1 General Definitions and Tools

1.1 NOTATIONS AND CONVENTIONS

1.1.1 Metric etc.

Minkowski Metric : $\eta^{\mu\nu} := \mathrm{diag}(+,-,-,-);\quad \epsilon^{0123}_{0123} := \pm 1$

Coordinates : $x^{\mu} := (t,x,y,z);\quad$ therefore $\partial_{\mu} = \left(\frac{\partial}{\partial t}, \nabla\right).$

Gamma Matrices : $\{\gamma^{\mu}, \gamma^{\nu}\} := 2\eta^{\mu\nu};\quad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}$

: therefore $\{\gamma^{\mu}, \gamma_5\} = 0, (\gamma_5)^2 = 1.$

Gamma Combinations : $1, \{\gamma^{\mu}\}, \{\sigma^{\mu\nu}\}, \{\gamma^{\mu}\gamma_5\}, \gamma_5;\quad \sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^{\mu}, \gamma^{\nu}] = 0 \Big/ \mathrm{i}\gamma^{\mu}\gamma^{\nu}$

Spinor ϵ and σ matrices : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$

: $(\sigma^{\mu})_{\alpha\dot{\beta}} := (1, \boldsymbol{\sigma})_{\alpha\dot{\beta}},\quad (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^{\mu})_{\beta\dot{\beta}} = (1, -\boldsymbol{\sigma})^{\dot{\alpha}\beta}.$

Pauli Matrices : $\sigma_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\quad \sigma_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\quad \sigma_2 = \left(\begin{smallmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{smallmatrix}\right),\quad \sigma_3 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),$

: $\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\quad \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),$

: $\sigma^{\mu} := (1, \boldsymbol{\sigma}),\quad \bar{\sigma}^{\mu} := (1, -\boldsymbol{\sigma}).$

Fourier Transformation : $\widetilde{f}(k) := \int \mathrm{d}^4x\; \mathrm{e}^{\mathrm{i}kx} f(x);\qquad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\; \mathrm{e}^{-\mathrm{i}kx}\widetilde{f}(k).$

1.1.2 Fields

Scalar : $(\partial^2 + m^2)\phi = 0;\quad \phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}px} + b_{\boldsymbol{p}}^{\dagger}\mathrm{e}^{\mathrm{i}px}\right]$

Dirac : $(\mathrm{i}\partial\!\!\!/ - m)\psi = 0;\quad \psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left[a_{\boldsymbol{p}}^s u^s(p)\mathrm{e}^{-\mathrm{i}px} + b_{\boldsymbol{p}}^{s\dagger}v^s(p)\mathrm{e}^{\mathrm{i}px}\right]$

Vector : $\partial^2 A^{\mu} = 0;\quad A^{\mu}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{r=0..3}\left[a_{\boldsymbol{p}}^r \epsilon^r(p)\mathrm{e}^{-\mathrm{i}px} + a_{\boldsymbol{p}}^{r\dagger}\epsilon^{r*}(p)\mathrm{e}^{\mathrm{i}px}\right]$

TODO: −**Goldstone; Gravitino**

1.1.3 Electromagnetism

Electromagnetic Fields : $A^{\mu} = (\phi, \boldsymbol{A})$ We can invert the signs, but cannot lower the index.

Maxwell Equations : $F_{\mu\nu} := \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu};\qquad \epsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma} = 0,\quad \partial_{\mu}F^{\mu\nu} = ej^{\nu}$

Our Old Language : $\nabla\cdot\boldsymbol{B} = 0,\; \nabla\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = 0;\quad \nabla\cdot\boldsymbol{E} = ej^0,\; (\nabla\times\boldsymbol{B})_i - \frac{\partial}{\partial t}E_i = ej^i.$

: $F_{\mu\nu} = \begin{pmatrix} 0 & & \boldsymbol{E} & \\ & 0 & -B_3 & B_2 \\ -\boldsymbol{E} & B_3 & 0 & -B_1 \\ & -B_2 & B_1 & 0 \end{pmatrix};\quad F_{\mu\nu}F^{\mu\nu} = -2\left(\|\boldsymbol{E}\|^2 - \|\boldsymbol{B}\|^2\right)$

1.2 Spinor Fields

Spinor $\quad\quad:\ \xi_\alpha, \quad\quad\quad\quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta;$ $\quad$ Lorentz tr. $:\ \xi_\alpha \mapsto \Lambda_\alpha{}^\beta\xi_\beta, \quad\quad \xi^\alpha \mapsto \xi^\beta\Lambda^{-1}{}_\beta{}^\alpha;$

$\quad\quad\quad\quad\quad:\ \bar\eta^{\dot\alpha} := (\eta^\alpha)^* \quad \bar\eta_{\dot\alpha} := (\eta_\alpha)^* \quad\quad\quad\quad\quad\quad\quad :\ \bar\eta^{\dot\alpha} \mapsto \Lambda^{\dagger-1\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta}, \quad \bar\eta_{\dot\alpha} \mapsto \bar\eta_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}.$

Kinetic term $:\ \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi \quad (=\mathrm{i}\,\eta\sigma^\mu\partial_\mu\bar\eta\ )$

Mass term $\quad:\ [\text{Majorana}]\ -\tfrac{1}{2}(m_{\mathrm M}\xi\xi + m_{\mathrm M}^*\bar\xi\bar\xi) \quad\quad [\text{Dirac}]\ -(m_{\mathrm D}\xi\eta + m_{\mathrm D}^*\bar\xi\bar\eta)$

Dirac fermion $\quad:\ \mathcal{L}_{\mathrm{Dirac}} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi + \mathrm{i}\,\eta\sigma^\mu\partial_\mu\bar\eta - m(\xi\eta + \bar\xi\bar\eta) = \overline\psi(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$

Majorana fermion $:\ \mathcal{L}_{\mathrm{Majorana}} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi - \tfrac{m}{2}(\xi\xi + \bar\xi\bar\xi) = \tfrac{1}{2}\overline\psi_{\mathrm M}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi_{\mathrm M}$

Charge conjugate $:\ \psi^{\mathrm C} := C(\overline\psi)^{\mathrm T} \quad [(\psi_{\mathrm M})^{\mathrm C} = \psi_{\mathrm M}]$

1.2.1 Chiral Notation (Peskin)

Gamma Matrices $:\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad P^{\mathrm R}_{\mathrm L} = \frac{1\pm\gamma_5}{2}.$

Fields $\quad\quad:\ \psi = \begin{pmatrix} \xi_\alpha \\ \bar\eta^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm L} \\ \psi_{\mathrm R} \end{pmatrix}; \quad \overline\psi = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\alpha & \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm R}^\dagger & \psi_{\mathrm L}^\dagger \end{pmatrix}; \quad \psi_{\mathrm M} = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}.$

$\quad\quad\quad\quad:\ u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}; \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}$

$\quad\quad\quad\quad:\ \left[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\right].$

Charge conj. $\quad:\ C := -\mathrm{i}\gamma^2\gamma^0 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \quad \begin{cases} -C = C^{-1} = C^\dagger = C^{\mathrm T} \\ C = C^* \end{cases}, \ C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm T}.$

$\quad\quad\quad\quad:\ \psi^{\mathrm C} = C(\overline\psi)^{\mathrm T} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} \eta_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}, \quad \overline{\psi^{\mathrm C}} = \psi^{\mathrm T}C = \mathrm{i}\overline\psi^*\gamma^2$

Weyl Equations $:\ \mathrm{i}\bar\sigma\cdot\partial\psi_{\mathrm L} = m\psi_{\mathrm R}; \quad \mathrm{i}\sigma\cdot\partial\psi_{\mathrm R} = m\psi_{\mathrm L}$

z-boost limit $\quad:\ $ Halt: $u^s = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}, v^s = \sqrt{m}\begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix};$

$\quad\quad\quad\quad:\ $ Slow: $\sqrt{p\cdot\sigma} \simeq \sqrt{m}(1 - \boldsymbol{v}\cdot\boldsymbol{\sigma}/2), \sqrt{p\cdot\bar\sigma} \simeq \sqrt{m}(1 + \boldsymbol{v}\cdot\boldsymbol{\sigma}/2);$

$\quad\quad\quad\quad:\ $ Extreme: $u^s = \sqrt{2E}\begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\xi^s \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\xi^s \end{pmatrix}, v^s = \sqrt{2E}\begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\eta^s \\ -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\eta^s \end{pmatrix}.$

1.2.2 Dirac Notation

Gamma Matrices $:\ \hat\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat\gamma^i = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \hat\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad\quad \hat P^{\mathrm R}_{\mathrm L} = \frac{1\pm\gamma_5}{2}.$

Fields $\quad\quad:\ \hat\psi = \begin{pmatrix} \psi_{\mathrm A} \\ \psi_{\mathrm B} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_{\mathrm L} + \psi_{\mathrm R} \\ -\psi_{\mathrm L} + \psi_{\mathrm R} \end{pmatrix}; \quad \hat\psi_{\mathrm M} = \begin{pmatrix} \psi_{\mathrm A} \\ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\psi_{\mathrm A}^* \end{pmatrix}.$

$\quad\quad\quad\quad:\ \hat{\overline\psi} = \hat\psi^\dagger\gamma^0 = \begin{pmatrix} \psi_{\mathrm A}^\dagger & -\psi_{\mathrm B}^\dagger \end{pmatrix}$

$\quad\quad\quad\quad:\ \hat u^s(p) = \begin{pmatrix} \sqrt{p^0 + m}\,\xi^s \\ \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\xi^s \end{pmatrix}; \quad \hat v^s(p) = \begin{pmatrix} -\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\eta^s \\ -\sqrt{p^0 + m}\,\eta^s \end{pmatrix}$

$\quad\quad\quad\quad:\ \left[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\right]$

Charge conj. : $\hat{C} = -\mathrm{i}\hat{\gamma}^2\hat{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ with $C = -C^{-1} = -C^\dagger$, $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$.

z-boost limit :   Halt: $\hat{u}^s = \sqrt{2m}\left(\begin{smallmatrix} \xi^s \\ 0 \end{smallmatrix}\right)$, $\hat{v}^s = -\sqrt{2m}\left(\begin{smallmatrix} 0 \\ \eta^s \end{smallmatrix}\right)$;

:   Slow: $\sqrt{p^0+m} \simeq \sqrt{2m}(1+\frac{v^2}{8})$, $\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}} \simeq \sqrt{\frac{m}{2}}(\boldsymbol{v}\cdot\boldsymbol{\sigma})$;

:   Extreme: $\hat{u}^s = \sqrt{E}\begin{pmatrix} \xi^s \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\xi^s \end{pmatrix}$, $\hat{v}^s = -\sqrt{E}\begin{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\eta^s \\ \eta^s \end{pmatrix}$

1.2.3   CPT transformations   Note that these expressions are valid under the above frameworks.

In the following, CP means "P, then C" **in algebraic sense**. Be careful to the order.

$$\psi(t,\boldsymbol{x}) \xrightarrow{\;P\;} \eta_P\gamma^0\psi(t,-\boldsymbol{x}) \qquad\qquad \overline{\psi} \xrightarrow{\;P\;} \eta_P^*\overline{\psi}\gamma^0$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{\;T\;} \eta_T C\gamma_5\psi(-t,\boldsymbol{x}) \qquad\qquad \overline{\psi} \xrightarrow{\;T\;} -\eta_T^*\overline{\psi}C\gamma_5$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{\;C\;} \eta_C C\overline{\psi}^{\mathrm{T}}(t,\boldsymbol{x}) = C\gamma^0\psi^* \qquad\qquad \overline{\psi} \xrightarrow{\;C\;} \eta_C^*\overline{\psi}^*\gamma^0 C = -\eta_C^*(C\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{\;CP\;} \eta_{CP}(\overline{\psi}\gamma^0 C)^{\mathrm{T}} \qquad\qquad \overline{\psi} \xrightarrow{\;CP\;} \eta_{CP}^*(C\gamma^0\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{\;CPT\;} (\overline{\psi}\gamma^0\gamma_5)^{\mathrm{T}} \qquad\qquad \overline{\psi} \xrightarrow{\;CPT\;} (\gamma^0\gamma_5\psi)^{\mathrm{T}}$$

Note that T-transformation is anti-unitary, and $\eta_{CPT} = 1$. Especially, photon is $(P,T,C) = (-,+,-)$.

	ϕ	A^μ	$\bar{\psi}\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$\bar{\psi}\gamma^\mu\gamma_5\psi$	$\mathrm{i}\bar{\psi}\gamma_5\psi$	∂_μ
P	ϕ	$-{+}{+}{+}A^\mu$	$+$	$+{-}{-}{-}$	$(+{-}{-}{-})(+{-}{-}{-})$	$-{+}{+}{+}$	$-$	$+{-}{-}{-}$
T	ϕ	$+{-}{-}{-}A^\mu$	$+$	$+{-}{-}{-}$	$-(+{-}{-}{-})(+{-}{-}{-})$	$+{-}{-}{-}$	$-$	$-{+}{+}{+}$
C	ϕ^*	$+A^{\mu*}$	$+$	$-$	$-$	$+$	$+$	$+$
CPT	ϕ^*	$-A^{\mu*}$	$+$	$-$	$+$	$-$	$+$	$-$

1.2.4   Noether current

Infinitesimal transformation   : $\phi(x) \mapsto \phi'(x) := \phi(x) + \alpha\Delta\phi(x)$

Correspondent transformation : $\alpha\Delta\mathcal{L} = \alpha\partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\dfrac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi$

:   So, defining $\alpha\partial_\mu\mathcal{J}^\mu(x) := \mathcal{L}'(x) - \mathcal{L}(x)$,

Noether current   : $j^\mu(x) := \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu$; $\partial_\mu j^\mu(x) = 0$

Noether charge   : $Q := \int j^0\mathrm{d}^3x$

Energy-momentum tensor   : $T^\mu{}_\nu = \partial_\mu\mathcal{L}(\partial_\mu\phi)\partial_\nu\phi - \mathcal{L}\delta_\nu^\mu$; $\mathcal{H} = T^{00}$, $\mathcal{P}^i = T^{0i}$.

: $T^\mu{}_\nu$ is the variation along μ in respect to the modification a^ν.

1.3 Feynman Rules

Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\overset{\frown}{\phi\ \phi} = \text{⊘} \dashleftarrow{p}\dashrightarrow \text{⊘} = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

$$\mathcal{L} \supset |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$$\overset{\frown}{\phi^*\ \phi} = \text{⊘} \dashleftarrow{p}\dashrightarrow \text{⊘} = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

(External lines equal to 1 in both cases.)

Dirac Fermion

$$\mathcal{L} \supset \overline{\psi}(\mathrm{i}\slashed{\partial} - m)\psi$$
$$= \mathrm{i}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \xi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi - m(\xi\chi + \bar{\xi}\bar{\chi})$$

Initial state

$$\overset{\frown}{\psi\ |\boldsymbol{p}, s\rangle} = \text{⊘} \longleftarrow{p} = u^s(p)$$

$$\overset{\frown}{\overline{\psi}\ |\boldsymbol{p}, s\rangle} = \text{⊘} \longrightarrow{p} = \bar{v}^s(p)$$

Final state

$$\langle \boldsymbol{p}, s| \overset{\frown}{\overline{\psi}} = \underset{p}{\longleftarrow} \text{⊘} = \bar{u}^s(p)$$

$$\langle \boldsymbol{p}, s| \overset{\frown}{\psi} = \underset{p}{\longrightarrow} \text{⊘} = v^s(p)$$

Propagator

$$\overset{\frown}{\psi\ \overline{\psi}} = \text{⊘} \longleftarrow{p} \text{⊘} = \frac{\mathrm{i}(\slashed{p} + m)}{p^2 - m^2 + \mathrm{i}\epsilon}$$

Majorana Fermion

$$\mathcal{L} \supset \frac{1}{2}\overline{\psi}(\mathrm{i}\slashed{\partial} - m)\psi$$
$$= \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda - \frac{m}{2}(\lambda\lambda + \bar{\lambda}\bar{\lambda})$$

Initial state

Abelian Gauge Theory (Photon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}\slashed{D}\psi + |D_\mu \phi|^2$$
$$(D_\mu = \partial_\mu - \mathrm{i}QA_\mu)$$

$$\overset{\frown}{A_\mu\ |\boldsymbol{p};\ \rangle} = \text{⊘}\wwwwww{\longleftarrow p} = \epsilon_\mu(p)$$

$$\langle \boldsymbol{p};\ |\overset{\frown}{A_\mu} = \wwwwww{\longleftarrow p}\text{⊘} = \epsilon_\mu{}^*(p)$$

$$\overset{\frown}{A_\mu\ A_\nu} = \text{⊘}\wwwwww{\longleftarrow p}\text{⊘} = \frac{-\mathrm{i}\eta_{\mu\nu}}{p^2 + \mathrm{i}\epsilon}$$

$$Q\overline{\psi}\slashed{A}\psi = \ \ {}^\mu = \mathrm{i}Q\gamma^\mu$$

$$\begin{array}{c}\mathrm{i}QA^\mu \phi^* \partial_\mu \phi \\ +\text{H.c.}\end{array} = \underset{p \to \quad q \to}{\ } = \mathrm{i}Q(p^\mu + q^\mu)$$

(Momentum must be taken along the arrow)

$$Q^2 A^2 |\phi|^2 = \ \ {}^\mu_\nu = 2\mathrm{i}Q^2 \eta^{\mu\nu}$$

Non-Abelian Gauge Theory (Gluon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}\slashed{D}\psi + |D_\mu \phi|^2$$
$$(D_\mu = \partial_\mu - \mathrm{i}gA_\mu)$$

$$\overset{\frown}{A_\mu^b\ |\boldsymbol{p};\ , a\rangle} = \text{⊘}\oooooooo{\longleftarrow p} = \epsilon_\mu(p)\delta^{ab}$$

$$\langle \boldsymbol{p};\ , a| \overset{\frown}{A_\mu^b} = \oooooooo{\longleftarrow p}\text{⊘} = \epsilon_\mu{}^*(p)\delta^{ab}$$

$$-gf^{abc}A^{\mu a}A^{\nu b}(\partial_\mu A_\nu^c) =$$

$$\begin{aligned}= gf^{abc}\big[&\eta^{\mu\nu}(p - q)^\rho \\ &+ \eta^{\nu\rho}(q - r)^\mu \\ &+ \eta^{\rho\mu}(r - p)^\nu \big]\end{aligned}$$

(Momentum are in incoming directions)

$$-\frac{1}{4}g^2 (f^{abe}A_\mu^a A_\nu^b)(f^{cde}A_\rho^c A_\sigma^d) =$$

$$\begin{aligned}= -\mathrm{i}g^2 \big[\ & \\ f^{abe}f^{cde}&(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ + f^{ace}f^{bde}&(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ + f^{ade}f^{bce}&(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})\big]\end{aligned}$$

TODO: vertex   lagrangian   ($n!$)i   )

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1.4 Field Calculation Techniques

1.4.1 Dirac Field Techniques

$$
\begin{aligned}
\text{Dirac Equations} \quad &: \quad (\not{p}-m)u^s(p)=0; \quad (\not{p}+m)v^s(p)=0 \\
&: \quad \bar{u}^s(p)(\not{p}-m)=0; \quad \bar{v}^s(p)(\not{p}+m)=0 \\
\text{Dirac Components} \quad &: \quad u^{r\dagger}(p)u^s(p)=2E_{\boldsymbol{p}}\delta^{rs}; \quad v^{r\dagger}(p)v^s(p)=2E_{\boldsymbol{p}}\delta^{rs} \\
&: \quad \bar{u}^r(p)u^s(p)=2m\delta^{rs}; \quad \bar{v}^r(p)v^s(p)=-2m\delta^{rs}; \quad \bar{u}^r(p)v^s(p)=\bar{v}^r(p)u^s(p)=0 \\
\text{Spin Sums} \quad &: \quad \sum_{\text{spin}}u^s(p)\bar{u}^s(p)=\not{p}+m; \quad \sum_{\text{spin}}v^s(p)\bar{v}^s(p)=\not{p}-m \\
\text{Chage Conj.} \quad &: \quad -C=C^{-1}=C^\dagger=C^{\mathrm{T}}, \quad C^{-1}\gamma^\mu C=-C\gamma^\mu C=-\gamma^{\mu\mathrm{T}}, \quad C^{-1}\gamma^0 C=-\gamma^0 \\
&: \quad C=C^* \quad\quad\quad\quad , \quad \psi^{\mathrm{C}}=C(\overline{\psi})^{\mathrm{T}}, \quad \overline{\psi}^{\mathrm{C}}=\psi^{\mathrm{T}}C \\
u\ \&\ v \quad &: \quad u^*=-\mathrm{i}\gamma^2 v; \quad v^{\mathrm{T}}=-\mathrm{i}u^\dagger\gamma^2; \quad\quad \overline{u}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}v_{\mathrm{A}} \\
&: \quad v^*=-\mathrm{i}\gamma^2 u; \quad u^{\mathrm{T}}=-\mathrm{i}v^\dagger\gamma^2; \quad\quad \overline{v}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}u_{\mathrm{A}}
\end{aligned}
$$

1.4.2 Polarization Sum

<u>Single photon case</u> $M=\epsilon_\mu^*(k)M^\mu$

When Ward identity $k_\mu M^\mu=0$ is valid,

$$
\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon_\mu^*(k)\epsilon_\nu(k)M^\mu M^{\nu *}=\eta_{\mu\nu}M^\mu M^{\nu *}. \tag{1.1}
$$

<u>Double photons case</u> $M=\epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu}=k_\nu' M^{\mu\nu}=0$ is valid,

$$
\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon_\mu^*(k)\epsilon_\rho(k)\epsilon_\nu'^*(k')\epsilon_\sigma'(k')M^{\mu\nu}M^{\rho\sigma *}=\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *}. \tag{1.2}
$$

See Sec. C.3 for verbose information.

1.4.3 Fierz transformations

For Dirac spinors a,b,c,d,

$$
\begin{aligned}
S(a,b;c,d) &:= (\bar{a}b)(\bar{c}d); \\
V(a,b;c,d) &:= (\bar{a}\gamma^\mu b)(\bar{c}\gamma_\mu d); \\
T(a,b;c,d) &:= \frac{1}{2}(\bar{a}\sigma^{\mu\nu}b)(\bar{c}\sigma_{\mu\nu}d); \\
A(a,b;c,d) &:= (\bar{a}\gamma^\mu\gamma_5 b)(\bar{c}\gamma_\mu\gamma_5 d); \\
P(a,b;c,d) &:= (\bar{a}\gamma_5 b)(\bar{c}\gamma_5 d);
\end{aligned}
\qquad
\begin{pmatrix} S(a,b;c,d) \\ V(a,b;c,d) \\ T(a,b;c,d) \\ A(a,b;c,d) \\ P(a,b;c,d) \end{pmatrix}
=\frac{1}{4}
\begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 4 & -2 & 0 & -2 & 4 \\ 6 & 0 & -2 & 0 & -6 \\ -4 & -2 & 0 & -2 & -4 \\ -1 & 1 & -1 & -1 & 1 \end{pmatrix}
\begin{pmatrix} S(a,d;c,b) \\ V(a,d;c,b) \\ T(a,d;c,b) \\ A(a,d;c,b) \\ P(a,d;c,b) \end{pmatrix}
$$

Also defining $V_{\mathrm{LR}}(a,b;c,d):=(\bar{a}\gamma^\mu P_{\mathrm{L}}b)(\bar{c}\gamma_\mu P_{\mathrm{R}}d)$ and so on,

$$
V_{\mathrm{LL}}(a,b;c,d)=-V_{\mathrm{LL}}(a,d;c,b) \qquad S_{\mathrm{RL}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{LR}}(a,d;b,c)-A_{\mathrm{LR}}(a,d;b,c)\right] \tag{1.3}
$$

$$
V_{\mathrm{RR}}(a,b;c,d)=-V_{\mathrm{RR}}(a,d;c,b) \qquad S_{\mathrm{LR}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{RL}}(a,d;b,c)-A_{\mathrm{RL}}(a,d;b,c)\right] \tag{1.4}
$$

Here we can create another equations using

$$
(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}; \qquad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{1.5}
$$

1.4.4 Gordon identity

For $P := p' + p$ and $q := p' - p$,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \qquad \bar{u}(p')\gamma^\mu v(p) = \bar{u}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \qquad (1.6)$$

$$\bar{v}(p')\gamma^\mu v(p) = -\bar{v}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]v(p) \qquad \bar{v}(p')\gamma^\mu u(p) = -\bar{v}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \qquad (1.7)$$

1.4.5 Color Sum

(Here T^a is **3** of SU(3). For other representations or gauge groups, see Sec. 1.9.)

$$\mathrm{Tr}(T^a T^b) := \frac{1}{2}\delta^{ab} \qquad \text{(That is, } T^a\text{'s are } \tfrac{1}{2}\times\text{Gell-Mann matrices.)} \qquad (1.8)$$

$$\sum_a T^a T^a = \frac{4}{3}\cdot\mathbf{1}, \qquad \sum_{c,d} f^{acd}f^{bcd} = 3\delta^{ab} \qquad \sum_a T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} - \frac{1}{6}\delta_{ij}\delta_{kl} \qquad (1.9)$$

$$\sum_a T^a T^b T^a = -\frac{1}{6}T^b \qquad \sum_{b,c} f^{abc}T^b T^c = \frac{3\mathrm{i}}{2}T^a \qquad f^{Dab}f^{EDc}+f^{Dca}f^{EDb} + f^{Dbc}f^{EDa} = 0 \qquad (1.10)$$

TODO: TODO:

- Majorana Ferminos
- Feynman Rules(A.1)

1.5 DIRAC'S GAMMA ALGEBRAS

1.5.1 Traces

$$\text{Tr(any odd \# of } \gamma\text{'s)} = 0 \tag{1.11}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} \tag{1.12}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \tag{1.13}$$

$$\text{Tr}(\gamma_5 \text{ and any odd \# of } \gamma\text{'s)} = 0 \tag{1.14}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma_5) = 0 \tag{1.15}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \tag{1.16}$$

Generally, for some γ-matrices $A, B, C, \ldots$,

$$\begin{aligned} \text{Tr}(ABCDEF\cdots) = {} & \eta^{AB}\,\text{Tr}(CDEF\cdots) - \eta^{AC}\,\text{Tr}(BDEF\cdots) \\ & + \eta^{AD}\,\text{Tr}(BCEF\cdots) - \eta^{AE}\,\text{Tr}(BCDF\cdots) + \cdots, \end{aligned} \tag{1.17}$$

$$\text{Tr}(ABCDEF\cdots\gamma_5) = \textcolor{red}{\textbf{Not Established.}} \tag{1.18}$$

To prove the second equation, we use following technique:

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \cdots) = \text{Tr}(\cdots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu); \qquad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \cdots \gamma_5) = \text{Tr}(\gamma_5 \cdots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu). \tag{1.19}$$

1.5.2 Contractions

$$\gamma^\mu \gamma_\mu = 4 \tag{1.20}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{1.21}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} \tag{1.22}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \tag{1.23}$$

Generally, for some γ-matrices $A, B, C, \ldots$,

$$\text{ODD \# :} \quad \gamma^\mu ABC \cdots \gamma_\mu = -2(\cdots CBA), \tag{1.24}$$

$$\text{EVEN \# :} \quad \gamma^\mu ABC \cdots \gamma_\mu = \text{Tr}(ABC\cdots) - \text{Tr}(ABC\cdots\gamma_5)\cdot\gamma_5. \tag{1.25}$$

Contractions in d-dimension

$$\gamma^\mu \gamma_\mu = d \tag{1.26}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu \tag{1.27}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho \tag{1.28}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{1.29}$$

Contractions of ϵ's

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu; \quad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \tag{1.30}$$

$$\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\alpha'\beta'\gamma'} = -\left(\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\alpha_{\beta'}\delta^\beta_{\gamma'}\delta^\gamma_{\alpha'} + \delta^\alpha_{\gamma'}\delta^\beta_{\alpha'}\delta^\gamma_{\beta'} - \delta^\alpha_{\alpha'}\delta^\beta_{\gamma'}\delta^\gamma_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\delta^\gamma_{\gamma'} - \delta^\alpha_{\gamma'}\delta^\beta_{\beta'}\delta^\gamma_{\alpha'}\right) \tag{1.31}$$

1.6 Miscellaneous Techniques

$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2$

$\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$

$\sqrt{p_\mu \sigma^\mu} = \dfrac{p_\mu \sigma^\mu + m}{\sqrt{2(m + p^0)}}$

$\sigma^i \sigma^j = \delta_{ij}\sigma^0 + \mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^\mu \sigma^\nu = \mathrm{i}\epsilon^{0\mu\nu\rho}\sigma^\rho + \delta_0^\mu \sigma^\nu + \delta_0^\nu \sigma^\mu - \eta^{\mu\nu}\sigma^0$

$[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^i, \sigma^j = 2\delta_{ij}$

1.7 Loop Integrals and Dimensional Regularization

1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots x_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{1.32}$$

$$\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{[xA_1 + (1-x)A_2]^2} \tag{1.33}$$

1.7.2 d-dimensional integrals in Minkowski space

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \tag{1.34}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \tag{1.35}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1} \tag{1.36}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \tag{1.37}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 2} \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{4} \tag{1.38}$$

Here we can use following expansions: $\qquad (\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = 1 - (d-4)\frac{\log \Delta}{2} + O\left((d-4)^2\right) \quad \text{around } d = 4, \tag{1.39}$$

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x) \quad \text{around } x = 0, \tag{1.40}$$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left[\frac{1}{x+n} - \gamma + \sum_{k=1}^n \frac{1}{k} + O(x+n)\right] \quad \text{around } x = -n. \tag{1.41}$$

and we get following expansion:

$$\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} = \frac{1}{(4\pi)^2} \left[\left(\frac{2}{4-d} - \gamma + \log 4\pi\right) - \log \Delta + O(4-d)\right]. \tag{1.42}$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S-matrix elements. To compute the S-matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \rightarrow \left(\frac{1}{\Delta - i\epsilon}\right)^{n - \frac{d}{2}}. \tag{1.43}$$

1.8 CROSS SECTIONS AND DECAY RATES

General expression   (The mass dimension of $\mathcal{M}$ is $2 - N_\mathrm{f}$ for $\mathrm{d}\sigma$ and $3 - N_\mathrm{f}$ for $\mathrm{d}\Gamma$.)

$$\mathrm{d}\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(p_A, p_B \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)} \left(p_A + p_B - \{p_f\} \right) \qquad (1.44)$$

$$\mathrm{d}\Gamma = \frac{1}{2m_A} \left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right] \left| \mathcal{M}(m_A \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)} \left(m_A - \{p_f\} \right) \quad \text{(in } A\text{-rest frame.)} \qquad (1.45)$$

2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} (2\pi)^4 \delta^{(4)} \left(E_\mathrm{cm} - (p_1 + p_2) \right) \qquad \text{(in center-of-mass frame)} \qquad (1.46)$$

$$= \int \frac{\mathrm{d}\Omega}{4\pi} \frac{1}{8\pi} \frac{2 \|\boldsymbol{p_1}\|}{E_\mathrm{cm}} \qquad (1.47)$$

$$= \frac{1}{8\pi} \sqrt{1 - \frac{2(m_1{}^2 + m_2{}^2)}{E_\mathrm{cm}^2} + \frac{(m_1{}^2 - m_2{}^2)^2}{E_\mathrm{cm}^4}} \quad \overset{m_2 = 0}{\longrightarrow} \quad \frac{1}{8\pi} \left(1 - \frac{m_1{}^2}{E_\mathrm{cm}{}^2} \right) \qquad (1.48)$$

Kinematics of Decay

$$K \to p_1 + p_2 \qquad \text{or} \qquad \begin{pmatrix} M \\ \boldsymbol{0} \end{pmatrix} \to \begin{pmatrix} \sqrt{p^2 + m_1{}^2} \\ \boldsymbol{p} \end{pmatrix} + \begin{pmatrix} \sqrt{p^2 + m_2{}^2} \\ -\boldsymbol{p} \end{pmatrix}; \qquad (1.49)$$

$$\|\boldsymbol{p}\|^2 = \frac{1}{4} \left[M^2 - 2\left(m_1{}^2 + m_2{}^2 \right) + \frac{\left(m_1{}^2 - m_2{}^2 \right)^2}{M^2} \right] \approx \left(\frac{M^2 - m_1{}^2}{2M} \right)^2$$
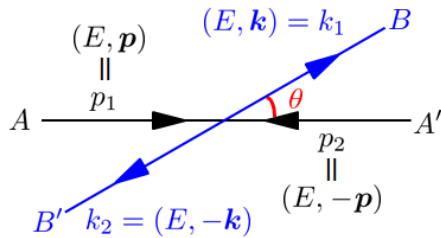
$$E_1 = \frac{M^2 + m_1{}^2 - m_2{}^2}{2M}, \qquad E_2 = \frac{M^2 + m_2{}^2 - m_1{}^2}{2M};$$

$$K \cdot p_1 = \frac{M^2 + m_1{}^2 - m_2{}^2}{2}, \qquad p_1 \cdot p_2 = \frac{M^2 - \left(m_1{}^2 + m_2{}^2 \right)}{2}.$$

Mandelstam Variables

$$\text{For } p_1 + p_2 \to k_1 + k_2 \text{ collision}, \qquad s = (p_1 + p_2)^2 = (k_1 + k_2)^2,$$
$$t = (p_1 - k_1)^2 = (p_2 - k_2)^2,$$
$$u = (p_1 - k_2)^2 = (p_2 - k_1)^2,$$
$$\text{and} \qquad s + t + u = p_1^2 + p_2^2 + k_1^2 + k_2^2 = \sum m^2.$$

Kinematics of Collision (Same Mass)

$$\|\boldsymbol{p}\|^2 = E^2 - m_A^2 \qquad \boldsymbol{p} \cdot \boldsymbol{k} = \|\boldsymbol{p}\| \, \|\boldsymbol{k}\| \cos\theta$$
$$\|\boldsymbol{k}\|^2 = E^2 - m_B^2$$

$$p_1 \cdot p_2 = s/2 - m_A^2 \qquad p_1 \cdot k_1 = p_2 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - t);$$
$$k_1 \cdot k_2 = s/2 - m_B^2 \qquad p_1 \cdot k_2 = p_1 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - u);$$

$$s = 4E^2,$$
$$(p_1 - p_2)^2 = -4(E^2 - m_A^2) \qquad t = -(2E^2 - m_A^2 - m_B^2) + 2\boldsymbol{p} \cdot \boldsymbol{k}$$
$$(k_1 - k_2)^2 = -4(E^2 - m_B^2) \qquad u = -(2E^2 - m_A^2 - m_B^2) - 2\boldsymbol{p} \cdot \boldsymbol{k}$$

1.9   -Mills Theory   (See App. C.5 for verbose notes.)

1.9.1   Non-Abelian gauge theory

$$[T^a, T^b] = \mathrm{i} f^{ab}{}_c T^c, \qquad 0 = f^D{}_{ab} f^E{}_{Dc} + f^D{}_{ca} f^E{}_{Db} + f^D{}_{bc} f^E{}_{Da}, \qquad \mathrm{D}_\mu = \partial_\mu - \mathrm{i} g A_\mu$$

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}, \qquad [\widetilde{T}^a]_i{}^j := T^{\mathrm{ad}\ a}{}_i{}^j := -\mathrm{i} f^{aij} \qquad\qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j := \delta_i^j \partial_\mu + g f^{iaj} A_\mu^a.$$

$$F_{\mu\nu} = \frac{\mathrm{i}}{g} [\mathrm{D}_\mu, \mathrm{D}_\nu] \qquad\qquad \mathrm{D}_\mu \phi = \partial_\mu \phi - \mathrm{i} g A_\mu^a (T_\phi^a \phi)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{\mathrm{i}} [A_\mu, A_\nu] \qquad\qquad \mathrm{D}_\mu F_{\mu\nu}{}^a = \partial_\mu F_{\mu\nu}^a + g f^{abc} A_\mu^b F_{\mu\nu}^c,$$

$$= \left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \right] T^a \qquad\qquad \left(\mathrm{D}_\mu F_{\nu\rho} = \partial_\mu \lambda - \mathrm{i} g [A_\mu, F_{\nu\rho}] \right).^{*1}$$

$$\phi \mapsto V\phi := \mathrm{e}^{\mathrm{i} g \theta} \phi \qquad\qquad A_\mu \mapsto V \left(A_\mu + \frac{\mathrm{i}}{g} \partial_\mu \right) V^{-1} \qquad\qquad F_{\mu\nu} \mapsto V F_{\mu\nu} V^{-1}$$

$$\phi^{a\prime} \simeq \phi + \mathrm{i} g \theta^a T^a \phi \qquad\qquad A_\mu^{a\prime} \simeq A_\mu^a + \partial_\mu \theta^a + g f^{abc} A_\mu^b \theta^c \qquad\qquad F_{\mu\nu}^a{}' \simeq F_{\mu\nu}^a + g f^{abc} F_{\mu\nu}^b \theta^c$$

$$\epsilon^{\mu\nu\rho\sigma} [\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\sigma]] = \epsilon^{\mu\nu\rho\sigma} \mathrm{D}_\nu F_{\rho\sigma} = 0.$$

Killing and Casimir   Here we have two constants which **depend on representation** r.

$$\mathrm{Tr}(T^a T^b) =: C(r) \delta^{ab} \quad \text{(Killing form)}, \qquad T^a T^a =: C_2(r) \cdot \mathbf{1} \quad \text{(quadratic Casimir operator)}, \qquad (1.50)$$

which satisfy

$$C(r) = \frac{d(r)}{d(\mathrm{ad})} C_2(r), \qquad\qquad T^a T^b T^a = \left[C_2(r) - \frac{1}{2} C_2(\mathrm{ad}) \right] T^b, \qquad (1.51)$$

$$f^{acd} f^{bcd} = C_2(\mathrm{ad}) \delta^{ab}, \qquad\qquad f^{abc} T^b T^c = \frac{\mathrm{i}}{2} C_2(\mathrm{ad}) T^a. \qquad (1.52)$$

<u>For SU(N)</u>   For its fundamental representation N with definition $C(N) := \frac{1}{2}$, we have

$$C(N) := \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}, \quad C(\mathrm{ad}) = C_2(\mathrm{ad}) = N; \quad (T^a)_{ij} (T^a)_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{\delta_{ij} \delta_{kl}}{N} \right).$$

1.9.2   Abelian gauge theory

In Abelian gauge theory, V and fields are always commutative, and thus we have charge freedom (Q).

$$\mathrm{D}_\mu \phi = (\partial_\mu - \mathrm{i} g A_\mu Q) \phi \qquad\qquad \phi \mapsto \mathrm{e}^{\mathrm{i} g Q \theta} \phi \qquad\qquad F_{\mu\nu} = \frac{\mathrm{i}}{g} [\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathrm{D}_\mu \lambda^a = \partial_\mu \lambda^a \qquad\qquad A_\mu \mapsto A_\mu + \partial_\mu \theta \qquad\qquad F_{\mu\nu} \mapsto F_{\mu\nu}$$

1.9.3   Lagrangian Block

$$\mathcal{L} \ni |\mathrm{D}_\mu \phi|^2 - m^2 |\phi|^2, \quad \overline{\psi}(\mathrm{i}\slashed{D} - m)\psi, \quad -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \left(= -\frac{1}{2} \mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu} \right), \quad \theta \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \qquad (1.53)$$

$$-\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a = -\frac{1}{2} \left[(\partial_\mu A_\nu^a)^2 + A_\mu^a \partial^\mu \partial^\nu A_\nu^a \right] - g f^{abc} A_\mu^a A_\nu^b \partial^\mu A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu} \qquad (1.54)$$

---

*1 Note that we can use any representation T^a but must the same ones for $A_\mu^a T^a$ and $\lambda^a T^a$.

2 Standard Model

Any representations assumed to be *normalized Hermitian*. Note that the $SU(2)$ **2** representation is

$$T^a = \frac{1}{2}\sigma^a; \qquad [T^a, T^b] = \mathrm{i}\epsilon^{abc}T^c; \qquad T^{\pm} := T^1 \pm \mathrm{i}T^2. \tag{2.1}$$

We use the following abridged notations:

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad\qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.2}$$

2.1 SYMMETRIES AND FIELDS

	$SU(3)_{\text{strong}}$	$SU(2)_{\text{weak}}$	$U(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_{\text{L}}Q_i$: Left-handed quarks	**3**	**2**	$1/6$
$P_{\text{L}}U_i$: Right-handed up-type quarks	**3**	**1**	$2/3$
$P_{\text{R}}D_i$: Right-handed down-type quarks	**3**	**1**	$-1/3$
$P_{\text{R}}L_i$: Left-handed leptons	**1**	**2**	$-1/2$
$P_{\text{R}}E_i$: Right-handed leptons	**1**	**1**	-1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	**1**	**2**	$1/2$
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	**8**	**1**	0
W : Weak bosons	**1**	**3**	0
B : B boson	**1**	**1**	0

Full Lagrangian $\quad \mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}$

$$\text{where} \quad \mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu \right) H \right|^2 - V(H), \tag{2.4}$$

$$\begin{aligned}
\mathcal{L}_{\text{matter}} = \ &\overline{Q}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu \right) P_{\text{L}}Q_i \\
&+ \overline{U}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \frac{2}{3}\mathrm{i}g_1 B_\mu \right) P_{\text{R}}U_i \\
&+ \overline{D}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu + \frac{1}{3}\mathrm{i}g_1 B_\mu \right) P_{\text{R}}D_i \\
&+ \overline{L}_i \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g_2 W_\mu + \frac{1}{2}\mathrm{i}g_1 B_\mu \right) P_{\text{L}}L_i \\
&+ \overline{E}_i \mathrm{i}\gamma^\mu \left(\partial_\mu + \mathrm{i}g_1 B_\mu \right) P_{\text{R}}E_i,
\end{aligned} \tag{2.5}$$

$$\mathcal{L} = \overline{U}_i(y_u)_{ij} H P_{\text{L}}Q_j - \overline{D}_i(y_d)_{ij} H^\dagger P_{\text{L}}Q_j - \overline{E}_i(y_e)_{ij} H^\dagger P_{\text{L}}L_j + \text{H.c.} \tag{2.6}$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix C.4.

Gauge Kinetic Terms

the gauge kinetic terms can be expanded as

$$
\begin{aligned}
\mathcal{L}_{\text{gauge}} = &-\frac{1}{4}(\partial B)(\partial B) \\
&-\frac{1}{4}(\partial W^a)(\partial W^a) - g_2 \epsilon^{abc}(\partial_\mu W_\nu^a) W^{\mu b} W^{\nu c} - \frac{g_2{}^2}{4}\left(\epsilon^{eab} W_\mu^a W_\nu^b\right)\left(\epsilon^{ecd} W^{c\mu} W^{d\nu}\right) \\
&-\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G_\nu^a) G^{\mu b} G^{\nu c} - \frac{g_3{}^2}{4}\left(f^{eab} G_\mu^a G_\nu^b\right)\left(f^{ecd} G^{c\mu} G^{d\nu}\right).
\end{aligned}
\tag{2.7}
$$

2.2 Higgs Mechanism

Higgs Potential

The (renormalizable) Higgs potential must be

$$
V(H) = -\mu^2(H^\dagger H) + \lambda\left(H^\dagger H\right)^2.
\tag{2.8}
$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field as

$$
H = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + \mathrm{i}\phi_2 \\ v + (h + \mathrm{i}\phi_3) \end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}.
\tag{2.9}
$$

Here h is the "Higgs boson," and ϕ_i are   –Goldstone bosons.

The Higgs potential becomes

$$
V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2 h^2,
\tag{2.10}
$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$
\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right)H\right|^2
\tag{2.11}
$$

$$
= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\left[g_2{}^2 W_1{}^2 + g_2{}^2 W_2{}^2 + (g_1 B - g_2 W_3)^2\right].
\tag{2.12}
$$

Redefining the gauge fields (with concerning the norms) as

$$
W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp \mathrm{i}W_\mu^2), \qquad\qquad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} := \begin{pmatrix} \cos\theta_{\text{w}} & -\sin\theta_{\text{w}} \\ \sin\theta_{\text{w}} & \cos\theta_{\text{w}} \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix},
\tag{2.13}
$$

where

$$
\tan\theta_{\text{w}} := \frac{g_1}{g_2}, \qquad e := -\frac{g_1 g_2}{\sqrt{g_1{}^2 + g_2{}^2}}; \qquad g_Z := \sqrt{g_1{}^2 + g_2{}^2};
\tag{2.14}
$$

$$
g_1 = \frac{|e|}{\cos\theta_{\text{w}}} = g_Z \sin\theta_{\text{w}}, \qquad g_2 = \frac{|e|}{\sin\theta_{\text{w}}} = g_Z \cos\theta_{\text{w}}.
\tag{2.15}
$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$
\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4}\left[g_2{}^2 W^{+\mu} W_\mu^- + \frac{g_Z{}^2}{2}Z^\mu Z_\mu\right].
\tag{2.16}
$$

Here we have omitted the   –Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan\theta_\mathrm{w} Z_\mu, \tag{2.17}$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}} \left(W_\mu^+ T^+ + W_\mu^- T^- \right) + \left(\frac{|e|}{\tan\theta_\mathrm{w}} Z_\mu + |e| A_\mu \right) T^3, \tag{2.18}$$

$$Z_\mu^0 := \frac{1}{\sqrt{g_1{}^2 + g_2{}^2}} (g_2 W_\mu^3 - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{g_1{}^2 + g_2{}^2}} (g_1 W_\mu^3 + g_2 B_\mu) \tag{2.19}$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \tag{2.20}$$

Gauge Term   The SU(2) gauge term is converted into

$$
\begin{aligned}
W^{a\mu\nu} W_{\mu\nu}^a = {}& (\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\
& - 4\mathrm{i}g \left[(\partial W^3)^{\mu\nu} W_\mu^+ W_\nu^- + (\partial W^+)^{\mu\nu} W_\mu^- W_\nu^3 + (\partial W^-)^{\mu\nu} W_\mu^3 W_\nu^+ \right] \\
& - 2g^2 (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \left(W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- - 2 W_\mu^3 W_\nu^3 W_\rho^+ W_\sigma^- \right),
\end{aligned}
$$

and therefore the final expression is

$$
\begin{aligned}
\mathcal{L}_{\text{gauge}} := {}& -\frac{1}{4} \left[G^{a\mu\nu} G_{\mu\nu}^a + (\partial Z)^{\mu\nu}(\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu}(\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu}(\partial W^-)_{\mu\nu} \right] \\
& + \frac{\mathrm{i}|e|}{\tan\theta_\mathrm{w}} \left[(\partial W^+)^{\mu\nu} W_\mu^- Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W_\nu^+ + (\partial Z)^{\mu\nu} W_\mu^+ W_\nu^- \right] \\
& + \mathrm{i}|e| \left[(\partial W^+)^{\mu\nu} W_\mu^- A_\nu + (\partial W^-)^{\mu\nu} A_\mu W_\nu^+ + (\partial A)^{\mu\nu} W_\mu^+ W_\nu^- \right] \\
& + (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \left[\frac{|e|^2}{2\sin^2\theta_\mathrm{w}} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- + \frac{|e|^2}{\tan^2\theta_\mathrm{w}} W_\mu^+ Z_\nu W_\rho^- Z_\sigma \right. \\
& \left. + \frac{|e|^2}{\tan\theta_\mathrm{w}} \left(W_\mu^+ Z_\nu W_\rho^- A_\sigma + W_\mu^+ A_\nu W_\rho^- Z_\sigma \right) + |e|^2 W_\mu^+ A_\nu W_\rho^- A_\sigma \right].
\end{aligned}
\tag{2.21}
$$

Term

$$
\begin{aligned}
\mathcal{L} \quad = {}& \overline{U} y_u H P_\mathrm{L} Q - \overline{D} y_d H^\dagger P_\mathrm{L} Q - \overline{E} y_e H^\dagger P_\mathrm{L} L + \text{H.c.} \\
= {}& \overline{U} y_u \epsilon^{\alpha\beta} H^\alpha P_\mathrm{L} Q^\beta - \overline{D} y_d H^{\dagger\alpha} P_\mathrm{L} Q^\alpha - \overline{E} y_e H^{\dagger\alpha} P_\mathrm{L} L^\alpha + \text{H.c.} \\
= {}& -\frac{v+h}{\sqrt{2}} \left(\overline{U} y_u P_\mathrm{L} Q^1 + \overline{D} y_d P_\mathrm{L} Q^2 + \overline{E} y_e P_\mathrm{L} L^2 \right) + \text{H.c.}
\end{aligned}
\tag{2.22}
$$

2.3   FULL LAGRANGIAN AFTER HIGGS MECHANISM

Now we have the following Lagrangian (with omitting P_L etc.):

$$
\mathcal{L} = \mathcal{L}_{\text{gauge}} + m_W{}^2 W^+ W^- + \frac{m_Z{}^2}{2} Z^2
$$

$$
\text{\color{orange}Higgs} \quad + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2} m_h{}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4
$$

$$
+ \frac{v g_2{}^2}{4} W^+ W^- h + \frac{v(g_1{}^2 + g_2{}^2)}{8} Z^2 h
$$

$$
+ \frac{g_2{}^2}{4} W^+ W^- h^2 + \frac{g_1{}^2 + g_2{}^2}{8} Z^2 h^2
$$

$$
- \left(\frac{1}{\sqrt{2}} h \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} h \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} h \bar{E} y_e L^2 + \text{ H.c.} \right)
$$

$$
\text{\color{orange}SU(3)} \quad + \bar{Q}\left(\mathrm{i}\slashed{\partial} + g_3 \slashed{G} \right) Q + \bar{U}\left(\mathrm{i}\slashed{\partial} + g_3 \slashed{G} \right) U + \bar{D}\left(\mathrm{i}\slashed{\partial} + g_3 \slashed{G} \right) D + \bar{L}\left(\mathrm{i}\slashed{\partial} \right) L + \bar{E}\left(\mathrm{i}\slashed{\partial} \right) E
$$

$$
\text{\color{orange}W} \quad + \bar{Q}\frac{g_2}{\sqrt{2}}\left(\slashed{W}^+ T^+ + \slashed{W}^- T^- \right) Q + \bar{L}\frac{g_2}{\sqrt{2}}\left(\slashed{W}^+ T^+ + \slashed{W}^- T^- \right) L
$$

$$
\text{\color{orange}A\&Z^0} \quad + \bar{Q}\left[\left(T^3 + \frac{1}{6} \right) |e| \slashed{A} + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) \slashed{Z}^0 \right] Q
$$

$$
+ \bar{U}\left(\frac{2}{3}|e|\slashed{A} - \frac{2|e|s}{3c}\slashed{Z} \right) U
$$

$$
+ \bar{D}\left(-\frac{1}{3}|e|\slashed{A} + \frac{|e|s}{3c}\slashed{Z} \right) D
$$

$$
+ \bar{L}\left[\left(T^3 - \frac{1}{2} \right) |e|\slashed{A} + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) \slashed{Z}^0 \right] L
$$

$$
+ \bar{E}\left(-|e|\slashed{A} + \frac{|e|s}{c}\slashed{Z} \right) E
$$

$$
- \left(\frac{1}{\sqrt{2}} v \bar{U} y_u Q^1 + \frac{1}{\sqrt{2}} v \bar{D} y_d Q^2 + \frac{1}{\sqrt{2}} v \bar{E} y_e L^2 + \text{ H.c.} \right) \tag{2.23}
$$

2.4   MASS EIGENSTATES

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the   matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := v y_\bullet / \sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \qquad (m_i \geq 0). \tag{2.24}
$$

Using this Ψ and Φ, we can rotate the basis as

$$
Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \qquad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \tag{2.25}
$$

and now we have the   terms in mass eigenstates as

$$
\mathcal{L} = -\left(1 + \frac{1}{v} h \right) \left[(m_u)_i \overline{U}_i P_\mathrm{L} Q_i^1 + (m_d)_i \overline{D}_i P_\mathrm{L} Q_i^2 + (m_e)_i \overline{E}_i P_\mathrm{L} L_i^2 + \text{H.c.} \right]. \tag{2.26}
$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark–W interactions do change drastically, as

$$\mathcal{L} \supset \overline{Q}\mathrm{i}\gamma^\mu \left(-\mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu \right) P_\mathrm{L} Q \tag{2.27}$$

$$= \overline{Q}\frac{g_2}{\sqrt{2}} \left(\slashed{W}^+ T^+ + \slashed{W}^- T^- \right) P_\mathrm{L} Q \quad + \quad (\text{interaction terms with } Z \text{ and } A) \tag{2.28}$$

$$\mapsto \frac{g_2}{\sqrt{2}} \begin{pmatrix} \overline{Q}^1 \Psi_u & \overline{Q}^2 \Psi_d \end{pmatrix} \begin{pmatrix} 0 & \slashed{W}^+ \\ \slashed{W}^- & 0 \end{pmatrix} P_\mathrm{L} \begin{pmatrix} \Psi_u^\dagger Q^1 \\ \Psi_d^\dagger Q^2 \end{pmatrix} + (\quad \cdots \quad) \tag{2.29}$$

$$= \frac{g_2}{\sqrt{2}} \left[\overline{Q}^2 \slashed{W}^- X P_\mathrm{L} Q^1 + \overline{Q}^1 \slashed{W}^+ X^\dagger P_\mathrm{L} Q^2 \right] + (\quad \cdots \quad), \tag{2.30}$$

where $X := \Psi_d \Psi_u^\dagger$ is a matrix, so-called the Cabbibo– – (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP-symmetry.

> In our notation, CP-transformation of a spinor is described as
>
> $$\mathscr{CP}(\psi) = -\mathrm{i}\eta^*(\overline{\psi}\gamma^2)^\mathrm{T}, \quad \mathscr{CP}(\overline{\psi}) = \mathrm{i}\eta(\gamma^2\psi)^\mathrm{T}, \tag{2.31}$$
>
> where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,
>
> $$\begin{aligned} \mathscr{CP}\left(\overline{Q}^2 \slashed{W}^- X P_\mathrm{L} Q^1 \right) &= (\gamma^2 Q^2)^\mathrm{T} \mathscr{P}(-\slashed{W}^+) X P_\mathrm{L} (\overline{Q}^1 \gamma^2)^\mathrm{T} \\ &= -W_\mu^{+P}(\gamma^2 Q^2)^\mathrm{T}(\overline{Q}^1 X^\mathrm{T} \gamma^2 P_\mathrm{L} \gamma^{\mu\mathrm{T}})^\mathrm{T} \\ &= (\overline{Q}^1 \slashed{W}^+ X^\mathrm{T} P_\mathrm{L} Q^2). \end{aligned} \tag{2.32}$$
>
> Therefore, we can see that the CP-symmetry is maintained if and only if $X^\mathrm{T} = X^\dagger$, that is, if and only if X is a real matrix.

Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gauge}}$$

$$+ \quad m_W{}^2 W^+ W^- \quad + \quad \frac{m_Z{}^2}{2} Z^2$$

$$- \quad \left(\bar{U} M_u P_{\text{L}} Q^1 + \bar{D} M_d P_{\text{L}} Q^2 + \bar{E} M_e P_{\text{L}} L^2 + \text{H.c.} \right)$$

Higgs Field
$$+ \quad \frac{1}{2} (\partial_\mu h)^2 \quad -\frac{1}{2} m_h{}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4} \lambda h^4$$

Higgs
$$+ \quad \frac{v g_2{}^2}{4} W^+ W^- h \quad + \quad \frac{v(g_1{}^2 + g_2{}^2)}{8} Z^2 h$$

$$+ \quad \frac{g_2{}^2}{4} W^+ W^- h^2 \quad + \quad \frac{g_1{}^2 + g_2{}^2}{8} Z^2 h^2$$

$$- \quad \left(\frac{1}{v} \bar{U} M_u P_{\text{L}} Q^1 h + \frac{1}{v} \bar{D} M_d P_{\text{L}} Q^2 h + \frac{1}{v} \bar{E} M_e P_{\text{L}} L^2 h + \text{H.c.} \right)$$

SU(3)
$$+ \quad \bar{Q} \left(\text{i} \slashed{\partial} + g_3 \slashed{G} \right) P_{\text{L}} Q \quad + \quad \bar{U} \left(\text{i} \slashed{\partial} + g_3 \slashed{G} \right) P_{\text{R}} U \quad + \quad \bar{D} \left(\text{i} \slashed{\partial} + g_3 \slashed{G} \right) P_{\text{R}} D$$

$$+ \quad \bar{L} \left(\text{i} \slashed{\partial} \right) P_{\text{L}} L \quad + \quad \bar{E} \left(\text{i} \slashed{\partial} \right) P_{\text{R}} E$$

W boson
$$+ \quad \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \slashed{W}^- X P_{\text{L}} Q^1 + \bar{Q}^1 \slashed{W}^+ X^\dagger P_{\text{L}} Q^2 \right] \qquad \textcolor{orange}{CP \text{ and flavor violating!}}$$

$$+ \quad \bar{L} \frac{g_2}{\sqrt{2}} \left(\slashed{W}^+ T^+ + \slashed{W}^- T^- \right) P_{\text{L}} L$$

A&Z⁰ boson
$$+ \quad \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| \slashed{A} + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) \slashed{Z}^0 \right] P_{\text{L}} Q$$

$$+ \quad \bar{U} \left(\frac{2}{3} |e| \slashed{A} - \frac{2|e|s}{3c} \slashed{Z} \right) P_{\text{R}} U$$

$$+ \quad \bar{D} \left(-\frac{1}{3} |e| \slashed{A} + \frac{|e|s}{3c} \slashed{Z} \right) P_{\text{R}} D$$

$$+ \quad \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| \slashed{A} + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) \slashed{Z}^0 \right] P_{\text{L}} L$$

$$+ \quad \bar{E} \left(-|e| \slashed{A} + \frac{|e|s}{c} \slashed{Z} \right) P_{\text{R}} E \tag{2.33}$$

2.5 Chiral Notation

Lagrangian　chiral

$$\mathcal{L} =(\text{Higgs terms}) + (\text{Gauge fields strength})$$

$$+ Q_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu} \left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \mathrm{i}g_2 W_{\mu} - \frac{1}{6}\mathrm{i}g_1 B_{\mu} \right) Q_{\mathrm{L}}$$

$$+ U_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu} \left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} - \frac{2}{3}\mathrm{i}g_1 B_{\mu} \right) U_{\mathrm{R}}$$

$$+ D_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu} \left(\partial_{\mu} - \mathrm{i}g_3 G_{\mu} + \frac{1}{3}\mathrm{i}g_1 B_{\mu} \right) D_{\mathrm{R}}$$

$$+ L_{\mathrm{L}}^{\dagger}\mathrm{i}\bar{\sigma}^{\mu} \left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu} + \frac{1}{2}\mathrm{i}g_1 B_{\mu} \right) L_{\mathrm{L}}$$

$$+ E_{\mathrm{R}}^{\dagger}\mathrm{i}\sigma^{\mu} \left(\partial_{\mu} + \mathrm{i}g_1 B_{\mu} \right) E_{\mathrm{R}}$$

$$- \left(U_{\mathrm{R}}^{\dagger}y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger}y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger}y_e H^{\dagger} L_{\mathrm{L}} + \text{H.c.} \right)$$

$$=(\text{Higgs terms}) + (\text{Gauge fields strength})$$

$$+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger}$$

$$+ g_3 \left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}} \right)$$

$$+ g_2 \left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}Q_{\mathrm{L}} + L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}L_{\mathrm{L}} \right)$$

$$+ g_1 \left(\frac{1}{6}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}Q_{\mathrm{L}} + \frac{2}{3}U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}U_{\mathrm{R}} - \frac{1}{3}D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}D_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}L_{\mathrm{L}} - E_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}E_{\mathrm{R}} \right)$$

$$- \left(U_{\mathrm{R}}^{\dagger}y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger}y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger}y_e H^{\dagger} L_{\mathrm{L}} + \text{H.c.} \right) \tag{2.34}$$

$$\mathcal{L} =(\text{Gauge bosons and Higgs})$$

$$+ \mathrm{i}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger}$$

$$+ g_3 \left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}} \right)$$

$$- m_u(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}}) - (\text{quarks}) - m_e(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}}) - (\text{leptons})$$

$$- \frac{m_u}{v}(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}})h - (\text{quarks}) - \frac{m_e}{v}(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}})h - (\text{leptons})$$

$$+ \frac{g_2}{\sqrt{2}} \left[\left(d_{\mathrm{L}}^{\dagger} \ s_{\mathrm{L}}^{\dagger} \ b_{\mathrm{L}}^{\dagger} \right) \bar{\sigma}^{\mu}W_{\mu}^{-}X \begin{pmatrix} u_{\mathrm{L}} \\ c_{\mathrm{L}} \\ t_{\mathrm{L}} \end{pmatrix} + \left(u_{\mathrm{L}}^{\dagger} \ c_{\mathrm{L}}^{\dagger} \ t_{\mathrm{L}}^{\dagger} \right) \bar{\sigma}^{\mu}W_{\mu}^{+}X^{\dagger} \begin{pmatrix} d_{\mathrm{L}} \\ s_{\mathrm{L}} \\ b_{\mathrm{L}} \end{pmatrix} \right]$$

$$+ \frac{g_2}{\sqrt{2}} \left[\nu_e^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{+}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{-}\nu_e \right]$$

$$+ |e| \left[\frac{2}{3}u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}u_{\mathrm{L}} - \frac{1}{3}d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}d_{\mathrm{L}} + \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}u_{\mathrm{R}} - \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}d_{\mathrm{R}} + (\text{quarks}) \right.$$

$$\left. - e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}e_{\mathrm{L}} - e_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}e_{\mathrm{R}} + (\text{leptons}) \right]$$

$$+ \frac{|e|s}{c} \left[\left(\frac{c^2}{2s^2} - \frac{1}{6} \right) u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}u_{\mathrm{L}} - \left(\frac{c^2}{2s^2} + \frac{1}{6} \right) d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}d_{\mathrm{L}} - \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}u_{\mathrm{R}} + \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}d_{\mathrm{R}} \right.$$

$$\left. + \left(\frac{c^2}{2s^2} + \frac{1}{2} \right) \nu_e^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2} \right) e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}e_{\mathrm{L}} + e_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}e_{\mathrm{R}} + (\text{others}) \right] \tag{2.35}$$

2.6   Values of SM Parameters   (Extracted from PDG 2010)

2.6.1   Experimental Values

Low energy values

$$\alpha_{\mathrm{EM}} = 1/137.035999679(94) \qquad G_{\mathrm{F}} = \frac{g_2{}^2}{4\sqrt{2}m_W{}^2} = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

Electroweak scale   These values are all in $\overline{\mathrm{MS}}$ scheme.

$$\alpha_{\mathrm{EM}}^{-1}(m_Z) = 127.925(16) \qquad\qquad m_W(m_W) = 80.399(23)\,\mathrm{GeV}$$
$$\alpha_{\mathrm{EM}}^{-1}(m_\tau) = 133.452(16) \qquad\qquad m_Z(m_Z) = 91.1876(21)\,\mathrm{GeV}$$
$$\alpha_{\mathrm{s}}(m_Z) = 0.1183(15) \qquad\qquad \sin^2\theta_{\mathrm{W}}(m_Z) = 0.23116(13)$$
$$\Gamma_{l^+l^-} = 83.984(86)\,\mathrm{MeV} \qquad\qquad \sin^2\theta_{\mathrm{eff}} = 0.23146(12)$$

Fundamental masses

$$e : 0.510998910(13)\,\mathrm{MeV} \qquad u : 1.7 \text{ to } 3.3\,\mathrm{MeV} \qquad d : 4.1 \text{ to } 5.8\,\mathrm{MeV}$$
$$\mu : 105.658367(4)\,\mathrm{MeV} \qquad c : 1.27^{+0.07}_{-0.09}\,\mathrm{GeV} \qquad s : 101^{+29}_{-21}\,\mathrm{MeV}$$
$$\tau : 1.77682(16)\,\mathrm{GeV} \qquad t : 172.0_{\pm 2.2}\,\mathrm{GeV} \qquad b : 4.19^{+0.18}_{-0.06}\,\mathrm{GeV}$$

$$\pi^\pm : 139.57018(35)\,\mathrm{MeV} \qquad K^\pm : 493.677(16)\,\mathrm{MeV} \qquad p : 938.27203(8)\,\mathrm{MeV}$$
$$\pi^0 : 134.9766(6)\,\mathrm{MeV} \qquad K^0 : 497.614(24)\,\mathrm{MeV} \qquad n : 939.565346(23)\,\mathrm{MeV}$$

Fundamental Lifetime   (also $c\tau$ for some particles)

$$\mu : 2.197034(21)\,\mu\mathrm{s} \quad (659\,\mathrm{m}) \qquad \pi^\pm : 2.6033(5) \times 10^{-8}\,\mathrm{s} \qquad K^\pm : 1.2380(21) \times 10^{-8}\,\mathrm{s}$$
$$\tau : 2.906(10) \times 10^{-13}\,\mathrm{s} \quad (87\,\mu\mathrm{m}) \qquad \pi^0 : 8.4(5) \times 10^{-17}\,\mathrm{s} \qquad K_{\mathrm{S}}^0 : 8.953(5) \times 10^{-11}\,\mathrm{s}$$
$$K_{\mathrm{L}}^0 : 5.116(20) \times 10^{-8}\,\mathrm{s}$$

CKM matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} 0.97428(15) & 0.2253(7) & 0.00347(16) \\ 0.2252(7) & 0.97345(16) & 0.0410(11) \\ 0.00862(26) & 0.0403(11) & 0.999152(45) \end{pmatrix} \sim \begin{pmatrix} 1-\epsilon^2 & \epsilon & \epsilon^4 \\ \epsilon & 1-\epsilon^2 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1-\epsilon^4 \end{pmatrix} \quad \text{for } \epsilon \sim 0.23 \tag{2.36}$$

2.6.2   Estimation of SM Parameters

For EW scale, we can estimate the values as

$$e \sim 0.313, \qquad g_1 \sim 0.358, \qquad g_2 \sim 0.651; \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246\,\mathrm{GeV} \tag{2.37}$$

Therefore   matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.98 \end{pmatrix}, \quad y_d \sim \begin{pmatrix} 3\times 10^{-5} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad y_e \sim \begin{pmatrix} 3\times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \tag{2.38}$$

Also, for $m_h \sim 120\,\mathrm{GeV}$, we can estimate the Higgs potential as $\mu \sim 85\,\mathrm{GeV}$ and $\lambda \sim 0.12$.

3 Supersymmetry for $\eta = \mathrm{diag}(+,-,-,-)$

3.1 SPINOR CONVENTION

ϵ tensor $\quad : \quad \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1 \qquad$ (definition)

Sum Rule $\quad : \quad {}^{\alpha}{}_{\alpha}$ and ${}_{\dot{\alpha}}{}^{\dot{\alpha}}$, except for $\quad \xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta}, \quad \xi^{\alpha} = \epsilon^{\alpha\beta}\xi_{\beta}, \quad \xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}, \quad \xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}.$

Lorentz $\quad : \quad \psi'_{\alpha} = \Lambda_{\alpha}{}^{\beta}\psi_{\beta}, \quad \bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dagger\dot{\beta}}{}_{\dot{\alpha}}, \quad \psi'^{\alpha} = \psi^{\beta}\Lambda^{-1}{}_{\beta}{}^{\alpha}, \quad \bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dagger\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}'^{\dot{\beta}}.$

σ matrices $\quad : \quad (\sigma^{\mu})_{\alpha\dot{\beta}} := (1,\boldsymbol{\sigma})_{\alpha\dot{\beta}}, \quad (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^{\mu})_{\beta\dot{\beta}} = (1,-\boldsymbol{\sigma})^{\dot{\alpha}\beta}.$

3.2 SPINOR CALCULATION CHEATSHEET

$$\eta = (+,-,-,-), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1; \qquad \textbf{Left Differential};$$
$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^{\alpha} := \epsilon^{\alpha\beta}\xi_{\beta}, \quad \xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta}, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$
$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \qquad \sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}, \qquad \sigma^{\mu} := (1,\boldsymbol{\sigma}), \quad \bar{\sigma}^{\mu} := (1,-\boldsymbol{\sigma})$$
$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} := \tfrac{1}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right)_{\alpha}{}^{\beta}, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \tfrac{1}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} = (\sigma^{\nu\mu})^{\dagger\dot{\alpha}}{}_{\dot{\beta}}.$$

$\theta^{\alpha}\theta^{\beta} = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \qquad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad (\theta\phi)(\theta\psi) = -\tfrac{1}{2}(\psi\phi)(\theta\theta) \qquad (\theta\sigma^{\nu}\bar{\theta})\theta^{\alpha} = \tfrac{1}{2}\theta\theta(\bar{\theta}\bar{\sigma}^{\nu})^{\alpha}$

$\theta_{\alpha}\theta_{\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \qquad (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\tfrac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta}) \qquad (\theta\sigma^{\nu}\bar{\theta})\bar{\theta}_{\dot{\alpha}} = -\tfrac{1}{2}\bar{\theta}\bar{\theta}(\theta\sigma^{\mu})_{\dot{\alpha}}$

$\theta^{\alpha}\theta_{\beta} = \tfrac{1}{2}\delta^{\alpha}_{\beta}\theta\theta \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\bar{\theta}\bar{\theta} \qquad (\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) = \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu} \qquad (\sigma^{\mu}\bar{\theta})_{\alpha}(\theta\sigma^{\nu}\bar{\theta}) = \tfrac{1}{2}(\sigma^{\mu}\bar{\sigma}^{\nu}\theta)_{\alpha}\bar{\theta}\bar{\theta}$

$\theta\sigma^{\mu}\bar{\sigma}^{\nu}\theta = \eta^{\mu\nu}\theta\theta \qquad \bar{\theta}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\theta} = \eta^{\mu\nu}\bar{\theta}\bar{\theta}$

$\sigma^{\mu}\bar{\sigma}^{\nu} = \eta^{\mu\nu} + 2\sigma^{\mu\nu} \qquad\qquad \sigma^{\mu}\bar{\sigma}^{\rho}\sigma^{\nu} + \sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\mu} = 2\left(\eta^{\mu\rho}\sigma^{\nu} + \eta^{\nu\rho}\sigma^{\mu} - \eta^{\mu\nu}\sigma^{\rho}\right)$

$\bar{\sigma}^{\mu}\sigma^{\nu} = \eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu} \qquad\qquad \bar{\sigma}^{\mu}\sigma^{\rho}\bar{\sigma}^{\nu} + \bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\mu} = 2\left(\eta^{\mu\rho}\bar{\sigma}^{\nu} + \eta^{\nu\rho}\bar{\sigma}^{\mu} - \eta^{\mu\nu}\bar{\sigma}^{\rho}\right)$

$\sigma^{\mu\nu} = -\sigma^{\nu\mu} \qquad\qquad\qquad \sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} - \sigma^{\rho}\bar{\sigma}^{\nu}\sigma^{\mu} = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\sigma}$

$\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu} \qquad\qquad\qquad \bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} - \bar{\sigma}^{\rho}\sigma^{\nu}\bar{\sigma}^{\mu} = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\sigma}$

$\mathrm{Tr}\,\bar{\sigma}^{\mu}\sigma^{\nu} = \mathrm{Tr}\,\sigma^{\mu}\bar{\sigma}^{\nu} = 2\eta^{\mu\nu} \qquad \mathrm{Tr}\,\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\tfrac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$

$\mathrm{Tr}\,\sigma^{\mu\nu} = \mathrm{Tr}\,\bar{\sigma}^{\mu\nu} = 0 \qquad\qquad \mathrm{Tr}\,\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} = -\tfrac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$

$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \qquad\qquad \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} - \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right]$

$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \qquad\qquad \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} + \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = \eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} - 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}}$

$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \qquad\qquad \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}} \qquad\qquad \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\mathrm{i}\sigma^{\mu\nu}$

$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_{\gamma}{}^{\beta}\epsilon_{\beta\alpha} \qquad\qquad \epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \qquad\qquad \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = -2\mathrm{i}\bar{\sigma}^{\mu\nu}$

$\bar{\xi}\bar{\sigma}^{\mu}\chi = -\chi\sigma^{\mu}\bar{\xi} = (\bar{\chi}\bar{\sigma}^{\mu}\xi)^{*} = -(\xi\sigma^{\mu}\bar{\chi})^{*} \qquad\qquad (\psi\phi)\chi_{\alpha} = -(\phi\chi)\psi_{\alpha} - (\chi\psi)\phi_{\alpha}$

$\xi\sigma^{\mu}\bar{\sigma}^{\nu}\chi = \chi\sigma^{\nu}\bar{\sigma}^{\mu}\xi = (\bar{\chi}\bar{\sigma}^{\nu}\sigma^{\mu}\bar{\xi})^{*} = (\bar{\xi}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\chi})^{*} \qquad\qquad (\psi\phi)\bar{\chi}_{\dot{\alpha}} = \tfrac{1}{2}(\phi\sigma^{\mu}\bar{\chi})(\psi\sigma_{\mu})_{\dot{\alpha}}$

$\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^{\beta}} = -\dfrac{\partial}{\partial\theta_{\alpha}} \qquad\qquad \dfrac{\partial}{\partial\theta^{\alpha}}\theta\theta = 2\theta_{\alpha} \qquad\qquad \epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^{\alpha}}\dfrac{\partial}{\partial\theta^{\beta}}\theta\theta = \dfrac{\partial}{\partial\theta_{\alpha}}\dfrac{\partial}{\partial\theta^{\alpha}}\theta\theta = 4$

$\epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_{\beta}} = -\dfrac{\partial}{\partial\theta^{\alpha}} \qquad\qquad \dfrac{\partial}{\partial\theta_{\alpha}}\theta\theta = -2\theta^{\alpha} \qquad\qquad \epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_{\alpha}}\dfrac{\partial}{\partial\theta_{\beta}}\theta\theta = \dfrac{\partial}{\partial\theta^{\alpha}}\dfrac{\partial}{\partial\theta_{\alpha}}\theta\theta = -4$

$\epsilon^{\dot{\alpha}\dot{\beta}}\dfrac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} = -\dfrac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \qquad\qquad \dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} = -2\bar{\theta}_{\dot{\alpha}} \qquad\qquad \epsilon_{\dot{\alpha}\dot{\beta}}\dfrac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\dfrac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} = \dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\dfrac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} = 4$

$\epsilon_{\dot{\alpha}\dot{\beta}}\dfrac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} = -\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \qquad\qquad \dfrac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} = 2\bar{\theta}^{\dot{\alpha}} \qquad\qquad \epsilon^{\dot{\alpha}\dot{\beta}}\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\dfrac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} = \dfrac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} = -4$

3.3 GENERAL RELATIONS   (Note: $P_\mu = i\partial_\mu$ in our convention.)

$$Q_\alpha := \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad D_\alpha := \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad y := x - i\theta\sigma\bar\theta,$$

$$\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad y^\dagger = x + i\theta\sigma\bar\theta$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{D_\alpha, \bar D_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad (\text{others}) = 0.$$

$$
\begin{array}{llll}
 & <x\text{-basis}> & <y\text{-basis}> & <y^\dagger\text{-basis}> \\
D_\alpha & = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu & = \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial y^\mu} & = \frac{\partial}{\partial\theta^\alpha} & (3.1) \\
\bar D_{\dot\alpha} & = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu & = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} & = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i(\theta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial(y^\dagger)^\mu} & (3.2) \\
D^\alpha & = -\frac{\partial}{\partial\theta_\alpha} + i(\bar\theta\bar\sigma^\mu)^\alpha\partial_\mu & = -\frac{\partial}{\partial\theta_\alpha} + 2i(\bar\theta\bar\sigma^\mu)^\alpha\frac{\partial}{\partial y^\mu} & = -\frac{\partial}{\partial\theta_\alpha} & (3.3) \\
\bar D^{\dot\alpha} & = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu & = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} & = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - 2i(\bar\sigma^\mu\theta)^{\dot\alpha}\frac{\partial}{\partial(y^\dagger)^\mu} & (3.4)
\end{array}
$$

$$\phi(y) = \phi(x) - i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y^\dagger) - 2i\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \quad (3.5)$$

$$\phi(y^\dagger) = \phi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y) + 2i\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y) \quad (3.6)$$

$$\phi(x) = \phi(y) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(y) = \phi(y^\dagger) - i\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \quad (3.7)$$

3.4 Chiral Superfields : $\bar{D}_{\dot\alpha}\Phi = 0$

Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{3.8}$$

$$= \phi(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) \tag{3.9}$$

$$= \phi(y^\dagger) - 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) + \sqrt{2}\theta\psi(y^\dagger) - \sqrt{2}\mathrm{i}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(y^\dagger) + \theta\theta F(y^\dagger) \tag{3.10}$$

$$\Phi^\dagger = \phi^*(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta\bar\theta F^*(y^\dagger) \tag{3.11}$$

$$= \phi^*(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \bar\theta\bar\theta F^*(x) \tag{3.12}$$

$$= \phi^*(y) + 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi^*(y) + \sqrt{2}\bar\theta\bar\psi(y) - \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(y) + \bar\theta\bar\theta F^*(y) \tag{3.13}$$

Product of Chiral Superfields

$$
\begin{aligned}
\Phi_i^\dagger\Phi_j(x,\theta,\bar\theta) =\ & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j \\
& - \mathrm{i}\theta\sigma^\mu\bar\theta\,(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j) + 2\bar\theta\bar\psi_i\theta\psi_j \\
& + \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\,(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j)\,\sigma^\mu\bar\theta + \sqrt{2}\theta\theta\bar\theta\bar\psi_iF_j \\
& - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\,(\partial_\mu\bar\psi_i\phi_j - \bar\psi_i\partial_\mu\phi_j) + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j \\
& + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \frac{1}{4}\phi_i^*\partial^2\phi_j - \frac{1}{4}\partial^2\phi_i^*\phi_j + \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j - \frac{\mathrm{i}}{2}\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_j + \frac{\mathrm{i}}{2}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\rightsquigarrow\ & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j \\
& - 2\mathrm{i}(\theta\sigma^\mu\bar\theta)(\phi_i^*\partial_\mu\phi_j) + \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar\theta\bar\sigma^\mu\psi_j + \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j \\
& + 2\bar\theta\bar\psi_i\theta\psi_j + \sqrt{2}\theta\theta\bar\theta\bar\psi_iF_j + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j \\
& + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned} \tag{3.15}
$$

$$\Phi_i\Phi_j(\text{in }y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta\,[\psi_i\phi_j + \phi_i\psi_j] + \theta\theta\,[\phi_iF_j + F_i\phi_j - \psi_i\psi_j] \tag{3.16}$$

$$
\begin{aligned}
\Phi_i\Phi_j\Phi_k(\text{in }y\text{-basis}) =\ & \phi_i\phi_j\phi_k + \sqrt{2}\theta\,[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k] \\
& + \theta\theta\,[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k]
\end{aligned} \tag{3.17}
$$

(Products of chiral superfields are still chiral superfields.)

$$\mathrm{e}^{\mathrm{i}k\Phi} = \mathrm{e}^{\mathrm{i}k\phi(y)}\left[1 + \mathrm{i}k\left(\sqrt{2}\theta\psi(y) + \theta\theta F(y)\right) + \frac{k^2}{2}\theta\theta\psi(y)\psi(y)\right] \tag{3.18}$$

Lagrangian Blocks

$$\mathcal{L}_{\text{kin.}} = \Phi_i^\dagger\Phi_j\Big|_{\theta\theta\bar\theta\bar\theta} \rightsquigarrow F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j \tag{3.19}$$

$$
\begin{aligned}
\mathcal{L}_{\text{super}} = W\Big|_{\theta\theta} + W^*\Big|_{\bar\theta\bar\theta} &= \int \mathrm{d}^2\theta\,[\lambda_i\Phi_i + m_{ij}\Phi_i\Phi_j + y_{ijk}\Phi_i\Phi_j\Phi_k] + \text{H.c.} \\
&= \lambda_iF_i + m_{ij}\,(\phi_iF_j + F_i\phi_j - \psi_i\psi_j) + y_{ijk}\Big[(F_i\phi_j\phi_k - \psi_i\psi_j\phi_k) + (jki \text{ and } kij \text{ terms})\Big]
\end{aligned} \tag{3.20}
$$

3.5 Vector Superfields and Gauge Theory : $V = V^\dagger$

3.5.1 Abelian Case — Field Construction

Explicit Expression

$$
\begin{aligned}
V &= C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar\theta\bar\chi(x) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(x) + \mathrm{i}N(x)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(x) - \mathrm{i}N(x)\right] - \theta\sigma^\mu\bar\theta A_\mu(x) \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(x) + \frac{1}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right] + \bar\theta\bar\theta\theta\left[\lambda(x) - \frac{1}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) - \frac{1}{2}\partial^2 C(x)\right] \quad (3.21) \\
&= C(y) + \mathrm{i}\theta\chi(y) - \mathrm{i}\bar\theta\bar\chi(y) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y) + \mathrm{i}N(y)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y) - \mathrm{i}N(y)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y) - \mathrm{i}\partial_\mu C(y)\right] \\
&\quad + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\left[\lambda(y) - \sigma^\mu\partial_\mu\bar\chi(y)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y) - \partial^2 C(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] \quad (3.22) \\
&= C(y^\dagger) + \mathrm{i}\theta\chi(y^\dagger) - \mathrm{i}\bar\theta\bar\chi(y^\dagger) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y^\dagger) + \mathrm{i}N(y^\dagger)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y^\dagger) - \mathrm{i}N(y^\dagger)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y^\dagger) + \mathrm{i}\partial_\mu C(y^\dagger)\right] \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(y^\dagger) + \bar\sigma^\mu\partial_\mu\chi(y^\dagger)\right] + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) - \partial^2 C(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] \quad (3.23)
\end{aligned}
$$

Supersymmetric Gauge Transformation : $V \to V + \Phi + \Phi^\dagger$

$$
\begin{array}{ll}
C \mapsto C + (\phi + \phi^*) & A_\mu \mapsto A_\mu + \mathrm{i}\partial_\mu(\phi - \phi^*) \\
\chi \mapsto \chi - \mathrm{i}\sqrt{2}\psi & \lambda \mapsto \lambda \\
M + \mathrm{i}N \mapsto M + \mathrm{i}N - 2\mathrm{i}F & D \mapsto D
\end{array} \quad (3.24)
$$

Wess-Zumino Gauge $\quad C = \chi = M = N = 0$

Fixing this gauge breaks SUSY, but still allows the usual gauge transformation

$$
A_\mu \mapsto A_\mu + \partial_\mu\alpha, \quad \lambda \mapsto \lambda, \quad D \mapsto D. \quad (3.25)
$$

$$
\begin{aligned}
V &= -\theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x) && V^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta A_\mu A^\mu \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y) + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] && V^3 = 0 \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y^\dagger) + \theta\theta\bar\theta\bar\lambda(y^\dagger) + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] &&
\end{aligned}
$$

$$
\mathrm{e}^{kV} = 1 - k\theta\sigma^\mu\bar\theta A_\mu(x) + k\theta\theta\bar\theta\bar\lambda + k\bar\theta\bar\theta\theta\lambda + \theta\theta\bar\theta\bar\theta\left[\frac{k}{2}D + \frac{k^2}{4}A_\mu A^\mu\right]
$$

Field Strength

$$
W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V, \quad \bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha}V; \qquad W_\alpha \mapsto W_\alpha \quad \text{(gauge invariant)} \quad (3.26)
$$

$$
\bar D_{\dot\beta} W_\alpha = D_\beta \bar W_{\dot\alpha} = 0; \qquad D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} \quad (3.27)
$$

$$
W_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \mathrm{i}\theta\theta\left[\sigma^\mu\partial_\mu\bar\lambda(y)\right]_\alpha \quad (3.28)
$$

$$
\bar W_{\dot\alpha} = \bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha}D(y^\dagger) + \mathrm{i}(\bar\theta\bar\sigma^{\mu\nu})_{\dot\alpha}F_{\mu\nu}(y^\dagger) - \mathrm{i}\bar\theta\bar\theta\left[\partial_\mu\lambda(y^\dagger)\sigma^\mu\right]_{\dot\alpha} \quad (3.29)
$$

$$
W^\alpha W_\alpha\big|_{\theta\theta} = -\frac{1}{4}\bar D\bar D W^\alpha D_\alpha V \rightsquigarrow -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + 2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 \quad (3.30)
$$

<u>Lagrangian Blocks</u> $\qquad$ ($\mathcal{L}_{\text{inv}}$ is SUSY- and gauge-invariant, while $\mathcal{L}_{\text{mass}}$ is not gauge-invariant.)

$$\mathcal{L}_{\text{inv}} = \frac{\tau}{4} W^\alpha W_\alpha \Big|_{\theta\theta} + \frac{\tau^*}{4} \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \Big|_{\bar\theta\bar\theta} \qquad \left(\text{with} \quad \tau := 1 + \frac{i\theta}{8\pi^2} \right)$$

$$\rightsquigarrow -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\lambda\sigma^\mu \partial_\mu \bar\lambda + \frac{1}{2} D^2 - \frac{\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{3.31}$$

$$\mathcal{L}_{\text{mass}} = m^2 V^2 \Big|_{\theta\theta\bar\theta\bar\theta}$$

$$= m^2 \left(\frac{1}{2} A^\mu A_\mu + i\bar\chi \bar\sigma^\mu \partial_\mu \chi - (\lambda\chi + \bar\lambda\bar\chi) + \frac{1}{2}(M^2 + N^2) + CD + \frac{1}{2}\partial_\mu C \partial^\mu C \right)$$

$$= \frac{1}{2}\partial_\mu C' \partial^\mu C' + i\bar\chi' \bar\sigma^\mu \partial_\mu \chi' + \frac{m^2}{2} A^\mu A_\mu - m(\lambda\chi' + \bar\lambda\bar\chi') + mC'D + \frac{m^2}{2}(M^2 + N^2) \tag{3.32}$$

3.5.2 Abelian Case — Gauge Theory

Here we turn on the coupling constant g. When $\Lambda = i\lambda(y) + \sqrt{2}\theta\xi(y) + \theta\theta K(y)$,

$$\mathcal{L} \ni \Phi^\dagger e^{2gqV} \Phi; \qquad \Phi \mapsto e^{iqg\Lambda}\Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger e^{-iqg\Lambda^\dagger}; \qquad 2V \mapsto 2V - i(\Lambda - \Lambda^\dagger) \tag{3.33}$$

$$\phi \mapsto e^{-qg\lambda}\phi \qquad\qquad\qquad C \mapsto C + \text{Re}\,\lambda \qquad M + iN \mapsto M + iN - K$$

$$\psi \mapsto e^{-qg\lambda}(\psi + iqg\phi \cdot \xi) \qquad\qquad \chi \mapsto \chi - \frac{1}{\sqrt{2}}\xi \qquad A_\mu \mapsto A_\mu - \partial_\mu(\text{Im}\,\lambda)$$

$$F \mapsto e^{-qg\lambda}\left(F + iqg\phi K - iqg\xi\psi + \frac{(qg)^2}{2}\xi\xi\phi \right) \qquad \lambda \mapsto \lambda \qquad\qquad D \mapsto D$$

(Very similar to the gauge transformations in Sec. 1.9.2.)

<u>Lagrangian block</u>

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4} W^\alpha W_\alpha \Big|_{\theta\theta} + \text{H.c.} \qquad \rightsquigarrow -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\lambda\sigma^\mu \partial_\mu \bar\lambda + \frac{1}{2} D^2$$

$$\mathcal{L}_{\text{chiral}} = \Phi^\dagger e^{2gqV} \Phi \Big|_{\theta\theta\bar\theta\bar\theta} \qquad\quad \rightsquigarrow F^* F + D_\mu \phi^* D^\mu \phi + i\bar\psi \bar\sigma^\mu D_\mu \psi + qgD\phi^*\phi - \sqrt{2}gq\left(\phi^*\lambda\psi + \phi\bar\lambda\bar\psi\right)$$

$$\mathcal{L}_{\cancel{CP}} = \frac{i\theta}{32\pi^2} W^\alpha W_\alpha \Big|_{\theta\theta} + \text{H.c.} \quad \rightsquigarrow -\frac{\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$\text{with} \quad D_\mu[\phi,\psi] = (\partial_\mu - igqA_\mu)[\phi,\psi], \qquad D_\mu\lambda = \partial_\mu\lambda,$$

3.5.3 Non-Abelian Case

T^a : generators (Hermitian);

$\text{Tr}\, T^a T^b = K\delta^{ab}\ (K > 0), \qquad [T^a, T^b] = if^{abc}T^c$ (f is anti-symmetric)

<u>Explicit Expression</u> $\quad$ Same as the Abelian case.

<u>Supersymmetric Gauge Transformation</u>

$$\mathcal{L} \ni \Phi^\dagger e^{2g\widetilde{V}} \Phi; \qquad \Phi \mapsto e^{ig\widetilde\Lambda}\Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger e^{-ig\widetilde\Lambda^\dagger}; \qquad e^{2g\widetilde{V}} \mapsto e^{ig\widetilde\Lambda^\dagger} e^{2g\widetilde{V}} e^{-ig\widetilde\Lambda} \tag{3.34}$$

$$\text{with} \quad \widetilde{V} := V^a T^a, \quad \widetilde\Lambda := \Lambda^a T^a, \quad \widetilde\Lambda^\dagger := (\Lambda^a)^\dagger T^a$$

$$2\widetilde{V} \mapsto 2\widetilde{V} - i(\widetilde\Lambda - \widetilde\Lambda^\dagger) - \frac{g}{2}\left([\widetilde\Lambda, \widetilde\Lambda^\dagger] + i[2\widetilde{V}, \widetilde\Lambda + \widetilde\Lambda^\dagger]\right) + \cdots \tag{3.35}$$

$$= \left[2V^a - i(\Lambda^a - \Lambda^{\dagger a}) + \frac{g}{2}\left(-i\Lambda^b \Lambda^{\dagger c} + 2V^b(\Lambda^c + \Lambda^{\dagger c})\right) f^{abc} + \cdots \right] T^a \tag{3.36}$$

We do not present the transformations of the components; note that λ and D are transformed in non-Abelian theories.

Wess-Zumino Gauge

$$V^a = -\theta\sigma^\mu\bar\theta A_\mu^a(x) + \theta\theta\bar\theta\bar\lambda^a(x) + \bar\theta\bar\theta\theta\lambda^a(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D^a(x) \tag{3.37}$$

$$\mathrm{e}^{kV^aT^a} = 1 + kV^aT^a + \frac{k^2}{4}\theta\theta\bar\theta\bar\theta A_\mu^a A^{b\mu}T^aT^b \tag{3.38}$$

Note that the lowest order term of the gauge transformation is independent of V, which guarantees that we can still take the Wess-Zumino gauge. The gauge transformation is restricted as $\mathrm{e}^{\mathrm{i}g\widetilde\Lambda}$, where $\Lambda^a = \xi^a(y) = \xi^a(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\xi^a(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\xi^a(x) : \xi \in \mathbb{R}$.

$$2V^e \mapsto 2\left(V^e - gf^{eab}\xi^aV^b + 6g^2f^{abc}f^{ade}V^b\xi^c\xi^d\right)$$
$$+ \theta\sigma^\mu\bar\theta\left(-2\partial_\mu\xi^e + gf^{eab}\xi^a\partial_\mu\xi^b + 4g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \tag{3.39}$$

$$A_\mu^e \mapsto A_\mu^e + gf^{eab}A_\mu^a\xi^b + 6g^2f^{abc}f^{ade}A_\mu^b\xi^c\xi^d$$
$$+ \left(\partial_\mu\xi^e - \frac{g}{2}f^{eab}\xi^a\partial_\mu\xi^b - 2g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \tag{3.40}$$

$$\lambda^e \mapsto \lambda^e + gf^{eab}\lambda^a\xi^b + 6g^2f^{abc}f^{ade}\lambda^b\xi^c\xi^d + \cdots \tag{3.41}$$

$$D^e \mapsto D^e + gf^{eab}D^a\xi^b + 6g^2f^{abc}f^{ade}D^b\xi^c\xi^d + \cdots \tag{3.42}$$

Note that C, χ, M and N are kept invariant automatically, for now we are under Wess-Zumino gauge.

Field Strength [*2]

$$\widetilde W_\alpha = -\frac{1}{8g}\bar D\bar D\mathrm{e}^{-2g\widetilde V}D_\alpha\mathrm{e}^{2g\widetilde V} \qquad \bar D_{\dot\beta}W_\alpha = 0 \qquad W_\alpha \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda}W_\alpha\mathrm{e}^{-\mathrm{i}g\Lambda^\dagger} \tag{3.43}$$

$$W_\alpha^a = \lambda_\alpha^a(y) + \theta_\alpha D^a(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) + \mathrm{i}\theta\theta(\sigma^\mu\mathrm{D}_\mu\bar\lambda^a(y))_\alpha \tag{3.44}$$

$$\mathrm{Tr}\,\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} = \mathrm{Tr}\left[DD + \mathrm{i}\lambda\sigma^\mu\mathrm{D}_\mu\bar\lambda + \mathrm{i}\mathrm{D}_\mu\bar\lambda\bar\sigma^\mu\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \tag{3.45}$$

$$= K\left[D^aD^a + \mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar\lambda^a + \mathrm{i}\mathrm{D}_\mu\bar\lambda^a\bar\sigma^\mu\lambda^a - \frac{1}{2}F^{\mu\nu\,a}F_{\mu\nu}^a + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^aF_{\rho\sigma}^a\right] \tag{3.46}$$

Lagrangian Block

$$\mathcal{L}_{\mathrm{gauge}} = \frac{1}{4K}\mathrm{Tr}\,\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} + \mathrm{H.c.} \qquad \rightsquigarrow -\frac{1}{4}F^{\mu\nu\,a}F_{\mu\nu}^a + \mathrm{i}\bar\lambda^a\bar\sigma^\mu\mathrm{D}_\mu\lambda^a + \frac{1}{2}D^aD^a \tag{3.47}$$

$$\mathcal{L}_{\cancel{CP}} = \frac{\mathrm{i}}{32K\pi^2}\mathrm{Tr}\,\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} + \mathrm{H.c.} \qquad \rightsquigarrow -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^aF_{\rho\sigma}^a \tag{3.48}$$

$$\mathcal{L}_{\mathrm{matter}} = \Phi_i^\dagger[\mathrm{e}^{2gV^aT^a}]_{ij}\Phi_j\Big|_{\theta\theta\bar\theta\bar\theta} \qquad \rightsquigarrow \mathrm{D}^\mu\phi_i^*\mathrm{D}_\mu\phi_i + \mathrm{i}\bar\psi_i\bar\sigma^\mu\mathrm{D}_\mu\psi_i + F_i^*F_i$$
$$+ gD^a(\phi^*T^a\phi) - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar\psi\bar\lambda T^a\phi\right) \tag{3.49}$$

$$\begin{aligned}
\mathrm{D}_\mu\phi_i &= \partial_\mu\phi - \mathrm{i}gA_\mu^a(T^a\phi)_i & F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu],\\
\mathrm{D}_\mu\phi_i^* &= \partial_\mu\phi + \mathrm{i}gA_\mu^a(\phi^*T^a)_i & \mathrm{D}_\mu\lambda^a &= \partial_\mu\lambda^a + gf^{abc}A_\mu^b\lambda^c,\\
\mathrm{D}_\mu\psi_i &= \partial_\mu\psi - \mathrm{i}gA_\mu^a(T^a\psi)_i & \mathrm{D}_\mu\bar\lambda^a &= \partial_\mu\bar\lambda^a + gf^{abc}A_\mu^b\bar\lambda^c,
\end{aligned}$$

[*2] Note the signs. $\bar D\bar D\mathrm{e}^{2gV}D_\alpha\mathrm{e}^{-2gV}$ is not gauge invariant! Also the curvature tensor and the covariant derivative is well-known ones: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$, and $\mathrm{D}_\mu\bar\lambda = \partial_\mu\bar\lambda - \mathrm{i}g[A_\mu, \bar\lambda] = \partial_\mu\bar\lambda + gf^{abc}A_\mu^b\bar\lambda^cT^a$.

3.6 Minimal Supersymmetric Standard Model

3.6.1 Definitions

Gauge Group

$\mathrm{SU(3)_{color}} \times \mathrm{SU(2)_{weak}} \times \mathrm{U(1)_Y} \quad (\times \mathrm{Z}_{2R} : R\text{-parity})$

Fields

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	**3**	**2**	1/6	1/3	
L_i		**2**	$-1/2$		1
$\bar{U}_i$	$\bar{\mathbf{3}}$		$-2/3$	$-1/3$	
$\bar{D}_i$	$\bar{\mathbf{3}}$		1/3	$-1/3$	
$\bar{E}_i$			1		-1
H_u		**2**	1/2		
H_d		**2**	$-1/2$		

Field	SU(3)	SU(2)	U(1)
g	**8**		
W		**3**	
B			

Superpotential

$$W_{\mathrm{RPC}} = \mu H_\mathrm{u} H_\mathrm{d} - y_{\mathrm{u}ij} H_\mathrm{u} Q_i \bar{U}_j + y_{\mathrm{d}ij} H_\mathrm{d} Q_i \bar{D}_j + y_{\mathrm{e}ij} H_\mathrm{d} L_i \bar{E}_j \tag{3.50}$$

$$W_{\mathrm{RPV}} = \mu_i H_\mathrm{u} L_i + \frac{1}{2}\lambda_{ijk} L_i L_j \bar{E}_k + \lambda'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2}\lambda''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \tag{3.51}$$

$$\text{(Here we define } \lambda_{ijk} = -\lambda_{jik} \text{ and } \lambda''_{ijk} = -\lambda''_{ikj}.\text{)}$$

~~SUSY~~-terms

$$V_{\mathrm{\sout{SUSY}}}^{\mathrm{RPC}} = \frac{1}{2}\left(M_3 \widetilde{g}\widetilde{g} + M_2 \widetilde{W}\widetilde{W} + M_1 \widetilde{B}\widetilde{B} + \mathrm{H.c.}\right) \tag{3.52}$$
$$+ \left(-H_\mathrm{u} Q A^u \bar{U} + H_\mathrm{d} Q A^d \bar{D} + H_\mathrm{d} L A^e \bar{E} + B H_\mathrm{u} H_\mathrm{d} + \mathrm{H.c.}\right)$$
$$+ \left(Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E} + m_{H_\mathrm{u}}^2 |H_\mathrm{u}|^2 + m_{H_\mathrm{d}}^2 |H_\mathrm{d}|^2\right)$$
$$+ \left(-H_\mathrm{d}^* Q C^u \bar{U} + H_\mathrm{u}^* Q C^d \bar{D} + H_\mathrm{u}^* L C^e \bar{E} + \mathrm{H.c.}\right)$$

$$V_{\mathrm{\sout{SUSY}}}^{\mathrm{RPV}} = \left(B_i H_\mathrm{u} L_i + \frac{1}{2}A_{ijk} L_i L_j \bar{E}_k + A'_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2}A''_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k + \mathrm{H.c.}\right) \tag{3.53}$$
$$+ \left(C^1_{ijk} L_i^* Q_j \bar{U}_k + C^2_i H_\mathrm{u}^* H_\mathrm{d} \bar{E}_i + C^3_{ijk} \bar{D}_i^* \bar{U}_j \bar{E}_k + \frac{1}{2}C^4_{ijk} \bar{D}_i^* Q_j Q_k + \mathrm{H.c.}\right)$$
$$+ \left(M_{Li}^2 H_\mathrm{d}^* L_i + \mathrm{H.c.}\right) \qquad\qquad \text{with } C^4_{ijk} = C^4_{ikj}.$$

3.6.2 Scalar Potential

F-terms

$$-F_{H_\mathrm{u}}{}^{a*} = \epsilon^{ab}\left(\mu H_\mathrm{d}{}^b - y_{\mathrm{u}ij}Q_i^{bx}\bar{U}_j^x + \mu_i L_i^b\right) \tag{3.54}$$

$$-F_{H_\mathrm{d}}{}^{a*} = \epsilon^{ab}\left(-\mu H_\mathrm{u}{}^b + y_{\mathrm{d}ij}Q_i^{bx}\bar{D}_j^x + y_{\mathrm{e}ij}L_i^b\bar{E}_j\right) \tag{3.55}$$

$$-F_{Q_i}^{ax*} = \epsilon^{ab}\left(y_{\mathrm{u}ij}H_\mathrm{u}{}^b\bar{U}_j^x - y_{\mathrm{d}ij}H_\mathrm{d}{}^b\bar{D}_j^x - \lambda'_{jik}L_i^b\bar{D}_j^x\right) \tag{3.56}$$

$$-F_{L_i}^{a*} = \epsilon^{ab}\left(-\mu_i H_\mathrm{u}{}^b - y_{\mathrm{e}ij}H_\mathrm{d}{}^b\bar{E}_j + \lambda_{ijk}L_j^b\bar{E}_k + \lambda'_{ijk}Q_j^{bx}\bar{D}_k^x\right) \tag{3.57}$$

$$-F_{\bar{U}_i}^{x*} = \left(-\epsilon^{ab}y_{\mathrm{u}ji}H_\mathrm{u}{}^a Q_j^{bx} + \frac{1}{2}\epsilon^{xyz}\lambda''_{ijk}\bar{D}_j^y\bar{D}_k^z\right) \tag{3.58}$$

$$-F_{\bar{D}_i}^{x*} = \left(\epsilon^{ab}y_{\mathrm{d}ji}H_\mathrm{d}{}^a Q_j^{bx} + \epsilon^{ab}\lambda'_{jki}L_j^a Q_k^{bx} + \epsilon^{yzx}\lambda''_{jki}\bar{U}_j^y\bar{D}_k^z\right) \tag{3.59}$$

$$-F_{\bar{E}i}{}^{*} = \left(\epsilon^{ab}y_{\mathrm{e}ji}H_\mathrm{d}{}^a L_j^b + \frac{1}{2}\epsilon^{ab}\lambda_{jki}L_j^a L_k^b\right) \tag{3.60}$$

D-terms

$$D_g{}^\alpha = -g_3\sum_{i=1}^3\left[\sum_{a=1,2}Q_i^{ax*}(T^\alpha)_{xy}Q_i^{ay} - \bar{U}_i^{x*}(T^\alpha)_{xy}\bar{U}_i^y - \bar{D}_i^{x*}(T^\alpha)_{xy}\bar{D}_i^y\right] \tag{3.61}$$

$$D_W{}^\alpha = -g_2\left[\sum_{i=1}^3\sum_{x=1}^3 Q_i^{ax*}(T^\alpha)_{ab}Q_i^{by} + \sum_{i=1}^3 L_i^{a*}(T^\alpha)_{ab}L_i^b + H_\mathrm{u}{}^{a*}(T^\alpha)_{ab}H_\mathrm{u}{}^b + H_\mathrm{d}{}^{a*}(T^\alpha)_{ab}H_\mathrm{d}{}^b\right] \tag{3.62}$$

$$D_B = -g_1\left[\frac{1}{6}|Q_i^{ax}|^2 - \frac{1}{2}|L_i^a|^2 - \frac{2}{3}|\bar{U}_i^x|^2 + \frac{1}{3}|\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2}|H_\mathrm{u}{}^a|^2 - \frac{1}{2}|H_\mathrm{d}{}^a|^2\right] \tag{3.63}$$

Full Scalar Potential

$$V = \sum|F_\bullet|^2 + \frac{1}{2}\sum|D_\bullet|^2 \tag{3.64}$$

3.6.3 Lagrangian (Verbose)

$$\mathcal{L}_{\mathrm{K;CP}} = -\frac{1}{4}F^{\mu\nu\,a}F_{\mu\nu}^a + \mathrm{D}^\mu\phi_i^*\mathrm{D}_\mu\phi_i + \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\mathrm{D}_\mu\psi_i + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^a - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar{\psi}\bar{\lambda}T^a\phi\right) \tag{3.65}$$

$$\mathcal{L}_{\mathrm{K;}\cancel{\mathrm{CP}}} = -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a \tag{3.66}$$

$$\mathcal{L}_{\mathrm{scalar}} = -\left(\sum V^F + \sum V^D + \sum V^{\cancel{SUSY}}\right) \tag{3.67}$$

$$\mathcal{L}_{\mathrm{S;fermi}}^{\mathrm{RPC}} = -\left(\mu\widetilde{H}_\mathrm{u}\widetilde{H}_\mathrm{d} - y_{\mathrm{u}ij}H_\mathrm{u}Q_i\bar{U}_j + y_{\mathrm{d}ij}H_\mathrm{d}Q_i\bar{D}_j + y_{\mathrm{e}ij}H_\mathrm{d}L_i\bar{E}_j + \cdots + \mathrm{H.c.}\right) \tag{3.68}$$

$$\mathcal{L}_{\mathrm{S;fermi}}^{\mathrm{RPV}} = -\left(\mu_i\widetilde{H}_\mathrm{u}L_i + \frac{1}{2}\lambda_{ijk}L_iL_j\bar{E}_k + \lambda'_{ijk}L_iQ_j\bar{D}_k + \frac{1}{2}\lambda''_{ijk}\bar{U}_i\bar{D}_j\bar{D}_k + \cdots + \mathrm{H.c.}\right) \tag{3.69}$$

Scalar Potential

$$V_{H_{\mathrm{u}}}^F = |\mu|^2 |H_{\mathrm{d}}|^2 + \sum_a \left(|Q^a y^u \bar{U}|^2 + |\mu_i L_i^a|^2 \right) \tag{3.70}$$
$$+ \left[\mu^* \mu_i H_{\mathrm{d}}^{\ *} L_i - \mu^* H_{\mathrm{d}}^{\ *} Q y^u \bar{U} - \mu_i^* L_i^* Q y^u \bar{U} + \mathrm{H.c.} \right]$$

$$V_{H_{\mathrm{d}}}^F = |\mu|^2 |H_{\mathrm{u}}|^2 + \sum_a \left(|Q^a y^d \bar{D}|^2 + |L^a y^e \bar{E}|^2 \right) \tag{3.71}$$
$$+ \left[-\mu^* H_{\mathrm{u}}^{\ *} Q y^d \bar{D} - \mu^* H_{\mathrm{u}}^{\ *} L y^e \bar{E} + (Q y^d \bar{D})^* (L y^e \bar{E}) + \mathrm{H.c.} \right]$$

$$V_Q^F = |H_{\mathrm{u}}|^2 |y_{ij}^u \bar{U}_j|^2 + |H_{\mathrm{d}}|^2 |y_{ij}^d \bar{D}_j|^2 + \lambda_{jik}'^* \lambda_{lim}' L_j^* L_l \bar{D}_k^* \bar{D}_m \tag{3.72}$$
$$+ \left[-y_{ij}^{u*} y_{ik}^d H_{\mathrm{u}}^{\ *} H_{\mathrm{d}} \bar{U}_j^* \bar{D}_k - y_{ij}^{u*} \lambda_{lim}' H_{\mathrm{u}}^{\ *} L_l \bar{U}_j^* \bar{D}_m + y_{ij}^{d*} \lambda_{lim}' H_{\mathrm{d}}^{\ *} L_l \bar{D}_j^* \bar{D}_m + \mathrm{H.c.} \right]$$

$$V_L^F = |\mu_i|^2 |H_{\mathrm{u}}|^2 + |H_{\mathrm{d}}|^2 (\bar{E}^* y^{e\dagger} y^e \bar{E}) + \lambda_{ijk}' \lambda_{ilm}' (Q_j^* \bar{D}_k^*) Q_l \bar{D}_m + \lambda_{ijk}^* \lambda_{ilm} L_j^* L_l \bar{E}_k^* \bar{E}_m \tag{3.73}$$
$$+ \left[\mu_i^* y_{ij}^e \bar{E}_j H_{\mathrm{u}}^{\ *} H_{\mathrm{d}} - \mu_i^* \lambda_{ijk}' H_{\mathrm{u}}^{\ *} Q_j \bar{D}_k - \mu_i^* \lambda_{ijk} H_{\mathrm{u}}^{\ *} L_j \bar{E}_k \right.$$
$$\left. - y_{ij}^{e*} \lambda_{ilm}' \bar{E}_j^* H_{\mathrm{d}}^{\ *} Q_l \bar{D}_m - y_{ij}^{e*} \lambda_{ilm} \bar{E}_j^* H_{\mathrm{d}}^{\ *} L_l \bar{E}_m + \lambda_{ijk}'^* \lambda_{ilm} \bar{D}_k^* Q_j^* L_l \bar{E}_m + \mathrm{H.c.} \right]$$

$$V_{\bar{U}}^F = y_{ji}^{u*} y_{ki}^u \epsilon^{ab} \epsilon^{cd} H_{\mathrm{u}}^{a*} H_{\mathrm{u}}^c Q_j^{b*} Q_k^d + \frac{1}{2} \lambda_{ijk}''^* \lambda_{ilm}'' (\bar{D}_j^* \bar{D}_l)(\bar{D}_k^* \bar{D}_m) - \left[y_{li}^{u*} \lambda_{ijk}'' H_{\mathrm{u}}^{\ *} Q_l^* \bar{D}_j \bar{D}_k + \mathrm{H.c.} \right] \tag{3.74}$$

$$V_{\bar{D}}^F = \epsilon^{ab} \epsilon^{cd} \left(y_{ji}^d H_{\mathrm{d}}^{\ a} + \lambda_{kji}' L_k^a \right)^* \left(y_{li}^d H_{\mathrm{d}}^{\ c} + \lambda_{mli}' L_m^c \right) Q_j^{b*} Q_l^d + \lambda_{jki}''^* \lambda_{lmi}'' \left(\bar{U}_j^* \bar{U}_l \ \bar{D}_k^* \bar{D}_m - \bar{U}_j^* \bar{D}_m \ \bar{D}_k^* \bar{U}_l \right)$$
$$+ \left[\lambda_{lmi}'' (y_{ji}^{d*} H_{\mathrm{d}}^{\ *} + \lambda_{kji}'^* L_k^*) Q_j^* \bar{U}_l \bar{D}_m + \mathrm{H.c.} \right] \tag{3.75}$$

$$V_{\bar{E}}^F = \epsilon^{ab} \epsilon^{cd} \left(y_{ji}^e H_{\mathrm{d}}^{\ a} + \frac{1}{2} \lambda_{kji} L_k^a \right)^* L_j^{*b} \left(y_{li}^e H_{\mathrm{d}}^{\ c} + \frac{1}{2} \lambda_{mli} L_m^c \right) L_l^d \tag{3.76}$$

$$V_g^D = \frac{g_3^2}{2} \left\{ \sum_{\alpha=1}^8 \sum_{i=1}^3 \left[\sum_{a=1,2} Q_i^{a*} (t^\alpha) Q_i^a - \bar{U}_i^* (t^\alpha) \bar{U}_i - \bar{D}_i^* (t^\alpha) \bar{D}_i \right] \right\}^2 \tag{3.77}$$

$$V_W^D = \frac{g_2^2}{2} \left[\sum_{i=1}^3 \sum_{x=1}^3 Q_i^{x*} (T^\alpha) Q_i^x + \sum_{i=1}^3 L_i^* (T^\alpha) L_i + H_{\mathrm{u}}^{\ *} (T^\alpha) H_{\mathrm{u}} + H_{\mathrm{d}}^{\ *} (T^\alpha) H_{\mathrm{d}} \right]^2 \tag{3.78}$$

$$V_B^D = \frac{g_1^2}{2} \left[\sum_i \left(\frac{1}{6} |Q_i|^2 - \frac{1}{2} |L_i|^2 - \frac{2}{3} |\bar{U}_i|^2 + \frac{1}{3} |\bar{D}_i|^2 + |\bar{E}_i|^2 \right) + \frac{1}{2} |H_{\mathrm{u}}|^2 - \frac{1}{2} |H_{\mathrm{d}}|^2 \right]^2 \tag{3.79}$$

$$V_{\mathrm{SUSY}}^{\mathrm{RPC}} = \frac{1}{2} \left(M_3 \widetilde{g} \widetilde{g} + M_2 \widetilde{W} \widetilde{W} + M_1 \widetilde{B} \widetilde{B} + \mathrm{H.c.} \right) \tag{3.80}$$
$$+ \left(-H_{\mathrm{u}} Q A^u \bar{U} + H_{\mathrm{d}} Q A^d \bar{D} + H_{\mathrm{d}} L A^e \bar{E} + B H_{\mathrm{u}} H_{\mathrm{d}} + \mathrm{H.c.} \right)$$
$$+ \left(Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E} + m_{H_{\mathrm{u}}}^2 |H_{\mathrm{u}}|^2 + m_{H_{\mathrm{d}}}^2 |H_{\mathrm{d}}|^2 \right)$$
$$+ \left(-H_{\mathrm{d}}^{\ *} Q C^u \bar{U} + H_{\mathrm{u}}^{\ *} Q C^d \bar{D} + H_{\mathrm{u}}^{\ *} L C^e \bar{E} + \mathrm{H.c.} \right)$$

$$V_{\mathrm{SUSY}}^{\mathrm{RPV}} = \left(B_i H_{\mathrm{u}} L_i + \frac{1}{2} A_{ijk} L_i L_j \bar{E}_k + A_{ijk}' L_i Q_j \bar{D}_k + \frac{1}{2} A_{ijk}'' \bar{U}_i \bar{D}_j \bar{D}_k + \mathrm{H.c.} \right) \tag{3.81}$$
$$+ \left(C_{ijk}^1 L_i^* Q_j \bar{U}_k + C_i^2 H_{\mathrm{u}}^{\ *} H_{\mathrm{d}} \bar{E}_i + C_{ijk}^3 \bar{D}_i^* \bar{U}_j \bar{E}_k + \frac{1}{2} C_{ijk}^4 \bar{D}_i^* Q_j Q_k + \mathrm{H.c.} \right)$$
$$+ \left(M_{Li}^2 H_{\mathrm{d}}^{\ *} L_i + \mathrm{H.c.} \right)$$

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Full Potential (Verbose)

<u>With R-parity</u>

$$
\begin{aligned}
V_{\text{full}}^{\text{RPC}} = \ &\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B} + \text{H.c.}\right) \\
&+ \left(Q^*m_Q^2 Q + L^*m_L^2 L + \bar{U}^*m_{\bar{U}}^2\bar{U} + \bar{D}^*m_{\bar{D}}^2\bar{D} + \bar{E}^*m_{\bar{E}}^2\bar{E}\right) \\
&+ \left(|\mu|^2 + m_{H_{\text{u}}}^2\right)|H_{\text{u}}|^2 + \left(|\mu|^2 + m_{H_{\text{d}}}^2\right)|H_{\text{d}}|^2 + \left(BH_{\text{u}}H_{\text{d}} + \text{H.c.}\right) \\
&+ \left[\left(-\mu^*y^u H_{\text{d}}^* - A^u H_{\text{u}} - C^u H_{\text{d}}^*\right)_{ij} Q_i\bar{U}_j + \text{H.c.}\right] \\
&+ \left[\left(-\mu^*y^d H_{\text{u}}^* + A^d H_{\text{d}} + C^d H_{\text{u}}^*\right)_{ij} Q_i\bar{D}_j + \text{H.c.}\right] \\
&+ \left[\left(-\mu^*y^e H_{\text{u}}^* + A^e H_{\text{d}} + C^e H_{\text{u}}^*\right)_{ij} L_i\bar{E}_j + \text{H.c.}\right] \\
&+ |H_{\text{u}}|^2|y_{ij}^u\bar{U}_j|^2 + |H_{\text{d}}|^2|y_{ij}^d\bar{D}_j|^2 + |H_{\text{d}}|^2|y_{ij}^e\bar{E}_j|^2 + |H_{\text{u}}|^2|Qy^u|^2 + |H_{\text{d}}|^2|Qy^d|^2 + |H_{\text{d}}|^2|Ly^e|^2 \\
&+ \sum_a\left(|Q^a y^u\bar{U}|^2 + |Q^a y^d\bar{D}|^2 + |L^a y^e\bar{E}|^2\right) \\
&+ \left[(Qy^d\bar{D})^*(Ly^e\bar{E}) - y_{ij}^{u*}y_{ik}^d H_{\text{u}}^* H_{\text{d}}\bar{U}_j^*\bar{D}_k + \text{H.c.}\right] \\
&- \left[y_{ik}^{u*}y_{jk}^u(H_{\text{u}}^*Q_j)(Q_i^*H_{\text{u}}) + y_{ik}^{d*}y_{jk}^d(H_{\text{d}}^*Q_j)(Q_i^*H_{\text{d}}) + y_{ik}^{e*}y_{jk}^e(H_{\text{d}}^*L_j)(L_i^*H_{\text{d}})\right] \\
&+ \frac{g_3^2}{2}\left\{\sum_{\alpha=1}^8\sum_{i=1}^3\left[\sum_{a=1,2}Q_i^{a*}(t^\alpha)Q_i^a - \bar{U}_i^*(t^\alpha)\bar{U}_i - \bar{D}_i^*(t^\alpha)\bar{D}_i\right]\right\}^2 \\
&+ \frac{g_2^2}{2}\left[\sum_{i=1}^3\sum_{x=1}^3 Q_i^{x*}(T^\alpha)Q_i^x + \sum_{i=1}^3 L_i^*(T^\alpha)L_i + H_{\text{u}}^*(T^\alpha)H_{\text{u}} + H_{\text{d}}^*(T^\alpha)H_{\text{d}}\right]^2 \\
&+ \frac{g_1^2}{2}\left[\sum_i\left(\frac{1}{6}|Q_i|^2 - \frac{1}{2}|L_i|^2 - \frac{2}{3}|\bar{U}_i|^2 + \frac{1}{3}|\bar{D}_i|^2 + |\bar{E}_i|^2\right) + \frac{1}{2}|H_{\text{u}}|^2 - \frac{1}{2}|H_{\text{d}}|^2\right]^2
\end{aligned}
\tag{3.82}
$$

<u>With Bilinear R-parity Violation</u>

$$
V_{H_{\text{u}}}^F += \sum_a|\mu_i L_i^a|^2 + \left[\mu^*\mu_i H_{\text{d}}^*L_i - \mu_i^*L_i^*Qy^u\bar{U} + \text{H.c.}\right]
\tag{3.83}
$$

$$
V_L^F += |\mu_i|^2|H_{\text{u}}|^2 + \left[\mu_i^*y_{ij}^e\bar{E}_j H_{\text{u}}^*H_{\text{d}} + \text{H.c.}\right]
\tag{3.84}
$$

$$
V_{\text{SUSY}}^{\text{RPV}} = \left(B_i H_{\text{u}}L_i + M_{Li}^2 H_{\text{d}}^*L_i + \text{H.c.}\right)
\tag{3.85}
$$

<u>With Trilinear leptonic R-parity Violation</u>

$$
V_Q^F += |\lambda'_{jik}L_j\bar{D}_k|^2 + \left[-y_{ij}^{u*}\lambda'_{lim}H_{\text{u}}^*L_l\bar{U}_j^*\bar{D}_m + y_{ij}^{d*}\lambda'_{lim}H_{\text{d}}^*L_l\bar{D}_j^*\bar{D}_m + \text{H.c.}\right]
\tag{3.86}
$$

$$
\begin{aligned}
V_L^F += \ &|\lambda'_{ijk}Q_j\bar{D}_k|^2 + |\lambda_{ijk}L_j\bar{E}_k|^2 \\
&+ \left[-y_{im}^{e*}\lambda'_{ijk}\bar{E}_m^*H_{\text{d}}^*Q_j\bar{D}_k - y_{im}^{e*}\lambda_{ijk}\bar{E}_m^*H_{\text{d}}^*L_j\bar{E}_k + \lambda'^*_{ijk}\lambda_{ilm}\bar{D}_k^*Q_j^*L_l\bar{E}_m + \text{H.c.}\right]
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
V_{\bar{D}}^F += \ &\lambda'^*_{kji}\lambda'_{mli}\left[(L_k^*L_m)(Q_j^*Q_l) - (L_k^*Q_l)(Q_j^*L_m)\right] \\
&+ \left\{y_{ji}^{d*}\lambda'_{mli}\left[(H_{\text{d}}^*L_m)(Q_jQ_l) - (H_{\text{d}}^*Q_l)(Q_j^*L_m)\right] + \text{H.c.}\right\}
\end{aligned}
\tag{3.88}
$$

$$
V_{\bar{E}}^F = \frac{1}{2}\lambda^*_{kji}\lambda_{mli}(L_k^*L_m)(L_j^*L_l) + \lambda^*_{kji}y_{li}^e\left[(L_k^*H_{\text{d}})(L_j^*L_l) + \text{H.c.}\right]
\tag{3.89}
$$

4 Supergravity

4.1 MINIMAL SUGRA LAGRANGIAN

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2}eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu \mathrm{D}_\rho\psi_\sigma \tag{4.1}$$

where

$$\mathrm{D}_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \qquad \left[\omega_\mu{}^{ab} : \text{``spin} \qquad \text{''}\right] \tag{4.2}$$

$$e := \det e_a{}^\mu \tag{4.3}$$

$$M := 1/\sqrt{8\pi G} \quad \text{(Reduced Planck mass)} \tag{4.4}$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \tag{4.5}$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \tag{4.6}$$

4.2 GENERAL SUGRA LAGRANGIAN

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \qquad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \qquad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \tag{4.7}$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
  - Real function of chiral multiplets.
  - In global SUSY, $\int \mathrm{d}^4\theta K$ yields kinetic terms of the chiral multiplet.
  - "Minimal Kähler" is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

  $$\int \mathrm{d}^4\theta \ \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \tag{4.8}$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
  - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
  - $(a), (b), \ldots$ are indices for adjoint representation of gauge group.
  - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$
\begin{aligned}
\mathcal{L} = {} & -\frac{1}{2}eR + eg_{ij^*}\mathrm{D}_\mu\phi^i\mathrm{D}^\mu\phi^{*j} - \frac{1}{2}eg^2 D_{(a)}D^{(a)} \\
& + \mathrm{i}eg_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu\mathrm{D}_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\mathrm{D}_\rho\psi_\sigma \\
& - \frac{1}{4}ef^{\mathrm{R}}{}_{(ab)}F^{(a)}_{\mu\nu}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f^{\mathrm{I}}{}_{(ab)}f^{(a)}_{\mu\nu}f^{(b)}_{\rho\sigma} \\
& + \frac{\mathrm{i}}{2}e\left[\lambda_{(a)}\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^{(a)}\right] - \frac{1}{2}f^{\mathrm{I}}{}_{(ab)}\mathrm{D}_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{1}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_{i^*}f^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F^{(b)}_{\mu\nu} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F^{(b)}_{\mu\nu} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{\mathrm{i}}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F^{(a)}_{\nu\rho} + \hat{F}^{(a)}_{\nu\rho}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f^{\mathrm{R}}{}_{(ab)} + f^{\mathrm{R}(cd)^{-1}}\partial_i f_{(bc)}\partial_{j^*}f^*_{(ad)}\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f^*_{(ab)}\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}(cd)^{-1}}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}(cd)^{-1}}\partial_{i^*}f^*_{(ac)}\partial_{j^*}f^*_{(bd)}\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f^*_{(cd)}\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{\mathrm{i}}{2}\sqrt{2}ee^{K/2}\left[D_i W\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}ee^{K/2}\left[\mathrm{D}_i D_j W\chi^i\chi^j + \mathrm{D}_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}ee^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_i W\partial_{j^*}f^*_{(ab)}\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - ee^K\left[g^{ij^*}(D_i W)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned}
\tag{4.9}
$$

A   Mathematics

A.1   ALGEBRA

A.1.1   Algebraic Structure

Semigroup   :   For $a, b \in A_{\text{set}}$, $ab \in A$; Associative.

Monoid   :   For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit.

Group   :   For $a, b \in A_{\text{set}}$, $ab \in A$; Associative, Unit, Inverse.

Module (/)   :   For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit, Inverse.

Semimodule   :   For $a, b \in A_{\text{set}}$, $a + b \in A$; Commutative, Associative, Unit.

Ring ( )   :   +: Module, ×: Semigroup(Monoid), Distributive.

Semiring   :   +: Semimodule, ×: Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.

Field   :   +: Module, ×: Commutative Monoid, a^{-1} but 0, $1 \neq 0$, Distributive.

Vector Space

Vector space on K :   For $v \in (V, +)_{\text{module}}$ and $k \in K_{\text{field}}$,

(K-module) : $kv \in (V, +)$; Compatible, Distributive, $1v = v$.

Norm   : $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$, $\|kx\| = k\,\|x\|$, $\|x + y\| \leq \|x\| + \|y\|$

Inner product   : $\langle x|x \rangle \geq 0$, $\langle x|x \rangle = 0 \Leftrightarrow x = 0$, $\langle x|y \rangle = \langle y|x \rangle$,

                : $\langle x + y|z \rangle = \langle x|z \rangle + \langle y|z \rangle$, $\langle kx|y \rangle = k\,\langle x|y \rangle$

K-algebra

K-algebra $C(V)$ vector V distributive

   $xy \in C(V)$; $(xy)z = x(yz)$, $(x + y)z = xz + yz$, $x(y + z) = xy + xz$, $k(xy) = (kx)y = x(ky)$.

A.1.2   Lie Algebra

Lie Algebra :   For a Finite-dimensional K-module $(A, +)$ and $x, y, z \in (A, +)$, $a, b \in K$,

            : $[u, v] \in (A, +)$ (Lie product), and

            :   Bilinear: $[ax + by, z] = a[x, z] + b[y, z]$, $[x, ay + bz] = a[x, y] + b[x, z]$,

            :   Alternating: $[x, x] = 0$ $(\Longrightarrow [x, y] = -[y, x])$,

            :   Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

   $[A, B] := AB - BA$ Lie algebra

A.1.3   Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.

For $C(V)$, **Clifford algebra** $(C(V), \theta)$ is defined as

- $C(V)$: K-algebra with 1 $(1x = x1 = x)$,
- $\theta : V \to C(V)$, homomorphism, $\theta(x)^2 = \langle x|x \rangle\, 1$,
- Any C': K-algebra with 1 and any homomorphism $\phi : V \to C'$ with $\phi(x)^2 = \langle x|x \rangle\, 1$, there's unique $\bar{\phi} : C \to C'$, homomorphism, $\bar{\phi}(1) = 1$.

The Gamma matrices form Clifford Algebra:

$$V : \mathbb{R}^4 \text{ with } \{G_0, G_1, G_2, G_3\}, \quad \langle G_0 | G_0 \rangle = 1, \langle G_i | G_i \rangle = -1; \qquad C(V) : M(n, \mathbb{C}).$$

A.1.4 Multilinear Algebra

K vector V V^*

Tensor Algebra $T(V)$: $f : V \to T(V)$ K-algebra K-algebra

A $g : V \to A$

$h : T(V) \to A$ s.t. $h \circ f = g$

Symmetric Algebra $S(V)$: $C(V)$ A K-algebra

Exterior Algebra $\wedge(V)$: $g(\cdots vv \cdots) = 0$

Tensor and Tensor Space

p Tensor   : $T^p(V) := V^{\otimes p} = V \otimes V \otimes \cdots \otimes V$

q Tensor   : $T_q(V) := (V^*)^{\otimes q} = V^* \otimes V^* \otimes \cdots \otimes V^*$

       Tensor   : $T_q^p(V) := T^p(V) \otimes T_q(V)$ V V^*

Tensor Space   : $T(V) := \oplus T_q^p(V)$

Tensor space $(+, \otimes)$

Contraction   : V^* $V \to K$ $T_q^p(V) \to T_{q-1}^{p-1}(V)$

: $T_2(V) \ni g_{ij} : V \times V \to \mathbb{R}$

: $T_2(V) \cong \mathrm{Hom}(V, V^*)$

: $g^{ij} : \mathrm{Hom}(V^*, V)$

Grassmann Operator

V Hilbert $V \ni v$

$$\|\psi v\|^2 = (\psi_{ab} v_b)^* (\psi_{ac} v_c) = v_b^* v_c (\psi_{ab})^* \psi_{ac} \geq 0 \qquad \therefore (\psi_{ab})^* \psi_{ac} = -(\psi_{ab}(\psi_{ac})^*)^* \qquad (A.1)$$

$$(ab)^\dagger = a^\dagger b^\dagger, \ (ab)^\mathrm{T} = -a^\mathrm{T} b^\mathrm{T}, \ (ab)^* = b^* a^*$$

TODO: ψ ......

A.1.5 Lie Group and Lie Algebra

- G **Lie group**   ... G C^∞ C^∞
- Lie G **COMPLEX Lie group**   ...

- Lie G G **Lie algebra** $\mathfrak{g}$
  - $\mathfrak{g}$ G vector
  - $\mathfrak{g}$ vector   Lie algebra
- G Lie
  - Lie B_i **structure constant** c $[B_i, B_j] = c_{ij}^k B_k$

* * *

- Compact Lie   Lie
- G **Linear group** $\mathrm{GL}(n; \mathbb{R})$
  - Lie n

○ Vector   **commutation relation** $[X, Y] = XY - YX$

• Lie $GL(n; \mathbb{C})$ Lie

     Lie $GL(n; \mathbb{R})$ Lie

A.1.6   Matrix Representation

• Lie G Lie G **generators**
• $GL(n; \mathbb{R})$ n
• Lie G $\{T_i\}$ 2 G

$$(x_1, \cdots, x_m) \mapsto e^{x_1 T_1 + \cdots + x_m T_m} \qquad\qquad (x_1, \cdots, x_m) \mapsto e^{x_1 T_1} \cdots e^{x_m T_m} \qquad (A.2)$$

• Lie G compact   ...
  1. G compact   **TODO:**
  2. G $\{T_i\}$ $\text{Tr}(T_i T_j) = k\,\delta_{ij}$ $k > 0$

• Compact G **unitary representation**
           Hermitian matrix T^i parameters $x^i \in \mathbb{R}$ G

$$e^{\mathrm{i} x^i T^i} \qquad (A.3)$$

A.1.7

Compact Lie V

Hermitian Representation

$$V = \exp(\mathrm{i} x^i T^i) \qquad \text{where} \quad T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R},$$
$$[T^i, T^j] = \mathrm{i} f^{ijk} T^k, \quad \text{Tr}(T^i T^j) = \lambda\,\delta^{ij} > 0; \qquad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \qquad \text{where} \quad R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R},$$
$$[R^i, R^j] = -f^{ijk} R^k, \quad \text{Tr}(R^i R^j) = -\lambda\,\delta^{ij} < 0; \qquad f \in \mathbb{R}$$

B Statistics

Histogram   Sturges $k \approx 1 + \log_2 n$ n:

$$\overline{x} := \frac{1}{n} \sum x_i, \qquad x_G := \left(\prod x_i \right)^{1/n}, \qquad x_H := n \left(\sum \frac{1}{x_i} \right)^{-1}; \qquad \text{(B.1)}$$

n mid-range   ......   (B.2)

$$d := \frac{1}{n} \sum |x_i - \overline{x}|, \qquad\qquad S^2 := \frac{1}{n} \sum (x_i - \overline{x})^2, \qquad C.V. := S_x / \overline{x}; \quad \text{(B.3)}$$

$$M.D. := \frac{1}{n^2} \sum_i \sum_j |x_i - x_j|, \quad \text{Gini} \quad G.I. := \frac{M.D.}{2\overline{x}} = \frac{1}{2n^2 \overline{x}} \sum_i \sum_j |x_i - x_j|; \qquad \text{(B.4)}$$

Entropy $H = -\sum p_i \log p_i$ p: ... $= 0 \le H \le 1 =$ (B.5)

......   (B.6)

$$C_{xy} := \frac{1}{n} \sum (x_i - \overline{x})(y_i - \overline{y}) \qquad\qquad r_{xy} := \frac{C_{xy}}{S_x S_y} \quad -1 \le r_{xy} \le 1, \qquad \text{(B.7)}$$

$$r_{12;3} := \frac{r_{12} - r_{13} r_{23}}{\sqrt{1 - r_{13}{}^2} \sqrt{1 - r_{23}{}^2}} \qquad\qquad r_h := \frac{1}{S_x} \sum_{i=1}^{n-h} \frac{(x_i - \overline{x})(x_i + h - \overline{x})}{n - h} \qquad \text{(B.8)}$$

... $\{R_i\}$ $\{R_i'\}$

$$\text{Spearman}: \ r_{\mathrm{S}} := 1 - \frac{6}{n(n^2 - 1)} \sum (R_i - R_i')^2 \qquad\qquad\qquad\qquad \text{(B.9)}$$

$$\text{Kendall}: \ r_{\mathrm{K}} := \frac{\sum G_{ij}}{n(n-1)/2} \qquad \text{where} \quad G_{ij} := (i, j) \qquad\qquad +1 \qquad\qquad -1 \quad \text{(B.10)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad\qquad\qquad\qquad\qquad\qquad \text{(B.11)}$$

Bayes   : $\{H_i\}$

$$P(A) = \sum P(A \cap H_i) \qquad\qquad P(H_i|A) = \frac{P(H_i) P(A|H_i)}{\sum_k P(H_k) P(A|H_k)}. \qquad \text{(B.12)}$$

ρ:

$$f(r) = \frac{\left(1 - \rho^2\right)^{(n-1)/2} \left(1 - r^2\right)^{(n-4)/2}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n-2}{2}\right)} \sum_{i=0}^{\infty} \frac{(2\rho r)^i}{i!} \left[\Gamma\left(\frac{n - 1 + i}{2} \right) \right]^2 \qquad \text{(B.13)}$$

Moment

Moment $\mu_r := \langle X^r \rangle$; $\mu_r' := \langle (X - \mu)^r \rangle$ moment $\alpha_r := \langle (X - \mu)^r \rangle / \sigma^r$ (B.14)

Moment $M_X(t) := \langle \exp(tX) \rangle \Longrightarrow \mu_r = \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} M_X(t)$ (B.15)

$$\mu := \mu_1; \qquad\qquad\qquad\qquad\qquad : \ \mu_2' = \langle X^2 \rangle - \mu^2 \tag{B.16}$$

$$\sigma^2 := \mu_2'; \qquad \sigma := \sqrt{\sigma^2}; \qquad \mu_3' = \langle X^3 \rangle - 3\mu\mu_2 + 2\mu^3 \tag{B.17}$$

$$C_{\text{skew}} := \alpha_3; \qquad C_{\text{kurt}} := \alpha_4 - 3 \qquad \mu_4' = \langle X^4 \rangle - 4\mu\mu_3 + 6\mu^2\mu_2 - 3\mu^4 \tag{B.18}$$

Chebyshev $P\Big(|X - \mu| \geq k\sigma \Big) \leq \dfrac{1}{k^2}.$

Stirling

$$\log n! = \left(n + \frac{1}{2} \right) \log n - n + \frac{\log 2\pi}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + \mathrm{O}(n^{-11}) \tag{B.19}$$

B.1

(A,B) $(M, N - M)$ n $(k, n - k)$

$$P_k = \frac{{}_M\mathrm{C}_k \ {}_{N-M}\mathrm{C}_{n-k}}{{}_N\mathrm{C}_n} \qquad E = np, \quad V = np(1-p)\frac{N-n}{N-1} \qquad (p := M/N) \tag{B.20}$$

              p n k Bernoulli

$$P_k = {}_n\mathrm{C}_k \cdot p^k (1-p)^{n-k} \qquad E = np, \quad V = np(1-p) \qquad (n = 1 : \text{Bernoulli} \quad) \tag{B.21}$$

Poisson $np = \lambda$ $n \to \infty, \ p \to 0$ $P_k = \dfrac{\mathrm{e}^{-\lambda} \lambda^k}{k!}, \quad E = V = \lambda.$

        p k

$$P_k = p(1-p)^k \qquad\qquad\qquad E = \frac{1-p}{p}, \quad V = \frac{1-p}{p^2}. \tag{B.22}$$

(Pascal   ) p n k ($n + k$)

$$P_k = {}_{n+k-1}\mathrm{C}_k \ p^n (1-p)^k \qquad\qquad E = \frac{k(1-p)}{p}, \quad V = \frac{k(1-p)}{p^2}. \tag{B.23}$$

$$P_k = \frac{1}{N}, \qquad E = \frac{N+1}{2}, \quad V = \frac{N^2 - 1}{12}. \tag{B.24}$$

B.2

$$N[\mu, \sigma] = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]; \qquad E = \mu, \quad V = \sigma^2. \tag{B.25}$$

$$Ex[\lambda] = Ga[\lambda, 1] = \lambda e^{-\lambda x} \quad (x \geq 0); \qquad E = \frac{1}{\lambda}, \quad V = \frac{1}{\lambda^2}. \tag{B.26}$$

Gamma

$$Ga[\lambda, \alpha] = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad (x \geq 0); \qquad E = \frac{\alpha}{\lambda}, \quad V = \frac{\alpha}{\lambda^2}. \tag{B.27}$$

χ^2

$$\chi^2[n] = Ga[1/2, n/2] = \frac{1}{\Gamma(n/2)} \sqrt{\frac{x^{n-2} e^{-x}}{2^n}} \quad (x \geq 0); \qquad E = n, \quad V = 2n. \tag{B.28}$$

Beta

$$Be[\alpha, \beta] = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} \quad (0 < x < 1), \qquad E = \frac{\alpha}{\alpha + \beta}, \quad V = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \tag{B.29}$$

$$\text{where} \quad B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{B.30}$$

Cauchy

$$f[\alpha, \lambda] = \frac{\alpha}{\pi} \left[(x-\lambda)^2 + \alpha^2\right]; \qquad E \quad V \tag{B.31}$$

$$\log x \qquad\qquad N[\mu, \sigma]$$

$$f[\mu, \sigma] = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} \exp \frac{-(\log x - \mu)^2}{2\sigma^2}; \qquad E = e^{\mu+\sigma^2/2}, \quad V = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}. \tag{B.32}$$

Pareto $\qquad\qquad E \quad V \qquad\qquad a > 1, a > 2$

$$f_{[a, x_0]} = \frac{a}{x_0} \left(\frac{x_0}{x}\right)^{a+1} \quad (x \geq x_0; \quad a > 0); \qquad E = \frac{ax_0}{a-1}, \qquad V = \frac{ax_0^2}{a-2} - \left(\frac{ax_0}{a-1}\right)^2. \tag{B.33}$$

Weibull $\qquad a > 0, b > 0$

$$f[a, b] = \frac{b}{a^b} x^{b-1} \exp\left[-\left(\frac{x}{a}\right)^b\right] \quad (x \geq 0); \qquad E = a\Gamma\left(1 + \frac{1}{b}\right), \quad V = a^2\left[\Gamma\left(1 + \frac{2}{b}\right) - \left[\Gamma\left(1 + \frac{1}{b}\right)\right]^2\right] \tag{B.34}$$

C   Verbose Notes

C.1   SPINOR FIELDS

C.1.1   Lorentz group and Lorentz algebra

Metric : $\eta = \mathrm{diag}(+1,-1,-1,-1), \qquad \eta = \mathrm{diag}(-1,+1,+1,+1).$

Lorentz transf. in $\mathbb{R}^{1,3}$: Linear transf. $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ which conserve x^2.

: $\implies \eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma.$ and form a group L.

: $(\implies \quad (\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^\alpha{}_\beta =: \Lambda_\nu{}^\mu \implies \quad \Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta^\rho_\nu)$

Disconnected parts of L : $L_0 := \left\{\det \Lambda = +1 \wedge \Lambda_0{}^0 > 0\right\} \quad L_P := \left\{\det \Lambda = -1 \wedge \Lambda_0{}^0 > 0\right\}$

: $L_T := \left\{\det \Lambda = +1 \wedge \Lambda_0{}^0 < 0\right\} \quad L_{PT} := \left\{\det \Lambda = -1 \wedge \Lambda_0{}^0 < 0\right\}$

: (L_0 is identical with $(\mathrm{SO}(1,3), \mathrm{SO}(3,1))$.)

Infinitesimal one in L_0 : $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ (for $\eta = \eta\Lambda\Lambda$)

$$\epsilon^\mu{}_\nu = \begin{pmatrix} 0 & \beta_x & \beta_y & \beta_z \\ \beta_x & 0 & -\theta_z & \theta_y \\ \beta_y & \theta_z & 0 & -\theta_x \\ \beta_y & -\theta_y & \theta_x & 0 \end{pmatrix} \qquad\qquad \boldsymbol{J} \qquad\qquad \boldsymbol{K}$$

$$\Lambda = \exp\left[\kappa \mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] \implies J_x = -\kappa\mathrm{i}\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}, \quad K_x = -\kappa\mathrm{i}\begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \tag{C.1}$$

$\kappa = \pm 1$ notation

$$\epsilon^\mu{}_\nu =: \mp\tfrac{\mathrm{i}}{2}\epsilon^{\rho\sigma}(J_{\rho\sigma})^\mu{}_\nu \qquad\qquad\qquad \epsilon_{\mu\nu}$$

$$\boldsymbol{\theta} = (+,-)(\epsilon^{23},\epsilon^{31},\epsilon^{12}), \qquad\qquad \boldsymbol{\beta} = (+,-)(\epsilon^{10},\epsilon^{20},\epsilon^{30}).$$
$$= (+,-)(\epsilon_{23},\epsilon_{31},\epsilon_{12}) \qquad\qquad = (-,+)(\epsilon_{10},\epsilon_{20},\epsilon_{30})$$

$J^{\rho\sigma}$ $\qquad\qquad (J_{\rho\sigma})^\mu{}_\nu = \pm\mathrm{i}\left(\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}\right)$

Lie

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mp\mathrm{i}(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}). \tag{C.2}$$

$$J_{10}{}^\mu{}_\nu = (\pm\mathrm{i}, \mp\mathrm{i})\begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}_{\mu\nu}, J_{23}{}^\mu{}_\nu = (\pm\mathrm{i}, \mp\mathrm{i})\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}_{\mu\nu}$$

κ $\qquad\qquad\qquad \boldsymbol{J}\ \boldsymbol{K}\quad J_{\rho\sigma}$

$$*\quad *\quad *$$

κ $\qquad\qquad (-,\ ),(+,\ ),(-,\ ),(+,\ )$

$$\boldsymbol{J} = (J_{23}, J_{31}, J_{12}), \quad \boldsymbol{K} = (J_{10}, J_{20}, J_{30}); \tag{C.3}$$

$$\Lambda = \exp\epsilon = \exp\left[\kappa\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] = \exp\left[\mp\mathrm{i}(\epsilon^{\rho\sigma}J_{\rho\sigma})/2\right]. \tag{C.4}$$

C.1.2 Lorentz group and $\mathrm{SL}(2,\mathbb{C})$

Lie　L_0　　　　Lie　$\mathrm{SL}(2,\mathbb{C})/\mathrm{Z}_2$

$$\mathfrak{sl}(2,\mathbb{C}) := \{a \in \mathfrak{gl}(2,\mathbb{C}) \mid \mathrm{Tr}(a) = 0\}, \qquad \mathrm{SL}(2,\mathbb{C}) := \{g \in \mathrm{GL}(2,\mathbb{C}) \mid \det(g) = 1\}. \tag{C.5}$$

$,\sigma^\mu$　　　　　　　　　$\sigma^\mu := (\alpha 1, \beta \boldsymbol{\sigma})$　　　　$(\alpha = \beta = \pm 1)$　$x^2 = (+-)\det(x_\mu \sigma^\mu)$

$$f^g : (x_\mu \sigma^\mu) \mapsto g(x_\mu \sigma^\mu)g^\dagger; \quad g \in \mathrm{SL}(2,\mathbb{C}) \tag{C.6}$$

x^2　　　　　　　　Lorentz

$$\mathrm{SL}(2,\mathbb{C}) \ni g = \exp(-\mathrm{i}a) \qquad x_\mu(g\sigma^\mu g^\dagger) = \Lambda_\mu{}^\nu x_\nu \sigma^\mu$$

$$\Lambda^\mu{}_\nu \sigma^\nu = g^{-1}\sigma^\mu (g^{-1})^\dagger \implies \epsilon^\mu{}_\nu \sigma^\nu = \mathrm{i}(a\sigma^\mu - \sigma^\mu a^\dagger) \tag{C.7}$$

$$g$$

$$g = \exp\left(-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma} - \frac{\alpha\beta}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right). \tag{C.8}$$

Lorentz

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k \tag{C.9}$$

$$\boldsymbol{A} := \frac{1}{2}(\boldsymbol{J} + \mathrm{i}\boldsymbol{K}), \qquad \boldsymbol{B} := \frac{1}{2}(\boldsymbol{J} - \mathrm{i}\boldsymbol{K}). \tag{C.10}$$

$$[A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0, \tag{C.11}$$

Lorentz　　　$\mathrm{SU}(2) \times \mathrm{SU}(2)$

C.2 Weyl Spinor

$\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$　　　　　$(1/2, 0)$　　　　　　　　　spinor ξ　$(0, 1/2)$
spinor $\bar{\xi}$

$$\xi \mapsto \left(1 - \frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\xi \qquad\qquad \bar{\xi} \mapsto \left(1 - \frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\bar{\xi}. \tag{C.12}$$

$\alpha\beta = 1$　　　　　g　　　　　spinor　　　　　　$\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta, \bar{\xi}^{\dot\alpha} \mapsto (g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}\bar{\xi}^{\dot\beta}$

$\xi^\alpha \chi_\alpha$　　　　$\bar{\xi}_{\dot\alpha}\bar{\chi}^{\dot\alpha}$　scalar　　　　　$E = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$　　　　　$-Eg^\mathrm{T}E = g^{-1}$

$$(\xi')^\alpha = \xi^\beta (g^{-1})_\beta{}^\alpha = -\xi^\beta(E_{\beta A}g_B{}^A E_{B\alpha}) \qquad \therefore (-E_{\gamma\alpha})(\xi')^\alpha = -g_\gamma{}^\beta(E_{\beta\delta}\xi^\delta). \tag{C.13}$$

$\epsilon^{12} = \epsilon_{21} = 1$　　　　$\xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta$　$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$

$\bar{\xi}'_{\dot\alpha} = -E(g^\dagger)^{-1}E\bar{\xi}_{\dot\beta}$　　　$(E^{\dot\alpha\dot\beta}\bar{\xi}_{\dot\beta})' = (g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}(E^{\dot\beta\dot\gamma}\bar{\xi}_{\dot\gamma})$　　　　$\epsilon_{12} = \epsilon_{\dot1\dot2}$　　　$\bar{\xi}_{\dot\alpha} := \epsilon_{\dot\alpha\dot\beta}\bar{\xi}^{\dot\beta}$

$$(\xi_a)^* = \bar{\xi}_{\dot\alpha} \qquad\qquad \xi^\dagger = \bar{\xi}$$

$x_\mu \sigma^\mu \mapsto x_\mu(g\sigma^\mu g^\dagger)$　　　　　$x_\mu(\xi^\alpha \sigma^\mu \bar{\chi}^{\dot\alpha})$　scalar　　　　　　$\sigma^\mu_{\alpha\dot\alpha}$

$x_\mu(\bar{\xi}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\chi_\alpha)$　scalar　　　　　$\bar{\sigma}$

$$\bar{\xi}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\chi_\alpha = -\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\chi^\beta\bar{\sigma}^{\mu\dot\alpha\alpha}\bar{\xi}^{\dot\beta} \qquad \therefore \sigma^\mu_{\beta\dot\beta} \propto \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar{\sigma}^{\mu\dot\alpha\alpha} \tag{C.14}$$

convention

$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta, \quad \xi^\alpha \mapsto \xi^\beta (g^{-1})_\beta{}^\alpha, \quad \bar{\xi}_{\dot{\alpha}} \mapsto \bar{\xi}_{\dot{\beta}}(g^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}} \quad \bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \quad \sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta} \quad \therefore \sigma^\mu := (\alpha 1, \beta\boldsymbol{\sigma}), \quad \bar{\sigma}^\mu := (\alpha 1, -\beta\boldsymbol{\sigma})$$

C.3 POLARIZATION SUM

Firstly we focus on the single photon case $M = \epsilon_\mu^*(k)M^\mu$. In this case, the replacement

$$\sum_{\text{pol.}} \epsilon_\mu \epsilon'_\nu \rightarrow \eta_{\mu\nu} \tag{C.15}$$

is valid. Let us prove this validity. First we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\nu(k)M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2, \tag{C.16}$$

while

$$\eta_{\mu\nu}M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2 \tag{C.17}$$

for Ward identity $k_\mu M^\mu = 0$. Now we can see the validity easily.

Next we think about the double photons case [*3] $M = \epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \qquad\qquad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \tag{C.18}$$
$$k' = (E, 0, 0, -E) \qquad\qquad \epsilon' = (0, \cos\theta, \sin\theta, 0) \oplus (0, -\sin\theta, \cos\theta, 0). \tag{C.19}$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon_\mu^*(k)\epsilon_\nu(k)\epsilon_\rho^*(k')\epsilon_\sigma(k')M^{\mu\rho}M^{\nu\sigma*} \tag{C.20}$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2 \tag{C.21}$$

$$\overset{?}{=} \eta^{\mu\nu}\eta^{\rho\sigma}M^{\mu\rho}M^{\nu\sigma*}. \tag{C.22}$$

Badly, our Ward identities

$$k_\mu \epsilon_\nu'^*(k')M^{\mu\nu} = \epsilon_\mu^*(k)k'_\nu M^{\mu\nu} = 0 \tag{C.23}$$

do not help us; now we want to be given

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0, \tag{C.24}$$

to recover validity of the replacement.

The difference between (C.23) and (C.24) is that the former considers (and tries to sum up) all polarizations but the latter does only physical ones. Actually, as long as we are summing up all polarizations, the replacement is still valid; these two conditions are equivalent because of cancellation of unphysical

---

[*3] This part is derived from   's notebook.

polarizations. However, once we restrict the polarizations (for example with using a relation $\epsilon \cdot k = 0$), we can no more use (C.24) and thus the replacement becomes invalid.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our latter conditions (C.24), and define $\widetilde{M}^{\mu\nu}$ and $\widetilde{M}$ as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \tag{C.25}$$

$$\widetilde{M} := \epsilon_\mu^*(k)\epsilon_\nu'^*(k')\widetilde{M}^{\mu\nu}. \tag{C.26}$$

Here $\widetilde{M}^{\mu\nu} \neq M^{\mu\nu}$ but $\widetilde{M} = M$; thus $\widetilde{M}$ satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$). However, we cannot utilize the replacement for $\widetilde{M}$, while it is valid for M. If you did the replacement, a wrong result would come out, like

$$\eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma*} = \eta_{\mu\rho}\eta_{\nu\sigma}\left(M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu\right)\left(M^{\rho\sigma*} + k^\rho p^{\sigma*} + p'^{\rho*}k'^\sigma\right) \tag{C.27}$$

$$= \sum_{\text{pol.}} |M|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H.c.}\right]. \tag{C.28}$$

After all, we have obtained following expression:

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} |\epsilon_\mu^*(k)\epsilon_\nu'^*(k')M^{\mu\nu}|^2 = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}$$

$$= \sum_{\text{pol.}} |\widetilde{M}|^2 = \sum_{\text{pol.}} |\epsilon_\mu^*(k)\epsilon_\nu'^*(k')\widetilde{M}^{\mu\nu}|^2 \neq \eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma*} = \sum_{\text{pol.}} |\widetilde{M}|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H.c.}\right]. \tag{C.29}$$

To check the Ward identity always helps us!

C.4   Phantom Terms in the Gauge Theory

You may think we forget to introduce $\overline{\psi}\gamma_5\psi$ $\overline{\psi}\gamma_5\slashed{D}\psi$ $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a$ $\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F_{\rho\sigma}^a$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_L and P_R,
- the last term is equivalent to the third term as

$$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F_{\rho\sigma}^a = \epsilon^{\mu\nu\rho\sigma}\frac{1}{2}[D_\mu, D_\nu]F_{\rho\sigma}^a = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a.$$

Therefore, we have to discuss only the ϵFF terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A_\mu^b A_\nu^c A_\rho^d A_\sigma^e = \epsilon^{\mu\nu\rho\sigma}\left(-f^{acd}f^{abe} - f^{adb}f^{ace}\right)A_\mu^b A_\nu^c A_\rho^d A_\sigma^e$$
$$= -2\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A_\mu^b A_\nu^c A_\rho^d A_\sigma^e \tag{C.30}$$
$$= 0,$$

$$\therefore \ \epsilon^{\nu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu^a \partial_\rho A_\sigma^a + 4g\epsilon^{\mu\nu\rho\sigma}f^{abc}A_\mu^a A_\nu^b \partial_\rho A_\sigma^c$$
$$= 2\partial_\mu G^\mu, \tag{C.31}$$

where G^μ is the Chern–Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma}\left(A_\nu^a \partial_\rho A_\sigma^a + \frac{1}{3}gf^{abc}A_\nu^a A_\rho^b A_\sigma^c\right) = \epsilon^{\mu\nu\rho\sigma}\left(A_\nu^a F_{\rho\sigma}^a - \frac{1}{3}gf^{abc}A_\nu^a A_\rho^b A_\sigma^c\right). \tag{C.32}$$

See Appendix C.7 for the instanton effect.

C.5   -Mills Theory

C.5.1   General Gauge Theory

For any Lie group G, we can consider "gauge transformation" $\phi(x) \mapsto V(x)\phi(x)$, where $V : \mathbb{R}^{1,3} \to G$. Also we can define a "connection field" $A_\mu(x)$ as:

$$\phi_\parallel(x + \mathrm{d}x) := \phi(x) + \mathrm{i}gA_\mu(x)\phi(x)\mathrm{d}x^\mu \qquad \text{s.t.} \quad \phi_\parallel(x + \mathrm{d}x) \mapsto V(x + \mathrm{d}x)\phi_\parallel(x + \mathrm{d}x). \tag{C.33}$$

Then the covariant derivative D_μ can be defined as

$$\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu := \Delta_{\mathrm{d}x}\phi(x) := \phi(x + \mathrm{d}x) - \phi_\parallel(x + \mathrm{d}x) \qquad \therefore \mathrm{D}_\mu := \partial_\mu - \mathrm{i}gA_\mu. \tag{C.34}$$

Note that $\Delta_{\mathrm{d}x}\phi(x) \mapsto V(x + \mathrm{d}x)\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu$, which means $\mathrm{D}_\mu\phi(x) \mapsto V(x)\mathrm{D}_\mu\phi(x)$. Now we can see

$$\phi \mapsto V\phi, \qquad\qquad A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1}, \qquad\qquad \mathrm{D}_\mu \mapsto V\mathrm{D}_\mu V^{-1}. \tag{C.35}$$

We can do another discussion: we can define D_μ as a kind of derivative which satisfies (C.35).

Next we introduce the curvature tensor, or "field strength" as

$$\Delta\phi(x) := \phi_\parallel^{xy}(x + \mathrm{d}x + \mathrm{d}y) - \phi_\parallel^{yx}(x + \mathrm{d}x + \mathrm{d}y) = [\mathrm{D}_\mu, \mathrm{D}_\nu]\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu =: -\mathrm{i}gF_{\mu\nu}\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu; \tag{C.36}$$

$$F_{\mu\nu}(x) := \frac{\mathrm{i}}{g}[\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}g[A_\mu(x), A_\nu(x)]. \tag{C.37}$$

$\Delta\phi(x)$ is transformed in terms of $V(x + \mathrm{d}x + \mathrm{d}y) \simeq V(x)$, thus $F_{\mu\nu}(x) \mapsto V(x)F_{\mu\nu}(x)V^{-1}(x)$.

C.5.2   Compact Gauge Theory

<u>Generators</u>   If the gauge group G is **compact**, it has a finite-dimensional unitary representation. The generators T_a can be taken to be Hermitian, and $V(x) = \exp[\mathrm{i}g\theta^a(x)T^a]$ for $\theta^a(x) \in \mathbb{R}$;

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_c T^c \quad (f \in \mathbb{R}) \qquad\qquad 0 = f^D{}_{ab}f^E{}_{Dc} + f^D{}_{ca}f^E{}_{Db} + f^D{}_{bc}f^E{}_{Da} \tag{C.38}$$

For the sake of the compactness Killing form is positive-definite, where we can normalize the generators as $\operatorname{Tr} T^a T^b = \frac{1}{2}\delta^{ab}$, and the structure constant f^{abc} would be totally antisymmetric.

<u>Adjoint Representations</u>

$$[\widetilde{T}^a]_i{}^j := -\mathrm{i}f^{aij}; \qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j := \delta_i^j\partial_\mu + gf^{iaj}A_\mu^a. \tag{C.39}$$

<u>Field Expansion</u>   In this *normalized Hermitian* basis, the relations would be[*4]

$$\phi' = \mathrm{e}^{\mathrm{i}gT^a\theta^a}\phi; \qquad\qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

$$A_\mu^{a}{}' \simeq A_\mu^a + \partial_\mu\theta^a + gf^{abc}A_\mu^b\theta^c \qquad\qquad F_{\mu\nu}^a{}' = [\mathrm{e}^{\mathrm{i}g\theta^c\widetilde{T}^c}]^{ab}F_{\mu\nu}^b$$

$$= A_\mu^a + (\widetilde{\mathrm{D}}_\mu\theta)^a, \qquad\qquad \simeq F_{\mu\nu}^a + gf^{abc}F_{\mu\nu}^b\theta^c.$$

<u>Covariant Derivative</u>   For a field λ^a under the adjoint representation,

$$(\mathrm{D}_\mu\lambda)^a = \partial_\mu\lambda^a + gf^{abc}A_\mu^b\lambda^c \qquad \text{or} \quad \mathrm{D}_\mu\lambda^a T^a = \partial_\mu\lambda^a T^a - \mathrm{i}g[A_\mu^b T^b, \lambda^a T^a] \text{ [*5]} \tag{C.40}$$

<u>Bianchi Equation</u>

$$\epsilon^{\mu\nu\rho\sigma}[\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\sigma]] = \epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma} = 0. \tag{C.41}$$

[*4] We can expand A_μ in T^a-basis, because it is induced by the gauge transformation.

[*5] Note that we can use any representation T^a but must the same ones for $A_\mu^a T^a$ and $\lambda^a T^a$.

C.6 Spinor

$\eta^{\mu\nu} = (-,+,+,+)$ case

Grassmann Number : $(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$

$\qquad\qquad\qquad\quad$: $\Longrightarrow$ for $a, b \in \mathbb{G}^\mathbb{R}$, $ab \in \mathrm{i}\mathbb{G}^\mathbb{R}$

γ matrix $\qquad\qquad$: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$

$\qquad\qquad\qquad\quad$: $\gamma^{\mu\nu} = \dfrac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$ $\quad$ etc...

$\qquad\qquad\qquad\quad$: $(\mathrm{i}\gamma^0)^\dagger := \mathrm{i}\gamma^0, \quad \gamma^{i\dagger} := \gamma^i$

Dirac Conjugate $\qquad$: $\bar{\psi} = \mathrm{i}\psi^\dagger\gamma^0$

TODO: SPINOR

C.7 Instanton

TODO: INSTANTON

D   Supersymmetry in the text by Wess & Bagger

D.1   Spinor Convention

ϵ tensor   : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)

Sum Rule   : $^{\alpha}_{\;\alpha}$ and $_{\dot{\alpha}}^{\;\dot{\alpha}}$, except for $\xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta}$, $\xi^{\alpha} = \epsilon^{\alpha\beta}\xi_{\beta}$, $\xi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}$, $\xi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta}}$.

Lorentz   : $\psi'_{\alpha} = \Lambda_{\alpha}^{\;\beta}\psi_{\beta}$, $\bar{\psi}'_{\dot{\alpha}} = \bar{\psi}_{\dot{\beta}}\Lambda^{\dagger\dot{\beta}}_{\;\;\dot{\alpha}}$, $\psi'^{\alpha} = \psi^{\beta}\Lambda^{-1}{}_{\beta}^{\;\alpha}$, $\bar{\psi}'^{\dot{\alpha}} = (\Lambda^{-1})^{\dagger\dot{\alpha}}_{\;\;\dot{\beta}}\bar{\psi}'^{\dot{\beta}}$.

σ matrices   : $(\sigma^{\mu})_{\alpha\dot{\beta}} := (-1, \boldsymbol{\sigma})_{\alpha\dot{\beta}}$, $(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^{\mu})_{\beta\dot{\beta}} = (-1, -\boldsymbol{\sigma})^{\dot{\alpha}\beta}$.

(See App. C.1.1 for a verbose explanation.)

D.2   Spinor Calculation Cheatsheet

$$\eta = (-, +, +, +), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1$$
$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^{\alpha} := \epsilon^{\alpha\beta}\xi_{\beta}, \quad \xi_{\alpha} = \epsilon_{\alpha\beta}\xi^{\beta}, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$
$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} \quad \sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}, \quad \sigma^{\mu} := (-1, \boldsymbol{\sigma}), \quad \bar{\sigma}^{\mu} := (-1, -\boldsymbol{\sigma})$$
$$(\sigma^{\mu\nu})_{\alpha}^{\;\beta} := \tfrac{1}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}^{\;\beta}, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\;\dot{\beta}} := \tfrac{1}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\;\dot{\beta}} = (\sigma^{\nu\mu})^{\dagger\dot{\alpha}}_{\;\dot{\beta}}.$$

$\theta^{\alpha}\theta^{\beta} = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta$ $\theta_{\alpha}\theta_{\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta$ $(\theta\phi)(\theta\psi) = -\tfrac{1}{2}(\psi\phi)(\theta\theta)$ $(\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) = -\tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}$

$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$ $\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$ $(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\tfrac{1}{2}(\bar{\psi}\bar{\phi})(\bar{\theta}\bar{\theta})$ $(\sigma^{\mu}\bar{\theta})_{\alpha}(\theta\sigma^{\nu}\bar{\theta}) = \tfrac{1}{2}(\sigma^{\mu}\bar{\sigma}^{\nu}\theta)_{\alpha}\bar{\theta}\bar{\theta}$

$\theta\sigma^{\mu}\bar{\sigma}^{\nu}\theta = -\eta^{\mu\nu}\theta\theta$ $\bar{\theta}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\theta} = -\eta^{\mu\nu}\bar{\theta}\bar{\theta}$ $(\theta\sigma^{\mu})_{\dot{\alpha}}(\theta\sigma^{\nu}\bar{\theta}) = \tfrac{1}{2}(\bar{\theta}\bar{\sigma}^{\nu}\sigma^{\mu})_{\dot{\alpha}}\theta\theta$

$\sigma^{\mu}\bar{\sigma}^{\nu} = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}$ $\sigma^{\mu}\bar{\sigma}^{\rho}\sigma^{\nu} + \sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\mu} = -2(\eta^{\mu\rho}\sigma^{\nu} + \eta^{\nu\rho}\sigma^{\mu} - \eta^{\mu\nu}\sigma^{\rho})$

$\bar{\sigma}^{\mu}\sigma^{\nu} = -\eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu}$ $\bar{\sigma}^{\mu}\sigma^{\rho}\bar{\sigma}^{\nu} + \bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\mu} = -2(\eta^{\mu\rho}\bar{\sigma}^{\nu} + \eta^{\nu\rho}\bar{\sigma}^{\mu} - \eta^{\mu\nu}\bar{\sigma}^{\rho})$

$\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ $\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} - \sigma^{\rho}\bar{\sigma}^{\nu}\sigma^{\mu} = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\sigma}$

$\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu}$ $\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} - \bar{\sigma}^{\rho}\sigma^{\nu}\bar{\sigma}^{\mu} = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\sigma}$

$\mathrm{Tr}\,\bar{\sigma}^{\mu}\sigma^{\nu} = \mathrm{Tr}\,\sigma^{\mu}\bar{\sigma}^{\nu} = -2\eta^{\mu\nu}$ $\mathrm{Tr}\,\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\tfrac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$

$\mathrm{Tr}\,\sigma^{\mu\nu} = \mathrm{Tr}\,\bar{\sigma}^{\mu\nu} = 0$ $\mathrm{Tr}\,\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} = -\tfrac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}$

$\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = -2\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}}$ $\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} - \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\right]$

$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma_{\mu\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$ $\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} + \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\sigma\nu})_{\dot{\alpha}\dot{\beta}}$

$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}_{\mu}^{\dot{\beta}\beta} = -2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$ $\epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}}$ $\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2\mathrm{i}\sigma^{\mu\nu}$

$\sigma^{\mu\nu}{}_{\alpha}^{\;\beta}\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_{\gamma}^{\;\beta}\epsilon_{\beta\alpha}$ $\epsilon_{\beta\alpha}\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}}$ $\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = 2\mathrm{i}\bar{\sigma}^{\mu\nu}$

$\bar{\xi}\bar{\sigma}^{\mu}\chi = -\chi\sigma^{\mu}\bar{\xi} = (\bar{\chi}\bar{\sigma}^{\mu}\xi)^{*} = -(\xi\sigma^{\mu}\bar{\chi})^{*}$ $(\psi\phi)\chi_{\alpha} = -(\phi\chi)\psi_{\alpha} - (\chi\psi)\phi_{\alpha}$

$\xi\sigma^{\mu}\bar{\sigma}^{\nu}\chi = \chi\sigma^{\nu}\bar{\sigma}^{\mu}\xi = (\bar{\chi}\bar{\sigma}^{\nu}\sigma^{\mu}\bar{\xi})^{*} = (\bar{\xi}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\chi})^{*}$ $(\psi\phi)\bar{\chi}_{\dot{\alpha}} = -\tfrac{1}{2}(\phi\sigma^{\mu}\bar{\chi})(\psi\sigma_{\mu})_{\dot{\alpha}}$

In the following equations, we chose left-differential notation.

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}} = -\frac{\partial}{\partial\theta_{\alpha}} \qquad \frac{\partial}{\partial\theta^{\alpha}}\theta\theta = 2\theta_{\alpha} \qquad \epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha}}\frac{\partial}{\partial\theta^{\beta}}\theta\theta = 4$$

$$\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{\beta}} = -\frac{\partial}{\partial\theta^{\alpha}} \qquad \frac{\partial}{\partial\theta_{\alpha}}\theta\theta = -2\theta^{\alpha} \qquad \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{\alpha}}\frac{\partial}{\partial\theta_{\beta}}\theta\theta = -4$$

$$\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}\bar{\theta} = -2\bar{\theta}_{\dot{\alpha}} \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} = 4$$

$$\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}\bar{\theta} = 2\bar{\theta}^{\dot{\alpha}} \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}\bar{\theta} = -4$$

D.3   Chiral Superfields : $\bar{D}_{\dot{\alpha}}\Phi = 0$

Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{D.1}$$

$$= \phi(x) + \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) + \sqrt{2}\theta\psi(x) + \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x) \tag{D.2}$$

$$\Phi^\dagger = \phi^*(x) - \mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^*(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x) \tag{D.3}$$

Product of Chiral Superfields

$$
\begin{aligned}
\Phi_i^\dagger\Phi_j(x,\theta,\bar{\theta}) \rightsquigarrow{} & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^*F_j + \bar{\theta}\bar{\theta}F_i^*\phi_j \\
& + 2\mathrm{i}(\theta\sigma^\mu\bar{\theta})(\phi_i^*\partial_\mu\phi_j) - \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar{\theta}\bar{\sigma}^\mu\psi_j - \sqrt{2}\mathrm{i}\bar{\theta}\bar{\theta}\theta\sigma^\mu\bar{\psi}_i\partial_\mu\phi_j \\
& + 2\bar{\theta}\bar{\psi}_i\theta\psi_j + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j + \sqrt{2}\bar{\theta}\bar{\theta}F_i^*\theta\psi_j \\
& + \theta\theta\bar{\theta}\bar{\theta}\left[F_i^*F_j - \partial^\mu\phi_i^*\partial_\mu\phi_j - \mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j\right]
\end{aligned} \tag{D.4}
$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right] \tag{D.5}$$

$$
\begin{aligned}
\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) = {} & \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right] \\
& + \theta\theta\left[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right]
\end{aligned} \tag{D.6}
$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

E Cherry on the Cake

Conversion of Units

$$1\,\text{GeV} = \frac{1}{6.5821 \times 10^{-25}\,\text{s}} = \frac{1}{2.086 \times 10^{-32}\,\text{yr}} = \frac{1}{0.19733\,\text{fm}} = 1.1605 \times 10^{13}\,\text{K} = 1.7827 \times 10^{-24}\,\text{g}$$

$$= \frac{1.519268 \times 10^{24}}{1\,\text{s}} = \frac{4.79 \times 10^{31}}{1\,\text{yr}} = \frac{5.0677}{1\,\text{fm}} \tag{E.1}$$

$$1\,\text{K} = 8.6173 \times 10^{-5}\,\text{eV} = \frac{1}{8.0591 \times 10^{-21}\,\text{s}} = \frac{1.2408 \times 10^{20}}{1\,\text{s}} \tag{E.2}$$

$$1\,\text{GeV}^2 = \frac{1}{0.38938\,\text{mbarn}} = \frac{2568.2}{1\,\text{barn}} \qquad (1\,\text{barn} = 10^{-28}\,\text{m}^2 = 100\,\text{fm}^2) \tag{E.3}$$

$$1\,\text{tropical yr} = 3.1557 \times 10^7\,\text{s}, \quad 1\,\text{sidereal yr} = 3.1558 \times 10^7\,\text{s};$$
$$1\,\text{s} = 3.1689 \times 10^{-8}\,\text{tr-yr} = 3.1688 \times 10^{-8}\,\text{sr-yr}. \tag{E.4}$$

Physical Constants

$$G_\text{F} = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5}\,\text{GeV}^{-2}, \qquad G_\text{N} = 6.70881(67) \times 10^{-39}\,\text{GeV}^{-2} \tag{E.5}$$

$$\sqrt{G_\text{N}} = 1.61624(8) \times 10^{-35}\,\text{m} = \frac{1}{1.22089(6) \times 10^{19}\,\text{GeV}} = \frac{1}{2.17644(11) \times 10^{-8}\,\text{kg}} \tag{E.6}$$

$$\sqrt{8\pi G_\text{N}} = 8.1026(4) \times 10^{-35}\,\text{m} = \frac{1}{2.4353(1) \times 10^{18}\,\text{GeV}} = \frac{1}{4.3413(2) \times 10^{-9}\,\text{kg}} \tag{E.7}$$

Component of Spinor in Weyl Representation
$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta,\ \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta},\ \chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta};\quad \epsilon^{12} = \epsilon^{\dot1\dot2} = 1,\ \epsilon_{12} = \epsilon_{\dot1\dot2} = -1.)$

$$\psi = \begin{pmatrix} \psi_\text{L} \\ \psi_\text{R} \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar\chi^{\dot1} \\ \bar\chi^{\dot2} \end{pmatrix} \quad \xrightarrow{\ C\ } \quad \psi^\text{c} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} -(\bar\chi^{\dot2})^* \\ (\bar\chi^{\dot1})^* \\ (\xi_2)^* \\ -(\xi_1)^* \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \bar\xi_{\dot2} \\ -\bar\xi_{\dot1} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix} \tag{E.8}$$

$$\overline\psi = \begin{pmatrix} \chi^\alpha & \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \chi^1 & \chi^2 & \bar\xi_{\dot1} & \bar\xi_{\dot2} \end{pmatrix} \quad \xrightarrow{\ C\ } \quad \overline{\psi^\text{c}} = \mathrm{i}\psi^\mathsf{T}\gamma^0\gamma^2 = \begin{pmatrix} \xi_2 & -\xi_1 & -\bar\chi^{\dot2} & \bar\chi^{\dot1} \end{pmatrix} = \begin{pmatrix} \xi^\alpha & \bar\chi_{\dot\alpha} \end{pmatrix} \tag{E.9}$$

$$A^\alpha B_\alpha = \overline\psi_{A^\text{c}} P_\text{L} \psi_B = \overline\psi_{B^\text{c}} P_\text{L} \psi_A \qquad\qquad \bar A_{\dot\alpha} \bar B^{\dot\alpha} = \overline\psi_A P_\text{R} \psi_{B^\text{c}} = \overline\psi_B P_\text{R} \psi_{A^\text{c}} \tag{E.10}$$

$$\overline\psi \slashed{A} \chi = \overline\psi_\text{L} \slashed{A} P_\text{L} \chi_\text{L} + \overline\psi_\text{R} \slashed{A} P_\text{R} \chi_\text{R}$$
$$= \overline\psi_\text{L} \slashed{A} P_\text{L} \chi_\text{L} - \overline\chi^\text{c}_\text{R} \slashed{A} P_\text{L} \psi^\text{c}_\text{R} \tag{E.11}$$