1 General Definitions and Tools

1.1 Notations and Conventions

1.1.1 Metric etc.

Minkowski Metric : $\eta^{\mu\nu} := \mathrm{diag}(+,-,-,-)$; $\epsilon^{0123}_{0123} := \pm 1$

Coordinates : $x^\mu := (t,x,y,z)$; therefore $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$.

Gamma Matrices : $\{\gamma^\mu,\gamma^\nu\} := 2\eta^{\mu\nu}$; $\gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \dfrac{-\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

: therefore $\{\gamma^\mu,\gamma_5\} = 0, (\gamma_5)^2 = 1$.

Gamma Combinations : $1, \{\gamma^\mu\}, \{\sigma^{\mu\nu}\}, \{\gamma^\mu\gamma_5\}, \gamma_5$; $\sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu] = 0 \big/ \mathrm{i}\gamma^\mu\gamma^\nu$

Spinor ϵ and σ matrices : $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$

: $(\sigma^\mu)_{\alpha\dot\beta} := (1,\boldsymbol{\sigma})_{\alpha\dot\beta}$, $(\bar\sigma^\mu)^{\dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1,-\boldsymbol{\sigma})^{\dot\alpha\beta}$.

> Pauli Matrices : $\sigma_0 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$,
>
> : $\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $\sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$,
>
> : $\sigma^\mu := (1,\boldsymbol{\sigma})$, $\bar\sigma^\mu := (1,-\boldsymbol{\sigma})$.

Fourier Transformation : $\widetilde{f}(k) := \displaystyle\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}kx} f(x)$; $f(x) = \displaystyle\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}kx}\widetilde{f}(k)$.

1.1.2 Fields

Scalar : $(\partial^2 + m^2)\phi = 0$; $\phi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left[a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}px}\right]$

Dirac : $(i\partial\!\!\!/ - m)\psi = 0$; $\psi(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left[a^s_{\boldsymbol{p}}u^s(p)\mathrm{e}^{-\mathrm{i}px} + b^{s\dagger}_{\boldsymbol{p}}v^s(p)\mathrm{e}^{\mathrm{i}px}\right]$

Vector : $\partial^2 A^\mu = 0$; $A^\mu(x) = \displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{r=0..3}\left[a^r_{\boldsymbol{p}}\epsilon^r(p)\mathrm{e}^{-\mathrm{i}px} + a^{r\dagger}_{\boldsymbol{p}}\epsilon^{r*}(p)\mathrm{e}^{\mathrm{i}px}\right]$

TODO: 南部–Goldstone; Gravitino

1.1.3 Electromagnetism

Electromagnetic Fields : $A^\mu = (\phi, \boldsymbol{A})$ 【We can invert the signs, but cannot lower the index.】

Maxwell Equations : $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$; $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$, $\partial_\mu F^{\mu\nu} = ej^\nu$

Our Old Language : $\nabla\cdot\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = 0$; $\nabla\cdot\boldsymbol{E} = ej^0$, $(\nabla\times\boldsymbol{B})_i - \frac{\partial}{\partial t}E_i = ej^i$.

: $F_{\mu\nu} = \begin{pmatrix} 0 & & \boldsymbol{E} & \\ & 0 & -B_3 & B_2 \\ -\boldsymbol{E} & B_3 & 0 & -B_1 \\ & -B_2 & B_1 & 0 \end{pmatrix}$; $F_{\mu\nu}F^{\mu\nu} = -2\left(\|\boldsymbol{E}\|^2 - \|\boldsymbol{B}\|^2\right)$

1.2 Spinor Fields

1.2.1 Lorentz transformation

Algebra : $[J_{\mu\nu}, J_{\rho\sigma}] = -\mathrm{i}(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma})$.

For vector : $V^\alpha \mapsto \Lambda^\alpha{}_\beta V^\beta = \exp(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu})^\alpha{}_\beta V^\beta$, where $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = \mathrm{i}(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha)$.

For 4-spinor : $\psi \mapsto \Lambda_{\frac{1}{2}}\psi = \exp(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu})$, where $S^{\mu\nu} := \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$, and $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. [*1]

For 2-spinor : $\xi \mapsto L\xi = \exp(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}s^{\mu\nu})\xi$, We will use $s^{0i} = \bar{s}^{i0} = -\frac{\mathrm{i}}{2}\sigma_i$,

$\bar\eta \mapsto \bar{L}\bar\eta = \exp(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\bar{s}^{\mu\nu})\bar\eta$; $s^{ij} = \bar{s}^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma_k$. (See Sec. C.2 for details.)

1.2.2 Weyl spinor

We use the convention where $L\xi \equiv L_\alpha{}^\beta\xi_\beta$ and $\bar{L}\bar\eta \equiv \Lambda^{\dagger-1\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta}$.

Spinor : ξ_α, $\xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta$; $\bar\eta^{\dot\alpha} := (\eta^\alpha)^*$, $\bar\eta_{\dot\alpha} := (\eta_\alpha)^*$;

Lorentz tr. : $\xi_\alpha \mapsto \Lambda_\alpha{}^\beta\xi_\beta$, $\xi^\alpha \mapsto \xi^\beta\Lambda^{-1}{}_\beta{}^\alpha$; $\bar\eta^{\dot\alpha} \mapsto \Lambda^{\dagger-1\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta}$, $\bar\eta_{\dot\alpha} \mapsto \bar\eta_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}$.

Kinetic term : $\mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi$ $(= \mathrm{i}\,\xi\sigma^\mu\partial_\mu\bar\xi\,)$

Mass term : [Majorana] $-\frac{1}{2}(m_\mathrm{M}\xi\xi + m_\mathrm{M}^*\bar\xi\bar\xi)$; [Dirac] $-(m_\mathrm{D}\xi\eta + m_\mathrm{D}^*\bar\xi\bar\eta)$

Dirac fermion : $\mathcal{L}_\mathrm{Dirac} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi + \mathrm{i}\,\eta\sigma^\mu\partial_\mu\bar\eta - m(\xi\eta + \bar\xi\bar\eta) = \overline{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$

Majorana fermion : $\mathcal{L}_\mathrm{Majorana} = \mathrm{i}\,\bar\xi\bar\sigma^\mu\partial_\mu\xi - \frac{m}{2}(\xi\xi + \bar\xi\bar\xi) = \frac{1}{2}\overline{\psi}_\mathrm{M}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi_\mathrm{M}$

1.2.3 Dirac spinor

$$\psi(x) \equiv \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol p}}}\sum_{s=1,2}\Big[a^s_{\boldsymbol p}u^s(p)\mathrm{e}^{-\mathrm{i}p\cdot x} + b^{s\dagger}_{\boldsymbol p}v^s(p)\mathrm{e}^{\mathrm{i}p\cdot x}\Big],\quad \text{where}\ \ (\not p - m)u^s(p) = (\not p + m)v^s(p) = 0.$$

- $\overline{\psi}(x)$ and $\psi^\mathrm{C}(x)$ should be defined as they properly respond to Lorentz transformation;
- We define matrix C such that $\psi^\mathrm{C} \equiv C(\overline{\psi})^\mathrm{T}$.

■Chiral notation (Peskin)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};\quad \begin{aligned}&(\gamma^\mu)^\mathrm{T} = \{\gamma^0, -\gamma^1, \gamma^2, -\gamma^3\} = \gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0,\\ &(\gamma^\mu)^* = \{\gamma^0, \gamma^1, -\gamma^2, \gamma^3\} = \gamma^2\gamma^\mu\gamma^2,\ (\gamma^\mu)^\dagger = \{\gamma^0, -\gamma^i\} = \gamma^0\gamma^\mu\gamma^0.\end{aligned}$$

Fields : $\psi = \begin{pmatrix} \xi_\alpha \\ \bar\eta^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix} \left[\Longleftarrow S^{\mu\nu} = \begin{pmatrix} s^{\mu\nu} & 0 \\ 0 & \bar{s}^{\mu\nu} \end{pmatrix}\right]$; $P^\mathrm{R}_\mathrm{L} = \dfrac{1 \pm \gamma_5}{2}$.

: $\overline{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \eta^\alpha & \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \psi_\mathrm{R}^\dagger & \psi_\mathrm{L}^\dagger \end{pmatrix} \left[\Longleftarrow \Lambda_{\frac{1}{2}}^{-1} = \gamma^0\Lambda_{\frac{1}{2}}^\dagger\gamma^0\right]$;

: $\psi^\mathrm{C} = \begin{pmatrix} \eta_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$ $\quad\therefore C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} = -\mathrm{i}\gamma^2\gamma^0.$ $\quad\begin{cases} C = C^* = -C^{-1} = -C^\dagger = -C^\mathrm{T} \\ C^{-1}\gamma^\mu C = -\gamma^{\mu\,\mathrm{T}} \end{cases}$

: $\psi^\mathrm{C} = C(\overline{\psi})^\mathrm{T} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} \eta_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$, $\overline{\psi}^\mathrm{C} = \psi^\mathrm{T}C = \mathrm{i}\overline{\psi}^*\gamma^2$. $\quad\psi_\mathrm{M} = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$.

components : $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar\sigma}\xi^s \end{pmatrix}$; $\quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}$.

ξ^s are SU(2) spinors composing a basis set; $\xi^{s\dagger}\xi^t = \delta^{st}$. We label $\eta^s = -\mathrm{i}\sigma_2(\xi^s)^*$ so that $v = C\overline{u}^\mathrm{T}$, $u = C\overline{v}^\mathrm{T}$, and

$$\psi^\mathrm{C}(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol p}}}\sum_{s=1,2}\Big[b^s_{\boldsymbol p}u^s(p)\mathrm{e}^{-\mathrm{i}p\cdot x} + a^{s\dagger}_{\boldsymbol p}v^s(p)\mathrm{e}^{\mathrm{i}p\cdot x}\Big].$$

Weyl eqs. : $\mathrm{i}\bar\sigma\cdot\partial\psi_\mathrm{L} = m\psi_\mathrm{R}$; $\quad\mathrm{i}\sigma\cdot\partial\psi_\mathrm{R} = m\psi_\mathrm{L}$

[*1] $\mathcal{J}^{\mu\nu}$, $S^{\mu\nu}$, and $s^{\mu\nu}$ are representations of Lorentz algebra; $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ is sufficient for $S^{\mu\nu}$ to follow the algebra.

z-boost limit : Halt: $u^s = \sqrt{m}\begin{pmatrix}\xi^s\\\xi^s\end{pmatrix}$, $v^s = \sqrt{m}\begin{pmatrix}\eta^s\\-\eta^s\end{pmatrix}$;

: Slow: $\sqrt{p\cdot\sigma} \simeq \sqrt{m}(1-\boldsymbol{v}\cdot\boldsymbol{\sigma}/2)$, $\sqrt{p\cdot\bar\sigma} \simeq \sqrt{m}(1+\boldsymbol{v}\cdot\boldsymbol{\sigma}/2)$;

: Extreme: $u^s = \sqrt{2E}\begin{pmatrix}\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\xi^s\\\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\xi^s\end{pmatrix}$, $v^s = \sqrt{2E}\begin{pmatrix}\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\eta^s\\-\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\eta^s\end{pmatrix}$.

CPT transformations

In the following, CP means "P, then C" **in algebraic sense**. Be careful on the order.

$$\psi(t,\boldsymbol{x}) \xrightarrow{P} \eta_P\gamma^0\psi(t,-\boldsymbol{x}) \qquad \overline{\psi} \xrightarrow{P} \eta_P^*\overline{\psi}\gamma^0$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{T} \eta_T C\gamma_5\psi(-t,\boldsymbol{x}) \qquad \overline{\psi} \xrightarrow{T} -\eta_T^*\overline{\psi}C\gamma_5$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{C} \eta_C C\overline{\psi}^{\mathrm{T}}(t,\boldsymbol{x}) = C\gamma^0\psi^* \qquad \overline{\psi} \xrightarrow{C} \eta_C^*\overline{\psi}^*\gamma^0 C = -\eta_C^*(C\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{CP} \eta_{CP}(\overline{\psi}\gamma^0 C)^{\mathrm{T}} \qquad \overline{\psi} \xrightarrow{CP} \eta_{CP}^*(C\gamma^0\psi)^{\mathrm{T}}$$

$$\psi(t,\boldsymbol{x}) \xrightarrow{CPT} (\overline{\psi}\gamma^0\gamma_5)^{\mathrm{T}} \qquad \overline{\psi} \xrightarrow{CPT} (\gamma^0\gamma_5\psi)^{\mathrm{T}}$$

Note that T-transformation is anti-unitary, and $\eta_{CPT}=1$. Especially, photon is $(P,T,C)=(-,+,-)$.

	ϕ	A^μ	$\bar\psi\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\sigma^{\mu\nu}\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$	$\mathrm{i}\bar\psi\gamma_5\psi$	∂_μ
P	ϕ	$-{+}{+}{+}A^\mu$	$+$	$+---$	$(+---)(+---)$	$-+++$	$-$	$+---$
T	ϕ	$+---A^\mu$	$+$	$+---$	$-(+---)(+---)$	$+---$	$-$	$-+++$
C	ϕ^*	$+A^{\mu*}$	$+$	$-$	$-$	$+$	$+$	$+$
CPT	ϕ^*	$-A^{\mu*}$	$+$	$-$	$+$	$-$	$+$	$-$

■Dirac Notation

Gamma Matrices : $\hat\gamma^0 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\hat\gamma^i = \begin{pmatrix}0&\boldsymbol{\sigma}\\-\boldsymbol{\sigma}&0\end{pmatrix}$, $\hat\gamma_5 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$; $\hat P^{\mathrm{R}}_{\mathrm{L}} = \dfrac{1\pm\gamma_5}{2}$.

: $\hat\sigma^{0i} = \begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}$, $\hat\sigma^{ij} = \epsilon^{ijk}\begin{pmatrix}\sigma^k&0\\0&\sigma^k\end{pmatrix}$.

Fields : $\hat\psi = \begin{pmatrix}\psi_{\mathrm{A}}\\\psi_{\mathrm{B}}\end{pmatrix} = \dfrac{1}{\sqrt2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\psi = \dfrac{1}{\sqrt2}\begin{pmatrix}\psi_{\mathrm{L}}+\psi_{\mathrm{R}}\\-\psi_{\mathrm{L}}+\psi_{\mathrm{R}}\end{pmatrix}$; $\hat\psi_{\mathrm{M}} = \begin{pmatrix}\psi_{\mathrm{A}}\\\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\psi_{\mathrm{A}}^*\end{pmatrix}$.

: $\hat{\overline{\psi}} = \hat\psi^\dagger\gamma^0 = \begin{pmatrix}\psi_{\mathrm{A}}^\dagger & -\psi_{\mathrm{B}}^\dagger\end{pmatrix}$

: $\hat u^s(p) = \begin{pmatrix}\sqrt{p^0+m}\\\xi^s\\\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\xi^s\end{pmatrix}$; $\hat v^s(p) = \begin{pmatrix}-\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}}\eta^s\\-\sqrt{p^0+m}\\\eta^s\end{pmatrix}$

: $\left[\eta^s = \xi^{-s} := -\mathrm{i}\sigma^2(\xi^s)^* = (\xi^2,-\xi^1)\right]$

Charge conj. : $\hat C = -\mathrm{i}\hat\gamma^2\hat\gamma^0 = \left(\begin{smallmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{smallmatrix}\right)$ with $C=-C^{-1}=-C^\dagger$, $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$.

z-boost limit : Halt: $\hat u^s = \sqrt{2m}\begin{pmatrix}\xi^s\\0\end{pmatrix}$, $\hat v^s = -\sqrt{2m}\begin{pmatrix}0\\\eta^s\end{pmatrix}$;

: Slow: $\sqrt{p^0+m} \simeq \sqrt{2m}(1+\frac{v^2}{8})$, $\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{p^0+m}} \simeq \sqrt{\frac{m}{2}}(\boldsymbol{v}\cdot\boldsymbol{\sigma})$;

: Extreme: $\hat u^s = \sqrt{E}\begin{pmatrix}\xi^s\\\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\xi^s\end{pmatrix}$, $\hat v^s = -\sqrt{E}\begin{pmatrix}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\eta^s\\\eta^s\end{pmatrix}$

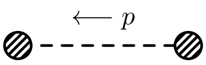
1.3 Feynman Rules

■Vertex Rule

$$\mathcal{L} \ni \lambda\times(\phi_1)^{n_1}\cdots(\phi_k)^{n_k} \Longrightarrow \mathrm{i}\lambda\prod_i(n_i!) \qquad \text{(See Sec. C.8 for details.)}$$

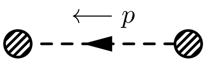
■Scalar Boson

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

$\overbracket{\phi\ \phi} =$ 

$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

$$\mathcal{L} \supset |\partial_\mu\phi|^2 - m^2|\phi|^2$$

$\overbracket{\phi^*\ \phi} =$ 

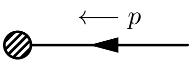
$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

(External lines equal to 1 in both cases.)

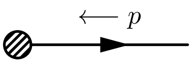
■Dirac Fermion

$$\mathcal{L} \supset \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi$$
$$= \mathrm{i}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \bar{\xi}\bar{\chi})$$

Initial state

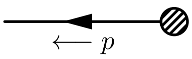
$\overbracket{\psi\ |\boldsymbol{p}, s\rangle} =$ 

$= u^s(p)$

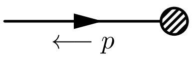
$\overbracket{\overline{\psi}\ |\boldsymbol{p}, s\rangle} =$ 

$= \overline{v}^s(p)$

Final state

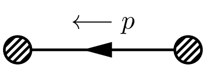
$\overbracket{\langle\boldsymbol{p}, s|\ \overline{\psi}} =$ 

$= \overline{u}^s(p)$

$\overbracket{\langle\boldsymbol{p}, s|\ \psi} =$ 

$= v^s(p)$

Propagator

$\overbracket{\psi\ \overline{\psi}} =$ 

$= \dfrac{\mathrm{i}(p\!\!\!/ + m)}{p^2 - m^2 + \mathrm{i}\epsilon}$

■Majorana Fermion

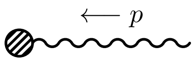
$$\mathcal{L} \supset \tfrac{1}{2}\overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi$$
$$= \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda - \tfrac{m}{2}(\lambda\lambda + \bar{\lambda}\bar{\lambda})$$

Initial state

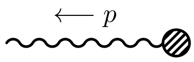
TODO: なんか２種類流儀があるっぽい

■Abelian Gauge Theory (Photon)

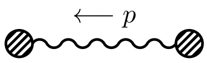
$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}D\!\!\!\!/\,\psi + |\mathrm{D}_\mu\phi|^2$$
$$(\mathrm{D}_\mu = \partial_\mu - \mathrm{i}QA_\mu)$$

$\overbracket{A_\mu\ |\boldsymbol{p}; あ\rangle} =$ 

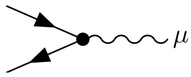
$= \epsilon^{あ}_\mu(p)$

$\overbracket{\langle\boldsymbol{p}; あ\,|\ A_\mu} =$ 

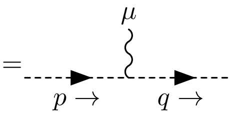
$= \epsilon^{あ*}_\mu(p)$

$\overbracket{A_\mu\ A_\nu} =$ 

$= \dfrac{-\mathrm{i}\eta_{\mu\nu}}{p^2 + \mathrm{i}\epsilon}$

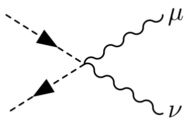
$Q\overline{\psi}A\!\!\!/\,\psi =$ 

$= \mathrm{i}Q\gamma^\mu$

$\mathrm{i}QA^\mu\phi^*\partial_\mu\phi$
+H.c. $=$ 

$= \mathrm{i}Q(p^\mu + q^\mu)$

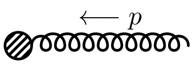
(Momentum must be taken along the arrow)

$Q^2A^2|\phi|^2 =$ 

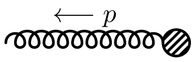
$= 2\mathrm{i}Q^2\eta^{\mu\nu}$

■Non-Abelian Gauge Theory (Gluon)

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\overline{\psi}D\!\!\!\!/\,\psi + |\mathrm{D}_\mu\phi|^2$$
$$(\mathrm{D}_\mu = \partial_\mu - \mathrm{i}gA_\mu)$$

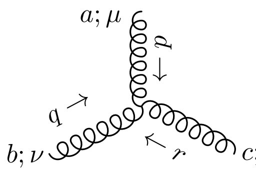
$\overbracket{A^b_\mu\ |\boldsymbol{p}; あ, a\rangle} =$ 

$= \epsilon^{あ}_\mu(p)\delta^{ab}$

$\overbracket{\langle\boldsymbol{p}; あ, a|\ A^b_\mu} =$ 

$= \epsilon^{あ*}_\mu(p)\delta^{ab}$

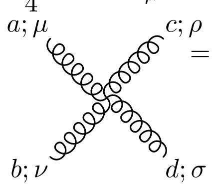
$-gf^{abc}A^{\mu a}A^{\nu b}(\partial_\mu A^c_\nu) =$



$$= gf^{abc}\left[\eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-r)^\mu + \eta^{\rho\mu}(r-p)^\nu\right]$$

(Momentum are in incoming directions)

$-\dfrac{1}{4}g^2(f^{abe}A^a_\mu A^b_\nu)(f^{cde}A^c_\rho A^d_\sigma) =$



$$= -\mathrm{i}g^2\big[\, f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})\big]$$

1.4 Field Calculation Techniques

1.4.1 Dirac Field Techniques

$$
\begin{array}{lll}
\text{Dirac Equations} & : & (\not{p}-m)u^s(p)=0; \quad (\not{p}+m)v^s(p)=0 \\
 & : & \bar{u}^s(p)(\not{p}-m)=0; \quad \bar{v}^s(p)(\not{p}+m)=0 \\
\text{Dirac Components} & : & u^{r\dagger}(p)u^s(p)=2E_{\boldsymbol{p}}\delta^{rs}; \quad v^{r\dagger}(p)v^s(p)=2E_{\boldsymbol{p}}\delta^{rs} \\
 & : & \bar{u}^r(p)u^s(p)=2m\delta^{rs}; \quad \bar{v}^r(p)v^s(p)=-2m\delta^{rs}; \quad \bar{u}^r(p)v^s(p)=\bar{v}^r(p)u^s(p)=0 \\
\text{Spin Sums} & : & \sum_{\text{spin}} u^s(p)\bar{u}^s(p)=\not{p}+m; \quad \sum_{\text{spin}} v^s(p)\bar{v}^s(p)=\not{p}-m \\
\text{Charge Conj.} & : & -C=C^{-1}=C^\dagger=C^{\mathrm{T}}, \quad C^{-1}\gamma^\mu C=-C\gamma^\mu C=-\gamma^{\mu\mathrm{T}}, \quad C^{-1}\gamma^0 C=-\gamma^0 \\
 & : & C=C^* \quad , \quad \psi^{\mathrm{C}}=C(\overline{\psi})^{\mathrm{T}}, \quad \overline{\psi}^{\mathrm{C}}=\psi^{\mathrm{T}}C \\
u\ \&\ v & : & u^*=-\mathrm{i}\gamma^2 v, \quad v^{\mathrm{T}}=-\mathrm{i}u^\dagger\gamma^2=\overline{u}C^{-1}, \quad v=C\overline{u}^{\mathrm{T}}; \quad \overline{v}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}v_{\mathrm{A}} \\
 & : & v^*=-\mathrm{i}\gamma^2 u, \quad u^{\mathrm{T}}=-\mathrm{i}v^\dagger\gamma^2=\overline{v}C^{-1}, \quad u=C\overline{v}^{\mathrm{T}}; \quad \overline{v}_{\mathrm{A}}P_{\mathrm{H}}u_{\mathrm{B}}=-\overline{v}_{\mathrm{B}}P_{\mathrm{H}}u_{\mathrm{A}}
\end{array}
$$

1.4.2 Polarization Sum

Single photon case $M=\epsilon^*_\mu(k)M^\mu$

When Ward identity $k_\mu M^\mu=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\nu(k)M^\mu M^{\nu*}=\eta_{\mu\nu}M^\mu M^{\nu*}. \tag{1.1}$$

Double photons case $M=\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}$

When $k_\mu M^{\mu\nu}=k'_\nu M^{\mu\nu}=0$ is valid,

$$\sum_{\text{pol.}}|M|^2=\sum_{\text{pol.}}\epsilon^*_\mu(k)\epsilon_\rho(k)\epsilon'^*_\nu(k')\epsilon'_\sigma(k')M^{\mu\nu}M^{\rho\sigma*}=\eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma*}. \tag{1.2}$$

【See Sec. C.3 for verbose information.】

1.4.3 Fierz transformations

For Dirac spinors a,b,c,d,

$$
\begin{aligned}
S(a,b;c,d)&:=(\bar{a}b)(\bar{c}d);\\
V(a,b;c,d)&:=(\bar{a}\gamma^\mu b)(\bar{c}\gamma_\mu d);\\
T(a,b;c,d)&:=\frac{1}{2}(\bar{a}\sigma^{\mu\nu}b)(\bar{c}\sigma_{\mu\nu}d);\\
A(a,b;c,d)&:=(\bar{a}\gamma^\mu\gamma_5 b)(\bar{c}\gamma_\mu\gamma_5 d);\\
P(a,b;c,d)&:=(\bar{a}\gamma_5 b)(\bar{c}\gamma_5 d);
\end{aligned}
\qquad
\begin{pmatrix} S(a,b;c,d)\\ V(a,b;c,d)\\ T(a,b;c,d)\\ A(a,b;c,d)\\ P(a,b;c,d)\end{pmatrix}
=\frac{1}{4}
\begin{pmatrix} 1&1&1&-1&-1\\ 4&-2&0&-2&4\\ 6&0&-2&0&-6\\ -4&-2&0&-2&-4\\ -1&1&-1&-1&1\end{pmatrix}
\begin{pmatrix} S(a,d;c,b)\\ V(a,d;c,b)\\ T(a,d;c,b)\\ A(a,d;c,b)\\ P(a,d;c,b)\end{pmatrix}
$$

Also defining $V_{\mathrm{LR}}(a,b;c,d):=(\bar{a}\gamma^\mu P_{\mathrm{L}}b)(\bar{c}\gamma_\mu P_{\mathrm{R}}d)$ and so on,

$$V_{\mathrm{LL}}(a,b;c,d)=-V_{\mathrm{LL}}(a,d;c,b) \qquad S_{\mathrm{RL}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{LR}}(a,d;b,c)-A_{\mathrm{LR}}(a,d;b,c)\right] \tag{1.3}$$

$$V_{\mathrm{RR}}(a,b;c,d)=-V_{\mathrm{RR}}(a,d;c,b) \qquad S_{\mathrm{LR}}(a,b;c,d)=\frac{1}{4}\left[V_{\mathrm{RL}}(a,d;b,c)-A_{\mathrm{RL}}(a,d;b,c)\right] \tag{1.4}$$

Here we can create another equations using

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}; \qquad (\bar{\sigma}^\mu)_{\alpha\beta}(\bar{\sigma}_\mu)_{\gamma\delta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{1.5}$$

1.4.4 Gordon identity

For $P := p' + p$ and $q := p' - p$,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \qquad \bar{u}(p')\gamma^\mu v(p) = \bar{u}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]v(p) \tag{1.6}$$

$$\bar{v}(p')\gamma^\mu v(p) = -\bar{v}(p')\left[\frac{P^\mu + \mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right]v(p) \qquad \bar{v}(p')\gamma^\mu u(p) = -\bar{v}(p')\left[\frac{q^\mu + \mathrm{i}\sigma^{\mu\nu}P_\nu}{2m}\right]u(p) \tag{1.7}$$

1.4.5 Color Sum

(Here T^a is **3** of SU(3). For other representations or gauge groups, see Sec. 1.9.)

$$\mathrm{Tr}(T^aT^b) := \frac{1}{2}\delta^{ab} \qquad \text{(That is, } T^a\text{'s are } \tfrac{1}{2}\times\text{Gell-Mann matrices.)} \tag{1.8}$$

$$\sum_a T^aT^a = \frac{4}{3}\cdot\mathbf{1}, \qquad \sum_{c,d} f^{acd}f^{bcd} = 3\delta^{ab} \qquad \sum_a T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} - \frac{1}{6}\delta_{ij}\delta_{kl} \tag{1.9}$$

$$\sum_a T^aT^bT^a = -\frac{1}{6}T^b \qquad \sum_{b,c} f^{abc}T^bT^c = \frac{3\mathrm{i}}{2}T^a \qquad f^{Dab}f^{EDc} + f^{Dca}f^{EDb} + f^{Dbc}f^{EDa} = 0 \tag{1.10}$$

1.5 Miscellaneous Techniques

$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2$; $\qquad (p\cdot\sigma)^* = \sigma_2(p\cdot\bar{\sigma})\sigma_2$

$\epsilon^{ab}\epsilon^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$

$\sqrt{p_\mu\sigma^\mu} = \dfrac{p_\mu\sigma^\mu + m}{\sqrt{2(m+p^0)}}$

$\sigma^i\sigma^j = \delta_{ij}\sigma^0 + \mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^\mu\sigma^\nu = \mathrm{i}\epsilon^{0\mu\nu\rho}\sigma^\rho + \delta^\mu_0\sigma^\nu + \delta^\nu_0\sigma^\mu - \eta^{\mu\nu}\sigma^0$

$[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon_{ijk}\sigma^k$

$\sigma^i, \sigma^j = 2\delta_{ij}$

1.5.1 Noether current

Infinitesimal transformation	:	$\phi(x) \mapsto \phi'(x) := \phi(x) + \alpha\Delta\phi(x)$
Correspondent transformation	:	$\alpha\Delta\mathcal{L} = \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\cancel{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)}\right]\Delta\phi$
	:	So, defining $\alpha\partial_\mu\mathcal{J}^\mu(x) := \mathcal{L}'(x) - \mathcal{L}(x)$,
Noether current	:	$j^\mu(x) := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu; \qquad \partial_\mu j^\mu(x) = 0$
Noether charge	:	$Q := \int j^0\mathrm{d}^3x$
Energy-momentum tensor	:	$T^\mu{}_\nu = \partial_\mu\mathcal{L}(\partial_\mu\phi)\partial_\nu\phi - \mathcal{L}\delta^\mu_\nu; \quad \mathcal{H} = T^{00}, \quad \mathcal{P}^i = T^{0i}$.
	:	$T^\mu{}_\nu$ is the variation along μ in respect to the modification a^ν.

TODO: TODO:

- Majorana Ferminos
- Feynman Rules(A.1)

1.6 Dirac's Gamma Algebras

1.6.1 Traces

$$\mathrm{Tr}(\text{any odd } \# \text{ of } \gamma\text{'s}) = 0, \tag{1.11}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \tag{1.12}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}), \tag{1.13}$$
$$\mathrm{Tr}(\gamma_5 \text{ and any odd } \# \text{ of } \gamma\text{'s}) = 0, \tag{1.14}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0, \tag{1.15}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}. \tag{1.16}$$

Because of $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{Tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu)$, for some γ-matrices $A, B, C, \ldots,$

$$\begin{aligned}\mathrm{Tr}(ABCDEF\cdots) = {} & \eta^{AB}\,\mathrm{Tr}(CDEF\cdots) - \eta^{AC}\,\mathrm{Tr}(BDEF\cdots) \\ & + \eta^{AD}\,\mathrm{Tr}(BCEF\cdots) - \eta^{AE}\,\mathrm{Tr}(BCDF\cdots) + \cdots.\end{aligned} \tag{1.17}$$

$\mathrm{Tr}(ABCDEF\cdots\gamma_5)$ can be calculated by utilizing the identity

$$\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho = \gamma_5\gamma^\mu\eta^{\nu\rho} - \gamma_5\gamma^\nu\eta^{\mu\rho} + \gamma_5\gamma^\rho\eta^{\mu\nu} - \mathrm{i}\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma. \tag{1.18}$$

1.6.2 Contractions

$$\gamma^\mu\gamma_\mu = 4 \tag{1.19}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{1.20}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} \tag{1.21}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{1.22}$$

$$\not{q}\not{q} = q^2 \tag{1.23}$$

Generally, for some γ-matrices $A, B, C, \ldots,$

$$\text{ODD } \# : \quad \gamma^\mu ABC\cdots\gamma_\mu = -2(\cdots CBA), \tag{1.24}$$
$$\text{EVEN } \# : \quad \gamma^\mu ABC\cdots\gamma_\mu = \mathrm{Tr}(ABC\cdots) - \mathrm{Tr}(ABC\cdots\gamma_5)\cdot\gamma_5. \tag{1.25}$$

Contractions in d-dimension

$$\gamma^\mu\gamma_\mu = d \tag{1.26}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu \tag{1.27}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho \tag{1.28}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{1.29}$$

Contractions of ϵ's

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24; \quad \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu_\nu; \quad \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \tag{1.30}$$
$$\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\alpha'\beta'\gamma'} = -\left(\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\alpha_{\beta'}\delta^\beta_{\gamma'}\delta^\gamma_{\alpha'} + \delta^\alpha_{\gamma'}\delta^\beta_{\alpha'}\delta^\gamma_{\beta'} - \delta^\alpha_{\alpha'}\delta^\beta_{\gamma'}\delta^\gamma_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\delta^\gamma_{\gamma'} - \delta^\alpha_{\gamma'}\delta^\beta_{\beta'}\delta^\gamma_{\alpha'}\right) \tag{1.31}$$

1.7 Loop Integrals and Dimensional Regularization

1.7.1 Feynman Parameters

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots x_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{1.32}$$

$$\frac{1}{A_1 A_2} = \int_0^1 \mathrm{d}x \frac{1}{[xA_1 + (1-x)A_2]^2} \tag{1.33}$$

1.7.2 d-dimensional integrals in Minkowski space

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{1.34}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.35}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} \mathrm{i}}{(4\pi)^{d/2}} \frac{\eta^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{1.36}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{1.37}$$

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-\Delta)^n} = \frac{(-1)^n \mathrm{i}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2}-2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{4} \tag{1.38}$$

Here we can use following expansions: $(\gamma \simeq 0.5772)$

$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - (d-4)\frac{\log\Delta}{2} + \mathrm{O}\left((d-4)^2\right) \quad \text{around } d=4, \tag{1.39}$$

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathrm{O}(x) \quad \text{around } x=0, \tag{1.40}$$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left[\frac{1}{x+n} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + \mathrm{O}(x+n)\right] \quad \text{around } x=-n. \tag{1.41}$$

and we get following expansion:

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2} \left[\left(\frac{2}{4-d} - \gamma + \log 4\pi\right) - \log\Delta + \mathrm{O}(4-d)\right]. \tag{1.42}$$

Usually this Δ is positive, but when Δ contains some timelike momenta, it becomes negative. Then these integrals acquire imaginary parts, which give the discontinuities of S-matrix elements. To compute the S-matrix in a physical region choose the correct branch

$$\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \to \left(\frac{1}{\Delta - \mathrm{i}\epsilon}\right)^{n-\frac{d}{2}}. \tag{1.43}$$

1.8 Cross Sections and Decay Rates

■General expression (The mass dimension of $\mathcal{M}$ is $2 - N_{\mathrm{f}}$ for $\mathrm{d}\sigma$ and $3 - N_{\mathrm{f}}$ for $\mathrm{d}\Gamma$.)

$$\mathrm{d}\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(p_A, p_B \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \{p_f\}\right) \tag{1.44}$$

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\left[\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right]\left|\mathcal{M}(m_A \to \{p_f\})\right|^2 (2\pi)^4 \delta^{(4)}\left(m_A - \{p_f\}\right) \quad \text{(in A-rest frame.)} \tag{1.45}$$

■2-body phase space in center-of-mass frame

$$\int \Pi_2 := \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4 \delta^{(4)}\left(E_{\mathrm{cm}} - (p_1 + p_2)\right) \qquad \text{(in center-of-mass frame)} \tag{1.46}$$

$$= \int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}\frac{2\left\|\boldsymbol{p_1}\right\|}{E_{\mathrm{cm}}} \tag{1.47}$$

$$= \frac{1}{8\pi}\sqrt{1 - \frac{2({m_1}^2 + {m_2}^2)}{E_{\mathrm{cm}}^2} + \frac{({m_1}^2 - {m_2}^2)^2}{E_{\mathrm{cm}}^4}} \quad \longrightarrow [m_2 = 0]\ \frac{1}{8\pi}\left(1 - \frac{{m_1}^2}{{E_{\mathrm{cm}}}^2}\right) \tag{1.48}$$

■Kinematics of Decay

$$K \to p_1 + p_2 \qquad \text{or} \qquad \begin{pmatrix} M \\ \boldsymbol{0} \end{pmatrix} \to \begin{pmatrix} \sqrt{p^2 + {m_1}^2} \\ \boldsymbol{p} \end{pmatrix} + \begin{pmatrix} \sqrt{p^2 + {m_2}^2} \\ -\boldsymbol{p} \end{pmatrix}; \tag{1.49}$$

$$\left\|\boldsymbol{p}\right\|^2 = \frac{1}{4}\left[M^2 - 2\left({m_1}^2 + {m_2}^2\right) + \frac{\left({m_1}^2 - {m_2}^2\right)^2}{M^2}\right] \approx \left(\frac{M^2 - {m_1}^2}{2M}\right)^2$$

$$E_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2M}, \qquad E_2 = \frac{M^2 + {m_2}^2 - {m_1}^2}{2M};$$

$$K \cdot p_1 = \frac{M^2 + {m_1}^2 - {m_2}^2}{2}, \qquad p_1 \cdot p_2 = \frac{M^2 - \left({m_1}^2 + {m_2}^2\right)}{2}.$$

■Mandelstam Variables

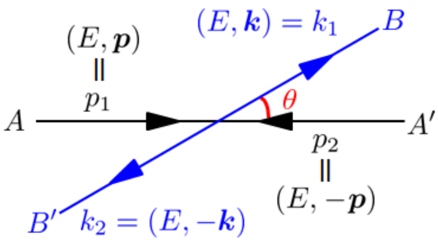
For $p_1 + p_2 \to k_1 + k_2$ collision,

$$\begin{aligned} s &= (p_1 + p_2)^2 = (k_1 + k_2)^2, \\ t &= (p_1 - k_1)^2 = (p_2 - k_2)^2, \\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2, \end{aligned}$$

and

$$s + t + u = p_1^2 + p_2^2 + k_1^2 + k_2^2 = \sum m^2.$$

■Kinematics of Collision (Same Mass)



$$\left\|\boldsymbol{p}\right\|^2 = E^2 - m_A^2 \qquad \boldsymbol{p}\cdot\boldsymbol{k} = \left\|\boldsymbol{p}\right\|\left\|\boldsymbol{k}\right\|\cos\theta$$

$$\left\|\boldsymbol{k}\right\|^2 = E^2 - m_B^2$$

$$p_1 \cdot p_2 = s/2 - m_A^2 \quad p_1 \cdot k_1 = p_2 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - t);$$

$$k_1 \cdot k_2 = s/2 - m_B^2 \quad p_1 \cdot k_2 = p_1 \cdot k_2 = \tfrac{1}{2}(m_A^2 + m_B^2 - u);$$

$$s = 4E^2,$$

$$(p_1 - p_2)^2 = -4(E^2 - m_A^2) \qquad t = -(2E^2 - m_A^2 - m_B^2) + 2\boldsymbol{p}\cdot\boldsymbol{k}$$

$$(k_1 - k_2)^2 = -4(E^2 - m_B^2) \qquad u = -(2E^2 - m_A^2 - m_B^2) - 2\boldsymbol{p}\cdot\boldsymbol{k}$$

Kinematics of Collision (initial massless)

$$s = 4E^2, \qquad (t,u) = -2E^2 + \frac{m_1^2 + m_2^2}{2} \pm 2\boldsymbol{p}\cdot\boldsymbol{k}, \qquad \|\boldsymbol{k}\|^2 = E^2 - \frac{m_1^2 + m_2^2}{2} + \frac{\left(m_1^2 - m_2^2\right)^2}{16E^2}$$

TODO: needs check

1.9 楊-Mills Theory

(See App. C.5 for verbose notes.)

1.9.1 Non-Abelian gauge theory

$$[T^a,T^b]=\mathrm{i}f^{ab}{}_cT^c,\qquad 0=f^D{}_{ab}f^E{}_{Dc}+f^D{}_{ca}f^E{}_{Db}+f^D{}_{bc}f^E{}_{Da},\qquad \mathrm{D}_\mu=\partial_\mu-\mathrm{i}gA_\mu$$

$$\operatorname{Tr}T^aT^b=\frac12\delta^{ab},\qquad [\widetilde{T}^a]_i{}^j:=T^{\mathrm{ad}\ a}{}_i{}^j:=-\mathrm{i}f^{aij}\qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j:=\delta_i^j\partial_\mu+gf^{iaj}A^a_\mu.$$

$$\begin{aligned}F_{\mu\nu}&=\frac{\mathrm{i}}{g}\left[\mathrm{D}_\mu,\mathrm{D}_\nu\right]\\&=\partial_\mu A_\nu-\partial_\nu A_\mu+\frac{g}{\mathrm{i}}\left[A_\mu,A_\nu\right]\\&=\Big[\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\Big]T^a\end{aligned}\qquad \begin{aligned}&\mathrm{D}_\mu\phi=\partial_\mu\phi-\mathrm{i}gA^a_\mu(T^a_\phi\phi)\\&\mathrm{D}_\mu F_{\mu\nu}{}^a=\partial_\mu F^a_{\mu\nu}+gf^{abc}A^b_\mu F^c_{\mu\nu},\\&\Big(\mathrm{D}_\mu F_{\nu\rho}=\partial_\mu\lambda-\mathrm{i}g[A_\mu,F_{\nu\rho}]\Big).^{*2}\end{aligned}$$

$$\phi\mapsto V\phi:=\mathrm{e}^{\mathrm{i}g\theta}\phi\qquad A_\mu\mapsto V\left(A_\mu+\frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1}\qquad F_{\mu\nu}\mapsto VF_{\mu\nu}V^{-1}$$

$$\phi^a{}'\simeq\phi+\mathrm{i}g\theta^aT^a\phi\qquad A^a_\mu{}'\simeq A^a_\mu+\partial_\mu\theta^a+gf^{abc}A^b_\mu\theta^c\qquad F^a_{\mu\nu}{}'\simeq F^a_{\mu\nu}+gf^{abc}F^b_{\mu\nu}\theta^c$$

$$\epsilon^{\mu\nu\rho\sigma}\left[\mathrm{D}_\nu,[\mathrm{D}_\rho,\mathrm{D}_\sigma]\right]=\epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma}=0.$$

■Killing and Casimir Here we have two constants which **depend on representation** r.

$$\operatorname{Tr}(T^aT^b)=:C(r)\delta^{ab}\quad\text{(Killing form)},\qquad T^aT^a=:C_2(r)\cdot\mathbf{1}\quad\text{(quadratic Casimir operator)},\tag{1.50}$$

which satisfy

$$C(r)=\frac{d(r)}{d(\mathrm{ad})}C_2(r),\qquad T^aT^bT^a=\left[C_2(r)-\frac12C_2(\mathrm{ad})\right]T^b,\tag{1.51}$$

$$f^{acd}f^{bcd}=C_2(\mathrm{ad})\delta^{ab},\qquad f^{abc}T^bT^c=\frac{\mathrm{i}}{2}C_2(\mathrm{ad})T^a.\tag{1.52}$$

For SU(N) For its fundamental representation N with definition $C(N):=\frac12$, we have

$$C(N):=\frac12,\quad C_2(N)=\frac{N^2-1}{2N},\quad C(\mathrm{ad})=C_2(\mathrm{ad})=N;\quad (T^a)_{ij}(T^a)_{kl}=\frac12\left(\delta_{il}\delta_{kj}-\frac{\delta_{ij}\delta_{kl}}{N}\right).$$

1.9.2 Abelian gauge theory

In Abelian gauge theory, V and fields are always commutative, and thus we have charge freedom (Q).

$$\mathrm{D}_\mu\phi=(\partial_\mu-\mathrm{i}gA_\mu Q)\,\phi\qquad \phi\mapsto\mathrm{e}^{igQ\theta}\phi\qquad F_{\mu\nu}=\frac{\mathrm{i}}{g}\left[\mathrm{D}_\mu,\mathrm{D}_\nu\right]=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$\mathrm{D}_\mu\lambda^a=\partial_\mu\lambda^a\qquad A_\mu\mapsto A_\mu+\partial_\mu\theta\qquad F_{\mu\nu}\mapsto F_{\mu\nu}$$

1.9.3 Lagrangian Block

$$\mathcal{L}\ni|\mathrm{D}_\mu\phi|^2-m^2|\phi|^2,\quad \overline{\psi}(\mathrm{i}\not{\mathrm{D}}-m)\psi,\quad -\frac14F^{\mu\nu}_aF^a_{\mu\nu}\ \left(=-\frac12\operatorname{Tr}F^{\mu\nu}F_{\mu\nu}\right),\quad \theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\tag{1.53}$$

$$-\frac14F^{\mu\nu}_aF^a_{\mu\nu}=-\frac12\left[(\partial_\mu A^a_\nu)^2+A^a_\mu\partial^\mu\partial^\nu A^a_\nu\right]-gf^{abc}A^a_\mu A^b_\nu\partial^\mu A^{c\nu}-\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}\tag{1.54}$$

---

*2 Note that we can use any representation T^a but must the same ones for $A^a_\mu T^a$ and λ^aT^a.

1.10 SU(2) Gauge Group

The SU(2) is defined as $[T^a, T^b] = \mathrm{i}\epsilon^{abc}T^c$. The representation whose generator is $\tau^a := \sigma^2/2$ is called the fundamental representation $\mathbf{2}$.

1.10.1 Gauge singlet of SU(2)

Consider the fields which transform under $\mathbf{2}$,

$$\phi \mapsto \phi + \mathrm{i}g\theta^a \left(\tau^a\phi\right). \tag{1.55}$$

For scalar fields ϕ and vector fields v, it is obvious that $\phi^\dagger\phi'(=\phi^*_a\phi'_a)$ and $v^\mu v'^*_\nu$ are gauge singlet.

Fermions are Lorentz symmetric under the form $\xi^\alpha\chi_\alpha$. So, in order to make a gauge and Lorentz invariant term, we have to build $\overline{\mathbf{2}}$ representation. It can be done with σ^2:

$$\begin{aligned} \delta(\sigma^2\xi^\alpha) = \sigma^2\cdot\mathrm{i}g\theta^a\left(\tau^a\xi^\alpha\right) = \mathrm{i}g\theta^a\left(\sigma^2\tau^a\sigma^2\right)\left(\sigma^2\xi^\alpha\right) = -\mathrm{i}g\theta^a\tau^{a\mathrm{T}}\left(\sigma^2\xi^\alpha\right), \\ \text{i.e.,}\quad (\xi^\alpha_a\sigma^2)^\mathrm{T} \mapsto (\xi^\alpha\sigma^2)^\mathrm{T} - \mathrm{i}g\theta^a\cdot(\xi^\alpha\sigma^2)^\mathrm{T}\tau^a \end{aligned} \tag{1.56}$$

Thus $\sigma^2\xi^\alpha$ follows $\overline{\mathbf{2}}$ representation, and $(\sigma^2\xi^\alpha)\chi_\alpha = \mathrm{i}\epsilon_{ab}(\xi_a)^\alpha(\chi_b)_\alpha$ is invariant under gauge and Lorentz transformation.

In summary, Weyl (Dirac) spinors ξ (ψ) in $\mathbf{2}$ representation composes a gauge singlet in the form of

$$\epsilon_{ab}\xi_a\xi'_b \qquad (\overline{\psi}\psi'). \tag{1.57}$$

1.10.2 $\mathbf{3}$ representation

Let us consider a field ϕ whose SU(2) charge is $\mathbf{3}$:[*3]

$$\phi \mapsto \exp\left(\mathrm{i}g\theta^A J^A\right)\phi \qquad \text{with}\quad [J^A]_{bc} = -\mathrm{i}\epsilon^{Abc}. \tag{1.58}$$

Then,

$$\phi^a\tau^a \mapsto \mathrm{e}^{\mathrm{i}g\theta^A\tau^a}\left(\phi^a\tau^a\right)\mathrm{e}^{-\mathrm{i}g\theta^A\tau^a}. \tag{1.59}$$

It is convenient to define $\sigma^\pm$ and $\phi^\pm$ as

$$\phi^\pm := \frac{1}{\sqrt{2}}(\phi^1 \mp \mathrm{i}\phi^2),\quad \sigma^+ := \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma^- := \begin{pmatrix}0&0\\1&0\end{pmatrix};\quad \phi^1 = \frac{\phi^+ + \phi^-}{\sqrt{2}}\quad \phi^2 = \frac{\mathrm{i}(\phi^+ - \phi^-)}{\sqrt{2}}; \tag{1.60}$$

$$\phi^a\tau^a = \frac{1}{2}\begin{pmatrix}\phi^3 & \sqrt{2}\phi^+\\ \sqrt{2}\phi^- & -\phi^3\end{pmatrix} = \frac{\sigma^+}{\sqrt{2}}\phi^+ + \frac{\sigma^-}{\sqrt{2}}\phi^- + \frac{\sigma^3}{2}\phi^3. \tag{1.61}$$

[*3] Note that the representation $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\right\}$ is equivalent to J^A.

2 Standard Model

Any representations assumed to be *normalized Hermitian.* Note that the SU(2) **2** representation is

$$T^a = \frac{1}{2}\sigma^a; \qquad \left[T^a, T^b\right] = \mathrm{i}\epsilon^{abc}T^c; \qquad T^\pm := T^1 \pm \mathrm{i}T^2. \tag{2.1}$$

We use the following abridged notations:

$$(\partial A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.2}$$

2.1 Symmetries and Fields

	$\mathrm{SU}(3)_{\mathrm{strong}}$	$\mathrm{SU}(2)_{\mathrm{weak}}$	$\mathrm{U}(1)_Y$
Matter Fields (Fermionic / Lorentz Spinor)			
$P_{\mathrm{L}}Q_i$: Left-handed quarks	**3**	**2**	1/6
$P_{\mathrm{L}}U_i$: Right-handed up-type quarks	**3**	**1**	2/3
$P_{\mathrm{R}}D_i$: Right-handed down-type quarks	**3**	**1**	−1/3
$P_{\mathrm{R}}L_i$: Left-handed leptons	**1**	**2**	−1/2
$P_{\mathrm{R}}E_i$: Right-handed leptons	**1**	**1**	−1
Higgs Field (Bosonic / Lorentz Scalar)			
H : Higgs	**1**	**2**	1/2
Gauge Fields (Bosonic / Lorentz Vector)			
G : Gluons	**8**	**1**	0
W : Weak bosons	**1**	**3**	0
B : B boson	**1**	**1**	0

■Full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{湯川}}$

where
$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)H\right|^2 - V(H), \tag{2.4}$$

$$\begin{aligned}\mathcal{L}_{\text{matter}} = {} & \overline{Q}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \frac{1}{6}\mathrm{i}g_1B_\mu\right)P_{\mathrm{L}}Q_i \\ & + \overline{U}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \frac{2}{3}\mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}U_i \\ & + \overline{D}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu + \frac{1}{3}\mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}D_i \\ & + \overline{L}_i\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)P_{\mathrm{L}}L_i \\ & + \overline{E}_i\mathrm{i}\gamma^\mu\left(\partial_\mu + \mathrm{i}g_1B_\mu\right)P_{\mathrm{R}}E_i,\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{\text{湯川}} = \overline{U}_i(y_u)_{ij}HP_{\mathrm{L}}Q_j - \overline{D}_i(y_d)_{ij}H^\dagger P_{\mathrm{L}}Q_j - \overline{E}_i(y_e)_{ij}H^\dagger P_{\mathrm{L}}L_j + \text{H.c.} \tag{2.6}$$

We have no freedom to add other terms into this Lagrangian of the gauge theory. See Appendix C.4.

■Gauge Kinetic Terms the gauge kinetic terms can be expanded as

$$\begin{aligned}\mathcal{L}_{\text{gauge}} =& -\frac{1}{4}(\partial B)(\partial B)\\ &-\frac{1}{4}(\partial W^a)(\partial W^a) - g_2\epsilon^{abc}(\partial_\mu W^a_\nu)W^{\mu b}W^{\nu c} - \frac{{g_2}^2}{4}\left(\epsilon^{eab}W^a_\mu W^b_\nu\right)\left(\epsilon^{ecd}W^{c\mu}W^{d\nu}\right)\\ &-\frac{1}{4}(\partial G^a)(\partial G^a) - g_3 f^{abc}(\partial_\mu G^a_\nu)G^{\mu b}G^{\nu c} - \frac{{g_3}^2}{4}\left(f^{eab}G^a_\mu G^b_\nu\right)\left(f^{ecd}G^{c\mu}G^{d\nu}\right).\end{aligned} \tag{2.7}$$

2.2 Higgs Mechanism

■Higgs Potential The (renormalizable) Higgs potential must be

$$V(H) = -\mu^2(H^\dagger H) + \lambda\left(H^\dagger H\right)^2. \tag{2.8}$$

for the SU(2), and $\lambda > 0$ in order not to run away the VEVs, while μ^2 is positive for the EWSB.

To discuss this clearly, let us *redefine* the Higgs field *by gauge-fixing* as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + \mathrm{i}\phi_2\\ v + (h + \mathrm{i}\phi_3)\end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.9}$$

Here h is the "Higgs boson," and ϕ_i are 南部–Goldstone bosons.

The Higgs potential becomes

$$V(h) = \frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3 + \mu^2 h^2, \tag{2.10}$$

and now we know the Higgs boson has acquired mass $m_h = \sqrt{2}\mu$. Also

$$\mathcal{L}_{\text{Higgs}} = \left|\left(\partial_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{2}\mathrm{i}g_1 B_\mu\right)H\right|^2 \tag{2.11}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\Big[{g_2}^2{W_1}^2 + {g_2}^2{W_2}^2 + (g_1 B - g_2 W_3)^2\Big]. \tag{2.12}$$

Redefining the gauge fields (with concerning the norms) as

$$W^\pm_\mu := \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i}W^2_\mu), \qquad \begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix} := \begin{pmatrix}\cos\theta_{\mathrm{w}} & -\sin\theta_{\mathrm{w}}\\ \sin\theta_{\mathrm{w}} & \cos\theta_{\mathrm{w}}\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}, \tag{2.13}$$

where

$$\tan\theta_{\mathrm{w}} := \frac{g_1}{g_2}, \qquad e := -\frac{g_1 g_2}{\sqrt{{g_1}^2 + {g_2}^2}}; \qquad g_Z := \sqrt{{g_1}^2 + {g_2}^2}; \tag{2.14}$$

$$g_1 = \frac{|e|}{\cos\theta_{\mathrm{w}}} = g_Z\sin\theta_{\mathrm{w}}, \qquad g_2 = \frac{|e|}{\sin\theta_{\mathrm{w}}} = g_Z\cos\theta_{\mathrm{w}}. \tag{2.15}$$

We obtain the following terms in $\mathcal{L}_{\text{Higgs}}$:

$$\mathcal{L}_{\text{Higgs}} \supset \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{4}\Big[{g_2}^2 W^{+\mu}W^-_\mu + \frac{{g_Z}^2}{2}Z^\mu Z_\mu\Big]. \tag{2.16}$$

Here we have omitted the 南部–Goldstone bosons.

Here we present another form:

$$g_1 B_\mu = |e| A_\mu - \tan\theta_\mathrm{w} Z_\mu, \tag{2.17}$$

$$g_2 W_\mu = \frac{g_2}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) + \left(\frac{|e|}{\tan\theta_\mathrm{w}} Z_\mu + |e| A_\mu\right) T^3, \tag{2.18}$$

$$Z^0_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_2 W^3_\mu - g_1 B_\mu), \quad A_\mu := \frac{1}{\sqrt{{g_1}^2 + {g_2}^2}}(g_1 W^3_\mu + g_2 B_\mu) \tag{2.19}$$

You can see the gauge bosons have acquired the masses

$$m_A = 0, \quad m_W := \frac{g_2}{2} v, \quad m_Z := \frac{g_Z}{2} v. \tag{2.20}$$

■Gauge Term The SU(2) gauge term is converted into

$$\begin{aligned} W^{a\mu\nu} W^a_{\mu\nu} = &(\partial W^3)(\partial W^3) + 2(\partial W^+)(\partial W^-) \\ &- 4\mathrm{i}g\left[(\partial W^3)^{\mu\nu} W^+_\mu W^-_\nu + (\partial W^+)^{\mu\nu} W^-_\mu W^3_\nu + (\partial W^-)^{\mu\nu} W^3_\mu W^+_\nu\right] \\ &- 2g^2(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\left(W^+_\mu W^+_\nu W^-_\rho W^-_\sigma - 2W^3_\mu W^3_\nu W^+_\rho W^-_\sigma\right), \end{aligned}$$

and therefore the final expression is

$$\begin{aligned} \mathcal{L}_\mathrm{gauge} := &-\frac{1}{4}\left[G^{a\mu\nu} G^a_{\mu\nu} + (\partial Z)^{\mu\nu}(\partial Z)_{\mu\nu} + (\partial A)^{\mu\nu}(\partial A)_{\mu\nu} + 2(\partial W^+)^{\mu\nu}(\partial W^-)_{\mu\nu}\right] \\ &+ \frac{\mathrm{i}|e|}{\tan\theta_\mathrm{w}}\Big[(\partial W^+)^{\mu\nu} W^-_\mu Z_\nu + (\partial W^-)^{\mu\nu} Z_\mu W^+_\nu + (\partial Z)^{\mu\nu} W^+_\mu W^-_\nu\Big] \\ &+ \mathrm{i}|e|\Big[(\partial W^+)^{\mu\nu} W^-_\mu A_\nu + (\partial W^-)^{\mu\nu} A_\mu W^+_\nu + (\partial A)^{\mu\nu} W^+_\mu W^-_\nu\Big] \\ &+ (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\Bigg[\frac{|e|^2}{2\sin^2\theta_\mathrm{w}} W^+_\mu W^+_\nu W^-_\rho W^-_\sigma + \frac{|e|^2}{\tan^2\theta_\mathrm{w}} W^+_\mu Z_\nu W^-_\rho Z_\sigma \\ &\qquad + \frac{|e|^2}{\tan\theta_\mathrm{w}}\left(W^+_\mu Z_\nu W^-_\rho A_\sigma + W^+_\mu A_\nu W^-_\rho Z_\sigma\right) + |e|^2 W^+_\mu A_\nu W^-_\rho A_\sigma\Bigg]. \end{aligned} \tag{2.21}$$

■湯川 Term

$$\begin{aligned} \mathcal{L}_{湯川} = &\overline{U} y_u H P_\mathrm{L} Q - \overline{D} y_d H^\dagger P_\mathrm{L} Q - \overline{E} y_e H^\dagger P_\mathrm{L} L + \mathrm{H.c.} \\ = &\overline{U} y_u \epsilon^{\alpha\beta} H^\alpha P_\mathrm{L} Q^\beta - \overline{D} y_d {H^\dagger}^\alpha P_\mathrm{L} Q^\alpha - \overline{E} y_e {H^\dagger}^\alpha P_\mathrm{L} L^\alpha + \mathrm{H.c.} \\ = &-\frac{v+h}{\sqrt{2}}\left(\overline{U} y_u P_\mathrm{L} Q^1 + \overline{D} y_d P_\mathrm{L} Q^2 + \overline{E} y_e P_\mathrm{L} L^2\right) + \mathrm{H.c.} \end{aligned} \tag{2.22}$$

2.3 Full Lagrangian After Higgs Mechanism

Now we have the following Lagrangian (with omitting P_{L} etc.):

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} + {m_W}^2 W^+W^- + \frac{{m_Z}^2}{2}Z^2 \\
\text{【Higgs】}\quad & + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}{m_h}^2h^2 - \sqrt{\frac{\lambda}{2}}m_h h^3 - \frac{1}{4}\lambda h^4 \\
& + \frac{v{g_2}^2}{4}W^+W^-h + \frac{v({g_1}^2+{g_2}^2)}{8}Z^2h \\
& + \frac{{g_2}^2}{4}W^+W^-h^2 + \frac{{g_1}^2+{g_2}^2}{8}Z^2h^2 \\
& - \left(\frac{1}{\sqrt{2}}h\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}h\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}h\bar{E}y_eL^2 + \text{H.c.}\right) \\
\text{【SU(3)】}\quad & + \bar{Q}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)Q + \bar{U}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)U + \bar{D}\left(\mathrm{i}\partial\!\!\!/ + g_3 G\!\!\!/\right)D + \bar{L}\left(\mathrm{i}\partial\!\!\!/\right)L + \bar{E}\left(\mathrm{i}\partial\!\!\!/\right)E \\
\text{【}W\text{】}\quad & + \bar{Q}\frac{g_2}{\sqrt{2}}\left(W\!\!\!\!/^{\,+}T^+ + W\!\!\!\!/^{\,-}T^-\right)Q + \bar{L}\frac{g_2}{\sqrt{2}}\left(W\!\!\!\!/^{\,+}T^+ + W\!\!\!\!/^{\,-}T^-\right)L \\
\text{【}A\&Z^0\text{】}\quad & + \bar{Q}\left[\left(T^3+\frac{1}{6}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right)Z\!\!\!/^0\right]Q \\
& + \bar{U}\left(\frac{2}{3}|e|A\!\!\!/ - \frac{2|e|s}{3c}Z\!\!\!/\right)U \\
& + \bar{D}\left(-\frac{1}{3}|e|A\!\!\!/ + \frac{|e|s}{3c}Z\!\!\!/\right)D \\
& + \bar{L}\left[\left(T^3-\frac{1}{2}\right)|e|A\!\!\!/ + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right)Z\!\!\!/^0\right]L \\
& + \bar{E}\left(-|e|A\!\!\!/ + \frac{|e|s}{c}Z\!\!\!/\right)E \\
\text{【湯川項】}\quad & - \left(\frac{1}{\sqrt{2}}v\bar{U}y_uQ^1 + \frac{1}{\sqrt{2}}v\bar{D}y_dQ^2 + \frac{1}{\sqrt{2}}v\bar{E}y_eL^2 + \text{H.c.}\right)
\end{aligned}
\tag{2.23}
$$

2.4 Mass Eigenstates

Here we will obtain the mass eigenstates of the fermions, by diagonalizing the 湯川 matrices.

We use the singular value decomposition method to mass matrices $Y_\bullet := vy_\bullet/\sqrt{2}$. Generally, any matrices can be transformed with two unitary matrices Ψ and Φ as

$$
Y = \Phi^\dagger \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \Psi =: \Phi^\dagger M \Psi \qquad (m_i \geq 0). \tag{2.24}
$$

Using this Ψ and Φ, we can rotate the basis as

$$
Q^1 \mapsto \Psi_u^\dagger Q^1, \quad Q^2 \mapsto \Psi_d^\dagger Q^2, \quad L \mapsto \Psi_e^\dagger L, \qquad U \mapsto \Phi_u^\dagger U, \quad D \mapsto \Phi_d^\dagger D, \quad E \mapsto \Phi_e^\dagger E, \tag{2.25}
$$

and now we have the 湯川 terms in mass eigenstates as

$$
\mathcal{L}_{\text{湯川}} = -\left(1+\frac{1}{v}h\right)\left[(m_u)_i\overline{U}_iP_{\mathrm{L}}Q_i^1 + (m_d)_i\overline{D}_iP_{\mathrm{L}}Q_i^2 + (m_e)_i\overline{E}_iP_{\mathrm{L}}L_i^2 + \text{H.c.}\right]. \tag{2.26}
$$

In the transformation from the gauge eigenstates to the mass eigenstates, almost all the terms in the Lagrangian are not modified. However, only the terms of quark–quark–W interactions do change drastically, as

$$\begin{aligned}
\mathcal{L} &\supset \overline{Q}\mathrm{i}\gamma^\mu\left(-\mathrm{i}g_2W_\mu - \frac{1}{6}\mathrm{i}g_1B_\mu\right)P_\mathrm{L}Q && (2.27)\\
&= \overline{Q}\frac{g_2}{\sqrt{2}}\left(\not{W}^+T^+ + \not{W}^-T^-\right)P_\mathrm{L}Q \quad + \quad \text{(interaction terms with } Z \text{ and } A) && (2.28)\\
&\mapsto \frac{g_2}{\sqrt{2}}\begin{pmatrix}\overline{Q}^1\Psi_u & \overline{Q}^2\Psi_d\end{pmatrix}\begin{pmatrix}0 & \not{W}^+\\ \not{W}^- & 0\end{pmatrix}P_\mathrm{L}\begin{pmatrix}\Psi_u^\dagger Q^1\\ \Psi_d^\dagger Q^2\end{pmatrix} + (\quad\cdots\quad) && (2.29)\\
&= \frac{g_2}{\sqrt{2}}\left[\overline{Q}^2\not{W}^-XP_\mathrm{L}Q^1 + \overline{Q}^1\not{W}^+X^\dagger P_\mathrm{L}Q^2\right] + (\quad\cdots\quad), && (2.30)
\end{aligned}$$

where $X := \Psi_d\Psi_u^\dagger$ is a matrix, so-called the Cabbibo–小林–益川 (CKM) matrix, which is *not* diagonal, and *not* real, generally. These terms violate the flavor symmetry of quarks, and even the CP-symmetry.

> In our notation, CP-transformation of a spinor is described as
>
> $$\mathscr{C}\mathscr{P}(\psi) = -\mathrm{i}\eta^*(\overline{\psi}\gamma^2)^\mathrm{T}, \quad \mathscr{C}\mathscr{P}\left(\overline{\psi}\right) = \mathrm{i}\eta(\gamma^2\psi)^\mathrm{T}, \tag{2.31}$$
>
> where η is a complex phase ($|\eta| = 1$). Under this transformation, those terms are transformed as, e.g.,
>
> $$\begin{aligned}
\mathscr{C}\mathscr{P}\left(\overline{Q}^2\not{W}^-XP_\mathrm{L}Q^1\right) &= (\gamma^2Q^2)^\mathrm{T}\mathscr{P}(-\not{W}^+)XP_\mathrm{L}(\overline{Q}^1\gamma^2)^\mathrm{T}\\
&= -W_\mu^{+P}(\gamma^2Q^2)^\mathrm{T}(\overline{Q}^1X^\mathrm{T}\gamma^2P_\mathrm{L}\gamma^{\mu\mathrm{T}})^\mathrm{T}\\
&= (\overline{Q}^1\not{W}^+X^\mathrm{T}P_\mathrm{L}Q^2).
\end{aligned} \tag{2.32}$$
>
> Therefore, we can see that the CP-symmetry is maintained if and only if $X^\mathrm{T} = X^\dagger$, that is, if and only if X is a real matrix.

以上より，標準模型の Lagrangian は

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_{\text{gauge}} \\
\text{【質量項】} \quad & + \; {m_W}^2 W^+ W^- \; + \; \frac{{m_Z}^2}{2} Z^2 \\
& - \; \left(\bar{U} M_u P_{\mathrm{L}} Q^1 \; + \; \bar{D} M_d P_{\mathrm{L}} Q^2 \; + \; \bar{E} M_e P_{\mathrm{L}} L^2 \; + \; \text{H.c.}\right) \\
\text{【 Higgs Field 】} \quad & + \; \frac{1}{2}(\partial_\mu h)^2 \quad - \frac{1}{2}{m_h}^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \\
\text{【 Higgs との結合】} \quad & + \; \frac{v {g_2}^2}{4} W^+ W^- h \; + \; \frac{v({g_1}^2 + {g_2}^2)}{8} Z^2 h \\
& + \; \frac{{g_2}^2}{4} W^+ W^- h^2 \; + \; \frac{{g_1}^2 + {g_2}^2}{8} Z^2 h^2 \\
& - \; \left(\frac{1}{v}\bar{U} M_u P_{\mathrm{L}} Q^1 h \; + \; \frac{1}{v}\bar{D} M_d P_{\mathrm{L}} Q^2 h \; + \; \frac{1}{v}\bar{E} M_e P_{\mathrm{L}} L^2 h \; + \; \text{H.c.}\right) \\
\text{【 SU(3) および微分項】} \quad & + \; \bar{Q}\left(\mathrm{i}\not{\partial} + g_3 \not{G}\right) P_{\mathrm{L}} Q \; + \; \bar{U}\left(\mathrm{i}\not{\partial} + g_3 \not{G}\right) P_{\mathrm{R}} U \; + \; \bar{D}\left(\mathrm{i}\not{\partial} + g_3 \not{G}\right) P_{\mathrm{R}} D \\
& + \; \bar{L}\left(\mathrm{i}\not{\partial}\right) P_{\mathrm{L}} L \; + \; \bar{E}\left(\mathrm{i}\not{\partial}\right) P_{\mathrm{R}} E \\
\text{【 } W \text{ boson 】} \quad & + \; \frac{g_2}{\sqrt{2}}\left[\bar{Q}^2 \not{W}^- X P_{\mathrm{L}} Q^1 + \bar{Q}^1 \not{W}^+ X^\dagger P_{\mathrm{L}} Q^2\right] \qquad \text{【} \leftarrow CP \text{ and flavor violating!】} \\
& + \; \bar{L}\frac{g_2}{\sqrt{2}}\left(\not{W}^+ T^+ + \not{W}^- T^-\right) P_{\mathrm{L}} L \\
\text{【 } A\&Z^0 \text{ boson 】} \quad & + \; \bar{Q}\left[\left(T^3 + \frac{1}{6}\right)|e|\not{A} + \left(\frac{|e|c}{s}T^3 - \frac{|e|s}{6c}\right)\not{Z}^0\right] P_{\mathrm{L}} Q \\
& + \; \bar{U}\left(\frac{2}{3}|e|\not{A} - \frac{2|e|s}{3c}\not{Z}\right) P_{\mathrm{R}} U \\
& + \; \bar{D}\left(-\frac{1}{3}|e|\not{A} + \frac{|e|s}{3c}\not{Z}\right) P_{\mathrm{R}} D \\
& + \; \bar{L}\left[\left(T^3 - \frac{1}{2}\right)|e|\not{A} + \left(\frac{|e|c}{s}T^3 + \frac{|e|s}{2c}\right)\not{Z}^0\right] P_{\mathrm{L}} L \\
& + \; \bar{E}\left(-|e|\not{A} + \frac{|e|s}{c}\not{Z}\right) P_{\mathrm{R}} E
\end{aligned}
\tag{2.33}
$$

となる。

2.5 Chiral Notation

In the chiral expression, the Lagrangian is written as

$$
\begin{aligned}
\mathcal{L} =&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ Q_\mathrm{L}^\dagger \mathrm{i}\bar\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \mathrm{i}g_2 W_\mu - \frac{1}{6}\mathrm{i}g_1 B_\mu\right) Q_\mathrm{L}\\
&+ U_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu - \frac{2}{3}\mathrm{i}g_1 B_\mu\right) U_\mathrm{R}\\
&+ D_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu - \mathrm{i}g_3 G_\mu + \frac{1}{3}\mathrm{i}g_1 B_\mu\right) D_\mathrm{R}\\
&+ L_\mathrm{L}^\dagger \mathrm{i}\bar\sigma^\mu \left(\partial_\mu - \mathrm{i}g_2 W_\mu + \frac{1}{2}\mathrm{i}g_1 B_\mu\right) L_\mathrm{L}\\
&+ E_\mathrm{R}^\dagger \mathrm{i}\sigma^\mu \left(\partial_\mu + \mathrm{i}g_1 B_\mu\right) E_\mathrm{R}\\
&- \left(U_\mathrm{R}^\dagger y_u H Q_\mathrm{L} + D_\mathrm{R}^\dagger y_d H^\dagger Q_\mathrm{L} + E_\mathrm{R}^\dagger y_e H^\dagger L_\mathrm{L} + \text{H.c.}\right)\\
=&(\text{Higgs terms}) + (\text{Gauge fields strength})\\
&+ \mathrm{i}Q_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu Q_\mathrm{L} + \mathrm{i}U_\mathrm{R}\bar\sigma^\mu\partial_\mu U_\mathrm{R}^\dagger + \mathrm{i}D_\mathrm{R}\bar\sigma^\mu\partial_\mu D_\mathrm{R}^\dagger + \mathrm{i}L_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu L_\mathrm{L} + \mathrm{i}E_\mathrm{R}\bar\sigma^\mu\partial_\mu E_\mathrm{R}^\dagger\\
&+ g_3\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu G_\mu Q_\mathrm{L} + U_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu U_\mathrm{R} + D_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu D_\mathrm{R}\right)\\
&+ g_2\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu Q_\mathrm{L} + L_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu L_\mathrm{L}\right)\\
&+ g_1\left(\frac{1}{6}Q_\mathrm{L}^\dagger\bar\sigma^\mu B_\mu Q_\mathrm{L} + \frac{2}{3}U_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu U_\mathrm{R} - \frac{1}{3}D_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu D_\mathrm{R} - \frac{1}{2}L_\mathrm{L}^\dagger\bar\sigma^\mu B_\mu L_\mathrm{L} - E_\mathrm{R}^\dagger\bar\sigma^\mu B_\mu E_\mathrm{R}\right)\\
&- \left(U_\mathrm{R}^\dagger y_u H Q_\mathrm{L} + D_\mathrm{R}^\dagger y_d H^\dagger Q_\mathrm{L} + E_\mathrm{R}^\dagger y_e H^\dagger L_\mathrm{L} + \text{H.c.}\right),
\end{aligned}
\tag{2.34}
$$

and finally we obtain

$$
\begin{aligned}
\mathcal{L} =&(\text{Gauge bosons and Higgs})\\
&+ \mathrm{i}Q_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu Q_\mathrm{L} + \mathrm{i}U_\mathrm{R}\bar\sigma^\mu\partial_\mu U_\mathrm{R}^\dagger + \mathrm{i}D_\mathrm{R}\bar\sigma^\mu\partial_\mu D_\mathrm{R}^\dagger + \mathrm{i}L_\mathrm{L}^\dagger\bar\sigma^\mu\partial_\mu L_\mathrm{L} + \mathrm{i}E_\mathrm{R}\bar\sigma^\mu\partial_\mu E_\mathrm{R}^\dagger\\
&+ g_3\left(Q_\mathrm{L}^\dagger\bar\sigma^\mu G_\mu Q_\mathrm{L} + U_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu U_\mathrm{R} + D_\mathrm{R}^\dagger\bar\sigma^\mu G_\mu D_\mathrm{R}\right)\\
&- m_u(u_\mathrm{R}^\dagger u_\mathrm{L} + u_\mathrm{L}^\dagger u_\mathrm{R}) - (\text{quarks}) - m_e(e_\mathrm{R}^\dagger e_\mathrm{L} + e_\mathrm{L}^\dagger e_\mathrm{R}) - (\text{leptons})\\
&- \frac{m_u}{v}(u_\mathrm{R}^\dagger u_\mathrm{L} + u_\mathrm{L}^\dagger u_\mathrm{R})h - (\text{quarks}) - \frac{m_e}{v}(e_\mathrm{R}^\dagger e_\mathrm{L} + e_\mathrm{L}^\dagger e_\mathrm{R})h - (\text{leptons})\\
&+ \frac{g_2}{\sqrt{2}}\left[\begin{pmatrix}d_\mathrm{L}^\dagger & s_\mathrm{L}^\dagger & b_\mathrm{L}^\dagger\end{pmatrix}\bar\sigma^\mu W_\mu^- X\begin{pmatrix}u_\mathrm{L}\\ c_\mathrm{L}\\ t_\mathrm{L}\end{pmatrix} + \begin{pmatrix}u_\mathrm{L}^\dagger & c_\mathrm{L}^\dagger & t_\mathrm{L}^\dagger\end{pmatrix}\bar\sigma^\mu W_\mu^+ X^\dagger\begin{pmatrix}d_\mathrm{L}\\ s_\mathrm{L}\\ b_\mathrm{L}\end{pmatrix}\right]\\
&+ \frac{g_2}{\sqrt{2}}\left[\nu_e^\dagger\bar\sigma^\mu W_\mu^+ e_\mathrm{L} + e_\mathrm{L}^\dagger\bar\sigma^\mu W_\mu^-\nu_e\right]\\
&+ |e|\left[\frac{2}{3}u_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu u_\mathrm{L} - \frac{1}{3}d_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu d_\mathrm{L} + \frac{2}{3}u_\mathrm{R}^\dagger\sigma^\mu A_\mu u_\mathrm{R} - \frac{1}{3}d_\mathrm{R}^\dagger\sigma^\mu A_\mu d_\mathrm{R} + (\text{quarks})\right.\\
&\qquad\left. - e_\mathrm{L}^\dagger\bar\sigma^\mu A_\mu e_\mathrm{L} - e_\mathrm{R}^\dagger\sigma^\mu A_\mu e_\mathrm{R} + (\text{leptons})\right]\\
&+ \frac{|e|s}{c}\left[\left(\frac{c^2}{2s^2} - \frac{1}{6}\right)u_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu u_\mathrm{L} - \left(\frac{c^2}{2s^2} + \frac{1}{6}\right)d_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu d_\mathrm{L} - \frac{2}{3}u_\mathrm{R}^\dagger\sigma^\mu Z_\mu u_\mathrm{R} + \frac{1}{3}d_\mathrm{R}^\dagger\sigma^\mu Z_\mu d_\mathrm{R}\right.\\
&\qquad\left. + \left(\frac{c^2}{2s^2} + \frac{1}{2}\right)\nu_e^\dagger\bar\sigma^\mu Z_\mu\nu_e - \left(\frac{c^2}{2s^2} - \frac{1}{2}\right)e_\mathrm{L}^\dagger\bar\sigma^\mu Z_\mu e_\mathrm{L} + e_\mathrm{R}^\dagger\sigma^\mu Z_\mu e_\mathrm{R} + (\text{others})\right].
\end{aligned}
\tag{2.35}
$$

2.6 Values of SM Parameters

(Extracted from PDG 2010 / 2012)

2.6.1 Experimental Values

Theoretical Parameters 【These values are all in $\overline{\mathrm{MS}}$ scheme.】

$$\alpha_{\mathrm{EM}}^{-1}(0) = 137.035999074(44) \qquad G_{\mathrm{F}} = \frac{{g_2}^2}{4\sqrt{2}{m_W}^2} = \frac{1}{\sqrt{2}v^2} = 1.1663787(6) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

$$\alpha_{\mathrm{EM}}^{-1}(m_Z) = 127.944(14) \qquad m_W(m_W) = 80.385(15)\,\mathrm{GeV} \qquad \Gamma_W \approx 2.085(42)\,\mathrm{GeV}$$

$$\alpha_{\mathrm{EM}}^{-1}(m_\tau) = 133.471(14) \qquad m_Z(m_Z) = 91.1876(21)\,\mathrm{GeV} \qquad \Gamma_Z \approx 2.4952(23)\,\mathrm{GeV}$$

$$\alpha_{\mathrm{s}}(m_Z) = 0.1184(7) \qquad \sin^2\theta_{\mathrm{W}}(m_Z) = 0.23116(12) \qquad \sin^2\theta_{\mathrm{eff}} = 0.23146(12)$$

Masses and Lifetimes 【t pole mass is the “MC mass”. Quark $\overline{\mathrm{MS}}$ mass at 1 GeV can be obtained by $\times 1.35$.】

$$e : 0.510998928(11)\,\mathrm{MeV} \qquad \mu : 105.6583715(35)\,\mathrm{MeV} \qquad \tau : 1.77682(16)\,\mathrm{GeV}$$

$$[\overline{\mathrm{MS}}\ (2\,\mathrm{GeV})] \quad u : 2.3^{+0.7}_{-0.5}\,\mathrm{MeV} \qquad [\overline{\mathrm{MS}}(m)] \quad c : 1.275(25)\,\mathrm{GeV} \qquad [\mathrm{pole}] \quad c : 1.67(7)\,\mathrm{GeV}$$

$$d : 4.8^{+0.7}_{-0.3}\,\mathrm{MeV} \qquad b : 4.18(3)\,\mathrm{GeV} \qquad b : 4.78(6)\,\mathrm{GeV}$$

$$s : 95 \pm 5\,\mathrm{MeV} \qquad t : 160^{+5}_{-4}\,\mathrm{GeV} \qquad t : 173.5(6)(8)\,\mathrm{GeV}$$

$$\pi^\pm : 139.57018(35)\,\mathrm{MeV} \qquad K^\pm : 493.677(16)\,\mathrm{MeV} \qquad p : 938.272046(21)\,\mathrm{MeV}$$

$$\pi^0 : 134.9766(6)\,\mathrm{MeV} \qquad K^0 : 497.614(24)\,\mathrm{MeV} \qquad n : 939.565379(21)\,\mathrm{MeV}$$

$$\mu : 2.1969811(22)\,\mu\mathrm{s}\ (659\,\mathrm{m}) \qquad \pi^\pm : 2.6033(5) \times 10^{-8}\,\mathrm{s} \qquad K^\pm : 1.2380(21) \times 10^{-8}\,\mathrm{s}\ (3.7\,\mathrm{m})$$

$$\tau : 2.906(10) \times 10^{-13}\,\mathrm{s}\ (87\,\mu\mathrm{m}) \qquad \pi^0 : 8.52(18) \times 10^{-17}\,\mathrm{s} \qquad K^0_{\mathrm{S}} : 8.9564(33) \times 10^{-11}\,\mathrm{s}\ (2.68\,\mathrm{cm})$$

$$K^0_{\mathrm{L}} : 5.116(21) \times 10^{-8}\,\mathrm{s}\ (15.3\,\mathrm{m})$$

Other Important Values

$$a_e = 11596521.8076(27) \times 10^{-10} \qquad d_e^{\mathrm{EDM}} < 10.5 \times 10^{-28} e\,\mathrm{cm} \qquad \mathrm{Br}(\tau \to e) = 17.83(4)\%$$

$$a_\mu = 11659209(6) \times 10^{-10} \qquad d_\mu^{\mathrm{EDM}} = -1(9) \times 10^{-20} e\,\mathrm{cm} \qquad \mathrm{Br}(\tau \to \mu) = 17.41(4)\%$$

$$\sin^2 2\theta_{12} = 0.857(24) \qquad \mathrm{Br}(\tau \to \mathrm{had}) \sim 64.8\%$$

$$\sin^2 2\theta_{23} > 0.95 \qquad \Delta m^2_{\nu 21} = 7.50(20) \times 10^{-5}\,\mathrm{eV}^2$$

$$\sin^2 2\theta_{13} = 0.098(13) \qquad \left|\Delta m^2_{\nu 32}\right| = 0.00232({}^{12}_{08})\,\mathrm{eV}^2$$

CKM matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} 0.97425(22) & 0.2252(9) & 0.0084(6) \\ 0.230(11) & 1.006(23) & 0.0429(26) \\ 0.00415(49) & 0.0409(11) & 0.89(7) \end{pmatrix} \approx \begin{pmatrix} 0.97427(15) & 0.22534(65) & 0.00351({}^{15}_{14}) \\ 0.22520(65) & 0.97344(16) & 0.0412({}^{11}_{05}) \\ 0.00867({}^{29}_{31}) & 0.0404({}^{11}_{05}) & 0.999146({}^{21}_{46}) \end{pmatrix}$$

$$\lambda = 0.22535(65), \quad A = 0.811^{+0.022}_{-0.012}, \quad \bar\rho = 0.131^{+0.026}_{-0.013}, \quad \bar\eta = 0.345^{+0.013}_{-0.014}; \quad J = (2.96^{+20}_{-16}) \times 10^{-5}$$

2.6.2 Estimation of SM Parameters

For EW scale, we can estimate the values as

$$|e| \sim 0.313, \qquad g_1 \sim 0.357, \qquad g_2 \sim 0.652, \qquad g_Z \sim 0.743; \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \sim 246\,\mathrm{GeV} \tag{2.36}$$

Therefore 湯川 matrices are (after diagonalization), since $vy/\sqrt{2} = M$,

$$y_u \sim \begin{pmatrix} 10^{-5} & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 0.997 \end{pmatrix} \quad y_d \sim \begin{pmatrix} 3\times 10^{-5} & 0 & 0 \\ 0 & 0.0005 & 0 \\ 0 & 0 & 0.02 \end{pmatrix} \quad y_e \sim \begin{pmatrix} 3\times 10^{-6} & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.01 \end{pmatrix} \tag{2.37}$$

Also, for $m_h \sim 125\,\mathrm{GeV}$, we can estimate the Higgs potential as $\mu \sim 88\,\mathrm{GeV}$ and $\lambda \sim 0.13$.

2.7 南部–Goldstone Boson

Defining the 南部–Goldstone mode as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}\sqrt{2}\phi^+(x) \\ v + h(x) + \mathrm{i}\phi_3(x) \end{pmatrix}, \qquad \text{where} \quad v = \sqrt{\frac{\mu^2}{\lambda}}, \tag{2.38}$$

The Higgs part of the Lagrangian turns out to be

$$\begin{aligned}
\mathcal{L} \supset{}& \left|\mathrm{D}_\mu H\right|^2 + \mu^2 \left|H\right|^2 - \lambda \left|H\right|^4 \\
={}& \frac{\lambda}{4}v^4 + \left(1 + \frac{2h}{v}\right)\left(m_W^2 W^{+\mu}W^-_\mu + \frac{m_Z^2}{2}Z^\mu Z_\mu\right) - \frac{m_h^2}{2}\left(h + \frac{h^2 + \phi_3^2 + 2\phi^+\phi^-}{2v}\right)^2 \\
&+ m_Z Z^\mu \partial_\mu \phi^3 + m_W \left[W^-_\mu \left(\partial^\mu - \mathrm{i}|e|A^\mu + \frac{\mathrm{i}|e|s}{c}Z^\mu\right)\phi^+ + \mathrm{H.c.}\right] \\
&+ \frac{1}{2}\left(\partial_\mu h - \frac{g_Z}{2}Z_\mu \phi_3 - \frac{g_2}{2}W^+_\mu \phi^- - \frac{g_2}{2}W^-_\mu \phi^+\right)^2 \\
&+ \frac{1}{2}\left(\partial_\mu \phi_3 + \frac{g_Z}{2}Z_\mu h - \frac{\mathrm{i}g_2}{2}W^+_\mu \phi^- + \frac{\mathrm{i}g_2}{2}W^-_\mu \phi^+\right)^2 \\
&+ \left|\left[\partial_\mu - \mathrm{i}|e|A_\mu - \frac{\mathrm{i}|e|s}{c}\left(\frac{c^2}{2s^2} - \frac{1}{2}\right)Z_\mu\right]\phi^+ + \frac{g_2}{2}(h + \mathrm{i}\phi^3)W^+_\mu\right|^2,
\end{aligned} \tag{2.39}$$

and the 湯川 interactions are

$$\begin{aligned}
\mathcal{L} \supset{}& \overline{U}y_u H P_\mathrm{L} Q - \overline{D}y_d H^\dagger P_\mathrm{L} Q - \overline{E}y_e H^\dagger P_\mathrm{L} L \\
={}& -\frac{v + h + \mathrm{i}\phi^3}{\sqrt{2}}\left(\overline{U}y_u P_\mathrm{L} Q^1 + \overline{D}y_d P_\mathrm{L} Q^2 + \overline{E}y_e P_\mathrm{L} L^2\right) \\
&- \mathrm{i}\left(\phi^+ \overline{U}y_u P_\mathrm{L} Q^2 + \phi^- \overline{D}y_d P_\mathrm{L} Q^1 + \phi^- \overline{E}y_e P_\mathrm{L} L^1\right).
\end{aligned} \tag{2.40}$$

3 Supersymmetry for $\eta = \mathrm{diag}(+,-,-,-)$

3.1 Spinor Convention

(See App. C.1.1 for a verbose explanation.)

ϵ tensor	:	$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1$ (definition)
Sum Rule	:	${}^{\alpha}{}_{\alpha}$ and ${}_{\dot\alpha}{}^{\dot\alpha}$, except for $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\xi^{\dot\beta}, \quad \xi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\xi_{\dot\beta}$.
Lorentz 変換	:	$\psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta, \quad \bar\psi'_{\dot\alpha} = \bar\psi_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}, \quad \psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha, \quad \bar\psi'^{\dot\alpha} = (\Lambda^{-1})^{\dagger\dot\alpha}{}_{\dot\beta}\bar\psi'^{\dot\beta}$.
σ matrices	:	$(\sigma^\mu)_{\alpha\dot\beta} := (1,\boldsymbol{\sigma})_{\alpha\dot\beta}, \quad (\bar\sigma^\mu)^{\dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1,-\boldsymbol{\sigma})^{\dot\alpha\beta}$.

3.2 Spinor Calculation Cheatsheet

$$\eta = (+,-,-,-), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1; \qquad \textbf{Left Differential};$$
$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot1\dot2} = \epsilon_{\dot2\dot1} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar\xi^{\dot\alpha} := \epsilon^{\dot\alpha\dot\beta}\bar\xi_{\dot\beta}, \quad \bar\xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta}$$
$$\bar\sigma^{\mu\dot\alpha\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} \qquad \sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\beta\beta}, \qquad \sigma^\mu := (1,\boldsymbol{\sigma}), \quad \bar\sigma^\mu := (1,-\boldsymbol{\sigma})$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta := \tfrac14\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} := \tfrac14\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)^{\dot\alpha}{}_{\dot\beta} = (\sigma^{\nu\mu})^{\dagger\dot\alpha}{}_{\dot\beta}.$$

$$\begin{array}{llll}
\theta^\alpha\theta^\beta = -\frac12\epsilon^{\alpha\beta}\theta\theta & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac12\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & (\theta\phi)(\theta\psi) = -\frac12(\psi\phi)(\theta\theta) & (\theta\sigma^\nu\bar\theta)\theta^\alpha = \frac12\theta\theta(\bar\theta\bar\sigma^\nu)^\alpha \\
\theta_\alpha\theta_\beta = \frac12\epsilon_{\alpha\beta}\theta\theta & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac12\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta & (\bar\theta\bar\phi)(\bar\theta\bar\psi) = -\frac12(\bar\psi\bar\phi)(\bar\theta\bar\theta) & (\theta\sigma^\nu\bar\theta)\bar\theta_{\dot\alpha} = -\frac12\bar\theta\bar\theta(\theta\sigma^\mu)_{\dot\alpha} \\
\theta^\alpha\theta_\beta = \frac12\delta^\alpha_\beta\theta\theta & \bar\theta_{\dot\alpha}\bar\theta^{\dot\beta} = \frac12\delta^{\dot\beta}_{\dot\alpha}\bar\theta\bar\theta & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac12\theta\theta\bar\theta\bar\theta\eta^{\mu\nu} & (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) = \frac12(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\bar\theta\bar\theta \\
\theta\sigma^\mu\bar\sigma^\nu\theta = \eta^{\mu\nu}\theta\theta & \bar\theta\bar\sigma^\mu\sigma^\nu\bar\theta = \eta^{\mu\nu}\bar\theta\bar\theta & &
\end{array}$$

$$\begin{array}{ll}
\sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar\sigma^\rho\sigma^\nu + \sigma^\nu\bar\sigma^\rho\sigma^\mu = 2\left(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho\right) \\
\bar\sigma^\mu\sigma^\nu = \eta^{\mu\nu} + 2\bar\sigma^{\mu\nu} & \bar\sigma^\mu\sigma^\rho\bar\sigma^\nu + \bar\sigma^\nu\sigma^\rho\bar\sigma^\mu = 2\left(\eta^{\mu\rho}\bar\sigma^\nu + \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\nu}\bar\sigma^\rho\right) \\
\sigma^{\mu\nu} = -\sigma^{\nu\mu} & \sigma^\mu\bar\sigma^\nu\sigma^\rho - \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\
\bar\sigma^{\mu\nu} = -\bar\sigma^{\nu\mu} & \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma \\
\mathrm{Tr}\,\bar\sigma^\mu\sigma^\nu = \mathrm{Tr}\,\sigma^\mu\bar\sigma^\nu = 2\eta^{\mu\nu} & \mathrm{Tr}\,\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\frac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\mathrm{Tr}\,\sigma^{\mu\nu} = \mathrm{Tr}\,\bar\sigma^{\mu\nu} = 0 & \mathrm{Tr}\,\bar\sigma^{\mu\nu}\bar\sigma^{\rho\sigma} = -\frac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\dot\beta\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} - \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + (\epsilon\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}\right] \\
\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} + \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = \eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} - 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar\sigma^{\sigma\nu})_{\dot\alpha\dot\beta}
\end{array}$$

$$\begin{array}{lll}
\bar\sigma^{\mu\dot\alpha\alpha}\bar\sigma^{\dot\beta\beta}_\mu = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} & \epsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\mathrm{i}\sigma^{\mu\nu} \\
\sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} = -2\mathrm{i}\bar\sigma^{\mu\nu}
\end{array}$$

$$\begin{array}{ll}
\bar\xi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\xi = (\bar\chi\bar\sigma^\mu\xi)^* = -(\xi\sigma^\mu\bar\chi)^* & (\psi\phi)\chi_\alpha = -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\
\xi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\xi = (\bar\chi\bar\sigma^\nu\sigma^\mu\bar\xi)^* = (\bar\xi\bar\sigma^\mu\sigma^\nu\bar\chi)^* & (\psi\phi)\bar\chi_{\dot\alpha} = \frac12(\phi\sigma^\mu\bar\chi)(\psi\sigma_\mu)_{\dot\alpha}
\end{array}$$

$$\begin{array}{lll}
\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\beta} = -\dfrac{\partial}{\partial\theta_\alpha} & \dfrac{\partial}{\partial\theta^\alpha}\theta\theta = 2\theta_\alpha & \epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta^\beta}\theta\theta = \dfrac{\partial}{\partial\theta_\alpha}\dfrac{\partial}{\partial\theta^\alpha}\theta\theta = 4 \\
\epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\beta} = -\dfrac{\partial}{\partial\theta^\alpha} & \dfrac{\partial}{\partial\theta_\alpha}\theta\theta = -2\theta^\alpha & \epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\alpha}\dfrac{\partial}{\partial\theta_\beta}\theta\theta = \dfrac{\partial}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta_\alpha}\theta\theta = -4 \\
\epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -2\bar\theta_{\dot\alpha} & \epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}}\bar\theta\bar\theta = \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 4 \\
\epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 2\bar\theta^{\dot\alpha} & \epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}}\bar\theta\bar\theta = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -4
\end{array}$$

3.3 General Relations

(Note: $P_\mu = \mathrm{i}\partial_\mu$ in our convention.)

$$Q_\alpha := \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad D_\alpha := \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \qquad y := x - \mathrm{i}\theta\sigma\bar\theta,$$

$$\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \qquad y^\dagger = x + \mathrm{i}\theta\sigma\bar\theta$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{D_\alpha, \bar D_{\dot\alpha}\} = 2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad (\text{others}) = 0.$$

$$\begin{array}{llll}
 & < x\text{-basis} > & < y\text{-basis} > & < y^\dagger\text{-basis} > \\
D_\alpha & = \dfrac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu & = \dfrac{\partial}{\partial\theta^\alpha} - 2\mathrm{i}(\sigma^\mu\bar\theta)_\alpha\dfrac{\partial}{\partial y^\mu} & = \dfrac{\partial}{\partial\theta^\alpha} \qquad (3.1)\\
\bar D_{\dot\alpha} & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} & = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2\mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\dfrac{\partial}{\partial(y^\dagger)^\mu} \qquad (3.2)\\
D^\alpha & = -\dfrac{\partial}{\partial\theta_\alpha} + \mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\partial_\mu & = -\dfrac{\partial}{\partial\theta_\alpha} + 2\mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\dfrac{\partial}{\partial y^\mu} & = -\dfrac{\partial}{\partial\theta_\alpha} \qquad (3.3)\\
\bar D^{\dot\alpha} & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} - \mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} & = \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} - 2\mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\dfrac{\partial}{\partial(y^\dagger)^\mu} \qquad (3.4)
\end{array}$$

$$\phi(y) = \phi(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y^\dagger) - 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \tag{3.5}$$

$$\phi(y^\dagger) = \phi(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) = \phi(y) + 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y) \tag{3.6}$$

$$\phi(x) = \phi(y) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(y) = \phi(y^\dagger) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) \tag{3.7}$$

3.4 Chiral Superfields : $\bar{D}_{\dot{\alpha}}\Phi = 0$

Explicit Expression

$$\begin{aligned}
\Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) && (3.8)\\
&= \phi(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) && (3.9)\\
&= \phi(y^\dagger) - 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(y^\dagger) - \theta\theta\bar\theta\bar\theta\partial^2\phi(y^\dagger) + \sqrt{2}\theta\psi(y^\dagger) - \sqrt{2}\mathrm{i}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(y^\dagger) + \theta\theta F(y^\dagger) && (3.10)\\
\Phi^\dagger &= \phi^*(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \bar\theta\bar\theta F^*(y^\dagger) && (3.11)\\
&= \phi^*(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \bar\theta\bar\theta F^*(x) && (3.12)\\
&= \phi^*(y) + 2\mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(y) - \theta\theta\bar\theta\bar\theta\partial^2\phi^*(y) + \sqrt{2}\bar\theta\bar\psi(y) - \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(y) + \bar\theta\bar\theta F^*(y) && (3.13)
\end{aligned}$$

Product of Chiral Superfields

$$\begin{aligned}
\Phi_i^\dagger\Phi_j(x,\theta,\bar\theta) = {}& \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j\\
&- \mathrm{i}\theta\sigma^\mu\bar\theta\left(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j\right) + 2\bar\theta\bar\psi_i\theta\psi_j\\
&+ \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\left(\phi_i^*\partial_\mu\psi_j - \partial_\mu\phi_i^*\psi_j\right)\sigma^\mu\bar\theta + \sqrt{2}\theta\theta\bar\theta\bar\psi_i F_j\\
&- \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\left(\partial_\mu\bar\psi_i\phi_j - \bar\psi_i\partial_\mu\phi_j\right) + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j && (3.14)\\
&+ \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \frac{1}{4}\phi_i^*\partial^2\phi_j - \frac{1}{4}\partial^2\phi_i^*\phi_j + \frac{1}{2}\partial_\mu\phi_i^*\partial_\mu\phi_j - \frac{\mathrm{i}}{2}\partial_\mu\bar\psi_i\bar\sigma^\mu\psi_j + \frac{\mathrm{i}}{2}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]\\
\rightsquigarrow{}& \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j\\
&- 2\mathrm{i}(\theta\sigma^\mu\bar\theta)(\phi_i^*\partial_\mu\phi_j) + \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar\theta\bar\sigma^\mu\psi_j + \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j && (3.15)\\
&+ 2\bar\theta\bar\psi_i\theta\psi_j + \sqrt{2}\theta\theta\bar\theta\bar\psi_i F_j + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j\\
&+ \theta\theta\bar\theta\bar\theta\left[F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]
\end{aligned}$$

$$\begin{aligned}
\Phi_i\Phi_j(\text{in } y\text{-basis}) &= \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right] && (3.16)\\
\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) &= \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right]\\
&\quad + \theta\theta\left[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right] && (3.17)
\end{aligned}$$

(Products of chiral superfields are still chiral superfields.)

$$\mathrm{e}^{\mathrm{i}k\Phi} = \mathrm{e}^{\mathrm{i}k\phi(y)}\left[1 + \mathrm{i}k\left(\sqrt{2}\theta\psi(y) + \theta\theta F(y)\right) + \frac{k^2}{2}\theta\theta\psi(y)\psi(y)\right] \tag{3.18}$$

Lagrangian Blocks

$$\mathcal{L}_{\text{kin.}} = \Phi_i^\dagger\Phi_j\Big|_{\theta\theta\bar\theta\bar\theta} \rightsquigarrow F_i^*F_j + \partial^\mu\phi_i^*\partial_\mu\phi_j + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j \tag{3.19}$$

$$\begin{aligned}
\mathcal{L}_{\text{super}} &= W\Big|_{\theta\theta} + W^*\Big|_{\bar\theta\bar\theta} = \int \mathrm{d}^2\theta\left[\lambda_i\Phi_i + m_{ij}\Phi_i\Phi_j + y_{ijk}\Phi_i\Phi_j\Phi_k\right] + \text{H.c.}\\
&= \lambda_iF_i + m_{ij}\left(\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right) + y_{ijk}\Big[(F_i\phi_j\phi_k - \psi_i\psi_j\phi_k) + (jki \text{ and } kij \text{ terms})\Big] && (3.20)
\end{aligned}$$

3.5 Vector Superfields and Gauge Theory : $V = V^\dagger$

3.5.1 Abelian Case — Field Construction

Explicit Expression

$$\begin{aligned}
V &= C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar\theta\bar\chi(x) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(x)+\mathrm{i}N(x)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(x)-\mathrm{i}N(x)\right] - \theta\sigma^\mu\bar\theta A_\mu(x) \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(x) + \frac12\bar\sigma^\mu\partial_\mu\chi(x)\right] + \bar\theta\bar\theta\theta\left[\lambda(x) - \frac12\sigma^\mu\partial_\mu\bar\chi(x)\right] + \frac12\theta\theta\bar\theta\bar\theta\left[D(x) - \frac12\partial^2 C(x)\right] \qquad (3.21) \\
&= C(y) + \mathrm{i}\theta\chi(y) - \mathrm{i}\bar\theta\bar\chi(y) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y)+\mathrm{i}N(y)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y)-\mathrm{i}N(y)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y) - \mathrm{i}\partial_\mu C(y)\right] \\
&\quad + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\left[\lambda(y) - \sigma^\mu\partial_\mu\bar\chi(y)\right] + \frac12\theta\theta\bar\theta\bar\theta\left[D(y) - \partial^2 C(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] \qquad (3.22) \\
&= C(y^\dagger) + \mathrm{i}\theta\chi(y^\dagger) - \mathrm{i}\bar\theta\bar\chi(y^\dagger) \\
&\quad + \frac{\mathrm{i}}{2}\theta\theta\left[M(y^\dagger)+\mathrm{i}N(y^\dagger)\right] - \frac{\mathrm{i}}{2}\bar\theta\bar\theta\left[M(y^\dagger)-\mathrm{i}N(y^\dagger)\right] - \theta\sigma^\mu\bar\theta\left[A_\mu(y^\dagger) + \mathrm{i}\partial_\mu C(y^\dagger)\right] \\
&\quad + \theta\theta\bar\theta\left[\bar\lambda(y^\dagger) + \bar\sigma^\mu\partial_\mu\chi(y^\dagger)\right] + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac12\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) - \partial^2 C(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] \qquad (3.23)
\end{aligned}$$

Supersymmetric Gauge Transformation : $V \to V + \Phi + \Phi^\dagger$

$$\begin{aligned}
C &\mapsto C + (\phi+\phi^*) & A_\mu &\mapsto A_\mu + \mathrm{i}\partial_\mu(\phi-\phi^*) \\
\chi &\mapsto \chi - \mathrm{i}\sqrt{2}\psi & \lambda &\mapsto \lambda \\
M+\mathrm{i}N &\mapsto M+\mathrm{i}N - 2\mathrm{i}F & D &\mapsto D
\end{aligned} \qquad (3.24)$$

Wess-Zumino Gauge $\quad C = \chi = M = N = 0$

Fixing this gauge breaks SUSY, but still allows the usual gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu\alpha, \quad \lambda \mapsto \lambda, \quad D \mapsto D. \qquad (3.25)$$

$$\begin{aligned}
V &= -\theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\lambda(x) + \frac12\theta\theta\bar\theta\bar\theta D(x) & V^2 &= \frac12\theta\theta\bar\theta\bar\theta A_\mu A^\mu \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y) + \theta\theta\bar\theta\bar\lambda(y) + \bar\theta\bar\theta\theta\lambda(y) + \frac12\theta\theta\bar\theta\bar\theta\left[D(y) - \mathrm{i}\partial_\mu A^\mu(y)\right] & V^3 &= 0 \\
&= -\theta\sigma^\mu\bar\theta A_\mu(y^\dagger) + \theta\theta\bar\theta\bar\lambda(y^\dagger) + \bar\theta\bar\theta\theta\lambda(y^\dagger) + \frac12\theta\theta\bar\theta\bar\theta\left[D(y^\dagger) + \mathrm{i}\partial_\mu A^\mu(y^\dagger)\right] \\
\mathrm{e}^{kV} &= 1 - k\theta\sigma^\mu\bar\theta A_\mu(x) + k\theta\theta\bar\theta\bar\lambda + k\bar\theta\bar\theta\theta\lambda + \theta\theta\bar\theta\bar\theta\left[\frac{k}{2}D + \frac{k^2}{4}A_\mu A^\mu\right]
\end{aligned}$$

Field Strength

$$W_\alpha = -\frac14\bar D\bar D D_\alpha V, \quad \bar W_{\dot\alpha} = -\frac14 DD\bar D_{\dot\alpha} V; \qquad W_\alpha \mapsto W_\alpha \quad \text{(gauge invariant)} \qquad (3.26)$$

$$\bar D_{\dot\beta} W_\alpha = D_\beta \bar W_{\dot\alpha} = 0; \qquad D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} \qquad (3.27)$$

$$W_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \mathrm{i}\theta\theta\left[\sigma^\mu\partial_\mu\bar\lambda(y)\right]_\alpha \qquad (3.28)$$

$$\bar W_{\dot\alpha} = \bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha} D(y^\dagger) + \mathrm{i}(\bar\theta\bar\sigma^{\mu\nu})_{\dot\alpha} F_{\mu\nu}(y^\dagger) - \mathrm{i}\bar\theta\bar\theta\left[\partial_\mu\lambda(y^\dagger)\sigma^\mu\right]_{\dot\alpha} \qquad (3.29)$$

$$W^\alpha W_\alpha\big|_{\theta\theta} = -\frac14\bar D\bar D W^\alpha D_\alpha V \rightsquigarrow -\frac12 F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + 2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 \qquad (3.30)$$

Lagrangian Blocks

($\mathcal{L}_{\text{inv}}$ is SUSY- and gauge-invariant, while $\mathcal{L}_{\text{mass}}$ is not gauge-invariant.)

$$\mathcal{L}_{\text{inv}} = \frac{\tau}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \frac{\tau^*}{4}\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|_{\bar\theta\bar\theta} \qquad \left(\text{with} \quad \tau := 1 + \frac{\mathrm{i}\theta}{8\pi^2}\right)$$
$$\rightsquigarrow -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2 - \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{3.31}$$
$$\mathcal{L}_{\text{mass}} = m^2V^2\Big|_{\theta\theta\bar\theta\bar\theta}$$
$$= m^2\left(\frac{1}{2}A^\mu A_\mu + \mathrm{i}\bar\chi\bar\sigma^\mu\partial_\mu\chi - (\lambda\chi + \bar\lambda\bar\chi) + \frac{1}{2}(M^2+N^2) + CD + \frac{1}{2}\partial_\mu C\partial^\mu C\right)$$
$$= \frac{1}{2}\partial_\mu C'\partial^\mu C' + \mathrm{i}\bar\chi'\bar\sigma^\mu\partial_\mu\chi' + \frac{m^2}{2}A^\mu A_\mu - m(\lambda\chi' + \bar\lambda\bar\chi') + mC'D + \frac{m^2}{2}(M^2+N^2) \tag{3.32}$$

3.5.2 Abelian Case — Gauge Theory

Here we turn on the coupling constant g. When $\Lambda = \mathrm{i}\lambda(y) + \sqrt{2}\theta\xi(y) + \theta\theta K(y)$,

$$\mathcal{L} \ni \Phi^\dagger \mathrm{e}^{2gqV}\Phi; \qquad \Phi \mapsto \mathrm{e}^{\mathrm{i}qg\Lambda}\Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger\mathrm{e}^{-\mathrm{i}qg\Lambda^\dagger}; \qquad 2V \mapsto 2V - \mathrm{i}(\Lambda - \Lambda^\dagger) \tag{3.33}$$

$$\begin{aligned}
&\phi \mapsto \mathrm{e}^{-qg\lambda}\phi && C \mapsto C + \operatorname{Re}\lambda && M + \mathrm{i}N \mapsto M + \mathrm{i}N - K\\
&\psi \mapsto \mathrm{e}^{-qg\lambda}(\psi + \mathrm{i}qg\phi\cdot\xi) && \chi \mapsto \chi - \tfrac{1}{\sqrt2}\xi && A_\mu \mapsto A_\mu - \partial_\mu(\operatorname{Im}\lambda)\\
&F \mapsto \mathrm{e}^{-qg\lambda}\left(F + \mathrm{i}qg\phi K - \mathrm{i}qg\xi\psi + \frac{(qg)^2}{2}\xi\xi\phi\right) && \lambda \mapsto \lambda && D \mapsto D
\end{aligned}$$

Lagrangian block

(Very similar to the gauge transformations in Sec. 1.9.2.)

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= \frac{1}{4}W^\alpha W_\alpha\Big|_{\theta\theta} + \text{H.c.} && \rightsquigarrow -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2\\
\mathcal{L}_{\text{chiral}} &= \Phi^\dagger\mathrm{e}^{2gqV}\Phi\Big|_{\theta\theta\bar\theta\bar\theta} && \rightsquigarrow F^*F + \mathrm{D}_\mu\phi^*\mathrm{D}^\mu\phi + \mathrm{i}\bar\psi\bar\sigma^\mu\mathrm{D}_\mu\psi + qgD\phi^*\phi - \sqrt{2}gq\left(\phi^*\lambda\psi + \phi\bar\lambda\bar\psi\right)\\
\mathcal{L}_{\cancel{CP}} &= \frac{\mathrm{i}\theta}{32\pi^2}W^\alpha W_\alpha\Big|_{\theta\theta} + \text{H.c.} && \rightsquigarrow -\frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\
& && \text{with} \quad \mathrm{D}_\mu[\phi,\psi] = (\partial_\mu - \mathrm{i}gqA_\mu)[\phi,\psi], \qquad \mathrm{D}_\mu\lambda = \partial_\mu\lambda,
\end{aligned}$$

3.5.3 Non-Abelian Case

T^a : generators (Hermitian);
$\operatorname{Tr} T^aT^b = K\delta^{ab}$ $(K>0)$, $\qquad [T^a, T^b] = \mathrm{i}f^{abc}T^c$ (f is anti-symmetric)

Explicit Expression Same as the Abelian case.

Supersymmetric Gauge Transformation

$$\mathcal{L} \ni \Phi^\dagger\mathrm{e}^{2g\widetilde{V}}\Phi; \qquad \Phi \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda}\Phi, \quad \Phi^\dagger \mapsto \Phi^\dagger\mathrm{e}^{-\mathrm{i}g\widetilde\Lambda^\dagger}; \qquad \mathrm{e}^{2g\widetilde V} \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda^\dagger}\mathrm{e}^{2g\widetilde V}\mathrm{e}^{-\mathrm{i}g\widetilde\Lambda} \tag{3.34}$$
$$\text{with } \widetilde V := V^aT^a, \quad \widetilde\Lambda := \Lambda^aT^a, \quad \widetilde\Lambda^\dagger := (\Lambda^a)^\dagger T^a$$

$$2\widetilde V \mapsto 2\widetilde V - \mathrm{i}(\widetilde\Lambda - \widetilde\Lambda^\dagger) - \frac{g}{2}\left([\widetilde\Lambda, \widetilde\Lambda^\dagger] + \mathrm{i}[2\widetilde V, \widetilde\Lambda + \widetilde\Lambda^\dagger]\right) + \cdots \tag{3.35}$$
$$= \left[2V^a - \mathrm{i}(\Lambda^a - \Lambda^{\dagger a}) + \frac{g}{2}\Big(-\mathrm{i}\Lambda^b\Lambda^{\dagger c} + 2V^b(\Lambda^c + \Lambda^{\dagger c})\Big)f^{abc} + \cdots\right]T^a \tag{3.36}$$

We do not present the transformations of the components; note that λ and D are transformed in non-Abelian theories.

Wess-Zumino Gauge

$$V^a = -\theta\sigma^\mu\bar\theta A^a_\mu(x) + \theta\theta\bar\theta\bar\lambda^a(x) + \bar\theta\bar\theta\theta\lambda^a(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D^a(x) \tag{3.37}$$

$$\mathrm{e}^{kV^aT^a} = 1 + kV^aT^a + \frac{k^2}{4}\theta\theta\bar\theta\bar\theta A^a_\mu A^{b\mu}T^aT^b \tag{3.38}$$

Note that the lowest order term of the gauge transformation is independent of V, which guarantees that we can still take the Wess-Zumino gauge. The gauge transformation is restricted as $\mathrm{e}^{\mathrm{i}g\widetilde\Lambda}$, where $\Lambda^a = \xi^a(y) = \xi^a(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\xi^a(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\xi^a(x) : \xi\in\mathbb{R}$.

$$\begin{aligned} 2V^e \mapsto\ & 2\left(V^e - gf^{eab}\xi^aV^b + 6g^2f^{abc}f^{ade}V^b\xi^c\xi^d\right) \\ & + \theta\sigma^\mu\bar\theta\left(-2\partial_\mu\xi^e + gf^{eab}\xi^a\partial_\mu\xi^b + 4g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \end{aligned} \tag{3.39}$$

$$\begin{aligned} A^e_\mu \mapsto\ & A^e_\mu + gf^{eab}A^a_\mu\xi^b + 6g^2f^{abc}f^{ade}A^b_\mu\xi^c\xi^d \\ & + \left(\partial_\mu\xi^e - \frac{g}{2}f^{eab}\xi^a\partial_\mu\xi^b - 2g^2f^{acd}f^{abe}\xi^b\xi^c\partial_\mu\xi^d\right) + \cdots \end{aligned} \tag{3.40}$$

$$\lambda^e \mapsto \lambda^e + gf^{eab}\lambda^a\xi^b + 6g^2f^{abc}f^{ade}\lambda^b\xi^c\xi^d + \cdots \tag{3.41}$$

$$D^e \mapsto D^e + gf^{eab}D^a\xi^b + 6g^2f^{abc}f^{ade}D^b\xi^c\xi^d + \cdots \tag{3.42}$$

Note that C, χ, M and N are kept invariant automatically, for now we are under Wess-Zumino gauge.

Field Strength [*4]

$$\widetilde W_\alpha = -\frac{1}{8g}\bar D\bar D\mathrm{e}^{-2g\widetilde V}D_\alpha\mathrm{e}^{2g\widetilde V} \qquad \bar D_{\dot\beta}W_\alpha = 0 \qquad W_\alpha \mapsto \mathrm{e}^{\mathrm{i}g\widetilde\Lambda}W_\alpha\mathrm{e}^{-\mathrm{i}g\Lambda^\dagger} \tag{3.43}$$

$$W^a_\alpha = \lambda^a_\alpha(y) + \theta_\alpha D^a(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) + \mathrm{i}\theta\theta(\sigma^\mu\mathrm{D}_\mu\bar\lambda^a(y))_\alpha \tag{3.44}$$

$$\operatorname{Tr}\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} = \operatorname{Tr}\left[DD + \mathrm{i}\lambda\sigma^\mu\mathrm{D}_\mu\bar\lambda + \mathrm{i}\mathrm{D}_\mu\bar\lambda\bar\sigma^\mu\lambda - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] \tag{3.45}$$

$$= K\left[D^aD^a + \mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar\lambda^a + \mathrm{i}\mathrm{D}_\mu\bar\lambda^a\bar\sigma^\mu\lambda^a - \frac{1}{2}F^{\mu\nu\,a}F^a_{\mu\nu} + \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\right] \tag{3.46}$$

Lagrangian Block

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4K}\operatorname{Tr}\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} + \text{H.c.} \qquad \rightsquigarrow -\frac{1}{4}F^{\mu\nu\,a}F^a_{\mu\nu} + \mathrm{i}\bar\lambda^a\bar\sigma^\mu\mathrm{D}_\mu\lambda^a + \frac{1}{2}D^aD^a \tag{3.47}$$

$$\mathcal{L}_{\cancel{CP}} = \frac{\mathrm{i}}{32K\pi^2}\operatorname{Tr}\widetilde W^\alpha\widetilde W_\alpha\Big|_{\theta\theta} + \text{H.c.} \qquad \rightsquigarrow -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.48}$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = \Phi^\dagger_i[\mathrm{e}^{2gV^aT^a}]_{ij}\Phi_j\Big|_{\theta\theta\bar\theta\bar\theta} \qquad & \rightsquigarrow \mathrm{D}^\mu\phi^*_i\mathrm{D}_\mu\phi_i + \mathrm{i}\bar\psi_i\bar\sigma^\mu\mathrm{D}_\mu\psi_i + F^*_iF_i \\ & + gD^a(\phi^*T^a\phi) - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar\psi\bar\lambda T^a\phi\right) \end{aligned} \tag{3.49}$$

$$\begin{aligned} \mathrm{D}_\mu\phi_i &= \partial_\mu\phi - \mathrm{i}gA^a_\mu(T^a\phi)_i & F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu], \\ \mathrm{D}_\mu\phi^*_i &= \partial_\mu\phi + \mathrm{i}gA^a_\mu(\phi^*T^a)_i & \mathrm{D}_\mu\lambda^a &= \partial_\mu\lambda^a + gf^{abc}A^b_\mu\lambda^c, \\ \mathrm{D}_\mu\psi_i &= \partial_\mu\psi - \mathrm{i}gA^a_\mu(T^a\psi)_i & \mathrm{D}_\mu\bar\lambda^a &= \partial_\mu\bar\lambda^a + gf^{abc}A^b_\mu\bar\lambda^c, \end{aligned}$$

[*4] Note the signs. $\bar D\bar D\mathrm{e}^{2gV}D_\alpha\mathrm{e}^{-2gV}$ is not gauge invariant! Also the curvature tensor and the covariant derivative is well-known ones: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$, and $\mathrm{D}_\mu\bar\lambda = \partial_\mu\bar\lambda - \mathrm{i}g[A_\mu, \bar\lambda] = \partial_\mu\bar\lambda + gf^{abc}A^b_\mu\bar\lambda^cT^a$.

3.6 Minimal Supersymmetric Standard Model

3.6.1 Definitions

Gauge Group and Superfields*[5]

$$\mathrm{SU}(3)_{\text{color}} \times \mathrm{SU}(2)_{\text{weak}} \times \mathrm{U}(1)_{\mathrm{Y}} \quad (\times \mathrm{Z}_{2R} : R\text{-parity}); \tag{3.50}$$

Field	SU(3)	SU(2)	U(1)	B	L
Q_i	$\mathbf{3}$	$\mathbf{2}$	1/6	1/3	
L_i		$\mathbf{2}$	−1/2		1
$\bar{U}_i$	$\bar{\mathbf{3}}$		−2/3	−1/3	
$\bar{D}_i$	$\bar{\mathbf{3}}$		1/3	−1/3	
$\bar{E}_i$			1		−1
H_u		$\mathbf{2}$	1/2		
H_d		$\mathbf{2}$	−1/2		

Field	SU(3)	SU(2)	U(1)
g	$\mathbf{8}$		
W		$\mathbf{3}$	
B			

Superpotential and ~~SUSY~~-terms

$$W_{\mathrm{RPC}} = \mu H_\mathrm{u} H_\mathrm{d} - y_{\mathrm{u}ij} \bar{U}_i H_\mathrm{u} Q_j + y_{\mathrm{d}ij} \bar{D}_i H_\mathrm{d} Q_j + y_{\mathrm{e}ij} \bar{E}_i H_\mathrm{d} L_j \tag{3.51}$$

$$W_{\mathrm{RPV}} = \mu_i H_\mathrm{u} L_i + \frac{1}{2}\lambda_{ijk} L_i L_j \bar{E}_k + \lambda'{}_{ijk} L_i Q_j \bar{D}_k + \frac{1}{2}\lambda''{}_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k \tag{3.52}$$

$$\mathcal{L}_{\cancel{\mathrm{SUSY}}} = -\frac{1}{2}\left(M_3 \widetilde{g}\widetilde{g} + M_2 \widetilde{w}\widetilde{w} + M_1 \widetilde{b}\widetilde{b} + \text{H.c.}\right) - V_{\cancel{\mathrm{SUSY}}}; \tag{3.53}$$

$$\begin{aligned} V_{\cancel{\mathrm{SUSY}}}^{\mathrm{RPC}} = &\left(\widetilde{q}^* m_Q^2 \widetilde{q} + \widetilde{l}^* m_L^2 \widetilde{l} + \widetilde{u}_\mathrm{R} m_{\bar{U}}^2 \widetilde{u}_\mathrm{R}^* + \widetilde{d}_\mathrm{R} m_{\bar{D}}^2 \widetilde{d}_\mathrm{R}^* + \widetilde{e}_\mathrm{R} m_{\bar{E}}^2 \widetilde{e}_\mathrm{R}^* + m_{H_\mathrm{u}}^2 |H_\mathrm{u}|^2 + m_{H_\mathrm{d}}^2 |H_\mathrm{d}|^2\right) \\ &+ \left(-\widetilde{u}_\mathrm{R}^* H_\mathrm{u} A^u \widetilde{q} + \widetilde{d}_\mathrm{R}^* H_\mathrm{d} A^d \widetilde{q} + \widetilde{e}_\mathrm{R}^* H_\mathrm{d} A^e \widetilde{l} + b H_\mathrm{u} H_\mathrm{d} + \text{H.c.}\right) m \\ &+ \left(-\widetilde{u}_\mathrm{R}^* H_\mathrm{d}^* C^u \widetilde{q} + \widetilde{d}_\mathrm{R}^* H_\mathrm{u}^* C^d \widetilde{q} + \widetilde{e}_\mathrm{R}^* H_\mathrm{u}^* C^e \widetilde{l} + \text{H.c.}\right) \end{aligned} \tag{3.54}$$

$$V_{\cancel{\mathrm{SUSY}}}^{\mathrm{RPV}} = \left(b_i H_\mathrm{u} \widetilde{l}_i + \frac{1}{2} A_{ijk} \widetilde{l}_i \widetilde{l}_j \widetilde{e}_{\mathrm{R}k}^* + A'_{ijk} \widetilde{l}_i \widetilde{q}_j \widetilde{d}_{\mathrm{R}k}^* + \frac{1}{2} A''_{ijk} \widetilde{u}_{\mathrm{R}i}^* \widetilde{d}_{\mathrm{R}j}^* \widetilde{d}_{\mathrm{R}k}^* + M_{Li}^2 H_\mathrm{d}^* \widetilde{l}_i + \text{H.c.}\right) \tag{3.55}$$

$$+ \left(C_{ijk}^1 \widetilde{l}_i^* \widetilde{q}_j \widetilde{u}_{\mathrm{R}k}^* + C_i^2 H_\mathrm{u}^* H_\mathrm{d} \widetilde{e}_{\mathrm{R}i}^* + C_{ijk}^3 \widetilde{d}_{\mathrm{R}i} \widetilde{u}_{\mathrm{R}j}^* \widetilde{e}_{\mathrm{R}k}^* + \frac{1}{2} C_{ijk}^4 \widetilde{d}_{\mathrm{R}i} \widetilde{q}_j \widetilde{q}_k + \text{H.c.}\right), \tag{3.56}$$

where we define $\lambda_{ijk} = -\lambda_{jik}$, $\lambda''_{ijk} = -\lambda''_{ikj}$, and $C^4_{ijk} = C^4_{ikj}$

*5 For left-handed fermions, the superfield will be written as, e.g., Q, and it contains a scalar component $\widetilde{q}$ and a chiral fermion Q_α. Their complex conjugates will be shown as $\widetilde{q}^*$ and $\bar{Q}_{\dot\alpha}$ as is done in the previous section.

For right-handed fermions such as $\bar{U}$, the superfield will be written as $\bar{U}$. Its scalar component is written as (or, in other words, equivalent to) $\widetilde{u}_\mathrm{R}^*$, and its fermionic one is U^c_α. *c is just a label;* does not mean charge or complex conjugation. Their complex conjugates are $\widetilde{u}_\mathrm{R}$ and $\bar{U}^\mathrm{c}_{\dot\alpha}$.

They form a Dirac fermion as $U = \begin{pmatrix} u_\alpha \\ \bar{U}^{\mathrm{c}\dot\alpha} \end{pmatrix} =: \begin{pmatrix} U_\mathrm{L} \\ U_\mathrm{R} \end{pmatrix}$; its *charge* conjugate is $U^\mathrm{C} = \begin{pmatrix} U^\mathrm{C}_\alpha \\ \bar{u}^{\dot\alpha} \end{pmatrix}$.

Majorana fermions are written as, e.g., $\widetilde{b} = \widetilde{b}^\mathrm{C} = \begin{pmatrix} \widetilde{b}_\alpha \\ \bar{\widetilde{b}}^{\dot\alpha} \end{pmatrix}$. Here $\bar{\widetilde{b}}_{\dot\alpha}$ is the complex conjugate of $\widetilde{b}_\alpha$.

scalars : $\widetilde{q}\,(\widetilde{u}_\mathrm{L}, \widetilde{d}_\mathrm{L}), \widetilde{u}_\mathrm{R}^*, \widetilde{d}_\mathrm{R}^*, \widetilde{l}\,(\widetilde{e}_\mathrm{L}, \widetilde{\nu}), \widetilde{e}_\mathrm{R}^*, H_\mathrm{u}\,(H_\mathrm{u}^+, H_\mathrm{u}^0), H_\mathrm{d}\,(H_\mathrm{d}^0, H_\mathrm{d}^-)$

$\widetilde{q}^*\,(\widetilde{u}_\mathrm{L}^*, \widetilde{d}_\mathrm{L}^*), \widetilde{u}_\mathrm{R}, \widetilde{d}_\mathrm{R}, \widetilde{l}^*\,(\widetilde{e}_\mathrm{L}^*, \widetilde{\nu}^*), \widetilde{e}_\mathrm{R}, H_\mathrm{u}^*, H_\mathrm{d}^*$

Diracs : $U\,(U_\mathrm{L}, U_\mathrm{R}), D\,(D_\mathrm{L}, D_\mathrm{R}), E\,(E_\mathrm{L}, E_\mathrm{R}), \nu$

Weyls : $Q\,(u, d), U^\mathrm{c}, D^\mathrm{c}, L\,(\nu, e), E^\mathrm{c}$

$\bar{Q}\,(\bar{u}, \bar{d}), \bar{U}^\mathrm{c}, \bar{D}^\mathrm{c}, \bar{L}\,(\bar{\nu}, \bar{e}), \bar{E}^\mathrm{c}$

$\widetilde{h}_\mathrm{u}\,(\widetilde{h}_\mathrm{u}^+, \widetilde{h}_\mathrm{u}^0), \widetilde{h}_\mathrm{d}\,(\widetilde{h}_\mathrm{d}^0, \widetilde{h}_\mathrm{d}^-), \widetilde{b}, \widetilde{w}, \widetilde{g}$

$\bar{\widetilde{h}}_\mathrm{u}\,(\bar{\widetilde{h}}_\mathrm{u}^+, \bar{\widetilde{h}}_\mathrm{u}^0), \bar{\widetilde{h}}_\mathrm{d}\,(\bar{\widetilde{h}}_\mathrm{d}^0, \bar{\widetilde{h}}_\mathrm{d}^-), \bar{\widetilde{b}}, \bar{\widetilde{w}}, \bar{\widetilde{g}}$

3.6.2 Lagrangian Build Block

$$\mathcal{L}_{\mathrm{K;\,CP}} = -\frac{1}{4}F^{\mu\nu\,a}F^{a}_{\mu\nu} + \mathrm{D}^{\mu}\phi_i^*\mathrm{D}_{\mu}\phi_i + \mathrm{i}\bar{\psi}_i\bar{\sigma}^{\mu}\mathrm{D}_{\mu}\psi_i + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^{\mu}\mathrm{D}_{\mu}\lambda^a - \sqrt{2}g\left(\phi^*T^a\psi\lambda + \bar{\psi}\bar{\lambda}T^a\phi\right) \tag{3.57}$$

$$\mathcal{L}^{\cancel{\mathrm{SUSY}}}_{\mathrm{gaugino}} = -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{w}\widetilde{w} + M_1\widetilde{b}\widetilde{b} + \mathrm{H.c.}\right) \tag{3.58}$$

$$\mathcal{L}_{\mathrm{scalar}} = -\left(\sum V^F + \sum V^D + \sum V_{\cancel{SUSY}}\right) \tag{3.59}$$

$$\mathcal{L}^{\mathrm{RPC}}_{\mathrm{S;\,fermi}} = -\left(\mu\widetilde{h}_{\mathrm{u}}\widetilde{h}_{\mathrm{d}} - y_{\mathrm{u}ij}U_i^{\mathrm{c}}H_{\mathrm{u}}Q_j + y_{\mathrm{d}ij}D_i^{\mathrm{c}}H_{\mathrm{d}}Q_j + y_{\mathrm{e}ij}E_i^{\mathrm{c}}H_{\mathrm{d}}L_j + \cdots + \mathrm{H.c.}\right) \tag{3.60}$$

$$\mathcal{L}^{\mathrm{RPV}}_{\mathrm{S;\,fermi}} = -\left(\mu_i\widetilde{h}_{\mathrm{u}}L_i + \frac{1}{2}\lambda_{ijk}L_iL_jE_k^{\mathrm{c}} + \lambda'{}_{ijk}L_iQ_jD_k^{\mathrm{c}} + \frac{1}{2}\lambda''{}_{ijk}U_i^{\mathrm{c}}D_j^{\mathrm{c}}D_k^{\mathrm{c}} + \cdots + \mathrm{H.c.}\right) \tag{3.61}$$

$$\mathcal{L}_{\mathrm{K;\,\cancel{CP}}} = -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.62}$$

The scalar potential is decomposed as

$$-F_{H_{\mathrm{u}}}{}^{a*} = \epsilon^{ab}\left(\mu H^b_{\mathrm{d}} - y_{\mathrm{u}ij}\bar{U}_i^x Q_j^{bx} + \mu_i L_i^b\right) \tag{3.63}$$

$$-F_{H_{\mathrm{d}}}{}^{a*} = \epsilon^{ab}\left(-\mu H^b_{\mathrm{u}} + y_{\mathrm{d}ij}\bar{D}_i^x Q_j^{bx} + y_{\mathrm{e}ij}\bar{E}_i L_j^b\right) \tag{3.64}$$

$$-F_{Q_i}{}^{ax*} = \epsilon^{ab}\left(y_{\mathrm{u}ji}H^b_{\mathrm{u}}\bar{U}_j^x - y_{\mathrm{d}ji}H^b_{\mathrm{d}}\bar{D}_j^x - \lambda'{}_{jik}L_i^b\bar{D}_j^x\right) \tag{3.65}$$

$$-F_{L_i}{}^{a*} = \epsilon^{ab}\left(-\mu_i H^b_{\mathrm{u}} - y_{\mathrm{e}ji}H^b_{\mathrm{d}}\bar{E}_j + \lambda_{ijk}L_j^b\bar{E}_k + \lambda'_{ijk}Q_j^{bx}\bar{D}_k^x\right) \tag{3.66}$$

$$-F_{\bar{U}i}{}^{x*} = \left(-\epsilon^{ab}y_{\mathrm{u}ij}H^a_{\mathrm{u}}Q_j^{bx} + \frac{1}{2}\epsilon^{xyz}\lambda''_{ijk}\bar{D}_j^y\bar{D}_k^z\right) \tag{3.67}$$

$$-F_{\bar{D}i}{}^{x*} = \left(\epsilon^{ab}y_{\mathrm{d}ij}H^a_{\mathrm{d}}Q_j^{bx} + \epsilon^{ab}\lambda'{}_{jki}L_j^aQ_k^{bx} + \epsilon^{yzx}\lambda''_{jki}\bar{U}_j^y\bar{D}_k^z\right) \tag{3.68}$$

$$-F_{\bar{E}i}{}^{*} = \left(\epsilon^{ab}y_{\mathrm{e}ij}H^a_{\mathrm{d}}L_j^b + \frac{1}{2}\epsilon^{ab}\lambda_{jki}L_j^aL_k^b\right) \tag{3.69}$$

and

$$D_g{}^{\alpha} = -g_3\sum_{i=1}^{3}\left[\sum_{a=1,2}Q_i^{ax*}(T^\alpha)_{xy}Q_i^{ay} - \bar{U}_i^{x*}(T^\alpha)_{xy}\bar{U}_i^y - \bar{D}_i^{x*}(T^\alpha)_{xy}\bar{D}_i^y\right] \tag{3.70}$$

$$D_W{}^{\alpha} = -g_2\left[\sum_{i=1}^{3}\sum_{x=1}^{3}Q_i^{ax*}(T^\alpha)_{ab}Q_i^{by} + \sum_{i=1}^{3}L_i^{a*}(T^\alpha)_{ab}L_i^b + H_{\mathrm{u}}^{a*}(T^\alpha)_{ab}H_{\mathrm{u}}^b + H_{\mathrm{d}}^{a*}(T^\alpha)_{ab}H_{\mathrm{d}}^b\right] \tag{3.71}$$

$$D_B = -g_1\left[\frac{1}{6}|Q_i^{ax}|^2 - \frac{1}{2}|L_i^a|^2 - \frac{2}{3}|\bar{U}_i^x|^2 + \frac{1}{3}|\bar{D}_i^x|^2 + |\bar{E}_i|^2 + \frac{1}{2}|H_{\mathrm{u}}^a|^2 - \frac{1}{2}|H_{\mathrm{d}}^a|^2\right]. \tag{3.72}$$

Here we use the superfield notation for simple appearance.

3.6.3 Scalar Potential (Verbose)

$$\begin{aligned}V_{H_\mathrm{u}}^F &= |\mu|^2|H_\mathrm{d}|^2 + \sum_a\Bigl(|\bar{U}y^uQ^a|^2 + |\mu_iL_i^a|^2\Bigr)\\ &\quad + \Bigl[\mu^*\mu_iH_\mathrm{d}^*L_i - \mu^*H_\mathrm{d}^*\bar{U}y^uQ - \mu_i^*L_i^*\bar{U}y^uQ + \mathrm{H.c.}\Bigr]\end{aligned} \tag{3.73}$$

$$\begin{aligned}V_{H_\mathrm{d}}^F &= |\mu|^2|H_\mathrm{u}|^2 + \sum_a\Bigl(|\bar{D}y^dQ^a|^2 + |\bar{E}y^eL^a|^2\Bigr)\\ &\quad + \Bigl[-\mu^*H_\mathrm{u}^*\bar{D}y^dQ - \mu^*H_\mathrm{u}^*\bar{E}y^eL + (\bar{D}y^dQ)^*(\bar{E}y^eL) + \mathrm{H.c.}\Bigr]\end{aligned} \tag{3.74}$$

$$\begin{aligned}V_Q^F &= |H_\mathrm{u}|^2|\bar{U}_iy_{ij}^u|^2 + |H_\mathrm{d}|^2|\bar{D}_iy_{ij}^d|^2 + \lambda'^*_{jik}\lambda'_{lim}L_j^*L_l\bar{D}_k^*\bar{D}_m\\ &\quad + \Bigl[-y_{ji}^{u*}y_{ki}^dH_\mathrm{u}^*H_\mathrm{d}\bar{U}_j^*\bar{D}_k - y_{ji}^{u*}\lambda'_{lim}H_\mathrm{u}^*L_l\bar{U}_j^*\bar{D}_m + y_{ji}^{d*}\lambda'_{lim}H_\mathrm{d}^*L_l\bar{D}_j^*\bar{D}_m + \mathrm{H.c.}\Bigr]\end{aligned} \tag{3.75}$$

$$\begin{aligned}V_L^F &= |\mu_i|^2|H_\mathrm{u}|^2 + |H_\mathrm{d}|^2(\bar{E}y^ey^{e\dagger}\bar{E}^*) + \lambda'_{ijk}\lambda'_{ilm}(Q_j^*\bar{D}_k^*)Q_l\bar{D}_m + \lambda^*_{ijk}\lambda_{ilm}L_j^*L_l\bar{E}_k^*\bar{E}_m\\ &\quad + \Bigl[\mu_i^*y_{ji}^e\bar{E}_jH_\mathrm{u}^*H_\mathrm{d} - \mu_i^*\lambda'_{ijk}H_\mathrm{u}^*Q_j\bar{D}_k - \mu_i^*\lambda_{ijk}H_\mathrm{u}^*L_j\bar{E}_k\\ &\qquad -y_{ji}^{e*}\lambda'_{ilm}\bar{E}_j^*H_\mathrm{d}^*Q_l\bar{D}_m - y_{ji}^{e*}\lambda_{ilm}\bar{E}_j^*H_\mathrm{d}^*L_l\bar{E}_m + \lambda'^*_{ijk}\lambda_{ilm}\bar{D}_k^*Q_j^*L_l\bar{E}_m + \mathrm{H.c.}\Bigr]\end{aligned} \tag{3.76}$$

$$V_{\bar{U}}^F = y_{ij}^{u*}y_{ik}^u\epsilon^{ab}\epsilon^{cd}H_\mathrm{u}^{a*}H_\mathrm{u}^cQ_j^{b*}Q_k^d + \frac{1}{2}\lambda''^*_{ijk}\lambda''_{ilm}(\bar{D}_j^*\bar{D}_l)(\bar{D}_k^*\bar{D}_m) - \Bigl[y_{il}^{u*}\lambda''_{ijk}H_\mathrm{u}^*Q_l^*\bar{D}_j\bar{D}_k + \mathrm{H.c.}\Bigr] \tag{3.77}$$

$$\begin{aligned}V_{\bar{D}}^F &= \epsilon^{ab}\epsilon^{cd}\left(y_{ij}^dH_\mathrm{d}^a + \lambda'_{kji}L_k^a\right)^*\left(y_{il}^dH_\mathrm{d}^c + \lambda'_{mli}L_m^c\right)Q_j^{b*}Q_l^d + \lambda''^*_{jki}\lambda''_{lmi}\left(\bar{U}_j^*\bar{U}_l\ \bar{D}_k^*\bar{D}_m - \bar{U}_j^*\bar{D}_m\ \bar{D}_k^*\bar{U}_l\right)\\ &\quad + \Bigl[\lambda''_{lmi}(y_{ij}^{d*}H_\mathrm{d}^* + \lambda'^*_{kji}L_k^*)Q_j^*\bar{U}_l\bar{D}_m + \mathrm{H.c.}\Bigr]\end{aligned} \tag{3.78}$$

$$V_{\bar{E}}^F = \epsilon^{ab}\epsilon^{cd}\left(y_{ij}^eH_\mathrm{d}^a + \frac{1}{2}\lambda_{kji}L_k^a\right)^*L_j^{*b}\left(y_{il}^eH_\mathrm{d}^c + \frac{1}{2}\lambda_{mli}L_m^c\right)L_l^d \tag{3.79}$$

$$V_g^D = \frac{g_3^2}{2}\left\{\sum_{\alpha=1}^{8}\sum_{i=1}^{3}\left[\sum_{a=1,2}Q_i^{a*}(t^\alpha)Q_i^a - \bar{U}_i^*(t^\alpha)\bar{U}_i - \bar{D}_i^*(t^\alpha)\bar{D}_i\right]\right\}^2 \tag{3.80}$$

$$V_W^D = \frac{g_2^2}{2}\left[\sum_{i=1}^{3}\sum_{x=1}^{3}Q_i^{x*}(T^\alpha)Q_i^x + \sum_{i=1}^{3}L_i^*(T^\alpha)L_i + H_\mathrm{u}^*(T^\alpha)H_\mathrm{u} + H_\mathrm{d}^*(T^\alpha)H_\mathrm{d}\right]^2 \tag{3.81}$$

$$V_B^D = \frac{g_1^2}{2}\left[\sum_i\left(\frac{1}{6}|Q_i|^2 - \frac{1}{2}|L_i|^2 - \frac{2}{3}|\bar{U}_i|^2 + \frac{1}{3}|\bar{D}_i|^2 + |\bar{E}_i|^2\right) + \frac{1}{2}|H_\mathrm{u}|^2 - \frac{1}{2}|H_\mathrm{d}|^2\right]^2 \tag{3.82}$$

$$\begin{aligned}V_{\cancel{\mathrm{SUSY}}}^{\mathrm{RPC}} &= \left(Q^*m_Q^2Q + L^*m_L^2L + \bar{U}^*m_{\bar{U}}^2\bar{U} + \bar{D}^*m_{\bar{D}}^2\bar{D} + \bar{E}^*m_{\bar{E}}^2\bar{E} + m_{H_\mathrm{u}}^2|H_\mathrm{u}|^2 + m_{H_\mathrm{d}}^2|H_\mathrm{d}|^2\right)\\ &\quad + \left(-\bar{U}H_\mathrm{u}A^uQ + \bar{D}H_\mathrm{d}A^dQ + \bar{E}H_\mathrm{d}A^eL + BH_\mathrm{u}H_\mathrm{d} + \mathrm{H.c.}\right)\\ &\quad + \left(-\bar{U}H_\mathrm{d}^*C^uQ + \bar{D}H_\mathrm{u}^*C^dQ + \bar{E}H_\mathrm{u}^*C^eL + \mathrm{H.c.}\right)\end{aligned} \tag{3.83}$$

$$\begin{aligned}V_{\cancel{\mathrm{SUSY}}}^{\mathrm{RPV}} &= \left(B_iH_\mathrm{u}L_i + \frac{1}{2}A_{ijk}L_iL_j\bar{E}_k + A'_{ijk}L_iQ_j\bar{D}_k + \frac{1}{2}A''_{ijk}\bar{U}_i\bar{D}_j\bar{D}_k + \mathrm{H.c.}\right)\\ &\quad + \left(C_{ijk}^1L_i^*Q_j\bar{U}_k + C_i^2H_\mathrm{u}^*H_\mathrm{d}\bar{E}_i + C_{ijk}^3\bar{D}_i^*\bar{U}_j\bar{E}_k + \frac{1}{2}C_{ijk}^4\bar{D}_i^*Q_jQ_k + \mathrm{H.c.}\right)\\ &\quad + \left(M_{Li}^2H_\mathrm{d}^*L_i + \mathrm{H.c.}\right)\end{aligned} \tag{3.84}$$

With R-parity

$$\begin{aligned}
V_{\text{full}}^{\text{RPC}} = &\left(Q^* m_Q^2 Q + L^* m_L^2 L + \bar{U}^* m_{\bar{U}}^2 \bar{U} + \bar{D}^* m_{\bar{D}}^2 \bar{D} + \bar{E}^* m_{\bar{E}}^2 \bar{E}\right) \\
&+ \left(|\mu|^2 + m_{H_{\text{u}}}^2\right)|H_{\text{u}}|^2 + \left(|\mu|^2 + m_{H_{\text{d}}}^2\right)|H_{\text{d}}|^2 + (BH_{\text{u}}H_{\text{d}} + \text{H.c.}) \\
&+ \left[(-\mu^* y^u H_{\text{d}}^* - A^u H_{\text{u}} - C^u H_{\text{d}}^*)_{ij}\,\bar{U}_i Q_j + \text{H.c.}\right] \\
&+ \left[\left(-\mu^* y^d H_{\text{u}}^* + A^d H_{\text{d}} + C^d H_{\text{u}}^*\right)_{ij}\bar{D}_i Q_j + \text{H.c.}\right] \\
&+ \left[(-\mu^* y^e H_{\text{u}}^* + A^e H_{\text{d}} + C^e H_{\text{u}}^*)_{ij}\,\bar{E}_i L_j + \text{H.c.}\right] \\
&+ |H_{\text{u}}|^2|\bar{U}y^u|^2 + |H_{\text{d}}|^2|\bar{D}y^d|^2 + |H_{\text{d}}|^2|\bar{E}y^e|^2 + |H_{\text{u}}|^2|y^u Q|^2 + |H_{\text{d}}|^2|y^d Q|^2 + |H_{\text{d}}|^2|y^e L|^2 \\
&+ \sum_a \left(|\bar{U}y^u Q^a|^2 + |\bar{D}y^d Q^a|^2 + |\bar{E}y^e L^a|^2\right) \\
&+ \left[(\bar{D}y^d Q)^*(\bar{E}y^e L) - y_{ji}^{u*} y_{ki}^d H_{\text{u}}^* H_{\text{d}} \bar{U}_j^* \bar{D}_k + \text{H.c.}\right] \\
&- \left[y_{ki}^{u*} y_{kj}^u (H_{\text{u}}^* Q_j)(Q_i^* H_{\text{u}}) + y_{ki}^{d*} y_{kj}^d (H_{\text{d}}^* Q_j)(Q_i^* H_{\text{d}}) + y_{ki}^{e*} y_{kj}^e (H_{\text{d}}^* L_j)(L_i^* H_{\text{d}})\right] \\
&+ \frac{g_3^2}{2}\left\{\sum_{\alpha=1}^{8}\sum_{i=1}^{3}\left[\sum_{a=1,2} Q_i^{a*}(t^\alpha)Q_i^a - \bar{U}_i^*(t^\alpha)\bar{U}_i - \bar{D}_i^*(t^\alpha)\bar{D}_i\right]\right\}^2 \\
&+ \frac{g_2^2}{2}\left[\sum_{i=1}^{3}\sum_{x=1}^{3} Q_i^{x*}(T^\alpha)Q_i^x + \sum_{i=1}^{3} L_i^*(T^\alpha)L_i + H_{\text{u}}^*(T^\alpha)H_{\text{u}} + H_{\text{d}}^*(T^\alpha)H_{\text{d}}\right]^2 \\
&+ \frac{g_1^2}{2}\left[\sum_i \left(\frac{1}{6}|Q_i|^2 - \frac{1}{2}|L_i|^2 - \frac{2}{3}|\bar{U}_i|^2 + \frac{1}{3}|\bar{D}_i|^2 + |\bar{E}_i|^2\right) + \frac{1}{2}|H_{\text{u}}|^2 - \frac{1}{2}|H_{\text{d}}|^2\right]^2
\end{aligned} \tag{3.85}$$

With Bilinear R-parity Violation

$$V_{H_{\text{u}}}^F \mathrel{+}= \sum_a |\mu_i L_i^a|^2 + \left[\mu^* \mu_i H_{\text{d}}^* L_i - \mu_i^* L_i^* \bar{U} y^u Q + \text{H.c.}\right] \tag{3.86}$$

$$V_L^F \mathrel{+}= |\mu_i|^2 |H_{\text{u}}|^2 + \left[\mu_i^* y_{ji}^e \bar{E}_j H_{\text{u}}^* H_{\text{d}} + \text{H.c.}\right] \tag{3.87}$$

$$V_{\cancel{\text{SUSY}}}^{\text{RPV}} = \left(B_i H_{\text{u}} L_i + M_{Li}^2 H_{\text{d}}^* L_i + \text{H.c.}\right) \tag{3.88}$$

With Trilinear leptonic R-parity Violation

$$V_Q^F \mathrel{+}= |\lambda'_{jik} L_j \bar{D}_k|^2 + \left[-y_{ji}^{u*} \lambda'_{lim} H_{\text{u}}^* L_l \bar{U}_j^* \bar{D}_m + y_{ji}^{d*} \lambda'_{lim} H_{\text{d}}^* L_l \bar{D}_j^* \bar{D}_m + \text{H.c.}\right] \tag{3.89}$$

$$\begin{aligned}
V_L^F \mathrel{+}= &|\lambda'_{ijk} Q_j \bar{D}_k|^2 + |\lambda_{ijk} L_j \bar{E}_k|^2 \\
&+ \left[-y_{mi}^{e*} \lambda'_{ijk} \bar{E}_m^* H_{\text{d}}^* Q_j \bar{D}_k - y_{mi}^{e*} \lambda_{ijk} \bar{E}_m^* H_{\text{d}}^* L_j \bar{E}_k + \lambda_{ijk}'^{*} \lambda_{ilm} \bar{D}_k^* Q_j^* L_l \bar{E}_m + \text{H.c.}\right]
\end{aligned} \tag{3.90}$$

$$\begin{aligned}
V_{\bar{D}}^F \mathrel{+}= &\lambda_{kji}'^{*} \lambda'_{mli}\left[(L_k^* L_m)(Q_j^* Q_l) - (L_k^* Q_l)(Q_j^* L_m)\right] \\
&+ \left\{y_{ij}^{d}{}^{*} \lambda'_{mli}\left[(H_{\text{d}}^* L_m)(Q_j Q_l) - (H_{\text{d}}^* Q_l)(Q_j^* L_m)\right] + \text{H.c.}\right\}
\end{aligned} \tag{3.91}$$

$$V_{\bar{E}}^F = \frac{1}{2}\lambda_{kji}^* \lambda_{mli} (L_k^* L_m)(L_j^* L_l) + \lambda_{kji}^* y_{ilp}^e \left[(L_k^* H_{\text{d}})(L_j^* L_l) + \text{H.c.}\right] \tag{3.92}$$

3.6.4 Full Lagrangian (in Gauge eigenstates)

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{\mu\nu\,a}W^{a}_{\mu\nu} - \frac{1}{4}G^{\mu\nu\,a}G^{a}_{\mu\nu} \quad \text{[*6]} \tag{3.93}$$

$$\begin{aligned}\mathcal{L}_{\text{gaugino}} &= -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{w}\widetilde{w} + M_1\widetilde{b}\widetilde{b} + \text{H.c.}\right)\\ &\quad + \mathrm{i}\bar{\widetilde{b}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{b}^a + \mathrm{i}\bar{\widetilde{w}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{w}^a + \mathrm{i}\bar{\widetilde{g}}^a\bar{\sigma}^\mu\partial_\mu\widetilde{g}^a + \mathrm{i}g_2\epsilon^{abc}\bar{\widetilde{w}}^a\bar{\sigma}^\mu W^b_\mu\widetilde{w}^c + \mathrm{i}g_3 f^{abc}\bar{\widetilde{g}}^a\bar{\sigma}^\mu G^b_\mu\widetilde{g}^c\end{aligned} \tag{3.94}$$

$$\mathcal{L}_{\cancel{CP}} = -\frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma} - \frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}W^a_{\mu\nu}W^a_{\rho\sigma} - \frac{1}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} \tag{3.95}$$

$$\begin{aligned}\mathcal{L}_{\text{scalar}} &= \Big[\big(\partial^\mu + \mathrm{i}g_3G^\mu + \mathrm{i}g_2W^\mu + \tfrac{1}{6}\mathrm{i}g_1B^\mu\big)\widetilde{q}_i^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \tfrac{1}{6}\mathrm{i}g_1B_\mu\big)\widetilde{q}_i\Big]\\ &\quad + \Big[\big(\partial^\mu + \mathrm{i}g_3G^\mu + \tfrac{2}{3}\mathrm{i}g_1B^\mu\big)\widetilde{u}_{\mathrm{R}i}^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_3G_\mu - \tfrac{2}{3}\mathrm{i}g_1B_\mu\big)\widetilde{u}_{\mathrm{R}i}\Big]\\ &\quad + \Big[\big(\partial^\mu + \mathrm{i}g_3G^\mu - \tfrac{1}{3}\mathrm{i}g_1B^\mu\big)\widetilde{d}_{\mathrm{R}i}^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_3G_\mu + \tfrac{1}{3}\mathrm{i}g_1B_\mu\big)\widetilde{d}_{\mathrm{R}i}\Big]\\ &\quad + \Big[\big(\partial^\mu + \mathrm{i}g_2W^\mu - \tfrac{1}{2}\mathrm{i}g_1B^\mu\big)\widetilde{l}_i^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_2W_\mu + \tfrac{1}{2}\mathrm{i}g_1B_\mu\big)\widetilde{l}_i\Big]\\ &\quad + \Big[\big(\partial^\mu - \mathrm{i}g_1B^\mu\big)\widetilde{e}_{\mathrm{R}i}^*\Big]\Big[\big(\partial_\mu + \mathrm{i}g_1B_\mu\big)\widetilde{e}_{\mathrm{R}i}\Big]\\ &\quad + \Big[\big(\partial^\mu + \mathrm{i}g_2W^\mu + \tfrac{1}{2}\mathrm{i}g_1B^\mu\big)H_{\mathrm{u}}^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_2W_\mu - \tfrac{1}{2}\mathrm{i}g_1B_\mu\big)H_{\mathrm{u}}\Big]\\ &\quad + \Big[\big(\partial^\mu + \mathrm{i}g_2W^\mu - \tfrac{1}{2}\mathrm{i}g_1B^\mu\big)H_{\mathrm{d}}^*\Big]\Big[\big(\partial_\mu - \mathrm{i}g_2W_\mu + \tfrac{1}{2}\mathrm{i}g_1B_\mu\big)H_{\mathrm{d}}\Big],\end{aligned} \tag{3.96}$$

$$\begin{aligned}\mathcal{L}_{\text{fermion}} &= \mathrm{i}\bar{Q}_i\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3G_\mu - \mathrm{i}g_2W_\mu - \frac{1}{6}\mathrm{i}g_1B_\mu\right)Q_i\\ &\quad + \mathrm{i}\bar{U}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3[-G_\mu^{\mathrm{T}}] + \frac{2}{3}\mathrm{i}g_1B_\mu\right)U_i^{\mathrm{c}} + \mathrm{i}\bar{D}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_3[-G_\mu^{\mathrm{T}}] - \frac{1}{3}\mathrm{i}g_1B_\mu\right)D_i^{\mathrm{c}}\\ &\quad + \mathrm{i}\bar{L}_i\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)L_i + \mathrm{i}\bar{E}_i^{\mathrm{c}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_1B_\mu\right)E_i^{\mathrm{c}}\\ &\quad + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{u}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{u}} + \mathrm{i}\bar{\widetilde{h}}_{\mathrm{d}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\widetilde{h}_{\mathrm{d}},\end{aligned} \tag{3.97}$$

$$\begin{aligned}\mathcal{L}_{\text{SFG}} &= -\sqrt{2}g_3\Big[\big(\widetilde{q}_i^*\tau^aQ_i + \widetilde{u}_{\mathrm{R}i}[-\tau^{a\,\mathrm{T}}]U_i^{\mathrm{c}} + \widetilde{d}_{\mathrm{R}i}[-\tau^{a\,\mathrm{T}}]D_i^{\mathrm{c}}\big)\widetilde{g}^a\\ &\qquad\qquad + \bar{\widetilde{g}}^a\big(\bar{Q}_i\tau^a\widetilde{q}_i + \bar{U}_i^{\mathrm{c}}[-\tau^{a\,\mathrm{T}}]\widetilde{u}_{\mathrm{R}i}^* + \bar{D}_i^{\mathrm{c}}[-\tau^{a\,\mathrm{T}}]\widetilde{d}_{\mathrm{R}i}^*\big)\Big]\\ &\quad - \sqrt{2}g_2\Big[\big(\widetilde{q}_i^*T^aQ_i + \widetilde{l}_i^*T^aL_i + H_{\mathrm{u}}^*T^a\widetilde{h}_{\mathrm{u}} + H_{\mathrm{d}}^*T^a\widetilde{h}_{\mathrm{d}}\big)\widetilde{w}^a\\ &\qquad\qquad + \bar{\widetilde{w}}^a\big(\bar{Q}_iT^a\widetilde{q}_i + \bar{L}_iT^a\widetilde{l}_i + \bar{\widetilde{h}}_{\mathrm{u}}T^aH_{\mathrm{u}} + \bar{\widetilde{h}}_{\mathrm{d}}T^aH_{\mathrm{d}}\big)\Big]\\ &\quad - \sqrt{2}g_1\Big[\big(\tfrac{1}{6}\widetilde{q}_i^*Q_i - \tfrac{2}{3}\widetilde{u}_{\mathrm{R}i}U_i^{\mathrm{c}} + \tfrac{1}{3}\widetilde{d}_{\mathrm{R}i}D_i^{\mathrm{c}} - \tfrac{1}{2}\widetilde{l}_i^*L_i + \widetilde{e}_{\mathrm{R}i}E_i^{\mathrm{c}} + \tfrac{1}{2}H_{\mathrm{u}}^*\widetilde{h}_{\mathrm{u}} - \tfrac{1}{2}H_{\mathrm{d}}^*\widetilde{h}_{\mathrm{d}}\big)\widetilde{b}\\ &\qquad\qquad + \bar{\widetilde{b}}\big(\tfrac{1}{6}\bar{Q}_i\widetilde{q}_i - \tfrac{2}{3}\bar{U}_i^{\mathrm{c}}\widetilde{u}_{\mathrm{R}i}^* + \tfrac{1}{3}\bar{D}_i^{\mathrm{c}}\widetilde{d}_{\mathrm{R}i}^* - \tfrac{1}{2}\bar{L}_i\widetilde{l}_i + \bar{E}_i^{\mathrm{c}}\widetilde{e}_{\mathrm{R}i}^* + \tfrac{1}{2}\bar{\widetilde{h}}_{\mathrm{u}}H_{\mathrm{u}} - \tfrac{1}{2}\bar{\widetilde{h}}_{\mathrm{d}}H_{\mathrm{d}}\big)\Big]\end{aligned} \tag{3.98}$$

$$\begin{aligned}\mathcal{L}_{\text{super}}^{\text{RPC}} &= -\mu\widetilde{h}_{\mathrm{u}}\widetilde{h}_{\mathrm{d}} + y_{\mathrm{u}ij}U_i^{\mathrm{c}}H_{\mathrm{u}}Q_j - y_{\mathrm{d}ij}D_i^{\mathrm{c}}H_{\mathrm{d}}Q_j - y_{\mathrm{e}ij}E_i^{\mathrm{c}}H_{\mathrm{d}}L_j\\ &\quad + y_{\mathrm{u}ij}U_i^{\mathrm{c}}\widetilde{h}_{\mathrm{u}}\widetilde{q}_j + y_{\mathrm{u}ij}\widetilde{u}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{u}}Q_j - y_{\mathrm{d}ij}D_i^{\mathrm{c}}\widetilde{h}_{\mathrm{d}}\widetilde{q}_j - y_{\mathrm{d}ij}\widetilde{d}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{d}}Q_j\\ &\quad - y_{\mathrm{e}ij}E_i^{\mathrm{c}}\widetilde{h}_{\mathrm{d}}\widetilde{l}_j - y_{\mathrm{e}ij}\widetilde{e}_{\mathrm{R}i}^*\widetilde{h}_{\mathrm{d}}L_j + \text{H.c.}\end{aligned} \tag{3.99}$$

$$\mathcal{L}_{\text{pot.}}^{\text{RPC}} = -(3.85)\Big[V_{\text{full}}^{\text{RPC}}\Big] \tag{3.100}$$

[*6] Further decomposed results are shown in *Standard Model* section, Eqs. (2.7) and (2.21).

Fermion Composition

$$\mathcal{L}_{\text{gaugino}} = \frac{1}{2}\bar{\tilde{b}}\left(\mathrm{i}\partial\!\!\!/ - M_1\right)\tilde{b} + \frac{1}{2}\bar{\tilde{w}}\left(\mathrm{i}D\!\!\!\!/ - M_2\right)\tilde{w} + \frac{1}{2}\bar{\tilde{g}}\left(\mathrm{i}D\!\!\!\!/ - M_3\right)\tilde{g} \tag{3.101}$$

$$\begin{aligned}
\mathcal{L}_{\text{fermion}} &= (2.5)\left[\mathcal{L}^{\text{SM}}_{\text{matter}}\right] \\
&\quad - \left(\mu\tilde{h}_{\mathrm{u}}\tilde{h}_{\mathrm{d}} + \text{H.c.}\right) + \mathrm{i}\bar{\tilde{h}}_{\mathrm{u}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}_{\mathrm{u}} + \mathrm{i}\bar{\tilde{h}}_{\mathrm{d}}\bar{\sigma}^\mu\left(\partial_\mu - \mathrm{i}g_2W_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}_{\mathrm{d}} \\
&= \mathcal{L}^{\text{SM}}_{\text{matter}} - \left[\mu\left(\tilde{h}^+_{\mathrm{u}}\tilde{h}^-_{\mathrm{d}} - \tilde{h}^0_{\mathrm{u}}\tilde{h}^0_{\mathrm{d}}\right) + \text{H.c.}\right] \\
&\quad + \mathrm{i}\bar{\tilde{h}}^+_{\mathrm{u}}\bar{\sigma}^\mu\left(\partial_\mu - \frac{1}{2}\mathrm{i}g_2W^3_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}^+_{\mathrm{u}} + \mathrm{i}\bar{\tilde{h}}^-_{\mathrm{d}}\bar{\sigma}^\mu\left(\partial_\mu + \frac{1}{2}\mathrm{i}g_2W^3_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}^-_{\mathrm{d}} \\
&\quad + \mathrm{i}\bar{\tilde{h}}^0_{\mathrm{u}}\bar{\sigma}^\mu\left(\partial_\mu + \frac{1}{2}\mathrm{i}g_2W^3_\mu - \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}^0_{\mathrm{u}} + \mathrm{i}\bar{\tilde{h}}^0_{\mathrm{d}}\bar{\sigma}^\mu\left(\partial_\mu - \frac{1}{2}\mathrm{i}g_2W^3_\mu + \frac{1}{2}\mathrm{i}g_1B_\mu\right)\tilde{h}^0_{\mathrm{d}} \\
&\quad + \frac{g_2}{\sqrt{2}}\left(\bar{\tilde{h}}^0_{\mathrm{u}}\bar{\sigma}^\mu W^-_\mu\tilde{h}^+_{\mathrm{u}} + \bar{\tilde{h}}^0_{\mathrm{d}}\bar{\sigma}^\mu W^+_\mu\tilde{h}^-_{\mathrm{d}} + \bar{\tilde{h}}^+_{\mathrm{u}}\bar{\sigma}^\mu W^+_\mu\tilde{h}^0_{\mathrm{u}} + \bar{\tilde{h}}^-_{\mathrm{d}}\bar{\sigma}^\mu W^-_\mu\tilde{h}^0_{\mathrm{d}}\right)
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
\mathcal{L}_{\text{SFG}} &= -\sqrt{2}g_3\Big[\tilde{u}^*_{\mathrm{L}i}\tau^a\left(\bar{\tilde{g}}^aP_{\mathrm{L}}U_i\right) + \tilde{d}^*_{\mathrm{L}i}\tau^a\left(\bar{\tilde{g}}^aP_{\mathrm{L}}D_i\right) - \left(\overline{U}_iP_{\mathrm{L}}\tilde{g}^a\right)\tau^a\tilde{u}_{\mathrm{R}i} - \left(\overline{D}_iP_{\mathrm{L}}\tilde{g}^a\right)\tau^a\tilde{d}_{\mathrm{R}i} \\
&\qquad + \left(\overline{U}_iP_{\mathrm{R}}\tilde{g}^a\right)\tau^a\tilde{u}_{\mathrm{L}i} + \left(\overline{D}_iP_{\mathrm{R}}\tilde{g}^a\right)\tau^a\tilde{d}_{\mathrm{L}i} - \tilde{u}^*_{\mathrm{R}i}\tau^a\left(\bar{\tilde{g}}^aP_{\mathrm{R}}U\right) - \tilde{d}^*_{\mathrm{R}i}\tau^a\left(\bar{\tilde{g}}^aP_{\mathrm{R}}D\right)\Big] \\
&\quad - g_2\Big[\tilde{w}^+\left(d_i\tilde{u}_{\mathrm{L}i}^* + e_i\tilde{\nu}^*_i + \tilde{h}^0_{\mathrm{u}}H_{\mathrm{u}}^{+*} + \tilde{h}^-_{\mathrm{d}}H_{\mathrm{d}}^{0*}\right) + \left(\tilde{d}_{\mathrm{L}i}\bar{U}_i + \tilde{e}_{\mathrm{L}i}\bar{\nu}_i + H^0_{\mathrm{u}}\bar{\tilde{h}}^+_{\mathrm{u}} + H^-_{\mathrm{d}}\bar{\tilde{h}}^0_{\mathrm{d}}\right)\bar{\tilde{w}}^- \\
&\qquad + \tilde{w}^-\left(u_i\tilde{d}_{\mathrm{L}i}^* + \nu_i\tilde{e}_{\mathrm{L}i}^* + \tilde{h}^+_{\mathrm{u}}H_{\mathrm{u}}^{0*} + \tilde{h}^0_{\mathrm{d}}H_{\mathrm{d}}^{-*}\right) + \left(\tilde{u}_{\mathrm{L}i}\bar{D}_i + \tilde{\nu}_i\bar{E}_i + H^+_{\mathrm{u}}\bar{\tilde{h}}^0_{\mathrm{u}} + H^0_{\mathrm{d}}\bar{\tilde{h}}^-_{\mathrm{d}}\right)\bar{\tilde{w}}^+\Big] \\
&\quad - \sqrt{2}g_2\Big[\tfrac{1}{2}\big(\tilde{u}_{\mathrm{L}i}^*u_i + \tilde{\nu}^*_i\nu_i + H_{\mathrm{u}}^{+*}\tilde{h}^+_{\mathrm{u}} + H_{\mathrm{d}}^{0*}\tilde{h}^0_{\mathrm{d}} - \tilde{d}_{\mathrm{L}i}^*d_i - \tilde{e}_{\mathrm{L}i}^*e_i - H_{\mathrm{u}}^{0*}\tilde{h}^0_{\mathrm{u}} - H_{\mathrm{d}}^{-*}\tilde{h}^-_{\mathrm{d}}\big)\tilde{w}^3 \\
&\qquad\qquad + \tfrac{1}{2}\bar{\tilde{w}}^a\big(\bar{U}_i\tilde{u}_{\mathrm{L}i} + \bar{\nu}_i\tilde{\nu}_i + \bar{\tilde{h}}^+_{\mathrm{u}}H^+_{\mathrm{u}} + \bar{\tilde{h}}^0_{\mathrm{d}}H^0_{\mathrm{d}} - \bar{D}_i\tilde{d}_{\mathrm{L}i} - \bar{E}_i\tilde{e}_{\mathrm{L}i} - \bar{\tilde{h}}^0_{\mathrm{u}}H^0_{\mathrm{u}} - \bar{\tilde{h}}^-_{\mathrm{d}}H^-_{\mathrm{d}}\big)\Big] \\
&\quad - \sqrt{2}g_1\Big[\tfrac{1}{6}\tilde{q}^*_i\bar{\tilde{b}}P_{\mathrm{L}}Q_i - \tfrac{2}{3}\tilde{u}_{\mathrm{R}i}\overline{U}_iP_{\mathrm{L}}\tilde{b} + \tfrac{1}{3}\tilde{d}_{\mathrm{R}i}\overline{D}_iP_{\mathrm{L}}\tilde{b} - \tfrac{1}{2}\tilde{l}^*_i\bar{\tilde{b}}P_{\mathrm{L}}L_i + \tilde{e}_{\mathrm{R}i}\overline{E}_iP_{\mathrm{L}}\tilde{b} \\
&\qquad\qquad + \tfrac{1}{6}\tilde{q}_i\overline{Q}_iP_{\mathrm{R}}\tilde{b} - \tfrac{2}{3}\tilde{u}^*_{\mathrm{R}i}\bar{\tilde{b}}P_{\mathrm{R}}U_i + \tfrac{1}{3}\tilde{d}^*_{\mathrm{R}i}\bar{\tilde{b}}P_{\mathrm{R}}D_i - \tfrac{1}{2}\tilde{l}_i\overline{L}_iP_{\mathrm{R}}\tilde{b} + \tilde{e}^*_{\mathrm{R}i}\bar{\tilde{b}}P_{\mathrm{R}}E_i \\
&\qquad\qquad + \tfrac{1}{2}H^*_{\mathrm{u}}\tilde{h}_{\mathrm{u}}\tilde{b} - \tfrac{1}{2}H^*_{\mathrm{d}}\tilde{h}_{\mathrm{d}}\tilde{b} + \tfrac{1}{2}H_{\mathrm{u}}\bar{\tilde{h}}_{\mathrm{u}}\bar{\tilde{b}} - \tfrac{1}{2}H_{\mathrm{d}}\bar{\tilde{h}}_{\mathrm{d}}\bar{\tilde{b}}\Big]
\end{aligned} \tag{3.103}$$

$$\begin{aligned}
\mathcal{L}_{\text{湯川}} &= \Big(-H^0_{\mathrm{u}}\overline{U}y_{\mathrm{u}}P_{\mathrm{L}}U - H^0_{\mathrm{d}}\overline{D}y_{\mathrm{d}}P_{\mathrm{L}}D - H^0_{\mathrm{d}}\overline{E}y_{\mathrm{e}}P_{\mathrm{L}}E \\
&\qquad + H^+_{\mathrm{u}}\overline{U}y_{\mathrm{u}}P_{\mathrm{L}}D + H^-_{\mathrm{d}}\overline{D}y_{\mathrm{d}}P_{\mathrm{L}}U + H^-_{\mathrm{d}}\overline{E}y_{\mathrm{e}}P_{\mathrm{L}}\nu + \text{H.c.}\Big) \\
&\quad + \tilde{u}_{\mathrm{L}j}\left(y_{\mathrm{d}ij}D^{\mathrm{c}}_i\tilde{h}^-_{\mathrm{d}} - y_{\mathrm{u}ij}U^{\mathrm{c}}_i\tilde{h}^0_{\mathrm{u}}\right) + \tilde{d}_{\mathrm{L}j}\left(y_{\mathrm{u}ij}U^{\mathrm{c}}_i\tilde{h}^+_{\mathrm{u}} - y_{\mathrm{d}ij}D^{\mathrm{c}}_i\tilde{h}^0_{\mathrm{d}}\right) + y_{\mathrm{e}ij}E^{\mathrm{c}}_i\tilde{h}^-_{\mathrm{d}}\tilde{\nu}_j - y_{\mathrm{e}ij}E^{\mathrm{c}}_i\tilde{h}^0_{\mathrm{d}}\tilde{e}_{\mathrm{L}j} \\
&\quad + y_{\mathrm{u}ij}\tilde{u}^*_{\mathrm{R}i}\left(\tilde{h}^+_{\mathrm{u}}d_j - \tilde{h}^0_{\mathrm{u}}u_j\right) + y_{\mathrm{d}ij}\tilde{d}^*_{\mathrm{R}i}\left(\tilde{h}^-_{\mathrm{d}}u_j - \tilde{h}^0_{\mathrm{d}}d_j\right) + y_{\mathrm{e}ij}\tilde{e}^*_{\mathrm{R}i}\left(\tilde{h}^-_{\mathrm{d}}\nu_j - \tilde{h}^0_{\mathrm{d}}e_j\right) \\
&\quad + \tilde{u}^*_{\mathrm{L}j}\left(y^*_{\mathrm{d}ij}\bar{D}^{\mathrm{c}}_i\bar{\tilde{h}}^-_{\mathrm{d}} - y^*_{\mathrm{u}ij}\bar{U}^{\mathrm{c}}_i\bar{\tilde{h}}^0_{\mathrm{u}}\right) + \tilde{d}^*_{\mathrm{L}j}\left(y^*_{\mathrm{u}ij}\bar{U}^{\mathrm{c}}_i\bar{\tilde{h}}^+_{\mathrm{u}} - y^*_{\mathrm{d}ij}\bar{D}^{\mathrm{c}}_i\bar{\tilde{h}}^0_{\mathrm{d}}\right) + \tilde{\nu}^*{}_jy^*_{\mathrm{e}ij}\bar{E}^{\mathrm{c}}_i\bar{\tilde{h}}^-_{\mathrm{d}} - \tilde{e}^*_{\mathrm{L}j}y^*_{\mathrm{e}ij}\bar{E}^{\mathrm{c}}_i\bar{\tilde{h}}^0_{\mathrm{d}} \\
&\quad + y^*_{\mathrm{u}ij}\tilde{u}_{\mathrm{R}i}\left(\bar{\tilde{h}}^+_{\mathrm{u}}\bar{d}_j - \bar{\tilde{h}}^0_{\mathrm{u}}\bar{u}_j\right) + y^*_{\mathrm{d}ij}\tilde{d}_{\mathrm{R}i}\left(\bar{\tilde{h}}^-_{\mathrm{d}}\bar{u}_j - \bar{\tilde{h}}^0_{\mathrm{d}}\bar{d}_j\right) + y^*_{\mathrm{e}ij}\tilde{e}_{\mathrm{R}i}\left(\bar{\tilde{h}}^-_{\mathrm{d}}\bar{\nu}_j - \bar{\tilde{h}}^0_{\mathrm{d}}\bar{e}_j\right)
\end{aligned} \tag{3.104}$$

and the rest part is $\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}^{\text{RPC}}_{\text{pot.}}$.

3.6.5 Mass matrices

$$m^2_{\tilde{t}} = \begin{pmatrix} m^2_{Q_3} + m^2_t + (\frac{1}{2} - \frac{2}{3}s^2_{\mathrm{w}})m^2_Z \cos 2\beta & m_t(A^*_t - \mu\cot\beta) \\ m_t(A_t - \mu\cot\beta) & m^2_{\bar{U}_3} + m^2_t + \frac{2}{3}s^2_{\mathrm{w}} m^2_Z \cos 2\beta \end{pmatrix}, \tag{3.105}$$

$$m^2_{\tilde{b}} = \begin{pmatrix} m^2_{Q_3} + m^2_b + (-\frac{1}{2} + \frac{1}{3}s^2_{\mathrm{w}})m^2_Z \cos 2\beta & m_b(A^*_b - \mu\tan\beta) \\ m_b(A_b - \mu\tan\beta) & m^2_{\bar{D}_3} + m^2_b - \frac{1}{3}s^2_{\mathrm{w}} m^2_Z \cos 2\beta \end{pmatrix}, \tag{3.106}$$

$$m^2_{\tilde{\tau}} = \begin{pmatrix} m^2_{L_3} + m^2_\tau + (-\frac{1}{2} + s^2_{\mathrm{w}})m^2_Z \cos 2\beta & m_\tau(A^*_\tau - \mu\tan\beta) \\ m_\tau(A_\tau - \mu\tan\beta) & m^2_{\bar{E}_3} + m^2_\tau - s^2_{\mathrm{w}} m^2_Z \cos 2\beta \end{pmatrix}. \tag{3.107}$$

3.7 minimal GMSB

$$W_{\text{mess}} = \lambda_i X \bar{\Phi}_i \Phi_i \rightsquigarrow \lambda_i (M_X + F_X \theta\theta) \Phi_i \Phi_i \quad : \quad M_i^{\text{F}} = \lambda_i M_X, \quad \left(M_i^{\text{S}}\right)^2 = \lambda_i^2 M_X^2 - \lambda_i F_X.$$

If F_i/M_i is universal and the messengers feel only a single X,[*7]

$$\Lambda := \frac{F_X}{M_X}, \quad M_{\text{mess}} := \lambda M_X, \qquad M_a(M_{\text{mess}}) \simeq \frac{\alpha_a}{4\pi}\Lambda_{\text{G}}, \qquad m^2(M_{\text{mess}}) \simeq 2\Lambda_{\text{S}}^2 \sum \frac{\alpha_a^2}{(4\pi)^2} C_a, \tag{3.109}$$

C_3 is 4/3 for colored particles, C_2 is 3/4 for **2**, and $C_1 = (5/3)Y^2$. Also the ~~SUSY~~ scales are

$$\Lambda_{\text{G}} := N_5 \frac{F}{M} = N_5 \Lambda, \qquad \Lambda_{\text{S}} := N_5 \frac{F^2}{M^2} = N_5 \Lambda^2; \qquad \text{where } N_5 \text{ is } (\#\mathbf{5}) + 3 \times (\#\mathbf{10}), \tag{3.110}$$

Note that $(\Lambda_{\text{G}}, \Lambda_{\text{S}})$ can be used as the defining parameters instead of (N_5, Λ).

The total SUSY-breaking F_{total} is different from that felt by messengers; $F_{\text{total}} =: F/k$ with $k < 1$.

$$m_{3/2} = \frac{F_{\text{total}}}{\sqrt{3} M_{\text{pl}}^{\text{R}}} = \frac{F/k}{\sqrt{3} M_{\text{pl}}^{\text{R}}} = \frac{M_{\text{mess}} \Lambda}{\lambda k \cdot \sqrt{3} M_{\text{pl}}^{\text{R}}}. \tag{3.111}$$

[*7] If non universal, Λ_{S} cannot be written in simple form, and

$$\Lambda_{\text{G}} = \sum n_i \frac{F_i}{M_i} \left(1 + \mathrm{O}\left(\frac{F_i^2}{M_i^4}\right)\right), \tag{3.108}$$

where n_i is twice of the Dynkin index; 1 for **5**, and 3 for **10**.

3.8 Gravitino and Goldstino

3.8.1 Supercurrent

$$S^\mu_\alpha = (\sigma^\nu\bar\sigma^\mu\chi_i)_\alpha \mathrm{D}_\nu\phi^{*i} + \mathrm{i}(\sigma^\mu\chi^{\dagger i})_\alpha W^*_\phi - \frac{1}{2\sqrt2}(\sigma^\nu\bar\sigma^\rho\sigma^\mu\lambda^{\dagger a})_\alpha F^a_{\nu\rho} + \frac{\mathrm{i}}{\sqrt2}g_a(\phi^*T^a\phi)(\sigma^\mu\lambda^{\dagger a})_\alpha$$

With a Majorana particle $X = \begin{pmatrix} X_\alpha \\ \bar X^{\dot\alpha}\end{pmatrix}$,

$$\begin{aligned}X^\alpha S^\mu_\alpha &= \mathrm{D}_\nu\phi^*(\overline{X}\gamma^\nu\gamma^\mu P_{\mathrm{L}}\chi) + \mathrm{i}(\overline{X}\gamma^\nu P_{\mathrm{R}}\chi)W^*_\phi - \frac{1}{4\sqrt2}F^a_{\nu\rho}\overline{X}[\gamma^\nu,\gamma^\rho]\gamma^\mu P_{\mathrm{R}}\lambda^a + \frac{\mathrm{i}g_a(\phi^*T^a\phi)}{\sqrt2}\overline{X}\gamma^\mu P_{\mathrm{R}}\lambda^a \\ &\overset{\text{h.c.}}{\rightsquigarrow} \mathrm{D}_\nu\phi(\overline{X}\gamma^\nu\gamma^\mu P_{\mathrm{R}}\chi) + \mathrm{i}(\overline{X}\gamma^\nu P_{\mathrm{L}}\chi)W_\phi - \frac{1}{4\sqrt2}F^a_{\nu\rho}\overline{X}[\gamma^\nu,\gamma^\rho]\gamma^\mu P_{\mathrm{L}}\lambda^a + \frac{\mathrm{i}g_a(\phi^*T^a\phi)}{2}\overline{X}\gamma^\mu P_{\mathrm{L}}\lambda^a\end{aligned}$$

where $W_\phi = \frac{\delta}{\delta\phi}W|_{\text{scalar}}$, $W_\phi = \frac{\delta}{\delta\phi^*}W^*|_{\text{scalar}}$.

3.8.2 Goldstino

$$\mathcal{L}_{\text{goldstino}} = i\widetilde{\mathcal{G}}^\dagger\bar\sigma^\mu\partial_\mu\widetilde{\mathcal{G}} - \frac{1}{F_{\text{total}}}\left[\widetilde{\mathcal{G}}\partial_\mu S^\mu + \text{H.c.}\right], \tag{3.112}$$

especially interaction terms contain

$$-\frac{1}{F_{\text{total}}}\left[\left(\partial_\nu\phi^*\cdot\overline{\chi}\gamma^\mu\gamma^\nu P_{\mathrm{L}}\partial_\mu\widetilde{\mathcal{G}} + \text{H.c.}\right) - \frac{1}{4\sqrt2}(\overline{\lambda}\gamma^\mu[\gamma^\nu,\gamma^\rho]\partial_\mu\widetilde{\mathcal{G}})F^a_{\nu\rho}\right]. \tag{3.113}$$

If all the particles are on-shell, this can be reduced to

$$\frac{1}{F_{\text{total}}}\left[(m_\chi^2 - m_\phi^2)\phi^*\overline{\chi}P_{\mathrm{L}}\widetilde{\mathcal{G}} + \frac{1}{4\sqrt2}(m_\lambda\overline{\lambda}[\gamma^\nu,\gamma^\rho]P_{\mathrm{R}}\widetilde{\mathcal{G}})F^a_{\nu\rho} + \text{H.c.}\right]. \tag{3.114}$$

TODO: 実はよくわかってない

3.8.3 Gravitino $\quad \psi = \begin{pmatrix}\psi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix}$

Lagrangian: $\mathcal{L} = -\frac12\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \frac{m_{3/2}}{4}\overline{\psi}_\mu[\gamma^\mu,\gamma^\nu]\psi_\nu - \frac{1}{\sqrt2 M_{\text{pl}}^{\mathrm{R}}}\left(\psi^\alpha_\mu S^\mu_\alpha + \text{H.c.}\right)$

EOM: $\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma + \frac{m_{3/2}}{2}[\gamma^\mu,\gamma^\nu]\psi_\nu = 0$ (Rarita–Schwinger eq.)

$\Longrightarrow m_{3/2}\left(\partial\!\!\!/\psi - \gamma^\nu\partial\!\!\!/\psi_\nu\right) = 0, \quad \left(\partial\!\!\!/\psi - \gamma^\nu\partial\!\!\!/\psi_\nu\right) - 3\mathrm{i}m_{3/2}\psi = 0.$

⟪if massive $\Longrightarrow \psi\!\!\!/ = 0, \quad \partial_\mu\psi^\mu = 0, \quad (\mathrm{i}\partial\!\!\!/ - m_{3/2})\psi_\mu = 0$ (Dirac eq.).⟫

Field: $\psi_\mu = \displaystyle\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol p}}}\sum_{s=\pm1/2,\pm3/2}\left[a^s_{\boldsymbol p}u^s_\mu(p)\mathrm{e}^{-\mathrm{i}px} + a^{s\dagger}_{\boldsymbol p}v^s_\mu(p)\mathrm{e}^{\mathrm{i}px}\right].$

Of course, u and v are as usual related as $v^s_\mu(p) = \left[u^s_\mu(p)\right]^{\mathrm{C}} = C\overline{u}^{\mathrm{T}}$.

Spin sum: $\Pi_{\mu\nu}(p) := \sum_s\psi^s_\mu(p)\overline{\psi}^s_\nu(p)$

$$= -(p\!\!\!/ + m_{3/2})\left(\eta^{\mu\nu} - \frac{p_\mu p_\nu}{m_{3/2}^2}\right) - \frac13\left(\gamma^\mu + \frac{p^\mu}{m_{3/2}}\right)(p\!\!\!/ - m_{3/2})\left(\gamma^\nu + \frac{p^\nu}{m_{3/2}}\right)$$

Obviously, $\gamma^\mu\Pi_{\mu\nu}(p) = p^\mu\Pi_{\mu\nu}(p) = (p\!\!\!/ - m_{3/2})\Pi_{\mu\nu}(p) = 0.$

Formulae: $k^\mu k^\nu\Pi_{\mu\nu}(p) = \frac23\left[\frac{(p\cdot k)^2}{m_{3/2}^2} - k^2\right](p\!\!\!/ + m_{3/2}), \quad \eta^{\mu\nu}\Pi_{\mu\nu}(p) = -2\left(p\!\!\!/ + m_{3/2}\right).$

4 Supergravity

4.1 Minimal SUGRA Lagrangian

Minimal SUGRA Lagrangian is constructed from supergravity multiplet $(e_a{}^\mu, \psi_\mu^\alpha, B_\mu, F_\phi)$.

$$\mathcal{L} = -\frac{M^2}{2} eR + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu \mathrm{D}_\rho\psi_\sigma \tag{4.1}$$

where

$$\mathrm{D}_\mu\psi_\nu := \partial_\mu\psi_\nu + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}\psi_\nu \qquad \left[\omega_\mu{}^{ab} : \text{“spin 接続”}\right] \tag{4.2}$$

$$e := \det e_a{}^\mu \tag{4.3}$$

$$M := 1/\sqrt{8\pi G} \quad \text{(Reduced Planck mass)} \tag{4.4}$$

$$R := e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} \tag{4.5}$$

$$R_{\mu\nu}{}^{ab} := \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b. \tag{4.6}$$

4.2 General SUGRA Lagrangian

The components of general SUGRA Lagrangian is

$$\Phi_i = (\phi_i, \chi_i^\alpha, F_i), \qquad V^{(a)} = (A_\mu^{(a)}, \lambda^{\alpha(a)}, D^{(a)}), \qquad G = (e_\mu{}^a, \psi_\mu^\alpha, B_\mu, F_\phi), \tag{4.7}$$

and described with following functions:

- Kähler potential $K(\Phi, \Phi^*)$
  - Real function of chiral multiplets.
  - In global SUSY, $\int \mathrm{d}^4\theta K$ yields kinetic terms of the chiral multiplet.
  - “Minimal Kähler” is (if no gauge interaction) $K = \Phi\Phi^\dagger$, which is

$$\int \mathrm{d}^4\theta \ \Phi\Phi^* = \partial_\mu\phi^*\partial_\mu\phi + \mathrm{i}\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^*F. \tag{4.8}$$

- Super Potential $W(\Phi)$
- Gauge kinetic term $f_{(a)(b)}(\Phi)$
  - Some function which satisfies $f_{(a)(b)} = f_{(b)(a)}$.
  - $(a), (b), \ldots$ are indices for adjoint representation of gauge group.
  - Minimal one is $f_{(a)(b)} \propto \delta_{(a)(b)}$.

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}eR + eg_{ij^*}\mathrm{D}_\mu\phi^i\mathrm{D}^\mu\phi^{*j} - \frac{1}{2}eg^2D_{(a)}D^{(a)} \\
& + \mathrm{i}eg_{ij^*}\bar{\chi}^j\bar{\sigma}^\mu\mathrm{D}_\mu\chi^i + e\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\mathrm{D}_\rho\psi_\sigma \\
& - \frac{1}{4}ef^{\mathrm{R}}{}_{(ab)}F^{(a)}_{\mu\nu}F^{\mu\nu(b)} + \frac{1}{8}e\epsilon^{\mu\nu\rho\sigma}f^{\mathrm{I}}{}_{(ab)}f^{(a)}_{\mu\nu}f^{(b)}_{\rho\sigma} \\
& + \frac{\mathrm{i}}{2}e\left[\lambda_{(a)}\sigma^\mu\mathrm{D}_\mu\bar{\lambda}^{(a)} + \bar{\lambda}_{(a)}\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^{(a)}\right] - \frac{1}{2}f^{\mathrm{I}}{}_{(ab)}\mathrm{D}_\mu\left[e\lambda^{(a)}\sigma^\mu\bar{\lambda}^{(b)}\right] \\
& + \sqrt{2}egg_{ij^*}X^{*j}_{(a)}\chi^i\lambda^{(a)} + \sqrt{2}egg_{ij^*}X^i_{(a)}\bar{\chi}^j\bar{\lambda}^{(a)} \\
& - \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_i f_{(ab)}D^{(a)}\chi^i\lambda^{(b)} + \frac{\mathrm{i}}{4}\sqrt{2}eg\partial_{i*}f^*_{(ab)}D^{(a)}\bar{\chi}^i\bar{\lambda}^{(b)} \\
& - \frac{1}{4}\sqrt{2}e\partial_i f_{(ab)}\chi^i\sigma^{\mu\nu}\lambda^{(a)}F^{(b)}_{\mu\nu} - \frac{1}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}F^{(b)}_{\mu\nu} \\
& + \frac{1}{2}egD_{(a)}\psi_\mu\sigma^\mu\bar{\lambda}^{(a)} - \frac{1}{2}egD_{(a)}\bar{\psi}_\mu\bar{\sigma}^\mu\lambda^{(a)} \\
& - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^{*j}\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \frac{1}{2}\sqrt{2}eg_{ij^*}\mathrm{D}_\nu\phi^i\bar{\chi}^j\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu \\
& - \frac{\mathrm{i}}{4}e\left[\psi_\mu\sigma^{\nu\rho}\sigma^\mu\bar{\lambda}_{(a)} + \bar{\psi}_\mu\bar{\sigma}^{\nu\rho}\bar{\sigma}^\mu\lambda_{(a)}\right]\left[F^{(a)}_{\nu\rho} + \hat{F}^{(a)}_{\nu\rho}\right] \\
& + \frac{1}{4}eg_{ij^*}\left[\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\sigma_\nu\bar{\psi}_\rho + \psi_\mu\sigma^\sigma\bar{\psi}^\mu\right]\chi^i\sigma_\sigma\bar{\chi}^i \\
& - \frac{1}{8}e\left[g_{ij^*}g_{kl^*} - 2R_{ij^*kl^*}\right]\chi^i\chi^k\bar{\chi}^j\bar{\chi}^l \\
& + \frac{1}{16}e\left[2g_{ij^*}f^{\mathrm{R}}{}_{(ab)} + f^{\mathrm{R}(cd)^{-1}}\partial_i f_{(bc)}\partial_{j^*}f^*_{(ad)}\right]\bar{\chi}^j\bar{\sigma}^\mu\chi^i\bar{\lambda}^{(a)}\bar{\sigma}_\mu\lambda^{(b)} \\
& + \frac{1}{8}e\nabla_i\partial_j f_{(ab)}\chi^i\chi^j\lambda^{(a)}\lambda^{(b)} + \frac{1}{8}e\nabla_{i^*}\partial_{j^*}f^*_{(ab)}\bar{\chi}^i\bar{\chi}^j\bar{\lambda}^{(a)}\bar{\lambda}^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}(cd)^{-1}}\partial_i f_{(ac)}\partial_j f_{(bd)}\chi^i\lambda^{(a)}\chi^j\lambda^{(b)} \\
& + \frac{1}{16}ef^{\mathrm{R}(cd)^{-1}}\partial_{i^*}f^*_{(ac)}\partial_{j^*}f^*_{(bd)}\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\chi}^j\bar{\lambda}^{(b)} \\
& - \frac{1}{16}eg^{ij^*}\partial_i f_{(ab)}\partial_{j^*}f^*_{(cd)}\lambda^{(a)}\lambda^{(b)}\bar{\lambda}^c\bar{\lambda}^{(d)} \\
& + \frac{3}{16}e\lambda_{(a)}\sigma^\mu\bar{\lambda}^{(a)}\lambda_{(b)}\sigma_\mu\bar{\lambda}^{(b)} \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_i f_{(ab)}\left[\chi^i\sigma^{\mu\nu}\lambda^{(a)}\psi_\mu\sigma_\nu\bar{\lambda}^{(b)} - \frac{1}{4}\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i\lambda^{(a)}\lambda^{(b)}\right] \\
& + \frac{\mathrm{i}}{4}\sqrt{2}e\partial_{i^*}f^*_{(ab)}\left[\bar{\chi}^i\bar{\sigma}^{\mu\nu}\bar{\lambda}^{(a)}\bar{\psi}_\mu\bar{\sigma}_\nu\lambda^{(b)} - \frac{1}{4}\psi_\mu\sigma^\mu\bar{\chi}^i\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^{K/2}\left[W^*\psi_\mu\sigma^{\mu\nu}\psi_\nu + W\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\right] \\
& + \frac{\mathrm{i}}{2}\sqrt{2}e\mathrm{e}^{K/2}\left[D_iW\chi^i\sigma^\mu\bar{\psi}_\mu + D_{i^*}W^*\bar{\chi}^i\bar{\sigma}^\mu\psi_\mu\right] \\
& - \frac{1}{2}e\mathrm{e}^{K/2}\left[\mathrm{D}_iD_jW\chi^i\chi^j + \mathrm{D}_{i^*}D_{j^*}W^*\bar{\chi}^i\bar{\chi}^j\right] \\
& + \frac{1}{4}e\mathrm{e}^{K/2}g^{ij^*}\left[D_{j^*}W^*\partial_i f_{(ab)}\lambda^{(a)}\lambda^{(b)} + D_iW\partial_{j^*}f^*_{(ab)}\bar{\lambda}^{(a)}\bar{\lambda}^{(b)}\right] \\
& - e\mathrm{e}^K\left[g^{ij^*}(D_iW)(D_{j^*}W^*) - 3W^*W\right]
\end{aligned}
\tag{4.9}
$$

付録 A　Mathematics

A.1　Algebra

A.1.1　Algebraic Structure

Semigroup	:	For $a, b \in A_{\mathrm{set}}$, $ab \in A$; Associative.
Monoid	:	For $a, b \in A_{\mathrm{set}}$, $ab \in A$; Associative, Unit.
Group	:	For $a, b \in A_{\mathrm{set}}$, $ab \in A$; Associative, Unit, Inverse.
Module (加群/可換群)	:	For $a, b \in A_{\mathrm{set}}$, $a + b \in A$; Commutative, Associative, Unit, Inverse.
Semimodule	:	For $a, b \in A_{\mathrm{set}}$, $a + b \in A$; Commutative, Associative, Unit.
Ring (環)	:	$+$: Module, $\times$: Semigroup(Monoid), Distributive.
Semiring	:	$+$: Semimodule, $\times$: Monoid, $0 \neq 1$, Distributive, $0 \times a = a \times 0 = 0$.
Field	:	$+$: Module, $\times$: Commutative Monoid, a^{-1} but 0, $1 \neq 0$, Distributive.

■Vector Space

Vector space on K	:	For $v \in (V, +)_{\mathrm{module}}$ and $k \in K_{\mathrm{field}}$,
(K-module)	:	$kv \in (V, +)$; Compatible, Distributive, $1v = v$.
Norm	:	$\lVert x \rVert \geq 0$, $\lVert x \rVert = 0 \Leftrightarrow x = 0$, $\lVert kx \rVert = k \lVert x \rVert$, $\lVert x + y \rVert \leq \lVert x \rVert + \lVert y \rVert$
Inner product	:	$\langle x \rangle x \geq 0$, $\langle x \rangle x = 0 \Leftrightarrow x = 0$,$\langle x \rangle y = \langle y \rangle x$,
	:	$\langle x + y \rangle z = \langle x \rangle z + \langle y \rangle z$, $\langle kx \rangle y = k \langle x \rangle y$

■K-algebra　K-algebra $C(V)$ とは，vector 空間 V に，distributive な乗法を入れたもの：

$$xy \in C(V); \quad (xy)z = x(yz),\ (x+y)z = xz + yz,\ x(y+z) = xy + xz,\ k(xy) = (kx)y = x(ky).$$

A.1.2　Lie Algebra

Lie Algebra :　For a Finite-dimensional K-module $(A, +)$ and $x, y, z \in (A, +)$, $a, b \in K$,

:　$[u, v] \in (A, +)$ (Lie product), and

:　Bilinear: $[ax + by, z] = a[x, z] + b[y, z]$, $[x, ay + bz] = a[x, y] + b[x, z]$,

:　Alternating: $[x, x] = 0 \quad (\Longrightarrow [x, y] = -[y, x])$,

:　Jacobi id.: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

則ち $[A, B] := AB - BA$ として閉じていれば Lie algebra となる。

A.1.3　Clifford Algebra

Here V is a vector space on K with inner product which need not be positive definite.
For $C(V)$, **Clifford algebra** $(C(V), \theta)$ is defined as

- $C(V)$: K-algebra with 1 $\quad (1x = x1 = x)$,
- $\theta : V \to C(V)$, homomorphism, $\theta(x)^2 = \langle x \rangle x1$,
- Any C': K-algebra with 1 and any homomorphism $\phi : V \to C'$ with $\phi(x)^2 = \langle x \rangle x1$, there's unique $\bar{\phi} : C \to C'$, homomorphism, $\bar{\phi}(1) = 1$.

> The Gamma matrices form Clifford Algebra:
>
> $$V : \mathbb{R}^4 \text{ with } \{G_0, G_1, G_2, G_3\}, \quad \langle G_0 \rangle G_0 = 1, \langle G_i \rangle G_i = -1; \qquad C(V) : M(n, \mathbb{C}).$$

A.1.4 Multilinear Algebra

可換環 K 上の vector 空間 V とその双対空間 V^* について：

Tensor Algebra $T(V)$: 線形写像 $f : V \to T(V)$ を持つ K-algebra であり，別の K-algebra A への線型写像 $g : V \to A$ が与えられたときに可換な準同型 $h : T(V) \to A$ s.t. $h \circ f = g$ が一意に存在するもの。

Symmetric Algebra $S(V)$: 上の定義で，$C(V)$ （および A）を可換 K-algebra としたもの。

Exterior Algebra $\wedge(V)$: 上の定義で，$g(\cdots vv \cdots) = 0$ を要請したもの。

■Tensor and Tensor Space

p 次反変 Tensor 積： $T^p(V) := V^{\otimes p} = V \otimes V \otimes \cdots \otimes V$

q 次共変 Tensor 積： $T_q(V) := (V^*)^{\otimes q} = V^* \otimes V^* \otimes \cdots \otimes V^*$

混合 Tensor 積 ： $T^p_q(V) := T^p(V) \otimes T_q(V)$ （ただし V と V^* の順序を変えたものは同型）

Tensor Space ： $T(V) := \oplus T^p_q(V)$

Tensor space の代数 $(+, \otimes)$ は環を為しており，また更に：

Contraction ： V^* は $V \to K$ なので，$T^p_q(V) \to T^{p-1}_{q-1}(V)$ が定義される。

内積 ： $T_2(V) \ni g_{ij} : V \times V \to \mathbb{R}$ （通常は対称にする。正定値としてもよい。）

添字の上げ下げ： $T_2(V) \cong \mathrm{Hom}(V, V^*)$（同型）なので，内積から誘導される。

： 同型なので，$g^{ij} : \mathrm{Hom}(V^*, V)$ は逆写像になる。

■Grassmann Operator

台集合 V を Hilbert 空間であるとする。$V \ni v$ について

$$\|\psi v\|^2 = (\psi_{ab}v_b)^*(\psi_{ac}v_c) = v_b^* v_c (\psi_{ab})^* \psi_{ac} \geq 0 \qquad \therefore (\psi_{ab})^*\psi_{ac} = -\left(\psi_{ab}(\psi_{ac})^*\right)^* \tag{A.1}$$

即ち反可換な作用素について $(ab)^\dagger = a^\dagger b^\dagger$, $(ab)^{\mathrm{T}} = -a^{\mathrm{T}} b^{\mathrm{T}}$, $(ab)^* = b^* a^*$ である。

TODO: ψ の正体がわからない……。。。

A.1.5 Lie Group and Lie Algebra

- 群 G が **Lie group** である … G が同時に C^∞ 多様体であり，積演算と逆元写像が共に C^∞ 級である。
- Lie 群 G が **COMPLEX Lie group** である … 積演算と逆元写像が共に正則写像である。

- Lie 群 G の単位元における接空間を，G **の Lie algebra** $\mathfrak{g}$ という。
  - $\mathfrak{g}$ は G の左不変な vector 場全体である。
  - $\mathfrak{g}$ は vector 場の括弧積の下で Lie algebra となる。
- G として有限次元 Lie 群を考えると，
  - その Lie 代数の基底 B_i に対して **structure constant** c が $[B_i, B_j] = c^k_{ij} B_k$ として定義できる。

* * *

- Compact Lie 群は線型 Lie 群である。
- G として **Linear group** $\mathrm{GL}(n; \mathbb{R})$ を考えると，
  - その Lie 代数は n 次実正方行列全体となる。
  - Vector 場の括弧積は **commutation relation** $[X, Y] = XY - YX$ となる。
- Lie 群は，$\mathrm{GL}(n; \mathbb{C})$ の部分 Lie 群と局所同型になるような位相群でかつ連結成分が高々可算個であるものである。

以下では，Lie 群として $\mathrm{GL}(n;\mathbb{R})$ の部分群を考えることにし，Lie 代数の元を行列により表現する。

A.1.6 Matrix Representation

- Lie 群 G の Lie 代数の基底の組を，G の **generators** と言う。
- $\mathrm{GL}(n;\mathbb{R})$ の元は n 次元行列で表せる。
- Lie 群 G の生成子 $\{T_i\}$ に対し，以下の 2 つは共に G の単位元近傍の局所座標系を与える。

$$(x_1,\cdots,x_m)\mapsto \mathrm{e}^{x_1T_1+\cdots+x_mT_m} \qquad\qquad (x_1,\cdots,x_m)\mapsto \mathrm{e}^{x_1T_1}\cdots\mathrm{e}^{x_mT_m} \tag{A.2}$$

- Lie 群 G が compact である $\cdots$
  1. 多様体 G が compact である。 **TODO: これは何故同値なのか？**
  2. G の生成子 $\{T_i\}$ を，$\mathrm{Tr}(T_iT_j)=k\,\delta_{ij}$ かつ $k>0$ となるように取り替えることができる。
     【この基底の下では構造定数が完全反対称になる。】
- Compact 群 G は，**unitary representation** を持つ。
  故に，単位元の近傍では有限個の **Hermitian matrix** T^i と parameters $x^i\in\mathbb{R}$ により，G の元を

$$\mathrm{e}^{\mathrm{i}x^iT^i} \tag{A.3}$$

  と表すことが出来る。

A.1.7 結論

Compact Lie 群の元のうち，単位元近傍にあるもの V は，

Hermitian Representation

$$V=\exp(\mathrm{i}x^iT^i) \qquad \text{where} \quad T^i:\text{Hermitian Matrix}, \quad x^i\in\mathbb{R},$$
$$[T^i,T^j]=\mathrm{i}f^{ijk}T^k, \quad \mathrm{Tr}(T^iT^j)=\lambda\,\delta^{ij}>0; \qquad f\in\mathbb{R}$$

Real Representation

$$V=\exp(x^iR^i) \qquad \text{where} \quad R^i:\text{Real Matrix}, \quad x^i\in\mathbb{R},$$
$$[R^i,R^j]=-f^{ijk}R^k, \quad \mathrm{Tr}(R^iR^j)=-\lambda\,\delta^{ij}<0; \qquad f\in\mathbb{R}$$

と表すことが出来る。

A.1.8 Lie Algebra and States

TODO: かきかけ Assume $|\psi\rangle\in\mathcal{H}, X_a\in\mathfrak{g}\Longrightarrow X_a|\psi\rangle\in\mathcal{H}$ for a Hilbert space $\mathcal{H}$ and Lie group G ($\mathfrak{g}$: its Lie algebra). If there is a subspace $\mathcal{H}\ni\mathcal{H}_1=\{|\psi_1\rangle,\ldots,|\psi_n\rangle\}$ which satisfies

付録 B　Statistics

Histogram の階級数についての Sturges の公式 $k \approx 1 + \log_2 n$ （n:観測値の数）

■分布の代表値

$$\text{算術平均 } \overline{x} := \frac{1}{n}\sum x_i,\quad \text{幾何平均 } x_G := \left(\prod x_i\right)^{1/n},\quad \text{調和平均 } x_H := n\left(\sum \frac{1}{x_i}\right)^{-1}; \tag{B.1}$$

$$\text{中央値，} n \text{ 分位点，最頻値，mid-range，}\cdots\cdots \tag{B.2}$$

■分布の散らばり

$$\text{平均偏差 } d := \frac{1}{n}\sum |x_i - \overline{x}|,\quad \text{標準偏差（分散）} S^2 := \frac{1}{n}\sum (x_i - \overline{x})^2,\quad \text{変動係数 } C.V. := S_x/\overline{x}; \tag{B.3}$$

$$\text{平均差 } M.D. := \frac{1}{n^2}\sum_i\sum_j |x_i - x_j|,\quad \text{Gini 係数 } G.I. := \frac{M.D.}{2\overline{x}} = \frac{1}{2n^2\overline{x}}\sum_i\sum_j |x_i - x_j|; \tag{B.4}$$

$$\text{Entropy } H = -\sum p_i \log p_i \quad (p\text{:相対頻度}) \quad \cdots \text{ 一ヶ所集中} = 0 \le H \le 1 = \text{等確率} \tag{B.5}$$

$$\text{範囲，四分位偏差，}\cdots\cdots \tag{B.6}$$

■相関を表す量

$$\text{共分散 } C_{xy} := \frac{1}{n}\sum (x_i - \overline{x})(y_i - \overline{y}) \qquad \text{相関係数 } r_{xy} := \frac{C_{xy}}{S_x S_y} \quad -1 \le r_{xy} \le 1, \text{線型不変} \tag{B.7}$$

$$\text{偏相関関数 } r_{12;3} := \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - {r_{13}}^2}\sqrt{1 - {r_{23}}^2}} \qquad \text{系列相関関数 } r_h := \frac{1}{S_x}\sum_{i=1}^{n-h}\frac{(x_i - \overline{x})(x_i + h - \overline{x})}{n-h} \tag{B.8}$$

順位相関関数 $\cdots$ 順位の組 $\{R_i\}$，$\{R'_i\}$ の間の相関

$$\text{Spearman : } r_{\mathrm{S}} := 1 - \frac{6}{n(n^2-1)}\sum (R_i - R'_i)^2 \quad \text{（通常の相関関数）} \tag{B.9}$$

$$\text{Kendall : } r_{\mathrm{K}} := \frac{\sum G_{ij}}{n(n-1)/2} \qquad \text{where} \quad G_{ij} := (i,j) \text{ に対して同順なら } +1\text{，逆順なら } -1 \tag{B.10}$$

■条件付き確率

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{B.11}$$

Bayes の定理: 事象 $\{H_i\}$ が互いに排反かつ全体を尽くしているとき，

$$P(A) = \sum P(A \cap H_i) \quad \text{によって} \quad P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\sum_k P(H_k)P(A|H_k)}. \tag{B.12}$$

相関係数の分布 （ρ:母集団の（真の）相関係数）

$$f(r) = \frac{\left(1-\rho^2\right)^{(n-1)/2}\left(1-r^2\right)^{(n-4)/2}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\sum_{i=0}^{\infty}\frac{(2\rho r)^i}{i!}\left[\Gamma\left(\frac{n-1+i}{2}\right)\right]^2 \tag{B.13}$$

■確率分布に対する Moment

$$\text{Moment } \mu_r := \langle X^r \rangle\,; \quad \mu'_r := \langle (X-\mu)^r \rangle \qquad \text{標準化 moment } \alpha_r := \langle (X-\mu)^r \rangle / \sigma^r \tag{B.14}$$

$$\text{Moment 母関数 } M_X(t) := \langle \exp(tX) \rangle \Longrightarrow \mu_r = \frac{\mathrm{d}^n}{\mathrm{d}t^n} M_X(t) \tag{B.15}$$

期待値 $\mu := \mu_1$;

分散 $\sigma^2 := \mu'_2$;　標準偏差 $\sigma := \sqrt{\sigma^2}$;

歪度 $C_{\text{skew}} := \alpha_3$;　尖度 $C_{\text{kurt}} := \alpha_4 - 3$

計算の便法:

$$\mu'_2 = \left\langle X^2 \right\rangle - \mu^2 \tag{B.16}$$

$$\mu'_3 = \left\langle X^3 \right\rangle - 3\mu\mu_2 + 2\mu^3 \tag{B.17}$$

$$\mu'_4 = \left\langle X^4 \right\rangle - 4\mu\mu_3 + 6\mu^2\mu_2 - 3\mu^4 \tag{B.18}$$

■Chebyshev の不等式　いかなる確率変数に対しても，$P\Big(|X-\mu| \geq k\sigma\Big) \leq \dfrac{1}{k^2}$.

■Stirling の公式

$$\log n! = \left(n + \frac{1}{2}\right)\log n - n + \frac{\log 2\pi}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + \mathrm{O}(n^{-11}) \tag{B.19}$$

B.1　離散型確率分布

■超幾何分布　(A,B) が $(M, N-M)$ 個あるとき，n 個取り出して $(k, n-k)$ 個である確率。非復元捕獲。

$$P_k = \frac{{}_M\mathrm{C}_k\ {}_{N-M}\mathrm{C}_{n-k}}{{}_N\mathrm{C}_n} \qquad E = np, \quad V = np(1-p)\frac{N-n}{N-1} \qquad (p := M/N) \tag{B.20}$$

■二項分布　確率 p で起きる事象が，n 回のうち k 回起こる確率。復元捕獲，Bernoulli 試行。

$$P_k = {}_n\mathrm{C}_k \cdot p^k (1-p)^{n-k} \qquad E = np, \quad V = np(1-p) \qquad (n = 1 : \text{Bernoulli 分布}) \tag{B.21}$$

■Poisson 分布　二項分布において $np = \lambda$ 一定で $n \to \infty$，$p \to 0$ として，$P_k = \dfrac{\mathrm{e}^{-\lambda}\lambda^k}{k!}$，　$E = V = \lambda$.

■幾何分布　確率 p の事象が起こるまでの**失敗**回数 k の分布。

$$P_k = p(1-p)^k \qquad E = \frac{1-p}{p}, \quad V = \frac{1-p}{p^2}. \tag{B.22}$$

■負の二項分布 (Pascal **分布**)　確率 p の事象が n 回起こるまでの**失敗**回数 k の分布。(試行は $n+k$ 回)

$$P_k = {}_{n+k-1}\mathrm{C}_k\ p^n (1-p)^k \qquad E = \frac{k(1-p)}{p}, \quad V = \frac{k(1-p)}{p^2}. \tag{B.23}$$

■一様分布

$$P_k = \frac{1}{N}, \qquad E = \frac{N+1}{2}, \quad V = \frac{N^2-1}{12}. \tag{B.24}$$

B.2 連続型確率分布

■正規分布

$$\mathrm{N}[\mu,\sigma] = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]; \qquad E=\mu, \quad V=\sigma^2. \tag{B.25}$$

■指数分布

$$\mathrm{Ex}[\lambda] = \mathrm{Ga}[\lambda,1] = \lambda\mathrm{e}^{-\lambda x} \quad (x \geq 0); \qquad E=\frac{1}{\lambda}, \quad V=\frac{1}{\lambda^2}. \tag{B.26}$$

■Gamma 分布

$$\mathrm{Ga}[\lambda,\alpha] = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\lambda x} \quad (x \geq 0); \qquad E=\frac{\alpha}{\lambda}, \quad V=\frac{\alpha}{\lambda^2}. \tag{B.27}$$

■χ^2 分布

$$\chi^2[n] = \mathrm{Ga}[1/2,n/2] = \frac{1}{\Gamma(n/2)}\sqrt{\frac{x^{n-2}\mathrm{e}^{-x}}{2^n}} \quad (x \geq 0); \qquad E=n, \quad V=2n. \tag{B.28}$$

■Beta 分布

$$\mathrm{Be}[\alpha,\beta] = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad (0<x<1), \qquad E=\frac{\alpha}{\alpha+\beta}, \quad V=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \tag{B.29}$$

$$\text{where} \quad B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{B.30}$$

■Cauchy 分布

$$f[\alpha,\lambda] = \frac{\alpha}{\pi}\left[(x-\lambda)^2+\alpha^2\right]; \qquad E \text{ と } V \text{ は定義されない。} \tag{B.31}$$

■対数正規分布　所得の分布。$\log x$ が正規分布 $N[\mu,\sigma]$ に従うとき,

$$f[\mu,\sigma] = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}\exp\frac{-(\log x-\mu)^2}{2\sigma^2}; \qquad E=\mathrm{e}^{\mu+\sigma^2/2}, \quad V=\mathrm{e}^{2\mu+2\sigma^2}-\mathrm{e}^{2\mu+\sigma^2}. \tag{B.32}$$

■Pareto 分布　高額所得の分布。E と V はそれぞれ $a>1$, $a>2$ でのみ定義される。

$$f_{[a,x_0]} = \frac{a}{x_0}\left(\frac{x_0}{x}\right)^{a+1} \quad (x \geq x_0; \quad a>0); \qquad E=\frac{ax_0}{a-1}, \qquad V=\frac{ax_0^2}{a-2}-\left(\frac{ax_0}{a-1}\right)^2. \tag{B.33}$$

■Weibull 分布　$a>0$, $b>0$ とする。

$$f[a,b] = \frac{b}{a^b}x^{b-1}\exp\left[-\left(\frac{x}{a}\right)^b\right] \quad (x \geq 0); \qquad E=a\Gamma\left(1+\frac{1}{b}\right), \quad V=a^2\left[\Gamma\Big(1+\frac{2}{b}\Big)-\left[\Gamma\Big(1+\frac{1}{b}\Big)\right]^2\right] \tag{B.34}$$

付録 C　Verbose Notes

C.1　Spinor Fields

C.1.1　Lorentz group and Lorentz algebra

Metric	:	$\eta = \mathrm{diag}(+1,-1,-1,-1), \qquad \eta = \mathrm{diag}(-1,+1,+1,+1).$
Lorentz transf. in $\mathbb{R}^{1,3}$	:	Linear transf. $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ which conserve x^2.
	:	$\Longrightarrow \eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma$. and form a group L.
	:	$(\Longrightarrow \quad (\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\alpha}\eta^{\mu\beta}\Lambda^\alpha{}_\beta =: \Lambda_\nu{}^\mu \quad \Longrightarrow \quad \Lambda^\mu{}_\nu\Lambda_\mu{}^\rho = \delta^\rho_\nu)$
Disconnected parts of L	:	$L_0 := \left\{\det\Lambda = +1 \wedge \Lambda_0{}^0 > 0\right\} \quad L_P := \left\{\det\Lambda = -1 \wedge \Lambda_0{}^0 > 0\right\}$
	:	$L_T := \left\{\det\Lambda = +1 \wedge \Lambda_0{}^0 < 0\right\} \quad L_{PT} := \left\{\det\Lambda = -1 \wedge \Lambda_0{}^0 < 0\right\}$
	:	(L_0 is identical with ($\mathrm{SO}(1,3), \mathrm{SO}(3,1)$).)
Infinitesimal one in L_0	:	$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ (for $\eta = \eta\Lambda\Lambda$)

微小変換は $\epsilon^\mu{}_\nu = \begin{pmatrix} 0 & \beta_x & \beta_y & \beta_z \\ \beta_x & 0 & -\theta_z & \theta_y \\ \beta_y & \theta_z & 0 & -\theta_x \\ \beta_y & -\theta_y & \theta_x & 0 \end{pmatrix}$ の形となっているので，回転生成子 $\boldsymbol{J}$ と加速生成子 $\boldsymbol{K}$ は

$$\Lambda = \exp\left[\kappa\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] \quad\Longrightarrow\quad J_x = -\kappa\mathrm{i}\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}, \quad K_x = -\kappa\mathrm{i}\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \tag{C.1}$$

の形である。ここで $\kappa = \pm 1$ は notation である。

一方，微小変換から生成子を $\epsilon^\mu{}_\nu =: \mp\frac{\mathrm{i}}{2}\epsilon^{\rho\sigma}(J_{\rho\sigma})^\mu{}_\nu$ と定義すると，計量によらずに $\epsilon_{\mu\nu}$ は反対称となり，

$$\begin{aligned} \boldsymbol{\theta} &= (+,-)(\epsilon^{23},\epsilon^{31},\epsilon^{12}), \\ &= (+,-)(\epsilon_{23},\epsilon_{31},\epsilon_{12}) \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{\beta} &= (+,-)(\epsilon^{10},\epsilon^{20},\epsilon^{30}). \\ &= (-,+)(\epsilon_{10},\epsilon_{20},\epsilon_{30}) \end{aligned}$$

で，従って $J^{\rho\sigma}$ も反対称。$(J_{\rho\sigma})^\mu{}_\nu = \pm\mathrm{i}\left(\delta^\mu_\rho\eta_{\sigma\nu} - \delta^\mu_\sigma\eta_{\rho\nu}\right)$ となり，交換関係が得られ，これが閉じているので Lie 代数であることもわかる：

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mp\mathrm{i}(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}). \tag{C.2}$$

生成子の具体形は計量に依存し，$J_{10}{}^\mu{}_\nu = (\pm\mathrm{i},\mp\mathrm{i})\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}_{\mu\nu}$，$J_{23}{}^\mu{}_\nu = (\pm\mathrm{i},\mp\mathrm{i})\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}_{\mu\nu}$ となる。よって，ここでの複号の取り方と κ および計量の定義によって，$\boldsymbol{J}\cdot\boldsymbol{K}$ と $J_{\rho\sigma}$ の対応が定まることになる。

* * *

κ と複号について $(-,\text{上})$,$(+,\text{下})$,$(-,\text{下})$,$(+,\text{上})$ を取れば

$$\boldsymbol{J} = (J_{23}, J_{31}, J_{12}), \quad \boldsymbol{K} = (J_{10}, J_{20}, J_{30}); \tag{C.3}$$

となる：

$$\Lambda = \exp\epsilon = \exp\left[\kappa\mathrm{i}(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\beta}\cdot\boldsymbol{K})\right] = \exp\left[\mp\mathrm{i}(\epsilon^{\rho\sigma}J_{\rho\sigma})/2\right]. \tag{C.4}$$

C.1.2 Lorentz group and $\mathrm{SL}(2,\mathbb{C})$

次に，連結 Lie 群 L_0 が，連結 Lie 群 $\mathrm{SL}(2,\mathbb{C})/\mathrm{Z}_2$ と同型であることを見る：

$$\mathfrak{sl}(2,\mathbb{C}) :=\{a\in\mathfrak{gl}(2,\mathbb{C})\mid \mathrm{Tr}(a)=0\},\qquad \mathrm{SL}(2,\mathbb{C}) :=\{g\in\mathrm{GL}(2,\mathbb{C})|\ \det(g)=1\}. \tag{C.5}$$

まず,σ^μ を（極めて一般的に） $\sigma^\mu := (\alpha 1, \beta\boldsymbol{\sigma})$ と定義する $(\alpha=\beta=\pm1)$。$x^2=(+-)\det(x_\mu\sigma^\mu)$ なので

$$f^g : (x_\mu\sigma^\mu)\mapsto g(x_\mu\sigma^\mu)g^\dagger;\quad g\in\mathrm{SL}(2,\mathbb{C}) \tag{C.6}$$

は x^2 を保存する。よって Lorentz 変換であり，生成子を比べることで局所同型だとわかる：

> $\mathrm{SL}(2,\mathbb{C})\ni g=\exp(-\mathrm{i}a)$ として $x_\mu(g\sigma^\mu g^\dagger)=\Lambda_\mu{}^\nu x_\nu\sigma^\mu$ を微小展開すると
>
> $$\Lambda^\mu{}_\nu\sigma^\nu = g^{-1}\sigma^\mu(g^{-1})^\dagger \quad\Longrightarrow\quad \epsilon^\mu{}_\nu\sigma^\nu=\mathrm{i}(a\sigma^\mu-\sigma^\mu a^\dagger) \tag{C.7}$$
>
> であり，ここから g がわかる：
>
> $$g=\exp\left(-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}-\frac{\alpha\beta}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right). \tag{C.8}$$

このことを別の観点から見る。Lorentz 群の生成子の交換関係を見ると，（正しく複号を取った場合）

$$[J_i,J_j]=\mathrm{i}\epsilon_{ijk}J_k,\qquad [J_i,K_j]=\mathrm{i}\epsilon_{ijk}K_k,\qquad [K_i,K_j]=-\mathrm{i}\epsilon_{ijk}J_k \tag{C.9}$$

となるので，

$$\boldsymbol{A}:=\frac{1}{2}(\boldsymbol{J}+\mathrm{i}\boldsymbol{K}),\qquad \boldsymbol{B}:=\frac{1}{2}(\boldsymbol{J}-\mathrm{i}\boldsymbol{K}). \tag{C.10}$$

と定義すると

$$[A_i,A_j]=\mathrm{i}\epsilon_{ijk}A_k,\qquad [B_i,B_j]=\mathrm{i}\epsilon_{ijk}B_k,\qquad [A_i,B_j]=0, \tag{C.11}$$

となり，Lorentz 群が $\mathrm{SU}(2)\times\mathrm{SU}(2)$ に分解できる。

C.2 Weyl Spinor

$\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ に対して $(1/2,0)$ 表現を為すものを左巻き spinor ξ，$(0,1/2)$ 表現を為すものを右巻き spinor $\bar\xi$ と定義する。

$$\xi\mapsto\left(1-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\xi \qquad\qquad \bar\xi\mapsto\left(1-\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\bar\xi. \tag{C.12}$$

$\alpha\beta=1$ とすると g は左巻き spinor の変換子となる。記号を $\xi_\alpha\mapsto g_\alpha{}^\beta\xi_\beta$, $\bar\xi^{\dot\alpha}\mapsto(g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}$ と定義する。

次に，$\xi^\alpha\chi_\alpha$ および $\bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha}$ が scalar となるようにしたい。$E=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ としておくと，$-Eg^{\mathrm{T}}E=g^{-1}$ より

$$(\xi')^\alpha=\xi^\beta(g^{-1})_\beta{}^\alpha=-\xi^\beta(E_{\beta A}g_B{}^AE_{B\alpha})\qquad\therefore(-E_{\gamma\alpha})(\xi')^\alpha=-g_\gamma{}^\beta(E_{\beta\delta}\xi^\delta). \tag{C.13}$$

よって，$\epsilon^{12}=\epsilon_{21}=1$ として $\xi^\alpha:=\epsilon^{\alpha\beta}\xi_\beta$，$\xi_\alpha=\epsilon_{\alpha\beta}\xi^\beta$ とすれば良い。

同様に，$\bar\xi'_{\dot\alpha}=-E(g^\dagger)^{-1}E\bar\xi_{\dot\beta}$ から $(E^{\dot\alpha\dot\beta}\bar\xi_{\dot\beta})'=(g^\dagger)^{-1\dot\alpha}{}_{\dot\beta}(E^{\dot\beta\dot\gamma}\bar\xi_{\dot\gamma})$ となる。$\epsilon_{12}=\epsilon_{\dot1\dot2}$ として $\bar\xi_{\dot\alpha}:=\epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\alpha}$ とするのが一般的である。また，このことから $(\xi_a)^*=\bar\xi_{\dot\alpha}$（或いは $\xi^\dagger=\bar\xi$）が分かる。

$x_\mu\sigma^\mu\mapsto x_\mu(g\sigma^\mu g^\dagger)$ であるので，$x_\mu(\xi^\alpha\sigma^\mu\bar\chi^{\dot\alpha})$ は scalar である。よって，$\sigma^\mu_{\alpha\dot\alpha}$ のように書ける。更にここで $x_\mu(\bar\xi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\chi_\alpha)$ も scalar となるように $\bar\sigma$ を定めよう。

$$\bar\xi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\chi_\alpha=-\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\chi^\beta\bar\sigma^{\mu\dot\alpha\alpha}\bar\xi^{\dot\beta}\qquad\therefore\sigma^\mu_{\beta\dot\beta}\propto\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\alpha\alpha} \tag{C.14}$$

であり，あとは convention である。

$$\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar{\xi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$$

$$\xi_\alpha \mapsto g_\alpha{}^\beta \xi_\beta, \qquad \xi^\alpha \mapsto \xi^\beta (g^{-1})_\beta{}^\alpha, \qquad \bar{\xi}_{\dot{\alpha}} \mapsto \bar{\xi}_{\dot{\beta}} (g^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}} \qquad \bar{\xi}^{\dot{\alpha}} \mapsto (g^\dagger)^{-1\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} \quad \sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta} \quad \therefore \sigma^\mu := (\alpha 1, \beta\boldsymbol{\sigma}), \quad \bar{\sigma}^\mu := (\alpha 1, -\beta\boldsymbol{\sigma})$$

C.3 Polarization Sum

Firstly we focus on the single photon case $M = \epsilon^*_\mu(k) M^\mu$. In this case, the replacement

$$\sum_{\text{pol.}} \epsilon_\mu \epsilon'_\nu \to \eta_{\mu\nu} \tag{C.15}$$

is valid. Let us prove this validity. First we set $k = (E, 0, 0, E)$, and $\epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0)$. Then

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^*_\mu(k)\epsilon_\nu(k) M^\mu M^{\nu *} = |M^1|^2 + |M^2|^2, \tag{C.16}$$

while

$$\eta_{\mu\nu} M^\mu M^{\nu *} = |M^1|^2 + |M^2|^2 \tag{C.17}$$

for Ward identity $k_\mu M^\mu = 0$. Now we can see the validity easily.

Next we think about the double photons case [*8] $M = \epsilon^*_\mu(k)\epsilon'^*_\nu(k') M^{\mu\nu}$. Here we set

$$k = (E, 0, 0, E) \qquad \epsilon = (0, 1, 0, 0) \oplus (0, 0, 1, 0) \tag{C.18}$$

$$k' = (E, 0, 0, -E) \qquad \epsilon' = (0, \cos\theta, \sin\theta, 0) \oplus (0, -\sin\theta, \cos\theta, 0). \tag{C.19}$$

Then doing some simple calculations, we can get

$$\sum_{\text{pol.}} |M|^2 = \sum_{\text{pol.}} \epsilon^*_\mu(k)\epsilon_\nu(k)\epsilon^*_\rho(k')\epsilon_\sigma(k') M^{\mu\rho} M^{\nu\sigma *} \tag{C.20}$$

$$= |M^{11}|^2 + |M^{12}|^2 + |M^{21}|^2 + |M^{22}|^2 \tag{C.21}$$

$$\stackrel{?}{=} \eta^{\mu\nu}\eta^{\rho\sigma} M^{\mu\rho} M^{\nu\sigma *}. \tag{C.22}$$

Badly, our Ward identities

$$k_\mu \epsilon'^*_\nu(k') M^{\mu\nu} = \epsilon^*_\mu(k) k'_\nu M^{\mu\nu} = 0 \tag{C.23}$$

do not help us; now we want to be given

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0, \tag{C.24}$$

to recover validity of the replacement.

The difference between (C.23) and (C.24) is that the former considers (and tries to sum up) all polarizations but the latter does only physical ones. Actually, as long as we are summing up all polarizations, the replacement is still valid; these two conditions are equivalent because of cancellation of unphysical polarizations. However, once we restrict the polarizations (for example with using a relation $\epsilon \cdot k = 0$), we can no more use (C.24) and thus the replacement becomes invalid.

*8 This part is derived from 濱口幸一's notebook.

Now let's check what is happening from another viewpoint. First we suppose M satisfies our latter conditions (C.24), and define $\widetilde{M}^{\mu\nu}$ and $\widetilde{M}$ as

$$\widetilde{M}^{\mu\nu} := M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu, \tag{C.25}$$
$$\widetilde{M} := \epsilon^*_\mu(k)\epsilon'^*_\nu(k')\widetilde{M}^{\mu\nu}. \tag{C.26}$$

Here $\widetilde{M}^{\mu\nu} \neq M^{\mu\nu}$ but $\widetilde{M} = M$; thus $\widetilde{M}$ satisfies Ward identities (since photon is massless and $\epsilon \cdot k = 0$). However, we cannot utilize the replacement for $\widetilde{M}$, while it is valid for M. If you did the replacement, a wrong result would come out, like

$$\begin{aligned}\eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} &= \eta_{\mu\rho}\eta_{\nu\sigma}\left(M^{\mu\nu} + k^\mu p^\nu + p'^\mu k'^\nu\right)\left(M^{\rho\sigma *} + k^\rho p^{\sigma *} + p'^{\rho *} k'^\sigma\right) &\text{(C.27)}\\ &= \sum_{\text{pol.}} |M|^2 + \left[(k\cdot p'^*)(k'\cdot p) + \text{H.c.}\right]. &\text{(C.28)}\end{aligned}$$

After all, we have obtained following expression:

$$\begin{aligned}&\sum_{\text{pol.}}|M|^2 = \sum_{\text{pol.}}|\epsilon^*_\mu(k)\epsilon'^*_\nu(k')M^{\mu\nu}|^2 = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma *}\\ &= \sum_{\text{pol.}}|\widetilde{M}|^2 = \sum_{\text{pol.}}|\epsilon^*_\mu(k)\epsilon'^*_\nu(k')\widetilde{M}^{\mu\nu}|^2 \neq \eta_{\mu\rho}\eta_{\nu\sigma}\widetilde{M}^{\mu\nu}\widetilde{M}^{\rho\sigma *} = \sum_{\text{pol.}}|\widetilde{M}|^2 + \left[(k\cdot p'^*)(k'\cdot p) + \text{H.c.}\right]. \quad \text{(C.29)}\end{aligned}$$

To check the Ward identity always helps us!

C.4 Phantom Terms in the Gauge Theory

You may think we forget to introduce $\overline{\psi}\gamma_5\psi$, $\overline{\psi}\gamma_5\not{D}\psi$, $\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma}$ terms, but being a bit careful,

- the first two terms are nonsense, for now we use P_L and P_R,
- the last term is equivalent to the third term as
$$\epsilon^{\mu\nu\rho\sigma}D_\mu D_\nu F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\frac{1}{2}[D_\mu, D_\nu]F^a_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}.$$

Therefore, we have to discuss only the ϵFF terms. If the gauge group is simple, we can take the structure constant as totally antisymmetric, which leads these terms to fall into surface terms as:

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma &= \epsilon^{\mu\nu\rho\sigma}\left(-f^{acd}f^{abe} - f^{adb}f^{ace}\right)A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\\ &= -2\epsilon^{\mu\nu\rho\sigma}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\\ &= 0,\end{aligned} \tag{C.30}$$

$$\begin{aligned}\therefore\ \epsilon^{\nu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^a_\nu \partial_\rho A^a_\sigma + 4g\epsilon^{\mu\nu\rho\sigma}f^{abc}A^a_\mu A^b_\nu \partial_\rho A^c_\sigma\\ &= 2\partial_\mu G^\mu,\end{aligned} \tag{C.31}$$

where G^μ is the Chern–Simons term which is defined as

$$G^\mu := 2\epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma + \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right) = \epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{1}{3}gf^{abc}A^a_\nu A^b_\rho A^c_\sigma\right). \tag{C.32}$$

See Appendix C.7 for the instanton effect.

C.5 楊-Mills Theory

C.5.1 General Gauge Theory

For any Lie group G, we can consider "gauge transformation" $\phi(x) \mapsto V(x)\phi(x)$, where $V : \mathbb{R}^{1,3} \to G$. Also we can define a "connection field" $A_\mu(x)$ as:

$$\phi_\parallel(x+\mathrm{d}x) := \phi(x) + \mathrm{i}gA_\mu(x)\phi(x)\mathrm{d}x^\mu \qquad \text{s.t.} \quad \phi_\parallel(x+\mathrm{d}x) \mapsto V(x+\mathrm{d}x)\phi_\parallel(x+\mathrm{d}x). \tag{C.33}$$

Then the covariant derivative D_μ can be defined as

$$\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu := \Delta_{\mathrm{d}x}\phi(x) := \phi(x+\mathrm{d}x) - \phi_\parallel(x+\mathrm{d}x) \qquad \therefore \mathrm{D}_\mu := \partial_\mu - \mathrm{i}gA_\mu. \tag{C.34}$$

Note that $\Delta_{\mathrm{d}x}\phi(x) \mapsto V(x+\mathrm{d}x)\mathrm{D}_\mu\phi(x)\mathrm{d}x^\mu$, which means $\mathrm{D}_\mu\phi(x) \mapsto V(x)\mathrm{D}_\mu\phi(x)$. Now we can see

$$\phi \mapsto V\phi, \qquad A_\mu \mapsto V\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)V^{-1}, \qquad \mathrm{D}_\mu \mapsto V\mathrm{D}_\mu V^{-1}. \tag{C.35}$$

We can do another discussion: we can define D_μ as a kind of derivative which satisfies (C.35).

Next we introduce the curvature tensor, or "field strength" as

$$\Delta\phi(x) := \phi_\parallel^{xy}(x+\mathrm{d}x+\mathrm{d}y) - \phi_\parallel^{yx}(x+\mathrm{d}x+\mathrm{d}y) = [\mathrm{D}_\mu, \mathrm{D}_\nu]\,\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu =: -\mathrm{i}gF_{\mu\nu}\phi(x)\mathrm{d}x^\mu\mathrm{d}y^\nu; \tag{C.36}$$

$$F_{\mu\nu}(x) := \frac{\mathrm{i}}{g}[\mathrm{D}_\mu, \mathrm{D}_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}g\left[A_\mu(x), A_\nu(x)\right]. \tag{C.37}$$

$\Delta\phi(x)$ is transformed in terms of $V(x+\mathrm{d}x+\mathrm{d}y) \simeq V(x)$, thus $F_{\mu\nu}(x) \mapsto V(x)F_{\mu\nu}(x)V^{-1}(x)$.

C.5.2 Compact Gauge Theory

Generators If the gauge group G is **compact**, it has a finite-dimensional unitary representation. The generators T_a can be taken to be Hermitian, and $V(x) = \exp\left[\mathrm{i}g\theta^a(x)T^a\right]$ for $\theta^a(x) \in \mathbb{R}$;

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_cT^c \quad (f \in \mathbb{R}) \qquad 0 = f^D{}_{ab}f^E{}_{Dc} + f^D{}_{ca}f^E{}_{Db} + f^D{}_{bc}f^E{}_{Da} \tag{C.38}$$

For the sake of the compactness Killing form is positive-definite, where we can normalize the generators as $\operatorname{Tr} T^aT^b = \frac{1}{2}\delta^{ab}$, and the structure constant f^{abc} would be totally antisymmetric.

Adjoint Representations

$$[\widetilde{T}^a]_i{}^j := -\mathrm{i}f^{aij}; \qquad [\widetilde{\mathrm{D}}_\mu]_i{}^j := \delta_i^j\partial_\mu + gf^{iaj}A_\mu^a. \tag{C.39}$$

Field Expansion In this *normalized Hermitian* basis, the relations would be[*9]

$$\begin{aligned} \phi' &= \mathrm{e}^{\mathrm{i}gT^a\theta^a}\phi; & F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\ A^{a}_\mu{}' &\simeq A^a_\mu + \partial_\mu\theta^a + gf^{abc}A^b_\mu\theta^c & F^{a}_{\mu\nu}{}' &= [\mathrm{e}^{\mathrm{i}g\theta^c\widetilde{T}^c}]^{ab}F^b_{\mu\nu} \\ &= A^a_\mu + (\widetilde{\mathrm{D}}_\mu\theta)^a, & &\simeq F^a_{\mu\nu} + gf^{abc}F^b_{\mu\nu}\theta^c. \end{aligned}$$

Covariant Derivative For a field λ^a under the adjoint representation,

$$(\mathrm{D}_\mu\lambda)^a = \partial_\mu\lambda^a + gf^{abc}A^b_\mu\lambda^c \qquad \text{or} \quad \mathrm{D}_\mu\lambda^aT^a = \partial_\mu\lambda^aT^a - \mathrm{i}g[A^b_\mu T^b, \lambda^aT^a] \tag{C.40}$$ [*10]

[*9] We can expand A_μ in T^a-basis, because it is induced by the gauge transformation.

[*10] Note that we can use any representation T^a but must the same ones for $A^a_\mu T^a$ and λ^aT^a.

Bianchi Equation

$$\epsilon^{\mu\nu\rho\sigma}\left[\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\sigma]\right] = \epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu F_{\rho\sigma} = 0. \tag{C.41}$$

Conjugate representation Consider a representation r and its generators $\{T_r^a\}$. Then $\{-(T_r^a)^*\}$ also satisfies the group structure. The corresponding representation is called the *conjugate representation* $\overline{r}$ of r:

$$T_{\overline{r}}^a := -(T_r^a)^*. \tag{C.42}$$

If $\overline{r}$ is equivalent to r, i.e. there exists an unitary matrix U such that $T_{\overline{r}}^a = U^\dagger(T_r^a)U$, the representation r is called a *real representation.* This U is unique up to a trivial scale factor, and either symmetric or anti-symmetric. If U is symmetric, the representation is called *real-positive*; otherwise *real-negative* or *pseudo-real.*

The **2** representation of SU(2) is pseudo-real because

$$T_{\overline{\mathbf{2}}}^a \equiv -(\sigma^a/2)^* = \sigma^2(\sigma^a/2)\sigma^2. \tag{C.43}$$

If a field X transforms under a representation r, its complex conjugate does under $\overline{r}$:

$$\delta X = \mathrm{i}g\theta^a(T_r^a X) \quad \Longrightarrow \quad \delta X^* = -\mathrm{i}g\theta^a(T_r^A X)^* = \mathrm{i}g\theta^a(T_{\overline{r}}^A X^*) \tag{C.44}$$

C.6 Spinor

■$\eta^{\mu\nu} = (-,+,+,+)$ case

Grassmann Number	:	$(ab)^\dagger = b^\dagger a^\dagger$ for $a, b \in \mathbb{G}$
	:	$\Longrightarrow$ for $a, b \in \mathbb{G}^{\mathbb{R}}$, $ab \in \mathrm{i}\mathbb{G}^{\mathbb{R}}$
γ matrix	:	$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}$
	:	$\gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$ etc...
	:	$(\mathrm{i}\gamma^0)^\dagger := \mathrm{i}\gamma^0, \quad {\gamma^i}^\dagger := \gamma^i$
Dirac Conjugate	:	$\bar{\psi} = \mathrm{i}\psi^\dagger\gamma^0$

TODO: SPINOR

C.7 Instanton

TODO: INSTANTON

C.8 Note on the symmetry factor in the Feynman Rules

If a vertex consists of different particles, the Feynman rule for the vertex is defined as $\mathrm{i}\lambda$, where λ is the coupling constant for the vertex in the Lagrangian. However, if a particle appears n-times in the coupling, we define the vertex as $(n!) \times \mathrm{i}\lambda$.

This factor $n!$ corresponds to the freedom of contraction, which we encounter when contracting the fields. For example, $\langle \phi_{k_1}\phi_{k_2}\phi_{k_3} | \left(\lambda\phi^3\psi\right) | \psi_p \rangle$ has 3! choice of the contraction; thus we define the vertex rule as $(3!)\mathrm{i}\lambda$, and the amplitude is evaluated as $6\mathrm{i}\lambda$.

For this multiplication, we have to consider a symmetry factor. $\langle\phi|(\lambda\phi^4)(\lambda'\phi^3\chi)|\chi\rangle$ has $4 \times 3!$ freedom, but the rule is already multiplied by $4! \times 3!$; thus we have to divide the amplitude by the symmetry factor ($= 6$) to obtain $24\mathrm{i}\lambda\lambda'$. Similarly, in $\langle\phi|\frac{1}{2}(\lambda\phi^4)(\lambda\phi^4)|\phi\rangle + \langle\phi|\frac{1}{2}(\lambda\phi^4)(\lambda\phi^4)|\phi\rangle = \langle\phi|(\lambda\phi^4)(\lambda\phi^4)|\phi\rangle$, the symmetry factor $6 = (4!)^2/(4\cdot 4\cdot 3!)$ appears. (Note that the 1/2 from the Fourier expansion cancels with the freedom of operators' order.)

付録 D　Supersymmetry in the text by Wess & Bagger

D.1　Spinor Convention

$$
\begin{array}{lcl}
\epsilon \text{ tensor} & : & \epsilon^{12} = \epsilon^{\dot 1\dot 2} = \epsilon_{21} = \epsilon_{\dot 2\dot 1} = 1 \qquad \text{(definition)} \\
\text{Sum Rule} & : & {}^{\alpha}{}_{\alpha} \text{ and } {}_{\dot\alpha}{}^{\dot\alpha}, \text{ except for} \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\xi^{\dot\beta}, \quad \xi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\xi_{\dot\beta}. \\
\text{Lorentz 変換} & : & \psi'_\alpha = \Lambda_\alpha{}^\beta\psi_\beta, \quad \bar\psi'_{\dot\alpha} = \bar\psi_{\dot\beta}\Lambda^{\dagger\dot\beta}{}_{\dot\alpha}, \quad \psi'^\alpha = \psi^\beta\Lambda^{-1}{}_\beta{}^\alpha, \quad \bar\psi'^{\dot\alpha} = (\Lambda^{-1})^{\dagger\dot\alpha}{}_{\dot\beta}\bar\psi'^{\dot\beta}. \\
\sigma \text{ matrices} & : & (\sigma^\mu)_{\alpha\dot\beta} := (-1, \boldsymbol{\sigma})_{\alpha\dot\beta}, \quad (\bar\sigma^\mu)^{\dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (-1, -\boldsymbol{\sigma})^{\dot\alpha\beta}.
\end{array}
$$

(See App. C.1.1 for a verbose explanation.)

D.2　Spinor Calculation Cheatsheet

$$
\boxed{
\begin{gathered}
\eta = (-,+,+,+), \qquad \epsilon^{0123} = -\epsilon_{0123} = 1 \\
\epsilon^{12} = \epsilon_{21} = \epsilon^{\dot 1\dot 2} = \epsilon_{\dot 2\dot 1} = 1, \quad \xi^\alpha := \epsilon^{\alpha\beta}\xi_\beta, \quad \xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \bar\xi^{\dot\alpha} := \epsilon^{\dot\alpha\dot\beta}\bar\xi_{\dot\beta}, \quad \bar\xi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta} \\
\bar\sigma^{\mu\dot\alpha\alpha} := \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} \qquad \sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\beta\beta}, \qquad \sigma^\mu := (-1, \boldsymbol{\sigma}), \quad \bar\sigma^\mu := (-1, -\boldsymbol{\sigma}) \\
(\sigma^{\mu\nu})_\alpha{}^\beta := \tfrac14\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} := \tfrac14\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)^{\dot\alpha}{}_{\dot\beta} = (\sigma^{\nu\mu})^{\dagger\dot\alpha}{}_{\dot\beta}.
\end{gathered}
}
$$

$$
\begin{array}{llll}
\theta^\alpha\theta^\beta = -\tfrac12\epsilon^{\alpha\beta}\theta\theta & \theta_\alpha\theta_\beta = \tfrac12\epsilon_{\alpha\beta}\theta\theta & (\theta\phi)(\theta\psi) = -\tfrac12(\psi\phi)(\theta\theta) & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = -\tfrac12\theta\theta\bar\theta\bar\theta\eta^{\mu\nu} \\
\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \tfrac12\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\tfrac12\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta & (\bar\theta\bar\phi)(\bar\theta\bar\psi) = -\tfrac12(\bar\psi\bar\phi)(\bar\theta\bar\theta) & (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) = \tfrac12(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\bar\theta\bar\theta \\
\theta\sigma^\mu\bar\sigma^\nu\theta = -\eta^{\mu\nu}\theta\theta & \bar\theta\bar\sigma^\mu\sigma^\nu\bar\theta = -\eta^{\mu\nu}\bar\theta\bar\theta & & (\theta\sigma^\mu)_{\dot\alpha}(\theta\sigma^\nu\bar\theta) = \tfrac12(\bar\theta\bar\sigma^\nu\sigma^\mu)_{\dot\alpha}\theta\theta
\end{array}
$$

$$
\begin{array}{ll}
\sigma^\mu\bar\sigma^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu} & \sigma^\mu\bar\sigma^\rho\sigma^\nu + \sigma^\nu\bar\sigma^\rho\sigma^\mu = -2\left(\eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho\right) \\
\bar\sigma^\mu\sigma^\nu = -\eta^{\mu\nu} + 2\bar\sigma^{\mu\nu} & \bar\sigma^\mu\sigma^\rho\bar\sigma^\nu + \bar\sigma^\nu\sigma^\rho\bar\sigma^\mu = -2\left(\eta^{\mu\rho}\bar\sigma^\nu + \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\nu}\bar\sigma^\rho\right) \\
\sigma^{\mu\nu} = -\sigma^{\nu\mu} & \sigma^\mu\bar\sigma^\nu\sigma^\rho - \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\
\bar\sigma^{\mu\nu} = -\bar\sigma^{\nu\mu} & \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = -2\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma \\
\operatorname{Tr}\bar\sigma^\mu\sigma^\nu = \operatorname{Tr}\sigma^\mu\bar\sigma^\nu = -2\eta^{\mu\nu} & \operatorname{Tr}\sigma^{\mu\nu}\sigma^{\rho\sigma} = -\tfrac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\operatorname{Tr}\sigma^{\mu\nu} = \operatorname{Tr}\bar\sigma^{\mu\nu} = 0 & \operatorname{Tr}\bar\sigma^{\mu\nu}\bar\sigma^{\rho\sigma} = -\tfrac12\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma} \\
\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu^{\dot\beta\beta} = -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} - \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = 2\left[(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + (\epsilon\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}\right] \\
\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\,\beta\dot\beta} = -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} + \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + 4\eta_{\rho\sigma}(\sigma^{\rho\mu}\epsilon)_{\alpha\beta}(\epsilon\bar\sigma^{\sigma\nu})_{\dot\alpha\dot\beta}
\end{array}
$$

$$
\begin{array}{lll}
\bar\sigma^{\mu\dot\alpha\alpha}\bar\sigma_\mu^{\dot\beta\beta} = -2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} & \epsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2\mathrm{i}\sigma^{\mu\nu} \\
\sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_\gamma{}^\beta\epsilon_{\beta\alpha} & \epsilon_{\beta\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta} & \epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} = 2\mathrm{i}\bar\sigma^{\mu\nu}
\end{array}
$$

$$
\begin{array}{ll}
\bar\xi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\xi = (\bar\chi\bar\sigma^\mu\xi)^* = -(\xi\sigma^\mu\bar\chi)^* & (\psi\phi)\chi_\alpha = -(\phi\chi)\psi_\alpha - (\chi\psi)\phi_\alpha \\
\xi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\xi = (\bar\chi\bar\sigma^\nu\sigma^\mu\bar\xi)^* = (\bar\xi\bar\sigma^\mu\sigma^\nu\bar\chi)^* & (\psi\phi)\bar\chi_{\dot\alpha} = -\tfrac12(\phi\sigma^\mu\bar\chi)(\psi\sigma_\mu)_{\dot\alpha}
\end{array}
$$

In the following equations, we chose left-differential notation.

$$
\begin{array}{lll}
\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\beta} = -\dfrac{\partial}{\partial\theta_\alpha} & \dfrac{\partial}{\partial\theta^\alpha}\theta\theta = 2\theta_\alpha & \epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta^\beta}\theta\theta = 4 \\
\epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\beta} = -\dfrac{\partial}{\partial\theta^\alpha} & \dfrac{\partial}{\partial\theta_\alpha}\theta\theta = -2\theta^\alpha & \epsilon_{\alpha\beta}\dfrac{\partial}{\partial\theta_\alpha}\dfrac{\partial}{\partial\theta_\beta}\theta\theta = -4 \\
\epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -2\bar\theta_{\dot\alpha} & \epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}}\bar\theta\bar\theta = 4 \\
\epsilon_{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta_{\dot\beta}} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} & \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 2\bar\theta^{\dot\alpha} & \epsilon^{\dot\alpha\dot\beta}\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}\dfrac{\partial}{\partial\bar\theta^{\dot\beta}}\bar\theta\bar\theta = -4
\end{array}
$$

D.3 Chiral Superfields : $\bar{D}_{\dot\alpha}\Phi = 0$

■Explicit Expression

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{D.1}$$

$$= \phi(x) + \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) + \sqrt{2}\theta\psi(x) + \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x) \tag{D.2}$$

$$\Phi^\dagger = \phi^*(x) - \mathrm{i}\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{\mathrm{i}}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \bar\theta\bar\theta F^*(x) \tag{D.3}$$

■Product of Chiral Superfields

$$\begin{aligned}\Phi_i^\dagger\Phi_j(x,\theta,\bar\theta) \rightsquigarrow\ & \phi_i^*\phi_j + \sqrt{2}\phi_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_i\phi_j + \theta\theta\phi_i^*F_j + \bar\theta\bar\theta F_i^*\phi_j \\ & + 2\mathrm{i}(\theta\sigma^\mu\bar\theta)(\phi_i^*\partial_\mu\phi_j) - \sqrt{2}\mathrm{i}\theta\theta(\partial_\mu\phi_i^*)\bar\theta\bar\sigma^\mu\psi_j - \sqrt{2}\mathrm{i}\bar\theta\bar\theta\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j \\ & + 2\bar\theta\bar\psi_i\theta\psi_j + \sqrt{2}\theta\theta\bar\theta\bar\psi_i F_j + \sqrt{2}\bar\theta\bar\theta F_i^*\theta\psi_j \\ & + \theta\theta\bar\theta\bar\theta\left[F_i^*F_j - \partial^\mu\phi_i^*\partial_\mu\phi_j - \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right]\end{aligned} \tag{D.4}$$

$$\Phi_i\Phi_j(\text{in } y\text{-basis}) = \phi_i\phi_j + \sqrt{2}\theta\left[\psi_i\phi_j + \phi_i\psi_j\right] + \theta\theta\left[\phi_iF_j + F_i\phi_j - \psi_i\psi_j\right] \tag{D.5}$$

$$\begin{aligned}\Phi_i\Phi_j\Phi_k(\text{in } y\text{-basis}) = {}& \phi_i\phi_j\phi_k + \sqrt{2}\theta\left[\psi_i\phi_j\phi_k + \phi_i\psi_j\phi_k + \phi_i\phi_j\psi_k\right] \\ & + \theta\theta\left[F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k - \psi_i\psi_j\phi_k - \psi_i\phi_j\psi_k - \phi_i\psi_j\psi_k\right]\end{aligned} \tag{D.6}$$

Note that products of chiral superfields $\Phi_1\Phi_2\cdots$ are again chiral superfields.

付録 E　Cherry on the Cake

■Conversion of Units

$$\begin{aligned}1\,\mathrm{GeV} &= \frac{1}{6.5821\times10^{-25}\,\mathrm{s}} = \frac{1}{2.086\times10^{-32}\,\mathrm{yr}} = \frac{1}{0.19733\,\mathrm{fm}} = 1.1605\times10^{13}\,\mathrm{K} = 1.7827\times10^{-24}\,\mathrm{g}\\ &= \frac{1.519268\times10^{24}}{1\,\mathrm{s}} = \frac{4.79\times10^{31}}{1\,\mathrm{yr}} = \frac{5.0677}{1\,\mathrm{fm}}\end{aligned} \tag{E.1}$$

$$1\,\mathrm{K} = 8.6173\times10^{-5}\,\mathrm{eV} = \frac{1}{8.0591\times10^{-21}\,\mathrm{s}} = \frac{1.2408\times10^{20}}{1\,\mathrm{s}} \tag{E.2}$$

$$1\,\mathrm{GeV}^{-2} = 3.8938\times10^{-4}\,\mathrm{barn} = \frac{1\,\mathrm{barn}}{2568.2} = 1.1683\times10^{-17}\,\mathrm{cm}^3/\mathrm{s} \qquad (1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2) \tag{E.3}$$

$$\begin{aligned}&1\,\text{tropical yr} = 3.1557\times10^{7}\,\mathrm{s}, \quad 1\,\text{sidereal yr} = 3.1558\times10^{7}\,\mathrm{s};\\ &1\,\mathrm{s} = 3.1689\times10^{-8}\,\text{tr-yr} = 3.1688\times10^{-8}\,\text{sr-yr}.\end{aligned} \tag{E.4}$$

■Physical Constants

$$G_\mathrm{F} = \frac{1}{\sqrt{2}v^2} = 1.16637(1)\times10^{-5}\,\mathrm{GeV}^{-2}, \qquad G_\mathrm{N} = 6.70881(67)\times10^{-39}\,\mathrm{GeV}^{-2} \tag{E.5}$$

$$\sqrt{G_\mathrm{N}} = 1.61624(8)\times10^{-35}\,\mathrm{m} = \frac{1}{1.22089(6)\times10^{19}\,\mathrm{GeV}} = \frac{1}{2.17644(11)\times10^{-8}\,\mathrm{kg}} \tag{E.6}$$

$$\sqrt{8\pi G_\mathrm{N}} = 8.1026(4)\times10^{-35}\,\mathrm{m} = \frac{1}{2.4353(1)\times10^{18}\,\mathrm{GeV}} = \frac{1}{4.3413(2)\times10^{-9}\,\mathrm{kg}} \tag{E.7}$$

■Component of Spinor in Weyl Representation

$(\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta,\ \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta},\ \chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta};\quad \epsilon^{12} = \epsilon^{\dot1\dot2} = 1,\ \epsilon_{12} = \epsilon_{\dot1\dot2} = -1.)$

$$\psi = \begin{pmatrix}\psi_\mathrm{L}\\ \psi_\mathrm{R}\end{pmatrix} = \begin{pmatrix}\xi_\alpha\\ \bar\chi^{\dot\alpha}\end{pmatrix} = \begin{pmatrix}\xi_1\\ \xi_2\\ \bar\chi^{\dot1}\\ \bar\chi^{\dot2}\end{pmatrix} \quad \xrightarrow{C} \quad \psi^\mathrm{c} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix}-(\bar\chi^{\dot2})^*\\ (\bar\chi^{\dot1})^*\\ (\xi_2)^*\\ -(\xi_1)^*\end{pmatrix} = \begin{pmatrix}-\chi^2\\ \chi^1\\ \bar\xi_{\dot2}\\ -\bar\xi_{\dot1}\end{pmatrix} = \begin{pmatrix}\chi_\alpha\\ \bar\xi^{\dot\alpha}\end{pmatrix} \tag{E.8}$$

$$\overline{\psi} = \begin{pmatrix}\chi^\alpha & \bar\xi_{\dot\alpha}\end{pmatrix} = \begin{pmatrix}\chi^1 & \chi^2 & \bar\xi_{\dot1} & \bar\xi_{\dot2}\end{pmatrix} \quad \xrightarrow{C} \quad \overline{\psi^\mathrm{c}} = \mathrm{i}\psi^\mathrm{T}\gamma^0\gamma^2 = \begin{pmatrix}\xi_2 & -\xi_1 & -\bar\chi^{\dot2} & \bar\chi^{\dot1}\end{pmatrix} = \begin{pmatrix}\xi^\alpha & \bar\chi_{\dot\alpha}\end{pmatrix} \tag{E.9}$$

$$A^\alpha B_\alpha = \overline{\psi}_{A^\mathrm{c}}P_\mathrm{L}\psi_B = \overline{\psi}_{B^\mathrm{c}}P_\mathrm{L}\psi_A \qquad \bar A_{\dot\alpha}\bar B^{\dot\alpha} = \overline{\psi}_A P_\mathrm{R}\psi_{B^\mathrm{c}} = \overline{\psi}_B P_\mathrm{R}\psi_{A^\mathrm{c}} \tag{E.10}$$

$$\begin{aligned}\overline{\psi}\not{A}\chi &= \overline{\psi}_\mathrm{L}\not{A}P_\mathrm{L}\chi_\mathrm{L} + \overline{\psi}_\mathrm{R}\not{A}P_\mathrm{R}\chi_\mathrm{R}\\ &= \overline{\psi}_\mathrm{L}\not{A}P_\mathrm{L}\chi_\mathrm{L} - \overline{\chi}{}^\mathrm{c}_\mathrm{R}\not{A}P_\mathrm{L}\psi^\mathrm{c}_\mathrm{R}\end{aligned} \tag{E.11}$$