Exercise 1

a) The given subset is linearly dependent.

$$\left.\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 2 & 6 \\ 3 & 1 & 8 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \quad \left.\begin{array}{ccc} \boxed{1} & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 4 & 8 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \quad \left.\begin{array}{ccc} 1 & -1 & 0 \\ 0 & \boxed{3} & 6 \\ 0 & 0 & 0 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array}$$

b) The given subset is linearly dependent.

$$\left.\begin{array}{ccc} 1 & 1 & 1 \\ t & t-1 & t+1 \\ t^2 & (t-1)^2 & (t+1)^2 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \quad \left.\begin{array}{ccc} \boxed{1} & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2t+1 & 2t+1 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \quad \left.\begin{array}{ccc} 1 & 1 & 1 \\ 0 & \boxed{\text{-1}} & 1 \\ 0 & 0 & 0 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array}$$

Exercise 2

a) First we show that
$$\forall\, \vec{u}, \vec{v} \in U \cap W : \vec{u} + \vec{v} \in U \cap W.$$

Since U is a subspace, for all elements $\vec{u_1}, \vec{u_2} \in U$ we know $\vec{u_1} + \vec{u_2} \in U$. The same holds for the vectors in W. We know that the vectors $\vec{u}$ and $\vec{v}$ are in U and W. Thus

$$\vec{u} + \vec{v} \in U \wedge \vec{u} + \vec{v} \in W \quad \Longrightarrow \quad \vec{u} + \vec{v} \in U \cap W.$$

Second we show that
$$\forall\, \vec{v} \in U \cap W : \forall \lambda \in V : \lambda \cdot \vec{v} \in U \cap W$$

We again know for all $\vec{u} \in U$ and $\lambda \in V$ that we have $\lambda \cdot \vec{u} \in U$ and analogously for W. Thus, since $\vec{v}$ is in both U and W the same reasoning as above applies.

b) First we show that
$$\forall\, \vec{v_1}, \vec{v_2} \in U + W : \vec{v_1} + \vec{v_2} \in U + W.$$

Since

$$\begin{aligned} \vec{v_1} &= \vec{u_1} + \vec{w_1} \quad (\vec{u_1} \in U, \vec{w_1} \in W) \\ \vec{v_2} &= \vec{u_2} + \vec{w_2} \quad (\vec{u_2} \in U, \vec{w_2} \in W) \end{aligned}$$

by definition of $U + W$ we can restate the condition to

$$\begin{aligned} \vec{v_1} + \vec{v_2} &\in U + W \\ \vec{u_1} + \vec{w_1} + \vec{u_2} + \vec{w_2} &\in U + W \\ \vec{u_1} + \vec{u_2} + \vec{w_1} + \vec{w_2} &\in U + W \end{aligned}$$

which is true since $\vec{u_1} + \vec{u_2} \in U$ and $\vec{w_1} + \vec{w_2} \in W$.

Second we show that
$$\forall\, \vec{v} \in U + W : \forall\, \lambda \in V : \lambda \cdot \vec{v} \in U + W$$

Since $\vec{v} = \vec{u} + \vec{w}$ (for some vectors $\vec{u}$ and $\vec{w}$ in U and W respectively) we can restate the above condition to

$$\begin{aligned} \lambda(\vec{u} + \vec{w}) &\in U + W \\ \lambda \cdot \vec{u} + \lambda \cdot \vec{w} &\in U + W \end{aligned}$$

which is true since $\lambda \cdot \vec{u} \in U$ and $\lambda \cdot \vec{w} \in W$. (We can decompose the multiplication with λ because V is a vector space.)

Exercise 3 We show that $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

First we show the „if": Assume $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$ which is a subspace. The reasoning for $W_2 \subseteq W_1$ is very similar.

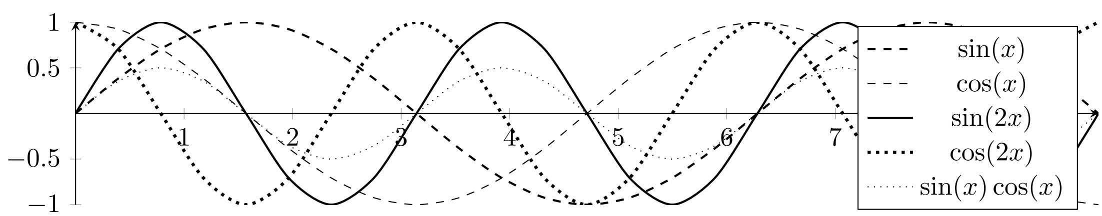
Second we show the „and only": Let $w_1 \in W_1$ such that $w_1 \notin W_2$ and $w_2 \in W_2$ with $w_2 \notin W_1$. We claim $w_1 + w_2 \notin W_1 \cap W_2$ — thus that $W_1 \cup W_2$ is not a subspace if neither is a subset of the other.

Proof by contradiction: Assume that $w_1 + w_2 \in W_1$. Since $-w_1$ is in W_1 (it is a subspace) we can construct the statement

$$\begin{aligned} (w_1 + w_2) - w_1 &\in W_1 \\ w_2 &\in W_1 \end{aligned}$$

which is a contradiction. Same goes for the assumption that $w_1 + w_2 \in W_2$.

Exercise 5



There is no way to add the functions together to get $f(x) = 0$.

Exercise 7 The base is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and the dimension of the linear hull is 3.