1 General Definitions and Tools

1.1 Notations and Conventions

◆1.1.1 Metric etc.

Minkowski Metric : $\eta^{\mu\nu} := \mathrm{diag}(+,-,-,-); \quad \epsilon^{0123}_{0123} := \pm 1$

Coordinates : $x^\mu := (t,x,y,z); \quad$ therefore $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$.

Gamma Matrices : $\{\gamma^\mu,\gamma^\nu\} := 2\eta^{\mu\nu}; \quad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

: therefore $\{\gamma^\mu,\gamma_5\} = 0, (\gamma_5)^2 = 1$.

Gamma Combinations : $1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5; \quad \sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu] = 0 \Big/ \mathrm{i}\gamma^\mu\gamma^\nu$

> Pauli Matrices : $\sigma_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \quad \sigma_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \quad \sigma_2 = \left(\begin{smallmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{smallmatrix}\right), \quad \sigma_3 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),$
>
> : $\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \quad \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),$
>
> : $\sigma^\mu := (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu := (1, -\boldsymbol{\sigma})$.

Fourier Transformation : $\widetilde{f}(k) := \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} f(x); \qquad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}kx}\widetilde{f}(k)$.

◆1.1.2 Fields

Scalar : $(\partial^2 + m^2)\phi = 0; \quad \phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}px}\right]$

Dirac : $(i\partial\!\!\!/ - m)\psi = 0; \quad \psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left[a^s_{\boldsymbol{p}}u^s(p)\mathrm{e}^{-\mathrm{i}px} + b^{s\dagger}_{\boldsymbol{p}}v^s(p)\mathrm{e}^{\mathrm{i}px}\right]$

Vector : $\partial^2 A^\mu = 0; \qquad A^\mu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{r=0..3}\left[a^r_{\boldsymbol{p}}\epsilon^r(p)\mathrm{e}^{-\mathrm{i}px} + a^{r\dagger}_{\boldsymbol{p}}\epsilon^{r*}(p)\mathrm{e}^{\mathrm{i}px}\right]$

TODO: 南部–Goldstone; Gravitino

◆1.1.3 Electromagnetism

Electromagnetic Fields : $A^\mu = (\phi, \boldsymbol{A})$ 【We can invert the signs, but cannot lower the index.】

Maxwell Equations : $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu; \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = ej^\nu$

Our Old Language : $\nabla\cdot\boldsymbol{B} = 0,\ \nabla\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = 0; \quad \nabla\cdot\boldsymbol{E} = ej^0,\ (\nabla\times\boldsymbol{B})_i - \frac{\partial}{\partial t}E_i = ej^i.$

: $F_{\mu\nu} = \begin{pmatrix} 0 & & \boldsymbol{E} & \\ & 0 & -B_3 & B_2 \\ -\boldsymbol{E} & B_3 & 0 & -B_1 \\ & -B_2 & B_1 & 0 \end{pmatrix}; \quad F_{\mu\nu}F^{\mu\nu} = -2\left(\|\boldsymbol{E}\|^2 - \|\boldsymbol{B}\|^2\right)$

1.2 Spinor Fields

1.3 Chiral Notation

Gamma Matrices : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad P^{\mathrm{R}}_{\mathrm{L}} = \dfrac{1 \pm \gamma_5}{2}.$

Dirac Field : $\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}; \quad \bar\psi = \psi^\dagger\gamma^0 = \begin{pmatrix} \psi_{\mathrm{R}}^\dagger & \psi_{\mathrm{L}}^\dagger \end{pmatrix}$

: $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar\sigma}\xi^s \end{pmatrix}; \; v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}$

: $\left[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\right]$

Weyl Equations : $\mathrm{i}\bar\sigma\cdot\partial\psi_L = m\psi_R; \quad \mathrm{i}\sigma\cdot\partial\psi_L = m\psi_L$

> <u>Chiral Notation</u>
>
> CPT transf. : $P\psi(t,\boldsymbol{x})P = \eta\gamma^0\psi(t,-\boldsymbol{x}) \quad (|\eta|^2 = 1)$
>
> : $T\psi(t,\boldsymbol{x})T = \gamma^1\gamma^3\psi(-t,\boldsymbol{x})$ (ignoring intrinsic phase)
>
> : $C\psi(t,\boldsymbol{x})C = -\mathrm{i}\gamma^2\psi^*(t,\boldsymbol{x}) = -\mathrm{i}(\bar\psi\gamma^0\gamma^2)^{\mathrm{T}} \quad ('')$
>
> : $\bar\psi \longrightarrow P : \eta^*\bar\psi\gamma^0 \quad T : -\bar\psi\gamma^1\gamma^3 \quad C : \mathrm{i}\bar\psi^*\gamma^2 = -\mathrm{i}(\gamma^0\gamma^2\psi)^{\mathrm{T}}$

◆1.3.1 CPT Table

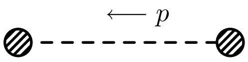
	ϕ	A^μ	$\bar\psi\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\sigma^{\mu\nu}\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$	$\mathrm{i}\bar\psi\gamma_5\psi$	∂_μ
P	$\eta\phi$	$\eta{-}{+}{+}{+}A^\mu$	$+$	$+---$	$(+---)(+---)$	$-+++$	$-$	$+---$
T	$\zeta\phi$	$\zeta{+}{-}{-}{-}A^\mu$	$+$	$+---$	$-(+---)(+---)$	$+---$	$-$	$-+++$
C	$\xi\phi^*$	$\xi{+}A^{\mu*}$	$+$	$-$	$-$	$+$	$+$	$+$

($\eta\zeta\xi = 1$; especially, photon A^μ is $(\eta,\zeta,\xi) = (-,+,-)$.)

1.4 Feynman Rules

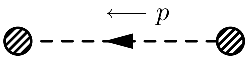
■Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

$\phi\ \phi =$ 

$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

$$\mathcal{L} \supset |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$\phi^*\ \phi =$ 

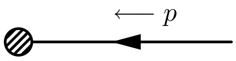
$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

(External lines equal to 1 in both cases.)

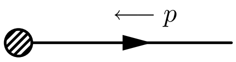
■Dirac Fermion

$$\mathcal{L} \supset \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi$$

Initial state

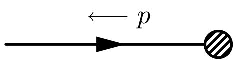
$\psi\ |\boldsymbol{p}, s\rangle =$ 

$= u^s(p)$

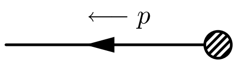
$\overline{\psi}\ |\boldsymbol{p}, s\rangle =$ 

$= \overline{v}^s(p)$

Final state

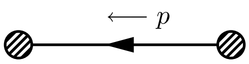
$\langle \boldsymbol{p}, s|\ \overline{\psi} =$ 

$= \overline{u}^s(p)$

$\langle \boldsymbol{p}, s|\ \psi =$ 

$= v^s(p)$

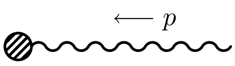
Propagator

$\psi\ \overline{\psi} =$ 

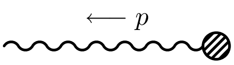
$= \dfrac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\epsilon}$

■Photon

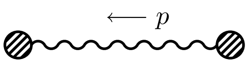
$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$A_\mu\ |\boldsymbol{p}; あ\rangle =$ 

$= \epsilon_\mu^{あ}(p)$

$\langle \boldsymbol{p}; あ|\ A_\mu =$ 

$= \epsilon_\mu^{あ*}(p)$

$A_\mu\ A_\nu =$ 

$= \dfrac{-\mathrm{i}\eta_{\mu\nu}}{p^2 + \mathrm{i}\epsilon}$

1.5 Dirac's Gamma Algebras

◆1.5.1 Traces

$$\begin{aligned}
\mathrm{Tr}(\text{any odd } \# \text{ of } \gamma\text{'s}) &= 0 &(1.1)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} &(1.2)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) &(1.3)\\
\mathrm{Tr}(\gamma_5 \text{ and any odd } \# \text{ of } \gamma\text{'s}) &= 0 &(1.4)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5) &= 0 &(1.5)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) &= -4i\epsilon^{\mu\nu\rho\sigma} &(1.6)
\end{aligned}$$

Generally, for some γ-matrices $A, B, C, \ldots,$

$$\begin{aligned}
\mathrm{Tr}(ABCDEF\cdots) &= \eta^{AB}\,\mathrm{Tr}(CDEF\cdots) - \eta^{AC}\,\mathrm{Tr}(BDEF\cdots)\\
&\quad + \eta^{AD}\,\mathrm{Tr}(BCEF\cdots) - \eta^{AE}\,\mathrm{Tr}(BCDF\cdots) + \cdots, \qquad (1.7)
\end{aligned}$$

$$\mathrm{Tr}(ABCDEF\cdots\gamma_5) = \textbf{Not Established.} \qquad (1.8)$$

To prove the second equation, we use following technique:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{Tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu); \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots\gamma_5) = \mathrm{Tr}(\gamma_5\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu). \qquad (1.9)$$

◆1.5.2 Contractions

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4 &(1.10)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu &(1.11)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} &(1.12)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu &(1.13)
\end{aligned}$$

Generally, for some γ-matrices $A, B, C, \ldots,$

$$\begin{aligned}
\text{ODD } \# : &\quad \gamma^\mu ABC\cdots\gamma_\mu = -2(\cdots CBA), &(1.14)\\
\text{EVEN } \# : &\quad \gamma^\mu ABC\cdots\gamma_\mu = \mathrm{Tr}(ABC\cdots) - \mathrm{Tr}(ABC\cdots\gamma_5)\cdot\gamma_5. &(1.15)
\end{aligned}$$

Contractions in d-dimension

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= d &(1.16)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(d-2)\gamma^\nu &(1.17)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho &(1.18)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma &(1.19)
\end{aligned}$$

Contractions of ϵ's

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu; \quad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \qquad (1.20)$$

$$\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\alpha'\beta'\gamma'} = -\left(\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\alpha_{\beta'}\delta^\beta_{\gamma'}\delta^\gamma_{\alpha'} + \delta^\alpha_{\gamma'}\delta^\beta_{\alpha'}\delta^\gamma_{\beta'} - \delta^\alpha_{\alpha'}\delta^\beta_{\gamma'}\delta^\gamma_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\delta^\gamma_{\gamma'} - \delta^\alpha_{\gamma'}\delta^\beta_{\beta'}\delta^\gamma_{\alpha'}\right) \qquad (1.21)$$

1.6 Miscellaneous Techniques

$$(p\cdot\sigma)(p\cdot\bar\sigma)=p^2$$
$$\epsilon^{ab}\epsilon^{cd}=\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc}$$
$$\sqrt{p_\mu\sigma^\mu}=\frac{p_\mu\sigma^\mu+m}{\sqrt{2(m+p^0)}}$$

◆1.6.1 Dirac Field Techniques

$$\begin{array}{lll}
\text{Dirac Equations} & : & (\not p-m)u^s(p)=0;\quad (\not p+m)v^s(p)=0\\
 & : & \bar u^s(p)(\not p-m)=0;\quad \bar v^s(p)(\not p+m)=0\\
\text{Dirac Components} & : & u^{r\dagger}(p)u^s(p)=2E_{\boldsymbol p}\delta^{rs};\quad v^{r\dagger}(p)v^s(p)=2E_{\boldsymbol p}\delta^{rs}\\
 & : & \bar u^r(p)u^s(p)=2m\delta^{rs};\quad \bar v^r(p)v^s(p)=-2m\delta^{rs};\quad \bar u^r(p)v^s(p)=\bar v^r(p)u^s(p)=0\\
\text{Spin Sums} & : & \sum_{\text{spin}}u^s(p)\bar u^s(p)=\not p+m;\quad \sum_{\text{spin}}v^s(p)\bar v^s(p)=\not p-m
\end{array}$$

◆1.6.2 Polarization Sum

Single photon case $M=\epsilon^*_\mu(k)M^\mu$

When Ward identity $k_\mu M^\mu=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu*}=\eta_{\mu\nu}M^\mu M^{\nu*}. \tag{1.22}$$

Double photons case $M=\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu}=k'_\nu M^{\mu\nu}=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\rho(k)\epsilon'^*_\nu(k')\epsilon'_\sigma(k')M^{\mu\nu}M^{\rho\sigma*}=\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \tag{1.23}$$

【See Sec. B.3 for verbose information.】

◆1.6.3 Fierz identities

For Dirac spinors a,b,c,d and their left-handed projections $a_{\mathrm L}:=P_{\mathrm L}a$ etc.,

$$(\bar a_{\mathrm L}\gamma^\mu b_{\mathrm L})(\bar c_{\mathrm L}\gamma_\mu d_{\mathrm L})=-(\bar a_{\mathrm L}\gamma^\mu d_{\mathrm L})(\bar c_{\mathrm L}\gamma_\mu b_{\mathrm L}) \tag{1.24}$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta};\qquad (\bar\sigma^\mu)_{\alpha\beta}(\bar\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{1.25}$$

◆1.6.4 Gordon identity

For $P:=p'+p$ and $q:=p'-p$,

$$\bar u(p')\gamma^\mu u(p)=\bar u(p')\left[\frac{P^\mu+\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\qquad \bar u(p')\gamma^\mu v(p)=\bar u(p')\left[\frac{q^\mu+\mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \tag{1.26}$$

$$\bar v(p')\gamma^\mu v(p)=-\bar v(p')\left[\frac{P^\mu+\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]v(p)\qquad \bar v(p')\gamma^\mu u(p)=-\bar v(p')\left[\frac{q^\mu+\mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \tag{1.27}$$

◆1.6.5 Gauge group algebra

For a gauge group G s.t.

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c, \tag{1.28}$$

we have two constants which **depend on representation** r.

$$\mathrm{Tr}(t^a t^b) =: C(r)\delta^{ab}; \qquad \mathrm{Tr}(t^a t^a) =: C_2(r)\cdot \mathbf{1} \quad \text{(quadratic Casimir operator)} \tag{1.29}$$

They satisfy

$$C(r) = \frac{d(r)}{d(\mathrm{Adj.})} C_2(r), \qquad t^a t^b t^a = \left[C_2(r) - \frac{1}{2} C_2(\mathrm{Adj.})\right] t^b. \tag{1.30}$$

$$f^{acd} f^{bcd} = C_2(\mathrm{Adj.})\delta^{ab}, \qquad f^{abc} t^b t^c = \frac{1}{2}\mathrm{i} C_2(\mathrm{Adj.}) t^a. \tag{1.31}$$

For $\mathrm{SU}(N)$ For $\mathrm{SU}(N)$ groups and its fundamental representation N, we have

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2-1}{2N}, \quad C(\mathrm{Adj.}) = C_2(\mathrm{Adj.}) = N; \quad (t^a)_{ij}(t^a)_{kj} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{\delta_{ij}\delta_{kl}}{N}\right).$$

1.7 Loop Integrals and Dimensional Regularization

◆1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots x_n \; \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{1.32}$$

$$\frac{1}{A_1 A_2} = \int_0^1 \mathrm{d}x \frac{1}{[xA_1 + (1-x)A_2]^2} \tag{1.33}$$

◆1.7.2 d-dimensional integrals in Minkowski space

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{1.34}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.35}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.36}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{1.37}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{4} \tag{1.38}$$

Here we can use following expansions: $(\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - (d-4)\frac{\log \Delta}{2} + \mathrm{O}\left((d-4)^2\right) \quad \text{around } d=4, \tag{1.39}$$

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathrm{O}(x) \quad \text{around } x = 0, \tag{1.40}$$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left[\frac{1}{x+n} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathrm{O}(x+n)\right] \quad \text{around } x = -n. \tag{1.41}$$

and we get following expansion:

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2} \left[\left(\frac{2}{4-d} - \gamma + \log 4\pi\right) - \log\Delta + \mathrm{O}(4-d)\right]. \tag{1.42}$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S-matrix elements. To compute the S-matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \to \left(\frac{1}{\Delta - \mathrm{i}\epsilon}\right)^{n-\frac{d}{2}}. \tag{1.43}$$

1.8 Cross Sections and Decay Rates

■General expression (The mass dimension of $\mathcal{M}$ is $2-N_\mathrm{f}$ for $\mathrm{d}\sigma$ and $3-N_\mathrm{f}$ for $\mathrm{d}\Gamma$.)

$$\mathrm{d}\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(p_A, p_B \to \{p_f\})\right|^2 (2\pi)^4\delta^{(4)}\left(p_A + p_B - \{p_f\}\right) \tag{1.44}$$

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(m_A \to \{p_f\})\right|^2 (2\pi)^4\delta^{(4)}\left(m_A - \{p_f\}\right) \quad \text{(in A-rest frame.)} \tag{1.45}$$

■2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int\frac{\mathrm{d}^3 p_1}{(2\pi)^3}\int\frac{\mathrm{d}^3 p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\delta^{(4)}\left(E_\mathrm{cm} - (p_1 + p_2)\right) \qquad \text{(in center-of-mass frame)} \tag{1.46}$$

$$= \int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}\frac{2\left\|\boldsymbol{p_1}\right\|}{E_\mathrm{cm}} \tag{1.47}$$

$$= \frac{1}{8\pi}\sqrt{1 - \frac{2({m_1}^2 + {m_2}^2)}{E_\mathrm{cm}^2} + \frac{({m_1}^2 - {m_2}^2)^2}{E_\mathrm{cm}^4}} \xrightarrow{m_2=0} \frac{1}{8\pi}\left(1 - \frac{{m_1}^2}{{E_\mathrm{cm}}^2}\right) \tag{1.48}$$

■Kinematics of Decay

$$K \to p_1 + p_2 \qquad \text{or} \qquad \begin{pmatrix} M \\ \boldsymbol{0}\end{pmatrix} \to \begin{pmatrix}\sqrt{p^2 + {m_1}^2} \\ \boldsymbol{p}\end{pmatrix} + \begin{pmatrix}\sqrt{p^2 + {m_2}^2} \\ -\boldsymbol{p}\end{pmatrix}; \tag{1.49}$$

$$\left\|\boldsymbol{p}\right\|^2 = \frac{1}{4}\left[M^2 - 2\left({m_1}^2 + {m_2}^2\right) + \frac{\left({m_1}^2 - {m_2}^2\right)^2}{M^2}\right] \approx \left(\frac{M^2 - {m_1}^2}{2M}\right)^2$$

$$E_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2M}, \qquad E_2 = \frac{M^2 + {m_2}^2 - {m_1}^2}{2M};$$

$$K\cdot p_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2}, \qquad p_1\cdot p_2 = \frac{M^2 - \left({m_1}^2 + {m_2}^2\right)}{2}.$$

- Fierz Transf.
- Noether current
- Majorana Ferminos
- Feynman Rules(A.1)

2 Standard Model

We use the following notations for the gauge group.

■Representation

$$\begin{aligned}
\mathrm{SU}(3): &\quad G_\mu = G_\mu^a \tau^a \ ; \quad \left[\tau^a, \tau^b\right] = \mathrm{i} f^{abc} \tau^c, \quad \mathrm{Tr}\left(\tau^a \tau^b\right) = \frac{1}{2}\delta^{ab}, \\
\mathrm{SU}(2): &\quad W_\mu = W_\mu^a T^a \ ; \quad \left[T^a, T^b\right] = \mathrm{i}\epsilon^{abc} T^c, \quad \mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}, \\
\mathrm{U}(1): &\quad B_\mu
\end{aligned}$$

■Field Strength

$$\begin{aligned}
F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} g [A_\mu, A_\nu] \\
&= \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c\right) T^a
\end{aligned}$$

■Abridged Notation

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

2.1 Symmetries and Fields

■Gauge Group

$$\mathrm{SU}(3)_{\mathrm{strong}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_Y \tag{2.1}$$

■Field Content

	$\mathrm{SU}(3)_{\mathrm{strong}}$	$\mathrm{SU}(2)_{\mathrm{weak}}$	$\mathrm{U}(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_{\mathrm{L}} Q_i$: Left-handed quarks	**3**	**2**	1/6
$P_{\mathrm{L}} U_i$: Right-handed up-type quarks	**3**	**1**	2/3
$P_{\mathrm{R}} D_i$: Right-handed down-type quarks	**3**	**1**	−1/3
$P_{\mathrm{R}} L_i$: Left-handed leptons	**1**	**2**	−1/2
$P_{\mathrm{R}} E_i$: Right-handed leptons	**1**	**1**	−1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	**1**	**2**	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	**8**	**1**	0
W : Weak bosons	**1**	**3**	0
B : B boson	**1**	**1**	0

■Full Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{湯川}}, \tag{2.2}$$

$$\text{where} \quad \mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^{a}_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^{a}_{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)H\right|^2 - V(H), \tag{2.4}$$

$$\begin{aligned}\mathcal{L}_{\text{matter}} = {} & \overline{Q}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \frac{1}{6}\mathrm{i}g_1B_\mu\right)P_{\mathrm{L}}Q_i \\ & + \overline{U}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \frac{2}{3}\mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}U_i \\ & + \overline{D}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu + \frac{1}{3}\mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}D_i \\ & + \overline{L}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)P_{\mathrm{L}}L_i \\ & + \overline{E}_i\mathrm{i}\gamma^\mu\left(\partial_\mu + \mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}E_i,\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{\text{湯川}} = -\overline{U}_i(y_u)_{ij}HP_{\mathrm{L}}Q_j + \overline{D}_i(y_d)_{ij}H^\dagger P_{\mathrm{L}}Q_j + \overline{E}_i(y_e)_{ij}H^\dagger P_{\mathrm{L}}L_j + \text{H. c.} \tag{2.6}$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix B.4.

■Gauge Kinetic Terms

the gauge kinetic terms can be expanded as

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = {} & -\frac{1}{4}(\partial B)(\partial B) \\ & -\frac{1}{4}(\partial W^a)(\partial W^a) - g_2\epsilon^{abc}(\partial_\mu W^a_\nu)W^{\mu b}W^{\nu c} - \frac{{g_2}^2}{4}\left(\epsilon^{eab}W^a_\mu W^b_\nu\right)\left(\epsilon^{ecd}W^{c\mu}W^{d\nu}\right) \\ & -\frac{1}{4}(\partial G^a)(\partial G^a) - g_3f^{abc}(\partial_\mu G^a_\nu)G^{\mu b}G^{\nu c} - \frac{{g_3}^2}{4}\left(f^{eab}G^a_\mu G^b_\nu\right)\left(f^{ecd}G^{c\mu}G^{d\nu}\right).\end{aligned} \tag{2.7}$$

2.2 Higgs Mechanism

■Higgs Potential

The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2(H^\dagger H) + \lambda\left(H^\dagger H\right)^2. \tag{2.8}$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + \mathrm{i}\phi_2 \\ v + (h + \mathrm{i}\phi_3)\end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.9}$$

Here h is the "Higgs boson," and ϕ_i are 南部–Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2h^2, \tag{2.10}$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right) H\right|^2 \tag{2.11}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\left[{g_2}^2{W_1}^2 + {g_2}^2{W_2}^2 + (g_1 B - g_2 W_3)^2\right]. \tag{2.12}$$

Redefining the gauge fields (with concerning the norms) as

$$W^\pm_\mu := \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}W^2_\mu), \qquad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} := \begin{pmatrix} \cos\theta_\mathrm{w} & -\sin\theta_\mathrm{w} \\ \sin\theta_\mathrm{w} & \cos\theta_\mathrm{w} \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \tag{2.13}$$

where

$$\tan\theta_\mathrm{w} := \frac{g_1}{g_2}, \qquad e := -\frac{g_1 g_2}{\sqrt{{g_1}^2 + {g_2}^2}}; \qquad g_Z := \sqrt{{g_1}^2 + {g_2}^2}; \tag{2.14}$$

$$g_1 = \frac{|e|}{\cos\theta_\mathrm{w}} = g_Z \sin\theta_\mathrm{w}, \qquad g_2 = \frac{|e|}{\sin\theta_\mathrm{w}} = g_Z \cos\theta_\mathrm{w}. \tag{2.15}$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4}\left[{g_2}^2 W^{+\mu} W^-_\mu + \frac{{g_Z}^2}{2} Z^\mu Z_\mu\right]. \tag{2.16}$$

Here we have omitted the 南部–Goldstone bosons.

> Here we present another form:
>
> $$g_1 B_\mu = |e| A_\mu - \tan\theta_\mathrm{w} Z_\mu, \tag{2.17}$$
>
> $$g_2 W_\mu = \frac{g_2}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) + \left(\frac{|e|}{\tan\theta_\mathrm{w}} Z_\mu + |e| A_\mu\right) T^3, \tag{2.18}$$
>
> $$Z^0_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_2 W^3_\mu - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_1 W^3_\mu + g_2 B_\mu) \tag{2.19}$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \tag{2.20}$$

■**Gauge Term** The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W^a_{\mu\nu} = &(\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ &- 4\mathrm{i}g\left[(\partial W^3)^{\mu\nu} W^+_\mu W^-_\nu + (\partial W^+)^{\mu\nu} W^-_\mu W^3_\nu + (\partial W^-)^{\mu\nu} W^3_\mu W^+_\nu\right] \\ &- 2g^2(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left(W^+_\mu W^+_\nu W^-_\rho W^-_\sigma - 2W^3_\mu W^3_\nu W^+_\rho W^-_\sigma\right), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_{\text{gauge}} := &-\frac{1}{4}\left[G^{a\mu\nu} G^a_{\mu\nu} + (\partial Z)^{\mu\nu}(\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu}(\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu}(\partial W^-)_{\mu\nu}\right] \\ &+ \frac{\mathrm{i}|e|}{\tan\theta_\mathrm{w}}\left[(\partial W^+)^{\mu\nu} W^-_\mu Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W^+_\nu + (\partial Z)^{\mu\nu} W^+_\mu W^-_\nu\right] \\ &+ \mathrm{i}|e|\left[(\partial W^+)^{\mu\nu} W^-_\mu A_\nu + (\partial W^-)^{\mu\nu} A_\mu W^+_\nu + (\partial A)^{\mu\nu} W^+_\mu W^-_\nu\right] \\ &+ (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left[\frac{|e|^2}{2\sin^2\theta_\mathrm{w}} W^+_\mu W^+_\nu W^-_\rho W^-_\sigma + \frac{|e|^2}{\tan^2\theta_\mathrm{w}} W^+_\mu Z_\nu W^-_\rho Z_\sigma\right. \\ &\qquad \left. + \frac{|e|^2}{\tan\theta_\mathrm{w}}\left(W^+_\mu Z_\nu W^-_\rho A_\sigma + W^+_\mu A_\nu W^-_\rho Z_\sigma\right) + |e|^2 W^+_\mu A_\nu W^-_\rho A_\sigma\right]. \end{aligned} \tag{2.21}$$

■湯川 Term

$$
\begin{aligned}
\mathcal{L}_{湯川} &= -\overline{U}y_u HP_{\mathrm{L}}Q + \overline{D}y_d H^{\dagger}P_{\mathrm{L}}Q + \overline{E}y_e H^{\dagger}P_{\mathrm{L}}L + \mathrm{H.\,c.} \\
&= -\overline{U}y_u\epsilon^{\alpha\beta}H^{\alpha}P_{\mathrm{L}}Q^{\beta} + \overline{D}y_d {H^{\dagger}}^{\alpha}P_{\mathrm{L}}Q^{\alpha} + \overline{E}y_e {H^{\dagger}}^{\alpha}P_{\mathrm{L}}L^{\alpha} + \mathrm{H.\,c.} \\
&= \frac{v+h}{\sqrt{2}}\left(\overline{U}y_u P_{\mathrm{L}}Q^1 + \overline{D}y_d P_{\mathrm{L}}Q^2 + \overline{E}y_e P_{\mathrm{L}}L^2\right) + \mathrm{H.\,c.}
\end{aligned} \tag{2.22}
$$

2.3 Full Lagrangian After Higgs Mechanism

Now we have the following Lagrangian (with omitting P_{L} etc.):

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} + {m_W}^2 W^+W^- + \frac{{m_Z}^2}{2}Z^2 \\
\text{【Higgs】} & + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}\mu^2 h^2 - \sqrt{\frac{\lambda}{2}}m_h h^3 - \frac{1}{4}\lambda h^4 \\
& + \frac{v{g_2}^2}{4}W^+W^-h + \frac{v({g_1}^2+{g_2}^2)}{8}Z^2 h \\
& + \frac{{g_2}^2}{4}W^+W^-h^2 + \frac{{g_1}^2+{g_2}^2}{8}Z^2h^2 \\
& + \frac{1}{\sqrt{2}}h\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}h\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}h\bar{E}y_eL^2 + \mathrm{H.\,c.} \\
\text{【SU(3)】} & + \bar{Q}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)Q + \bar{U}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)U + \bar{D}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)D + \bar{L}\left(\mathrm{i}\partial\!\!\!/\right)L + \bar{E}\left(\mathrm{i}\partial\!\!\!/\right)E \\
\text{【}W\text{】} & + \bar{Q}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+T^+ + W\!\!\!/^-T^-\right)Q + \bar{L}\frac{g_2}{\sqrt{2}}\left(W\!\!\!/^+T^+ + W\!\!\!/^-T^-\right)L \\
\text{【}A\&Z^0\text{】} & + \bar{Q}\left[\left(T^3+\frac{1}{6}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right)Z\!\!\!/^0\right]Q \\
& + \bar{U}\left(\frac{2}{3}|e|A\!\!\!/ - \frac{2|e|s}{3c}Z\!\!\!/\right)U \\
& + \bar{D}\left(-\frac{1}{3}|e|A\!\!\!/ + \frac{|e|s}{3c}Z\!\!\!/\right)D \\
& + \bar{L}\left[\left(T^3-\frac{1}{2}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right)Z\!\!\!/^0\right]L \\
& + \bar{E}\left(-|e|A\!\!\!/ + \frac{|e|s}{c}Z\!\!\!/\right)E \\
\text{【湯川項】} & + \frac{1}{\sqrt{2}}v\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}v\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}v\bar{E}y_eL^2 + \mathrm{H.\,c.}
\end{aligned} \tag{2.23}
$$

2.4 Mass Eigenstates

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := vy_\bullet/\sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \qquad (m_i \geq 0). \tag{2.24}
$$

Using this Ψ and Φ, we can rotate the basis as

$$Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \qquad\qquad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \tag{2.25}$$

and now we have the 湯川 terms in mass eigenstates as

$$\mathcal{L}_{\text{湯川}} = \left(1 + \frac{1}{v}h\right)\left[(m_u)_i \overline{U}_i P_{\mathrm{L}} Q_i^1 + (m_d)_i \overline{D}_i P_{\mathrm{L}} Q_i^2 + (m_e)_i \overline{E}_i P_{\mathrm{L}} L_i^2 + \text{H. c.}\right]. \tag{2.26}$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark–W interactions do change drastically, as

$$\begin{aligned}
\mathcal{L} &\supset \overline{Q}\mathrm{i}\gamma^\mu\left(-\mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) P_{\mathrm{L}} Q \qquad (2.27)\\
&= \overline{Q}\frac{g_2}{\sqrt{2}}\left(\not{W}^+ T^+ + \not{W}^- T^-\right) P_{\mathrm{L}} Q \quad + \quad \text{(interaction terms with } Z \text{ and } A\text{)} \qquad (2.28)\\
&\mapsto \frac{g_2}{\sqrt{2}}\begin{pmatrix}\overline{Q}^1\Psi_u & \overline{Q}^2\Psi_d\end{pmatrix}\begin{pmatrix}0 & \not{W}^+\\ \not{W}^- & 0\end{pmatrix} P_{\mathrm{L}}\begin{pmatrix}\Psi_u^\dagger Q^1\\ \Psi_d^\dagger Q^2\end{pmatrix} + (\quad\cdots\quad) \qquad (2.29)\\
&= \frac{g_2}{\sqrt{2}}\left[\overline{Q}^2\not{W}^- X P_{\mathrm{L}} Q^1 + \overline{Q}^1\not{W}^+ X^\dagger P_{\mathrm{L}} Q^2\right] + (\quad\cdots\quad), \qquad (2.30)
\end{aligned}$$

where $X := \Psi_d \Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP-symmetry.

> In our notation, CP-transformation of a spinor is described as
>
> $$\mathscr{C}\mathscr{P}(\psi) = -\mathrm{i}\eta^*(\overline{\psi}\gamma^2)^{\mathrm{T}}, \quad \mathscr{C}\mathscr{P}(\overline{\psi}) = \mathrm{i}\eta(\gamma^2\psi)^{\mathrm{T}}, \tag{2.31}$$
>
> where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,
>
> $$\begin{aligned}
\mathscr{C}\mathscr{P}\left(\overline{Q}^2\not{W}^- X P_{\mathrm{L}} Q^1\right) &= (\gamma^2 Q^2)^{\mathrm{T}}\mathscr{P}(-\not{W}^+) X P_{\mathrm{L}}(\overline{Q}^1\gamma^2)^{\mathrm{T}}\\
&= -W_\mu^{+P}(\gamma^2 Q^2)^{\mathrm{T}}(\overline{Q}^1 X^{\mathrm{T}}\gamma^2 P_{\mathrm{L}}\gamma^{\mu\mathrm{T}})^{\mathrm{T}}\\
&= (\overline{Q}^1\not{W}^+ X^{\mathrm{T}} P_{\mathrm{L}} Q^2).
\end{aligned} \tag{2.32}$$
>
> Therefore, we can see that the CP-symmetry is maintained if and only if $X^{\mathrm{T}} = X^\dagger$, that is, if and only if X is a real matrix.

以上より，標準模型の Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} \\
\text{【質量項】} \quad & + \; {m_W}^2 W^+ W^- \quad + \; \frac{{m_Z}^2}{2} Z^2 \\
& + \; \bar{U} M_u P_{\mathrm{L}} Q^1 \quad + \; \bar{D} M_d P_{\mathrm{L}} Q^2 \quad + \; \bar{E} M_e P_{\mathrm{L}} L^2 \quad + \; \mathrm{H.\,c.} \\
\text{【Higgs Field】} \quad & + \; \frac{1}{2}(\partial_\mu h)^2 \quad -\frac{1}{2}\mu^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
\text{【Higgs との結合】} \quad & + \; \frac{v{g_2}^2}{4} W^+ W^- h \quad + \; \frac{v({g_1}^2 + {g_2}^2)}{8} Z^2 h \\
& + \; \frac{{g_2}^2}{4} W^+ W^- h^2 \quad + \; \frac{{g_1}^2 + {g_2}^2}{8} Z^2 h^2 \\
& + \; \frac{1}{v} \bar{U} M_u P_{\mathrm{L}} Q^1 h \quad + \; \frac{1}{v} \bar{D} M_d P_{\mathrm{L}} Q^h 2 \quad + \; \frac{1}{v} \bar{E} M_e P_{\mathrm{L}} L^2 h \quad + \; \mathrm{H.\,c.} \\
\text{【SU(3) および微分項】} \quad & + \; \bar{Q} \left(\mathrm{i}\not{\partial} + g_3 \not{G} \right) P_{\mathrm{L}} Q \quad + \; \bar{U} \left(\mathrm{i}\not{\partial} + g_3 \not{G} \right) P_{\mathrm{R}} U \quad + \; \bar{D} \left(\mathrm{i}\not{\partial} + g_3 \not{G} \right) P_{\mathrm{R}} D \\
& + \; \bar{L} \left(\mathrm{i}\not{\partial} \right) P_{\mathrm{L}} L \quad + \; \bar{E} \left(\mathrm{i}\not{\partial} \right) P_{\mathrm{R}} E \\
\text{【W boson】} \quad & + \; \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \not{W}^- X P_{\mathrm{L}} Q^1 + \bar{Q}^1 \not{W}^+ X^\dagger P_{\mathrm{L}} Q^2 \right] \qquad \text{【$\leftarrow$ CP and flavor violating!】} \\
& + \; \bar{L} \frac{g_2}{\sqrt{2}} \left(\not{W}^+ T^+ + \not{W}^- T^- \right) P_{\mathrm{L}} L \\
\text{【A\&Z^0 boson】} \quad & + \; \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| \not{A} + \left(\frac{|e| c}{s} T^3 - \frac{|e| s}{6c} \right) \not{Z}^0 \right] P_{\mathrm{L}} Q \\
& + \; \bar{U} \left(\frac{2}{3} |e| \not{A} - \frac{2|e| s}{3c} \not{Z} \right) P_{\mathrm{R}} U \\
& + \; \bar{D} \left(-\frac{1}{3} |e| \not{A} + \frac{|e| s}{3c} \not{Z} \right) P_{\mathrm{R}} D \\
& + \; \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| \not{A} + \left(\frac{|e| c}{s} T^3 + \frac{|e| s}{2c} \right) \not{Z}^0 \right] P_{\mathrm{L}} L \\
& + \; \bar{E} \left(-|e| \not{A} + \frac{|e| s}{c} \not{Z} \right) P_{\mathrm{R}} E
\end{aligned}
\tag{2.33}
$$

となる。

2.5 Chiral Notation

以上の Lagrangian を chiral 表示で表すと，まず最初は

$$
\begin{aligned}
\mathcal{L} =&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ Q_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \mathrm{i}g_2 W_{\mu} - \frac{1}{6}\mathrm{i}g_1 B_{\mu}\right)Q_{\mathrm{L}}\\
&+ U_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \frac{2}{3}\mathrm{i}g_1 B_{\mu}\right)U_{\mathrm{R}}\\
&+ D_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} + \frac{1}{3}\mathrm{i}g_1 B_{\mu}\right)D_{\mathrm{R}}\\
&+ L_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu} + \frac{1}{2}\mathrm{i}g_1 B_{\mu}\right)L_{\mathrm{L}}\\
&+ E_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu}\left(\partial_{\mu} + \mathrm{i}g_1 B_{\mu}\right)E_{\mathrm{R}}\\
&+ U_{\mathrm{R}}^{\dagger}y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger}y_d H^{\dagger}Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger}y_e H^{\dagger}L_{\mathrm{L}} + \text{H. c.}\\
=&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger}\\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right)\\
&+ g_2\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}Q_{\mathrm{L}} + L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}L_{\mathrm{L}}\right)\\
&+ g_1\left(\frac{1}{6}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}Q_{\mathrm{L}} + \frac{2}{3}U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}U_{\mathrm{R}} - \frac{1}{3}D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}D_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}L_{\mathrm{L}} - E_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}E_{\mathrm{R}}\right)\\
&+ U_{\mathrm{R}}^{\dagger}y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger}y_d H^{\dagger}Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger}y_e H^{\dagger}L_{\mathrm{L}} + \text{H. c.}
\end{aligned}
\tag{2.34}
$$

であり，そして最終的には

$$
\begin{aligned}
\mathcal{L} =&(\text{Gauge bosons and Higgs})\\
&+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger}\\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right)\\
&+ m_u(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}}) + (\text{quarks}) + m_e(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}}) + (\text{leptons})\\
&+ \frac{m_u}{v}(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}})h + (\text{quarks}) + \frac{m_e}{v}(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}})h + (\text{leptons})\\
&+ \frac{g_2}{\sqrt{2}}\left[\left(d_{\mathrm{L}}^{\dagger}\ s_{\mathrm{L}}^{\dagger}\ b_{\mathrm{L}}^{\dagger}\right)\bar{\sigma}^{\mu}W_{\mu}^{-}X\begin{pmatrix}u_{\mathrm{L}}\\ c_{\mathrm{L}}\\ t_{\mathrm{L}}\end{pmatrix} + \left(u_{\mathrm{L}}^{\dagger}\ c_{\mathrm{L}}^{\dagger}\ t_{\mathrm{L}}^{\dagger}\right)\bar{\sigma}^{\mu}W_{\mu}^{+}X^{\dagger}\begin{pmatrix}d_{\mathrm{L}}\\ s_{\mathrm{L}}\\ b_{\mathrm{L}}\end{pmatrix}\right]\\
&+ \frac{g_2}{\sqrt{2}}\left[\nu_e^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{+}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{-}\nu_e\right]\\
&+ |e|\left[\frac{2}{3}u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}u_{\mathrm{L}} - \frac{1}{3}d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}d_{\mathrm{L}} + \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}u_{\mathrm{R}} - \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}d_{\mathrm{R}} + (\text{quarks})\right.\\
&\qquad\left. - e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}e_{\mathrm{L}} - e_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}e_{\mathrm{R}} + (\text{leptons})\right]\\
&+ \frac{|e|s}{c}\left[\left(\frac{c^2}{2s^2} - \frac{1}{6}\right)u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}u_{\mathrm{L}} - \left(\frac{c^2}{2s^2} + \frac{1}{6}\right)d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}d_{\mathrm{L}} - \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}u_{\mathrm{R}} + \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}d_{\mathrm{R}}\right.\\
&\qquad\left. + \left(\frac{c^2}{2s^2} + \frac{1}{2}\right)\nu_e^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2}\right)e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}e_{\mathrm{L}} + e_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}e_{\mathrm{R}} + (\text{others})\right]
\end{aligned}
\tag{2.35}
$$

となる。

2.6 Values of SM Parameters

◆2.6.1 Experimental Values

Low energy values

$$\alpha_{\mathrm{EM}} = 1/137.035999679(94) \qquad G_{\mathrm{F}} = 1.166367(5) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

Electroweak scale 【These values are all in $\overline{\mathrm{MS}}$ scheme.】

$$\begin{aligned} \alpha_{\mathrm{EM}}^{-1}(m_Z) &= 127.925(16) & m_W(m_W) &= 80.398(25)\,\mathrm{GeV} \\ \alpha_{\mathrm{EM}}^{-1}(m_\tau) &= 133.452(16) & m_Z(m_Z) &= 91.1876(21)\,\mathrm{GeV} \\ \alpha_{\mathrm{s}}(m_Z) &= 0.1176(20) & \sin^2\theta_{\mathrm{W}}(m_Z) &= 0.23119(14) \\ \Gamma_{l^+l^-} &= 83.984(86)\,\mathrm{MeV} & \sin^2\theta_{\mathrm{eff}} &= 0.23149(13) \end{aligned}$$

Fundamental masses

$$\begin{array}{lll} e : 0.510998910(13)\,\mathrm{MeV} & u : 1.5 \text{ to } 3.3\,\mathrm{MeV} & d : 3.5 \text{ to } 6.0\,\mathrm{MeV} \\ \mu : 105.658367(4)\,\mathrm{MeV} & c : 1.27^{+0.07}_{-0.11}\,\mathrm{GeV} & s : 104^{+26}_{-34}\,\mathrm{MeV} \\ \tau : 1.77784(17)\,\mathrm{GeV} & t : 171.2_{\pm 2.1}\,\mathrm{GeV} & b : 4.20^{+0.17}_{-0.07}\,\mathrm{GeV} \end{array}$$

$$\begin{array}{lll} \pi^\pm : 139.57018(35)\,\mathrm{MeV} & K^\pm : 493.677(16)\,\mathrm{MeV} & p : 938.27203(8)\,\mathrm{MeV} \\ \pi^0 : 139.766(6)\,\mathrm{MeV} & K^0 : 497.614(24)\,\mathrm{MeV} & n : 939.56536(8)\,\mathrm{MeV} \end{array}$$

Fundamental Lifetime (also $c\tau$ for some particles)

$$\begin{array}{lll} \mu : 2.197019(21)\,\mu\mathrm{s} \quad (658\,\mathrm{m}) & \pi^\pm : 2.6033(5) \times 10^{-8}\,\mathrm{s} & K^\pm : 1.2380(21) \times 10^{-8}\,\mathrm{s} \\ \tau : 2.906(10) \times 10^{-13}\,\mathrm{s} \quad (87\,\mu\mathrm{m}) & \pi^0 : 8.4(6) \times 10^{-17}\,\mathrm{s} & K^0_{\mathrm{S}} : 8.953(5) \times 10^{-11}\,\mathrm{s} \\ & & K^0_{\mathrm{L}} : 5.116(20) \times 10^{-8}\,\mathrm{s} \end{array}$$

CKM matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} 0.97419(22) & 0.2257(10) & 0.00359(16) \\ 0.2256(10) & 0.97334(23) & 0.0415(11) \\ 0.00874(37) & 0.0407(10) & 0.999133(44) \end{pmatrix} \sim \begin{pmatrix} 1-\epsilon^2 & \epsilon & \epsilon^4 \\ \epsilon & 1-\epsilon^2 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1-\epsilon^4 \end{pmatrix} \quad \text{for } \epsilon \sim 0.23 \tag{2.36}$$

◆2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$e \sim 0.313, \qquad g_1 \sim 0.358, \qquad g_2 \sim 0.651; \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246\,\mathrm{GeV} \tag{2.37}$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.98 \end{pmatrix}, \quad y_d \sim \begin{pmatrix} 3\times 10^{-5} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad y_e \sim \begin{pmatrix} 3\times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \tag{2.38}$$

Also, for $m_h \sim 120\,\mathrm{GeV}$, we can estimate the Higgs potential as $\mu \sim 85\,\mathrm{GeV}$ and $\lambda \sim 0.12$.

3 楊-Mills Theory

3.1 U(1) Theory

◆3.1.1 General SU(N)

$$\mathcal{L} = \tag{3.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{\mathrm{i}}[A_\mu, A_\nu] \tag{3.2}$$

$$\tag{3.3}$$

■Gauge Transformation

For any Lie group G,

$$V : \mathbb{R}^{1,3} \to G \tag{3.4}$$

$$A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1} \tag{3.5}$$

$$F_{\mu\nu} \mapsto V F_{\mu\nu} V^{-1} \tag{3.6}$$

If the gauge group G is **compact**, it has a finite-dimensional unitary representation.

$$\text{For} \quad t^a : \text{hermitian representation,} \tag{3.7}$$

$$[t^a, t^b] = \mathrm{i}f^{ab}{}_c t^c \quad \text{and} \quad f \in \mathbb{R} \tag{3.8}$$

$$0 = f^D{}_{ab} f^E{}_{Dc} + f^D{}_{ca} f^E{}_{Db} + f^D{}_{bc} f^E{}_{Da} \tag{3.9}$$

$$V = \exp\left[\mathrm{i}\alpha^a T^a\right] \quad \text{for} \quad \alpha^a \in \mathbb{R} \tag{3.10}$$

$$\tag{3.11}$$

with generators $\{T^a\}$ written in an hermitian representation,

$$A_\mu \mapsto V\left(A_\mu + \frac{1}{g}(\partial_\mu \alpha^a)T^a\right)V^{-1} \tag{3.12}$$

$$\simeq A_\mu + \frac{1}{g}(\partial_\mu \alpha^a)T^a + i\alpha^a A_\mu^b [T^a, T^b] \tag{3.13}$$

$$D_\mu = \partial_\mu - \mathrm{i}g A_\mu^a T^a \qquad \text{(for appropriate representation)} \tag{3.14}$$

4 Spinor

■$\eta^{\mu\nu} = (-,+,+,+)$ case

Grassmann Number	: $(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$
	: $\Longrightarrow$ for $a, b \in \mathbb{G}^\mathbb{R}$, $ab \in \mathrm{i}\mathbb{G}^\mathbb{R}$
γ matrix	: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$
	: $\gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$ etc...
	: $(\mathrm{i}\gamma^0)^\dagger := \mathrm{i}\gamma^0, \quad {\gamma^i}^\dagger := \gamma^i$
Dirac Conjugate	: $\bar{\psi} = \mathrm{i}\psi^\dagger\gamma^0$

5 Supersymmetry

5.1 General Relations

■Coordinates

$$y := x + \mathrm{i}\theta\sigma\bar{\theta}, \qquad y^{+} := x - \mathrm{i}\theta\sigma\bar{\theta} \tag{5.1}$$

5.2 Conventions

Martin (SUSY Primer)	$\epsilon_\alpha = \sqrt{2}\xi_\alpha$	Wess&Bagger (SUSY and SUGRA)
$\delta\phi = \epsilon\psi$	same	$\delta A = \sqrt{2}\xi\psi$
$\delta\psi_\alpha = \mathrm{i}(\sigma^\mu\epsilon^\dagger)_\alpha \mathrm{D}_\mu\phi + \epsilon_\alpha F$	same	$\delta\psi = \mathrm{i}\sqrt{2}\sigma^\mu\bar{\xi}\mathrm{D}_\mu A + \sqrt{2}\xi F$
$\delta F = \mathrm{i}\epsilon^\dagger\bar{\sigma}^\mu\mathrm{D}_\mu\psi + \sqrt{2}g(T^a\phi)\epsilon^\dagger\lambda^{\dagger a}$		
$\delta A_\mu = \frac{1}{\sqrt{2}}\left(\epsilon^\dagger\bar{\sigma}^\mu\lambda^a + \lambda^{\dagger a}\bar{\sigma}^\mu\epsilon\right)$	$\lambda_{\mathrm{M}} = \mathrm{i}\lambda_{\mathrm{WB}}$	$\delta v_\mu = \mathrm{i}\left(\xi^\dagger\bar{\sigma}^\mu\lambda^a - \lambda^\dagger\bar{\sigma}^\mu\xi\right)$
$\delta\lambda = \frac{\mathrm{i}}{2\sqrt{2}}(\sigma^\mu\sigma^\nu\epsilon)F_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon D$		
$\delta D = \frac{\mathrm{i}}{\sqrt{2}}\left(\epsilon^\dagger\bar{\sigma}^\mu\mathrm{D}_\mu\lambda - \mathrm{D}_\mu\lambda^\dagger\bar{\sigma}^\mu\epsilon\right)$		

5.3 Chiral Superfields

■Definition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{5.2}$$

■Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{5.3}$$

$$= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x) \tag{5.4}$$

$$\Phi^\dagger = \phi^*(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{\mathrm{i}}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x) \tag{5.5}$$

■Changing Bases

$$\phi(y) = \phi(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \tag{5.6}$$

$$= \phi(y^+) + 2\mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y^+) + \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y^+) \tag{5.7}$$

$$\phi(y^+) = \phi(x) - \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \tag{5.8}$$

$$= \phi(y) - 2\mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y) + \theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y) \tag{5.9}$$

$$\phi(x) = \phi(y) - \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y) \tag{5.10}$$

$$= \phi(y^+) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(y^+) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(y^+) \tag{5.11}$$

■Product of Chiral Superfields

$$
\begin{aligned}
\Phi_i^\dagger\Phi_j(\text{in } x\text{-basis}) = {} & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j \\
& + \mathrm{i}\theta\sigma^\mu\bar\theta\left(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j\right) + 2\bar\theta\bar\psi_i\theta\psi_j \\
& - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\left(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j\right)\sigma^\mu\bar\theta + \sqrt{2}\theta\theta\bar\theta\bar\psi_i F_j \\
& + \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\left(\partial_\mu\bar\psi_i\phi_j - \bar\psi_i\partial_\mu\phi_j\right) + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j \\
& + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j + \frac{1}{4}\phi_i^*\partial^2\phi_j + \frac{1}{4}\partial^2\phi_i^*\phi_j - \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j + \frac{\mathrm{i}}{2}\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_j - \frac{\mathrm{i}}{2}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\rightsquigarrow {} & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j \\
& + \mathrm{i}\theta\sigma^\mu\bar\theta\left(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j\right) + 2\bar\theta\bar\psi_i\theta\psi_j \\
& + \sqrt{2}\theta\theta\bar\theta\left(\bar\psi_i F_j - \mathrm{i}\bar\sigma^\mu\psi_j\partial_\mu\phi_i^*\right) + \sqrt{2}\bar\theta\bar\theta\theta\left(\psi_j F_i^* - \mathrm{i}\sigma^\mu\bar\psi_i\partial_\mu\phi_j\right) \\
& + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \partial_\mu\phi_i^*\partial_\mu\phi_j - \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned}
\tag{5.13}
$$

$$
\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_i F_j + F_i\phi_j - \psi_i\psi_j\right] \tag{5.14}
$$

$$
\begin{aligned}
\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) = {} & \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right] \\
& + \theta\theta\left[F_i\phi_j\phi_k + \phi_i F_j\phi_k + \phi_i\phi_j F_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right]
\end{aligned}
\tag{5.15}
$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

■Superpotential

$$
W = \int \mathrm{d}^2\theta\left[\lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k\right] \tag{5.16}
$$

$$
\begin{aligned}
&= \lambda_i F_i + \frac{1}{2}m_{ij}\left(\phi_i F_j + F_i\phi_j - \psi_i\psi_j\right) \\
&\quad + \frac{1}{3}y_{ijk}\left(F_i\phi_j\phi_k + \phi_i F_j\phi_k + \phi_i\phi_j F_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right)
\end{aligned}
\tag{5.17}
$$

(In x-basis, since we have omitted all $\bar\theta$s.)

5.4 Vector Superfields

◆5.4.1 Abelian Case

■General Definitions

Vector Superfields	:	$V = V^\dagger$
Gauge Transf.	:	$V \dashrightarrow V + \Phi + \Phi^\dagger$
Field Strength	:	$W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V$; $\bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha}V$
Lagrangian	:	$\mathcal{L} = \frac{1}{4}\left(W^\alpha W_\alpha\Big\vert_{\theta\theta} + \bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big\vert_{\bar\theta\bar\theta}\right)$

■Explicit Expression

$$\begin{aligned}
V = {} & C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar{\theta}\bar{\chi}(x) \\
& + \frac{\mathrm{i}}{2}\theta\theta\left[M(x) + \mathrm{i}N(x)\right] - \frac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\left[M(x) - \mathrm{i}N(x)\right] - \theta\sigma^\mu\bar{\theta}A_\mu(x) \\
& + \mathrm{i}\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right] - \mathrm{i}\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] \\
& + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\partial^2 C(x)\right]
\end{aligned} \tag{5.18}$$

In Wess-Zumino gauge,

$$V \dashrightarrow -\theta\sigma^\mu\bar{\theta}A_\mu(x) + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{5.19}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y) + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}(y) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] \tag{5.20}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y^+) + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}(y^+) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda(y^+) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y^+) + \mathrm{i}\partial_\mu A^\mu(y^+)\right] \tag{5.21}$$

■Field Strength

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\mathrm{e}^{2gV}D_\alpha\mathrm{e}^{-2gV} \qquad \text{where} \quad V = V^a T^a \tag{5.22}$$

■Field Strength

Defining all component fields as including generators and coupling constants,

$$\begin{aligned}
&\mathrm{e}^{\pm 2gV^aT^a} \to \mathrm{e}^{\pm 2V} \\
&\qquad = 1 \mp 2\theta\sigma^\mu\bar{\theta}A_\mu(x) \mp 2\mathrm{i}\left[\bar{\theta}\bar{\theta}\theta\lambda(x) - \theta\theta\bar{\theta}\bar{\lambda}(x)\right] + \theta\theta\bar{\theta}\bar{\theta}\left[-A^\mu(x)A_\mu(x) - D(x)\right] \qquad (5.23)
\end{aligned}$$

Therefore, in y^+-basis,

$$\begin{aligned}
D_\alpha \mathrm{e}^{-2V} &= \frac{\partial}{\partial\theta^\alpha}\left\{1 + 2\theta\sigma^\mu\bar{\theta}A_\mu + 2\mathrm{i}\left[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}\right] + \theta\theta\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right]\right\} \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2\mathrm{i}\bar{\theta}\bar{\theta}\lambda_\alpha - 4\mathrm{i}\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] \qquad (5.24)
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^{2V}D_\alpha \mathrm{e}^{-2V} &= \left\{1 - 2\theta\sigma^\mu\bar{\theta}A_\mu - 2\mathrm{i}\left[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}\right]\right\}D_\alpha \mathrm{e}^{-2V} \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2\mathrm{i}\bar{\theta}\bar{\theta}\lambda_\alpha - 4\mathrm{i}\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] \\
&\quad - 2\theta\sigma^\mu\bar{\theta}A_\mu\left[2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 4\mathrm{i}\theta_\alpha\bar{\theta}\bar{\lambda}\right] \\
&\quad + 2\mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}\cdot 2(\sigma^\mu\bar{\theta})_\alpha A_\mu \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 4\mathrm{i}\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] + 2\mathrm{i}\bar{\theta}\bar{\theta}\lambda_\alpha \\
&\quad - 2A_\mu A_\nu\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4\mathrm{i}A_\mu\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y^+\text{-basis)} \qquad (5.25) \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 2\mathrm{i}\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\nu\bar{\sigma}^\mu)^\gamma\partial_\nu A_\mu - 4\mathrm{i}\theta_\alpha\bar{\theta}\bar{\lambda} - 2\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] + 2\mathrm{i}\bar{\theta}\bar{\theta}\lambda_\alpha \\
&\quad - 2A_\mu A_\nu\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4\mathrm{i}A_\mu\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y\text{-basis)} \qquad (5.26)
\end{aligned}$$

In y-basis, $\bar{D}\bar{D} = 4\cdot\dfrac{\partial}{\partial(\bar{\theta}\bar{\theta})}$. Therefore,

$$\begin{aligned}
W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\mathrm{e}^{2V}D_\alpha \mathrm{e}^{-2V} \qquad (5.27) \\
&= 2(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad - 2\theta_\alpha\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] - 2\mathrm{i}\lambda_\alpha + 4\mathrm{i}A_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y\text{-basis)} \qquad (5.28) \\
&= 2(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad - 2\theta_\alpha\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] - 2\mathrm{i}\lambda_\alpha + 4\mathrm{i}A_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \\
&\quad - \mathrm{i}\theta\theta(\sigma^\nu\bar{\sigma}^\mu\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A_\mu A_\nu + A_\mu\partial_\rho A_\nu + \mathrm{i}\partial_\nu\partial_\rho A_\mu) \\
&\quad + \mathrm{i}\theta\theta(\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A^\mu A_\mu + A^\mu\partial_\rho A_\mu - \partial_\rho D - \mathrm{i}\partial_\mu\partial_\rho A^\mu) \quad \text{(in } x\text{-basis)} \qquad (5.29) \\
W^\alpha &= 2(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)(\theta\sigma^\mu\bar{\sigma}^\nu)^\alpha - 2\theta\theta(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu)^\alpha \\
&\quad - 2\theta^\alpha\left[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu\right] - 2\mathrm{i}\lambda^\alpha - 4\mathrm{i}A_\mu\theta\theta(\bar{\lambda}\bar{\sigma}^\mu)^\alpha \quad \text{(in } y\text{-basis)} \qquad (5.30)
\end{aligned}$$

$$
\begin{aligned}
W^\alpha W_\alpha\Big|_{\theta\theta} &= 4(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)(A_\rho A_\sigma + \mathrm{i}\partial_\sigma A_\rho)(\theta\sigma^\mu\bar\sigma^\nu\sigma^\sigma\bar\sigma^\rho\theta) \\
&\quad - 4(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu](\theta\sigma^\mu\bar\sigma^\nu\theta) \\
&\quad + 4\mathrm{i}\theta\theta(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda) \\
&\quad - 4[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu](A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu)(\theta\sigma^\nu\bar\sigma^\mu\theta) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu] \\
&\quad + 4\mathrm{i}[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu]\theta\lambda \\
&\quad - 4\mathrm{i}(\lambda\sigma^\nu\bar\sigma^\mu\theta)(A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu) \\
&\quad - 4\mathrm{i}\theta\theta(\lambda\sigma^\mu\partial_\mu\bar\lambda) \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \\
&= 4(\theta\sigma^\mu\bar\sigma^\nu\sigma^\sigma\bar\sigma^\rho\theta)[A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu][A_\rho A_\sigma + \mathrm{i}\partial_\sigma A_\rho] \\
&\quad - 4(\theta\sigma^\mu\bar\sigma^\nu\theta)\Big\{A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu, A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu\Big\} \\
&\quad + 4\mathrm{i}\theta\theta(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu] \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \quad \text{(in y-basis)} \\
&\rightsquigarrow 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})[A_\mu A_\nu + \mathrm{i}\partial_\nu A_\mu][A_\rho A_\sigma + \mathrm{i}\partial_\sigma A_\rho] \\
&\quad - 4\Big\{A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu, A^\nu A_\nu + \mathrm{i}\partial_\nu A^\nu\Big\} \\
&\quad + 4[A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\partial_\mu A^\mu] \\
&\quad + 4\mathrm{i}(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha - 8A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \\
&= -2F^{\mu\nu}F_{\mu\nu} + 4(D + 2\mathrm{i}\partial_\mu A^\mu)(D + 2\mathrm{i}\partial_\mu A^\mu) \\
&\quad + 4\mathrm{i}(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha - 8A_\mu(\bar\lambda\bar\sigma^\mu\lambda), \qquad (5.31) \\
\bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big|_{\bar\theta\bar\theta} &= -2F^{\mu\nu}F_{\mu\nu} + 4(D - 2\mathrm{i}\partial_\mu A^\mu)(D - 2\mathrm{i}\partial_\mu A^\mu) \\
&\quad + 4\mathrm{i}(\partial_\mu\lambda\sigma^\mu\bar\lambda - \bar\lambda\bar\sigma^\mu\partial_\mu\lambda) + 8\bar\lambda^\alpha A_\mu(\bar\sigma^\mu\lambda)_\alpha - 8A_\mu(\lambda\sigma^\mu\bar\lambda), \qquad (5.32)
\end{aligned}
$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu] \tag{5.33}$$

Defining the generators as [*1]

$$T^{a\dagger} = T^a, \qquad [T^a, T^b] = \mathrm{i}f^{abc}T^c, \qquad \mathrm{Tr}\,T^aT^b = k\delta^{ab} \quad (k>0), \qquad \mathrm{Tr}(\{T^a, T^b\}T^c) = 0, \tag{5.34}$$

$$
\begin{aligned}
\frac{1}{16kg^2}\,\mathrm{Tr}\left(W^\alpha W_\alpha\Big|_{\theta\theta} + \bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right) = &-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}D^aD^a - 2\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} \\
&-\frac{1}{2}\mathrm{i}\left(\lambda\sigma^\mu D_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu D_\mu\lambda\right) \qquad (5.35)
\end{aligned}
$$

Here

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu \tag{5.36}$$
$$D_\mu\lambda^a = \partial_\mu\lambda^a - gf^{abc}A^b_\mu\lambda^c \tag{5.37}$$
$$D_\mu\bar\lambda^a = \partial_\mu\bar\lambda^a - gf^{abc}A^b_\mu\bar\lambda^c \tag{5.38}$$

[*1] The last one is anomaly-free condition, yielding $\mathrm{Tr}\,T^aT^bT^c = \mathrm{i}kgf^{abc}/2$.

■Interaction Terms

$$\mathrm{e}^{-2gV^aT^a} = 1 + 2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a + 2\mathrm{i}g\bar{\theta}\bar{\theta}\theta\lambda^a T^a - 2\mathrm{i}g\theta\theta\bar{\theta}\bar{\lambda}^a T^a - \theta\theta\bar{\theta}\bar{\theta}\left[g^2 A^{a\mu}A^b_\mu T^a T^b + gD^a T^a\right] \tag{5.39}$$

$$\begin{aligned}
\Phi^\dagger \mathrm{e}^{-2gV^aT^a}\Phi\Big|_{\theta^4} &= \phi^*\theta\theta\bar{\theta}\bar{\theta}\left[-g^2 A^{a\mu}A^b_\mu T^a T^b - gD^a T^a\right]\phi \\
&\quad + \sqrt{2}\phi^*\left[2\mathrm{i}g\bar{\theta}\bar{\theta}\theta\lambda^a T^a\right]\theta\psi + \sqrt{2}\bar{\theta}\bar{\psi}\left[-2\mathrm{i}g\theta\theta\bar{\theta}\bar{\lambda}^a T^a\right]\phi \\
&\quad + \mathrm{i}\theta\sigma^\mu\bar{\theta}\left[\phi^*(2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a)\partial_\mu\phi - \partial_\mu\phi^*(2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a)\phi\right] \\
&\quad + 2\bar{\theta}\bar{\psi}\left[2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a\right]\theta\psi \\
&\quad + \theta\theta\bar{\theta}\bar{\theta}\left[F^*F + \frac{1}{4}\phi^*\partial^2\phi + \frac{1}{4}\partial^2\phi^*\phi - \frac{1}{2}\partial_\mu\phi^*\partial_\mu\phi + \frac{\mathrm{i}}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{\mathrm{i}}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right] \\
&\rightsquigarrow F^*F - D_\mu\phi^* D^\mu\phi - \mathrm{i}\bar{\psi}\bar{\sigma}^\mu D_\mu\psi \\
&\quad - gD^a(\phi^*T^a\phi) - \sqrt{2}\mathrm{i}g\lambda^a(\phi^*T^a\psi) + \sqrt{2}\mathrm{i}g\bar{\lambda}^a(\bar{\psi}T^a\phi)
\end{aligned} \tag{5.40}$$

TODO: 計量により符号が異なるという噂……

$$D_\mu\phi = (\partial_\mu - \mathrm{i}gA^{a\mu}T^a)\phi \tag{5.41}$$
$$D_\mu\phi^* = \partial_\mu\phi^* + \mathrm{i}gA^a_\mu(\phi^*T^a) \tag{5.42}$$
$$D_\mu\psi = (\partial_\mu - \mathrm{i}gA^a_\mu T^a)\psi \tag{5.43}$$

5.5 Minimal Supersymmetric Standard Model

◆5.5.1 Definitions

■Gauge Group

$\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_{\mathrm{Y}} \quad (\times \mathrm{Z}_{2R} : R\text{-parity})$

■Fields

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	**3**	**2**	1/6	1/3	
L_i		**2**	−1/2		1
$\bar{U}_i$	$\bar{\mathbf{3}}$		−2/3	−1/3	
$\bar{D}_i$	$\bar{\mathbf{3}}$		1/3	−1/3	
$\bar{E}_i$			1		−1
H_u		**2**	1/2		
H_d		**2**	−1/2		

Field	SU(3)	SU(2)	U(1)
g	**8**		
W		**3**	
B			

■Superpotential

$$W_{\mathrm{RPC}} = \mu H_\mathrm{u} H_\mathrm{d} + y_{\mathrm{u}ij} H_\mathrm{u} Q_i \bar{U}_j + y_{\mathrm{d}ij} H_\mathrm{d} Q_i \bar{D}_j + y_{\mathrm{e}ij} H_\mathrm{d} L_i \bar{E}_j \tag{5.44}$$

$$W_{\mathrm{RPV}} = \mu_i H_\mathrm{u} L_i + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'{}_{ijk} L_i Q_j \bar{D}_k + \lambda''{}_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \tag{5.45}$$

(Here we define $\lambda_{ijk} = -\lambda_{jik}$ and $\lambda''_{ijk} = \lambda''_{ikj}$.)

◆5.5.2 Scalar Potential

■F-terms

$$-F_{H_\mathrm{u}}{}^{a*} = \epsilon^{ab}\left(\mu H_\mathrm{d}{}^b + y_{\mathrm{u}ij} Q_i^{bx} \bar{U}_j^x + \mu_i L_i^b\right) \tag{5.46}$$

$$-F_{H_\mathrm{d}}{}^{a*} = \epsilon^{ab}\left(-\mu H_\mathrm{u}{}^b + y_{\mathrm{d}ij} Q_i^{bx} \bar{D}_j^x + y_{\mathrm{e}ij} L_i^b \bar{E}_j\right) \tag{5.47}$$

$$-F_{Q_i^{ax*}} = \epsilon^{ab}\left(-y_{\mathrm{u}ij} H_\mathrm{u}{}^b \bar{U}_j^x - y_{\mathrm{d}ij} H_\mathrm{d}{}^b \bar{D}_j^x - \lambda'{}_{jik} L_i^b \bar{D}_j^x\right) \tag{5.48}$$

$$-F_{L_i^{a*}} = \epsilon^{ab}\left(-\mu_i H_\mathrm{u}{}^b - y_{\mathrm{e}ij} H_\mathrm{d}{}^b \bar{E}_j + 2\lambda_{ijk} L_j^b \bar{E}_k + \lambda'_{ijk} Q_j^{bx} \bar{D}_k^x\right) \tag{5.49}$$

$$-F_{\bar{U}_i^{x*}} = \left(\epsilon^{ab} y_{\mathrm{u}ji} H_\mathrm{u}{}^a Q_j^{bx} + \epsilon^{xyz} \lambda''_{ijk} \bar{D}_j^y \bar{D}_k^z\right) \tag{5.50}$$

$$-F_{\bar{D}_i^{x*}} = \left(\epsilon^{ab} y_{\mathrm{d}ji} H_\mathrm{d}{}^a Q_j^{bx} + \epsilon^{ab} \lambda'{}_{jki} L_j^a Q_k^{bx} + 2\epsilon^{yzx} \lambda''_{jki} \bar{U}_j^y \bar{D}_k^z\right) \tag{5.51}$$

$$-F_{\bar{E}_i}{}^* = \left(\epsilon^{ab} y_{\mathrm{e}ji} H_\mathrm{d}{}^a L_j^b + \epsilon^{ab} \lambda_{jki} L_j^a L_k^b\right) \tag{5.52}$$

■D-terms

$$D_g{}^\alpha = -g_3 \sum_{i=1}^{3}\left[\sum_{a=1,2} Q_i^{ax*}(T^\alpha)_{xy} Q_i^{ay} - \bar{U}_i^{x*}(T^\alpha)_{xy} \bar{U}_i^y - \bar{D}_i^{x*}(T^\alpha)_{xy} \bar{D}_i^y\right] \tag{5.53}$$

$$D_W{}^\alpha = -g_2\left[\sum_{i=1}^{3}\sum_{x=1}^{3} Q_i^{ax*}(T^\alpha)_{ab} Q_i^{by} + \sum_{i=1}^{3} L_i^{a*}(T^\alpha)_{ab} L_i^b + H_\mathrm{u}{}^{a*}(T^\alpha)_{ab} H_\mathrm{u}{}^b + H_\mathrm{d}{}^{a*}(T^\alpha)_{ab} H_\mathrm{d}{}^b\right] \tag{5.54}$$

$$D_B = -g_1\left[\frac{1}{6}|Q_i^{ax}|^2 - \frac{1}{2}|L_i^a|^2 - \frac{2}{3}|\bar{U}_i^x|^2 + \frac{1}{3}|\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2}|H_\mathrm{u}{}^a|^2 - \frac{1}{2}|H_\mathrm{d}{}^a|^2\right] \tag{5.55}$$

■Full Scalar Potential

$$V = \sum |F_\bullet|^2 + \frac{1}{2}\sum |D_\bullet|^2 \tag{5.56}$$

6 Supergravity

6.1 Minimal SUGRA Lagrangian

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2} eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu \mathrm{D}_\rho\psi_\sigma \tag{6.1}$$

where

$$\mathrm{D}_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \qquad \left[\omega_\mu{}^{ab} : \text{“spin 接続”}\right] \tag{6.2}$$

$$e := \det e_a{}^\mu \tag{6.3}$$

$$M := 1/\sqrt{8\pi G} \quad \text{(Reduced Planck mass)} \tag{6.4}$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \tag{6.5}$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \tag{6.6}$$

6.2 General SUGRA Lagrangian

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \qquad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \qquad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \tag{6.7}$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
  - Real function of chiral multiplets.
  - In global SUSY, $\int \mathrm{d}^4\theta K$ yields kinetic terms of the chiral multiplet.
  - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int \mathrm{d}^4\theta\ \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \tag{6.8}$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
  - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
  - $(a), (b), \ldots$ are indices for adjoint representation of gauge group.
  - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}\mathrm{D}_\mu\phi^i\mathrm{D}^\mu\phi^{*j} - \frac{1}{2}eg^2D_{(a)}D^{(a)} \\
& + \mathrm{i}eg_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu\mathrm{D}_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\mathrm{D}_\rho\psi_\sigma \\
& - \frac{1}{4}ef^{\mathrm{R}}{}_{(ab)}F^{(a)}_{\mu\nu}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f^{\mathrm{I}}{}_{(ab)}f^{(a)}_{\mu\nu}f^{(b)}_{\rho\sigma} \\
& + \frac{\mathrm{i}}{2}e\left[\lambda_{(a)}\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^{(a)}\right] - \frac{1}{2}f^{\mathrm{I}}{}_{(ab)}\mathrm{D}_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_{i*}f^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F^{(b)}_{\mu\nu} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F^{(b)}_{\mu\nu} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{\mathrm{i}}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F^{(a)}_{\nu\rho} + \widehat{F}^{(a)}_{\nu\rho}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f^{\mathrm{R}}{}_{(ab)} + f^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(bc)}\partial_{j^*}f^*_{(ad)}\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f^*_{(ab)}\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_{i^*}f^*_{(ac)}\partial_{j^*}f^*_{(bd)}\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f^*_{(cd)}\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{\mathrm{i}}{2}\sqrt{2}e\mathrm{e}^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}e\mathrm{e}^{K/2}\left[\mathrm{D}_iD_jW\chi^i\chi^j + \mathrm{D}_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}e\mathrm{e}^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f^*_{(ab)}\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned} \tag{6.9}
$$

付録 A　Mathematics

A.1　Algebra

◆A.1.1　Algebraic Structure

Semigroup	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative.
Monoid	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit.
Group	:	For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit, Inverse.
Module (加群/可換群)	:	For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit, Inverse.
Semimodule	:	For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit.
Ring (環)	:	$+$: Module, $\times$: Semigroup(Monoid), Distributive.
Semiring	:	$+$: Semimodule, $\times$: Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.
Field	:	$+$: Module, $\times$: Commutative Monoid, a^{-1} but 0, $1 \neq 0$, Distributive.

◆A.1.2　Vector Space

Vector space on K	:	For $v \in (V, +)_{\text{module}}$ and $k \in K_{\text{field}}$,
(K-module)	:	$kv \in (V, +)$; Compatible, Distributive, $1v = v$.
Norm	:	$\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|kx\| = k\,\|x\|$, $\|x + y\| \leq \|x\| + \|y\|$
Inner product	:	$\langle x\|x\rangle \geq 0$, $\langle x\|x\rangle = 0 \Leftrightarrow x = 0$,$\langle x\|y\rangle = \langle y\|x\rangle$,
	:	$\langle x + y\|z\rangle = \langle x\|z\rangle + \langle y\|z\rangle$, $\langle kx\|y\rangle = k\,\langle x\|y\rangle$

◆A.1.3　Lie Algebra

Lie Algebra	:	For a Finite-dimensional K-module $(A, +)$ and $x, y, z \in (A, +)$, $a, b \in K$,
	:	$[u, v] \in (A, +)$ (Lie product), and
	:	Bilinear: $[ax + by, z] = a[x, z] + b[y, z]$, $[x, ay + bz] = a[x, y] + b[x, z]$,
	:	Alternating: $[x, x] = 0$　$(\Longrightarrow [x, y] = -[y, x])$,
	:	Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

◆A.1.4　Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.

- For V, **K-algebra** $C(V)$ is defined by introducing "product" $xy \in V$
  s.t. $(xy)z = x(yz)$, $(x + y)z = xz + yz$, $x(y + z) = xy + xz$, $k(xy) = (kx)y = x(ky)$.
- For V, **Clifford algebra** $(C(V), \theta)$ is defined as
  - $C(V)$: K-algebra with 1　$(1x = x1 = x)$,
  - $\theta : V \to C(V)$, homomorphism, $\theta(x)^2 = \langle x|x\rangle\, 1$,
  - Any C': K-algebra with 1 and any homomorphism $\phi : V \to C'$ with $\phi(x)^2 = \langle x|x\rangle\, 1$, there's unique $\bar{\phi} : C \to C'$, homomorphism, $\bar{\phi}(1) = 1$.

> The Gamma matrices form Clifford Algebra:
>
> $$V : \mathbb{R}^5 \text{ with } \mathbf{2}^{\{G_0, G_1, G_2, G_3\}}, \quad \langle G_0|G_0\rangle = 1, \langle G_i|G_i\rangle = -1; \qquad C(V) : M(n, \mathbb{C}).$$
>
> TODO: なんか絶対違う

◆A.1.5　Lie Group and Lie Algebra

- 群 G が **Lie group** である … G が同時に C^∞ 多様体であり，積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が **COMPLEX Lie group** である … 積演算と逆元写像が共に正則写像である。

- Lie 群 G の単位元における接空間を，**G の Lie algebra $\mathfrak{g}$** という。
  - $\mathfrak{g}$ は G の左不変な vector 場全体である。
  - $\mathfrak{g}$ は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると，
  - その Lie 代数の基底 B_i に対して **structure constant** c が $[B_i, B_j] = c_{ij}^k B_k$ として定義できる。

＊　＊　＊

- Compact Lie 群は線型 Lie 群である。
- G として **Linear group** $\mathrm{GL}(n;\mathbb{R})$ を考えると，
  - その Lie 代数は n 次実正方行列全体となる。
  - Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は，$\mathrm{GL}(n;\mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では，Lie 群として $\mathrm{GL}(n;\mathbb{R})$ の部分群を考えることにし，Lie 代数の元を行列により表現する。

◆A.1.6　Matrix Representation

- Lie 群 G の Lie 代数の基底の組を，G の **generators** と言う。
- $\mathrm{GL}(n;\mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し，以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1 + \cdots + x_m T_m} \qquad (x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1} \cdots \mathrm{e}^{x_m T_m} \tag{A.1}$$

- Lie 群 G が compact である …
  1. 多様体 G が compact である。 **TODO: これは何故同値なのか？**
  2. G の生成子 $\{T_i\}$ を，$\mathrm{Tr}(T_i T_j) = k\,\delta_{ij}$ かつ $k > 0$ となるように取り替えることができる。
     【この基底の下では構造定数が完全反対称になる。】
- Compact 群 G は，**unitary representation** を持つ。
  故に，単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i \in \mathbb{R}$ により，G の元を

$$\mathrm{e}^{\mathrm{i}x^i T^i} \tag{A.2}$$

  と表すことが出来る。

◆A.1.7　結論

Compact Lie 群の元のうち，単位元近傍にあるもの V は，

Hermitian Representation

$$V = \exp(\mathrm{i}x^i T^i) \qquad \text{where} \quad T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R},$$
$$[T^i, T^j] = \mathrm{i} f^{ijk} T^k, \quad \mathrm{Tr}(T^i T^j) = \lambda\,\delta^{ij} > 0; \qquad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \qquad \text{where} \quad R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R},$$
$$[R^i, R^j] = -f^{ijk} R^k, \quad \mathrm{Tr}(R^i R^j) = -\lambda\,\delta^{ij} < 0; \qquad f \in \mathbb{R}$$

と表すことが出来る。

付録 B　Verbose Notes

B.1　Spinor Fields

◆B.1.1　Lorentz group and Lorentz algebra

Metric	:	$\eta = \mathrm{diag}(+1,-1,-1,-1), \qquad \eta = \mathrm{diag}(-1,+1,+1,+1).$
Lorentz transf. in $\mathbb{R}^{1,3}$	:	Linear transf. $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ which conserve x^2.
	:	$\Longrightarrow \eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma$. and form a group L.
Disconnected parts of L	:	$L_0 := \left\{\det\Lambda = +1 \wedge {\Lambda_0}^0 > 0\right\} \quad L_P := \left\{\det\Lambda = -1 \wedge {\Lambda_0}^0 > 0\right\}$
	:	$L_T := \left\{\det\Lambda = +1 \wedge {\Lambda_0}^0 < 0\right\} \quad L_{PT} := \left\{\det\Lambda = -1 \wedge {\Lambda_0}^0 < 0\right\}$
	:	(L_0 is identical with $\mathrm{SO}(1,3)/\mathrm{SO}(3,1)$.)
Infinitesimal one in L_0	:	$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ (for $\eta = \eta\Lambda\Lambda$)
Generators	:	$x'^\mu = x^\mu + \epsilon^\mu{}_\nu x^\nu =: \left(\delta^\mu_\nu \mp \frac{\mathrm{i}}{2}\epsilon_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu\right)x^\nu \quad$ with $J^{\rho\sigma} = -J^{\sigma\rho}$
	:	$\Longrightarrow (J_{\rho\sigma})^\mu{}_\nu = \pm\mathrm{i}\left(\delta^\mu_\rho\eta_{\sigma\nu} - \delta^\mu_\sigma\eta_{\rho\nu}\right)$,
Lorentz algebra	:	therefore $[J_{\mu\nu}, J_{\rho\sigma}] = \mp\mathrm{i}(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma})$

Here we physicists derive the algebra from the property of the Lorentz transformation, but we can start from the albegra like mathematicians.

TODO: J の符号確認

◆B.1.2　Lorentz group and $\mathrm{SL}(2,\mathbb{C})$

Here we will explain that L_0 is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\mathrm{Z}_2$, which is defined as

$$\mathfrak{sl}(2,\mathbb{C}) :=\{a \in \mathfrak{gl}(2,\mathbb{C}) \mid \mathrm{Tr}(a) = 0\}, \qquad \mathrm{SL}(2,\mathbb{C}) :=\{g \in \mathrm{GL}(2,\mathbb{C}) | \det(g) = 1\}. \tag{B.1}$$

We can build a one-to-one correspondent between x^μ and an Hermitian matrix

$$A := t\cdot 1 + \boldsymbol{x}\cdot\boldsymbol{\sigma} = \begin{pmatrix} t+z & x-\mathrm{i}y \\ x+\mathrm{i}y & t-z\end{pmatrix}. \tag{B.2}$$

Then, we can define Lorentz transformations f^g from $g \in \mathrm{SL}(2,\mathbb{C})$ as

$$f^g : A \mapsto g^\dagger A g, \tag{B.3}$$

and thus we can induce Lorentz transformations $\Lambda(g)$ from g.

Actually, $\Lambda(g)$ covers only L_0, not whole L. I do not provide the proof, but since $\mathrm{SL}(2,\mathbb{C})$ is a connected group, it is clear that it can cover only the connected part of Lorentz group.

> I omit the proof here, but in principle, it can be done by direct calculation. For instance, $\Lambda^0{}_0 > 0$ can be proved as follows:
> Since $t = \frac{1}{2}\mathrm{Tr}(t\cdot 1 + \boldsymbol{x}\cdot\boldsymbol{\sigma})$, $\Lambda^0{}_0$ is nothing but $\frac{1}{2t}\mathrm{Tr}\left[g^\dagger\left(\begin{smallmatrix} t & 0 \\ 0 & t\end{smallmatrix}\right)g\right]$. Therefore
>
> $$\Lambda^0{}_0 = \frac{1}{2}\mathrm{Tr}\left(g^\dagger g\right) = \frac{1}{2}\left(|g_{11}| + |g_{12}| + |g_{21}| + |g_{22}|\right) > 0. \tag{B.4}$$
>
> This shows $\Lambda(g)$ are orthochronous.

As an element of $\mathfrak{su}(2)$ can be expressed by the 3 bases known as Pauli matrices,

The Lorentz group has 6 generators $J_{\rho\sigma}$, and these corresponds to the 6 elements of $\mathfrak{sl}(2,\mathbb{C})$. These can be broken down into three "rotation generators" $\boldsymbol{J}$, and three "boost generators" $\boldsymbol{K}$ as follows:

$$J_i := (J_{23}, J_{31}, J_{12}) \qquad K_i := (J_{10}, J_{20}, J_{30}) \qquad \text{for} \quad i = 1, 2, 3.^{*2} \tag{B.5}$$

With this definition, the algebra is, regardless of signature of the metric, as follows:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k. \tag{B.6}$$

Then, we define $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\boldsymbol{A} := \frac{1}{2}(\boldsymbol{J} + \mathrm{i}\boldsymbol{K}), \qquad \boldsymbol{B} := \frac{1}{2}(\boldsymbol{J} - \mathrm{i}\boldsymbol{K}). \tag{B.7}$$

Now the commutation relations are

$$[A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0, \tag{B.8}$$

which means we have reduced the Lorentz group into $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

* * *

In some neighborhood of the identity, the elements Λ of Lorentz group can be described as

$$\Lambda = \exp(\mp\frac{\mathrm{i}}{2}\epsilon_{\rho\sigma}J^{\rho\sigma}), \tag{B.9}$$

using the (antisymmetric) generators $J^{\rho\sigma}$. Here, defining $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ as

$$\pm\theta_i := (\epsilon_{23}, \epsilon_{31}, \epsilon_{12}) \qquad \pm\beta_i := (\epsilon_{10}, \epsilon_{20}, \epsilon_{30}) \qquad \text{for} \quad i = 1, 2, 3, \tag{B.10}$$

the element is denoted as:

$$\Lambda = \exp\left[\mathrm{i}\left(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\omega}\cdot\boldsymbol{K}\right)\right]. \tag{B.11}$$

B.2 Weyl Spinor

Now we introduce Weyl spinors. First, I de

B.3 Polarization Sum

Firstly we focus on the single photon case $M = \epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$. Here we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu *} = |M^1|^2 + |M^2|^2, \tag{B.12}$$

while

$$\eta_{\mu\nu}M^\mu M^{\nu *} = |M^1|^2 + |M^2|^2 \tag{B.13}$$

for Ward identity $k_\mu M^\mu = 0$. Therefore the replacement

$$\sum_{\text{pol.}} \epsilon_\mu\epsilon'_\nu \to \eta_{\mu\nu} \tag{B.14}$$

*2 For your information, $J_{23} = J^{23}$ and $K_{01} = -K^{01}$, regardless of the definition of the metic.

is valid.

Secondly we think about the double photons case [*3] $M = \epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \qquad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \tag{B.15}$$

$$k' = (E, 0, 0, -E) \qquad \epsilon' = (0, \cos\theta, \sin\theta, 0) \oplus (0, -\sin\theta, \cos\theta, 0). \tag{B.16}$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \tag{B.17}$$

Nevertheless, naïve replacement does not work, because our Ward identities

$$k_\mu \epsilon_\nu'^*(k')M^{\mu\nu} = \epsilon_\mu^*(k)k_\nu' M^{\mu\nu} = 0 \tag{B.18}$$

obviously does not help us. If we can omit ϵs from these identities, that is if

$$k_\mu M^{\mu\nu} = k_\nu' M^{\mu\nu} = 0, \tag{B.19}$$

we can recover validity of the replacement:

$$\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *} = -\eta_{\nu\sigma}\left(M^{1\nu}M^{1\sigma *} + M^{2\nu}M^{2\sigma *}\right) \tag{B.20}$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \tag{B.21}$$

Then what's happening? Why this replacement is not valid? Actually our new conditions (B.19) seem to guarantee that we are summing not only "physical" but also "unphysical" polarizations. Meanwhile if we use some physical condition such as $\epsilon \cdot k = 0$, (B.19) break down while Ward identities (B.18) are still valid.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our new conditions (B.19), and define $\widetilde{M}^{\mu\nu}$ and $\widetilde{M}$ as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \tag{B.22}$$

$$\widetilde{M} := \epsilon_\mu^*(k)\epsilon_\nu'^*(k')\widetilde{M}^{\mu\nu}. \tag{B.23}$$

This alternative amplitude satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$), and furthermore $\widetilde{M} = M$. Therefore $\widetilde{M}$ is physically identical to M. However technically these are very different, just because we cannot perform our "naïve replacement" for this $\widetilde{M}$:

$$\eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \eta_{\mu\rho}\eta_{\nu\sigma}\left(M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu\right)\left(M^{\rho\sigma *} + k^\rho p^{\sigma *} + p'^{\rho *} k'^\sigma\right) \tag{B.24}$$

$$= \sum_{\text{pol.}} |M|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H. c.}\right]. \tag{B.25}$$

After all, we have obtained following expression:

$$\begin{aligned}
\sum_{\text{pol.}} |\widetilde{M}|^2 &= \sum_{\text{pol.}} |M|^2 \qquad (\text{Furthermore } \widetilde{M} = M) \\
&= \sum_{\text{pol.}} |\epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}|^2 = \sum_{\text{pol.}} |\epsilon_\mu^*(k)\epsilon_\nu'^*(k')\widetilde{M}^{\mu\nu}|^2 \\
&= \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *} \\
&\neq \eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \sum_{\text{pol.}} |\widetilde{M}|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H. c.}\right].
\end{aligned} \tag{B.26}$$

[*3] This part is derived from 濱口幸一's notebook.

B.4 Phantom Terms in the Gauge Theory

You may think we forget to introduce $\overline{\psi}\gamma_5\psi$, $\overline{\psi}\gamma_5\not{D}\psi$, $\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma}$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_{L} and P_{R},
- the last term is equivalent to the third term as

$$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\frac{1}{2}[D_\mu, D_\nu]F^a_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}.$$

Therefore, we have to discuss only the ϵFF terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma &= \epsilon^{\mu\nu\rho\sigma}\left(-f^{acd}f^{abe} - f^{adb}f^{ace}\right)A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\ &= -2\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma \\ &= 0, \end{aligned} \tag{B.27}$$

$$\begin{aligned} \therefore\ \epsilon^{\nu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^a_\nu \partial_\rho A^a_\sigma + 4g\epsilon^{\mu\nu\rho\sigma}f^{abc}A^a_\mu A^b_\nu \partial_\rho A^c_\sigma \\ &= 2\partial_\mu G^\mu, \end{aligned} \tag{B.28}$$

where G^μ is the Chern–Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma + \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) = \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right). \tag{B.29}$$

See Appendix B.5 for the instanton effect.

B.5 Instanton

TODO: Postpone...

付録 C　Cherry on the Cake

■Conversion of Units

$$1\,\mathrm{GeV} = \frac{1}{6.5821\times10^{-25}\,\mathrm{s}} = \frac{1}{2.086\times10^{-32}\,\mathrm{yr}} = \frac{1}{0.19733\,\mathrm{fm}} = 1.2244\times10^{13}\,\mathrm{K} = 1.7827\times10^{-24}\,\mathrm{g}$$

$$= \frac{1.519268\times10^{24}}{1\,\mathrm{s}} = \frac{4.79\times10^{31}}{1\,\mathrm{yr}} = \frac{5.0677}{1\,\mathrm{fm}} \tag{C.1}$$

$$1\,\mathrm{K} = 8.167\times10^{-5}\,\mathrm{eV} = \frac{1}{8.0591\times10^{-21}\,\mathrm{s}} = \frac{1.2408\times10^{20}}{1\,\mathrm{s}} \tag{C.2}$$

$$1\,\mathrm{GeV}^2 = \frac{1}{0.38938\,\mathrm{mbarn}} = \frac{2568.2}{1\,\mathrm{barn}} \qquad (1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2) \tag{C.3}$$

$$\begin{aligned} &1\,\text{tropical yr} = 3.1557\times10^{7}\,\mathrm{s}, \quad 1\,\text{sidereal yr} = 3.1558\times10^{7}\,\mathrm{s};\\ &1\,\mathrm{s} = 3.1689\times10^{-8}\,\text{tr-yr} = 3.1688\times10^{-8}\,\text{sr-yr}. \end{aligned} \tag{C.4}$$

■Physical Constants

$$G_{\mathrm{F}} = \frac{1}{\sqrt{2}v^2} = 1.16637(1)\times10^{-5}\,\mathrm{GeV}^{-2}, \qquad G_{\mathrm{N}} = 6.70881(67)\times10^{-39}\,\mathrm{GeV}^{-2} \tag{C.5}$$

$$\sqrt{G_{\mathrm{N}}} = 1.61624(8)\times10^{-35}\,\mathrm{m} = \frac{1}{1.22089(6)\times10^{19}\,\mathrm{GeV}} = \frac{1}{2.17644(11)\times10^{-8}\,\mathrm{kg}} \tag{C.6}$$

$$\sqrt{8\pi G_{\mathrm{N}}} = 8.1026(4)\times10^{-35}\,\mathrm{m} = \frac{1}{2.4353(1)\times10^{18}\,\mathrm{GeV}} = \frac{1}{4.3413(2)\times10^{-9}\,\mathrm{kg}} \tag{C.7}$$

■Component of Spinor in Weyl Representation

$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta,\ \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta},\ \chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta};\quad \epsilon^{12} = \epsilon^{\dot1\dot2} = 1,\ \epsilon_{12} = \epsilon_{\dot1\dot2} = -1.)$

$$\psi = \begin{pmatrix}\psi_{\mathrm{L}}\\ \psi_{\mathrm{R}}\end{pmatrix} = \begin{pmatrix}\xi_\alpha\\ \bar\chi^{\dot\alpha}\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\\ \bar\chi^{\dot1}\\ \bar\chi^{\dot2}\end{pmatrix} \quad \xrightarrow{C} \quad \psi^{\mathrm{c}} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix}-(\bar\chi^{\dot2})^*\\ (\bar\chi^{\dot1})^*\\ (\xi_2)^*\\ -(\xi_1)^*\end{pmatrix} = \begin{pmatrix}-\chi^2\\ \chi^1\\ \bar\xi_{\dot2}\\ -\bar\xi_{\dot1}\end{pmatrix} = \begin{pmatrix}\chi_\alpha\\ \bar\xi^{\dot\alpha}\end{pmatrix} \tag{C.8}$$

$$\overline{\psi} = \begin{pmatrix}\chi^\alpha & \bar\xi_{\dot\alpha}\end{pmatrix} = \begin{pmatrix}\chi^1 & \chi^2 & \bar\xi_{\dot1} & \bar\xi_{\dot2}\end{pmatrix} \quad \xrightarrow{C} \quad \overline{\psi^{\mathrm{c}}} = \mathrm{i}\psi^{\mathrm{T}}\gamma^0\gamma^2 = \begin{pmatrix}\xi_2 & -\xi_1 & -\bar\chi^{\dot2} & \bar\chi^{\dot1}\end{pmatrix} = \begin{pmatrix}\xi^\alpha & \bar\chi_{\dot\alpha}\end{pmatrix} \tag{C.9}$$

$$A^\alpha B_\alpha = \overline{\psi}_{A^{\mathrm{c}}}P_{\mathrm{L}}\psi_B = \overline{\psi}_{B^{\mathrm{c}}}P_{\mathrm{L}}\psi_A \qquad \bar A_{\dot\alpha}\bar B^{\dot\alpha} = \overline{\psi}_A P_{\mathrm{R}}\psi_{B^{\mathrm{c}}} = \overline{\psi}_B P_{\mathrm{R}}\psi_{A^{\mathrm{c}}} \tag{C.10}$$