i \not{\partial}) E \\
\text{[W]} & + \bar{Q} \frac{g_2}{\sqrt{2}} (W^+ T^+ + W^- T^-) Q + \bar{L} \frac{g_2}{\sqrt{2}} (W^+ T^+ + W^- T^-) L \\
\text{[A\&Z}^0\text{]} & + \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| A + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) Z^0 \right] Q \\
& + \bar{U} \left(\frac{2}{3} |e| A - \frac{2|e|s}{3c} Z \right) U \\
& + \bar{D} \left(-\frac{1}{3} |e| A + \frac{|e|s}{3c} Z \right) D \\
& + \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| A + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) Z^0 \right] L \\
& + \bar{E} \left(-|e| A + \frac{|e|s}{c} Z \right) E \\
\text{[湯川項]} & - \left(\frac{1}{\sqrt{2}} v \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} v \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} v \bar{E} y_e L^2 + \text{H.c.} \right)
\end{aligned} \tag{2.23}$$

2.4 MASS EIGENSTATES

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := v y_\bullet / \sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \quad (m_i \geq 0). \tag{2.24}$$

Using this Ψ and Φ , we can rotate the basis as

$$Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \quad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \tag{2.25}$$

and now we have the 湯川 terms in mass eigenstates as

$$\mathcal{L}_{\text{湯川}} = - \left(1 + \frac{1}{v} h \right) \left[(m_u)_i \bar{U}_i P_L Q_i^1 + (m_d)_i \bar{D}_i P_L Q_i^2 + (m_e)_i \bar{E}_i P_L L_i^2 + \text{H.c.} \right]. \tag{2.26}$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark– W interactions do change drastically, as

$$\mathcal{L} \supset \bar{Q} i \gamma^\mu \left(-i g_2 W_\mu - \frac{1}{6} i g_1 B_\mu \right) P_L Q \quad (2.27)$$

$$= \bar{Q} \frac{g_2}{\sqrt{2}} \left(W^+ T^+ + W^- T^- \right) P_L Q \quad + \quad (\text{interaction terms with } Z \text{ and } A) \quad (2.28)$$

$$\mapsto \frac{g_2}{\sqrt{2}} \begin{pmatrix} \bar{Q}^1 \Psi_u & \bar{Q}^2 \Psi_d \end{pmatrix} \begin{pmatrix} 0 & W^+ \\ W^- & 0 \end{pmatrix} P_L \begin{pmatrix} \Psi_u^\dagger Q^1 \\ \Psi_d^\dagger Q^2 \end{pmatrix} + (\dots) \quad (2.29)$$

$$= \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 W^- X P_L Q^1 + \bar{Q}^1 W^+ X^\dagger P_L Q^2 \right] + (\dots), \quad (2.30)$$

where $X := \Psi_d \Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP -symmetry.

In our notation, CP -transformation of a spinor is described as

$$\mathcal{CP}(\psi) = -i\eta^* (\bar{\psi} \gamma^2)^\top, \quad \mathcal{CP}(\bar{\psi}) = i\eta (\gamma^2 \psi)^\top, \quad (2.31)$$

where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,

$$\begin{aligned} \mathcal{CP}(\bar{Q}^2 W^- X P_L Q^1) &= (\gamma^2 Q^2)^\top \mathcal{P}(-W^+) X P_L (\bar{Q}^1 \gamma^2)^\top \\ &= -W_\mu^{+P} (\gamma^2 Q^2)^\top (\bar{Q}^1 X^\top \gamma^2 P_L \gamma^{\mu\top})^\top \\ &= (\bar{Q}^1 W^+ X^\top P_L Q^2). \end{aligned} \quad (2.32)$$

Therefore, we can see that the CP -symmetry is maintained if and only if $X^\top = X^\dagger$, that is, if and only if X is a real matrix.

以上より, 標準模型の Lagrangian は

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{gauge}} \\
& \text{【質量項】} + m_W^2 W^+ W^- + \frac{m_Z^2}{2} Z^2 \\
& - (\bar{U} M_u P_L Q^1 + \bar{D} M_d P_L Q^2 + \bar{E} M_e P_L L^2 + \text{H.c.}) \\
& \text{【Higgs Field】} + \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} m_h^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4} \lambda h^4 \\
& \text{【Higgs との結合】} + \frac{v g_2^2}{4} W^+ W^- h + \frac{v (g_1^2 + g_2^2)}{8} Z^2 h \\
& + \frac{g_2^2}{4} W^+ W^- h^2 + \frac{g_1^2 + g_2^2}{8} Z^2 h^2 \\
& - \left(\frac{1}{v} \bar{U} M_u P_L Q^1 h + \frac{1}{v} \bar{D} M_d P_L Q^2 h + \frac{1}{v} \bar{E} M_e P_L L^2 h + \text{H.c.} \right) \\
& \text{【SU(3) および微分項】} + \bar{Q} (i \not{\partial} + g_3 \not{G}) P_L Q + \bar{U} (i \not{\partial} + g_3 \not{G}) P_R U + \bar{D} (i \not{\partial} + g_3 \not{G}) P_R D \\
& + \bar{L} (i \not{\partial}) P_L L + \bar{E} (i \not{\partial}) P_R E \\
& \text{【W boson】} + \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \not{W}^- X P_L Q^1 + \bar{Q}^1 \not{W}^+ X^\dagger P_L Q^2 \right] \quad \text{【← CP and flavor violating!】} \\
& + \bar{L} \frac{g_2}{\sqrt{2}} \left(\not{W}^+ T^+ + \not{W}^- T^- \right) P_L L \\
& \text{【A\&Z^0 boson】} + \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| \not{A} + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) \not{Z}^0 \right] P_L Q \\
& + \bar{U} \left(\frac{2}{3} |e| \not{A} - \frac{2|e|s}{3c} \not{Z} \right) P_R U \\
& + \bar{D} \left(-\frac{1}{3} |e| \not{A} + \frac{|e|s}{3c} \not{Z} \right) P_R D \\
& + \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| \not{A} + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) \not{Z}^0 \right] P_L L \\
& + \bar{E} \left(-|e| \not{A} + \frac{|e|s}{c} \not{Z} \right) P_R E
\end{aligned} \tag{2.33}$$

となる。

2.5 CHIRAL NOTATION

以上の Lagrangian を chiral 表示で表すと、まず最初は

$$\begin{aligned}
\mathcal{L} = & (\text{Higgs terms}) + (\text{Gauge fields strength}) \\
& + Q_L^\dagger i\bar{\sigma}^\mu \left(\partial_\mu - ig_3 G_\mu - ig_2 W_\mu - \frac{1}{6}ig_1 B_\mu \right) Q_L \\
& + U_R^\dagger i\sigma^\mu \left(\partial_\mu - ig_3 G_\mu - \frac{2}{3}ig_1 B_\mu \right) U_R \\
& + D_R^\dagger i\sigma^\mu \left(\partial_\mu - ig_3 G_\mu + \frac{1}{3}ig_1 B_\mu \right) D_R \\
& + L_L^\dagger i\bar{\sigma}^\mu \left(\partial_\mu - ig_2 W_\mu + \frac{1}{2}ig_1 B_\mu \right) L_L \\
& + E_R^\dagger i\sigma^\mu (\partial_\mu + ig_1 B_\mu) E_R \\
& - \left(U_R^\dagger y_u H Q_L + D_R^\dagger y_d H^\dagger Q_L + E_R^\dagger y_e H^\dagger L_L + \text{H.c.} \right) \\
= & (\text{Higgs terms}) + (\text{Gauge fields strength}) \\
& + iQ_L^\dagger \bar{\sigma}^\mu \partial_\mu Q_L + iU_R^\dagger \bar{\sigma}^\mu \partial_\mu U_R + iD_R^\dagger \bar{\sigma}^\mu \partial_\mu D_R + iL_L^\dagger \bar{\sigma}^\mu \partial_\mu L_L + iE_R^\dagger \bar{\sigma}^\mu \partial_\mu E_R \\
& + g_3 \left(Q_L^\dagger \bar{\sigma}^\mu G_\mu Q_L + U_R^\dagger \bar{\sigma}^\mu G_\mu U_R + D_R^\dagger \bar{\sigma}^\mu G_\mu D_R \right) \\
& + g_2 \left(Q_L^\dagger \bar{\sigma}^\mu W_\mu Q_L + L_L^\dagger \bar{\sigma}^\mu W_\mu L_L \right) \\
& + g_1 \left(\frac{1}{6}Q_L^\dagger \bar{\sigma}^\mu B_\mu Q_L + \frac{2}{3}U_R^\dagger \bar{\sigma}^\mu B_\mu U_R - \frac{1}{3}D_R^\dagger \bar{\sigma}^\mu B_\mu D_R - \frac{1}{2}L_L^\dagger \bar{\sigma}^\mu B_\mu L_L - E_R^\dagger \bar{\sigma}^\mu B_\mu E_R \right) \\
& - \left(U_R^\dagger y_u H Q_L + D_R^\dagger y_d H^\dagger Q_L + E_R^\dagger y_e H^\dagger L_L + \text{H.c.} \right) \tag{2.34}
\end{aligned}$$

であり、そして最終的には

$$\begin{aligned}
\mathcal{L} = & (\text{Gauge bosons and Higgs}) \\
& + iQ_L^\dagger \bar{\sigma}^\mu \partial_\mu Q_L + iU_R^\dagger \bar{\sigma}^\mu \partial_\mu U_R + iD_R^\dagger \bar{\sigma}^\mu \partial_\mu D_R + iL_L^\dagger \bar{\sigma}^\mu \partial_\mu L_L + iE_R^\dagger \bar{\sigma}^\mu \partial_\mu E_R \\
& + g_3 \left(Q_L^\dagger \bar{\sigma}^\mu G_\mu Q_L + U_R^\dagger \bar{\sigma}^\mu G_\mu U_R + D_R^\dagger \bar{\sigma}^\mu G_\mu D_R \right) \\
& - m_u (u_R^\dagger u_L + u_L^\dagger u_R) - (\text{quarks}) - m_e (e_R^\dagger e_L + e_L^\dagger e_R) - (\text{leptons}) \\
& - \frac{m_u}{v} (u_R^\dagger u_L + u_L^\dagger u_R) h - (\text{quarks}) - \frac{m_e}{v} (e_R^\dagger e_L + e_L^\dagger e_R) h - (\text{leptons}) \\
& + \frac{g_2}{\sqrt{2}} \left[\begin{pmatrix} d_L^\dagger & s_L^\dagger & b_L^\dagger \end{pmatrix} \bar{\sigma}^\mu W_\mu^- X \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} + \begin{pmatrix} u_L^\dagger & c_L^\dagger & t_L^\dagger \end{pmatrix} \bar{\sigma}^\mu W_\mu^+ X^\dagger \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \right] \\
& + \frac{g_2}{\sqrt{2}} \left[\nu_e^\dagger \bar{\sigma}^\mu W_\mu^+ e_L + e_L^\dagger \bar{\sigma}^\mu W_\mu^- \nu_e \right] \\
& + |e| \left[\frac{2}{3} u_L^\dagger \bar{\sigma}^\mu A_\mu u_L - \frac{1}{3} d_L^\dagger \bar{\sigma}^\mu A_\mu d_L + \frac{2}{3} u_R^\dagger \sigma^\mu A_\mu u_R - \frac{1}{3} d_R^\dagger \sigma^\mu A_\mu d_R + (\text{quarks}) \right. \\
& \quad \left. - e_L^\dagger \bar{\sigma}^\mu A_\mu e_L - e_R^\dagger \sigma^\mu A_\mu e_R + (\text{leptons}) \right] \\
& + \frac{|e|s}{c} \left[\left(\frac{c^2}{2s^2} - \frac{1}{6} \right) u_L^\dagger \bar{\sigma}^\mu Z_\mu u_L - \left(\frac{c^2}{2s^2} + \frac{1}{6} \right) d_L^\dagger \bar{\sigma}^\mu Z_\mu d_L - \frac{2}{3} u_R^\dagger \sigma^\mu Z_\mu u_R + \frac{1}{3} d_R^\dagger \sigma^\mu Z_\mu d_R \right. \\
& \quad \left. + \left(\frac{c^2}{2s^2} + \frac{1}{2} \right) \nu_e^\dagger \bar{\sigma}^\mu Z_\mu \nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2} \right) e_L^\dagger \bar{\sigma}^\mu Z_\mu e_L + e_R^\dagger \sigma^\mu Z_\mu e_R + (\text{others}) \right] \tag{2.35}
\end{aligned}$$

となる。

2.6 VALUES OF SM PARAMETERS (Extracted from PDG 2010)

2.6.1 Experimental Values

Low energy values

$$\alpha_{\text{EM}} = 1/137.035999679(94) \quad G_F = \frac{g_2^2}{4\sqrt{2}m_W^2} = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}$$

Electroweak scale [These values are all in $\overline{\text{MS}}$ scheme.]

$$\begin{aligned} \alpha_{\text{EM}}^{-1}(m_Z) &= 127.925(16) & m_W(m_W) &= 80.399(23) \text{ GeV} \\ \alpha_{\text{EM}}^{-1}(m_\tau) &= 133.452(16) & m_Z(m_Z) &= 91.1876(21) \text{ GeV} \\ \alpha_s(m_Z) &= 0.1183(15) & \sin^2 \theta_W(m_Z) &= 0.23116(13) \\ \Gamma_{l+l-} &= 83.984(86) \text{ MeV} & \sin^2 \theta_{\text{eff}} &= 0.23146(12) \end{aligned}$$

Fundamental masses

$$\begin{array}{lll} e : 0.510998910(13) \text{ MeV} & u : 1.7 \text{ to } 3.3 \text{ MeV} & d : 4.1 \text{ to } 5.8 \text{ MeV} \\ \mu : 105.658367(4) \text{ MeV} & c : 1.27_{-0.09}^{+0.07} \text{ GeV} & s : 101_{-21}^{+29} \text{ MeV} \\ \tau : 1.77682(16) \text{ GeV} & t : 172.0_{\pm 2.2} \text{ GeV} & b : 4.19_{-0.06}^{+0.18} \text{ GeV} \\ \\ \pi^\pm : 139.57018(35) \text{ MeV} & K^\pm : 493.677(16) \text{ MeV} & p : 938.27203(8) \text{ MeV} \\ \pi^0 : 134.9766(6) \text{ MeV} & K^0 : 497.614(24) \text{ MeV} & n : 939.565346(23) \text{ MeV} \end{array}$$

Fundamental Lifetime (also $c\tau$ for some particles)

$$\begin{array}{lll} \mu : 2.197034(21) \mu\text{s} \quad (659 \text{ m}) & \pi^\pm : 2.6033(5) \times 10^{-8} \text{ s} & K^\pm : 1.2380(21) \times 10^{-8} \text{ s} \\ \tau : 2.906(10) \times 10^{-13} \text{ s} \quad (87 \mu\text{m}) & \pi^0 : 8.4(5) \times 10^{-17} \text{ s} & K_S^0 : 8.953(5) \times 10^{-11} \text{ s} \\ & & K_L^0 : 5.116(20) \times 10^{-8} \text{ s} \end{array}$$

CKM matrix

$$V_{\text{CKM}} = \begin{pmatrix} 0.97428(15) & 0.2253(7) & 0.00347(16) \\ 0.2252(7) & 0.97345(16) & 0.0410(11) \\ 0.00862(26) & 0.0403(11) & 0.999152(45) \end{pmatrix} \sim \begin{pmatrix} 1 - \epsilon^2 & \epsilon & \epsilon^4 \\ \epsilon & 1 - \epsilon^2 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 - \epsilon^4 \end{pmatrix} \quad \text{for } \epsilon \sim 0.23 \quad (2.36)$$

2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$e \sim 0.313, \quad g_1 \sim 0.358, \quad g_2 \sim 0.651, \quad g_Z \sim 0.743; \quad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246 \text{ GeV} \quad (2.37)$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.98 \end{pmatrix}, \quad y_d \sim \begin{pmatrix} 3 \times 10^{-5} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad y_e \sim \begin{pmatrix} 3 \times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \quad (2.38)$$

Also, for $m_h \sim 120 \text{ GeV}$, we can estimate the Higgs potential as $\mu \sim 85 \text{ GeV}$ and $\lambda \sim 0.12$.

3 Supersymmetry for $\eta = \text{diag}(+, -, -, -)$

3.1 SPINOR CONVENTION

(See App. C.1.1 for a verbose explanation.)

ϵ tensor : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)

Sum Rule : α_α and $\dot{\alpha}^{\dot{\alpha}}$, except for $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$, $\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta$, $\xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}$, $\xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}$.

Lorentz 変換 : $\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$, $\bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dot{\beta}}{}_{\dot{\alpha}}$, $\psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha$, $\bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}$.

σ matrices : $(\sigma^\mu)_{\alpha\dot{\beta}} := (1, \sigma)_{\alpha\dot{\beta}}$, $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (1, -\sigma)^{\dot{\alpha}\beta}$.

3.2 SPINOR CALCULATION CHEATSHEET

$$\begin{aligned} \eta = (+, -, -, -), \quad \epsilon^{0123} = -\epsilon_{0123} = 1; \quad \text{Left Differential;} \\ \epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}, \quad \sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}, \quad \sigma^\mu := (1, \sigma), \quad \bar{\sigma}^\mu := (1, -\sigma) \\ (\sigma^{\mu\nu})_\alpha{}^\beta := \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = (\sigma^{\nu\mu})^{\dot{\alpha}}{}_{\dot{\beta}}. \end{aligned}$$

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & (\theta\phi)(\theta\psi) &= -\frac{1}{2}(\psi\phi)(\theta\theta) & (\theta\sigma^\nu\bar{\theta})\theta^\alpha &= \frac{1}{2}\theta\theta(\bar{\theta}\bar{\sigma}^\nu)^\alpha \\ \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) &= -\frac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta}) & (\theta\sigma^\nu\bar{\theta})\bar{\theta}_{\dot{\alpha}} &= -\frac{1}{2}\bar{\theta}\bar{\theta}(\theta\sigma^\mu)_{\dot{\alpha}} \\ \theta^\alpha\theta_\beta &= \frac{1}{2}\delta_\beta^\alpha\theta\theta & \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\delta_{\dot{\alpha}}^{\dot{\beta}}\bar{\theta}\bar{\theta} & (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu} & (\sigma^\mu\bar{\theta})_\alpha(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha\bar{\theta}\bar{\theta} \\ \theta\sigma^\mu\bar{\sigma}^\nu\theta &= \eta^{\mu\nu}\theta\theta & \bar{\theta}\bar{\sigma}^\mu\sigma^\nu\bar{\theta} &= \eta^{\mu\nu}\bar{\theta}\bar{\theta} \end{aligned}$$

$$\begin{aligned} \sigma^\mu\bar{\sigma}^\nu &= \eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar{\sigma}^\rho\sigma^\nu + \sigma^\nu\bar{\sigma}^\rho\sigma^\mu &= 2(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \\ \bar{\sigma}^\mu\sigma^\nu &= \eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu} & \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu &= 2(\eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\nu}\bar{\sigma}^\rho) \\ \sigma^{\mu\nu} &= -\sigma^{\nu\mu} & \sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu &= 2i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\ \bar{\sigma}^{\mu\nu} &= -\bar{\sigma}^{\nu\mu} & \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu &= -2i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \\ \text{Tr } \bar{\sigma}^\mu\sigma^\nu &= \text{Tr } \sigma^\mu\bar{\sigma}^\nu = 2\eta^{\mu\nu} & \text{Tr } \sigma^{\mu\nu}\sigma^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \\ \text{Tr } \sigma^{\mu\nu} &= \text{Tr } \bar{\sigma}^{\mu\nu} = 0 & \text{Tr } \bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \\ \sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu &= 2\delta_{\dot{\alpha}}^{\dot{\beta}}\delta_\alpha^\beta & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} - \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right] \\ \sigma^\mu_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} &= 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} + \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= \eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}_{\mu\dot{\beta}\beta} &= 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} & \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} &= 2i\sigma^{\mu\nu} \\ \sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} &= \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} & \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} &= -2i\bar{\sigma}^{\mu\nu} \\ \bar{\xi}\bar{\sigma}^\mu\chi &= -\chi\sigma^\mu\bar{\xi} = (\bar{\chi}\bar{\sigma}^\mu\xi)^* = -(\xi\sigma^\mu\bar{\chi})^* & (\psi\phi)\chi_\alpha &= -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\ \xi\sigma^\mu\bar{\sigma}^\nu\chi &= \chi\sigma^\nu\bar{\sigma}^\mu\xi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\xi})^* = (\bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^* & (\psi\phi)\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot{\alpha}} \end{aligned}$$

$$\begin{aligned} \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} &= -\frac{\partial}{\partial\theta_\alpha} & \frac{\partial}{\partial\theta^\alpha}\theta\theta &= 2\theta_\alpha & \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta &= \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}\theta\theta = 4 \\ \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} &= -\frac{\partial}{\partial\theta^\alpha} & \frac{\partial}{\partial\theta_\alpha}\theta\theta &= -2\theta^\alpha & \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta &= \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\alpha}\theta\theta = -4 \\ \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= -2\bar{\theta}_{\dot{\alpha}} & \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} &= \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} = 4 \\ \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= 2\bar{\theta}^{\dot{\alpha}} & \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} = -4 \end{aligned}$$

3.3 GENERAL RELATIONS (Note: $P_\mu = i\partial_\mu$ in our convention.)

$$\begin{aligned}
Q_\alpha &:= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, & D_\alpha &:= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, & y &:= x - i\theta\sigma\bar{\theta}, \\
\bar{Q}_{\dot{\alpha}} &:= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu, & \bar{D}_{\dot{\alpha}} &:= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu, & y^\dagger &:= x + i\theta\sigma\bar{\theta} \\
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= -2i\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu, & \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= 2i\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu, & (\text{others}) &= 0.
\end{aligned}$$

$$\begin{array}{ccc}
< x\text{-basis} > & & < y\text{-basis} > & & < y^\dagger\text{-basis} > \\
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu &= \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial y^\mu} &= \frac{\partial}{\partial\theta^\alpha} & (3.1)
\end{array}$$

$$\begin{array}{ccc}
\bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\frac{\partial}{\partial(y^\dagger)^\mu} & (3.2)
\end{array}$$

$$\begin{array}{ccc}
D^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + i(\bar{\theta}\sigma^\mu)^\alpha\partial_\mu &= -\frac{\partial}{\partial\theta_\alpha} + 2i(\bar{\theta}\sigma^\mu)^\alpha\frac{\partial}{\partial y^\mu} &= -\frac{\partial}{\partial\theta_\alpha} & (3.3)
\end{array}$$

$$\begin{array}{ccc}
\bar{D}^{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i(\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - 2i(\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\frac{\partial}{\partial(y^\dagger)^\mu} & (3.4)
\end{array}$$

$$\phi(y) = \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) = \phi(y^\dagger) - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y^\dagger) - \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y^\dagger) \quad (3.5)$$

$$\phi(y^\dagger) = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) = \phi(y) + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y) - \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y) \quad (3.6)$$

$$\phi(x) = \phi(y) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y) = \phi(y^\dagger) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y^\dagger) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y^\dagger) \quad (3.7)$$

3.4 CHIRAL SUPERFIELDS : $\bar{D}_\alpha \Phi = 0$

Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad (3.8)$$

$$= \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x) \quad (3.9)$$

$$= \phi(y^\dagger) - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y^\dagger) - \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y^\dagger) + \sqrt{2}\theta\psi(y^\dagger) - \sqrt{2}i\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(y^\dagger) + \theta\theta F(y^\dagger) \quad (3.10)$$

$$\Phi^\dagger = \phi^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger) \quad (3.11)$$

$$= \phi^*(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x) \quad (3.12)$$

$$= \phi^*(y) + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(y) - \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(y) + \sqrt{2}\bar{\theta}\bar{\psi}(y) - \sqrt{2}i\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(y) + \bar{\theta}\bar{\theta}F^*(y) \quad (3.13)$$

Product of Chiral Superfields

$$\begin{aligned} \Phi_i^\dagger\Phi_j(x, \theta, \bar{\theta}) &= \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\ &\quad - i\theta\sigma^\mu\bar{\theta}(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j) + 2\bar{\theta}\bar{\psi}_i\theta\psi_j \\ &\quad + \frac{i}{\sqrt{2}}\theta\theta(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j)\sigma^\mu\bar{\theta} + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j \\ &\quad - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu(\partial_\mu\bar{\psi}_i\phi_j - \bar{\psi}_i\partial_\mu\phi_j) + \sqrt{2}\bar{\theta}\bar{\theta}F_i^*\theta\psi_j \\ &\quad + \theta\theta\bar{\theta}\bar{\theta}\left[F_i^*F_j - \frac{1}{4}\phi_i^*\partial^2\phi_j - \frac{1}{4}\partial^2\phi_i^*\phi_j + \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j - \frac{i}{2}\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_j + \frac{i}{2}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j\right] \end{aligned} \quad (3.14)$$

$$\begin{aligned} &\rightsquigarrow \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\ &\quad - 2i(\theta\sigma^\mu\bar{\theta})(\phi_i^*\partial_\mu\phi_j) + \sqrt{2}i\theta\theta(\partial_\mu\phi_i^*)\bar{\theta}\bar{\sigma}^\mu\psi_j + \sqrt{2}i\bar{\theta}\bar{\theta}\theta\sigma^\mu\bar{\psi}_i\partial_\mu\phi_j \\ &\quad + 2\bar{\theta}\bar{\psi}_i\theta\psi_j + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j + \sqrt{2}\bar{\theta}\bar{\theta}F_i^*\theta\psi_j \\ &\quad + \theta\theta\bar{\theta}\bar{\theta}[F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j] \end{aligned} \quad (3.15)$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta[\psi_i\phi_j + \phi_i\psi_j] + \theta\theta[\phi_iF_j + F_i\phi_j - \psi_i\psi_j] \quad (3.16)$$

$$\begin{aligned} \Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k] \\ &\quad + \theta\theta[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k] \end{aligned} \quad (3.17)$$

(Products of chiral superfields are still chiral superfields.)

$$e^{ik\Phi} = e^{ik\phi(y)} \left[1 + ik \left(\sqrt{2}\theta\psi(y) + \theta\theta F(y) \right) + \frac{k^2}{2}\theta\theta\psi(y)\psi(y) \right] \quad (3.18)$$

Lagrangian Blocks

$$\mathcal{L}_{\text{kin.}} = \Phi_i^\dagger\Phi_j \Big|_{\theta\theta\bar{\theta}\bar{\theta}} \rightsquigarrow F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j \quad (3.19)$$

$$\begin{aligned} \mathcal{L}_{\text{super}} &= W \Big|_{\theta\theta} + W^* \Big|_{\bar{\theta}\bar{\theta}} = \int d^2\theta [\lambda_i\Phi_i + m_{ij}\Phi_i\Phi_j + y_{ijk}\Phi_i\Phi_j\Phi_k] + \text{H.c.} \\ &= \lambda_i F_i + m_{ij}(\phi_i F_j + F_i \phi_j - \psi_i \psi_j) + y_{ijk}[(F_i \phi_j \phi_k - \psi_i \psi_j \phi_k) + (jki \text{ and } kij \text{ terms})] \end{aligned} \quad (3.20)$$

3.5 VECTOR SUPERFIELDS AND GAUGE THEORY : $V = V^\dagger$

3.5.1 Abelian Case — Field Construction

Explicit Expression

$$\begin{aligned}
 V &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
 &\quad + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] - \theta\sigma^\mu\bar{\theta}A_\mu(x) \\
 &\quad + \theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right] + \bar{\theta}\bar{\theta}\theta\left[\lambda(x) - \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) - \frac{1}{2}\partial^2C(x)\right] \quad (3.21)
 \end{aligned}$$

$$\begin{aligned}
 &= C(y) + i\theta\chi(y) - i\bar{\theta}\bar{\chi}(y) \\
 &\quad + \frac{i}{2}\theta\theta[M(y) + iN(y)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(y) - iN(y)] - \theta\sigma^\mu\bar{\theta}[A_\mu(y) - i\partial_\mu C(y)] \\
 &\quad + \theta\theta\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\theta[\lambda(y) - \sigma^\mu\partial_\mu\bar{\chi}(y)] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y) - \partial^2C(y) - i\partial_\mu A^\mu(y)] \quad (3.22)
 \end{aligned}$$

$$\begin{aligned}
 &= C(y^\dagger) + i\theta\chi(y^\dagger) - i\bar{\theta}\bar{\chi}(y^\dagger) \\
 &\quad + \frac{i}{2}\theta\theta[M(y^\dagger) + iN(y^\dagger)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(y^\dagger) - iN(y^\dagger)] - \theta\sigma^\mu\bar{\theta}[A_\mu(y^\dagger) + i\partial_\mu C(y^\dagger)] \\
 &\quad + \theta\theta\bar{\theta}[\bar{\lambda}(y^\dagger) + \bar{\sigma}^\mu\partial_\mu\chi(y^\dagger)] + \bar{\theta}\bar{\theta}\theta[\lambda(y^\dagger) + \frac{1}{2}\partial^2C(y^\dagger) + i\partial_\mu A^\mu(y^\dagger)] \quad (3.23)
 \end{aligned}$$

Supersymmetric Gauge Transformation : $V \rightarrow V + \Phi + \Phi^\dagger$

$$\begin{aligned}
 C &\mapsto C + (\phi + \phi^*) & A_\mu &\mapsto A_\mu + i\partial_\mu(\phi - \phi^*) \\
 \chi &\mapsto \chi - i\sqrt{2}\psi & \lambda &\mapsto \lambda \\
 M + iN &\mapsto M + iN - 2iF & D &\mapsto D
 \end{aligned} \quad (3.24)$$

Wess-Zumino Gauge $C = \chi = M = N = 0$

Fixing this gauge breaks SUSY, but still allows the usual gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu\alpha, \quad \lambda \mapsto \lambda, \quad D \mapsto D. \quad (3.25)$$

$$\begin{aligned}
 V &= -\theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) & V^2 &= \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}A_\mu A^\mu \\
 &= -\theta\sigma^\mu\bar{\theta}A_\mu(y) + \theta\theta\bar{\theta}\bar{\lambda}(y) + \bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y) - i\partial_\mu A^\mu(y)] & V^3 &= 0 \\
 &= -\theta\sigma^\mu\bar{\theta}A_\mu(y^\dagger) + \theta\theta\bar{\theta}\bar{\lambda}(y^\dagger) + \bar{\theta}\bar{\theta}\theta\lambda(y^\dagger) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y^\dagger) + i\partial_\mu A^\mu(y^\dagger)] \\
 e^{kV} &= 1 - k\theta\sigma^\mu\bar{\theta}A_\mu(x) + k\theta\theta\bar{\theta}\bar{\lambda} + k\bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\theta}\left[\frac{k}{2}D + \frac{k^2}{4}A_\mu A^\mu\right]
 \end{aligned}$$

Field Strength

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V; \quad W_\alpha \mapsto W_\alpha \quad (\text{gauge invariant}) \quad (3.26)$$

$$\bar{D}_{\dot{\beta}}W_\alpha = D_\beta\bar{W}_{\dot{\alpha}} = 0; \quad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \quad (3.27)$$

$$W_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + i\theta\theta[\sigma^\mu\partial_\mu\bar{\lambda}(y)]_\alpha \quad (3.28)$$

$$\bar{W}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}(y^\dagger) + \bar{\theta}_{\dot{\alpha}}D(y^\dagger) + i(\bar{\theta}\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}F_{\mu\nu}(y^\dagger) - i\bar{\theta}\bar{\theta}[\partial_\mu\lambda(y^\dagger)\sigma^\mu]_{\dot{\alpha}} \quad (3.29)$$

$$W^\alpha W_\alpha|_{\theta\theta} = -\frac{1}{4}\bar{D}\bar{D}W^\alpha D_\alpha V \rightsquigarrow -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 \quad (3.30)$$

Lagrangian Blocks(L_{inv} is SUSY- and gauge-invariant, while L_{mass} is not gauge-invariant.)

$$\begin{aligned} \mathcal{L}_{\text{inv}} &= \frac{\tau}{4} W^\alpha W_\alpha \Big|_{\theta\theta} + \frac{\tau^*}{4} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} \quad \left(\text{with } \tau := 1 + \frac{i\theta}{8\pi^2} \right) \\ &\rightsquigarrow -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + \frac{1}{2} D^2 - \frac{\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \end{aligned} \quad (3.31)$$

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= m^2 V^2 \Big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ &= m^2 \left(\frac{1}{2} A^\mu A_\mu + i\bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi - (\lambda\chi + \bar{\lambda}\bar{\chi}) + \frac{1}{2}(M^2 + N^2) + CD + \frac{1}{2} \partial_\mu C \partial^\mu C \right) \\ &= \frac{1}{2} \partial_\mu C' \partial^\mu C' + i\bar{\chi}' \bar{\sigma}^\mu \partial_\mu \chi' + \frac{m^2}{2} A^\mu A_\mu - m(\lambda\chi' + \bar{\lambda}\bar{\chi}') + mC'D + \frac{m^2}{2}(M^2 + N^2) \end{aligned} \quad (3.32)$$

3.5.2 Abelian Case — Gauge TheoryHere we turn on the coupling constant g . When $\Lambda = i\lambda(y) + \sqrt{2}\theta\xi(y) + \theta\theta K(y)$,

$$\mathcal{L} \ni \Phi^\dagger e^{2gqV} \Phi; \quad \Phi \mapsto e^{iqg\Lambda} \Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger e^{-iqg\Lambda^\dagger}; \quad 2V \mapsto 2V - i(\Lambda - \Lambda^\dagger) \quad (3.33)$$

$$\begin{aligned} \phi &\mapsto e^{-qg\lambda} \phi & C &\mapsto C + \text{Re } \lambda & M + iN &\mapsto M + iN - K \\ \psi &\mapsto e^{-qg\lambda} (\psi + iqq\phi \cdot \xi) & \chi &\mapsto \chi - \frac{1}{\sqrt{2}}\xi & A_\mu &\mapsto A_\mu - \partial_\mu (\text{Im } \lambda) \\ F &\mapsto e^{-qg\lambda} \left(F + iqq\phi K - iqq\xi\psi + \frac{(qq)^2}{2} \xi\xi\phi \right) & \lambda &\mapsto \lambda & D &\mapsto D \end{aligned}$$

(Very similar to the gauge transformations in Sec. 1.9.2.)

Lagrangian block

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= \frac{1}{4} W^\alpha W_\alpha \Big|_{\theta\theta} + \text{H.c.} & \rightsquigarrow & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + \frac{1}{2} D^2 \\ \mathcal{L}_{\text{chiral}} &= \Phi^\dagger e^{2gqV} \Phi \Big|_{\theta\theta\bar{\theta}\bar{\theta}} & \rightsquigarrow & F^* F + D_\mu \phi^* D^\mu \phi + i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + qqD\phi^* \phi - \sqrt{2}qq(\phi^* \lambda\psi + \phi\bar{\lambda}\bar{\psi}) \\ \mathcal{L}_{\mathcal{CP}} &= \frac{i\theta}{32\pi^2} W^\alpha W_\alpha \Big|_{\theta\theta} + \text{H.c.} & \rightsquigarrow & -\frac{\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ & & \text{with } D_\mu[\phi, \psi] &= (\partial_\mu - igqA_\mu)[\phi, \psi], \quad D_\mu \lambda = \partial_\mu \lambda, \end{aligned}$$

3.5.3 Non-Abelian Case

$$\begin{aligned} T^a &: \text{generators (Hermitian);} \\ \text{Tr } T^a T^b &= K\delta^{ab} \quad (K > 0), \quad [T^a, T^b] = if^{abc}T^c \quad (f \text{ is anti-symmetric}) \end{aligned}$$

Explicit Expression Same as the Abelian case.Supersymmetric Gauge Transformation

$$\begin{aligned} \mathcal{L} \ni \Phi^\dagger e^{2g\tilde{V}} \Phi; \quad \Phi &\mapsto e^{ig\tilde{\Lambda}} \Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger e^{-ig\tilde{\Lambda}^\dagger}; \quad e^{2g\tilde{V}} \mapsto e^{ig\tilde{\Lambda}^\dagger} e^{2g\tilde{V}} e^{-ig\tilde{\Lambda}} \\ &\text{with } \tilde{V} := V^a T^a, \quad \tilde{\Lambda} := \Lambda^a T^a, \quad \tilde{\Lambda}^\dagger := (\Lambda^a)^\dagger T^a \end{aligned} \quad (3.34)$$

$$2\tilde{V} \mapsto 2\tilde{V} - i(\tilde{\Lambda} - \tilde{\Lambda}^\dagger) - \frac{g}{2} \left([\tilde{\Lambda}, \tilde{\Lambda}^\dagger] + i[2\tilde{V}, \tilde{\Lambda} + \tilde{\Lambda}^\dagger] \right) + \dots \quad (3.35)$$

$$= \left[2V^a - i(\Lambda^a - \Lambda^{\dagger a}) + \frac{g}{2} \left(-i\Lambda^b \Lambda^{\dagger c} + 2V^b (\Lambda^c + \Lambda^{\dagger c}) \right) f^{abc} + \dots \right] T^a \quad (3.36)$$

We do not present the transformations of the components; note that λ and D are transformed in non-Abelian theories.

Wess-Zumino Gauge

$$V^a = -\theta\sigma^\mu\bar{\theta}A_\mu^a(x) + \theta\theta\bar{\theta}\bar{\lambda}^a(x) + \bar{\theta}\bar{\theta}\theta\lambda^a(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D^a(x) \quad (3.37)$$

$$e^{kV^aT^a} = 1 + kV^aT^a + \frac{k^2}{4}\theta\theta\bar{\theta}\bar{\theta}A_\mu^aA^\mu T^aT^b \quad (3.38)$$

Note that the lowest order term of the gauge transformation is independent of V , which guarantees that we can still take the Wess-Zumino gauge. The gauge transformation is restricted as $e^{ig\tilde{\Lambda}}$, where $\Lambda^a = \xi^a(y) = \xi^a(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\xi^a(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\xi^a(x) : \xi \in \mathbb{R}$.

$$2V^e \mapsto 2(V^e - gf^{eab}\xi^aV^b + 6g^2f^{abc}f^{ade}V^b\xi^c\xi^d) + \theta\sigma^\mu\bar{\theta}(-2\partial_\mu\xi^e + gf^{eab}\xi^a\partial_\mu\xi^b + 4g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d) + \dots \quad (3.39)$$

$$A_\mu^e \mapsto A_\mu^e + gf^{eab}A_\mu^a\xi^b + 6g^2f^{abc}f^{ade}A_\mu^b\xi^c\xi^d + \left(\partial_\mu\xi^e - \frac{g}{2}f^{eab}\xi^a\partial_\mu\xi^b - 2g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \dots \quad (3.40)$$

$$\lambda^e \mapsto \lambda^e + gf^{eab}\lambda^a\xi^b + 6g^2f^{abc}f^{ade}\lambda^b\xi^c\xi^d + \dots \quad (3.41)$$

$$D^e \mapsto D^e + gf^{eab}D^a\xi^b + 6g^2f^{abc}f^{ade}D^b\xi^c\xi^d + \dots \quad (3.42)$$

Note that C, χ, M and N are kept invariant automatically, for now we are under Wess-Zumino gauge.

Field Strength ^{*2}

$$\widetilde{W}_\alpha = -\frac{1}{8g}\bar{D}\bar{D}e^{-2g\tilde{V}}D_\alpha e^{2g\tilde{V}} \quad \bar{D}_{\dot{\beta}}W_\alpha = 0 \quad W_\alpha \mapsto e^{ig\tilde{\Lambda}}W_\alpha e^{-ig\Lambda^\dagger} \quad (3.43)$$

$$W_\alpha^a = \lambda_\alpha^a(y) + \theta_\alpha D^a(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) + i\theta\theta(\sigma^\mu D_\mu\bar{\lambda}^a(y))_\alpha \quad (3.44)$$

$$\text{Tr}\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} = \text{Tr}\left[DD + i\lambda\sigma^\mu D_\mu\bar{\lambda} + iD_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \quad (3.45)$$

$$= K\left[D^aD^a + i\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + iD_\mu\bar{\lambda}^a\bar{\sigma}^\mu\lambda^a - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}^a + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^aF_{\rho\sigma}^a\right] \quad (3.46)$$

Lagrangian Block

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4K}\text{Tr}\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} + \text{H.c.} \quad \rightsquigarrow -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}^a + i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2}D^aD^a \quad (3.47)$$

$$\mathcal{L}_{\mathcal{CP}} = \frac{i}{32K\pi^2}\text{Tr}\widetilde{W}^\alpha\widetilde{W}_\alpha\Big|_{\theta\theta} + \text{H.c.} \quad \rightsquigarrow -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^aF_{\rho\sigma}^a \quad (3.48)$$

$$\mathcal{L}_{\text{matter}} = \Phi_i^\dagger[e^{2gV^aT^a}]_{ij}\Phi_j\Big|_{\theta\theta\bar{\theta}\bar{\theta}} \rightsquigarrow D^\mu\phi_i^*D_\mu\phi_i + i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i + F_i^*F_i + gD^a(\phi^*T^a\phi) - \sqrt{2}g(\phi^*T^a\psi\lambda + \bar{\psi}\bar{\lambda}T^a\phi) \quad (3.49)$$

$$D_\mu\phi_i = \partial_\mu\phi - igA_\mu^a(T^a\phi)_i$$

$$D_\mu\phi_i^* = \partial_\mu\phi + igA_\mu^a(\phi^*T^a)_i$$

$$D_\mu\psi_i = \partial_\mu\psi - igA_\mu^a(T^a\psi)_i$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

$$D_\mu\lambda^a = \partial_\mu\lambda^a + gf^{abc}A_\mu^b\lambda^c,$$

$$D_\mu\bar{\lambda}^a = \partial_\mu\bar{\lambda}^a + gf^{abc}A_\mu^b\bar{\lambda}^c,$$

^{*2} Note the signs. $\bar{D}\bar{D}e^{2gV}D_\alpha e^{-2gV}$ is not gauge invariant! Also the curvature tensor and the covariant derivative is well-known ones: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, and $D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - ig[A_\mu, \bar{\lambda}] = \partial_\mu\bar{\lambda} + gf^{abc}A_\mu^b\bar{\lambda}^cT^a$.

3.6 MINIMAL SUPERSYMMETRIC STANDARD MODEL

3.6.1 Definitions

■ Gauge Group

$$\text{SU}(3)_{\text{color}} \times \text{SU}(2)_{\text{weak}} \times \text{U}(1)_Y \quad (\times \text{Z}_{2R} : R\text{-parity})$$

■ Fields

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	3	2	1/6	1/3	
L_i		2	-1/2		1
\bar{U}_i	$\bar{3}$		-2/3	-1/3	
\bar{D}_i	$\bar{3}$		1/3	-1/3	
\bar{E}_i			1		-1
H_u		2	1/2		
H_d		2	-1/2		

Field	SU(3)	SU(2)	U(1)
g	8		
W		3	
B			

■ Superpotential

$$W_{\text{RPC}} = \mu H_u H_d - y_{u_{ij}} H_u Q_i \bar{U}_j + y_{d_{ij}} H_d Q_i \bar{D}_j + y_{e_{ij}} H_d L_i \bar{E}_j \quad (3.50)$$

$$W_{\text{RPV}} = \mu_i H_u L_i + \frac{1}{2} \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2} \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \quad (3.51)$$

(Here we define $\lambda_{ijk} = -\lambda_{jik}$ and $\lambda''_{ijk} = -\lambda''_{ikj}$.)

■ SUSY-terms

$$V_{\text{SUSY}}^{\text{RPC}} = \frac{1}{2} \left(M_3 \widetilde{g\widetilde{g}} + M_2 \widetilde{W\widetilde{W}} + M_1 \widetilde{B\widetilde{B}} + \text{H.c.} \right) \quad (3.52)$$

$$\begin{aligned} &+ (-H_u Q A^u \bar{U} + H_d Q A^d \bar{D} + H_d L A^e \bar{E} + B H_u H_d + \text{H.c.}) \\ &+ (Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E} + m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2) \\ &+ (-H_d^* Q C^u \bar{U} + H_u^* Q C^d \bar{D} + H_u^* L C^e \bar{E} + \text{H.c.}) \\ V_{\text{SUSY}}^{\text{RPV}} = &\left(B_i H_u L_i + \frac{1}{2} A_{ijk} L_i L_j \bar{E}_k + A'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2} A''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k + \text{H.c.} \right) \\ &+ \left(C_{ijk}^1 L_i^* Q_j \bar{U}_k + C_i^2 H_u^* H_d \bar{E}_i + C_{ijk}^3 \bar{D}_i^* \bar{U}_j \bar{E}_k + \frac{1}{2} C_{ijk}^4 \bar{D}_i^* Q_j Q_k + \text{H.c.} \right) \\ &+ (M_{L_i}^2 H_d^* L_i + \text{H.c.}) \end{aligned} \quad (3.53)$$

with $C_{ijk}^4 = C_{ikj}^4$.

3.6.2 Scalar Potential

■ F -terms

$$-F_{H_u}{}^{a*} = \epsilon^{ab} \left(\mu H_d{}^b - y_{u_{ij}} Q_i^{bx} \bar{U}_j^x + \mu_i L_i^b \right) \quad (3.54)$$

$$-F_{H_d}{}^{a*} = \epsilon^{ab} \left(-\mu H_u{}^b + y_{d_{ij}} Q_i^{bx} \bar{D}_j^x + y_{e_{ij}} L_i^b \bar{E}_j \right) \quad (3.55)$$

$$-F_{Q_i}{}^{ax*} = \epsilon^{ab} \left(y_{u_{ij}} H_u{}^b \bar{U}_j^x - y_{d_{ij}} H_d{}^b \bar{D}_j^x - \lambda'_{jik} L_i^b \bar{D}_j^x \right) \quad (3.56)$$

$$-F_{L_i}{}^{a*} = \epsilon^{ab} \left(-\mu_i H_u{}^b - y_{e_{ij}} H_d{}^b \bar{E}_j + \lambda_{ijk} L_j^b \bar{E}_k + \lambda'_{ijk} Q_j^{bx} \bar{D}_k^x \right) \quad (3.57)$$

$$-F_{\bar{U}_i}{}^{x*} = \left(-\epsilon^{ab} y_{u_{ji}} H_u{}^a Q_j^{bx} + \frac{1}{2} \epsilon^{xyz} \lambda''_{ijk} \bar{D}_j^y \bar{D}_k^z \right) \quad (3.58)$$

$$-F_{\bar{D}_i}{}^{x*} = \left(\epsilon^{ab} y_{d_{ji}} H_d{}^a Q_j^{bx} + \epsilon^{ab} \lambda'_{jki} L_j^a Q_k^{bx} + \epsilon^{yzx} \lambda''_{jki} \bar{U}_j^y \bar{D}_k^z \right) \quad (3.59)$$

$$-F_{\bar{E}_i}{}^* = \left(\epsilon^{ab} y_{e_{ji}} H_d{}^a L_j^b + \frac{1}{2} \epsilon^{ab} \lambda_{jki} L_j^a L_k^b \right) \quad (3.60)$$

■ D -terms

$$D_g{}^\alpha = -g_3 \sum_{i=1}^3 \left[\sum_{a=1,2} Q_i^{ax*} (T^\alpha)_{xy} Q_i^{ay} - \bar{U}_i^{x*} (T^\alpha)_{xy} \bar{U}_i^y - \bar{D}_i^{x*} (T^\alpha)_{xy} \bar{D}_i^y \right] \quad (3.61)$$

$$D_W{}^\alpha = -g_2 \left[\sum_{i=1}^3 \sum_{x=1}^3 Q_i^{ax*} (T^\alpha)_{ab} Q_i^{by} + \sum_{i=1}^3 L_i^{a*} (T^\alpha)_{ab} L_i^b + H_u{}^{a*} (T^\alpha)_{ab} H_u{}^b + H_d{}^{a*} (T^\alpha)_{ab} H_d{}^b \right] \quad (3.62)$$

$$D_B = -g_1 \left[\frac{1}{6} |Q_i^{ax}|^2 - \frac{1}{2} |L_i^a|^2 - \frac{2}{3} |\bar{U}_i^x|^2 + \frac{1}{3} |\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2} |H_u{}^a|^2 - \frac{1}{2} |H_d{}^a|^2 \right] \quad (3.63)$$

■ Full Scalar Potential

$$V = \sum |F_\bullet|^2 + \frac{1}{2} \sum |D_\bullet|^2 \quad (3.64)$$

3.6.3 Lagrangian (Verbose)

$$\mathcal{L}_{K; \text{CP}} = -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a + D^\mu \phi_i^* D_\mu \phi_i + i \bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i + i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a - \sqrt{2} g (\phi^* T^a \psi \lambda + \bar{\psi} \bar{\lambda} T^a \phi) \quad (3.65)$$

$$\mathcal{L}_{K; \cancel{\text{CP}}} = -\frac{1}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \quad (3.66)$$

$$\mathcal{L}_{\text{scalar}} = - \left(\sum V^F + \sum V^D + \sum V^{SU(3)_Y} \right) \quad (3.67)$$

$$\mathcal{L}_{S; \text{fermi}}^{\text{RPC}} = - \left(\mu \widetilde{H}_u \widetilde{H}_d - y_{u_{ij}} H_u Q_i \bar{U}_j + y_{d_{ij}} H_d Q_i \bar{D}_j + y_{e_{ij}} H_d L_i \bar{E}_j + \dots + \text{H.c.} \right) \quad (3.68)$$

$$\mathcal{L}_{S; \text{fermi}}^{\text{RPV}} = - \left(\mu_i \widetilde{H}_u L_i + \frac{1}{2} \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2} \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k + \dots + \text{H.c.} \right) \quad (3.69)$$

■ Scalar Potential

$$V_{H_u}^F = |\mu|^2 |H_d|^2 + \sum \left(|Q^a y^u \bar{U}|^2 + |\mu_i L_i^a|^2 \right) + \left[\mu^* \mu_i H_d^{*a} L_i - \mu^* H_d^* Q y^u \bar{U} - \mu_i^* L_i^* Q y^u \bar{U} + \text{H.c.} \right] \quad (3.70)$$

$$V_{H_d}^F = |\mu|^2 |H_u|^2 + \sum \left(|Q^a y^d \bar{D}|^2 + |L^a y^e \bar{E}|^2 \right) + \left[-\mu^* H_u^* Q y^d \bar{D} - \mu^* H_u^* L y^e \bar{E} + (Q y^d \bar{D})^* (L y^e \bar{E}) + \text{H.c.} \right] \quad (3.71)$$

$$V_Q^F = |H_u|^2 |y_{ij}^u \bar{U}_j|^2 + |H_d|^2 |y_{ij}^d \bar{D}_j|^2 + \lambda_{ijk}^* \lambda_{ilm}^* L_j^* L_l \bar{D}_k^* \bar{D}_m + \left[-y_{ij}^{u*} y_{ik}^d H_u^* H_d \bar{U}_j^* \bar{D}_k - y_{ij}^{u*} \lambda_{ilm}^* H_u^* L_l \bar{U}_j^* \bar{D}_m + y_{ij}^{d*} \lambda_{ilm}^* H_d^* L_l \bar{D}_j^* \bar{D}_m + \text{H.c.} \right] \quad (3.72)$$

$$V_L^F = |\mu_i|^2 |H_u|^2 + |H_d|^2 (\bar{E}^* y^{e\dagger} y^e \bar{E}) + \lambda_{ijk}^* \lambda_{ilm}^* (Q_j^* \bar{D}_k^*) Q_l \bar{D}_m + \lambda_{ijk}^* \lambda_{ilm}^* L_j^* L_l \bar{E}_k^* \bar{E}_m + \left[\mu_i^* y_{ij}^e \bar{E}_j H_u^* H_d - \mu_i^* \lambda_{ijk}^* H_u^* Q_j \bar{D}_k - \mu_i^* \lambda_{ijk}^* H_u^* L_j \bar{E}_k - y_{ij}^{e*} \lambda_{ilm}^* \bar{E}_j^* H_d^* Q_l \bar{D}_m - y_{ij}^{e*} \lambda_{ilm}^* \bar{E}_j^* H_d^* L_l \bar{E}_m + \lambda_{ijk}^* \lambda_{ilm}^* \bar{D}_k^* Q_j^* L_l \bar{E}_m + \text{H.c.} \right] \quad (3.73)$$

$$V_{\bar{U}}^F = y_{ji}^{u*} y_{ki}^u \epsilon^{ab} \epsilon^{cd} H_u^{a*} H_u^c Q_j^{b*} Q_k^d + \frac{1}{2} \lambda_{ijk}^* \lambda_{ilm}^* (\bar{D}_j^* \bar{D}_l) (\bar{D}_k^* \bar{D}_m) - \left[y_{li}^{u*} \lambda_{ijk}^* H_u^* Q_l^* \bar{D}_j \bar{D}_k + \text{H.c.} \right] \quad (3.74)$$

$$V_{\bar{D}}^F = \epsilon^{ab} \epsilon^{cd} (y_{ji}^d H_d^a + \lambda_{kji}^* L_k^a)^* (y_{li}^d H_d^c + \lambda_{mli}^* L_m^c) Q_j^{b*} Q_l^d + \lambda_{jki}^* \lambda_{lmi}^* (\bar{U}_j^* \bar{U}_l \bar{D}_k^* \bar{D}_m - \bar{U}_j^* \bar{D}_m \bar{D}_k^* \bar{U}_l) + \left[\lambda_{lmi}^* (y_{ji}^{d*} H_d^* + \lambda_{kji}^* L_k^*) Q_j^* \bar{U}_l \bar{D}_m + \text{H.c.} \right] \quad (3.75)$$

$$V_{\bar{E}}^F = \epsilon^{ab} \epsilon^{cd} \left(y_{ji}^e H_d^a + \frac{1}{2} \lambda_{kji}^* L_k^a \right)^* L_j^{*b} \left(y_{li}^e H_d^c + \frac{1}{2} \lambda_{mli}^* L_m^c \right) L_l^d \quad (3.76)$$

$$V_g^D = \frac{g_3^2}{2} \left\{ \sum_{\alpha=1}^8 \sum_{i=1}^3 \left[\sum_{a=1,2} Q_i^{a*}(t^\alpha) Q_i^a - \bar{U}_i^*(t^\alpha) \bar{U}_i - \bar{D}_i^*(t^\alpha) \bar{D}_i \right] \right\}^2 \quad (3.77)$$

$$V_W^D = \frac{g_2^2}{2} \left[\sum_{i=1}^3 \sum_{x=1}^3 Q_i^{x*}(T^\alpha) Q_i^x + \sum_{i=1}^3 L_i^*(T^\alpha) L_i + H_u^*(T^\alpha) H_u + H_d^*(T^\alpha) H_d \right]^2 \quad (3.78)$$

$$V_B^D = \frac{g_1^2}{2} \left[\sum_i \left(\frac{1}{6} |Q_i|^2 - \frac{1}{2} |L_i|^2 - \frac{2}{3} |\bar{U}_i|^2 + \frac{1}{3} |\bar{D}_i|^2 + |\bar{E}_i|^2 \right) + \frac{1}{2} |H_u|^2 - \frac{1}{2} |H_d|^2 \right]^2 \quad (3.79)$$

$$V_{\text{SUSY}}^{\text{RPC}} = \frac{1}{2} \left(M_3 \widetilde{g\widetilde{g}} + M_2 \widetilde{W\widetilde{W}} + M_1 \widetilde{B\widetilde{B}} + \text{H.c.} \right) + (-H_u Q A^u \bar{U} + H_d Q A^d \bar{D} + H_d L A^e \bar{E} + B H_u H_d + \text{H.c.}) + (Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E} + m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2) + (-H_d^* Q C^u \bar{U} + H_u^* Q C^d \bar{D} + H_u^* L C^e \bar{E} + \text{H.c.}) \quad (3.80)$$

$$V_{\text{SUSY}}^{\text{RPV}} = \left(B_i H_u L_i + \frac{1}{2} A_{ijk} L_i L_j \bar{E}_k + A'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2} A''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k + \text{H.c.} \right) + \left(C_{ijk}^1 L_i^* Q_j \bar{U}_k + C_i^2 H_u^* H_d \bar{E}_i + C_{ijk}^3 \bar{D}_i^* \bar{U}_j \bar{E}_k + \frac{1}{2} C_{ijk}^4 \bar{D}_i^* Q_j Q_k + \text{H.c.} \right) + (M_{L_i}^2 H_d^* L_i + \text{H.c.}) \quad (3.81)$$

■ Full Potential (Verbose)

With R -parity

$$\begin{aligned}
V_{\text{full}}^{\text{RPC}} = & \frac{1}{2} \left(M_3 \widetilde{g} \widetilde{g} + M_2 \widetilde{W} \widetilde{W} + M_1 \widetilde{B} \widetilde{B} + \text{H.c.} \right) \\
& + (Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E}) \\
& + (|\mu|^2 + m_{H_u}^2) |H_u|^2 + (|\mu|^2 + m_{H_d}^2) |H_d|^2 + (B H_u H_d + \text{H.c.}) \\
& + \left[(-\mu^* y^u H_d^* - A^u H_u - C^u H_d^*)_{ij} Q_i \bar{U}_j + \text{H.c.} \right] \\
& + \left[(-\mu^* y^d H_u^* + A^d H_d + C^d H_u^*)_{ij} Q_i \bar{D}_j + \text{H.c.} \right] \\
& + \left[(-\mu^* y^e H_u^* + A^e H_d + C^e H_u^*)_{ij} L_i \bar{E}_j + \text{H.c.} \right] \\
& + |H_u|^2 |y_{ij}^u \bar{U}_j|^2 + |H_d|^2 |y_{ij}^d \bar{D}_j|^2 + |H_d|^2 |y_{ij}^e \bar{E}_j|^2 + |H_u|^2 |Q y^u|^2 + |H_d|^2 |Q y^d|^2 + |H_d|^2 |L y^e|^2 \\
& + \sum_a \left(|Q^a y^u \bar{U}|^2 + |Q^a y^d \bar{D}|^2 + |L^a y^e \bar{E}|^2 \right) \\
& + \left[(Q y^d \bar{D})^* (L y^e \bar{E}) - y_{ij}^{u*} y_{ik}^d H_u^* H_d \bar{U}_j \bar{D}_k + \text{H.c.} \right] \\
& - \left[y_{ik}^{u*} y_{jk}^u (H_u^* Q_j) (Q_i^* H_u) + y_{jk}^{d*} y_{ik}^d (H_d^* Q_j) (Q_i^* H_d) + y_{jk}^{e*} y_{ik}^e (H_d^* L_j) (L_i^* H_d) \right] \\
& + \frac{g_3^2}{2} \left\{ \sum_{\alpha=1}^8 \sum_{i=1}^3 \left[\sum_{a=1,2} Q_i^{a*} (t^\alpha) Q_i^a - \bar{U}_i^* (t^\alpha) \bar{U}_i - \bar{D}_i^* (t^\alpha) \bar{D}_i \right] \right\}^2 \\
& + \frac{g_2^2}{2} \left[\sum_{i=1}^3 \sum_{x=1}^3 Q_i^{x*} (T^\alpha) Q_i^x + \sum_{i=1}^3 L_i^* (T^\alpha) L_i + H_u^* (T^\alpha) H_u + H_d^* (T^\alpha) H_d \right]^2 \\
& + \frac{g_1^2}{2} \left[\sum_i \left(\frac{1}{6} |Q_i|^2 - \frac{1}{2} |L_i|^2 - \frac{2}{3} |\bar{U}_i|^2 + \frac{1}{3} |\bar{D}_i|^2 + |\bar{E}_i|^2 \right) + \frac{1}{2} |H_u|^2 - \frac{1}{2} |H_d|^2 \right]^2
\end{aligned} \tag{3.82}$$

With Bilinear R -parity Violation

$$V_{H_u}^F = \sum_a |\mu_i L_i^a|^2 + \left[\mu^* \mu_i H_d^* L_i - \mu_i^* L_i^* Q y^u \bar{U} + \text{H.c.} \right] \tag{3.83}$$

$$V_L^F = |\mu_i|^2 |H_u|^2 + \left[\mu_i^* y_{ij}^e \bar{E}_j H_u^* H_d + \text{H.c.} \right] \tag{3.84}$$

$$V_{\text{SUSY}}^{\text{RPV}} = (B_i H_u L_i + M_{L_i}^2 H_d^* L_i + \text{H.c.}) \tag{3.85}$$

With Trilinear leptonic R -parity Violation

$$V_Q^F = |\lambda'_{jik} L_j \bar{D}_k|^2 + \left[-y_{ij}^{u*} \lambda'_{lim} H_u^* L_l \bar{U}_j \bar{D}_m + y_{ij}^{d*} \lambda'_{lim} H_d^* L_l \bar{D}_j \bar{D}_m + \text{H.c.} \right] \tag{3.86}$$

$$\begin{aligned}
V_L^F = & |\lambda'_{ijk} Q_j \bar{D}_k|^2 + |\lambda_{ijk} L_j \bar{E}_k|^2 \\
& + \left[-y_{im}^{e*} \lambda'_{ijk} \bar{E}_m^* H_d^* Q_j \bar{D}_k - y_{im}^e \lambda_{ijk} \bar{E}_m^* H_d^* L_j \bar{E}_k + \lambda_{ijk}'^* \lambda_{ilm} \bar{D}_k^* Q_j^* L_l \bar{E}_m + \text{H.c.} \right]
\end{aligned} \tag{3.87}$$

$$\begin{aligned}
V_D^F = & \lambda_{kji}'^* \lambda_{mli}' \left[(L_k^* L_m) (Q_j^* Q_l) - (L_k^* Q_l) (Q_j^* L_m) \right] \\
& + \left\{ y_{ji}^{d*} \lambda_{mli}' \left[(H_d^* L_m) (Q_j Q_l) - (H_d^* Q_l) (Q_j^* L_m) \right] + \text{H.c.} \right\}
\end{aligned} \tag{3.88}$$

$$V_E^F = \frac{1}{2} \lambda_{kji}'^* \lambda_{mli}' (L_k^* L_m) (L_j^* L_l) + \lambda_{kji}' y_{li}^e \left[(L_k^* H_d) (L_j^* L_l) + \text{H.c.} \right] \tag{3.89}$$

4 Supergravity

4.1 MINIMAL SUGRA LAGRANGIAN

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2}eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu D_\rho\psi_\sigma \quad (4.1)$$

where

$$D_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \quad [\omega_\mu{}^{ab} : \text{“spin 接続”}] \quad (4.2)$$

$$e := \det e_a{}^\mu \quad (4.3)$$

$$M := 1/\sqrt{8\pi G} \quad (\text{Reduced Planck mass}) \quad (4.4)$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \quad (4.5)$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \quad (4.6)$$

4.2 GENERAL SUGRA LAGRANGIAN

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \quad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \quad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \quad (4.7)$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
 - Real function of chiral multiplets.
 - In global SUSY, $\int d^4\theta K$ yields kinetic terms of the chiral multiplet.
 - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int d^4\theta \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \quad (4.8)$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
 - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
 - $(a), (b), \dots$ are indices for adjoint representation of gauge group.
 - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}D_\mu\phi^iD^\mu\phi^{*j} - \frac{1}{2}eg^2D_{(a)}D^{(a)} \\
& + ie g_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu D_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu D_\rho\psi_\sigma \\
& - \frac{1}{4}ef_{(ab)}^R F_{\mu\nu}^{(a)}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f_{(ab)}^I f_{\mu\nu}^{(a)}f_{\rho\sigma}^{(b)} \\
& + \frac{i}{2}e\left[\lambda_{(a)}\sigma^\mu D_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu D_\mu\lambda^{(a)}\right] - \frac{1}{2}f_{(ab)}^I D_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X_{(a)}^{*j}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X_{(a)}^i\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{i}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{i}{4}\sqrt{2}eg\partial_{i^*}f_{(ab)}^*D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F_{\mu\nu}^{(b)} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f_{(ab)}^*\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F_{\mu\nu}^{(b)} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}D_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}D_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{i}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F_{\nu\rho}^{(a)} + \hat{F}_{\nu\rho}^{(a)}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[i\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f_{(ab)}^R + f^{R(cd)-1}\partial_i f_{(bc)}\partial_{j^*}f_{(ad)}^*\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f_{(ab)}^*\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{R(cd)-1}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{R(cd)-1}\partial_{i^*}f_{(ac)}^*\partial_{j^*}f_{(bd)}^*\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f_{(cd)}^*\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{i}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{i}{4}\sqrt{2}e\partial_{i^*}f_{(ab)}^*\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{i}{2}\sqrt{2}ee^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}ee^{K/2}\left[D_iD_jW\chi^i\chi^j + D_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}ee^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f_{(ab)}^*\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned} \tag{4.9}$$

付録 A Mathematics

A.1 ALGEBRA

A.1.1 Algebraic Structure

Semigroup	: For $a, b \in A_{\text{set}}, ab \in A$; Associative.
Monoid	: For $a, b \in A_{\text{set}}, ab \in A$; Associative, Unit.
Group	: For $a, b \in A_{\text{set}}, ab \in A$; Associative, Unit, Inverse.
Module (加群/可換群)	: For $a, b \in A_{\text{set}}, a + b \in A$; Commutative, Associative, Unit, Inverse.
Semimodule	: For $a, b \in A_{\text{set}}, a + b \in A$; Commutative, Associative, Unit.
Ring (環)	: $+$: Module, \times : Semigroup(Monoid), Distributive.
Semiring	: $+$: Semimodule, \times : Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.
Field	: $+$: Module, \times : Commutative Monoid, a^{-1} but $0, 1 \neq 0$, Distributive.

■Vector Space

Vector space on K	: For $v \in (V, +)_{\text{module}}$ and $k \in K_{\text{field}}$, (K -module) : $kv \in (V, +)$; Compatible, Distributive, $1v = v$.
Norm	: $\ x\ \geq 0, \ x\ = 0 \Leftrightarrow x = 0, \ kx\ = k\ x\ , \ x + y\ \leq \ x\ + \ y\ $
Inner product	: $\langle x x \rangle \geq 0, \langle x x \rangle = 0 \Leftrightarrow x = 0, \langle x y \rangle = \langle y x \rangle,$: $\langle x + y z \rangle = \langle x z \rangle + \langle y z \rangle, \langle kx y \rangle = k\langle x y \rangle$

■ K -algebra

K -algebra $C(V)$ とは, vector 空間 V に, distributive な乗法を入れたもの :

$$xy \in C(V); \quad (xy)z = x(yz), (x + y)z = xz + yz, x(y + z) = xy + xz, k(xy) = (kx)y = x(ky).$$

A.1.2 Lie Algebra

Lie Algebra	: For a Finite-dimensional K -module $(A, +)$ and $x, y, z \in (A, +), a, b \in K,$: $[u, v] \in (A, +)$ (Lie product), and : Bilinear: $[ax + by, z] = a[x, z] + b[y, z], [x, ay + bz] = a[x, y] + b[x, z],$: Alternating: $[x, x] = 0 \quad (\Rightarrow [x, y] = -[y, x]),$: Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$
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則ち $[A, B] := AB - BA$ として閉じていれば Lie algebra となる。

A.1.3 Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.

For $C(V)$, **Clifford algebra** $(C(V), \theta)$ is defined as

- $C(V)$: K -algebra with $1 \quad (1x = x1 = x),$
- $\theta : V \rightarrow C(V)$, homomorphism, $\theta(x)^2 = \langle x|x \rangle 1,$
- Any C' : K -algebra with 1 and any homomorphism $\phi : V \rightarrow C'$ with $\phi(x)^2 = \langle x|x \rangle 1$, there's unique $\bar{\phi} : C \rightarrow C'$, homomorphism, $\bar{\phi}(1) = 1.$

The Gamma matrices form Clifford Algebra:

$$V : \mathbb{R}^4 \text{ with } \{G_0, G_1, G_2, G_3\}, \quad \langle G_0 | G_0 \rangle = 1, \langle G_i | G_i \rangle = -1; \quad C(V) : M(n, \mathbb{C}).$$

A.1.4 Multilinear Algebra

可換環 K 上の vector 空間 V とその双対空間 V^* について :

Tensor Algebra $T(V)$: 線形写像 $f : V \rightarrow T(V)$ を持つ K -algebra であり, 別の K -algebra A への線型写像 $g : V \rightarrow A$ が与えられたときに可換な準同型 $h : T(V) \rightarrow A$ s.t. $h \circ f = g$ が一意に存在するもの。

Symmetric Algebra $S(V)$: 上の定義で, $C(V)$ (および A) を可換 K -algebra としたもの。

Exterior Algebra $\wedge(V)$: 上の定義で, $g(\cdots vv \cdots) = 0$ を要請したもの。

Tensor and Tensor Space

p 次反変 Tensor 積 : $T^p(V) := V^{\otimes p} = V \otimes V \otimes \cdots \otimes V$

q 次共変 Tensor 積 : $T_q(V) := (V^*)^{\otimes q} = V^* \otimes V^* \otimes \cdots \otimes V^*$

混合 Tensor 積 : $T_q^p(V) := T^p(V) \otimes T_q(V)$ (ただし V と V^* の順序を変えたものは同型)

Tensor Space : $T(V) := \bigoplus_q T_q(V)$

Tensor space の代数 $(+, \otimes)$ は環を為しており, また更に :

Contraction : V^* は $V \rightarrow K$ なので, $T_q^p(V) \rightarrow T_{q-1}^{p-1}(V)$ が定義される。

内積 : $T_2(V) \ni g_{ij} : V \times V \rightarrow \mathbb{R}$ (通常は対称にする。正定値としてもよい。)

添字の上げ下げ : $T_2(V) \cong \text{Hom}(V, V^*)$ (同型) なので, 内積から誘導される。

: 同型なので, $g^{ij} : \text{Hom}(V^*, V)$ は逆写像になる。

Grassmann Operator

台集合 V を Hilbert 空間であるとする。 $V \ni v$ について

$$\|\psi v\|^2 = (\psi_{ab} v_b)^* (\psi_{ac} v_c) = v_b^* v_c (\psi_{ab})^* \psi_{ac} \geq 0 \quad \therefore (\psi_{ab})^* \psi_{ac} = -(\psi_{ab} (\psi_{ac})^*)^* \quad (\text{A.1})$$

即ち反可換な作用素について $(ab)^\dagger = a^\dagger b^\dagger$, $(ab)^T = -a^T b^T$, $(ab)^* = b^* a^*$ である。

TODO: ψ の正体がわからない.....

A.1.5 Lie Group and Lie Algebra

- 群 G が **Lie group** である $\cdots G$ が同時に C^∞ 多様体であり, 積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が **COMPLEX Lie group** である \cdots 積演算と逆元写像が共に正則写像である。
- Lie 群 G の単位元における接空間を, G の **Lie algebra** \mathfrak{g} という。
 - \mathfrak{g} は G の左不変な vector 場全体である。
 - \mathfrak{g} は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると,
 - その Lie 代数の基底 B_i に対して **structure constant** c が $[B_i, B_j] = c_{ij}^k B_k$ として定義できる。

* * *

- Compact Lie 群は線型 Lie 群である。
- G として **Linear group** $\text{GL}(n; \mathbb{R})$ を考えると,
 - その Lie 代数は n 次実正方行列全体となる。

- Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は, $GL(n; \mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では, Lie 群として $GL(n; \mathbb{R})$ の部分群を考えることにし, Lie 代数の元を行列により表現する。

A.1.6 Matrix Representation

- Lie 群 G の Lie 代数の基底の組を, G の **generators** と言う。
- $GL(n; \mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し, 以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1, \dots, x_m) \mapsto e^{x_1 T_1 + \dots + x_m T_m} \quad (x_1, \dots, x_m) \mapsto e^{x_1 T_1} \dots e^{x_m T_m} \quad (\text{A.2})$$

- Lie 群 G が compact である ...
 1. 多様体 G が compact である。 **TODO: これは何故同値なのか?**
 2. G の生成子 $\{T_i\}$ を, $\text{Tr}(T_i T_j) = k \delta_{ij}$ かつ $k > 0$ となるように取り替えることができる。
【この基底の下では構造定数が完全反対称になる。】
- Compact 群 G は, **unitary representation** を持つ。
故に, 単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i \in \mathbb{R}$ により, G の元を

$$e^{ix^i T^i} \quad (\text{A.3})$$

と表すことが出来る。

A.1.7 結論

Compact Lie 群の元のうち, 単位元近傍にあるもの V は,

Hermitian Representation

$$V = \exp(ix^i T^i) \quad \text{where } T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R}, \\ [T^i, T^j] = if^{ijk} T^k, \quad \text{Tr}(T^i T^j) = \lambda \delta^{ij} > 0; \quad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \quad \text{where } R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R}, \\ [R^i, R^j] = -f^{ijk} R^k, \quad \text{Tr}(R^i R^j) = -\lambda \delta^{ij} < 0; \quad f \in \mathbb{R}$$

と表すことが出来る。

付録 B Statistics

Histogram の階級数についての Sturges の公式 $k \approx 1 + \log_2 n$ (n :観測値の数)

■分布の代表値

$$\text{算術平均 } \bar{x} := \frac{1}{n} \sum x_i, \quad \text{幾何平均 } x_G := \left(\prod x_i \right)^{1/n}, \quad \text{調和平均 } x_H := n \left(\sum \frac{1}{x_i} \right)^{-1}; \quad (\text{B.1})$$

$$\text{中央値, } n \text{ 分位点, 最頻値, mid-range, } \dots \quad (\text{B.2})$$

■分布の散らばり

$$\text{平均偏差 } d := \frac{1}{n} \sum |x_i - \bar{x}|, \quad \text{標準偏差 (分散) } S^2 := \frac{1}{n} \sum (x_i - \bar{x})^2, \quad \text{変動係数 } C.V. := S_x / \bar{x}; \quad (\text{B.3})$$

$$\text{平均差 } M.D. := \frac{1}{n^2} \sum_i \sum_j |x_i - x_j|, \quad \text{Gini 係数 } G.I. := \frac{M.D.}{2\bar{x}} = \frac{1}{2n^2\bar{x}} \sum_i \sum_j |x_i - x_j|; \quad (\text{B.4})$$

$$\text{Entropy } H = - \sum p_i \log p_i \quad (p: \text{相対頻度}) \quad \dots \quad \text{一ヶ所集中} = 0 \leq H \leq 1 = \text{等確率} \quad (\text{B.5})$$

$$\text{範囲, 四分位偏差, } \dots \quad (\text{B.6})$$

■相関を表す量

$$\text{共分散 } C_{xy} := \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}) \quad \text{相関係数 } r_{xy} := \frac{C_{xy}}{S_x S_y} \quad -1 \leq r_{xy} \leq 1, \text{線型不変} \quad (\text{B.7})$$

$$\text{偏相関関数 } r_{12;3} := \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}} \quad \text{系列相関関数 } r_h := \frac{1}{S_x} \sum_{i=1}^{n-h} \frac{(x_i - \bar{x})(x_{i+h} - \bar{x})}{n-h} \quad (\text{B.8})$$

順位相関関数 \dots 順位の組 $\{R_i\}, \{R'_i\}$ の間の相関

$$\text{Spearman: } r_S := 1 - \frac{6}{n(n^2 - 1)} \sum (R_i - R'_i)^2 \quad (\text{通常の相関関数}) \quad (\text{B.9})$$

$$\text{Kendall: } r_K := \frac{\sum G_{ij}}{n(n-1)/2} \quad \text{where } G_{ij} := (i, j) \text{ に対して同順なら } +1, \text{逆順なら } -1 \quad (\text{B.10})$$

■条件付き確率

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (\text{B.11})$$

Bayes の定理: 事象 $\{H_i\}$ が互いに排反かつ全体を尽くしているとき,

$$P(A) = \sum P(A \cap H_i) \quad \text{よって} \quad P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\sum_k P(H_k)P(A|H_k)}. \quad (\text{B.12})$$

相関係数の分布 (ρ :母集団の(真の)相関係数)

$$f(r) = \frac{(1 - \rho^2)^{(n-1)/2} (1 - r^2)^{(n-4)/2}}{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2}) \Gamma(\frac{n-2}{2})} \sum_{i=0}^{\infty} \frac{(2\rho r)^i}{i!} \left[\Gamma\left(\frac{n-1+i}{2}\right) \right]^2 \quad (\text{B.13})$$

■確率分布に対する Moment

$$\text{Moment } \mu_r := \langle X^r \rangle; \quad \mu'_r := \langle (X - \mu)^r \rangle \quad \text{標準化 moment } \alpha_r := \langle (X - \mu)^r \rangle / \sigma^r \quad (\text{B.14})$$

$$\text{Moment 母関数 } M_X(t) := \langle \exp(tX) \rangle \implies \mu_r = \frac{d^r}{dt^r} M_X(t) \quad (\text{B.15})$$

$$\text{期待値 } \mu := \mu_1; \quad \text{計算の便法: } \mu'_2 = \langle X^2 \rangle - \mu^2 \quad (\text{B.16})$$

$$\text{分散 } \sigma^2 := \mu'_2; \quad \text{標準偏差 } \sigma := \sqrt{\sigma^2}; \quad \mu'_3 = \langle X^3 \rangle - 3\mu\mu_2 + 2\mu^3 \quad (\text{B.17})$$

$$\text{歪度 } C_{\text{skew}} := \alpha_3; \quad \text{尖度 } C_{\text{kurt}} := \alpha_4 - 3 \quad \mu'_4 = \langle X^4 \rangle - 4\mu\mu_3 + 6\mu^2\mu_2 - 3\mu^4 \quad (\text{B.18})$$

■Chebyshev の不等式 いかなる確率変数に対しても, $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

■Stirling の公式

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \frac{\log 2\pi}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + O(n^{-11}) \quad (\text{B.19})$$

B.1 離散型確率分布

■超幾何分布 (A,B) が $(M, N - M)$ 個あるとき, n 個取り出して $(k, n - k)$ 個である確率。非復元捕獲。

$$P_k = \frac{{}^M C_k {}^{N-M} C_{n-k}}{{}^N C_n} \quad E = np, \quad V = np(1-p) \frac{N-n}{N-1} \quad (p := M/N) \quad (\text{B.20})$$

■二項分布 確率 p で起きる事象が, n 回のうち k 回起こる確率。復元捕獲, Bernoulli 試行。

$$P_k = {}_n C_k \cdot p^k (1-p)^{n-k} \quad E = np, \quad V = np(1-p) \quad (n = 1 : \text{Bernoulli 分布}) \quad (\text{B.21})$$

■Poisson 分布 二項分布において $np = \lambda$ 一定で $n \rightarrow \infty, p \rightarrow 0$ として, $P_k = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E = V = \lambda$.

■幾何分布 確率 p の事象が起こるまでの**失敗**回数 k の分布。

$$P_k = p(1-p)^k \quad E = \frac{1-p}{p}, \quad V = \frac{1-p}{p^2}. \quad (\text{B.22})$$

■負の二項分布 (Pascal 分布) 確率 p の事象が n 回起こるまでの**失敗**回数 k の分布。(試行は $n + k$ 回)

$$P_k = {}_{n+k-1} C_k p^n (1-p)^k \quad E = \frac{k(1-p)}{p}, \quad V = \frac{k(1-p)}{p^2}. \quad (\text{B.23})$$

■一様分布

$$P_k = \frac{1}{N}, \quad E = \frac{N+1}{2}, \quad V = \frac{N^2-1}{12}. \quad (\text{B.24})$$

B.2 連続型確率分布

■正規分布

$$N[\mu, \sigma] = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x-\mu)^2}{2\sigma^2} \right]; \quad E = \mu, \quad V = \sigma^2. \quad (\text{B.25})$$

■指数分布

$$\text{Ex}[\lambda] = \text{Ga}[\lambda, 1] = \lambda e^{-\lambda x} \quad (x \geq 0); \quad E = \frac{1}{\lambda}, \quad V = \frac{1}{\lambda^2}. \quad (\text{B.26})$$

■Gamma 分布

$$\text{Ga}[\lambda, \alpha] = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad (x \geq 0); \quad E = \frac{\alpha}{\lambda}, \quad V = \frac{\alpha}{\lambda^2}. \quad (\text{B.27})$$

■ χ^2 分布

$$\chi^2[n] = \text{Ga}[1/2, n/2] = \frac{1}{\Gamma(n/2)} \sqrt{\frac{x^{n-2} e^{-x}}{2^n}} \quad (x \geq 0); \quad E = n, \quad V = 2n. \quad (\text{B.28})$$

■Beta 分布

$$\text{Be}[\alpha, \beta] = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} \quad (0 < x < 1), \quad E = \frac{\alpha}{\alpha + \beta}, \quad V = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}. \quad (\text{B.29})$$

$$\text{where } B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\text{B.30})$$

■Cauchy 分布

$$f[\alpha, \lambda] = \frac{\alpha}{\pi} [(x - \lambda)^2 + \alpha^2]^{-1}; \quad E \text{ と } V \text{ は定義されない。} \quad (\text{B.31})$$

■対数正規分布 所得の分布。 $\log x$ が正規分布 $N[\mu, \sigma]$ に従うとき,

$$f[\mu, \sigma] = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} \exp \frac{-(\log x - \mu)^2}{2\sigma^2}; \quad E = e^{\mu + \sigma^2/2}, \quad V = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}. \quad (\text{B.32})$$

■Pareto 分布 高所得の分布。 E と V はそれぞれ $a > 1$, $a > 2$ でのみ定義される。

$$f_{[a, x_0]} = \frac{a}{x_0} \left(\frac{x_0}{x} \right)^{a+1} \quad (x \geq x_0; \quad a > 0); \quad E = \frac{ax_0}{a-1}, \quad V = \frac{ax_0^2}{a-2} - \left(\frac{ax_0}{a-1} \right)^2. \quad (\text{B.33})$$

■Weibull 分布 $a > 0$, $b > 0$ とする。

$$f[a, b] = \frac{b}{a^b} x^{b-1} \exp \left[-\left(\frac{x}{a} \right)^b \right] \quad (x \geq 0); \quad E = a\Gamma \left(1 + \frac{1}{b} \right), \quad V = a^2 \left[\Gamma \left(1 + \frac{2}{b} \right) - \left[\Gamma \left(1 + \frac{1}{b} \right) \right]^2 \right] \quad (\text{B.34})$$

付録 C Verbose Notes

C.1 SPINOR FIELDS

C.1.1 Lorentz group and Lorentz algebra

$$\begin{aligned}
\text{Metric} & : \quad \eta = \text{diag}(+1, -1, -1, -1), \quad \eta = \text{diag}(-1, +1, +1, +1). \\
\text{Lorentz transf. in } \mathbb{R}^{1,3} & : \quad \text{Linear transf. } x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu \text{ which conserve } x^2. \\
& : \quad \Rightarrow \eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma. \text{ and form a group } L. \\
& : \quad (\Rightarrow \quad (\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\alpha} \eta^{\mu\beta} \Lambda^\alpha{}_\beta =: \Lambda_\nu{}^\mu \Rightarrow \quad \Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta_\nu^\rho) \\
\text{Disconnected parts of } L & : \quad L_0 := \{\det \Lambda = +1 \wedge \Lambda_0^0 > 0\} \quad L_P := \{\det \Lambda = -1 \wedge \Lambda_0^0 > 0\} \\
& : \quad L_T := \{\det \Lambda = +1 \wedge \Lambda_0^0 < 0\} \quad L_{PT} := \{\det \Lambda = -1 \wedge \Lambda_0^0 < 0\} \\
& : \quad (L_0 \text{ is identical with } (\text{SO}(1,3), \text{SO}(3,1)).) \\
\text{Infinitesimal one in } L_0 & : \quad \Lambda^\mu{}_\nu = \delta_\nu^\mu + \epsilon^\mu{}_\nu \text{ where } \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \text{ (for } \eta = \eta\Lambda\Lambda)
\end{aligned}$$

微小変換は $\epsilon^\mu{}_\nu = \begin{pmatrix} 0 & \beta_x & \beta_y & \beta_z \\ \beta_x & 0 & -\theta_z & \theta_y \\ \beta_y & \theta_z & 0 & -\theta_x \\ \beta_z & -\theta_y & \theta_x & 0 \end{pmatrix}$ の形となっているので、回転生成子 \mathbf{J} と加速生成子 \mathbf{K} は

$$\Lambda = \exp[\kappa i(\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\beta} \cdot \mathbf{K})] \Rightarrow J_x = -\kappa i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K_x = -\kappa i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.1})$$

の形である。ここで $\kappa = \pm 1$ は notation である。

一方、微小変換から生成子を $\epsilon^\mu{}_\nu = \mp \frac{1}{2} \epsilon^{\rho\sigma} (J_{\rho\sigma})^\mu{}_\nu$ と定義すると、計量によらずに $\epsilon_{\mu\nu}$ は反対称となり、

$$\begin{aligned}
\boldsymbol{\theta} &= (+, -)(\epsilon^{23}, \epsilon^{31}, \epsilon^{12}), & \boldsymbol{\beta} &= (+, -)(\epsilon^{10}, \epsilon^{20}, \epsilon^{30}). \\
&= (+, -)(\epsilon_{23}, \epsilon_{31}, \epsilon_{12}) & &= (-, +)(\epsilon_{10}, \epsilon_{20}, \epsilon_{30})
\end{aligned}$$

で、従って $J^{\rho\sigma}$ も反対称。 $(J_{\rho\sigma})^\mu{}_\nu = \pm i(\delta_\rho^\mu \eta_{\sigma\nu} - \delta_\sigma^\mu \eta_{\rho\nu})$ となり、交換関係が得られ、これが閉じているので Lie 代数であることもわかる：

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mp i(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma}). \quad (\text{C.2})$$

生成子の具体形は計量に依存し、 $J_{10}^\mu{}_\nu = (\pm i, \mp i) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$, $J_{23}^\mu{}_\nu = (\pm i, \mp i) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}$ となる。よって、ここでの複号の取り方と κ および計量の定義によって、 $\mathbf{J} \cdot \mathbf{K}$ と $J_{\rho\sigma}$ の対応が定まることになる。

* * *

κ と複号について $(-, \text{上}), (+, \text{下}), (-, \text{下}), (+, \text{上})$ を取れば

$$\mathbf{J} = (J_{23}, J_{31}, J_{12}), \quad \mathbf{K} = (J_{10}, J_{20}, J_{30}); \quad (\text{C.3})$$

となる：

$$\Lambda = \exp \epsilon = \exp[\kappa i(\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\beta} \cdot \mathbf{K})] = \exp[\mp i(\epsilon^{\rho\sigma} J_{\rho\sigma})/2]. \quad (\text{C.4})$$

C.1.2 Lorentz group and $\text{SL}(2, \mathbb{C})$

次に、連結 Lie 群 L_0 が、連結 Lie 群 $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ と同型であることを見る：

$$\mathfrak{sl}(2, \mathbb{C}) := \{a \in \mathfrak{gl}(2, \mathbb{C}) \mid \text{Tr}(a) = 0\}, \quad \text{SL}(2, \mathbb{C}) := \{g \in \text{GL}(2, \mathbb{C}) \mid \det(g) = 1\}. \quad (\text{C.5})$$

まず、 σ^μ を（極めて一般的に） $\sigma^\mu := (\alpha 1, \beta \sigma)$ と定義する（ $\alpha = \beta = \pm 1$ ）。 $x^2 = (+ -) \det(x_\mu \sigma^\mu)$ なので

$$f^g : (x_\mu \sigma^\mu) \mapsto g(x_\mu \sigma^\mu)g^\dagger; \quad g \in \text{SL}(2, \mathbb{C}) \quad (\text{C.6})$$

は x^2 を保存する。よって Lorentz 変換であり、生成子を比べることで局所同型だとわかる：

$$\begin{aligned} \text{SL}(2, \mathbb{C}) \ni g = \exp(-ia) \text{ として } x_\mu (g \sigma^\mu g^\dagger) &= \Lambda_\mu{}^\nu x_\nu \sigma^\mu \text{ を微小展開すると} \\ \Lambda^\mu{}_\nu \sigma^\nu &= g^{-1} \sigma^\mu (g^{-1})^\dagger \implies \epsilon^\mu{}_\nu \sigma^\nu = i(a \sigma^\mu - \sigma^\mu a^\dagger) \end{aligned} \quad (\text{C.7})$$

であり、ここから g がわかる：

$$g = \exp \left(-\frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \frac{\alpha \beta}{2} \beta \cdot \boldsymbol{\sigma} \right). \quad (\text{C.8})$$

このことを別の観点から見る。Lorentz 群の生成子の交換関係を見ると、（正しく複号を取った場合）

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \quad (\text{C.9})$$

となるので、

$$\mathbf{A} := \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} := \frac{1}{2}(\mathbf{J} - i\mathbf{K}). \quad (\text{C.10})$$

と定義すると

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0, \quad (\text{C.11})$$

となり、Lorentz 群が $\text{SU}(2) \times \text{SU}(2)$ に分解できる。

C.2 WEYL SPINOR

$\text{SU}(2)_A \times \text{SU}(2)_B$ に対して $(1/2, 0)$ 表現を為すものを左巻き spinor ξ , $(0, 1/2)$ 表現を為すものを右巻き spinor $\bar{\xi}$ と定義する。

$$\xi \mapsto \left(1 - \frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \frac{1}{2} \beta \cdot \boldsymbol{\sigma}\right) \xi, \quad \bar{\xi} \mapsto \left(1 - \frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} + \frac{1}{2} \beta \cdot \boldsymbol{\sigma}\right) \bar{\xi}. \quad (\text{C.12})$$

$\alpha\beta = 1$ とすると g は左巻き spinor の変換子となる。記号を $\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta$, $\bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}}$ と定義する。

次に、 $\xi^\alpha \chi_\alpha$ および $\bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ が scalar となるようにしたい。 $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ としておくと、 $-Eg^T E = g^{-1}$ より

$$(\xi')^\alpha = \xi^\beta (g^{-1})_\beta{}^\alpha = -\xi^\beta (E_{\beta A} g_B{}^A E_{B\alpha}) \quad \therefore (-E_{\gamma\alpha})(\xi')^\alpha = -g_\gamma{}^\beta (E_{\beta\delta} \xi^\delta). \quad (\text{C.13})$$

よって、 $\epsilon^{12} = \epsilon_{21} = 1$ として $\xi^\alpha := \epsilon^{\alpha\beta} \xi_\beta$, $\xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta$ とすれば良い。

同様に、 $\bar{\xi}'_{\dot{\alpha}} = -E(g^\dagger)^{-1} E \bar{\xi}_{\dot{\beta}}$ から $(E^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}})' = (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} (E^{\dot{\beta}\gamma} \bar{\xi}_{\dot{\gamma}})$ となる。 $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}}$ として $\bar{\xi}_{\dot{\alpha}} := \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{\dot{\beta}}$ とするのが一般的である。また、このことから $(\xi_a)^* = \bar{\xi}_{\dot{a}}$ （或いは $\xi^\dagger = \bar{\xi}$ ）が分かる。

$x_\mu \sigma^\mu \mapsto x_\mu (g \sigma^\mu g^\dagger)$ であるので、 $x_\mu (\xi^\alpha \sigma^\mu \bar{\chi}^{\dot{\alpha}})$ は scalar である。よって、 $\sigma_{\alpha\dot{\alpha}}^\mu$ のように書ける。更にここで $x_\mu (\bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \chi_\alpha)$ も scalar となるように $\bar{\sigma}$ を定めよう。

$$\bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \chi_\alpha = -\epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \chi^\beta \bar{\sigma}^{\mu\dot{\alpha}\alpha} \bar{\xi}^{\dot{\beta}} \quad \therefore \sigma_{\beta\dot{\beta}}^\mu \propto \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \quad (\text{C.14})$$

であり、あとは convention である。

$$\begin{aligned}
\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}} \\
\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta, \quad \xi^\alpha \mapsto \xi^\beta (g^{-1})_\beta{}^\alpha, \quad \bar{\xi}_{\dot{\alpha}} \mapsto \bar{\xi}_{\dot{\beta}} (g^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}, \quad \bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} \\
\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\beta\dot{\beta}}^\mu, \quad \sigma_{\alpha\dot{\alpha}}^\mu = \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\beta}, \quad \therefore \sigma^\mu := (\alpha 1, \beta \sigma), \quad \bar{\sigma}^\mu := (\alpha 1, -\beta \sigma)
\end{aligned}$$

C.3 POLARIZATION SUM

Firstly we focus on the single photon case $M = \epsilon_\mu^*(k)M^\mu$. In this case, the replacement

$$\sum_{\text{pol.}} \epsilon_\mu \epsilon'_\nu \rightarrow \eta_{\mu\nu} \quad (\text{C.15})$$

is valid. Let us prove this validity. First we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k) \epsilon_\nu(k) M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2, \quad (\text{C.16})$$

while

$$\eta_{\mu\nu} M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2 \quad (\text{C.17})$$

for Ward identity $k_\mu M^\mu = 0$. Now we can see the validity easily.

Next we think about the double photons case ^{*3} $M = \epsilon_\mu^*(k) \epsilon'_\nu(k') M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \quad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \quad (\text{C.18})$$

$$k' = (E, 0, 0, -E) \quad \epsilon' = (0, \cos \theta, \sin \theta, 0) \oplus (0, -\sin \theta, \cos \theta, 0). \quad (\text{C.19})$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k) \epsilon_\nu(k) \epsilon_\rho^*(k') \epsilon_\sigma(k') M^{\mu\rho} M^{\nu\sigma*} \quad (\text{C.20})$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2 \quad (\text{C.21})$$

$$\stackrel{?}{=} \eta^{\mu\nu} \eta^{\rho\sigma} M^{\mu\rho} M^{\nu\sigma*}. \quad (\text{C.22})$$

Badly, our Ward identities

$$k_\mu \epsilon_\nu^*(k') M^{\mu\nu} = \epsilon_\mu^*(k) k'_\nu M^{\mu\nu} = 0 \quad (\text{C.23})$$

do not help us; now we want to be given

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0, \quad (\text{C.24})$$

to recover validity of the replacement.

The difference between (C.23) and (C.24) is that the former considers (and tries to sum up) all polarizations but the latter does only physical ones. Actually, as long as we are summing up all polarizations, the replacement is still valid; these two conditions are equivalent because of cancellation of unphysical polarizations. However, once we restrict the polarizations (for example with using a relation $\epsilon \cdot k = 0$), we can no more use (C.24) and thus the replacement becomes invalid.

^{*3} This part is derived from 濱口幸一's notebook.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our latter conditions (C.24), and define $\widetilde{M}^{\mu\nu}$ and \widetilde{M} as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \quad (\text{C.25})$$

$$\widetilde{M} := \epsilon_\mu^*(k) \epsilon_\nu'^*(k') \widetilde{M}^{\mu\nu}. \quad (\text{C.26})$$

Here $\widetilde{M}^{\mu\nu} \neq M^{\mu\nu}$ but $\widetilde{M} = M$; thus \widetilde{M} satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$). However, we cannot utilize the replacement for \widetilde{M} , while it is valid for M . If you did the replacement, a wrong result would come out, like

$$\eta_{\mu\rho} \eta_{\nu\sigma} \widetilde{M}^{\mu\nu} \widetilde{M}^{\rho\sigma*} = \eta_{\mu\rho} \eta_{\nu\sigma} (M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu) (M^{\rho\sigma*} + k^\rho p^{\sigma*} + p'^{\rho*} k'^{\sigma*}) \quad (\text{C.27})$$

$$= \sum_{\text{pol.}} |M|^2 + [(k \cdot p'^*)(k' \cdot p) + \text{H.c.}]. \quad (\text{C.28})$$

After all, we have obtained following expression:

$$\begin{aligned} \sum_{\text{pol.}} |M|^2 &= \sum_{\text{pol.}} |\epsilon_\mu^*(k) \epsilon_\nu'^*(k') M^{\mu\nu}|^2 = \eta_{\mu\rho} \eta_{\nu\sigma} M^{\mu\nu} M^{\rho\sigma*} \\ &= \sum_{\text{pol.}} |\widetilde{M}|^2 = \sum_{\text{pol.}} |\epsilon_\mu^*(k) \epsilon_\nu'^*(k') \widetilde{M}^{\mu\nu}|^2 \neq \eta_{\mu\rho} \eta_{\nu\sigma} \widetilde{M}^{\mu\nu} \widetilde{M}^{\rho\sigma*} = \sum_{\text{pol.}} |\widetilde{M}|^2 + [(k \cdot p'^*)(k' \cdot p) + \text{H.c.}]. \end{aligned} \quad (\text{C.29})$$

To check the Ward identity always helps us!

C.4 PHANTOM TERMS IN THE GAUGE THEORY

You may think we forget to introduce $\bar{\psi}\gamma_5\psi$, $\bar{\psi}\gamma_5\not{D}\psi$, $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$, $\epsilon^{\mu\nu\rho\sigma} D_\mu D_\nu F_{\rho\sigma}^a$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_L and P_R ,
- the last term is equivalent to the third term as

$$\epsilon^{\mu\nu\rho\sigma} D_\mu D_\nu F_{\rho\sigma}^a = \epsilon^{\mu\nu\rho\sigma} \frac{1}{2} [D_\mu, D_\nu] F_{\rho\sigma}^a = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a.$$

Therefore, we have to discuss only the ϵFF terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e &= \epsilon^{\mu\nu\rho\sigma} (-f^{acd} f^{abe} - f^{adb} f^{ace}) A_\mu^b A_\nu^c A_\rho^d A_\sigma^e \\ &= -2\epsilon^{\mu\nu\rho\sigma} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e \\ &= 0, \end{aligned} \quad (\text{C.30})$$

$$\begin{aligned} \therefore \epsilon^{\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} &= 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu^a \partial_\rho A_\sigma^a + 4g\epsilon^{\mu\nu\rho\sigma} f^{abc} A_\mu^a A_\nu^b \partial_\rho A_\sigma^c \\ &= 2\partial_\mu G^\mu, \end{aligned} \quad (\text{C.31})$$

where G^μ is the Chern–Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a \partial_\rho A_\sigma^a + \frac{1}{3} g f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right) = \epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a F_{\rho\sigma}^a - \frac{1}{3} g f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right). \quad (\text{C.32})$$

See Appendix C.7 for the instanton effect.

C.5 楊-MILLS THEORY

C.5.1 General Gauge Theory

For any Lie group G , we can consider “gauge transformation” $\phi(x) \mapsto V(x)\phi(x)$, where $V : \mathbb{R}^{1,3} \rightarrow G$. Also we can define a “connection field” $A_\mu(x)$ as:

$$\phi_{\parallel}(x + dx) := \phi(x) + igA_\mu(x)\phi(x)dx^\mu \quad \text{s.t.} \quad \phi_{\parallel}(x + dx) \mapsto V(x + dx)\phi_{\parallel}(x + dx). \quad (\text{C.33})$$

Then the covariant derivative D_μ can be defined as

$$D_\mu\phi(x)dx^\mu := \Delta_{dx}\phi(x) := \phi(x + dx) - \phi_{\parallel}(x + dx) \quad \therefore D_\mu := \partial_\mu - igA_\mu. \quad (\text{C.34})$$

Note that $\Delta_{dx}\phi(x) \mapsto V(x + dx)D_\mu\phi(x)dx^\mu$, which means $D_\mu\phi(x) \mapsto V(x)D_\mu\phi(x)$. Now we can see

$$\phi \mapsto V\phi, \quad A_\mu \mapsto V \left(A_\mu + \frac{i}{g} \partial_\mu \right) V^{-1}, \quad D_\mu \mapsto VD_\mu V^{-1}. \quad (\text{C.35})$$

We can do another discussion: we can define D_μ as a kind of derivative which satisfies (C.35).

Next we introduce the curvature tensor, or “field strength” as

$$\Delta\phi(x) := \phi_{\parallel}^{xy}(x + dx + dy) - \phi_{\parallel}^{yx}(x + dx + dy) = [D_\mu, D_\nu]\phi(x)dx^\mu dy^\nu =: -igF_{\mu\nu}\phi(x)dx^\mu dy^\nu; \quad (\text{C.36})$$

$$F_{\mu\nu}(x) := \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]. \quad (\text{C.37})$$

$\Delta\phi(x)$ is transformed in terms of $V(x + dx + dy) \simeq V(x)$, thus $F_{\mu\nu}(x) \mapsto V(x)F_{\mu\nu}(x)V^{-1}(x)$.

C.5.2 Compact Gauge Theory

Generators If the gauge group G is **compact**, it has a finite-dimensional unitary representation. The generators T_a can be taken to be Hermitian, and $V(x) = \exp[ig\theta^a(x)T^a]$ for $\theta^a(x) \in \mathbb{R}$;

$$[T^a, T^b] = if^{ab}{}_c T^c \quad (f \in \mathbb{R}) \quad 0 = f^D{}_{ab}f^E{}_{Dc} + f^D{}_{ca}f^E{}_{Db} + f^D{}_{bc}f^E{}_{Da} \quad (\text{C.38})$$

For the sake of the compactness Killing form is positive-definite, where we can normalize the generators as $\text{Tr } T^a T^b = \frac{1}{2}\delta^{ab}$, and the structure constant f^{abc} would be totally antisymmetric.

Adjoint Representations

$$[\tilde{T}^a]_i{}^j := -if^{aij}; \quad [\tilde{D}_\mu]_i{}^j := \delta_i^j \partial_\mu + gf^{iaj} A_\mu^a. \quad (\text{C.39})$$

Field Expansion In this *normalized Hermitian* basis, the relations would be^{*4}

$$\begin{aligned} \phi' &= e^{igT^a\theta^a}\phi; & F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \\ A_\mu^{a'} &\simeq A_\mu^a + \partial_\mu\theta^a + gf^{abc}A_\mu^b\theta^c & F_{\mu\nu}^{a'} &= [e^{ig\theta^c\tilde{T}^c}]^{ab}F_{\mu\nu}^b \\ &= A_\mu^a + (\tilde{D}_\mu\theta)^a, & &\simeq F_{\mu\nu}^a + gf^{abc}F_{\mu\nu}^b\theta^c. \end{aligned}$$

Covariant Derivative For a field λ^a under the adjoint representation,

$$(D_\mu\lambda)^a = \partial_\mu\lambda^a + gf^{abc}A_\mu^b\lambda^c \quad \text{or} \quad D_\mu\lambda^a T^a = \partial_\mu\lambda^a T^a - ig[A_\mu^b T^b, \lambda^a T^a] \quad ^{*5} \quad (\text{C.40})$$

Bianchi Equation

$$\epsilon^{\mu\nu\rho\sigma} [D_\nu, [D_\rho, D_\sigma]] = \epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0. \quad (\text{C.41})$$

^{*4} We can expand A_μ in T^a -basis, because it is induced by the gauge transformation.

^{*5} Note that we can use any representation T^a but must the same ones for $A_\mu^a T^a$ and $\lambda^a T^a$.

C.6 SPINOR

■ $\eta^{\mu\nu} = (-, +, +, +)$ case

Grassmann Number : $(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$

: \implies for $a, b \in \mathbb{G}^{\mathbb{R}}, ab \in i\mathbb{G}^{\mathbb{R}}$

γ matrix : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$
 : $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ etc...
 : $(i\gamma^0)^\dagger := i\gamma^0, \quad \gamma^{i\dagger} := \gamma^i$

Dirac Conjugate : $\bar{\psi} = i\psi^\dagger \gamma^0$

TODO: SPINOR

C.7 INSTANTON

TODO: INSTANTON

付録 D Supersymmetry in the text by Wess & Bagger

D.1 SPINOR CONVENTION

ϵ tensor : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)
 Sum Rule : α_α and $\dot{\alpha}_{\dot{\alpha}}$, except for $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$, $\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta$, $\xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}$, $\xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}$.
 Lorentz 変換 : $\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$, $\bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dot{\beta}}{}_{\dot{\alpha}}$, $\psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha$, $\bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}$.
 σ matrices : $(\sigma^\mu)_{\alpha\dot{\beta}} := (-1, \sigma)_{\alpha\dot{\beta}}$, $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} = (-1, -\sigma)^{\dot{\alpha}\beta}$.

(See App. C.1.1 for a verbose explanation.)

D.2 SPINOR CALCULATION CHEATSHEET

$$\eta = (-, +, +, +), \quad \epsilon^{0123} = -\epsilon_{0123} = 1$$

$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}, \quad \sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\beta\dot{\beta}}, \quad \sigma^\mu := (-1, \sigma), \quad \bar{\sigma}^\mu := (-1, -\sigma)$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = (\sigma^{\nu\mu})^{\dot{\alpha}}{}_{\dot{\beta}}.$$

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta & (\theta\phi)(\theta\psi) &= -\frac{1}{2}(\psi\phi)(\theta\theta) & (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) &= -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu} \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) &= -\frac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta}) & (\sigma^\mu\bar{\theta})_\alpha(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\bar{\theta}\bar{\theta} \\ \theta\sigma^\mu\bar{\sigma}^\nu\theta &= -\eta^{\mu\nu}\theta\theta & \bar{\theta}\bar{\sigma}^\mu\sigma^\nu\bar{\theta} &= -\eta^{\mu\nu}\bar{\theta}\bar{\theta} & & & (\theta\sigma^\mu)_\alpha(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}(\bar{\theta}\bar{\sigma}^\nu\sigma^\mu)_\alpha{}^\beta\bar{\theta}\bar{\theta} \end{aligned}$$

$$\begin{aligned} \sigma^\mu\bar{\sigma}^\nu &= -\eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar{\sigma}^\rho\sigma^\nu + \sigma^\nu\bar{\sigma}^\rho\sigma^\mu &= -2(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \\ \bar{\sigma}^\mu\sigma^\nu &= -\eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu} & \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu &= -2(\eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\nu}\bar{\sigma}^\rho) \\ \sigma^{\mu\nu} &= -\sigma^{\nu\mu} & \sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu &= 2i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\ \bar{\sigma}^{\mu\nu} &= -\bar{\sigma}^{\nu\mu} & \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu &= -2i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \\ \text{Tr } \bar{\sigma}^\mu\sigma^\nu &= \text{Tr } \sigma^\mu\bar{\sigma}^\nu = -2\eta^{\mu\nu} & \text{Tr } \sigma^{\mu\nu}\sigma^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \\ \text{Tr } \sigma^{\mu\nu} &= \text{Tr } \bar{\sigma}^{\mu\nu} = 0 & \text{Tr } \bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \\ \sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu &= -2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}} & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} - \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= 2\left[(\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right] \\ \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\mu\beta\dot{\beta}} &= -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} + \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 4\eta_{\rho\sigma}(\sigma^{\rho\mu})_{\alpha\beta}(\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_\mu &= -2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} & \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} &= -2i\sigma^{\mu\nu} \\ \sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} &= \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} & \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} &= 2i\bar{\sigma}^{\mu\nu} \\ \bar{\xi}\bar{\sigma}^\mu\chi &= -\chi\sigma^\mu\bar{\xi} = (\bar{\chi}\bar{\sigma}^\mu\xi)^* = -(\xi\sigma^\mu\bar{\chi})^* & (\psi\phi)\chi_\alpha &= -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\ \xi\sigma^\mu\bar{\sigma}^\nu\chi &= \chi\sigma^\nu\bar{\sigma}^\mu\xi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\xi})^* = (\bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^* & (\psi\phi)\bar{\chi}_{\dot{\alpha}} &= -\frac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot{\alpha}} \end{aligned}$$

In the following equations, we chose left-differential notation.

$$\begin{aligned} \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} &= -\frac{\partial}{\partial\theta_\alpha} & \frac{\partial}{\partial\theta^\alpha}\theta\theta &= 2\theta_\alpha & \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta &= 4 \\ \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} &= -\frac{\partial}{\partial\theta^\alpha} & \frac{\partial}{\partial\theta_\alpha}\theta\theta &= -2\theta^\alpha & \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta &= -4 \\ \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= -2\bar{\theta}_{\dot{\alpha}} & \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} &= 4 \\ \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= 2\bar{\theta}^{\dot{\alpha}} & \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} &= -4 \end{aligned}$$

D.3 CHIRAL SUPERFIELDS : $\bar{D}_{\dot{\alpha}}\Phi = 0$

■Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad (\text{D.1})$$

$$= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x) \quad (\text{D.2})$$

$$\Phi^\dagger = \phi^*(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x) \quad (\text{D.3})$$

■Product of Chiral Superfields

$$\begin{aligned} \Phi_i^\dagger\Phi_j(x, \theta, \bar{\theta}) \rightsquigarrow & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\ & + 2i(\theta\sigma^\mu\bar{\theta})(\phi_i^*\partial_\mu\phi_j) - \sqrt{2}i\theta\theta(\partial_\mu\phi_i^*)\bar{\theta}\bar{\sigma}^\mu\psi_j - \sqrt{2}i\bar{\theta}\bar{\theta}\theta\sigma^\mu\bar{\psi}_i\partial_\mu\phi_j \\ & + 2\bar{\theta}\bar{\psi}_i\theta\psi_j + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j + \sqrt{2}\bar{\theta}\bar{\theta}F_i^*\theta\psi_j \\ & + \theta\theta\bar{\theta}\bar{\theta}[F_i^*F_j - \partial^\mu\phi_i^*\partial_\mu\phi_j - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j] \end{aligned} \quad (\text{D.4})$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta[\psi_i\phi_j + \phi_i\psi_j] + \theta\theta[\phi_iF_j + F_i\phi_j - \psi_i\psi_j] \quad (\text{D.5})$$

$$\begin{aligned} \Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) = & \phi_i\phi_j\phi_k + \sqrt{2}\theta[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k] \\ & + \theta\theta[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k] \end{aligned} \quad (\text{D.6})$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

付録 E Cherry on the Cake

■Conversion of Units

$$1 \text{ GeV} = \frac{1}{6.5821 \times 10^{-25} \text{ s}} = \frac{1}{2.086 \times 10^{-32} \text{ yr}} = \frac{1}{0.19733 \text{ fm}} = 1.1605 \times 10^{13} \text{ K} = 1.7827 \times 10^{-24} \text{ g}$$

$$= \frac{1.519268 \times 10^{24}}{1 \text{ s}} = \frac{4.79 \times 10^{31}}{1 \text{ yr}} = \frac{5.0677}{1 \text{ fm}} \quad (\text{E.1})$$

$$1 \text{ K} = 8.6173 \times 10^{-5} \text{ eV} = \frac{1}{8.0591 \times 10^{-21} \text{ s}} = \frac{1.2408 \times 10^{20}}{1 \text{ s}} \quad (\text{E.2})$$

$$1 \text{ GeV}^2 = \frac{1}{0.38938 \text{ mbarn}} = \frac{2568.2}{1 \text{ barn}} \quad (1 \text{ barn} = 10^{-28} \text{ m}^2 = 100 \text{ fm}^2) \quad (\text{E.3})$$

$$1 \text{ tropical yr} = 3.1557 \times 10^7 \text{ s}, \quad 1 \text{ sidereal yr} = 3.1558 \times 10^7 \text{ s};$$

$$1 \text{ s} = 3.1689 \times 10^{-8} \text{ tr-yr} = 3.1688 \times 10^{-8} \text{ sr-yr}. \quad (\text{E.4})$$

■Physical Constants

$$G_F = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}, \quad G_N = 6.70881(67) \times 10^{-39} \text{ GeV}^{-2} \quad (\text{E.5})$$

$$\sqrt{G_N} = 1.61624(8) \times 10^{-35} \text{ m} = \frac{1}{1.22089(6) \times 10^{19} \text{ GeV}} = \frac{1}{2.17644(11) \times 10^{-8} \text{ kg}} \quad (\text{E.6})$$

$$\sqrt{8\pi G_N} = 8.1026(4) \times 10^{-35} \text{ m} = \frac{1}{2.4353(1) \times 10^{18} \text{ GeV}} = \frac{1}{4.3413(2) \times 10^{-9} \text{ kg}} \quad (\text{E.7})$$

■Component of Spinor in Weyl Representation

$$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \chi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dot{\beta}}, \chi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}}; \quad \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = 1, \epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -1.)$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\chi}^{\dot{1}} \\ \bar{\chi}^{\dot{2}} \end{pmatrix} \xrightarrow{C} \psi^c = -i\gamma^2\psi^* = \begin{pmatrix} -(\bar{\chi}^{\dot{2}})^* \\ (\bar{\chi}^{\dot{1}})^* \\ (\xi_2)^* \\ -(\xi_1)^* \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \bar{\xi}_2 \\ -\bar{\xi}_1 \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{E.8})$$

$$\bar{\psi} = (\chi^\alpha \quad \bar{\xi}_{\dot{\alpha}}) = (\chi^1 \chi^2 \bar{\xi}_{\dot{1}} \bar{\xi}_{\dot{2}}) \xrightarrow{C} \bar{\psi}^c = i\psi^T\gamma^0\gamma^2 = (\xi_2 \ -\xi_1 \ -\bar{\chi}^{\dot{2}} \ \bar{\chi}^{\dot{1}}) = (\xi^\alpha \quad \bar{\chi}_{\dot{\alpha}}) \quad (\text{E.9})$$

$$A^\alpha B_\alpha = \bar{\psi}_{A^c} P_L \psi_B = \bar{\psi}_{B^c} P_L \psi_A \quad \bar{A}_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} = \bar{\psi}_A P_R \psi_{B^c} = \bar{\psi}_B P_R \psi_{A^c} \quad (\text{E.10})$$

$$\bar{\psi} \not{A} \chi = \bar{\psi}_L \not{A} P_L \chi_L + \bar{\psi}_R \not{A} P_R \chi_R$$

$$= \bar{\psi}_L \not{A} P_L \chi_L - \bar{\chi}_R^c \not{A} P_L \psi_R^c \quad (\text{E.11})$$