Laboratory Work: Optimal control

Zein Alabedeen Barhoum (345094)

Question 1:

1.1

The Hamiltonian is given as $H=-F_0(x,y,t)+\phi^T \dot{x}$, by substituting $F_0(x,u,t)=u^2(t)$ and $\dot{x}^T=[\dot{x}_1 \quad \dot{x}_2]$; therefore, we can write the Hamiltonian as:

$$H=-u^2+\phi^T \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=-u^2+\phi_1 \dot{x}_1+\phi_2 \dot{x}_2$$

But the plant's model is given as:

$$\dot{x}=Ax+Bu=\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} x+\begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$\dot{x}_1=-x_1+x_2+2u$$
$$\dot{x}_2=x_1+u$$

Therefore, the Hamiltonian is written as:

$$H=-u^2+\phi_1(-x_1+x_2+2u)+\phi_2(x_1+u)$$

1.2

The Euler-Lagrange equations are as follows,

$$\dot{\phi}_i=-\frac{\delta H}{\delta x_i}$$
$$\frac{\delta H}{\delta u}=0$$

By differentiating we get,

$$\frac{\delta H}{\delta x_1}=-\phi_1+\phi_2 \rightarrow \dot{\phi}_1=\phi_1-\phi_2$$
$$\frac{\delta H}{\delta x_2}=\phi_1 \rightarrow \dot{\phi}_2=-\phi_1$$
$$\frac{\delta H}{\delta u}=-2u+2\phi_1+1\phi_2 \rightarrow u=\frac{2\phi_1+\phi_2}{2}$$

By substituting the expression of u in the plant's model, we get:

$$\dot{x}_1=-x_1+x_2+2\phi_1+\phi_2$$
$$\dot{x}_2=x_1+\phi_1+0.5\phi_2$$

Combining the expressions of $\dot{\phi}_1, \dot{\phi}_2, \dot{x}_1, \dot{x}_2$ we get the following differential equation system:

$$\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 2 & 1 & -1 & 1\\ 1 & 0.5 & 1 & 0\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ x_1\\ x_2\end{bmatrix}$$

The boundary conditions are: $x_2(0)=0, x_1(\pi)=1, x_2(\pi)=0$, we see that's there are only three conditions, therefore, we can write these conditions as:

$$g_1=x_2(0)=0\ g_2=x_1(\pi)-1=0\ g_3=x_2(\pi)=0$$

By adding them we get $G=v_1 g_1+v_2 g_2+v_3 g_3=v_1 x_2(0)+v_2 x_1(\pi)-v_2+v_3 x_2(\pi)$

By applying the transversality conditions, we get

$$\phi_1(0)=-\frac{\delta G}{\delta x_1(0)}=0$$
$$\phi_2(0)=-\frac{\delta G}{\delta x_2(0)}=-v_1$$
$$\phi_1(\pi)=\frac{\delta G}{\delta x_1(\pi)}=v_2$$
$$\phi_2(\pi)=\frac{\delta G}{\delta x_2(\pi)}=v_3$$

We can see that we got a new condition $\dot{\phi}_1(0)=0$ which complete the required 4 conditions to definitely solve a system of 4 differential equations. The full condition set is

$$\phi_1(0)=0, x_2(0)=0, x_1(\pi)=1, x_2(\pi)=0$$

1.3

The following MATLAB code was created to solve the differential equation system,

```
syms f1(t) f2(t) x1(t) x2(t)
A = [1,-1,0,0;
    -1,0,0,0;
     2,1,-1,1;
     1,0.5,1,0];

odes = [diff(f1);diff(f2);diff(x1);diff(x2)] == A * [f1;f2;x1;x2];

conds = [f1(0);x2(0);x1(pi);x2(pi)] == [0;0;1;0];

[f1s(t),f2s(t),x1s(t),x2s(t)] = dsolve(odes,conds);

u(t) = simplify(f1s + 0.5*f2s);
pretty(u(t))
matlabFunction(u)
```

The resultant signal $u(t)$ is the following:

$$u(t)=\frac{e^{\frac{t}{2}+\pi}\left((1+\sqrt{5})e^{\frac{3\sqrt{5}\pi}{2}-\frac{\pi}{2}}+(\sqrt{5}-1)e^{\frac{\sqrt{5}\pi}{2}-\frac{\pi}{2}+\sqrt{5}t}+(1-\sqrt{5})e^{\frac{3\sqrt{5}\pi}{2}-\frac{\pi}{2}+\sqrt{5}t}+(-1-\sqrt{5})e^{\frac{\sqrt{5}\pi}{2}-\frac{\pi}{2}}\right)}{(5+2\sqrt{5})e^{\pi+\frac{\sqrt{5}t}{2}}+(-5+2\sqrt{5})e^{\pi+2\sqrt{5}\pi+\frac{\sqrt{5}t}{2}}+4\sqrt{5}e^{(1+\sqrt{5})\pi+\frac{\sqrt{5}t}{2}}-8\sqrt{5}e^{\sqrt{5}(\frac{t}{2}+\pi)}}$$

1.4

The following Simulink model was created to simulate the system with the derived input

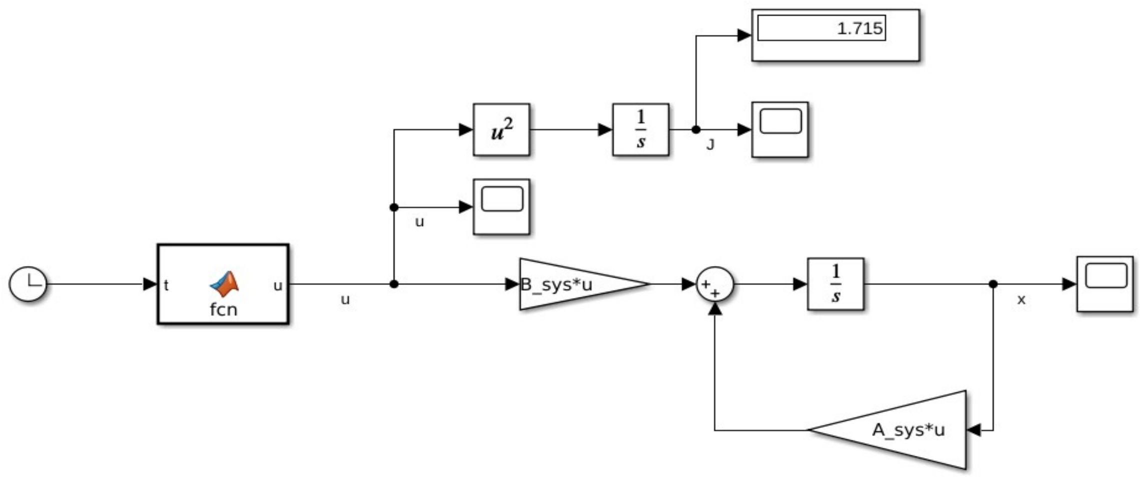

Figure 1: Simulink Model for Question 1

The system matrices were set as follows;

```
A_sys = [-1,1;1,0];
B_sys = [2;1];
```

The function $u(t)$ was generated automatically using the matlabFunction method in MATLAB which generates a MATLAB function from the symbolic function calculated before.

In terms of the initial conditions, we have the initial condition $x_2(0)=0$; however, we saw that we don’t have $x_1(0)$; but we already found the function $x_1(t)$ and by substituting $t=0$ as follows

```
>> double(subs(x1s,t,0))
```

We get that $x_1(0)=-0.9243$

The results of the simulation in terms of $x_1(t), x_2(t), u(t)$ and the cost function are shown in Figures 2,3,4.

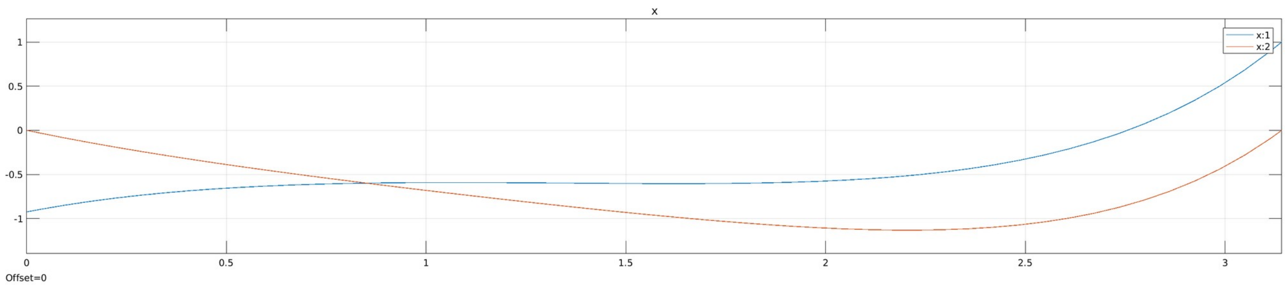

Figure 2: State (x)

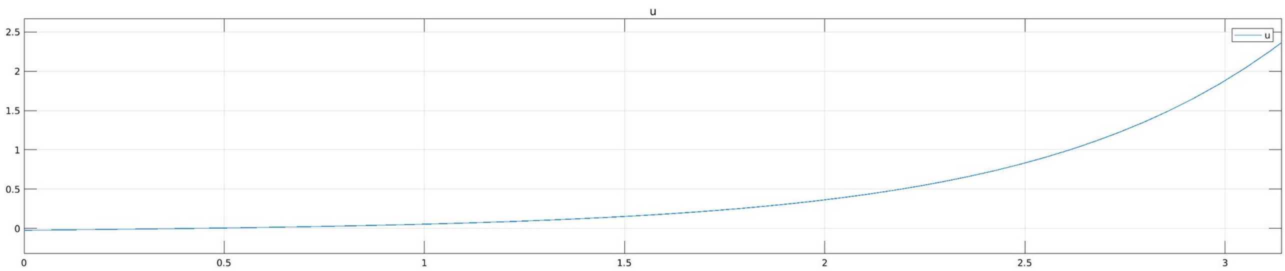

Figure 3: Input (u)

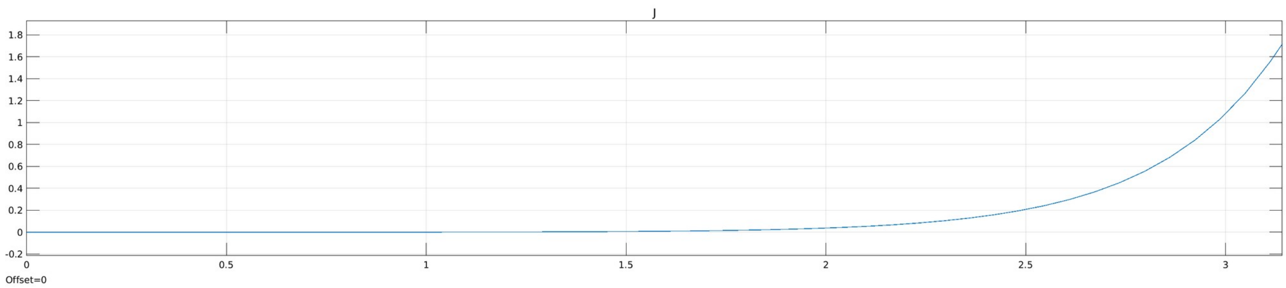

Figure 4: Cost Function (J)

We see that the the state variables satisfy the initial conditions where

$$x_2(0)=0, x_1(\pi)=1, x_2(\pi)=0$$

We see that the cost function $J=\int_0^{\pi} u^2 = 1.715$

Question 2:

2.1

First, we need to calculate the matrix P that satisfies the Riccati equation :

$$A^T P + P A + Q - P B r^{-1} B^T P = 0$$

We have the following values known

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} Q = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} r = 2$$

Therefore, the only variable in the former equation is P . The following MATLAB code was used to calculate P .

```
A = [-1,1;1,0];
B = [2;1];
Q = [3,0;0,5];
R = 2;

[K,P,poles] = lqr(A,B,Q,R);

P
```

The result is:

$$P = \begin{bmatrix} 0.8332 & 0.2977 \\ 0.2997 & 2.7498 \end{bmatrix}$$

We can check if the equation is satisfied, we use the following code

```
cond = A.'*P+P*A+Q -P*B*(R^-1)*B'*P ==0;
assert(cond(1) ==0)
assert(cond(2) ==0)
assert(cond(3) ==0)
assert(cond(4) ==0)
```

All conditions are satisfied ; therefore, the matrix P satisfies the Riccati equation.

2.2

The optimal gain can be calculated as:

$$K = r^{-1} B^T P = 0.5 * [2 \quad 1] \begin{bmatrix} 0.8332 & 0.2977 \\ 0.2997 & 2.7498 \end{bmatrix} = [0.9821 \quad 1.6726]$$

The same matrix was derived from the lqr function.

Therefore, the controller structure is:

$$u = -Kx = [0.9821 \quad 1.6726] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.9821 x_1 + 1.6726 x_2$$

The following Simulink model was used:

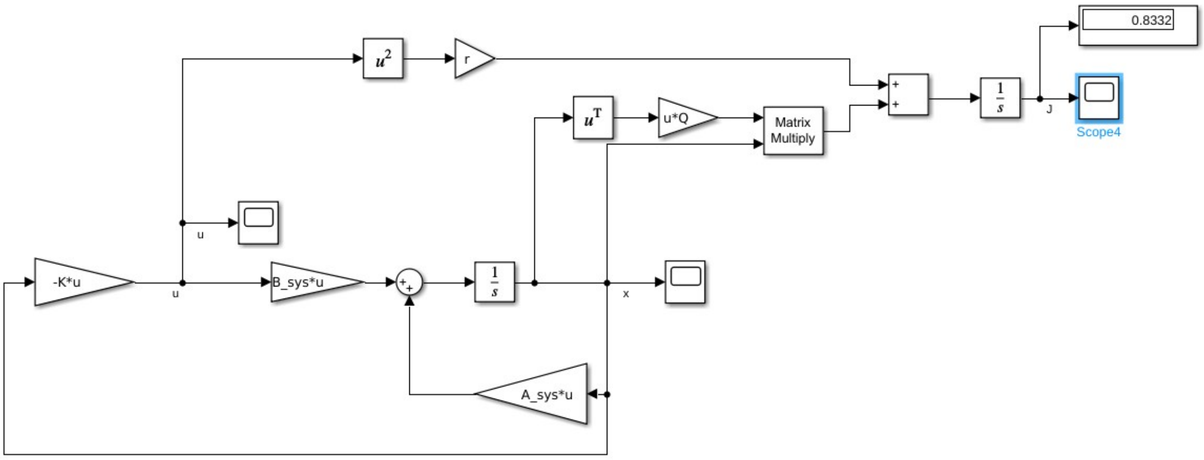

Figure 5: Simulink Model for Question 2

The system matrices and the control gain were defined as shown below

```
A_sys = [-1,1;1,0];
B_sys = [2;1];

K = [0.9821,1.6726];
r = 2;
Q = [3,0;0,5];
```

The results $x_1(t), x_2(t), u(t), J$ are shown in Figures 6,7,8.

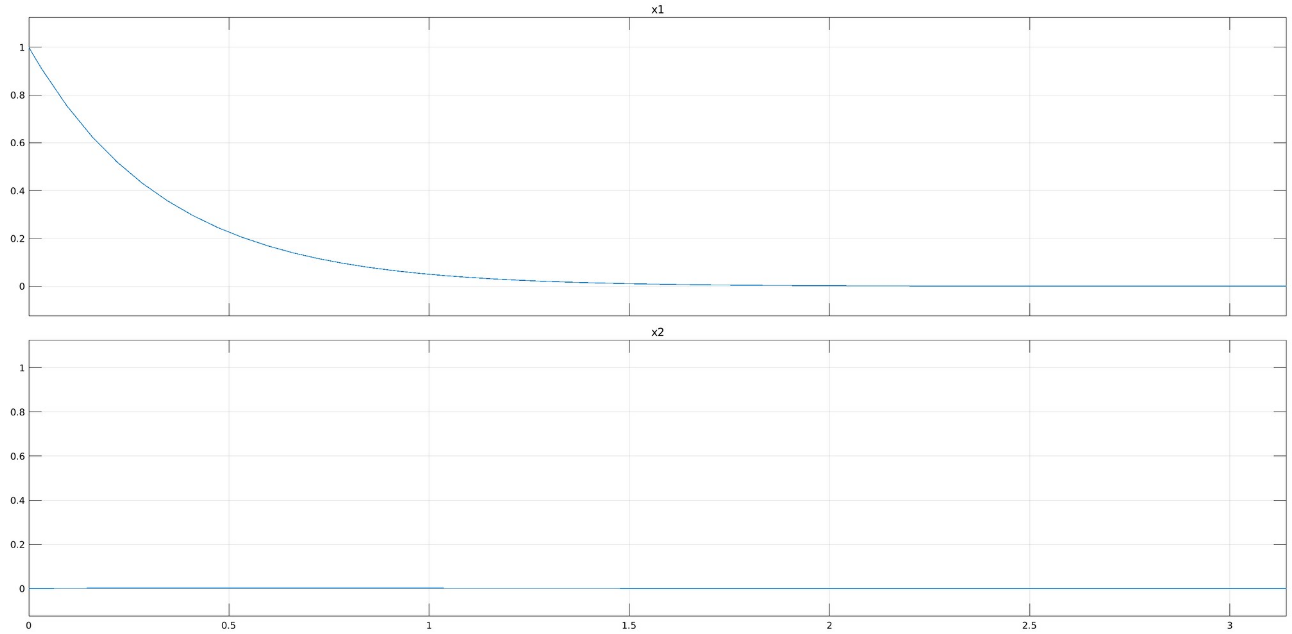

Figure 6: State x1 , x2

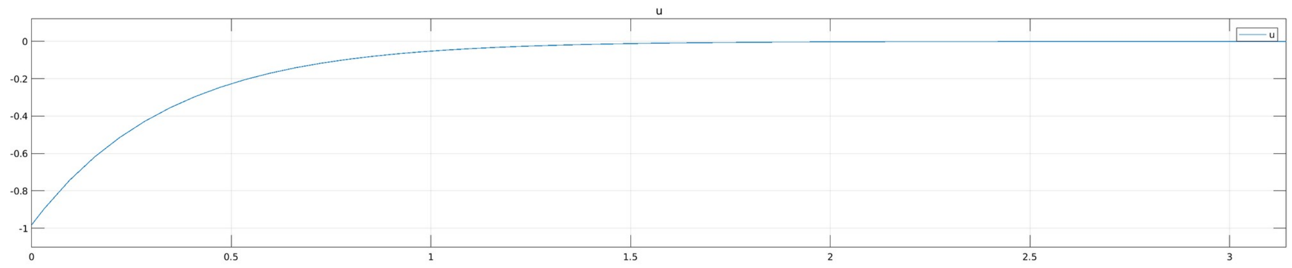

Figure 7: Control signal u

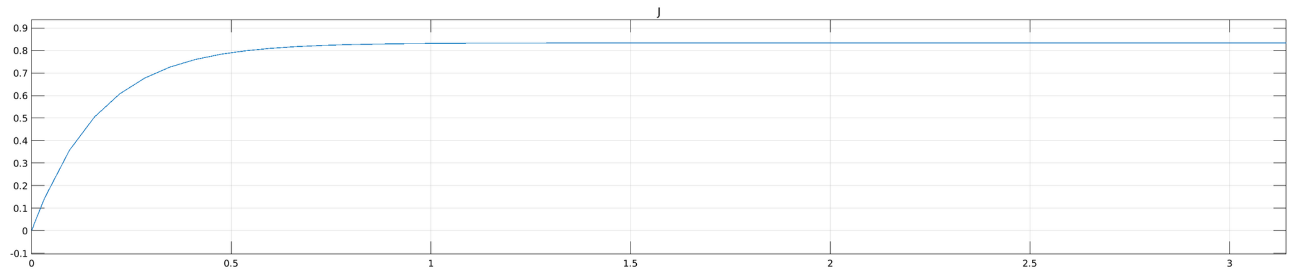

Figure 8: Cost function J

We see that the steady state value of the cost function is **0.8332** .