

1 General Definitions and Tools

1.1 NOTATIONS AND CONVENTIONS

1.1.1 Metric etc.

$$\begin{aligned}
 \text{Minkowski Metric} & : \quad \eta^{\mu\nu} := \text{diag}(+, -, -, -); \quad \epsilon_{0123}^{0123} := \pm 1 \\
 \text{Coordinates} & : \quad x^\mu := (t, x, y, z); \quad \text{therefore } \partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right). \\
 \text{Gamma Matrices} & : \quad \{\gamma^\mu, \gamma^\nu\} := 2\eta^{\mu\nu}; \quad \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\
 & : \quad \text{therefore } \{\gamma^\mu, \gamma_5\} = 0, (\gamma_5)^2 = 1.
 \end{aligned}$$

$$\text{Gamma Combinations} : \quad 1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5; \quad \sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 0 / i\gamma^\mu\gamma^\nu$$

$$\begin{aligned}
 \text{Pauli Matrices} : \quad \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
 : \quad \sigma_+ &= \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
 : \quad \sigma^\mu &:= (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu := (1, -\boldsymbol{\sigma}).
 \end{aligned}$$

$$\text{Fourier Transformation} : \quad \tilde{f}(k) := \int d^4x \, e^{ikx} f(x); \quad f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k).$$

1.1.2 Fields

$$\text{Scalar} : \quad (\partial^2 + m^2)\phi = 0; \quad \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx} \right]$$

$$\text{Dirac} : \quad (i\not{\partial} - m)\psi = 0; \quad \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left[a_{\mathbf{p}}^s u^s(p) e^{-ipx} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ipx} \right]$$

$$\text{Vector} : \quad \partial^2 A^\mu = 0; \quad A^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=0..3} \left[a_{\mathbf{p}}^r \epsilon^r(p) e^{-ipx} + a_{\mathbf{p}}^{r\dagger} \epsilon^{r*}(p) e^{ipx} \right]$$

TODO: 南部-Goldstone; Gravitino

1.1.3 Electromagnetism

$$\text{Electromagnetic Fields} : \quad A^\mu = (\phi, \mathbf{A}) \quad \text{【We can invert the signs, but cannot lower the index.】}$$

$$\text{Maxwell Equations} : \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu; \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = ej^\nu$$

$$\begin{aligned}
 \text{Our Old Language} & : \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0; \quad \nabla \cdot \mathbf{E} = ej^0, \quad (\nabla \times \mathbf{B})_i - \frac{\partial}{\partial t} E_i = ej^i. \\
 & : \quad F_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{E} \\ 0 & -B_3 & B_2 \\ -\mathbf{E} & B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 & 0 \end{pmatrix}; \quad F_{\mu\nu} F^{\mu\nu} = -2 \left(\|\mathbf{E}\|^2 - \|\mathbf{B}\|^2 \right)
 \end{aligned}$$

1.2 SPINOR FIELDS

1.3 CHIRAL NOTATION

Gamma Matrices : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad P_L^R = \frac{1 \pm \gamma_5}{2}.$

Dirac Field : $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}; \quad \bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}$
: $u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}; \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix}$
: $[\eta^s = \xi^{-s} := -i\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)]$

Weyl Equations : $i\bar{\sigma} \cdot \partial \psi_L = m\psi_R; \quad i\sigma \cdot \partial \psi_L = m\psi_L$

Chiral Notation

CPT transf. : $P\psi(t, \mathbf{x})P = \eta\gamma^0\psi(t, -\mathbf{x}) \quad (|\eta|^2 = 1)$
: $T\psi(t, \mathbf{x})T = \gamma^1\gamma^3\psi(-t, \mathbf{x}) \quad (\text{ignoring intrinsic phase})$
: $C\psi(t, \mathbf{x})C = -i\gamma^2\psi^*(t, \mathbf{x}) = -i(\bar{\psi}\gamma^0\gamma^2)^T \quad (")$
: $\bar{\psi} \longrightarrow P : \eta^* \bar{\psi} \gamma^0 \quad T : -\bar{\psi} \gamma^1 \gamma^3 \quad C : i\bar{\psi}^* \gamma^2 = -i(\gamma^0 \gamma^2 \psi)^T$

1.3.1 CPT Table

	ϕ	A^μ	$\bar{\psi}\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$\bar{\psi}\gamma^\mu\gamma_5\psi$	$i\bar{\psi}\gamma_5\psi$	∂_μ
P	$\eta\phi$	$\eta-++A^\mu$	+	+	(+---)(+---)	-+++	-	+
T	$\zeta\phi$	$\zeta+---A^\mu$	+	+	-(+---)(+---)	+---	-	-+++
C	$\xi\phi^*$	$\xi+A^{\mu*}$	+	-	-	+	+	+

($\eta\zeta\xi = 1$; especially, photon A^μ is $(\eta, \zeta, \xi) = (-, +, -)$.)

1.4 FEYNMAN RULES

Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2$$

$$\overline{\phi} \phi = \text{diagram with two shaded circles connected by a dashed line} \xleftarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\mathcal{L} \supset |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$$\overbrace{\phi^* \phi} = \text{diagram with two shaded circles connected by a dashed line with an arrow pointing left labeled } p = \frac{i}{p^2 - m^2 + i\epsilon}$$

(External lines equal to 1 in both cases.)

Dirac Fermion

$$\mathcal{L} \supset \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi$$

Initial state

$$\overbrace{\psi}^{\text{in}} |p, s\rangle = \text{diagram of a particle with momentum } p \text{ and spin } s \text{ entering a vertex} = u^s(p)$$

$$\overline{\psi} |p, s\rangle = \text{diagram of an incoming fermion line with momentum } p \text{ and spin } s = \overline{v}^s(p)$$

Final state

$$\langle \overline{p}, s | \overline{\psi} = \text{---} \blacktriangleright \text{---} \textcircled{///} = \overline{u}^s(p)$$

$$\langle \overbrace{p, s} \mid \psi = \text{---} \blacktriangleleft \text{---} \bullet = v^s(p)$$

Propagator

$$\overline{\psi} \psi = \text{diagram} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

Photon

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$\overbrace{A_\mu}^{\text{ghost}} |p; \text{a}\rangle = \text{ghost} \text{ line with } p \leftarrow = \epsilon_\mu^{\text{a}}(p)$$

$$\langle p; \overline{\mathfrak{a}} | A_\mu = \text{wavy line with arrow } p \text{ pointing left} \text{ connected to a shaded circle} = \epsilon_\mu^{\overline{\mathfrak{a}*}}(p)$$

$$A_\mu \overbrace{A_\nu} = \text{diagram with two shaded vertices and a wavy line} \xleftarrow{p} = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$$

1.5 DIRAC'S GAMMA ALGEBRAS

1.5.1 Traces

$$\text{Tr}(\text{any odd \# of } \gamma\text{'s}) = 0 \quad (1.1)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} \quad (1.2)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}) \quad (1.3)$$

$$\text{Tr}(\gamma_5 \text{ and any odd \# of } \gamma\text{'s}) = 0 \quad (1.4)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma_5) = 0 \quad (1.5)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma} \quad (1.6)$$

Generally, for some γ -matrices A, B, C, \dots ,

$$\begin{aligned} \text{Tr}(ABCDEF \dots) &= \eta^{AB} \text{Tr}(CDEF \dots) - \eta^{AC} \text{Tr}(BDEF \dots) \\ &\quad + \eta^{AD} \text{Tr}(BCEF \dots) - \eta^{AE} \text{Tr}(BCDF \dots) + \dots, \end{aligned} \quad (1.7)$$

$$\text{Tr}(ABCDEF \dots \gamma_5) = \text{Not Established.} \quad (1.8)$$

To prove the second equation, we use following technique:

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots) = \text{Tr}(\dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu); \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots \gamma_5) = \text{Tr}(\gamma_5 \dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu). \quad (1.9)$$

1.5.2 Contractions

$$\gamma^\mu \gamma_\mu = 4 \quad (1.10)$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \quad (1.11)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} \quad (1.12)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \quad (1.13)$$

Generally, for some γ -matrices A, B, C, \dots ,

$$\text{ODD \# : } \gamma^\mu ABC \dots \gamma_\mu = -2(\dots CBA), \quad (1.14)$$

$$\text{EVEN \# : } \gamma^\mu ABC \dots \gamma_\mu = \text{Tr}(ABC \dots) - \text{Tr}(ABC \dots \gamma_5) \cdot \gamma_5. \quad (1.15)$$

Contractions in d -dimension

$$\gamma^\mu \gamma_\mu = d \quad (1.16)$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu \quad (1.17)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu \gamma^\rho \quad (1.18)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma \quad (1.19)$$

Contractions of ϵ 's

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu} \epsilon_{\alpha\beta\gamma\nu} = -6\delta_\nu^\mu; \quad \epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} = -2(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu) \quad (1.20)$$

$$\epsilon^{\mu\alpha\beta\gamma} \epsilon_{\mu\alpha'\beta'\gamma'} = - \left(\delta_\alpha^\alpha \delta_{\beta'}^\beta \delta_{\gamma'}^\gamma + \delta_{\beta'}^\alpha \delta_\gamma^\beta \delta_{\alpha'}^\gamma + \delta_{\gamma'}^\alpha \delta_\alpha^\beta \delta_{\beta'}^\gamma - \delta_\alpha^\alpha \delta_{\gamma'}^\beta \delta_{\beta'}^\gamma - \delta_{\beta'}^\alpha \delta_\alpha^\beta \delta_{\gamma'}^\gamma - \delta_{\gamma'}^\alpha \delta_{\beta'}^\beta \delta_{\alpha'}^\gamma \right) \quad (1.21)$$

1.6 MISCELLANEOUS TECHNIQUES

$$\begin{aligned}(p \cdot \sigma)(p \cdot \bar{\sigma}) &= p^2 \\ \epsilon^{ab}\epsilon^{cd} &= \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \\ \sqrt{p_\mu\sigma^\mu} &= \frac{p_\mu\sigma^\mu + m}{\sqrt{2(m+p^0)}}\end{aligned}$$

1.6.1 Dirac Field Techniques

$$\begin{aligned}\text{Dirac Equations} &: (\not{p} - m)u^s(p) = 0; \quad (\not{p} + m)v^s(p) = 0 \\ &: \bar{u}^s(p)(\not{p} - m) = 0; \quad \bar{v}^s(p)(\not{p} + m) = 0 \\ \text{Dirac Components} &: u^{r\dagger}(p)u^s(p) = 2E_p\delta^{rs}; \quad v^{r\dagger}(p)v^s(p) = 2E_p\delta^{rs} \\ &: \bar{u}^r(p)u^s(p) = 2m\delta^{rs}; \quad \bar{v}^r(p)v^s(p) = -2m\delta^{rs}; \quad \bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0 \\ \text{Spin Sums} &: \sum_{\text{spin}} u^s(p)\bar{u}^s(p) = \not{p} + m; \quad \sum_{\text{spin}} v^s(p)\bar{v}^s(p) = \not{p} - m\end{aligned}$$

1.6.2 Polarization Sum

Single photon case $M = \epsilon_\mu^*(k)M^\mu$

When Ward identity $k_\mu M^\mu = 0$ is valid,

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\nu(k)M^\mu M^{\nu*} = \eta_{\mu\nu}M^\mu M^{\nu*}. \quad (1.22)$$

Double photons case $M = \epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0$ is valid,

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\rho(k)\epsilon_\nu'^*(k')\epsilon_\sigma'(k')M^{\mu\nu}M^{\rho\sigma*} = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \quad (1.23)$$

【See Sec. B.3 for verbose information.】

1.6.3 Fierz identities

For Dirac spinors a, b, c, d and their left-handed projections $a_L := P_L a$ etc.,

$$(\bar{a}_L \gamma^\mu b_L)(\bar{c}_L \gamma_\mu d_L) = -(\bar{a}_L \gamma^\mu d_L)(\bar{c}_L \gamma_\mu b_L) \quad (1.24)$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}; \quad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \quad (1.25)$$

1.6.4 Gordon identity

For $P := p' + p$ and $q := p' - p$,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu + i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \quad \bar{u}(p')\gamma^\mu v(p) = \bar{u}(p')\left[\frac{q^\mu + i\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \quad (1.26)$$

$$\bar{v}(p')\gamma^\mu v(p) = -\bar{v}(p')\left[\frac{P^\mu + i\sigma^{\mu\nu}q_\nu}{2m}\right]v(p) \quad \bar{v}(p')\gamma^\mu u(p) = -\bar{v}(p')\left[\frac{q^\mu + i\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \quad (1.27)$$

1.6.5 Gauge group algebra

For a gauge group G s.t.

$$[t^a, t^b] = i f^{abc} t^c, \quad (1.28)$$

we have two constants which **depend on representation r** .

$$\text{Tr}(t^a t^b) =: C(r) \delta^{ab}; \quad \text{Tr}(t^a t^a) =: C_2(r) \cdot \mathbf{1} \quad (\text{quadratic Casimir operator}) \quad (1.29)$$

They satisfy

$$C(r) = \frac{d(r)}{d(\text{Adj.})} C_2(r), \quad t^a t^b t^a = \left[C_2(r) - \frac{1}{2} C_2(\text{Adj.}) \right] t^b. \quad (1.30)$$

$$f^{acd} f^{bcd} = C_2(\text{Adj.}) \delta^{ab}, \quad f^{abc} t^b t^c = \frac{1}{2} i C_2(\text{Adj.}) t^a. \quad (1.31)$$

For $\text{SU}(N)$ For $\text{SU}(N)$ groups and its fundamental representation N , we have

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}, \quad C(\text{Adj.}) = C_2(\text{Adj.}) = N; \quad (t^a)_{ij} (t^a)_{kj} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{\delta_{ij} \delta_{kl}}{N} \right).$$

1.7 LOOP INTEGRALS AND DIMENSIONAL REGULARIZATION

1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots x_n \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \quad (1.32)$$

$$\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{[x A_1 + (1-x) A_2]^2} \quad (1.33)$$

1.7.2 d -dimensional integrals in Minkowski space

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n i \Gamma(n - \frac{d}{2})}{(4\pi)^{d/2} \Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \quad (1.34)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (1.35)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \quad (1.36)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \quad (1.37)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \frac{\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}}{4} \quad (1.38)$$

Here we can use following expansions: $(\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = 1 - (d-4) \frac{\log \Delta}{2} + \mathcal{O}((d-4)^2) \quad \text{around } d = 4, \quad (1.39)$$

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x) \quad \text{around } x = 0, \quad (1.40)$$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left[\frac{1}{x+n} - \gamma + \sum_{k=1}^n \frac{1}{k} + \mathcal{O}(x+n) \right] \quad \text{around } x = -n. \quad (1.41)$$

and we get following expansion:

$$\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = \frac{1}{(4\pi)^2} \left[\left(\frac{2}{4-d} - \gamma + \log 4\pi \right) - \log \Delta + \mathcal{O}(4-d) \right]. \quad (1.42)$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S -matrix elements. To compute the S -matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \rightarrow \left(\frac{1}{\Delta - i\epsilon}\right)^{n - \frac{d}{2}}. \quad (1.43)$$

1.8 CROSS SECTIONS AND DECAY RATES

General expression (The mass dimension of \mathcal{M} is $2 - N_f$ for $d\sigma$ and $3 - N_f$ for $d\Gamma$.)

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left[\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(p_A, p_B \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \{p_f\}) \quad (1.44)$$

$$d\Gamma = \frac{1}{2m_A} \left[\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(m_A \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(m_A - \{p_f\}) \quad (\text{in } A\text{-rest frame.}) \quad (1.45)$$

2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} (2\pi)^4 \delta^{(4)}(E_{\text{cm}} - (p_1 + p_2)) \quad (\text{in center-of-mass frame}) \quad (1.46)$$

$$= \int \frac{d\Omega}{4\pi} \frac{1}{8\pi} \frac{2\|\mathbf{p}_1\|}{E_{\text{cm}}} \quad (1.47)$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{E_{\text{cm}}^2} + \frac{(m_1^2 - m_2^2)^2}{E_{\text{cm}}^4}} \xrightarrow{m_2=0} \frac{1}{8\pi} \left(1 - \frac{m_1^2}{E_{\text{cm}}^2} \right) \quad (1.48)$$

Kinematics of Decay

$$K \rightarrow p_1 + p_2 \quad \text{or} \quad \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{p^2 + m_1^2} \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} \sqrt{p^2 + m_2^2} \\ -\mathbf{p} \end{pmatrix}; \quad (1.49)$$

$$\|\mathbf{p}\|^2 = \frac{1}{4} \left[M^2 - 2(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{M^2} \right] \approx \left(\frac{M^2 - m_1^2}{2M} \right)^2$$

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M};$$

$$K \cdot p_1 = \frac{M^2 + m_1^2 - m_2^2}{2}, \quad p_1 \cdot p_2 = \frac{M^2 - (m_1^2 + m_2^2)}{2}.$$

- Fierz Transf.
- Noether current
- Majorana Fermions
- Feynman Rules(A.1)

2 Standard Model

We use the following notations for the gauge group.

Representation

$$\begin{aligned} \text{SU}(3) : \quad G_\mu &= G_\mu^a \tau^a ; \quad [\tau^a, \tau^b] = i f^{abc} \tau^c, \quad \text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}, \\ \text{SU}(2) : \quad W_\mu &= W_\mu^a T^a ; \quad [T^a, T^b] = i \epsilon^{abc} T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \\ \text{U}(1) : \quad B_\mu & \end{aligned}$$

Field Strength

$$\begin{aligned} F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] \\ &= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c) T^a \end{aligned}$$

Abridged Notation

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

2.1 SYMMETRIES AND FIELDS

Gauge Group

$$\text{SU}(3)_{\text{strong}} \times \text{SU}(2)_{\text{weak}} \times \text{U}(1)_Y \quad (2.1)$$

Field Content

	$\text{SU}(3)_{\text{strong}}$	$\text{SU}(2)_{\text{weak}}$	$\text{U}(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_L Q_i$: Left-handed quarks	3	2	1/6
$P_L U_i$: Right-handed up-type quarks	3	1	2/3
$P_R D_i$: Right-handed down-type quarks	3	1	-1/3
$P_R L_i$: Left-handed leptons	1	2	-1/2
$P_R E_i$: Right-handed leptons	1	1	-1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	1	2	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	8	1	0
W : Weak bosons	1	3	0
B : B boson	1	1	0

Full Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{湯川}}, \quad (2.2)$$

$$\text{where } \mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W_{\mu\nu}^a - \frac{1}{4}G^{a\mu\nu}G_{\mu\nu}^a \quad (2.3)$$

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_\mu - ig_2 W_\mu - \frac{1}{2}ig_1 B_\mu \right) H \right|^2 - V(H), \quad (2.4)$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = & \bar{Q}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu - ig_2 W_\mu - \frac{1}{6}ig_1 B_\mu \right) P_L Q_i \\ & + \bar{U}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu - \frac{2}{3}ig_1 B_\mu \right) P_R U_i \\ & + \bar{D}_i i\gamma^\mu \left(\partial_\mu - ig_3 G_\mu + \frac{1}{3}ig_1 B_\mu \right) P_R D_i \\ & + \bar{L}_i i\gamma^\mu \left(\partial_\mu - ig_2 W_\mu + \frac{1}{2}ig_1 B_\mu \right) P_L L_i \\ & + \bar{E}_i i\gamma^\mu (\partial_\mu + ig_1 B_\mu) P_R E_i, \end{aligned} \quad (2.5)$$

$$\mathcal{L}_{\text{湯川}} = -\bar{U}_i (y_u)_{ij} H P_L Q_j + \bar{D}_i (y_d)_{ij} H^\dagger P_L Q_j + \bar{E}_i (y_e)_{ij} H^\dagger P_L L_j + \text{H. c.} \quad (2.6)$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix B.4.

Gauge Kinetic Terms

the gauge kinetic terms can be expanded as

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = & -\frac{1}{4}(\partial B)(\partial B) \\ & -\frac{1}{4}(\partial W^a)(\partial W^a) - g_2 \epsilon^{abc} (\partial_\mu W_\nu^a) W^{\mu b} W^{\nu c} - \frac{g_2^2}{4} (\epsilon^{eab} W_\mu^a W_\nu^b) (\epsilon^{ecd} W^{c\mu} W^{d\nu}) \\ & -\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc} (\partial_\mu G_\nu^a) G^{\mu b} G^{\nu c} - \frac{g_3^2}{4} (f^{eab} G_\mu^a G_\nu^b) (f^{ecd} G^{c\mu} G^{d\nu}). \end{aligned} \quad (2.7)$$

2.2 HIGGS MECHANISM

Higgs Potential

The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2 (H^\dagger H) + \lambda (H^\dagger H)^2. \quad (2.8)$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ v + (h + i\phi_3) \end{pmatrix}, \quad \text{where } v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (2.9)$$

Here h is the ‘‘Higgs boson,’’ and ϕ_i are 南部-Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2} h^4 + \frac{\mu^2}{v} h^3 + \mu^2 h^2, \quad (2.10)$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_\mu - ig_2 W_\mu - \frac{1}{2}ig_1 B_\mu \right) H \right|^2 \quad (2.11)$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8} \left[g_2^2 W_1^2 + g_2^2 W_2^2 + (g_1 B - g_2 W_3)^2 \right]. \quad (2.12)$$

Redefining the gauge fields (with concerning the norms) as

$$W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} := \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \quad (2.13)$$

where

$$\tan \theta_w := \frac{g_1}{g_2}, \quad e := -\frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}; \quad g_Z := \sqrt{g_1^2 + g_2^2}; \quad (2.14)$$

$$g_1 = \frac{|e|}{\cos \theta_w} = g_Z \sin \theta_w, \quad g_2 = \frac{|e|}{\sin \theta_w} = g_Z \cos \theta_w. \quad (2.15)$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4} \left[g_2^2 W^{+\mu} W_\mu^- + \frac{g_Z^2}{2} Z^\mu Z_\mu \right]. \quad (2.16)$$

Here we have omitted the 南部-Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan \theta_w Z_\mu, \quad (2.17)$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) + \left(\frac{|e|}{\tan \theta_w} Z_\mu + |e| A_\mu \right) T^3, \quad (2.18)$$

$$Z_\mu^0 := \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_2 W_\mu^3 - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_1 W_\mu^3 + g_2 B_\mu) \quad (2.19)$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2}v, \quad m_Z := \frac{g_Z}{2}v. \quad (2.20)$$

Gauge Term The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W_{\mu\nu}^a &= (\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ &\quad - 4ig \left[(\partial W^3)^{\mu\nu} W_\mu^+ W_\nu^- + (\partial W^+)^{\mu\nu} W_\mu^- W_\nu^3 + (\partial W^-)^{\mu\nu} W_\mu^3 W_\nu^+ \right] \\ &\quad - 2g^2 (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) (W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- - 2W_\mu^3 W_\nu^3 W_\rho^+ W_\sigma^-), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &:= -\frac{1}{4} \left[G^{a\mu\nu} G_{\mu\nu}^a + (\partial Z)^{\mu\nu} (\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu} (\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu} (\partial W^-)_{\mu\nu} \right] \\ &\quad + \frac{i|e|}{\tan \theta_w} \left[(\partial W^+)^{\mu\nu} W_\mu^- Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W_\nu^+ + (\partial Z)^{\mu\nu} W_\mu^+ W_\nu^- \right] \\ &\quad + i|e| \left[(\partial W^+)^{\mu\nu} W_\mu^- A_\nu + (\partial W^-)^{\mu\nu} A_\mu W_\nu^+ + (\partial A)^{\mu\nu} W_\mu^+ W_\nu^- \right] \\ &\quad + (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) \left[\frac{|e|^2}{2 \sin^2 \theta_w} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- + \frac{|e|^2}{\tan^2 \theta_w} W_\mu^+ Z_\nu W_\rho^- Z_\sigma \right. \\ &\quad \left. + \frac{|e|^2}{\tan \theta_w} (W_\mu^+ Z_\nu W_\rho^- A_\sigma + W_\mu^+ A_\nu W_\rho^- Z_\sigma) + |e|^2 W_\mu^+ A_\nu W_\rho^- A_\sigma \right]. \end{aligned} \quad (2.21)$$

湯川 Term

$$\begin{aligned}
\mathcal{L}_{\text{湯川}} &= -\bar{U}y_u H P_L Q + \bar{D}y_d H^\dagger P_L Q + \bar{E}y_e H^\dagger P_L L + \text{H. c.} \\
&= -\bar{U}y_u \epsilon^{\alpha\beta} H^\alpha P_L Q^\beta + \bar{D}y_d H^{\dagger\alpha} P_L Q^\alpha + \bar{E}y_e H^{\dagger\alpha} P_L L^\alpha + \text{H. c.} \\
&= \frac{v+h}{\sqrt{2}} (\bar{U}y_u P_L Q^1 + \bar{D}y_d P_L Q^2 + \bar{E}y_e P_L L^2) + \text{H. c.}
\end{aligned} \tag{2.22}$$

2.3 FULL LAGRANGIAN AFTER HIGGS MECHANISM

Now we have the following Lagrangian (with omitting P_L etc.):

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + m_W^2 W^+ W^- + \frac{m_Z^2}{2} Z^2 \\
\text{【Higgs】} &+ \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}\mu^2 h^2 - \sqrt{\frac{\lambda}{2}}m_h h^3 - \frac{1}{4}\lambda h^4 \\
&+ \frac{v g_2^2}{4} W^+ W^- h + \frac{v(g_1^2 + g_2^2)}{8} Z^2 h \\
&+ \frac{g_2^2}{4} W^+ W^- h^2 + \frac{g_1^2 + g_2^2}{8} Z^2 h^2 \\
&+ \frac{1}{\sqrt{2}} h \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} h \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} h \bar{E} y_e L^2 + \text{H. c.} \\
\text{【SU(3)】} &+ \bar{Q} (i\not{\partial} + g_3 \not{G}) Q + \bar{U} (i\not{\partial} + g_3 \not{G}) U + \bar{D} (i\not{\partial} + g_3 \not{G}) D + \bar{L} (i\not{\partial}) L + \bar{E} (i\not{\partial}) E \\
\text{【W】} &+ \bar{Q} \frac{g_2}{\sqrt{2}} (W^+ T^+ + W^- T^-) Q + \bar{L} \frac{g_2}{\sqrt{2}} (W^+ T^+ + W^- T^-) L \\
\text{【A\&Z^0】} &+ \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| A + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) Z^0 \right] Q \\
&+ \bar{U} \left(\frac{2}{3} |e| A - \frac{2|e|s}{3c} Z \right) U \\
&+ \bar{D} \left(-\frac{1}{3} |e| A + \frac{|e|s}{3c} Z \right) D \\
&+ \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| A + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) Z^0 \right] L \\
&+ \bar{E} \left(-|e| A + \frac{|e|s}{c} Z \right) E \\
\text{【湯川項】} &+ \frac{1}{\sqrt{2}} v \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} v \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} v \bar{E} y_e L^2 + \text{H. c.}
\end{aligned} \tag{2.23}$$

2.4 MASS EIGENSTATES

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := v y_\bullet / \sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \quad (m_i \geq 0). \tag{2.24}$$

Using this Ψ and Φ , we can rotate the basis as

$$Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \quad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \quad (2.25)$$

and now we have the 湯川 terms in mass eigenstates as

$$\mathcal{L}_{\text{湯川}} = \left(1 + \frac{1}{v}h\right) [(m_u)_i \bar{U}_i P_L Q_i^1 + (m_d)_i \bar{D}_i P_L Q_i^2 + (m_e)_i \bar{E}_i P_L L_i^2 + \text{H. c.}]. \quad (2.26)$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark– W interactions do change drastically, as

$$\mathcal{L} \supset \bar{Q} i \gamma^\mu \left(-ig_2 W_\mu - \frac{1}{6} ig_1 B_\mu \right) P_L Q \quad (2.27)$$

$$= \bar{Q} \frac{g_2}{\sqrt{2}} \left(W^+ T^+ + W^- T^- \right) P_L Q + (\text{interaction terms with } Z \text{ and } A) \quad (2.28)$$

$$\mapsto \frac{g_2}{\sqrt{2}} \begin{pmatrix} \bar{Q}^1 \Psi_u & \bar{Q}^2 \Psi_d \end{pmatrix} \begin{pmatrix} 0 & W^+ \\ W^- & 0 \end{pmatrix} P_L \begin{pmatrix} \Psi_u^\dagger Q^1 \\ \Psi_d^\dagger Q^2 \end{pmatrix} + (\dots) \quad (2.29)$$

$$= \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 W^- X P_L Q^1 + \bar{Q}^1 W^+ X^\dagger P_L Q^2 \right] + (\dots), \quad (2.30)$$

where $X := \Psi_d \Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP -symmetry.

In our notation, CP -transformation of a spinor is described as

$$\mathcal{CP}(\psi) = -i\eta^*(\bar{\psi}\gamma^2)^\text{T}, \quad \mathcal{CP}(\bar{\psi}) = i\eta(\gamma^2\psi)^\text{T}, \quad (2.31)$$

where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,

$$\begin{aligned} \mathcal{CP}(\bar{Q}^2 W^- X P_L Q^1) &= (\gamma^2 Q^2)^\text{T} \mathcal{P}(-W^+) X P_L (\bar{Q}^1 \gamma^2)^\text{T} \\ &= -W_\mu^{+P} (\gamma^2 Q^2)^\text{T} (\bar{Q}^1 X^\text{T} \gamma^2 P_L \gamma^\mu)^\text{T} \\ &= (\bar{Q}^1 W^+ X^\text{T} P_L Q^2). \end{aligned} \quad (2.32)$$

Therefore, we can see that the CP -symmetry is maintained if and only if $X^\text{T} = X^\dagger$, that is, if and only if X is a real matrix.

以上より，標準模型の Lagrangian は

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{gauge}} \\
& \text{【質量項】} + m_W^2 W^+ W^- + \frac{m_Z^2}{2} Z^2 \\
& + \bar{U} M_u P_L Q^1 + \bar{D} M_d P_L Q^2 + \bar{E} M_e P_L L^2 + \text{H. c.} \\
& \text{【Higgs Field】} + \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} \mu^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4} \lambda h^4 \\
& \text{【Higgs との結合】} + \frac{v g_2^2}{4} W^+ W^- h + \frac{v (g_1^2 + g_2^2)}{8} Z^2 h \\
& + \frac{g_2^2}{4} W^+ W^- h^2 + \frac{g_1^2 + g_2^2}{8} Z^2 h^2 \\
& + \frac{1}{v} \bar{U} M_u P_L Q^1 h + \frac{1}{v} \bar{D} M_d P_L Q^2 h + \frac{1}{v} \bar{E} M_e P_L L^2 h + \text{H. c.} \\
& \text{【SU(3) および微分項】} + \bar{Q} (\mathrm{i}\not{\partial} + g_3 \not{G}) P_L Q + \bar{U} (\mathrm{i}\not{\partial} + g_3 \not{G}) P_R U + \bar{D} (\mathrm{i}\not{\partial} + g_3 \not{G}) P_R D \\
& + \bar{L} (\mathrm{i}\not{\partial}) P_L L + \bar{E} (\mathrm{i}\not{\partial}) P_R E \\
& \text{【W boson】} + \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \not{W}^- X P_L Q^1 + \bar{Q}^1 \not{W}^+ X^\dagger P_L Q^2 \right] \quad \text{【CP and flavor violating!】} \\
& + \bar{L} \frac{g_2}{\sqrt{2}} \left(\not{W}^+ T^+ + \not{W}^- T^- \right) P_L L \\
& \text{【A\&Z}^0 \text{ boson】} + \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| \not{A} + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) \not{Z}^0 \right] P_L Q \\
& + \bar{U} \left(\frac{2}{3} |e| \not{A} - \frac{2|e|s}{3c} \not{Z} \right) P_R U \\
& + \bar{D} \left(-\frac{1}{3} |e| \not{A} + \frac{|e|s}{3c} \not{Z} \right) P_R D \\
& + \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| \not{A} + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) \not{Z}^0 \right] P_L L \\
& + \bar{E} \left(-|e| \not{A} + \frac{|e|s}{c} \not{Z} \right) P_R E
\end{aligned} \tag{2.33}$$

となる。

2.5 CHIRAL NOTATION

以上の Lagrangian を chiral 表示で表すと、まず最初は

$$\begin{aligned}
\mathcal{L} = & (\text{Higgs terms}) + (\text{Gauge fields strength}) \\
& + Q_L^\dagger i\bar{\sigma}^\mu \left(\partial_\mu - ig_3 G_\mu - ig_2 W_\mu - \frac{1}{6} ig_1 B_\mu \right) Q_L \\
& + U_R^\dagger i\sigma^\mu \left(\partial_\mu - ig_3 G_\mu - \frac{2}{3} ig_1 B_\mu \right) U_R \\
& + D_R^\dagger i\sigma^\mu \left(\partial_\mu - ig_3 G_\mu + \frac{1}{3} ig_1 B_\mu \right) D_R \\
& + L_L^\dagger i\bar{\sigma}^\mu \left(\partial_\mu - ig_2 W_\mu + \frac{1}{2} ig_1 B_\mu \right) L_L \\
& + E_R^\dagger i\sigma^\mu (\partial_\mu + ig_1 B_\mu) E_R \\
& + U_R^\dagger y_u H Q_L + D_R^\dagger y_d H^\dagger Q_L + E_R^\dagger y_e H^\dagger L_L + \text{H. c.} \\
= & (\text{Higgs terms}) + (\text{Gauge fields strength}) \\
& + iQ_L^\dagger \bar{\sigma}^\mu \partial_\mu Q_L + iU_R^\dagger \bar{\sigma}^\mu \partial_\mu U_R + iD_R^\dagger \bar{\sigma}^\mu \partial_\mu D_R + iL_L^\dagger \bar{\sigma}^\mu \partial_\mu L_L + iE_R^\dagger \bar{\sigma}^\mu \partial_\mu E_R \\
& + g_3 \left(Q_L^\dagger \bar{\sigma}^\mu G_\mu Q_L + U_R^\dagger \bar{\sigma}^\mu G_\mu U_R + D_R^\dagger \bar{\sigma}^\mu G_\mu D_R \right) \\
& + g_2 \left(Q_L^\dagger \bar{\sigma}^\mu W_\mu Q_L + L_L^\dagger \bar{\sigma}^\mu W_\mu L_L \right) \\
& + g_1 \left(\frac{1}{6} Q_L^\dagger \bar{\sigma}^\mu B_\mu Q_L + \frac{2}{3} U_R^\dagger \bar{\sigma}^\mu B_\mu U_R - \frac{1}{3} D_R^\dagger \bar{\sigma}^\mu B_\mu D_R - \frac{1}{2} L_L^\dagger \bar{\sigma}^\mu B_\mu L_L - E_R^\dagger \bar{\sigma}^\mu B_\mu E_R \right) \\
& + U_R^\dagger y_u H Q_L + D_R^\dagger y_d H^\dagger Q_L + E_R^\dagger y_e H^\dagger L_L + \text{H. c.} \tag{2.34}
\end{aligned}$$

であり、そして最終的には

$$\begin{aligned}
\mathcal{L} = & (\text{Gauge bosons and Higgs}) \\
& + iQ_L^\dagger \bar{\sigma}^\mu \partial_\mu Q_L + iU_R^\dagger \bar{\sigma}^\mu \partial_\mu U_R + iD_R^\dagger \bar{\sigma}^\mu \partial_\mu D_R + iL_L^\dagger \bar{\sigma}^\mu \partial_\mu L_L + iE_R^\dagger \bar{\sigma}^\mu \partial_\mu E_R \\
& + g_3 \left(Q_L^\dagger \bar{\sigma}^\mu G_\mu Q_L + U_R^\dagger \bar{\sigma}^\mu G_\mu U_R + D_R^\dagger \bar{\sigma}^\mu G_\mu D_R \right) \\
& + m_u (u_R^\dagger u_L + u_L^\dagger u_R) + (\text{quarks}) + m_e (e_R^\dagger e_L + e_L^\dagger e_R) + (\text{leptons}) \\
& + \frac{m_u}{v} (u_R^\dagger u_L + u_L^\dagger u_R) h + (\text{quarks}) + \frac{m_e}{v} (e_R^\dagger e_L + e_L^\dagger e_R) h + (\text{leptons}) \\
& + \frac{g_2}{\sqrt{2}} \left[(d_L^\dagger \ s_L^\dagger \ b_L^\dagger) \bar{\sigma}^\mu W_\mu^- X \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} + (u_L^\dagger \ c_L^\dagger \ t_L^\dagger) \bar{\sigma}^\mu W_\mu^+ X^\dagger \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \right] \\
& + \frac{g_2}{\sqrt{2}} \left[\nu_e^\dagger \bar{\sigma}^\mu W_\mu^+ e_L + e_L^\dagger \bar{\sigma}^\mu W_\mu^- \nu_e \right] \\
& + |e| \left[\frac{2}{3} u_L^\dagger \bar{\sigma}^\mu A_\mu u_L - \frac{1}{3} d_L^\dagger \bar{\sigma}^\mu A_\mu d_L + \frac{2}{3} u_R^\dagger \sigma^\mu A_\mu u_R - \frac{1}{3} d_R^\dagger \sigma^\mu A_\mu d_R + (\text{quarks}) \right. \\
& \quad \left. - e_L^\dagger \bar{\sigma}^\mu A_\mu e_L - e_R^\dagger \sigma^\mu A_\mu e_R + (\text{leptons}) \right] \\
& + \frac{|e|s}{c} \left[\left(\frac{c^2}{2s^2} - \frac{1}{6} \right) u_L^\dagger \bar{\sigma}^\mu Z_\mu u_L - \left(\frac{c^2}{2s^2} + \frac{1}{6} \right) d_L^\dagger \bar{\sigma}^\mu Z_\mu d_L - \frac{2}{3} u_R^\dagger \sigma^\mu Z_\mu u_R + \frac{1}{3} d_R^\dagger \sigma^\mu Z_\mu d_R \right. \\
& \quad \left. + \left(\frac{c^2}{2s^2} + \frac{1}{2} \right) \nu_e^\dagger \bar{\sigma}^\mu Z_\mu \nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2} \right) e_L^\dagger \bar{\sigma}^\mu Z_\mu e_L + e_R^\dagger \sigma^\mu Z_\mu e_R + (\text{others}) \right] \tag{2.35}
\end{aligned}$$

となる。

2.6 VALUES OF SM PARAMETERS

2.6.1 Experimental Values

Low energy values

$$\alpha_{\text{EM}} = 1/137.035999679(94) \quad G_F = 1.166367(5) \times 10^{-5} \text{ GeV}^{-2}$$

Electroweak scale 【These values are all in $\overline{\text{MS}}$ scheme.】

$$\begin{aligned} \alpha_{\text{EM}}^{-1}(m_Z) &= 127.925(16) & m_W(m_W) &= 80.398(25) \text{ GeV} \\ \alpha_{\text{EM}}^{-1}(m_\tau) &= 133.452(16) & m_Z(m_Z) &= 91.1876(21) \text{ GeV} \\ \alpha_s(m_Z) &= 0.1176(20) & \sin^2 \theta_W(m_Z) &= 0.23119(14) \\ \Gamma_{l+l-} &= 83.984(86) \text{ MeV} & \sin^2 \theta_{\text{eff}} &= 0.23149(13) \end{aligned}$$

Fundamental masses

$$\begin{aligned} e &: 0.510998910(13) \text{ MeV} & u &: 1.5 \text{ to } 3.3 \text{ MeV} & d &: 3.5 \text{ to } 6.0 \text{ MeV} \\ \mu &: 105.658367(4) \text{ MeV} & c &: 1.27_{-0.11}^{+0.07} \text{ GeV} & s &: 104_{-34}^{+26} \text{ MeV} \\ \tau &: 1.77784(17) \text{ GeV} & t &: 171.2_{\pm 2.1} \text{ GeV} & b &: 4.20_{-0.07}^{+0.17} \text{ GeV} \\ \pi^\pm &: 139.57018(35) \text{ MeV} & K^\pm &: 493.677(16) \text{ MeV} & p &: 938.27203(8) \text{ MeV} \\ \pi^0 &: 139.766(6) \text{ MeV} & K^0 &: 497.614(24) \text{ MeV} & n &: 939.56536(8) \text{ MeV} \end{aligned}$$

Fundamental Lifetime (also $c\tau$ for some particles)

$$\begin{aligned} \mu &: 2.197019(21) \mu\text{s} \quad (658 \text{ m}) & \pi^\pm &: 2.6033(5) \times 10^{-8} \text{ s} & K^\pm &: 1.2380(21) \times 10^{-8} \text{ s} \\ \tau &: 2.906(10) \times 10^{-13} \text{ s} \quad (87 \mu\text{m}) & \pi^0 &: 8.4(6) \times 10^{-17} \text{ s} & K_S^0 &: 8.953(5) \times 10^{-11} \text{ s} \\ & & & & K_L^0 &: 5.116(20) \times 10^{-8} \text{ s} \end{aligned}$$

CKM matrix

$$V_{\text{CKM}} = \begin{pmatrix} 0.97419(22) & 0.2257(10) & 0.00359(16) \\ 0.2256(10) & 0.97334(23) & 0.0415(11) \\ 0.00874(37) & 0.0407(10) & 0.999133(44) \end{pmatrix} \sim \begin{pmatrix} 1 - \epsilon^2 & \epsilon & \epsilon^4 \\ \epsilon & 1 - \epsilon^2 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 - \epsilon^4 \end{pmatrix} \quad \text{for } \epsilon \sim 0.23 \quad (2.36)$$

2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$e \sim 0.313, \quad g_1 \sim 0.358, \quad g_2 \sim 0.651; \quad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246 \text{ GeV} \quad (2.37)$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.98 \end{pmatrix}, \quad y_d \sim \begin{pmatrix} 3 \times 10^{-5} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad y_e \sim \begin{pmatrix} 3 \times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \quad (2.38)$$

Also, for $m_h \sim 120 \text{ GeV}$, we can estimate the Higgs potential as $\mu \sim 85 \text{ GeV}$ and $\lambda \sim 0.12$.

3 楊-Mills Theory

3.1 U(1) THEORY

3.1.1 General SU(N)

$$\mathcal{L} = \quad (3.1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{i}[A_\mu, A_\nu] \quad (3.2)$$

$$(3.3)$$

Gauge Transformation

For any Lie group G ,

$$V : \mathbb{R}^{1,3} \rightarrow G \quad (3.4)$$

$$A_\mu \mapsto V \left(A_\mu + \frac{i}{g} \partial_\mu \right) V^{-1} \quad (3.5)$$

$$F_{\mu\nu} \mapsto V F_{\mu\nu} V^{-1} \quad (3.6)$$

If the gauge group G is **compact**, it has a finite-dimensional unitary representation.

$$\text{For } t^a : \text{hermitian representation,} \quad (3.7)$$

$$[t^a, t^b] = i f^{ab}{}_c t^c \quad \text{and} \quad f \in \mathbb{R} \quad (3.8)$$

$$0 = f^D{}_{ab} f^E{}_{Dc} + f^D{}_{ca} f^E{}_{Db} + f^D{}_{bc} f^E{}_{Da} \quad (3.9)$$

$$V = \exp[i\alpha^a T^a] \quad \text{for } \alpha^a \in \mathbb{R} \quad (3.10)$$

$$(3.11)$$

with generators $\{T^a\}$ written in an hermitian representation,

$$A_\mu \mapsto V \left(A_\mu + \frac{1}{g} (\partial_\mu \alpha^a) T^a \right) V^{-1} \quad (3.12)$$

$$\simeq A_\mu + \frac{1}{g} (\partial_\mu \alpha^a) T^a + i\alpha^a A_\mu^b [T^a, T^b] \quad (3.13)$$

$$D_\mu = \partial_\mu - ig A_\mu^a T^a \quad (\text{for appropriate representation}) \quad (3.14)$$

4 Spinor

$\eta^{\mu\nu} = (-, +, +, +)$ case

Grassmann Number : $(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$

: \implies for $a, b \in \mathbb{G}^{\mathbb{R}}, ab \in i\mathbb{G}^{\mathbb{R}}$

γ matrix : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$

: $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ etc...

: $(i\gamma^0)^\dagger := i\gamma^0, \quad \gamma^{i\dagger} := \gamma^i$

Dirac Conjugate : $\bar{\psi} = i\psi^\dagger \gamma^0$

5 Supersymmetry for $\eta = \text{diag}(+, -, -, -)$

5.1 SPINOR CONVENTION

ϵ tensor : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)
 Sum Rule : ${}^\alpha_\alpha$ and ${}^{\dot{\alpha}}_{\dot{\alpha}}$, except for $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$, $\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta$, $\xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}$, $\xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}$.
 Lorentz 变换 : $\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$, $\bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dot{\beta}}{}_{\dot{\alpha}}$, $\psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha$, $\bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}$.
 σ matrices : $(\sigma^\mu)_{\alpha\dot{\beta}} := (1, \sigma)_{\alpha\dot{\beta}}$, $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\beta\alpha}(\sigma^\mu)_{\alpha\dot{\beta}} = (1, -\sigma)^{\dot{\alpha}\beta}$.

(See App. B.1.1 for a verbose explanation.)

5.2 SPINOR CALCULATION CHEATSHEET

$$\begin{aligned}
 \eta &= (+, -, -, -), & \epsilon^{0123} &= -\epsilon_{0123} = 1 \\
 \epsilon^{12} &= \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, & \xi_\alpha &:= \epsilon_{\alpha\beta}\xi^\beta, & \xi_\alpha &= \epsilon_{\alpha\beta}\xi^\beta, & \bar{\xi}_{\dot{\alpha}} &:= \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}, & \bar{\xi}_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}} \\
 \bar{\sigma}^{\mu\dot{\alpha}\alpha} &:= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}, & \sigma^\mu_{\alpha\dot{\alpha}} &= \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}, & \sigma^\mu &:= (1, \sigma), & \bar{\sigma}^\mu &:= (1, -\sigma) \\
 (\sigma^{\mu\nu})_{\alpha}{}^\beta &:= \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_{\alpha}{}^\beta, & (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &:= \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = (\sigma^{\nu\mu})^{\dagger\dot{\alpha}}{}_{\dot{\beta}}.
 \end{aligned}$$

$$\begin{aligned}
 \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta & (\theta\phi)(\theta\psi) &= -\frac{1}{2}(\psi\phi)(\theta\theta) & (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) &= \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu} \\
 \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} & (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) &= -\frac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta}) \\
 \sigma^\mu\bar{\sigma}^\nu &= \eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar{\sigma}^\rho\sigma^\nu + \sigma^\nu\bar{\sigma}^\rho\sigma^\mu &= 2(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \\
 \bar{\sigma}^\mu\sigma^\nu &= \eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu} & \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu &= 2(\eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\nu}\bar{\sigma}^\rho) \\
 \sigma^{\mu\nu} &= -\sigma^{\nu\mu} & \sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu &= 2i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\
 \bar{\sigma}^{\mu\nu} &= -\bar{\sigma}^{\nu\mu} & \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu &= -2i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \\
 \text{Tr } \bar{\sigma}^\mu\sigma^\nu &= \text{Tr } \sigma^\mu\bar{\sigma}^\nu = 2\eta^{\mu\nu} & \text{Tr } \sigma^{\mu\nu}\sigma^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \\
 \text{Tr } \sigma^{\mu\nu} &= \text{Tr } \bar{\sigma}^{\mu\nu} = 0 & \text{Tr } \bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} &= -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \\
 \sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu &= 2\delta_{\alpha}^{\dot{\beta}}\delta_{\dot{\alpha}}^\beta & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} - \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= 2\left[(\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right] \\
 \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\mu\beta\dot{\beta}} &= 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} & \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} + \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} &= \eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - 4\eta_{\rho\sigma}(\sigma^{\rho\mu})_{\alpha\beta}(\epsilon\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}} \\
 \bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_\mu &= 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} & \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} &= 2i\sigma^{\mu\nu} \\
 \sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} &= \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\sigma}^\mu_{\dot{\beta}\dot{\alpha}} & \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} &= -2i\bar{\sigma}^{\mu\nu} \\
 \xi\bar{\sigma}^\mu\chi &= -\chi\sigma^\mu\bar{\xi} = (\bar{\chi}\bar{\sigma}^\mu\xi)^* = -(\xi\sigma^\mu\bar{\chi})^* & (\psi\phi)\chi_\alpha &= -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\
 \xi\sigma^\mu\bar{\sigma}^\nu\chi &= \chi\sigma^\nu\bar{\sigma}^\mu\xi = (\bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\xi})^* = (\bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi})^* & (\psi\phi)\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma^\mu)_{\dot{\alpha}}
 \end{aligned}$$

In the following equations, we chose left-differential notation.

$$\begin{aligned}
 \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} &= -\frac{\partial}{\partial\theta_\alpha} & \frac{\partial}{\partial\theta^\alpha}\theta\theta &= 2\theta_\alpha & \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta &= 4 \\
 \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} &= -\frac{\partial}{\partial\theta^\alpha} & \frac{\partial}{\partial\theta_\alpha}\theta\theta &= -2\theta^\alpha & \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta &= -4 \\
 \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= -2\bar{\theta}_{\dot{\alpha}} & \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} &= 4 \\
 \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} & \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} &= 2\bar{\theta}^{\dot{\alpha}} & \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} &= -4
 \end{aligned}$$

5.3 GENERAL RELATIONS

$$\begin{aligned}
Q_\alpha &:= \frac{\partial}{\partial \theta^\alpha} - \mathrm{i}(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, & D_\alpha &:= \frac{\partial}{\partial \theta^\alpha} + \mathrm{i}(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, & y &:= x + \mathrm{i}\theta\sigma\bar{\theta}, \\
\bar{Q}_{\dot{\alpha}} &:= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \mathrm{i}(\theta\sigma^\mu)_{\dot{\alpha}} \partial_\mu, & \bar{D}_{\dot{\alpha}} &:= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \mathrm{i}(\theta\sigma^\mu)_{\dot{\alpha}} \partial_\mu, & y^\dagger &:= x - \mathrm{i}\theta\sigma\bar{\theta} \\
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\mathrm{i}\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, & \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= -2\mathrm{i}\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, & (\text{others}) &= 0. \\
\bar{D}_\alpha(y, \theta, \bar{\theta}) &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} & D_\alpha(y, \theta, \bar{\theta}) &= \frac{\partial}{\partial \theta^\alpha} + 2\mathrm{i}(\sigma^\mu \bar{\theta}) \frac{\partial}{\partial y^\mu} \\
D_\alpha(y^\dagger, \theta, \bar{\theta}) &= \frac{\partial}{\partial \theta^\alpha} & \bar{D}_{\dot{\alpha}}(y^\dagger, \theta, \bar{\theta}) &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - 2\mathrm{i}(\theta\sigma^\mu) \frac{\partial}{\partial (y^\dagger)^\mu}
\end{aligned}$$

5.4 CONVENTIONS

Martin (SUSY Primer)	$\epsilon_\alpha = \sqrt{2}\xi_\alpha$	Wess&Bagger (SUSY and SUGRA)
$\delta\phi = \epsilon\psi$	same	$\delta A = \sqrt{2}\xi\psi$
$\delta\psi_\alpha = \mathrm{i}(\sigma^\mu \epsilon^\dagger)_\alpha D_\mu \phi + \epsilon_\alpha F$	same	$\delta\psi = \mathrm{i}\sqrt{2}\sigma^\mu \bar{\xi} D_\mu A + \sqrt{2}\xi F$
$\delta F = \mathrm{i}\epsilon^\dagger \bar{\sigma}^\mu D_\mu \psi + \sqrt{2}g(T^a \phi)\epsilon^\dagger \lambda^{\dagger a}$		
$\delta A_\mu = \frac{1}{\sqrt{2}}(\epsilon^\dagger \bar{\sigma}^\mu \lambda^a + \lambda^{\dagger a} \bar{\sigma}^\mu \epsilon)$	$\lambda_M = \mathrm{i}\lambda_{\text{WB}}$	$\delta v_\mu = \mathrm{i}(\xi^\dagger \bar{\sigma}^\mu \lambda^a - \lambda^{\dagger a} \bar{\sigma}^\mu \xi)$
$\delta\lambda = \frac{\mathrm{i}}{2\sqrt{2}}(\sigma^\mu \sigma^\nu \epsilon)F_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon D$		
$\delta D = \frac{\mathrm{i}}{\sqrt{2}}(\epsilon^\dagger \bar{\sigma}^\mu D_\mu \lambda - D_\mu \lambda^{\dagger} \bar{\sigma}^\mu \epsilon)$		

5.5 CHIRAL SUPERFIELDS

Definition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad (5.1)$$

Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad (5.2)$$

$$= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\theta}\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) \quad (5.3)$$

$$\Phi^\dagger = \phi^*(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x) \quad (5.4)$$

Product of Chiral Superfields

$$\begin{aligned} \Phi_i^\dagger\Phi_j(x, \theta, \bar{\theta}) &= \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\ &\quad + i\theta\sigma^\mu\bar{\theta}(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j) + 2\bar{\theta}\bar{\psi}_i\theta\psi_j \\ &\quad - \frac{i}{\sqrt{2}}\theta\theta(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j)\sigma^\mu\bar{\theta} + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j \\ &\quad + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu(\partial_\mu\bar{\psi}_i\phi_j - \bar{\psi}_i\partial_\mu\phi_j) + \sqrt{2}\bar{\theta}\bar{\theta}F_i^*\theta\psi_j \end{aligned} \quad (5.5)$$

$$\begin{aligned} &+ \theta\theta\bar{\theta}\bar{\theta}\left[F_i^*F_j - \frac{1}{4}\phi_i^*\partial^2\phi_j - \frac{1}{4}\partial^2\phi_i^*\phi_j + \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j + \frac{i}{2}\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_j - \frac{i}{2}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j\right] \\ &\rightsquigarrow \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\ &\quad + i\theta\sigma^\mu\bar{\theta}(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j) + 2\bar{\theta}\bar{\psi}_i\theta\psi_j \\ &\quad + \sqrt{2}\theta\theta\bar{\theta}(\bar{\psi}_iF_j - i\bar{\sigma}^\mu\psi_j\partial_\mu\phi_i^*) + \sqrt{2}\bar{\theta}\bar{\theta}\theta(\psi_jF_i^* - i\sigma^\mu\bar{\psi}_i\partial_\mu\phi_j) \\ &\quad + \theta\theta\bar{\theta}\bar{\theta}[F_i^*F_j + \partial_\mu\phi_i^*\partial_\mu\phi_j - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j] \end{aligned} \quad (5.6)$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta[\psi_i\phi_j + \phi_i\psi_j] + \theta\theta[\phi_iF_j + F_i\phi_j - \psi_i\psi_j] \quad (5.7)$$

$$\begin{aligned} \Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k] \\ &\quad + \theta\theta[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k] \end{aligned} \quad (5.8)$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

Superpotential

$$W = \int d^2\theta \left[\lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k \right] \quad (5.9)$$

$$\begin{aligned} &= \lambda_iF_i + \frac{1}{2}m_{ij}(\phi_iF_j + F_i\phi_j - \psi_i\psi_j) \\ &\quad + \frac{1}{3}y_{ijk}(F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k) \end{aligned} \quad (5.10)$$

(In x -basis, since we have omitted all $\bar{\theta}$ s.)

5.6 VECTOR SUPERFIELDS

5.6.1 Abelian Case

General Definitions

Vector Superfields : $V = V^\dagger$

Gauge Transf. : $V \dashrightarrow V + \Phi + \Phi^\dagger$

Field Strength : $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V$; $\bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V$

Lagrangian : $\mathcal{L} = \frac{1}{4} \left(W^\alpha W_\alpha \Big|_{\theta\theta} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} \right)$

Explicit Expression

$$\begin{aligned}
V = & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
& + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] - \theta\sigma^\mu\bar{\theta}A_\mu(x) \\
& + i\theta\bar{\theta}\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right] - i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] \\
& + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\partial^2 C(x)\right]
\end{aligned} \tag{5.11}$$

In Wess-Zumino gauge,

$$V \dashrightarrow -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\bar{\theta}\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{5.12}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y) + i\theta\bar{\theta}\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y) - i\partial_\mu A^\mu(y)] \tag{5.13}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y^+) + i\theta\bar{\theta}\bar{\theta}\bar{\lambda}(y^+) - i\bar{\theta}\bar{\theta}\theta\lambda(y^+) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(y^+) + i\partial_\mu A^\mu(y^+)] \tag{5.14}$$

Field Strength

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{2gV}D_\alpha e^{-2gV} \quad \text{where} \quad V = V^a T^a \tag{5.15}$$

Field Strength

Defining all component fields as including generators and coupling constants,

$$\begin{aligned} e^{\pm 2gV^a T^a} &\rightarrow e^{\pm 2V} \\ &= 1 \mp 2\theta\sigma^\mu\bar{\theta}A_\mu(x) \mp 2i[\bar{\theta}\bar{\theta}\theta\lambda(x) - \theta\theta\bar{\theta}\bar{\lambda}(x)] + \theta\theta\bar{\theta}\bar{\theta}[-A^\mu(x)A_\mu(x) - D(x)] \end{aligned} \quad (5.16)$$

Therefore, in y^+ -basis,

$$\begin{aligned} D_\alpha e^{-2V} &= \frac{\partial}{\partial\theta^\alpha} \left\{ 1 + 2\theta\sigma^\mu\bar{\theta}A_\mu + 2i[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}] + \theta\theta\bar{\theta}\bar{\theta}[A^\mu A_\mu - D - i\partial_\mu A^\mu] \right\} \\ &= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2i\bar{\theta}\bar{\theta}\lambda_\alpha - 4i\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}[A^\mu A_\mu - D - i\partial_\mu A^\mu] \end{aligned} \quad (5.17)$$

$$\begin{aligned} e^{2V} D_\alpha e^{-2V} &= \left\{ 1 - 2\theta\sigma^\mu\bar{\theta}A_\mu - 2i[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}] \right\} D_\alpha e^{-2V} \\ &= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2i\bar{\theta}\bar{\theta}\lambda_\alpha - 4i\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}[A^\mu A_\mu - D - i\partial_\mu A^\mu] \\ &\quad - 2\theta\sigma^\mu\bar{\theta}A_\mu [2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 4i\theta_\alpha\bar{\theta}\bar{\lambda}] \\ &\quad + 2i\theta\bar{\theta}\bar{\theta}\bar{\lambda} \cdot 2(\sigma^\mu\bar{\theta})_\alpha A_\mu \\ &= \textcolor{red}{2(\sigma^\mu\bar{\theta})_\alpha A_\mu} - \textcolor{red}{4i\theta_\alpha\bar{\theta}\bar{\lambda}} + 2\theta_\alpha\bar{\theta}\bar{\theta}[A^\mu A_\mu - D - i\partial_\mu A^\mu] + 2i\bar{\theta}\bar{\theta}\lambda_\alpha \\ &\quad - 2A_\mu A_\nu \bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4iA_\mu\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad (\text{in } y^+\text{-basis}) \end{aligned} \quad (5.18)$$

$$\begin{aligned} &= \textcolor{blue}{2(\sigma^\mu\bar{\theta})_\alpha A_\mu} - \textcolor{blue}{2i\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\nu\bar{\sigma}^\mu)^\gamma\partial_\nu A_\mu} - \textcolor{blue}{4i\theta_\alpha\bar{\theta}\bar{\lambda}} - \textcolor{blue}{2\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha} \\ &\quad + 2\theta_\alpha\bar{\theta}\bar{\theta}[A^\mu A_\mu - D - i\partial_\mu A^\mu] + 2i\bar{\theta}\bar{\theta}\lambda_\alpha \\ &\quad - 2A_\mu A_\nu \bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4iA_\mu\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad (\text{in } y\text{-basis}) \end{aligned} \quad (5.19)$$

In y -basis, $\bar{D}\bar{D} = 4 \cdot \frac{\partial}{\partial(\bar{\theta}\bar{\theta})}$. Therefore,

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{2V}D_\alpha e^{-2V} \quad (5.20)$$

$$\begin{aligned} &= 2(A_\mu A_\nu + i\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\ &\quad - 2\theta_\alpha[A^\mu A_\mu - D - i\partial_\mu A^\mu] - 2i\lambda_\alpha + 4iA_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \quad (\text{in } y\text{-basis}) \end{aligned} \quad (5.21)$$

$$\begin{aligned} &= 2(A_\mu A_\nu + i\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\ &\quad - 2\theta_\alpha[A^\mu A_\mu - D - i\partial_\mu A^\mu] - 2i\lambda_\alpha + 4iA_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \\ &\quad - i\theta(\sigma^\nu\bar{\sigma}^\mu\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A_\mu A_\nu + A_\mu\partial_\rho A_\nu + i\partial_\nu\partial_\rho A_\mu) \\ &\quad + i\theta\theta(\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A^\mu A_\mu + A^\mu\partial_\rho A_\mu - \partial_\rho D - i\partial_\mu\partial_\rho A^\mu) \quad (\text{in } x\text{-basis}) \end{aligned} \quad (5.22)$$

$$\begin{aligned} W^\alpha &= 2(A_\mu A_\nu + i\partial_\nu A_\mu)(\theta\sigma^\mu\bar{\sigma}^\nu)^\alpha - 2\theta\theta(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu)^\alpha \\ &\quad - 2\theta^\alpha[A^\mu A_\mu - D - i\partial_\mu A^\mu] - 2i\lambda^\alpha - 4iA_\mu\theta\theta(\bar{\lambda}\bar{\sigma}^\mu)^\alpha \quad (\text{in } y\text{-basis}) \end{aligned} \quad (5.23)$$

$$\begin{aligned}
W^\alpha W_\alpha \Big|_{\theta\theta} &= 4(A_\mu A_\nu + i\partial_\nu A_\mu)(A_\rho A_\sigma + i\partial_\sigma A_\rho)(\theta\sigma^\mu\bar{\sigma}^\nu\sigma^\sigma\bar{\sigma}^\rho\theta) \\
&\quad - 4(A_\mu A_\nu + i\partial_\nu A_\mu)[A^\mu A_\mu - D - i\partial_\mu A^\mu](\theta\sigma^\mu\bar{\sigma}^\nu\theta) \\
&\quad + 4i\theta\theta(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda) \\
&\quad - 4[A^\mu A_\mu - D - i\partial_\mu A^\mu](A_\mu A_\nu + i\partial_\nu A_\mu)(\theta\sigma^\nu\bar{\sigma}^\mu\theta) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - i\partial_\mu A^\mu][A^\mu A_\mu - D - i\partial_\mu A^\mu] \\
&\quad + 4i[A^\mu A_\mu - D - i\partial_\mu A^\mu]\theta\lambda \\
&\quad - 4i(\lambda\sigma^\nu\bar{\sigma}^\mu\theta)(A_\mu A_\nu + i\partial_\nu A_\mu) \\
&\quad - 4i\theta\theta(\lambda\sigma^\mu\partial_\mu\bar{\lambda}) \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar{\lambda})_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar{\lambda}\bar{\sigma}^\mu\lambda) \\
&= 4(\theta\sigma^\mu\bar{\sigma}^\nu\sigma^\sigma\bar{\sigma}^\rho\theta)[A_\mu A_\nu + i\partial_\nu A_\mu][A_\rho A_\sigma + i\partial_\sigma A_\rho] \\
&\quad - 4(\theta\sigma^\mu\bar{\sigma}^\nu\theta)\left\{A^\mu A_\mu - D - i\partial_\mu A^\mu, A_\mu A_\nu + i\partial_\nu A_\mu\right\} \\
&\quad + 4i\theta\theta(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar{\lambda}) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - i\partial_\mu A^\mu][A^\mu A_\mu - D - i\partial_\mu A^\mu] \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar{\lambda})_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar{\lambda}\bar{\sigma}^\mu\lambda) \quad (\text{in } y\text{-basis}) \\
&\rightsquigarrow 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})[A_\mu A_\nu + i\partial_\nu A_\mu][A_\rho A_\sigma + i\partial_\sigma A_\rho] \\
&\quad - 4\left\{A^\mu A_\mu - D - i\partial_\mu A^\mu, A^\nu A_\nu + i\partial_\nu A^\nu\right\} \\
&\quad + 4[A^\mu A_\mu - D - i\partial_\mu A^\mu][A^\mu A_\mu - D - i\partial_\mu A^\mu] \\
&\quad + 4i(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar{\lambda}) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar{\lambda})_\alpha - 8A_\mu(\bar{\lambda}\bar{\sigma}^\mu\lambda) \\
&= -2F^{\mu\nu}F_{\mu\nu} + 4(D + 2i\partial_\mu A^\mu)(D + 2i\partial_\mu A^\mu) \\
&\quad + 4i(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar{\lambda}) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar{\lambda})_\alpha - 8A_\mu(\bar{\lambda}\bar{\sigma}^\mu\lambda), \tag{5.24}
\end{aligned}$$

$$\begin{aligned}
\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} &= -2F^{\mu\nu}F_{\mu\nu} + 4(D - 2i\partial_\mu A^\mu)(D - 2i\partial_\mu A^\mu) \\
&\quad + 4i(\partial_\mu\lambda\sigma^\mu\bar{\lambda} - \bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda) + 8\bar{\lambda}^\alpha A_\mu(\bar{\sigma}^\mu\lambda)_\alpha - 8A_\mu(\lambda\sigma^\mu\bar{\lambda}), \tag{5.25}
\end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{5.26}$$

Defining the generators as ^{*1}

$$T^{a\dagger} = T^a, \quad [T^a, T^b] = if^{abc}T^c, \quad \text{Tr } T^a T^b = k\delta^{ab} \quad (k > 0), \quad \text{Tr}(\{T^a, T^b\}T^c) = 0, \tag{5.27}$$

$$\begin{aligned}
\frac{1}{16kg^2} \text{Tr} \left(W^\alpha W_\alpha \Big|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} \right) &= -\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a + \frac{1}{2}D^a D^a - 2\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} \\
&\quad - \frac{1}{2}i(\lambda\sigma^\mu D_\mu\bar{\lambda} + \bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda) \tag{5.28}
\end{aligned}$$

Here

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc}A_\mu^b A_\nu^c \tag{5.29}$$

$$D_\mu\lambda^a = \partial_\mu\lambda^a - gf^{abc}A_\mu^b\lambda^c \tag{5.30}$$

$$D_\mu\bar{\lambda}^a = \partial_\mu\bar{\lambda}^a - gf^{abc}A_\mu^b\bar{\lambda}^c \tag{5.31}$$

^{*1} The last one is anomaly-free condition, yielding $\text{Tr } T^a T^b T^c = ikgf^{abc}/2$.

Interaction Terms

$$e^{-2gV^aT^a} = 1 + 2g\theta\sigma^\mu\bar{\theta}A_\mu^aT^a + 2ig\bar{\theta}\bar{\theta}\theta\lambda^aT^a - 2ig\theta\theta\bar{\theta}\bar{\lambda}^aT^a - \theta\theta\bar{\theta}\bar{\theta} [g^2A^{a\mu}A_\mu^bT^aT^b + gD^aT^a] \quad (5.32)$$

$$\begin{aligned} \Phi^\dagger e^{-2gV^aT^a} \Phi \Big|_{\theta^4} &= \phi^* \theta\theta\bar{\theta}\bar{\theta} [-g^2A^{a\mu}A_\mu^bT^aT^b - gD^aT^a] \phi \\ &\quad + \sqrt{2}\phi^* [2ig\bar{\theta}\bar{\theta}\theta\lambda^aT^a] \theta\psi + \sqrt{2}\bar{\theta}\bar{\psi} [-2ig\theta\theta\bar{\theta}\bar{\lambda}^aT^a] \phi \\ &\quad + i\theta\sigma^\mu\bar{\theta} [\phi^* (2g\theta\sigma^\mu\bar{\theta}A_\mu^aT^a) \partial_\mu\phi - \partial_\mu\phi^* (2g\theta\sigma^\mu\bar{\theta}A_\mu^aT^a) \phi] \\ &\quad + 2\bar{\theta}\bar{\psi} [2g\theta\sigma^\mu\bar{\theta}A_\mu^aT^a] \theta\psi \\ &\quad + \theta\theta\bar{\theta}\bar{\theta} \left[F^*F + \frac{1}{4}\phi^*\partial^2\phi + \frac{1}{4}\partial^2\phi^*\phi - \frac{1}{2}\partial_\mu\phi^*\partial_\mu\phi + \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \right] \\ &\rightsquigarrow F^*F - D_\mu\phi^*D^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi \\ &\quad - gD^a(\phi^*T^a\phi) - \sqrt{2}ig\lambda^a(\phi^*T^a\psi) + \sqrt{2}ig\bar{\lambda}^a(\bar{\psi}T^a\phi) \end{aligned} \quad (5.33)$$

TODO: 計量により符号が異なるという噂.....

$$D_\mu\phi = (\partial_\mu - igA^{a\mu}T^a)\phi \quad (5.34)$$

$$D_\mu\phi^* = \partial_\mu\phi^* + igA_\mu^a(\phi^*T^a) \quad (5.35)$$

$$D_\mu\psi = (\partial_\mu - igA_\mu^aT^a)\psi \quad (5.36)$$

5.7 MINIMAL SUPERSYMMETRIC STANDARD MODEL

5.7.1 Definitions

Gauge Group

$$\text{SU}(3)_{\text{color}} \times \text{SU}(2)_{\text{weak}} \times \text{U}(1)_Y \quad (\times \text{Z}_{2R} : R\text{-parity})$$

Fields

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	3	2	1/6	1/3	
L_i		2	-1/2		1
\bar{U}_i	$\bar{3}$		-2/3	-1/3	
\bar{D}_i	$\bar{3}$		1/3	-1/3	
\bar{E}_i			1		-1
H_u		2	1/2		
H_d		2	-1/2		

Field	SU(3)	SU(2)	U(1)
g	8		
W		3	
B			

Superpotential

$$W_{\text{RPC}} = \mu H_u H_d + y_{u_{ij}} H_u Q_i \bar{U}_j + y_{d_{ij}} H_d Q_i \bar{D}_j + y_{e_{ij}} H_d L_i \bar{E}_j \quad (5.37)$$

$$W_{\text{RPV}} = \mu_i H_u L_i + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \quad (5.38)$$

(Here we define $\lambda_{ijk} = -\lambda_{jik}$ and $\lambda''_{ijk} = \lambda''_{ikj}$.)

5.7.2 Scalar Potential

F -terms

$$-F_{H_u}{}^{a*} = \epsilon^{ab} \left(\mu H_d{}^b + y_{u_{ij}} Q_i^{bx} \bar{U}_j^x + \mu_i L_i^b \right) \quad (5.39)$$

$$-F_{H_d}{}^{a*} = \epsilon^{ab} \left(-\mu H_u{}^b + y_{d_{ij}} Q_i^{bx} \bar{D}_j^x + y_{e_{ij}} L_i^b \bar{E}_j \right) \quad (5.40)$$

$$-F_{Q_i}{}^{ax*} = \epsilon^{ab} \left(-y_{u_{ij}} H_u{}^b \bar{U}_j^x - y_{d_{ij}} H_d{}^b \bar{D}_j^x - \lambda'_{jik} L_i^b \bar{D}_j^x \right) \quad (5.41)$$

$$-F_{L_i}{}^{a*} = \epsilon^{ab} \left(-\mu_i H_u{}^b - y_{e_{ij}} H_d{}^b \bar{E}_j + 2\lambda_{ijk} L_j^b \bar{E}_k + \lambda'_{ijk} Q_j^{bx} \bar{D}_k^x \right) \quad (5.42)$$

$$-F_{\bar{U}_i}{}^{x*} = (\epsilon^{ab} y_{u_{ji}} H_u{}^a Q_j^{bx} + \epsilon^{xyz} \lambda''_{ijk} \bar{D}_j^y \bar{D}_k^z) \quad (5.43)$$

$$-F_{\bar{D}_i}{}^{x*} = (\epsilon^{ab} y_{d_{ji}} H_d{}^a Q_j^{bx} + \epsilon^{ab} \lambda'_{jki} L_j^a Q_k^{bx} + 2\epsilon^{xyz} \lambda''_{jki} \bar{U}_j^y \bar{D}_k^z) \quad (5.44)$$

$$-F_{\bar{E}_i}{}^{*} = (\epsilon^{ab} y_{e_{ji}} H_d{}^a L_j^b + \epsilon^{ab} \lambda_{jki} L_j^a L_k^b) \quad (5.45)$$

D -terms

$$D_g{}^\alpha = -g_3 \sum_{i=1}^3 \left[\sum_{a=1,2} Q_i^{ax*} (T^\alpha)_{xy} Q_i^{ay} - \bar{U}_i^{x*} (T^\alpha)_{xy} \bar{U}_i^y - \bar{D}_i^{x*} (T^\alpha)_{xy} \bar{D}_i^y \right] \quad (5.46)$$

$$D_W{}^\alpha = -g_2 \left[\sum_{i=1}^3 \sum_{x=1}^3 Q_i^{ax*} (T^\alpha)_{ab} Q_i^{by} + \sum_{i=1}^3 L_i^{a*} (T^\alpha)_{ab} L_i^b + H_u{}^{a*} (T^\alpha)_{ab} H_u{}^b + H_d{}^{a*} (T^\alpha)_{ab} H_d{}^b \right] \quad (5.47)$$

$$D_B = -g_1 \left[\frac{1}{6} |Q_i^{ax}|^2 - \frac{1}{2} |L_i^a|^2 - \frac{2}{3} |\bar{U}_i^x|^2 + \frac{1}{3} |\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2} |H_u{}^a|^2 - \frac{1}{2} |H_d{}^a|^2 \right] \quad (5.48)$$

Full Scalar Potential

$$V = \sum |F_\bullet|^2 + \frac{1}{2} \sum |D_\bullet|^2 \quad (5.49)$$

6 Supergravity

6.1 MINIMAL SUGRA LAGRANGIAN

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2}eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu D_\rho\psi_\sigma \quad (6.1)$$

where

$$D_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \quad [\omega_\mu{}^{ab} : \text{“spin 接続”}] \quad (6.2)$$

$$e := \det e_a{}^\mu \quad (6.3)$$

$$M := 1/\sqrt{8\pi G} \quad (\text{Reduced Planck mass}) \quad (6.4)$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \quad (6.5)$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \quad (6.6)$$

6.2 GENERAL SUGRA LAGRANGIAN

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \quad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \quad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \quad (6.7)$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
 - Real function of chiral multiplets.
 - In global SUSY, $\int d^4\theta K$ yields kinetic terms of the chiral multiplet.
 - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int d^4\theta \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \quad (6.8)$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
 - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
 - $(a), (b), \dots$ are indices for adjoint representation of gauge group.
 - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}D_\mu\phi^iD^\mu\phi^{*j} - \frac{1}{2}eg^2D_{(a)}D^{(a)} \\
& + ie g_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu D_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu D_\rho\psi_\sigma \\
& - \frac{1}{4}ef_{(ab)}^R F_{\mu\nu}^{(a)}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f_{(ab)}^I f_{\mu\nu}^{(a)}f_{\rho\sigma}^{(b)} \\
& + \frac{i}{2}e\left[\lambda_{(a)}\sigma^\mu D_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu D_\mu\lambda^{(a)}\right] - \frac{1}{2}f_{(ab)}^I D_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X_{(a)}^{*j}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X_{(a)}^i\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{i}{4}\sqrt{2}eg\partial_if_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{i}{4}\sqrt{2}eg\partial_{i^*}f_{(ab)}^*D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_if_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F_{\mu\nu}^{(b)} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f_{(ab)}^*\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F_{\mu\nu}^{(b)} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}D_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}D_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{i}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F_{\nu\rho}^{(a)} + \widehat{F}_{\nu\rho}^{(a)}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[i\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f_{(ab)}^R + f^{R(cd)-1}\partial_if_{(bc)}\partial_{j^*}f_{(ad)}^*\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_jf_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f_{(ab)}^*\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{R(cd)-1}\partial_if_{(ac)}\partial_jf_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{R(cd)-1}\partial_{i^*}f_{(ac)}^*\partial_{j^*}f_{(bd)}^*\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_if_{(ab)}\partial_{j^*}f_{(cd)}^*\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{i}{4}\sqrt{2}e\partial_if_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{i}{4}\sqrt{2}e\partial_{i^*}f_{(ab)}^*\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{i}{2}\sqrt{2}ee^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}ee^{K/2}\left[D_iD_jW\chi^i\chi^j + D_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}ee^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_if_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f_{(ab)}^*\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned} \tag{6.9}$$

付録 A Mathematics

A.1 ALGEBRA

A.1.1 Algebraic Structure

Semigroup	: For $a, b \in A_{\text{set}}, ab \in A$; Associative.
Monoid	: For $a, b \in A_{\text{set}}, ab \in A$; Associative, Unit.
Group	: For $a, b \in A_{\text{set}}, ab \in A$; Associative, Unit, Inverse.
Module (加群/可換群)	: For $a, b \in A_{\text{set}}, a + b \in A$; Commutative, Associative, Unit, Inverse.
Semimodule	: For $a, b \in A_{\text{set}}, a + b \in A$; Commutative, Associative, Unit.
Ring (環)	: $+$: Module, \times : Semigroup(Monoid), Distributive.
Semiring	: $+$: Semimodule, \times : Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.
Field	: $+$: Module, \times : Commutative Monoid, a^{-1} but $0, 1 \neq 0$, Distributive.

Vector Space

Vector space on K	: For $v \in (V, +)_{\text{module}}$ and $k \in K_{\text{field}}$, (K -module) : $kv \in (V, +)$; Compatible, Distributive, $1v = v$.
Norm	: $\ x\ \geq 0, \ x\ = 0 \Leftrightarrow x = 0, \ kx\ = k\ x\ , \ x + y\ \leq \ x\ + \ y\ $
Inner product	: $\langle x x \rangle \geq 0, \langle x x \rangle = 0 \Leftrightarrow x = 0, \langle x y \rangle = \langle y x \rangle,$: $\langle x + y z \rangle = \langle x z \rangle + \langle y z \rangle, \langle kx y \rangle = k\langle x y \rangle$

K -algebra

K -algebra $C(V)$ とは, vector 空間 V に, distributive な乗法を入れたもの:

$$xy \in C(V); \quad (xy)z = x(yz), (x + y)z = xz + yz, x(y + z) = xy + xz, k(xy) = (kx)y = x(ky).$$

A.1.2 Lie Algebra

Lie Algebra	: For a Finite-dimensional K -module $(A, +)$ and $x, y, z \in (A, +), a, b \in K,$: $[u, v] \in (A, +)$ (Lie product), and : Bilinear: $[ax + by, z] = a[x, z] + b[y, z], [x, ay + bz] = a[x, y] + b[x, z],$ 則 ち : Alternating: $[x, x] = 0 \quad (\Rightarrow [x, y] = -[y, x]),$: Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$
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$[A, B] := AB - BA$ として閉じていれば Lie algebra となる。

A.1.3 Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.

For $C(V)$, **Clifford algebra** $(C(V), \theta)$ is defined as

- $C(V)$: K -algebra with $1 \quad (1x = x1 = x),$
- $\theta : V \rightarrow C(V)$, homomorphism, $\theta(x)^2 = \langle x|x \rangle 1,$
- Any C' : K -algebra with 1 and any homomorphism $\phi : V \rightarrow C'$ with $\phi(x)^2 = \langle x|x \rangle 1$, there's unique $\bar{\phi} : C \rightarrow C'$, homomorphism, $\bar{\phi}(1) = 1.$

The Gamma matrices form Clifford Algebra:

$$V : \mathbb{R}^4 \text{ with } \{G_0, G_1, G_2, G_3\}, \quad \langle G_0 | G_0 \rangle = 1, \langle G_i | G_i \rangle = -1; \quad C(V) : M(n, \mathbb{C}).$$

A.1.4 Multilinear Algebra

可換環 K 上の vector 空間 V とその双対空間 V^* について :

Tensor Algebra $T(V)$: 線形写像 $f : V \rightarrow T(V)$ を持つ K -algebra であり, 別の K -algebra A への線型写像 $g : V \rightarrow A$ が与えられたときに可換な準同型 $h : T(V) \rightarrow A$ s.t. $h \circ f = g$ が一意に存在するもの。

Symmetric Algebra $S(V)$: 上の定義で, $C(V)$ (および A) を可換 K -algebra としたもの。

Exterior Algebra $\wedge(V)$: 上の定義で, $g(\cdots vv \cdots) = 0$ を要請したもの。

Tensor and Tensor Space

p 次反変 Tensor 積 : $T^p(V) := V^{\otimes p} = V \otimes V \otimes \cdots \otimes V$

q 次共変 Tensor 積 : $T_q(V) := (V^*)^{\otimes q} = V^* \otimes V^* \otimes \cdots \otimes V^*$

混合 Tensor 積 : $T_q^p(V) := T^p(V) \otimes T_q(V)$ (ただし V と V^* の順序を変えたものは同型)

Tensor Space : $T(V) := \oplus T_q(V)$

Tensor space の代数 $(+, \otimes)$ は環を為しており, また更に :

Contraction : V^* は $V \rightarrow K$ なので, $T_q^p(V) \rightarrow T_{q-1}^{p-1}(V)$ が定義される。

内積 : $T_2(V) \ni g_{ij} : V \times V \rightarrow \mathbb{R}$ (通常は対称にする。正定値としてもよい。)

添字の上げ下げ : $T_2(V) \cong \text{Hom}(V, V^*)$ (同型) なので, 内積から誘導される。

: 同型なので, $g^{ij} : \text{Hom}(V^*, V)$ は逆写像になる。

Grassmann Operator

台集合 V を Hilbert 空間であるとする。 $V \ni v$ について

$$\|\psi v\|^2 = (\psi_{ab} v_b)^* (\psi_{ac} v_c) = v_b^* v_c (\psi_{ab})^* \psi_{ac} \geq 0 \quad \therefore (\psi_{ab})^* \psi_{ac} = -(\psi_{ab} (\psi_{ac})^*)^* \quad (\text{A.1})$$

即ち反可換な作用素について $(ab)^\dagger = a^\dagger b^\dagger$, $(ab)^T = -a^T b^T$, $(ab)^* = b^* a^*$ である。

TODO: ψ の正体がわからない.....

A.1.5 Lie Group and Lie Algebra

- 群 G が Lie group である ... G が同時に C^∞ 多様体であり, 積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が COMPLEX Lie group である ... 積演算と逆元写像が共に正則写像である。
- Lie 群 G の単位元における接空間を, G の Lie algebra \mathfrak{g} という。
 - \mathfrak{g} は G の左不変な vector 場全体である。
 - \mathfrak{g} は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると,
 - その Lie 代数の基底 B_i に対して structure constant c が $[B_i, B_j] = c_{ij}^k B_k$ として定義できる。

* * *

- Compact Lie 群は線型 Lie 群である。
- G として Linear group $\text{GL}(n; \mathbb{R})$ を考えると,
 - その Lie 代数は n 次実正方行列全体となる。

- Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は, $GL(n; \mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では, Lie 群として $GL(n; \mathbb{R})$ の部分群を考えることにし, Lie 代数の元を行列により表現する。

A.1.6 Matrix Representation

- Lie 群 G の Lie 代数の基底の組を, G の **generators** と言う。
- $GL(n; \mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し, 以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1, \dots, x_m) \mapsto e^{x_1 T_1 + \dots + x_m T_m} \quad (x_1, \dots, x_m) \mapsto e^{x_1 T_1} \dots e^{x_m T_m} \quad (\text{A.2})$$

- Lie 群 G が compact である ...
 1. 多様体 G が compact である。 **TODO: これは何故同値なのか?**
 2. G の生成子 $\{T_i\}$ を, $\text{Tr}(T_i T_j) = k \delta_{ij}$ かつ $k > 0$ となるように取り替えることができる。

【この基底の下では構造定数が完全反対称になる。】

- Compact 群 G は, **unitary representation** を持つ。

故に, 単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i \in \mathbb{R}$ により, G の元を

$$e^{ix^i T^i} \quad (\text{A.3})$$

と表すことが出来る。

A.1.7 結論

Compact Lie 群の元のうち, 単位元近傍にあるもの V は,

Hermitian Representation

$$V = \exp(ix^i T^i) \quad \text{where} \quad T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R}, \\ [T^i, T^j] = if^{ijk} T^k, \quad \text{Tr}(T^i T^j) = \lambda \delta^{ij} > 0; \quad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \quad \text{where} \quad R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R}, \\ [R^i, R^j] = -f^{ijk} R^k, \quad \text{Tr}(R^i R^j) = -\lambda \delta^{ij} < 0; \quad f \in \mathbb{R}$$

と表すことが出来る。

付録 B Verbose Notes

B.1 SPINOR FIELDS

B.1.1 Lorentz group and Lorentz algebra

Metric : $\eta = \text{diag}(+1, -1, -1, -1)$, $\eta = \text{diag}(-1, +1, +1, +1)$.

Lorentz transf. in $\mathbb{R}^{1,3}$: Linear transf. $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ which conserve x^2 .

: $\Rightarrow \eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$. and form a group L .

: $(\Rightarrow (\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\alpha} \eta^{\mu\beta} \Lambda^\alpha{}_\beta =: \Lambda_\nu{}^\mu \Rightarrow \Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta_\nu^\rho)$

Disconnected parts of L : $L_0 := \{\det \Lambda = +1 \wedge \Lambda_0^0 > 0\}$ $L_P := \{\det \Lambda = -1 \wedge \Lambda_0^0 > 0\}$

: $L_T := \{\det \Lambda = +1 \wedge \Lambda_0^0 < 0\}$ $L_{PT} := \{\det \Lambda = -1 \wedge \Lambda_0^0 < 0\}$

: $(L_0 \text{ is identical with } (\text{SO}(1,3), \text{SO}(3,1)).)$

Infinitesimal one in L_0 : $\Lambda^\mu{}_\nu = \delta_\nu^\mu + \epsilon^\mu{}_\nu$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ (for $\eta = \eta\Lambda\Lambda$)

微小変換は $\epsilon^\mu{}_\nu = \begin{pmatrix} 0 & \beta_x & \beta_y & \beta_z \\ \beta_x & 0 & -\theta_z & \theta_y \\ \beta_y & \theta_z & 0 & -\theta_x \\ \beta_z & -\theta_y & \theta_x & 0 \end{pmatrix}$ の形となっているので、回転生成子 J と加速生成子 K は

$$\Lambda = \exp[\kappa i(\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\beta} \cdot \mathbf{K})] \Rightarrow J_x = -\kappa i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K_x = -\kappa i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{B.1})$$

の形である。ここで $\kappa = \pm 1$ は notation である。

一方、微小変換から生成子を $\epsilon^\mu{}_\nu = \mp \frac{i}{2} \epsilon^{\rho\sigma} (J_{\rho\sigma})^\mu{}_\nu$ と定義すると、計量によらずに $\epsilon_{\mu\nu}$ は反対称となり、

$$\begin{aligned} \boldsymbol{\theta} &= (+, -)(\epsilon^{23}, \epsilon^{31}, \epsilon^{12}), & \boldsymbol{\beta} &= (+, -)(\epsilon^{10}, \epsilon^{20}, \epsilon^{30}). \\ &= (+, -)(\epsilon_{23}, \epsilon_{31}, \epsilon_{12}) & &= (-, +)(\epsilon_{10}, \epsilon_{20}, \epsilon_{30}) \end{aligned}$$

で、従って $J^{\rho\sigma}$ も反対称。 $(J_{\rho\sigma})^\mu{}_\nu = \pm i(\delta_\rho^\mu \eta_{\sigma\nu} - \delta_\sigma^\mu \eta_{\rho\nu})$ となり、交換関係が得られ、これが閉じているので Lie 代数であることもわかる：

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mp i(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma}). \quad (\text{B.2})$$

生成子の具体形は計量に依存し、 $J_{10}{}^\mu{}_\nu = (\pm i, \mp i) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$, $J_{23}{}^\mu{}_\nu = (\pm i, \mp i) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\mu\nu}$ となる。
よって、ここでの複号の取り方と κ および計量の定義によって、 $\mathbf{J} \cdot \mathbf{K}$ と $J_{\rho\sigma}$ の対応が定まることになる。

* * *

κ と複号について $(-, \text{上}), (+, \text{下}), (-, \text{下}), (+, \text{上})$ を取れば

$$\mathbf{J} = (J_{23}, J_{31}, J_{12}), \quad \mathbf{K} = (J_{10}, J_{20}, J_{30}); \quad (\text{B.3})$$

となる：

$$\Lambda = \exp \epsilon = \exp[\kappa i(\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\beta} \cdot \mathbf{K})] = \exp[\mp i(\epsilon^{\rho\sigma} J_{\rho\sigma})/2]. \quad (\text{B.4})$$

B.1.2 Lorentz group and $\text{SL}(2, \mathbb{C})$

次に，連結 Lie 群 L_0 が，連結 Lie 群 $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ と同型であることを見る：

$$\mathfrak{sl}(2, \mathbb{C}) := \{a \in \mathfrak{gl}(2, \mathbb{C}) \mid \text{Tr}(a) = 0\}, \quad \text{SL}(2, \mathbb{C}) := \{g \in \text{GL}(2, \mathbb{C}) \mid \det(g) = 1\}. \quad (\text{B.5})$$

まず， σ^μ を（極めて一般的に） $\sigma^\mu := (\alpha 1, \beta \sigma)$ と定義する（ $\alpha = \beta = \pm 1$ ）。 $x^2 = (+ -) \det(x_\mu \sigma^\mu)$ なので

$$f^g : (x_\mu \sigma^\mu) \mapsto g(x_\mu \sigma^\mu) g^\dagger; \quad g \in \text{SL}(2, \mathbb{C}) \quad (\text{B.6})$$

は x^2 を保存する。よって Lorentz 変換であり，生成子を比べることで局所同型だとわかる：

$$\begin{aligned} \text{SL}(2, \mathbb{C}) \ni g = \exp(-ia) \text{ として } x_\mu (g \sigma^\mu g^\dagger) &= \Lambda_\mu{}^\nu x_\nu \sigma^\mu \text{ を微小展開すると} \\ \Lambda^\mu{}_\nu \sigma^\nu &= g^{-1} \sigma^\mu (g^{-1})^\dagger \implies \epsilon^\mu{}_\nu \sigma^\nu = i(a \sigma^\mu - \sigma^\mu a^\dagger) \end{aligned} \quad (\text{B.7})$$

であり，ここから g がわかる：

$$g = \exp \left(-\frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \frac{\alpha \beta}{2} \beta \cdot \boldsymbol{\sigma} \right). \quad (\text{B.8})$$

このことを別の観点から見る。Lorentz 群の生成子の交換関係を見ると，（正しく複号を取った場合）

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \quad (\text{B.9})$$

となるので，

$$\mathbf{A} := \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} := \frac{1}{2}(\mathbf{J} - i\mathbf{K}). \quad (\text{B.10})$$

と定義すると

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0, \quad (\text{B.11})$$

となり，Lorentz 群が $\text{SU}(2) \times \text{SU}(2)$ に分解できる。

B.2 WEYL SPINOR

$\text{SU}(2)_A \times \text{SU}(2)_B$ に対して $(1/2, 0)$ 表現を為すものを左巻き spinor ξ ， $(0, 1/2)$ 表現を為すものを右巻き spinor $\bar{\xi}$ と定義する。

$$\xi \mapsto \left(1 - \frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} - \frac{1}{2} \beta \cdot \boldsymbol{\sigma}\right) \xi, \quad \bar{\xi} \mapsto \left(1 - \frac{i}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma} + \frac{1}{2} \beta \cdot \boldsymbol{\sigma}\right) \bar{\xi}. \quad (\text{B.12})$$

$\alpha\beta = 1$ とすると g は左巻き spinor の変換子となる。記号を $\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta$ ， $\bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}}$ と定義する。

次に， $\xi^\alpha \chi_\alpha$ および $\bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$ が scalar となるようにしたい。 $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ としておくと， $-E g^T E = g^{-1}$ より

$$(\xi')^\alpha = \xi^\beta (g^{-1})_\beta{}^\alpha = -\xi^\beta (E_{\beta A} g_B{}^A E_{B\alpha}) \quad \therefore (-E_{\gamma\alpha})(\xi')^\alpha = -g_\gamma{}^\beta (E_{\beta\delta} \xi^\delta). \quad (\text{B.13})$$

よって， $\epsilon^{12} = \epsilon_{21} = 1$ として $\xi^\alpha := \epsilon^{\alpha\beta} \xi_\beta$ ， $\xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta$ とすれば良い。

同様に， $\bar{\xi}'_{\dot{\alpha}} = -E(g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}_{\dot{\beta}}$ から $(E^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}})' = (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} (E^{\dot{\beta}\dot{\gamma}} \bar{\xi}_{\dot{\gamma}})$ となる。 $\epsilon_{12} = \epsilon_{1\dot{2}}$ として $\bar{\xi}_{\dot{\alpha}} := \epsilon_{\dot{\alpha}\beta} \bar{\xi}^{\dot{\alpha}}$ とするのが一般的である。また，このことから $(\xi_a)^* = \bar{\xi}_{\dot{a}}$ （或いは $\xi^\dagger = \bar{\xi}$ ）が分かる。

$x_\mu \sigma^\mu \mapsto x_\mu (g \sigma^\mu g^\dagger)$ であるので， $x_\mu (\xi^\alpha \sigma^\mu \bar{\chi}^{\dot{\alpha}})$ は scalar である。よって， $\sigma^\mu_{\alpha\dot{\alpha}}$ のように書ける。更にここで $x_\mu (\bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \chi_\alpha)$ も scalar となるように $\bar{\sigma}$ を定めよう。

$$\bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \chi_\alpha = -\epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \chi^\beta \bar{\sigma}^{\mu\dot{\alpha}\alpha} \bar{\xi}^{\dot{\beta}} \quad \therefore \sigma^\mu_{\beta\dot{\beta}} \propto \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \quad (\text{B.14})$$

であり，あとは convention である。

$$\begin{aligned} \epsilon^{12} = \epsilon_{21} = \epsilon^{i\dot{2}} = \epsilon_{2\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta} \xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{\dot{\beta}} \\ \xi_\alpha \mapsto g_\alpha^\beta \xi_\beta, \quad \xi^\alpha \mapsto \xi^\beta (g^{-1})_\beta^\alpha, \quad \bar{\xi}_{\dot{\alpha}} \mapsto \bar{\xi}_{\dot{\beta}} (g^\dagger)^{\dot{\beta}}_{\dot{\alpha}}, \quad \bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\beta\dot{\beta}}^\mu, \quad \sigma_{\alpha\dot{\alpha}}^\mu = \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\beta}, \quad \therefore \sigma^\mu := (\alpha 1, \beta \sigma), \quad \bar{\sigma}^\mu := (\alpha 1, -\beta \sigma) \end{aligned}$$

B.3 POLARIZATION SUM

Firstly we focus on the single photon case $M = \epsilon_\mu^*(k) \epsilon_\nu'^*(k') M^{\mu\nu}$. Here we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k) \epsilon_\nu(k) M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2, \quad (\text{B.15})$$

while

$$\eta_{\mu\nu} M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2 \quad (\text{B.16})$$

for Ward identity $k_\mu M^\mu = 0$. Therefore the replacement

$$\sum_{\text{pol.}} \epsilon_\mu \epsilon_\nu' \rightarrow \eta_{\mu\nu} \quad (\text{B.17})$$

is valid.

Secondly we think about the double photons case ^{*2} $M = \epsilon_\mu^*(k) \epsilon_\nu'^*(k') M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \quad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \quad (\text{B.18})$$

$$k' = (E, 0, 0, -E) \quad \epsilon' = (0, \cos \theta, \sin \theta, 0) \oplus (0, -\sin \theta, \cos \theta, 0). \quad (\text{B.19})$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \quad (\text{B.20})$$

Nevertheless, naïve replacement does not work, because our Ward identities

$$k_\mu \epsilon_\nu'^*(k') M^{\mu\nu} = \epsilon_\mu^*(k) k'_\nu M^{\mu\nu} = 0 \quad (\text{B.21})$$

obviously does not help us. If we can omit ϵ s from these identities, that is if

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0, \quad (\text{B.22})$$

we can recover validity of the replacement:

$$\eta_{\mu\rho} \eta_{\nu\sigma} M^{\mu\nu} M^{\rho\sigma*} = -\eta_{\nu\sigma} (M^{1\nu} M^{1\sigma*} + M^{2\nu} M^{2\sigma*}) \quad (\text{B.23})$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \quad (\text{B.24})$$

Then what's happening? Why this replacement is not valid? Actually our new conditions (B.22) seem to guarantee that we are summing not only “physical” but also “unphysical” polarizations. Meanwhile if

^{*2} This part is derived from 濱口幸一's notebook.

we use some physical condition such as $\epsilon \cdot k = 0$, (B.22) break down while Ward identities (B.21) are still valid.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our new conditions (B.22), and define $\widetilde{M}^{\mu\nu}$ and \widetilde{M} as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \quad (\text{B.25})$$

$$\widetilde{M} := \epsilon_\mu^*(k) \epsilon_\nu'^*(k') \widetilde{M}^{\mu\nu}. \quad (\text{B.26})$$

This alternative amplitude satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$), and furthermore $\widetilde{M} = M$. Therefore \widetilde{M} is physically identical to M . However technically these are very different, just because we cannot perform our “naïve replacement” for this \widetilde{M} :

$$\eta_{\mu\rho} \eta_{\nu\sigma} \widetilde{M}^{\mu\nu} \widetilde{M}^{\rho\sigma*} = \eta_{\mu\rho} \eta_{\nu\sigma} (M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu) (M^{\rho\sigma*} + k^\rho p^{\sigma*} + p'^\rho k'^{\sigma*}) \quad (\text{B.27})$$

$$= \sum_{\text{pol.}} |M|^2 + [(k \cdot p')(k' \cdot p) + \text{H. c.}]. \quad (\text{B.28})$$

After all, we have obtained following expression:

$$\begin{aligned} \sum_{\text{pol.}} |\widetilde{M}|^2 &= \sum_{\text{pol.}} |M|^2 \quad (\text{Furthermore } \widetilde{M} = M) \\ &= \sum_{\text{pol.}} |\epsilon_\mu^*(k) \epsilon_\nu'^*(k') M^{\mu\nu}|^2 = \sum_{\text{pol.}} |\epsilon_\mu^*(k) \epsilon_\nu'^*(k') \widetilde{M}^{\mu\nu}|^2 \\ &= \eta_{\mu\rho} \eta_{\nu\sigma} M^{\mu\nu} M^{\rho\sigma*} \\ &\neq \eta_{\mu\rho} \eta_{\nu\sigma} \widetilde{M}^{\mu\nu} \widetilde{M}^{\rho\sigma*} = \sum_{\text{pol.}} |\widetilde{M}|^2 + [(k \cdot p')(k' \cdot p) + \text{H. c.}]. \end{aligned} \quad (\text{B.29})$$

B.4 PHANTOM TERMS IN THE GAUGE THEORY

You may think we forget to introduce $\bar{\psi} \gamma_5 \psi$, $\bar{\psi} \gamma_5 \not{D} \psi$, $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$, $\epsilon^{\mu\nu\rho\sigma} D_\mu D_\nu F_{\rho\sigma}^a$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_L and P_R ,
- the last term is equivalent to the third term as

$$\epsilon^{\mu\nu\rho\sigma} D_\mu D_\nu F_{\rho\sigma}^a = \epsilon^{\mu\nu\rho\sigma} \frac{1}{2} [D_\mu, D_\nu] F_{\rho\sigma}^a = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a.$$

Therefore, we have to discuss only the $\epsilon F F$ terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e &= \epsilon^{\mu\nu\rho\sigma} (-f^{acd} f^{abe} - f^{adb} f^{ace}) A_\mu^b A_\nu^c A_\rho^d A_\sigma^e \\ &= -2\epsilon^{\mu\nu\rho\sigma} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e \\ &= 0, \end{aligned} \quad (\text{B.30})$$

$$\begin{aligned} \therefore \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a &= 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu^a \partial_\rho A_\sigma^a + 4g\epsilon^{\mu\nu\rho\sigma} f^{abc} A_\mu^a A_\nu^b \partial_\rho A_\sigma^c \\ &= 2\partial_\mu G^\mu, \end{aligned} \quad (\text{B.31})$$

where G^μ is the Chern-Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a \partial_\rho A_\sigma^a + \frac{1}{3} g f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right) = \epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a F_{\rho\sigma}^a - \frac{1}{3} g f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right). \quad (\text{B.32})$$

See Appendix B.5 for the instanton effect.

B.5 INSTANTON

TODO: Postpone...

付録 C Cherry on the Cake

Conversion of Units

$$1 \text{ GeV} = \frac{1}{6.5821 \times 10^{-25} \text{ s}} = \frac{1}{2.086 \times 10^{-32} \text{ yr}} = \frac{1}{0.19733 \text{ fm}} = 1.2244 \times 10^{13} \text{ K} = 1.7827 \times 10^{-24} \text{ g}$$

$$= \frac{1.519268 \times 10^{24}}{1 \text{ s}} = \frac{4.79 \times 10^{31}}{1 \text{ yr}} = \frac{5.0677}{1 \text{ fm}} \quad (\text{C.1})$$

$$1 \text{ K} = 8.167 \times 10^{-5} \text{ eV} = \frac{1}{8.0591 \times 10^{-21} \text{ s}} = \frac{1.2408 \times 10^{20}}{1 \text{ s}} \quad (\text{C.2})$$

$$1 \text{ GeV}^2 = \frac{1}{0.38938 \text{ mbarn}} = \frac{2568.2}{1 \text{ barn}} \quad (1 \text{ barn} = 10^{-28} \text{ m}^2 = 100 \text{ fm}^2) \quad (\text{C.3})$$

$$1 \text{ tropical yr} = 3.1557 \times 10^7 \text{ s}, \quad 1 \text{ sidereal yr} = 3.1558 \times 10^7 \text{ s};$$

$$1 \text{ s} = 3.1689 \times 10^{-8} \text{ tr-yr} = 3.1688 \times 10^{-8} \text{ sr-yr}. \quad (\text{C.4})$$

Physical Constants

$$G_F = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}, \quad G_N = 6.70881(67) \times 10^{-39} \text{ GeV}^{-2} \quad (\text{C.5})$$

$$\sqrt{G_N} = 1.61624(8) \times 10^{-35} \text{ m} = \frac{1}{1.22089(6) \times 10^{19} \text{ GeV}} = \frac{1}{2.17644(11) \times 10^{-8} \text{ kg}} \quad (\text{C.6})$$

$$\sqrt{8\pi G_N} = 8.1026(4) \times 10^{-35} \text{ m} = \frac{1}{2.4353(1) \times 10^{18} \text{ GeV}} = \frac{1}{4.3413(2) \times 10^{-9} \text{ kg}} \quad (\text{C.7})$$

Component of Spinor in Weyl Representation

$$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \chi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dot{\beta}}, \chi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}}; \quad \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = 1, \epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -1.)$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\chi}^1 \\ \bar{\chi}^2 \end{pmatrix} \xrightarrow{C} \psi^c = -i\gamma^2\psi^* = \begin{pmatrix} -(\bar{\chi}^2)^* \\ (\bar{\chi}^1)^* \\ (\xi_2)^* \\ -(\xi_1)^* \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \bar{\xi}_2 \\ -\bar{\xi}_1 \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{C.8})$$

$$\bar{\psi} = (\chi^\alpha \quad \bar{\xi}_{\dot{\alpha}}) = (\chi^1 \quad \chi^2 \quad \bar{\xi}_1 \quad \bar{\xi}_2) \xrightarrow{C} \bar{\psi}^c = i\psi^\dagger\gamma^0\gamma^2 = (\xi_2 \quad -\xi_1 \quad -\bar{\chi}^2 \quad \bar{\chi}^1) = (\xi^\alpha \quad \bar{\chi}_{\dot{\alpha}}) \quad (\text{C.9})$$

$$A^\alpha B_\alpha = \bar{\psi}_{A^c} P_L \psi_B = \bar{\psi}_{B^c} P_L \psi_A \quad \bar{A}_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} = \bar{\psi}_A P_R \psi_{B^c} = \bar{\psi}_B P_R \psi_{A^c} \quad (\text{C.10})$$