My HoTT notes

12 June 2020 - ongoing

$$\texttt{Searchable}\ A :\equiv \prod p : A \to \mathbf{2}. \left(\sum x : A. p\ x = \texttt{tt}\right) + \prod x : A. p\ x = \texttt{ff}$$

In words: the type A is searchable if for every boolean predicate p we can find an element of A that satisfies p or else prove that no element of A satisfies p.

Theorem 0.1. The type `Searchable` A can be of any h-level.

Proof. $p\ x = \texttt{tt}$ and $p\ x = \texttt{ff}$ are propositions, the summands are disjoint, and dependent products preserve h-level (well, unless the domain is $\mathbf{0}$), so the h-level is determined by A.

If A is contractible, then `Searchable` A is contractible.

If A is a proposition, then `Searchable` A is a proposition.

If A is of h-level n, then so is `Searchable` A.

Hope my mental techniques for h-levels work :)

□

$$\texttt{MerelySearchable}\ A :\equiv \forall p : A \to \mathbf{2}. (\exists x : A. p\ x = \texttt{tt}) \vee \forall x : A. p\ x = \texttt{ff}$$

Theorem 0.2. `isProp` (`MerelySearchable`A)

Proof. We use the truncated logic. □

Theorem 0.3. Searchable types include: $\mathbf{0}, \mathbf{1}, \mathbf{2}$, all finite types and the interval.

If A and B are searchable, so are $A \times B$ and $A + B$.

If A is searchable and $B : A \to \mathcal{U}$ is a family of searchable types, $\sum x : A. B\ x$ is also searchable.

Proof. For finite types: we assume that "finite" means "there is a list of all the elements". So, use it to check every element, and poof – done.

The interval is contractible, thus equivalent to $\mathbf{1}$, so it's searchable.

To search $A + B$ first search A, then search B.

To search $A \times B$ and $\sum x : A.B\ x$ search for an a, and when looking for it, search for a b that can be paired with it. □

Theorem 0.4. Searchability of $\mathbb{N}$ is taboo.

In general, preservation of searchability for $\prod$, $\mathtt{W}$ and $\mathtt{M}$ should be taboo.

Proof. $\mathbb{N}$ is the primordial searchability taboo – how can we search infinitely many numbers in a finite time (besides checking every next number twice as fast as the previous one)?

$\mathtt{W}$ can be used to define $\mathbb{N}$, so its searchability should be taboo.

□