1 Kinematics

1.1 Fundamentals

$$\int \mathrm{d}\Pi = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\,\delta\left((p^0)^2 - m^2 - \|\boldsymbol{p}\|^2\right)\Big|_{p^0\geq 0}$$

Källén function

$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx = (x-y-z)^2-4yz;\quad \lambda(s,m_1^2,m_2^2) = s^2\,\lambda(1,m_1^2/s,m_2^2/s);$
$\lambda(1;\alpha_1^2,\alpha_2^2) = (1-(\alpha_1+\alpha_2)^2)(1-(\alpha_1-\alpha_2)^2) = (1+\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2)(1+\alpha_1-\alpha_2)(1-\alpha_1+\alpha_2).$

$$\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = \sqrt{1-\frac{2(m_1^2+m_2^2)}{s}+\frac{(m_1^2-m_2^2)^2}{s^2}};\quad m_1=m_2:\ \sqrt{1-\frac{4m^2}{s}},\quad m_2=0:\ \frac{s-m_1^2}{s}. \tag{1.1}$$

Two-body kinematics $\quad s=(p_1+p_2)^2$

In CM frame, $s=(E_1+E_2)^2$ and $p = \sqrt{\frac{s}{4}}\,\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right);\quad E_1 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}}.$

For $m_1=m_2$, $E=\frac{\sqrt{s}}{2}$ and $p=\sqrt{\frac{s}{4}}\sqrt{1-\frac{4m^2}{s}} = \frac{\sqrt{s-4m^2}}{2}$; $v=\frac{p}{E}=\sqrt{1-\frac{4m^2}{s}}$.

For $m_2=0$, $p=\frac{\sqrt{s}}{2}\left(1-\frac{m_1^2}{s}\right)=E_2$ and $E_1=\frac{\sqrt{s}}{2}\left(1+\frac{m_1^2}{s}\right)$.

Two-body phase space

$\int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2(\text{Lorentz invariant}) = \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,f(p_1,p_2,p_1^\mu p_{2\mu}) = \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,f(p_1,p_2,\cos\theta_{12}).$

Rewriting with $E_\pm = E_1\pm E_2$ and $s=(p_1+p_2)^2 = m_1^2+m_2^2+2E_1E_2-2\|\boldsymbol{p}_1\|\|\boldsymbol{p}_2\|\cos\theta_{12}$,

$\int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2 = \int \frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3}\frac{1}{2E_1 2E_2} = \int \frac{(4\pi)\,\mathrm{d}p_1\,p_1^2}{(2\pi)^3}\frac{(2\pi)\,\mathrm{d}p_2\,p_2^2\,\mathrm{d}\cos\theta_{12}}{(2\pi)^3}\frac{1}{2E_1 2E_2} = \int \frac{\mathrm{d}E_+\,\mathrm{d}E_-\,\mathrm{d}s}{128\pi^4},$

where the Jacobian is $\left|\frac{\mathrm{d}(E_+,E_-,s)}{\mathrm{d}(p_1,p_2,\cos\theta_{12})}\right| = \frac{4p_1^2p_2^2}{E_1E_2}$, or $\left|\frac{\mathrm{d}(E_1,E_2,s)}{\mathrm{d}(p_1,p_2,\cos\theta_{12})}\right| = \frac{2p_1^2p_2^2}{E_1E_2}$.

As $\cos\theta_{12} = \frac{E_+^2-E_-^2+2\left(m_1^2+m_2^2-s\right)}{\sqrt{(E_++E_-)^2-4m_1^2}\sqrt{(E_+-E_-)^2-4m_2^2}}$ is restricted as $[-1,1]$, E_- is bounded as

$\left|E_- - \frac{m_1^2-m_2^2}{s}E_+\right| \leq \sqrt{E_+^2-s}\cdot\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = 2p\sqrt{\frac{E_+^2-s}{s}}$; using these bounds,

$$\int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2 = \frac{1}{128\pi^4}\int_{(m_1+m_2)^2}^\infty \mathrm{d}s\int_{\sqrt{s}}^\infty \mathrm{d}E_+\int_{\min}^{\max}\mathrm{d}E_-. \tag{1.2}$$

Two-body phase space with momentum conservation

$\int \mathrm{d}\Pi^{(2)} := \int \mathrm{d}\Pi_1\,\mathrm{d}\Pi_2\,(2\pi)^4\,\delta^{(4)}(P_0-p_1-p_2) = \int \frac{\mathrm{d}p_1\,\mathrm{d}\Omega\,p_1^2}{16\pi^2}\frac{\delta(E_0-\sqrt{m_1^2+p_1^2}-\sqrt{m_2^2+\|\boldsymbol{P}_0-\boldsymbol{p}_1\|^2})}{E_1E_2};$

$p_1 = \frac{(E_0^2+m_1^2-m_2^2-P_0^2)P_0\cos\theta_1 + E_0\sqrt{\lambda(E_0^2,m_1^2,m_2^2)+P_0^4-2P_0^2(E_0^2+m_1^2-2m_1^2\cos^2\theta_1-m_2^2)}}{2(E_0^2-P_0^2\cos^2\theta_1)}$

and $\int \mathrm{d}\Pi^{(2)} = \frac{1}{8\pi}\int \mathrm{d}\cos\theta_1\frac{p_1^2}{E_0p_1-P_0E_1\cos\theta_1}$;

$$\text{CM }(E_0=\sqrt{s}):\quad \int \mathrm{d}\Pi^{(2)} = \int\frac{\mathrm{d}\cos\theta_1}{8\pi}\frac{p}{\sqrt{s}};\quad p=\frac{\sqrt{s}}{2}\,\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right),\quad E_1=\frac{s+m_1^2-m_2^2}{2\sqrt{s}}. \tag{1.3}$$

1.2 Decay rate and Cross section

Rate $\mathrm{d}R = \mathrm{d}\Pi^{(n)} \left|\mathcal{M}\right|^2 (2\pi)^4\, \delta^{(4)}(p_\mathrm{i} - p_\mathrm{f})$ is Lorentz invariant; density is given by $\rho = 2E$.
Decay rate: $R = \dfrac{N}{VT} =: \rho_1^\mathrm{rest}\Gamma = 2m\Gamma$ (defined at the rest frame);
Cross section: $R = \dfrac{N}{VT} =: v_\mathrm{rel}\rho_1\rho_2 \dfrac{p_1\cdot p_2}{E_1E_2}\sigma =: v_\mathrm{Møl}\rho_1\rho_2\sigma$ (defined as Lorentz invariant).
Note that ρ/E is Lorentz invariant; the factor is introduced as $(p_1\cdot p_2)/E_1E_2 = 1$ at $\boldsymbol{p}_1 = 0$ or $\boldsymbol{p}_2 = 0$.

$$\text{relative velocity:}\quad v_\mathrm{rel} = \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}{p_1\cdot p_2} = \frac{\sqrt{\|\boldsymbol{v}_1-\boldsymbol{v}_2\|^2 - \|\boldsymbol{v}_1\times\boldsymbol{v}_2\|^2}}{1-\boldsymbol{v}_1\cdot\boldsymbol{v}_2} = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{s-(m_1^2+m_2^2)}, \tag{1.4}$$

$$\text{Møller parameter:}\quad v_\mathrm{Møl} := \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}{E_1E_2} \quad\rightsquigarrow |v_1 - v_2| \text{ if } \boldsymbol{v}_1 \mathbin{/\!/} \boldsymbol{v}_2; \tag{1.5}$$

[mass dimension of $\mathcal{M}$ is $4 - N_\mathrm{i} - N_\mathrm{f}$]

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\left[\prod_\mathrm{f}\frac{\mathrm{d}^3p_\mathrm{f}}{(2\pi)^3}\frac{1}{2E_\mathrm{f}}\right]\left|\mathcal{M}(m_A\to\{p_\mathrm{f}\})\right|^2(2\pi)^4\,\delta^{(4)}\left(m_A - \{p_\mathrm{f}\}\right), \tag{1.6}$$

$$\mathrm{d}\sigma = \frac{1}{2E_A2E_B\,v_\mathrm{Møl}}\left[\prod_\mathrm{f}\frac{\mathrm{d}^3p_\mathrm{f}}{(2\pi)^3}\frac{1}{2E_\mathrm{f}}\right]\left|\mathcal{M}(p_A,p_B\to\{p_\mathrm{f}\})\right|^2(2\pi)^4\,\delta^{(4)}\left(p_A + p_B - \{p_\mathrm{f}\}\right). \tag{1.7}$$

Two-body kinematics in CM frame

$$\int \mathrm{d}\Pi^{(2)} := \int\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\,\delta^{(4)}(\sqrt{s}-p_1-p_2) \qquad (\sqrt{s} = E_\mathrm{CM} \text{ or } M_\mathrm{mother})$$

$$= \frac{\|\boldsymbol{p}\|}{4\pi\sqrt{s}}\int\frac{\mathrm{d}\Omega}{4\pi} = \frac{\|\boldsymbol{p}\|}{8\pi\sqrt{s}}\int \mathrm{d}\cos\theta \tag{1.8}$$

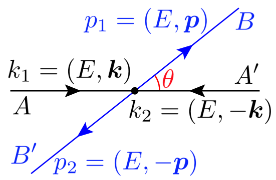
$$\frac{\|\boldsymbol{p}\|}{8\pi\sqrt{s}} = \frac{1}{8\pi\sqrt{s}}\cdot\frac{\sqrt{s}}{2}\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = \frac{1}{16\pi}\sqrt{1-\frac{2(m_1^2+m_2^2)}{s}+\frac{(m_1^2-m_2^2)^2}{s^2}} \tag{1.9}$$

$$E_1 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}} \qquad E_2 = \frac{s-m_1^2+m_2^2}{2\sqrt{s}} \qquad p_1\cdot p_2 = \frac{s-(m_1^2+m_2^2)}{2} \tag{1.10}$$

Mandelstam variables For $(k_1,k_2)\to(p_3,p_4)$ collision,

$$s = (k_1+k_2)^2 = (p_3+p_4)^2,\quad t = (p_3-k_1)^2 = (p_4-k_2)^2,\quad u = (p_3-k_2)^2 = (p_4-k_1)^2;$$
$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$$

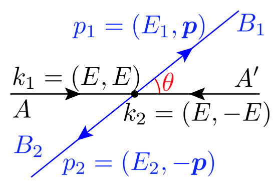
For same-mass collision, $s = (2E)^2 = E_\mathrm{CM}^2$ and



$$t = m_A^2 + m_B^2 - s/2 + 2kp\cos\theta \qquad (k_1-k_2)^2 = 4m_A^2 - s$$
$$u = m_A^2 + m_B^2 - s/2 - 2kp\cos\theta \qquad (p_1-p_2)^2 = 4m_B^2 - s$$

$$k = \frac{\sqrt{s-4m_A^2}}{2} \qquad k_1\cdot k_2 = \frac{s}{2} - m_A^2 \qquad k_1\cdot p_1 = k_2\cdot p_2 = \frac{m_A^2+m_B^2-t}{2}$$
$$p = \frac{\sqrt{s-4m_B^2}}{2} \qquad p_1\cdot p_2 = \frac{s}{2} - m_B^2 \qquad k_1\cdot p_2 = k_1\cdot p_2 = \frac{m_A^2+m_B^2-u}{2}$$

For collision with initial mass ignored,



$$t = (m_1^2+m_2^2-s)/2 + p\sqrt{s}\cos\theta$$
$$u = (m_1^2+m_2^2-s)/2 - p\sqrt{s}\cos\theta$$