1 General Definitions and Tools

1.1 Notations and Conventions

◆1.1.1 Metric etc.

Minkowski Metric : $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$

Coordinates : $x^\mu = (t, x, y, z)$; Therefore $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$

Gamma Matrices : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$; $\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3$

Gamma Combinations : $1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5$; $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] = 0 \Big/ \mathrm{i}\,\gamma^\mu\gamma^\nu$

◆1.1.2 Fields

Klein-Gordon Equation : $|\partial_\mu\phi|^2 - m^2|\phi|^2 = 0$

Klein-Gordon Field : $\phi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}\mathrm{e}^{-\,\mathrm{i}\,px} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{\,\mathrm{i}\,px}\right]$

Dirac Equation : $(i\partial\!\!\!/ - m)\psi(x) = 0$

Dirac Field : $\psi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left[a^s_{\boldsymbol{p}}u^s(p)\mathrm{e}^{-\,\mathrm{i}\,px} + b^{s\dagger}_{\boldsymbol{p}}v^s(p)\mathrm{e}^{\,\mathrm{i}\,px}\right]$

Gauge Boson : $(\partial^2 + m^2)A^\mu(x) = 0$ (Real Klein-Gordon Equation)

(Before Gauge Fixing) : $A^\mu(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{r=0,1,2,3}\left[a^r_{\boldsymbol{p}}\epsilon^r(p)\mathrm{e}^{-\,\mathrm{i}\,px} + a^{r\dagger}_{\boldsymbol{p}}\epsilon^{r*}(p)\mathrm{e}^{\,\mathrm{i}\,px}\right]$

南部-Goldstone Boson : **TODO: .**

> Chiral Notation
>
> Gamma Matrices : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$; $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
>
> Dirac Field : $\psi = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix}$; $\bar\psi = \psi^\dagger\gamma^0 = \begin{pmatrix} \psi^\dagger_\mathrm{R} & \psi^\dagger_\mathrm{L} \end{pmatrix}$
>
> : $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar\sigma}\xi^s \end{pmatrix}$; $v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}$
>
> : $\left[\eta^s = \xi^{-s} := -\,\mathrm{i}\,\sigma^2(\xi^s)^* = (\xi^2, -\xi^1)\right]$
>
> Weyl Equations : $\mathrm{i}\,\bar\sigma\cdot\partial\psi_L = m\psi_R$; $\mathrm{i}\,\sigma\cdot\partial\psi_L = m\psi_L$
>
> CPT transf. : $P\psi(t,\boldsymbol{x})P = \eta\gamma^0\psi(t,-\boldsymbol{x})$ $(|\eta|^2 = 1)$
>
> : $T\psi(t,\boldsymbol{x})T = \gamma^1\gamma^3\psi(-t,\boldsymbol{x})$ (ignoring intrinsic phase)
>
> : $C\psi(t,\boldsymbol{x})C = -\,\mathrm{i}\,\gamma^2\psi^*(t,\boldsymbol{x}) = -\,\mathrm{i}\,(\bar\psi\gamma^0\gamma^2)^\mathrm{T}$ (〃)
>
> : $\bar\psi \longrightarrow P: \eta^*\bar\psi\gamma^0 \quad T: -\bar\psi\gamma^1\gamma^3 \quad C: \mathrm{i}\,\bar\psi^*\gamma^2 = -\,\mathrm{i}\,(\gamma^0\gamma^2\psi)^\mathrm{T}$

Electromagnetic Fields : $A^\mu = (\phi, \boldsymbol{A})$ 【We can invert the signs, but cannot lower the index.】

$$F_{\mu\nu} = \begin{pmatrix} 0 & & \boldsymbol{E} & \\ & 0 & -B_3 & B_2 \\ -\boldsymbol{E} & B_3 & 0 & -B_1 \\ & -B_2 & B_1 & 0 \end{pmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\|\boldsymbol{E}\|^2 - \|\boldsymbol{B}\|^2\right)$$

◆1.1.3 CPT Table

	ϕ	A^μ	$\bar\psi\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\sigma^{\mu\nu}\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$	$\mathrm{i}\,\bar\psi\gamma_5\psi$	∂_μ
P	$\eta\phi$	$\eta{-}{+}{+}{+}A^\mu$	$+$	$+---$	$(+---)(+---)$	$-+++$	$-$	$+---$
T	$\zeta\phi$	$\zeta{+}{-}{-}{-}A^\mu$	$+$	$+---$	$-(+---)(+---)$	$+---$	$-$	$-+++$
C	$\xi\phi^*$	$\xi{+}A^{\mu*}$	$+$	$-$	$-$	$+$	$+$	$+$

($\eta\zeta\xi = 1$; especially, photon A^μ is $(\eta, \zeta, \xi) = (-, +, -)$.)

1.2 Dirac's Gamma Algebras

◆1.2.1 Traces

$$\begin{aligned}
\mathrm{Tr}(\text{any odd } \# \text{ of } \gamma\text{'s}) &= 0 && (1.1)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} && (1.2)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) && (1.3)\\
\mathrm{Tr}(\gamma_5 \text{ and any odd } \# \text{ of } \gamma\text{'s}) &= 0 && (1.4)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5) &= 0 && (1.5)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) &= -4\,\mathrm{i}\,\epsilon^{\mu\nu\rho\sigma} && (1.6)
\end{aligned}$$

Generally, for some γ-matrices $A, B, C, \ldots$,

$$\begin{aligned}
\mathrm{Tr}(ABCDEF\cdots) &= \eta^{AB}\,\mathrm{Tr}(CDEF\cdots) - \eta^{AC}\,\mathrm{Tr}(BDEF\cdots)\\
&\quad + \eta^{AD}\,\mathrm{Tr}(BCEF\cdots) - \eta^{AE}\,\mathrm{Tr}(BCDF\cdots) + \cdots, && (1.7)
\end{aligned}$$

$$\mathrm{Tr}(ABCDEF\cdots\gamma_5) = \textbf{Not Established.} \quad (1.8)$$

To prove the second equation, we use following technique:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{Tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu); \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots\gamma_5) = \mathrm{Tr}(\gamma_5\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu). \quad (1.9)$$

◆1.2.2 Contractions

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4 && (1.10)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu && (1.11)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} && (1.12)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu && (1.13)
\end{aligned}$$

Generally, for some γ-matrices $A, B, C, \ldots$,

$$\begin{aligned}
\text{ODD } \# : &\quad \gamma^\mu ABC\cdots\gamma_\mu = -2(\cdots CBA), && (1.14)\\
\text{EVEN } \# : &\quad \gamma^\mu ABC\cdots\gamma_\mu = \mathrm{Tr}(ABC\cdots) - \mathrm{Tr}(ABC\cdots\gamma_5)\cdot\gamma_5. && (1.15)
\end{aligned}$$

Contractions in d-dimension

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= d && (1.16)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(d-2)\gamma^\nu && (1.17)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho && (1.18)\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma && (1.19)
\end{aligned}$$

Contractions of ϵ's

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu; \quad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \quad (1.20)$$

$$\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\alpha'\beta'\gamma'} = -\left(\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\alpha_{\beta'}\delta^\beta_{\gamma'}\delta^\gamma_{\alpha'} + \delta^\alpha_{\gamma'}\delta^\beta_{\alpha'}\delta^\gamma_{\beta'} - \delta^\alpha_{\alpha'}\delta^\beta_{\gamma'}\delta^\gamma_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\delta^\gamma_{\gamma'} - \delta^\alpha_{\gamma'}\delta^\beta_{\beta'}\delta^\gamma_{\alpha'}\right) \quad (1.21)$$

1.3 Miscellaneous Techniques

$(p\cdot\sigma)(p\cdot\bar\sigma) = p^2$

◆1.3.1 Dirac Field Techniques

Dirac Equations	:	$(\not{p}-m)u^s(p)=0;\quad (\not{p}+m)v^s(p)=0$
	:	$\bar{u}^s(p)(\not{p}-m)=0;\quad \bar{v}^s(p)(\not{p}+m)=0$
Dirac Components	:	$u^{r\dagger}(p)u^s(p)=2E_{\boldsymbol{p}}\delta^{rs};\quad v^{r\dagger}(p)v^s(p)=2E_{\boldsymbol{p}}\delta^{rs}$
	:	$\bar{u}^r(p)u^s(p)=2m\delta^{rs};\quad \bar{v}^r(p)v^s(p)=-2m\delta^{rs};\quad \bar{u}^r(p)v^s(p)=\bar{v}^r(p)u^s(p)=0$
Spin Sums	:	$\sum_{\text{spin}} u^s(p)\bar{u}^s(p)=\not{p}+m;\quad \sum_{\text{spin}} v^s(p)\bar{v}^s(p)=\not{p}-m$

◆1.3.2 Polarization Sum

Single photon case $\quad M=\epsilon^*_\mu(k)M^\mu$

When Ward identity $k_\mu M^\mu=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu*}=\eta_{\mu\nu}M^\mu M^{\nu*}. \tag{1.22}$$

Double photons case $\quad M=\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu}=k'_\nu M^{\mu\nu}=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\rho(k)\epsilon'^*_\nu(k')\epsilon'_\sigma(k')M^{\mu\nu}M^{\rho\sigma*}=\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \tag{1.23}$$

【See Sec. A.1 for verbose information.】

◆1.3.3 Fierz identities

For Dirac spinors a,b,c,d and their left-handed projections $a_\mathrm{L}:=P_\mathrm{L}a$ etc.,

$$(\bar{a}_\mathrm{L}\gamma^\mu b_\mathrm{L})(\bar{c}_\mathrm{L}\gamma_\mu d_\mathrm{L})=-(\bar{a}_\mathrm{L}\gamma^\mu d_\mathrm{L})(\bar{c}_\mathrm{L}\gamma_\mu b_\mathrm{L}) \tag{1.24}$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta};\qquad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{1.25}$$

◆1.3.4 Gordon identity

For $P:=p'+p$ and $q:=p'-p$,

$$\bar{u}(p')\gamma^\mu u(p)=\bar{u}(p')\left[\frac{P^\mu+\mathrm{i}\,\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\qquad \bar{u}(p')\gamma^\mu v(p)=\bar{u}(p')\left[\frac{q^\mu+\mathrm{i}\,\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \tag{1.26}$$

$$\bar{v}(p')\gamma^\mu v(p)=-\bar{v}(p')\left[\frac{P^\mu+\mathrm{i}\,\sigma^{\mu\nu}q_\nu}{2m}\right]v(p)\qquad \bar{v}(p')\gamma^\mu u(p)=-\bar{v}(p')\left[\frac{q^\mu+\mathrm{i}\,\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \tag{1.27}$$

◆1.3.5 Gauge group algebra

For a gauge group G s.t.

$$[t^a,t^b]=\mathrm{i}\,f^{abc}t^c, \tag{1.28}$$

we have two constants which **depend on representation** r.

$$\mathrm{Tr}(t^at^b)=:C(r)\delta^{ab};\qquad \mathrm{Tr}(t^at^a)=:C_2(r)\cdot\mathbf{1}\quad\text{(quadratic Casimir operator)} \tag{1.29}$$

They satisfy

$$C(r) = \frac{d(r)}{d(\mathrm{Adj.})} C_2(r), \qquad t^a t^b t^a = \left[C_2(r) - \frac{1}{2} C_2(\mathrm{Adj.}) \right] t^b. \tag{1.30}$$

$$f^{acd} f^{bcd} = C_2(\mathrm{Adj.}) \delta^{ab}, \qquad f^{abc} t^b t^c = \frac{1}{2} \mathrm{i}\, C_2(\mathrm{Adj.}) t^a. \tag{1.31}$$

<u>For SU(N)</u> For SU(N) groups and its fundamental representation N, we have

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2-1}{2N}, \quad C(\mathrm{Adj.}) = C_2(\mathrm{Adj.}) = N; \quad (t^a)_{ij} (t^a)_{kj} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{\delta_{ij} \delta_{kl}}{N} \right).$$

1.4 Loop Integrals and Dimensional Regularization

◆1.4.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots x_n \; \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{1.32}$$

$$\frac{1}{A_1 A_2} = \int_0^1 \mathrm{d}x \frac{1}{[x A_1 + (1-x) A_2]^2} \tag{1.33}$$

◆1.4.2 d-dimensional integrals in Minkowski space

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n \,\mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{1.34}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \,\mathrm{i}}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.35}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \,\mathrm{i}}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.36}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2-\Delta)^n} = \frac{(-1)^n \,\mathrm{i}}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{1.37}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-\Delta)^n} = \frac{(-1)^n \,\mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{4} \tag{1.38}$$

Here we can use following expansions: $\quad (\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - (d-4)\frac{\log\Delta}{2} + \mathrm{O}\left((d-4)^2\right) \quad \text{around } d=4, \tag{1.39}$$

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathrm{O}(x) \quad \text{around } x=0, \tag{1.40}$$

$$\Gamma(x) = \frac{(-1)^n}{n!}\left[\frac{1}{x+n} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + \mathrm{O}(x+n)\right] \quad \text{around } x=-n. \tag{1.41}$$

and we get following expansion:

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left[\left(\frac{2}{4-d} - \gamma + \log 4\pi\right) - \log\Delta + \mathrm{O}(4-d)\right]. \tag{1.42}$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S-matrix elements. To compute the S-matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \to \left(\frac{1}{\Delta - \mathrm{i}\,\epsilon}\right)^{n-\frac{d}{2}}. \tag{1.43}$$

1.5 Cross Sections and Decay Rates

■General expression

$$\mathrm{d}\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(p_A, p_B \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \{p_f\}\right) \tag{1.44}$$

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(m_A \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(m_A - \{p_f\}\right) \quad \text{(in A-rest frame.)} \tag{1.45}$$

■2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4 \delta^{(4)}\left(E_{\mathrm{cm}} - (p_1 + p_2)\right) \qquad \text{(in center-of-mass frame)} \tag{1.46}$$

$$= \int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}\frac{2\left\|\boldsymbol{p_1}\right\|}{E_{\mathrm{cm}}} \tag{1.47}$$

$$= \frac{1}{8\pi}\sqrt{1 - \frac{2({m_1}^2 + {m_2}^2)}{E_{\mathrm{cm}}^2} + \frac{({m_1}^2 - {m_2}^2)^2}{E_{\mathrm{cm}}^4}} \quad \xrightarrow{m_2=0} \quad \frac{1}{8\pi}\left(1 - \frac{{m_1}^2}{{E_{\mathrm{cm}}}^2}\right) \tag{1.48}$$

- Fierz Transf.
- Noether current
- Majorana Ferminos
- Feynman Rules(A.1)

2 Standard Model

■Gauge Fields We use following notations for the Gauge Group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$:

$$\begin{aligned}\mathrm{SU}(3):&\quad G_\mu=G^a_\mu\tau^a\ ;\quad \left[\tau^a,\tau^b\right]=\mathrm{i}\,f^{abc}\tau^c,\quad \mathrm{Tr}\left(\tau^a\tau^b\right)=\frac{1}{2}\delta^{ab},\\ \mathrm{SU}(2):&\quad W_\mu=W^a_\mu T^a\ ;\quad \left[T^a,T^b\right]=\mathrm{i}\,\epsilon^{abc}T^c,\quad \mathrm{Tr}\left(T^aT^b\right)=\frac{1}{2}\delta^{ab},\\ \mathrm{U}(1):&\quad B_\mu.\end{aligned}$$

We will denotes projection operators $P_{\mathrm{L}}^{\mathrm{R}}:=\dfrac{1\pm\gamma_5}{2}$ explicitly in following expressions.

2.1 Full Lagrangian

The Lagrangian of the Standard Model, which conserves $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge symmetry, is

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}B^{\mu\nu}B_{\mu\nu}-\frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu}-\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu}\\ &+\left|\left(\partial_\mu-\mathrm{i}\,g_2W_\mu-\frac{1}{2}\mathrm{i}\,g_1B_\mu\right)\phi\right|^2-V(\phi)\\ &+\bar{Q}\,\mathrm{i}\,\gamma^\mu\left(\partial_\mu-\mathrm{i}\,g_3G_\mu-\mathrm{i}\,g_2W_\mu-\frac{1}{6}\mathrm{i}\,g_1B_\mu\right)P_{\mathrm{L}}Q\\ &+\bar{U}\,\mathrm{i}\,\gamma^\mu\left(\partial_\mu-\mathrm{i}\,g_3G_\mu-\frac{2}{3}\mathrm{i}\,g_1B_\mu\right)P_{\mathrm{R}}U\\ &+\bar{D}\,\mathrm{i}\,\gamma^\mu\left(\partial_\mu-\mathrm{i}\,g_3G_\mu+\frac{1}{3}\mathrm{i}\,g_1B_\mu\right)P_{\mathrm{R}}D\\ &+\bar{L}\,\mathrm{i}\,\gamma^\mu\left(\partial_\mu-\mathrm{i}\,g_2W_\mu+\frac{1}{2}\mathrm{i}\,g_1B_\mu\right)P_{\mathrm{L}}L\\ &+\bar{E}\,\mathrm{i}\,\gamma^\mu\left(\partial_\mu+\mathrm{i}\,g_1B_\mu\right)P_{\mathrm{R}}E\\ &-\bar{U}y_uHP_{\mathrm{L}}Q+\bar{D}y_dH^\dagger P_{\mathrm{L}}Q+\bar{E}y_eH^\dagger P_{\mathrm{L}}L+\mathrm{H.\,c.}.\end{aligned}\tag{2.1}$$

(Note that $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-\mathrm{i}\,g[A_\mu,A_\nu]=\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\right)T^a$.)

> **<u>これ以外の項が加わらない理由</u>**
>
> 次元勘定だけからは，これ以外に $\bar\psi\gamma_5\psi$，$\bar\psi\gamma_5\not{D}\psi$，$\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$，$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma}$ が付け加わりそうに見える。しかし，
>
> $$\begin{aligned}\left(\bar\psi\gamma_5\psi\right)^*&=-\bar\psi\gamma_5\psi\\ \left(\bar\psi\gamma_5\not{D}\psi\right)^*&=-\bar\psi\gamma_5\not{D}\psi\end{aligned}$$
>
> であるのでこの 2 つの項は Lagrangian には寄与せず，また
>
> $$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma}=\epsilon^{\mu\nu\rho\sigma}\frac{1}{2}[D_\mu,D_\nu]F^a_{\rho\sigma}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$
>
> である。ところが ϵFF の項は $\psi_{\mathrm{R}}\mapsto\mathrm{e}^{\mathrm{i}\,\alpha(x)}\psi_{\mathrm{R}}$ の変換によって打ち消すことができる。これは場の再定義なのか，U(1) gauge 固定なのか。**<u>TODO:</u> よくわかっていない。**

■Gauge Kinetic Terms

Using an abridged notation

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.2}$$

the gauge kinetic terms is

$$\mathcal{L}_{B;\mathrm{kin}} = -\frac{1}{4}(\partial B)(\partial B) \tag{2.3}$$

$$\mathcal{L}_{W;\mathrm{kin}} = -\frac{1}{4}(\partial W^a)(\partial W^a) - g_2\epsilon^{abc}(\partial_\mu W^a_\nu)W^{\mu b}W^{\nu c} - \frac{{g_2}^2}{4}\left(\epsilon^{eab}W^a_\mu W^b_\nu\right)\left(\epsilon^{ecd}W^{c\mu}W^{d\nu}\right) \tag{2.4}$$

$$\mathcal{L}_{G;\mathrm{kin}} = -\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G^a_\nu)G^{\mu b}G^{\nu c} - \frac{{g_3}^2}{4}\left(f^{eab}G^a_\mu G^b_\nu\right)\left(f^{ecd}G^{c\mu}G^{d\nu}\right). \tag{2.5}$$

2.2 Higgs Mechanism

Higgs 場を，真空期待値が $\langle\phi\rangle = \frac{1}{\sqrt{2}}\binom{0}{v}$ $(v \in \mathbb{R})$ となるように再定義する。このとき Higgs 場の 4 自由度のうち 3 つは南部-Goldstone 粒子となり，Higgs 場は

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \qquad (h(x) \text{ is a real scalar field.}) \tag{2.6}$$

と書ける。Lagrangian は

$$\begin{aligned}\mathcal{L}_{\mathrm{Higgs}} =& \left|\left(\partial_\mu - \mathrm{i}\, g_2 W_\mu - \frac{1}{2}\mathrm{i}\, g_1 B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}\right|^2 \\ =& \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\left[{g_2}^2{W_1}^2 + {g_2}^2{W_2}^2 + (g_1 B - g_2 W_3)^2\right]\end{aligned} \tag{2.7}$$

となる。更に場の norm に注意しながら gauge 場を再定義する。

$$W^\pm_\mu := \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}\, W^2_\mu), \quad Z^0_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_2 W^3_\mu - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_1 W^3_\mu + g_2 B_\mu) \tag{2.8}$$

その結果

$$\mathcal{L}_{\mathrm{Higgs}} = \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\left[{g_2}^2 W^+ W^- + {g_2}^2 W^- W^+ + ({g_1}^2 + {g_2}^2)(Z^0)^2\right], \tag{2.9}$$

$$g_1 B_\mu = |e|A - \frac{|e|s}{c}Z, \tag{2.10}$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) + \left(\frac{|e|c}{s}Z^0_\mu + |e|A_\mu\right)T^3 \tag{2.11}$$

となる。ここで $T^\pm = T^1 \pm \mathrm{i}\, T^2$ であり，また Weinberg 角 θ_w と素電荷 e を導入した：

$$|e| := \frac{g_1 g_2}{\sqrt{{g_1}^2 + {g_2}^2}}; \qquad s := \sin\theta_\mathrm{w}, \quad c := \cos\theta_\mathrm{w}; \qquad g_1 = \frac{|e|}{c}, \quad g_2 = \frac{|e|}{s}. \tag{2.12}$$

■Higgs potential

$\mathrm{SU}(2)_\mathrm{L}$ restricts Higgs term as

$$V(\phi) = -\mu^2(\phi^\dagger\phi) + \lambda|\phi^\dagger\phi|^2. \tag{2.13}$$

Therefore, using the VEV relation $v^2 = \mu^2/\lambda$, the potential after Higgs gets VEV is

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2 h^2, \tag{2.14}$$

and Higgs mass is given by $m_h = \sqrt{2}\mu$.

■Gauge 項

Gauge 項は

$$
\begin{aligned}
\mathcal{L}_{\mathrm{gauge}} =& -\frac{1}{4}\left[(\partial W^3)(\partial W^3)+(\partial B)(\partial B)+2(\partial W^+)(\partial W^-)+G^{a\mu\nu}G^a_{\mu\nu}\right] \\
&+\frac{\mathrm{i}\,g_2}{2}\Big[(W^+W^-)(\partial W^3)+(W^3W^+)(\partial W^-)+(W^-W^3)(\partial W^+)\Big] \\
&+\frac{{g_2}^2}{4}\Big[(W^+W^-)(W^+W^-)+2(W^3W^+)(W^-W^3)\Big] \\
=& -\frac{1}{4}\left[(\partial Z^0)(\partial Z^0)+(\partial A)(\partial A)+2(\partial W^+)(\partial W^-)+G^{a\mu\nu}G^a_{\mu\nu}\right] \\
&+\frac{\mathrm{i}\,|e|c}{2s}\Big[W^+W^-(\partial Z^0)+Z^0W^+(\partial W^-)+W^-Z^0(\partial W^+)\Big] \\
&+\frac{\mathrm{i}\,|e|}{2}\Big[W^+W^-(\partial A)+AW^+(\partial W^-)+W^-A(\partial W^+)\Big] \\
&+\frac{|e|^2}{4s^2}W^+W^+W^-W^-+\frac{|e|^2c^2}{2s^2}W^+W^-Z^0Z^0 \\
&+\frac{|e|^2c}{s}W^+W^-Z^0A+\frac{|e|^2}{2}W^+W^-AA
\end{aligned}
\tag{2.15}
$$

となる。

■湯川項

湯川項は，SU(2) の脚を露わに書くと

$$
\begin{aligned}
\mathcal{L}_{\text{湯川}} =& -\bar{U}y_uHP_{\mathrm{L}}Q+\bar{D}y_dH^\dagger P_{\mathrm{L}}Q+\bar{E}y_eH^\dagger P_{\mathrm{L}}L+\mathrm{H.\ c.} \\
=& -\bar{U}y_u\epsilon^{\alpha\beta}H^\alpha P_{\mathrm{L}}Q^\beta+\bar{D}y_d{H^\dagger}^\alpha P_{\mathrm{L}}Q^\alpha+\bar{E}y_e{H^\dagger}^\alpha P_{\mathrm{L}}L^\alpha+\mathrm{H.\ c.} \\
=& \frac{v+h}{\sqrt{2}}\left(\bar{U}y_uP_{\mathrm{L}}Q^1+\bar{D}y_dP_{\mathrm{L}}Q^2+\bar{E}y_eP_{\mathrm{L}}L^2\right)+\mathrm{H.\ c.}
\end{aligned}
\tag{2.16}
$$

となる。

2.3 Full Lagrangian After Higgs Mechanism

ここでは簡単のため P_L などを省略する。全 Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_\text{gauge} \\
& + \quad {m_W}^2 W^+W^- \quad + \quad \frac{{m_Z}^2}{2} Z^2 \\
\text{【Higgs Field】} \quad & + \quad (\partial_\mu h)^2 \quad -\frac{1}{2}\mu^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
& + \quad \frac{v{g_2}^2}{4} W^+W^- h \quad + \quad \frac{v({g_1}^2 + {g_2}^2)}{8} Z^2 h \\
& + \quad \frac{{g_2}^2}{4} W^+W^- h^2 \quad + \quad \frac{{g_1}^2 + {g_2}^2}{8} Z^2 h^2 \\
& + \quad \frac{1}{\sqrt{2}} h\bar{U} y_u Q^1 \quad + \quad \frac{1}{\sqrt{2}} h\bar{D} y_d Q^2 \quad + \quad \frac{1}{\sqrt{2}} h\bar{E} y_e L^2 \quad + \quad \text{H. c.} \\
\text{【SU(3) および微分項】} \quad & + \quad \bar{Q}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) Q \quad + \quad \bar{U}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) U \quad + \quad \bar{D}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right) D \\
& + \quad \bar{L}\left(\mathrm{i}\partial\!\!\!/\right) L \quad + \quad \bar{E}\left(\mathrm{i}\partial\!\!\!/\right) E \\
\text{【W boson】} \quad & + \quad \bar{Q}\frac{g_2}{\sqrt{2}}\left(W\!\!\!\!/^{\,+} T^+ + W\!\!\!\!/^{\,-} T^-\right) Q \quad + \quad \bar{L}\frac{g_2}{\sqrt{2}}\left(W\!\!\!\!/^{\,+} T^+ + W\!\!\!\!/^{\,-} T^-\right) L \\
\text{【A\&Z^0 boson】} \quad & + \quad \bar{Q}\left[\left(T^3 + \frac{1}{6}\right)|e| A\!\!\!/ + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right) Z\!\!\!/^0\right] Q \\
& + \quad \bar{U}\left(\frac{2}{3}|e| A\!\!\!/ - \frac{2|e|s}{3c} Z\!\!\!/\right) U \\
& + \quad \bar{D}\left(-\frac{1}{3}|e| A\!\!\!/ + \frac{|e|s}{3c} Z\!\!\!/\right) D \\
& + \quad \bar{L}\left[\left(T^3 - \frac{1}{2}\right)|e| A\!\!\!/ + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right) Z\!\!\!/^0\right] L \\
& + \quad \bar{E}\left(-|e| A\!\!\!/ + \frac{|e|s}{c} Z\!\!\!/\right) E \\
\text{【湯川項】} \quad & + \quad \frac{1}{\sqrt{2}} v\bar{U} y_u Q^1 \quad + \quad \frac{1}{\sqrt{2}} v\bar{D} y_d Q^2 \quad + \quad \frac{1}{\sqrt{2}} v\bar{E} y_e L^2 \quad + \quad \text{H. c.}
\end{aligned}
\tag{2.17}
$$

である。ただしここで

$$
m_W := \frac{g_2 v}{2}, \quad m_Z := \frac{\sqrt{{g_1}^2 + {g_2}^2}}{2} v \tag{2.18}
$$

を導入した。

2.4 Mass Eigenstates

湯川行列を対角化し，質量の固有状態を得ることを考える。

湯川行列 $Y := vy/\sqrt{2}$ に対して特異値分解[*1]を行う。即ち，2 つの unitary 行列 Ψ, Φ および $m_i \geq 0$ によって

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \tag{2.19}
$$

[*1] $A^\dagger A$ および $AA^\dagger$ は Hermite 行列であるため，unitary 行列により対角化可能であり，固有値は全て非負である。

と展開する。これを用いて

$$Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \qquad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E \tag{2.20}$$

と置き換えてやることによって，湯川項が対角化できる：

$$\mathcal{L}_{\text{湯川}} = \bar{U} M_u Q^1 + D M_d Q^2 + E M_e L^2 + \mathrm{H.\,c.} \tag{2.21}$$

しかし，Q^1 と Q^2 を別の方法で変換したため，W boson との結合の項に歪みが生じる：

$$\mathcal{L}_{QQW} = \frac{g_2}{\sqrt{2}} \begin{pmatrix} \bar{Q}^1 \Psi_u & \bar{Q}^2 \Psi_d \end{pmatrix} \begin{pmatrix} 0 & \slashed{W}^+ \\ \slashed{W}^- & 0 \end{pmatrix} \begin{pmatrix} \Psi_u^\dagger Q^1 \\ \Psi_d^\dagger Q^2 \end{pmatrix} \tag{2.22}$$

$$= \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 \slashed{W}^- X Q^1 + \bar{Q}^1 \slashed{W}^+ X^\dagger Q^2 \right] \tag{2.23}$$

ここで $X := \Psi_d \Psi_u^\dagger$ である。

この項は明らかに flavor violating であり，また CP violating でもある。

> CP 変換により，spinor は
>
> $$\psi \mapsto -\mathrm{i}\,\eta^* (\bar{\psi}\gamma^2)^{\mathrm{T}}, \quad \bar{\psi} \mapsto \mathrm{i}\,\eta (\gamma^2 \psi)^{\mathrm{T}} \tag{2.24}$$
>
> のように変換される。
>
> $\mathcal{L}_{QQW}$ 以外の項は，例えば
>
> $$\begin{aligned} \bar{Q}(\mathrm{i}\,\slashed{\partial}) P_{\mathrm{L}} Q \mapsto & (\gamma^2 Q)^{\mathrm{T}} (\mathrm{i}\,\slashed{\partial}^P) P_{\mathrm{L}} (\bar{Q}\gamma^2)^{\mathrm{T}} \\ = & \ \mathrm{i}\,(\gamma^2 Q)^{\mathrm{T}} (\partial_\mu^P \bar{Q} \gamma^2 P_{\mathrm{L}} \gamma^{\mu\mathrm{T}})^{\mathrm{T}} \\ = & -\mathrm{i}\,(\partial_\mu^P \bar{Q} \gamma^2 P_{\mathrm{L}} \gamma^{\mu\mathrm{T}} \gamma^2 Q) \\ = & \ \mathrm{i}\,(\bar{Q}\gamma^2 P_{\mathrm{L}} (\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2)\gamma^2 \partial_\mu^P Q) \\ = & \ \mathrm{i}\,(\bar{Q}\gamma^\mu \partial_\mu P_{\mathrm{L}} Q) \end{aligned}$$
>
> のように CP 不変であるが，$\mathcal{L}_{QQW}$ の項は
>
> $$\begin{aligned} \bar{Q}^2 \slashed{W}^- X P_{\mathrm{L}} Q^1 \mapsto & (\gamma^2 Q^2)^{\mathrm{T}} (-\slashed{W}^{+P}) X P_{\mathrm{L}} (\bar{Q}^1 \gamma^2)^{\mathrm{T}} \\ = & -W_\mu^{+P} (\gamma^2 Q^2)^{\mathrm{T}} (\bar{Q}^1 X^{\mathrm{T}} \gamma^2 P_{\mathrm{L}} \gamma^{\mu\mathrm{T}})^{\mathrm{T}} \\ = & (\bar{Q}^1 X^{\mathrm{T}} \slashed{W}^+ P_{\mathrm{L}} Q^2) \end{aligned}$$
>
> のように変換する。CP 変換により結合定数の符号が変わるべきであることも踏まえると CP の保存は $X^{\mathrm{T}} = X^\dagger$ と同値である。即ち CP の保存は，X が実行列であることと同値である。

以上より，標準模型の Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} \\
\text{【質量項】} \quad & + \quad {m_W}^2 W^+ W^- \quad + \quad \frac{{m_Z}^2}{2} Z^2 \\
& + \quad \bar{U} M_u P_{\mathrm{L}} Q^1 \quad + \quad \bar{D} M_d P_{\mathrm{L}} Q^2 \quad + \quad \bar{E} M_e P_{\mathrm{L}} L^2 \quad + \quad \text{H. c.} \\
\text{【Higgs Field】} \quad & + \quad (\partial_\mu h)^2 \quad -\frac{1}{2}\mu^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
\text{【Higgs との結合】} \quad & + \quad \frac{v {g_2}^2}{4} W^+ W^- h \quad + \quad \frac{v({g_1}^2 + {g_2}^2)}{8} Z^2 h \\
& + \quad \frac{{g_2}^2}{4} W^+ W^- h^2 \quad + \quad \frac{{g_1}^2 + {g_2}^2}{8} Z^2 h^2 \\
& + \quad \frac{1}{v} \bar{U} M_u P_{\mathrm{L}} Q^1 h \quad + \quad \frac{1}{v} \bar{D} M_d P_{\mathrm{L}} Q^h 2 \quad + \quad \frac{1}{v} \bar{E} M_e P_{\mathrm{L}} L^2 h \quad + \quad \text{H. c.} \\
\text{【SU(3) および微分項】} \quad & + \quad \bar{Q} \left(\mathrm{i} \partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{L}} Q \quad + \quad \bar{U} \left(\mathrm{i} \partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{R}} U \quad + \quad \bar{D} \left(\mathrm{i} \partial\!\!\!/ + g_3 G\!\!\!/ \right) P_{\mathrm{R}} D \\
& + \quad \bar{L} \left(\mathrm{i} \partial\!\!\!/ \right) P_{\mathrm{L}} L \quad + \quad \bar{E} \left(\mathrm{i} \partial\!\!\!/ \right) P_{\mathrm{R}} E \\
\text{【W boson】} \quad & + \quad \frac{g_2}{\sqrt{2}} \left[\bar{Q}^2 W\!\!\!/^- X P_{\mathrm{L}} Q^1 + \bar{Q}^1 W\!\!\!/^+ X^\dagger P_{\mathrm{L}} Q^2 \right] \qquad \text{【$\leftarrow$ CP and flavor violating!】} \\
& + \quad \bar{L} \frac{g_2}{\sqrt{2}} \left(W\!\!\!/^+ T^+ + W\!\!\!/^- T^- \right) P_{\mathrm{L}} L \\
\text{【A\&Z^0 boson】} \quad & + \quad \bar{Q} \left[\left(T^3 + \frac{1}{6} \right) |e| A\!\!\!/ + \left(\frac{|e|c}{s} T^3 - \frac{|e|s}{6c} \right) Z\!\!\!/^0 \right] P_{\mathrm{L}} Q \\
& + \quad \bar{U} \left(\frac{2}{3} |e| A\!\!\!/ - \frac{2|e|s}{3c} Z\!\!\!/ \right) P_{\mathrm{R}} U \\
& + \quad \bar{D} \left(-\frac{1}{3} |e| A\!\!\!/ + \frac{|e|s}{3c} Z\!\!\!/ \right) P_{\mathrm{R}} D \\
& + \quad \bar{L} \left[\left(T^3 - \frac{1}{2} \right) |e| A\!\!\!/ + \left(\frac{|e|c}{s} T^3 + \frac{|e|s}{2c} \right) Z\!\!\!/^0 \right] P_{\mathrm{L}} L \\
& + \quad \bar{E} \left(-|e| A\!\!\!/ + \frac{|e|s}{c} Z\!\!\!/ \right) P_{\mathrm{R}} E
\end{aligned}
\tag{2.25}
$$

となる。

2.5 Chiral Notation

以上の Lagrangian を chiral 表示で表すと，まず最初は

$$
\begin{aligned}
\mathcal{L} =&(\text{Higgs terms}) + (\text{Gauge fields strength}) \\
&+ Q_{\mathrm{L}}^{\dagger}\,\mathrm{i}\,\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}\,g_3 G_{\mu} - \mathrm{i}\,g_2 W_{\mu} - \frac{1}{6}\,\mathrm{i}\,g_1 B_{\mu}\right) Q_{\mathrm{L}} \\
&+ U_{\mathrm{R}}^{\dagger}\,\mathrm{i}\,\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}\,g_3 G_{\mu} - \frac{2}{3}\,\mathrm{i}\,g_1 B_{\mu}\right) U_{\mathrm{R}} \\
&+ D_{\mathrm{R}}^{\dagger}\,\mathrm{i}\,\sigma^{\mu}\left(\partial_{\mu} - \mathrm{i}\,g_3 G_{\mu} + \frac{1}{3}\,\mathrm{i}\,g_1 B_{\mu}\right) D_{\mathrm{R}} \\
&+ L_{\mathrm{L}}^{\dagger}\,\mathrm{i}\,\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}\,g_2 W_{\mu} + \frac{1}{2}\,\mathrm{i}\,g_1 B_{\mu}\right) L_{\mathrm{L}} \\
&+ E_{\mathrm{R}}^{\dagger}\,\mathrm{i}\,\sigma^{\mu}\left(\partial_{\mu} + \mathrm{i}\,g_1 B_{\mu}\right) E_{\mathrm{R}} \\
&+ U_{\mathrm{R}}^{\dagger} y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger} y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger} y_e H^{\dagger} L_{\mathrm{L}} + \text{H. c.} \\
=&(\text{Higgs terms}) + (\text{Gauge fields strength}) \\
&+ \mathrm{i}\,Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}\,U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}\,D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}\,L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}\,E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger} \\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right) \\
&+ g_2\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}Q_{\mathrm{L}} + L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}L_{\mathrm{L}}\right) \\
&+ g_1\left(\frac{1}{6}Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}Q_{\mathrm{L}} + \frac{2}{3}U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}U_{\mathrm{R}} - \frac{1}{3}D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}D_{\mathrm{R}} - \frac{1}{2}L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}L_{\mathrm{L}} - E_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}B_{\mu}E_{\mathrm{R}}\right) \\
&+ U_{\mathrm{R}}^{\dagger} y_u H Q_{\mathrm{L}} + D_{\mathrm{R}}^{\dagger} y_d H^{\dagger} Q_{\mathrm{L}} + E_{\mathrm{R}}^{\dagger} y_e H^{\dagger} L_{\mathrm{L}} + \text{H. c.}
\end{aligned} \tag{2.26}
$$

であり，そして最終的には

$$
\begin{aligned}
\mathcal{L} =&(\text{Gauge bosons and Higgs}) \\
&+ \mathrm{i}\,Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}Q_{\mathrm{L}} + \mathrm{i}\,U_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}U_{\mathrm{R}}^{\dagger} + \mathrm{i}\,D_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}D_{\mathrm{R}}^{\dagger} + \mathrm{i}\,L_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}L_{\mathrm{L}} + \mathrm{i}\,E_{\mathrm{R}}\bar{\sigma}^{\mu}\partial_{\mu}E_{\mathrm{R}}^{\dagger} \\
&+ g_3\left(Q_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}Q_{\mathrm{L}} + U_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}U_{\mathrm{R}} + D_{\mathrm{R}}^{\dagger}\bar{\sigma}^{\mu}G_{\mu}D_{\mathrm{R}}\right) \\
&+ m_u(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}}) + (\text{quarks}) + m_e(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}}) + (\text{leptons}) \\
&+ \frac{m_u}{v}(u_{\mathrm{R}}^{\dagger}u_{\mathrm{L}} + u_{\mathrm{L}}^{\dagger}u_{\mathrm{R}})h + (\text{quarks}) + \frac{m_e}{v}(e_{\mathrm{R}}^{\dagger}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}e_{\mathrm{R}})h + (\text{leptons}) \\
&+ \frac{g_2}{\sqrt{2}}\left[\begin{pmatrix} d_{\mathrm{L}}^{\dagger} & s_{\mathrm{L}}^{\dagger} & b_{\mathrm{L}}^{\dagger}\end{pmatrix}\bar{\sigma}^{\mu}W_{\mu}^{-}X\begin{pmatrix} u_{\mathrm{L}} \\ c_{\mathrm{L}} \\ t_{\mathrm{L}}\end{pmatrix} + \begin{pmatrix} u_{\mathrm{L}}^{\dagger} & c_{\mathrm{L}}^{\dagger} & t_{\mathrm{L}}^{\dagger}\end{pmatrix}\bar{\sigma}^{\mu}W_{\mu}^{+}X^{\dagger}\begin{pmatrix} d_{\mathrm{L}} \\ s_{\mathrm{L}} \\ b_{\mathrm{L}}\end{pmatrix}\right] \\
&+ \frac{g_2}{\sqrt{2}}\left[\nu_e^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{+}e_{\mathrm{L}} + e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}W_{\mu}^{-}\nu_e\right] \\
&+ |e|\left[\frac{2}{3}u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}u_{\mathrm{L}} - \frac{1}{3}d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}d_{\mathrm{L}} + \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}u_{\mathrm{R}} - \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}d_{\mathrm{R}} + (\text{quarks})\right. \\
&\qquad\left. - e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}A_{\mu}e_{\mathrm{L}} - e_{\mathrm{R}}^{\dagger}\sigma^{\mu}A_{\mu}e_{\mathrm{R}} + (\text{leptons})\right] \\
&+ \frac{|e|s}{c}\left[\left(\frac{c^2}{2s^2} - \frac{1}{6}\right)u_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}u_{\mathrm{L}} - \left(\frac{c^2}{2s^2} + \frac{1}{6}\right)d_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}d_{\mathrm{L}} - \frac{2}{3}u_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}u_{\mathrm{R}} + \frac{1}{3}d_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}d_{\mathrm{R}}\right. \\
&\qquad\left. + \left(\frac{c^2}{2s^2} + \frac{1}{2}\right)\nu_e^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2}\right)e_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}Z_{\mu}e_{\mathrm{L}} + e_{\mathrm{R}}^{\dagger}\sigma^{\mu}Z_{\mu}e_{\mathrm{R}} + (\text{others})\right]
\end{aligned} \tag{2.27}
$$

となる。

2.6 Values of SM Parameters

◆2.6.1 Experimental Values

Low energy values

$$\alpha_{\text{EM}} = 1/137.035999679(94) \qquad G_{\text{F}} = 1.166367(5) \times 10^{-5}\,\text{GeV}^{-2}$$

Electroweak scale 【These values are all in $\overline{\text{MS}}$ scheme.】

$$\begin{aligned} \alpha_{\text{EM}}^{-1}(m_Z) &= 127.925(16) & m_W(m_W) &= 80.398(25)\,\text{GeV} \\ \alpha_{\text{EM}}^{-1}(m_\tau) &= 133.452(16) & m_Z(m_Z) &= 91.1876(21)\,\text{GeV} \\ \alpha_{\text{s}}(m_Z) &= 0.1176(20) & \sin^2\theta_{\text{W}}(m_Z) &= 0.23119(14) \\ \Gamma_{l^+l^-} &= 83.984(86)\,\text{MeV} & \sin^2\theta_{\text{eff}} &= 0.23149(13) \end{aligned}$$

Fundamental masses

$$\begin{array}{lll} e : 0.510998910(13)\,\text{MeV} & u : 1.5 \text{ to } 3.3\,\text{MeV} & d : 3.5 \text{ to } 6.0\,\text{MeV} \\ \mu : 105.658367(4)\,\text{MeV} & c : 1.27^{+0.07}_{-0.11}\,\text{GeV} & s : 104^{+26}_{-34}\,\text{MeV} \\ \tau : 1.77784(17)\,\text{GeV} & t : 171.2_{\pm 2.1}\,\text{GeV} & b : 4.20^{+0.17}_{-0.07}\,\text{GeV} \end{array}$$

$$\begin{array}{lll} \pi^\pm : 139.57018(35)\,\text{MeV} & K^\pm : 493.677(16)\,\text{MeV} & p : 938.27203(8)\,\text{MeV} \\ \pi^0 : 139.766(6)\,\text{MeV} & K^0 : 497.614(24)\,\text{MeV} & n : 939.56536(8)\,\text{MeV} \end{array}$$

Fundamental Lifetime (also $c\tau$ for some particles)

$$\begin{array}{lll} \mu : 2.197019(21)\,\mu\text{s} \quad (658\,\text{m}) & \pi^\pm : 2.6033(5) \times 10^{-8}\,\text{s} & K^\pm : 1.2380(21) \times 10^{-8}\,\text{s} \\ \tau : 2.906(10) \times 10^{-13}\,\text{s} \quad (87\,\mu\text{m}) & \pi^0 : 8.4(6) \times 10^{-17}\,\text{s} & K^0_{\text{S}} : 8.953(5) \times 10^{-11}\,\text{s} \\ & & K^0_{\text{L}} : 5.116(20) \times 10^{-8}\,\text{s} \end{array}$$

CKM matrix

$$V_{\text{CKM}} = \begin{pmatrix} 0.97419(22) & 0.2257(10) & 0.00359(16) \\ 0.2256(10) & 0.97334(23) & 0.0415(11) \\ 0.00874(37) & 0.0407(10) & 0.999133(44) \end{pmatrix} \sim \begin{pmatrix} 1-\epsilon^2 & \epsilon & \epsilon^4 \\ \epsilon & 1-\epsilon^2 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1-\epsilon^4 \end{pmatrix} \quad \text{for } \epsilon \sim 0.23 \tag{2.28}$$

◆2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$e \sim 0.313, \qquad g_1 \sim 0.358, \qquad g_2 \sim 0.651; \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246\,\text{GeV} \tag{2.29}$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.98 \end{pmatrix}, \quad y_d \sim \begin{pmatrix} 3\times10^{-5} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}, \quad y_e \sim \begin{pmatrix} 3\times10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \tag{2.30}$$

Also, for $m_h \sim 120\,\text{GeV}$, we can estimate the Higgs potential as $\mu \sim 85\,\text{GeV}$ and $\lambda \sim 0.12$.

3 楊-Mills Theory

3.1 U(1) Theory

◆3.1.1 General SU(()N)

$$\mathcal{L} = \tag{3.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{\mathrm{i}}[A_\mu, A_\nu] \tag{3.2}$$

$$\tag{3.3}$$

■Gauge Transformation

For any Lie group G,

$$V : \mathbb{R}^{1,3} \to G \tag{3.4}$$

$$A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1} \tag{3.5}$$

$$F_{\mu\nu} \mapsto V F_{\mu\nu} V^{-1} \tag{3.6}$$

If the gauge group G is **compact**, it has a finite-dimensional unitary representation.

$$\text{For} \quad t^a : \text{hermitian representation}, \tag{3.7}$$

$$[t^a, t^b] = \mathrm{i}\, f^{ab}{}_c t^c \quad \text{and} \quad f \in \mathbb{R} \tag{3.8}$$

$$0 = f^D{}_{ab} f^E{}_{Dc} + f^D{}_{ca} f^E{}_{Db} + f^D{}_{bc} f^E{}_{Da} \tag{3.9}$$

$$V = \exp\left[\mathrm{i}\,\alpha^a T^a\right] \quad \text{for} \quad \alpha^a \in \mathbb{R} \tag{3.10}$$

$$\tag{3.11}$$

with generators $\{T^a\}$ written in an hermitian representation,

$$A_\mu \mapsto V\left(A_\mu + \frac{1}{g}(\partial_\mu \alpha^a)T^a\right)V^{-1} \tag{3.12}$$

$$\simeq A_\mu + \frac{1}{g}(\partial_\mu \alpha^a)T^a + i\alpha^a A^b_\mu [T^a, T^b] \tag{3.13}$$

$$D_\mu = \partial_\mu - \mathrm{i}\, g A^a_\mu T^a \qquad \text{(for appropriate representation)} \tag{3.14}$$

4 Spinor

■$\eta^{\mu\nu} = (-,+,+,+)$ case

Grassmann Number	: $(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$
	: $\Longrightarrow$ for $a, b \in \mathbb{G}^{\mathbb{R}}$, $ab \in \mathrm{i}\,\mathbb{G}^{\mathbb{R}}$
γ matrix	: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$
	: $\gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$ etc...
	: $(\mathrm{i}\,\gamma^0)^\dagger := \mathrm{i}\,\gamma^0, \quad {\gamma^i}^\dagger := \gamma^i$
Dirac Conjugate	: $\bar{\psi} = \mathrm{i}\,\psi^\dagger\gamma^0$

5 Mathematics

5.1 Group Theory

◆5.1.1 Lie Group and Lie Algebra

- G が **group** である … 積が定義されており，積閉・単位元・逆元の 3 条件を満たす。
- 群 G が **Lie group** である … G が同時に C^∞ 多様体であり，積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が **COMPLEX Lie group** である … 積演算と逆元写像が共に正則写像である。

- Vector 空間 g が **Lie algebra** である … 括弧積が定義されており，線型性・反対称性・Jacobi 恒等式を満たす。
- Lie 群 G の単位元における接空間を，G **の Lie algebra** $\mathfrak{g}$ という。
  - $\mathfrak{g}$ は G の左不変な vector 場全体である。
  - $\mathfrak{g}$ は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると，
  - その Lie 代数の基底 B_i に対して **structure constant** c が $[B_i, B_j] = c_{ij}^k B_k$ として定義できる。

* * *

- Compact Lie 群は線型 Lie 群である。
- G として **Linear group** $\mathrm{GL}(n;\mathbb{R})$ を考えると，
  - その Lie 代数は n 次実正方行列全体となる。
  - Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は，$\mathrm{GL}(n;\mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では，Lie 群として $\mathrm{GL}(n;\mathbb{R})$ の部分群を考えることにし，Lie 代数の元を行列により表現する。

◆5.1.2 Matrix Representation

- Lie 群 G の Lie 代数の基底の組を，G の **generators** と言う。
- $\mathrm{GL}(n;\mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し，以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1 + \cdots + x_m T_m} \qquad\qquad (x_1, \cdots, x_m) \mapsto \mathrm{e}^{x_1 T_1} \cdots \mathrm{e}^{x_m T_m} \tag{5.1}$$

- Lie 群 G が compact である …
  1. 多様体 G が compact である。 **<u>TODO</u>: これは何故同値なのか？**
  2. G の生成子 $\{T_i\}$ を，$\mathrm{Tr}(T_i T_j) = k\,\delta_{ij}$ かつ $k > 0$ となるように取り替えることができる。
     【この基底の下では構造定数が完全反対称になる。】
- Compact 群 G は，**unitary representation** を持つ。
  故に，単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i \in \mathbb{R}$ により，G の元を

$$\mathrm{e}^{\,\mathrm{i}\,x^i T^i} \tag{5.2}$$

  と表すことが出来る。

◆5.1.3　結論

Compact Lie 群の元のうち，単位元近傍にあるもの V は,

Hermitian Representation

$$V = \exp(\mathrm{i}\, x^i T^i) \qquad \text{where} \quad T^i : \text{Hermitian Matrix}, \quad x^i \in \mathbb{R},$$
$$[T^i, T^j] = \mathrm{i}\, f^{ijk} T^k, \quad \mathrm{Tr}(T^i T^j) = \lambda\, \delta^{ij} > 0; \qquad f \in \mathbb{R}$$

Real Representation

$$V = \exp(x^i R^i) \qquad \text{where} \quad R^i : \text{Real Matrix}, \quad x^i \in \mathbb{R},$$
$$[R^i, R^j] = -f^{ijk} R^k, \quad \mathrm{Tr}(R^i R^j) = -\lambda\, \delta^{ij} < 0; \qquad f \in \mathbb{R}$$

と表すことが出来る。

6 Supersymmetry

6.1 General Relations

■Coordinates

$$y := x + \mathrm{i}\,\theta\sigma\bar{\theta}, \qquad\qquad y^{+} := x - \mathrm{i}\,\theta\sigma\bar{\theta} \tag{6.1}$$

6.2 Chiral Superfields

■Definition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{6.2}$$

■Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{6.3}$$

$$= \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \theta\theta F(x) \tag{6.4}$$

$$\Phi^{\dagger} = \phi^{*}(x) - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi^{*}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi^{*}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{\mathrm{i}}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^{\mu}\partial_{\mu}\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^{*}(x) \tag{6.5}$$

■Changing Bases

$$\phi(y) = \phi(x) + \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(x) \tag{6.6}$$

$$= \phi(y^{+}) + 2\,\mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(y^{+}) + \theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(y^{+}) \tag{6.7}$$

$$\phi(y^{+}) = \phi(x) - \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(x) \tag{6.8}$$

$$= \phi(y) - 2\,\mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(y) + \theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(y) \tag{6.9}$$

$$\phi(x) = \phi(y) - \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(y) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(y) \tag{6.10}$$

$$= \phi(y^{+}) + \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(y^{+}) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^{2}\phi(y^{+}) \tag{6.11}$$

■Product of Chiral Superfields

$$\begin{aligned}
\Phi_i^{\dagger}\Phi_j(\text{in } x\text{-basis}) = {} & \phi_i^{*}\phi_j + \sqrt{2}\phi_i^{*}\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^{*}F_j + \bar{\theta}\bar{\theta}F_i^{*}\phi_j \\
& + \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\left(\phi_i^{*}\partial_{\mu}\phi_j - \partial_{\mu}\phi_i^{*}\phi_j\right) + 2\bar{\theta}\bar{\psi}_i\theta\psi_j \\
& - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\left(\phi_i^{*}\partial_{\mu}\psi_j - \partial_{\mu}\phi_i^{*}\psi_j\right)\sigma^{\mu}\bar{\theta} + \sqrt{2}\theta\theta\bar{\theta}\bar{\psi}_iF_j \\
& + \frac{\mathrm{i}}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^{\mu}\left(\partial_{\mu}\bar{\psi}_i\phi_j - \bar{\psi}_i\partial_{\mu}\phi_j\right) + \sqrt{2}\bar{\theta}\bar{\theta}F_i^{*}\theta\psi_j \\
& + \theta\theta\bar{\theta}\bar{\theta}\left[F_i^{*}F_j + \frac{1}{4}\phi_i^{*}\partial^{2}\phi_j + \frac{1}{4}\partial^{2}\phi_i^{*}\phi_j - \frac{1}{2}\partial_{\mu}\phi_i^{*}\partial_{\mu}\phi_j + \frac{\mathrm{i}}{2}\partial_{\mu}\bar{\psi}_i\bar{\sigma}^{\mu}\psi_j - \frac{\mathrm{i}}{2}\bar{\psi}_i\bar{\sigma}^{\mu}\partial_{\mu}\psi_j\right]
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\rightsquigarrow {} & \phi_i^{*}\phi_j + \sqrt{2}\phi_i^{*}\theta\psi_j + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta\theta\phi_i^{*}F_j + \bar{\theta}\bar{\theta}F_i^{*}\phi_j \\
& + \mathrm{i}\,\theta\sigma^{\mu}\bar{\theta}\left(\phi_i^{*}\partial_{\mu}\phi_j - \partial_{\mu}\phi_i^{*}\phi_j\right) + 2\bar{\theta}\bar{\psi}_i\theta\psi_j \\
& + \sqrt{2}\theta\theta\bar{\theta}\left(\bar{\psi}_iF_j - \mathrm{i}\,\bar{\sigma}^{\mu}\psi_j\partial_{\mu}\phi_i^{*}\right) + \sqrt{2}\bar{\theta}\bar{\theta}\theta\left(\psi_jF_i^{*} - \mathrm{i}\,\sigma^{\mu}\bar{\psi}_i\partial_{\mu}\phi_j\right) \\
& + \theta\theta\bar{\theta}\bar{\theta}\left[F_i^{*}F_j - \partial_{\mu}\phi_i^{*}\partial_{\mu}\phi_j - \mathrm{i}\,\bar{\psi}_i\bar{\sigma}^{\mu}\partial_{\mu}\psi_j\right]
\end{aligned} \tag{6.13}$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_i F_j + F_i\phi_j - \psi_i\psi_j\right] \tag{6.14}$$

$$\begin{aligned}\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right] \\ &\quad + \theta\theta\left[F_i\phi_j\phi_k + \phi_i F_j\phi_k + \phi_i\phi_j F_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right]\end{aligned} \tag{6.15}$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

■Superpotential

$$W = \int \mathrm{d}^2\theta \left[\lambda_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k\right] \tag{6.16}$$

$$\begin{aligned}&= \lambda_i F_i + \frac{1}{2}m_{ij}\left(\phi_i F_j + F_i\phi_j - \psi_i\psi_j\right) \\ &\quad + \frac{1}{3}y_{ijk}\left(F_i\phi_j\phi_k + \phi_i F_j\phi_k + \phi_i\phi_j F_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right)\end{aligned} \tag{6.17}$$

(In x-basis, since we have omitted all $\bar{\theta}$s.)

6.3 Vector Superfields

◆6.3.1 Abelian Case

■General Definitions

Vector Superfields	:	$V = V^\dagger$
Gauge Transf.	:	$V \dashrightarrow V + \Phi + \Phi^\dagger$
Field Strength	:	$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V;\quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V$
Lagrangian	:	$\mathcal{L} = \frac{1}{4}\left(W^\alpha W_\alpha\Big\vert_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\Big\vert_{\bar{\theta}\bar{\theta}}\right)$

■Explicit Expression

$$\begin{aligned}V &= C(x) + \mathrm{i}\,\theta\chi(x) - \mathrm{i}\,\bar{\theta}\bar{\chi}(x) \\ &\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(x) + \mathrm{i}\,N(x)\right] - \frac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\left[M(x) - \mathrm{i}\,N(x)\right] - \theta\sigma^\mu\bar{\theta}A_\mu(x) \\ &\quad + \mathrm{i}\,\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right] - \mathrm{i}\,\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] \\ &\quad + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\partial^2 C(x)\right]\end{aligned} \tag{6.18}$$

In Wess-Zumino gauge,

$$V \dashrightarrow -\theta\sigma^\mu\bar{\theta}A_\mu(x) + \mathrm{i}\,\theta\theta\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\,\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{6.19}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y) + \mathrm{i}\,\theta\theta\bar{\theta}\bar{\lambda}(y) - \mathrm{i}\,\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y) - \mathrm{i}\,\partial_\mu A^\mu(y)\right] \tag{6.20}$$

$$= -\theta\sigma^\mu\bar{\theta}A_\mu(y^+) + \mathrm{i}\,\theta\theta\bar{\theta}\bar{\lambda}(y^+) - \mathrm{i}\,\bar{\theta}\bar{\theta}\theta\lambda(y^+) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y^+) + \mathrm{i}\,\partial_\mu A^\mu(y^+)\right] \tag{6.21}$$

■Field Strength

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\mathrm{e}^{2gV}D_\alpha\mathrm{e}^{-2gV} \qquad \text{where} \quad V = V^a T^a \tag{6.22}$$

■Field Strength

Defining all component fields as including generators and coupling constants,

$$\begin{aligned}
\mathrm{e}^{\pm 2gV^aT^a} &\to \mathrm{e}^{\pm 2V} \\
&= 1 \mp 2\theta\sigma^\mu\bar{\theta}A_\mu(x) \mp 2\,\mathrm{i}\,\left[\bar{\theta}\bar{\theta}\theta\lambda(x) - \theta\theta\bar{\theta}\bar{\lambda}(x)\right] + \theta\theta\bar{\theta}\bar{\theta}\left[-A^\mu(x)A_\mu(x) - D(x)\right] \qquad (6.23)
\end{aligned}$$

Therefore, in y^+-basis,

$$\begin{aligned}
D_\alpha \mathrm{e}^{-2V} &= \frac{\partial}{\partial\theta^\alpha}\Big\{1 + 2\theta\sigma^\mu\bar{\theta}A_\mu + 2\,\mathrm{i}\,\left[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}\right] + \theta\theta\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right]\Big\} \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2\,\mathrm{i}\,\bar{\theta}\bar{\theta}\lambda_\alpha - 4\,\mathrm{i}\,\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] \qquad (6.24)
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^{2V}D_\alpha \mathrm{e}^{-2V} &= \Big\{1 - 2\theta\sigma^\mu\bar{\theta}A_\mu - 2\,\mathrm{i}\,\left[\bar{\theta}\bar{\theta}\theta\lambda - \theta\theta\bar{\theta}\bar{\lambda}\right]\Big\}D_\alpha \mathrm{e}^{-2V} \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu + 2\,\mathrm{i}\,\bar{\theta}\bar{\theta}\lambda_\alpha - 4\,\mathrm{i}\,\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] \\
&\quad - 2\theta\sigma^\mu\bar{\theta}A_\mu\left[2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 4\,\mathrm{i}\,\theta_\alpha\bar{\theta}\bar{\lambda}\right] \\
&\quad + 2\,\mathrm{i}\,\theta\theta\bar{\theta}\bar{\lambda}\cdot 2(\sigma^\mu\bar{\theta})_\alpha A_\mu \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 4\,\mathrm{i}\,\theta_\alpha\bar{\theta}\bar{\lambda} + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] + 2\,\mathrm{i}\,\bar{\theta}\bar{\theta}\lambda_\alpha \\
&\quad - 2A_\mu A_\nu\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4\,\mathrm{i}\,A_\mu\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y^+\text{-basis)} \qquad (6.25) \\
&= 2(\sigma^\mu\bar{\theta})_\alpha A_\mu - 2\,\mathrm{i}\,\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\nu\bar{\sigma}^\mu)^\gamma\partial_\nu A_\mu - 4\,\mathrm{i}\,\theta_\alpha\bar{\theta}\bar{\lambda} - 2\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad + 2\theta_\alpha\bar{\theta}\bar{\theta}\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] + 2\,\mathrm{i}\,\bar{\theta}\bar{\theta}\lambda_\alpha \\
&\quad - 2A_\mu A_\nu\bar{\theta}\bar{\theta}\epsilon_{\alpha\gamma}(\theta\sigma^\mu\bar{\sigma}^\nu)^\gamma - 4\,\mathrm{i}\,A_\mu\theta\theta\bar{\theta}\bar{\theta}(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y\text{-basis)} \qquad (6.26)
\end{aligned}$$

In y-basis, $\bar{D}\bar{D} = 4\cdot\dfrac{\partial}{\partial(\bar{\theta}\bar{\theta})}$. Therefore,

$$\begin{aligned}
W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\mathrm{e}^{2V}D_\alpha \mathrm{e}^{-2V} \qquad (6.27) \\
&= 2(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad - 2\theta_\alpha\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] - 2\,\mathrm{i}\,\lambda_\alpha + 4\,\mathrm{i}\,A_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \quad \text{(in } y\text{-basis)} \qquad (6.28) \\
&= 2(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)(\sigma^\nu\bar{\sigma}^\mu\theta)_\alpha + 2\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \\
&\quad - 2\theta_\alpha\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] - 2\,\mathrm{i}\,\lambda_\alpha + 4\,\mathrm{i}\,A_\mu\theta\theta(\sigma^\mu\bar{\lambda})_\alpha \\
&\quad - \mathrm{i}\,\theta\theta(\sigma^\nu\bar{\sigma}^\mu\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A_\mu A_\nu + A_\mu\partial_\rho A_\nu + \mathrm{i}\,\partial_\nu\partial_\rho A_\mu) \\
&\quad + \mathrm{i}\,\theta\theta(\sigma^\rho\bar{\theta})_\alpha(\partial_\rho A^\mu A_\mu + A^\mu\partial_\rho A_\mu - \partial_\rho D - \mathrm{i}\,\partial_\mu\partial_\rho A^\mu) \quad \text{(in } x\text{-basis)} \qquad (6.29) \\
W^\alpha &= 2(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)(\theta\sigma^\mu\bar{\sigma}^\nu)^\alpha - 2\theta\theta(\partial_\mu\bar{\lambda}\bar{\sigma}^\mu)^\alpha \\
&\quad - 2\theta^\alpha\left[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu\right] - 2\,\mathrm{i}\,\lambda^\alpha - 4\,\mathrm{i}\,A_\mu\theta\theta(\bar{\lambda}\bar{\sigma}^\mu)^\alpha \quad \text{(in } y\text{-basis)} \qquad (6.30)
\end{aligned}$$

$$
\begin{aligned}
W^\alpha W_\alpha\Big|_{\theta\theta} &= 4(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)(A_\rho A_\sigma + \mathrm{i}\,\partial_\sigma A_\rho)(\theta\sigma^\mu\bar\sigma^\nu\sigma^\sigma\bar\sigma^\rho\theta) \\
&\quad - 4(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu](\theta\sigma^\mu\bar\sigma^\nu\theta) \\
&\quad + 4\,\mathrm{i}\,\theta\theta(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda) \\
&\quad - 4[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu](A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu)(\theta\sigma^\nu\bar\sigma^\mu\theta) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu] \\
&\quad + 4\,\mathrm{i}\,[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu]\theta\lambda \\
&\quad - 4\,\mathrm{i}\,(\lambda\sigma^\nu\bar\sigma^\mu\theta)(A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu) \\
&\quad - 4\,\mathrm{i}\,\theta\theta(\lambda\sigma^\mu\partial_\mu\bar\lambda) \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \\
&= 4(\theta\sigma^\mu\bar\sigma^\nu\sigma^\sigma\bar\sigma^\rho\theta)[A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu][A_\rho A_\sigma + \mathrm{i}\,\partial_\sigma A_\rho] \\
&\quad - 4(\theta\sigma^\mu\bar\sigma^\nu\theta)\Big\{A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu, A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu\Big\} \\
&\quad + 4\,\mathrm{i}\,\theta\theta(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) \\
&\quad + 4\theta\theta[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu] \\
&\quad + 8\theta\theta\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha \\
&\quad - 8\theta\theta A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \quad \text{(in } y\text{-basis)} \\
&\rightsquigarrow 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})[A_\mu A_\nu + \mathrm{i}\,\partial_\nu A_\mu][A_\rho A_\sigma + \mathrm{i}\,\partial_\sigma A_\rho] \\
&\quad - 4\Big\{A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu, A^\nu A_\nu + \mathrm{i}\,\partial_\nu A^\nu\Big\} \\
&\quad + 4[A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu][A^\mu A_\mu - D - \mathrm{i}\,\partial_\mu A^\mu] \\
&\quad + 4\,\mathrm{i}\,(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha - 8A_\mu(\bar\lambda\bar\sigma^\mu\lambda) \\
&= -2F^{\mu\nu}F_{\mu\nu} + 4(D + 2\,\mathrm{i}\,\partial_\mu A^\mu)(D + 2\,\mathrm{i}\,\partial_\mu A^\mu) \\
&\quad + 4\,\mathrm{i}\,(\partial_\mu\bar\lambda\bar\sigma^\mu\lambda - \lambda\sigma^\mu\partial_\mu\bar\lambda) + 8\lambda^\alpha A_\mu(\sigma^\mu\bar\lambda)_\alpha - 8A_\mu(\bar\lambda\bar\sigma^\mu\lambda), \qquad (6.31)\\
\bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big|_{\bar\theta\bar\theta} &= -2F^{\mu\nu}F_{\mu\nu} + 4(D - 2\,\mathrm{i}\,\partial_\mu A^\mu)(D - 2\,\mathrm{i}\,\partial_\mu A^\mu) \\
&\quad + 4\,\mathrm{i}\,(\partial_\mu\lambda\sigma^\mu\bar\lambda - \bar\lambda\bar\sigma^\mu\partial_\mu\lambda) + 8\bar\lambda^\alpha A_\mu(\bar\sigma^\mu\lambda)_\alpha - 8A_\mu(\lambda\sigma^\mu\bar\lambda), \qquad (6.32)
\end{aligned}
$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}\,[A_\mu, A_\nu] \tag{6.33}$$

Defining the generators as [*2]

$$T^{a\dagger} = T^a, \qquad [T^a, T^b] = \mathrm{i}\,f^{abc}T^c, \qquad \mathrm{Tr}\,T^aT^b = k\delta^{ab} \quad (k>0), \qquad \mathrm{Tr}(\{T^a,T^b\}T^c) = 0, \tag{6.34}$$

$$
\begin{aligned}
\frac{1}{16kg^2}\,\mathrm{Tr}\left(W^\alpha W_\alpha\Big|_{\theta\theta} + \bar W_{\dot\alpha}\bar W^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right) &= -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}D^aD^a - 2\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} \\
&\quad - \frac{1}{2}\,\mathrm{i}\,\left(\lambda\sigma^\mu D_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu D_\mu\lambda\right) \qquad (6.35)
\end{aligned}
$$

Here

$$
\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu && (6.36)\\
D_\mu\lambda^a &= \partial_\mu\lambda^a - gf^{abc}A^b_\mu\lambda^c && (6.37)\\
D_\mu\bar\lambda^a &= \partial_\mu\bar\lambda^a - gf^{abc}A^b_\mu\bar\lambda^c && (6.38)
\end{aligned}
$$

[*2] The last one is anomaly-free condition, yielding $\mathrm{Tr}\,T^aT^bT^c = \mathrm{i}\,kgf^{abc}/2$.

■Interaction Terms

$$\mathrm{e}^{-2gV^aT^a} = 1 + 2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a + 2\,\mathrm{i}\,g\bar{\theta}\bar{\theta}\theta\lambda^a T^a - 2\,\mathrm{i}\,g\theta\theta\bar{\theta}\bar{\lambda}^a T^a - \theta\theta\bar{\theta}\bar{\theta}\left[g^2 A^{a\mu}A^b_\mu T^a T^b + gD^aT^a\right] \tag{6.39}$$

$$\begin{aligned}
\Phi^\dagger \mathrm{e}^{-2gV^aT^a}\Phi\Big|_{\theta^4} &= \phi^*\theta\theta\bar{\theta}\bar{\theta}\left[-g^2A^{a\mu}A^b_\mu T^aT^b - gD^aT^a\right]\phi \\
&\quad + \sqrt{2}\phi^*\left[2\,\mathrm{i}\,g\bar{\theta}\bar{\theta}\theta\lambda^aT^a\right]\theta\psi + \sqrt{2}\bar{\theta}\bar{\psi}\left[-2\,\mathrm{i}\,g\theta\theta\bar{\theta}\bar{\lambda}^aT^a\right]\phi \\
&\quad + \,\mathrm{i}\,\theta\sigma^\mu\bar{\theta}\left[\phi^*(2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a)\partial_\mu\phi - \partial_\mu\phi^*(2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a)\phi\right] \\
&\quad + 2\bar{\theta}\bar{\psi}\left[2g\theta\sigma^\mu\bar{\theta}A^a_\mu T^a\right]\theta\psi \\
&\quad + \theta\theta\bar{\theta}\bar{\theta}\left[F^*F + \frac{1}{4}\phi^*\partial^2\phi + \frac{1}{4}\partial^2\phi^*\phi - \frac{1}{2}\partial_\mu\phi^*\partial_\mu\phi + \frac{\mathrm{i}}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{\mathrm{i}}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right] \\
&\rightsquigarrow F^*F - D_\mu\phi^*D^\mu\phi - \,\mathrm{i}\,\bar{\psi}\bar{\sigma}^\mu D_\mu\psi \\
&\quad - gD^a(\phi^*T^a\phi) - \sqrt{2}\,\mathrm{i}\,g\lambda^a(\phi^*T^a\psi) + \sqrt{2}\,\mathrm{i}\,g\bar{\lambda}^a(\bar{\psi}T^a\phi)
\end{aligned} \tag{6.40}$$

$$D_\mu\phi = (\partial_\mu - \,\mathrm{i}\,gA^{a\mu}T^a)\phi \tag{6.41}$$

$$D_\mu\phi^* = \partial_\mu\phi^* + \,\mathrm{i}\,gA^a_\mu(\phi^*T^a) \tag{6.42}$$

$$D_\mu\psi = (\partial_\mu - \,\mathrm{i}\,gA^a_\mu T^a)\psi \tag{6.43}$$

6.4 Minimal Supersymmetric Standard Model

◆6.4.1 Definitions

■Gauge Group

$\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_{\mathrm{Y}} \quad (\times \mathrm{Z}_{2R} : R\text{-parity})$

■Fields

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$1/3$	
L_i		$\mathbf{2}$	$-1/2$		1
$\bar{U}_i$	$\bar{\mathbf{3}}$		$-2/3$	$-1/3$	
$\bar{D}_i$	$\bar{\mathbf{3}}$		$1/3$	$-1/3$	
$\bar{E}_i$			1		-1
H_{u}		$\mathbf{2}$	$1/2$		
H_{d}		$\mathbf{2}$	$-1/2$		

Field	SU(3)	SU(2)	U(1)
g	$\mathbf{8}$		
W		$\mathbf{3}$	
B			

■Superpotential

$$W_{\mathrm{RPC}} = \mu H_{\mathrm{u}} H_{\mathrm{d}} + y_{\mathrm{u}ij} H_{\mathrm{u}} Q_i \bar{U}_j + y_{\mathrm{d}ij} H_{\mathrm{d}} Q_i \bar{D}_j + y_{\mathrm{e}ij} H_{\mathrm{d}} L_i \bar{E}_j \tag{6.44}$$

$$W_{\mathrm{RPV}} = \mu_i H_{\mathrm{u}} L_i + \lambda_{ijk} L_i L_j \bar{E}_k + \lambda'{}_{ijk} L_i Q_j \bar{D}_k + \lambda''{}_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \tag{6.45}$$

(Here we define $\lambda_{ijk} = -\lambda_{jik}$ and $\lambda''_{ijk} = \lambda''_{ikj}$.)

◆6.4.2 Scalar Potential

■F-terms

$$-F_{H_{\mathrm{u}}}{}^{a*} = \epsilon^{ab} \left(\mu H_{\mathrm{d}}{}^b + y_{\mathrm{u}ij} Q_i^{bx} \bar{U}_j^x + \mu_i L_i^b \right) \tag{6.46}$$

$$-F_{H_{\mathrm{d}}}{}^{a*} = \epsilon^{ab} \left(-\mu H_{\mathrm{u}}{}^b + y_{\mathrm{d}ij} Q_i^{bx} \bar{D}_j^x + y_{\mathrm{e}ij} L_i^b \bar{E}_j \right) \tag{6.47}$$

$$-F_{Q_i^{ax*}} = \epsilon^{ab} \left(-y_{\mathrm{u}ij} H_{\mathrm{u}}{}^b \bar{U}_j^x - y_{\mathrm{d}ij} H_{\mathrm{d}}{}^b \bar{D}_j^x - \lambda'{}_{jik} L_i^b \bar{D}_j^x \right) \tag{6.48}$$

$$-F_{L_i^{a*}} = \epsilon^{ab} \left(-\mu_i H_{\mathrm{u}}{}^b - y_{\mathrm{e}ij} H_{\mathrm{d}}{}^b \bar{E}_j + 2\lambda_{ijk} L_j^b \bar{E}_k + \lambda'_{ijk} Q_j^{bx} \bar{D}_k^x \right) \tag{6.49}$$

$$-F_{\bar{U}_i^{x*}} = \left(\epsilon^{ab} y_{\mathrm{u}ji} H_{\mathrm{u}}{}^a Q_j^{bx} + \epsilon^{xyz} \lambda''_{ijk} \bar{D}_j^y \bar{D}_k^z \right) \tag{6.50}$$

$$-F_{\bar{D}_i^{x*}} = \left(\epsilon^{ab} y_{\mathrm{d}ji} H_{\mathrm{d}}{}^a Q_j^{bx} + \epsilon^{ab} \lambda'{}_{jki} L_j^a Q_k^{bx} + 2\epsilon^{yzx} \lambda''_{jki} \bar{U}_j^y \bar{D}_k^z \right) \tag{6.51}$$

$$-F_{\bar{E}_i}{}^{*} = \left(\epsilon^{ab} y_{\mathrm{e}ji} H_{\mathrm{d}}{}^a L_j^b + \epsilon^{ab} \lambda_{jki} L_j^a L_k^b \right) \tag{6.52}$$

■D-terms

$$D_g{}^\alpha = -g_3 \sum_{i=1}^{3} \left[\sum_{a=1,2} Q_i^{ax*} (T^\alpha)_{xy} Q_i^{ay} - \bar{U}_i^{x*} (T^\alpha)_{xy} \bar{U}_i^y - \bar{D}_i^{x*} (T^\alpha)_{xy} \bar{D}_i^y \right] \tag{6.53}$$

$$D_W{}^\alpha = -g_2 \left[\sum_{i=1}^{3} \sum_{x=1}^{3} Q_i^{ax*} (T^\alpha)_{ab} Q_i^{by} + \sum_{i=1}^{3} L_i^{a*} (T^\alpha)_{ab} L_i^b + H_{\mathrm{u}}{}^{a*} (T^\alpha)_{ab} H_{\mathrm{u}}{}^b + H_{\mathrm{d}}{}^{a*} (T^\alpha)_{ab} H_{\mathrm{d}}{}^b \right] \tag{6.54}$$

$$D_B = -g_1 \left[\frac{1}{6} |Q_i^{ax}|^2 - \frac{1}{2} |L_i^a|^2 - \frac{2}{3} |\bar{U}_i^x|^2 + \frac{1}{3} |\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2} |H_{\mathrm{u}}{}^a|^2 - \frac{1}{2} |H_{\mathrm{d}}{}^a|^2 \right] \tag{6.55}$$

■Full Scalar Potential

$$V = \sum |F_\bullet|^2 + \frac{1}{2} \sum |D_\bullet|^2 \tag{6.56}$$

7 Supergravity

7.1 Minimal SUGRA Lagrangian

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2} eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu \mathrm{D}_\rho\psi_\sigma \tag{7.1}$$

where

$$\mathrm{D}_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \qquad \left[\omega_\mu{}^{ab} : \text{“spin 接続”}\right] \tag{7.2}$$

$$e := \det e_a{}^\mu \tag{7.3}$$

$$M := 1/\sqrt{8\pi G} \quad \text{(Reduced Planck mass)} \tag{7.4}$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \tag{7.5}$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \tag{7.6}$$

7.2 General SUGRA Lagrangian

The components of general SUGRA Lagrangian is

$$\Phi_i = (\psi_i, \chi_i^\alpha, F_i), \qquad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \qquad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \tag{7.7}$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
  - Real function of chiral multiplets.
  - In global SUSY, $\int \mathrm{d}^4\theta K$ yields kinetic terms of the chiral multiplet.
  - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int \mathrm{d}^4\theta\ \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + \mathrm{i}\,\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \tag{7.8}$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
  - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
  - $(a), (b), \ldots$ are indices for adjoint representation of gauge group.
  - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}\mathrm{D}_\mu\phi^i\mathrm{D}^\mu\phi^{*j} - \frac{1}{2}eg^2 D_{(a)}D^{(a)} \\
& + \mathrm{i}\,eg_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu\mathrm{D}_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\mathrm{D}_\rho\psi_\sigma \\
& - \frac{1}{4}ef^{\mathrm{R}}{}_{(ab)}F^{(a)}_{\mu\nu}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f^{\mathrm{I}}{}_{(ab)}f^{(a)}_{\mu\nu}f^{(b)}_{\rho\sigma} \\
& + \frac{\mathrm{i}}{2}e\left[\lambda_{(a)}\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^{(a)}\right] - \frac{1}{2}f^{\mathrm{I}}{}_{(ab)}\mathrm{D}_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_{i*}f^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F^{(b)}_{\mu\nu} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F^{(b)}_{\mu\nu} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{\mathrm{i}}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F^{(a)}_{\nu\rho} + \hat{F}^{(a)}_{\nu\rho}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[\mathrm{i}\,\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f^{\mathrm{R}}{}_{(ab)} + f^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(bc)}\partial_{j^*}f^*_{(ad)}\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f^*_{(ab)}\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}\,(cd)^{-1}}\partial_{i^*}f^*_{(ac)}\partial_{j^*}f^*_{(bd)}\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f^*_{(cd)}\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{\mathrm{i}}{2}\sqrt{2}e\mathrm{e}^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}e\mathrm{e}^{K/2}\left[\mathrm{D}_iD_jW\chi^i\chi^j + \mathrm{D}_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}e\mathrm{e}^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f^*_{(ab)}\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned} \tag{7.9}
$$

付録 A　Verbose Notes

A.1　Polarization Sum

Firstly we focus on the single photon case $M = \epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$. Here we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2, \tag{A.1}$$

while

$$\eta_{\mu\nu}M^\mu M^{\nu*} = |M^1|^2 + |M^2|^2 \tag{A.2}$$

for Ward identity $k_\mu M^\mu = 0$. Therefore the replacement

$$\sum_{\text{pol.}} \epsilon_\mu \epsilon'_\nu \to \eta_{\mu\nu} \tag{A.3}$$

is valid.

Secondly we think about the double photons case [*3] $M = \epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \qquad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \tag{A.4}$$

$$k' = (E, 0, 0, -E) \qquad \epsilon' = (0, \cos\theta, \sin\theta, 0) \oplus (0, -\sin\theta, \cos\theta, 0). \tag{A.5}$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \tag{A.6}$$

Nevertheless, naïve replacement does not work, because our Ward identities

$$k_\mu \epsilon'^*_\nu(k')M^{\mu\nu} = \epsilon^*_\mu(k)k'_\nu M^{\mu\nu} = 0 \tag{A.7}$$

obviously does not help us. If we can omit ϵs from these identities, that is if

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0, \tag{A.8}$$

we can recover validity of the replacement:

$$\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*} = -\eta_{\nu\sigma}\left(M^{1\nu}M^{1\sigma*} + M^{2\nu}M^{2\sigma*}\right) \tag{A.9}$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2. \tag{A.10}$$

Then what's happening? Why this replacement is not valid? Actually our new conditions (A.8) seem to guarantee that we are summing not only "physical" but also "unphysical" polarizations. Meanwhile if we use some physical condition such as $\epsilon \cdot k = 0$, (A.8) break down while Ward identities (A.7) are still valid.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our new conditions (A.8), and define $\widetilde{M}^{\mu\nu}$ and $\widetilde{M}$ as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \tag{A.11}$$

$$\widetilde{M} := \epsilon^*_\mu(k)\epsilon'^*_\nu(k')\widetilde{M}^{\mu\nu}. \tag{A.12}$$

[*3] This part is derived from 濱口幸一's notebook.

This alternative amplitude satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$), and furthermore $\widetilde{M} = M$. Therefore $\widetilde{M}$ is physically identical to M. However technically these are very different, just because we cannot perform our "naïve replacement" for this $\widetilde{M}$:

$$\eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \eta_{\mu\rho}\eta_{\nu\sigma}\left(M^{\mu\nu} + k^{\mu}p^{\nu} + p'^{\mu}k'^{\nu}\right)\left(M^{\rho\sigma *} + k^{\rho}p^{\sigma *} + p'^{\rho *}k'^{\sigma}\right) \tag{A.13}$$

$$= \sum_{\text{pol.}} |M|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H. c.}\right]. \tag{A.14}$$

After all, we have obtained following expression:

$$\begin{aligned}
\sum_{\text{pol.}} |\widetilde{M}|^2 &= \sum_{\text{pol.}} |M|^2 \qquad (\text{Furthermore } \widetilde{M} = M) \\
&= \sum_{\text{pol.}} |\epsilon^*_{\mu}(k)\epsilon'^*_{\nu}(k')M^{\mu\nu}|^2 = \sum_{\text{pol.}} |\epsilon^*_{\mu}(k)\epsilon'^*_{\nu}(k')\widetilde{M}^{\mu\nu}|^2 \\
&= \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *} \\
&\neq \eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \sum_{\text{pol.}} |\widetilde{M}|^2 + \left[(k \cdot p'^*)(k' \cdot p) + \text{H. c.}\right].
\end{aligned} \tag{A.15}$$