MAE 507 Engineering Analysis 1

Project 1 Report

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Abstract

The purpose of this project was to investigate both analytically and numerically the Duffing equation, which is an equation used to model damped and driven oscillators. This model highlights the chaotic behavior of these responses when specified for unstable amplitudes and frequencies. To approach this problem, the Duffing equation was coded in MATLAB and specified for various constants. Then, the driving force and frequency were changed for several states and the responses of the equation were graphed. These numerical results were compared to the analytically solved results as shown in the discussion section. It was found that the system was stable from a given periodic driving force (F_0) between 0 and 0.37. The jump phenomenon observed in the Duffing equation occurred at a periodic driving force of about 0.37, and complete and random chaos was present in the phase space and time series at 0.50. At a periodic driving force of 0.65, the system reached stability once again and created repeatable oscillations. These numerical results support the analytical derivations and clearly show that the Duffing equation has a variety of periodic and chaotic solutions.

Introduction

The Duffing equation, named after George Duffing (1861–1944), is a non-linear second-order differential equation that is used to model damped and driven oscillators. This is given in **Equation 1** below:

$$mx'' + rx' + \alpha x + \beta x^3 = F_0 cos\Omega t \qquad (1)$$

Where the (unknown) function ***x = x(t)*** *is the displacement at time* ***t****,* ***x'*** *is the first derivative of* ***x*** *with respect to time, i.e. velocity, and* ***x''*** *is the second time-derivative of* ***x****, i.e. acceleration. The numbers* ***r, α, β,*** F_0*, and* ***Ω*** *are given constants and represent the amount of damping* ***r****, linear stiffness* ***α****, amount of non-linearity in the restoring force* ***β****, amplitude of the periodic driving force* F_0*, and the angular frequency of the periodic driving force* ***Ω****.*

Equation 1 describes the motion of a damped, forced, mechanical oscillator of mass *m* having a nonlinear spring (meaning that spring force is not *kx* but rather $\alpha x + \beta x^3$). It represents a more complex model of an oscillator vs just simple harmonic motion (which corresponds to the case r = β= 0); in physical terms, it models situations like an elastic pendulum whose spring's stiffness does not quite obey Hooke's law. **Figure 1** shows the results as ***β*** is varied (a harmonic result, or x(t) = A$\cos\Omega t$, is achieved when β = 0):

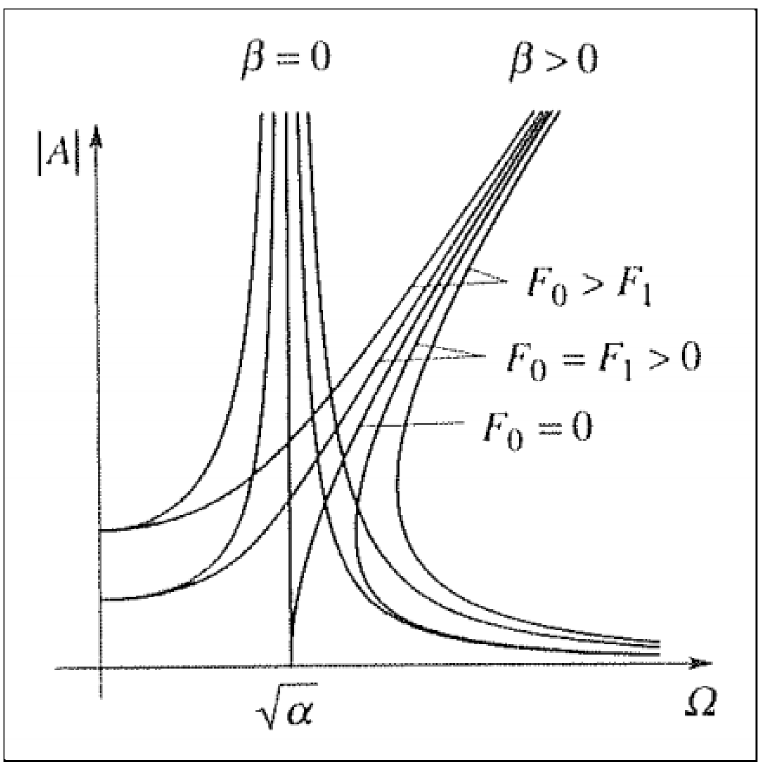

Figure 1: Amplitude Response Curves (undampened)

The Duffing equation is an example of a dynamical system that exhibits chaotic behavior. This means that the Duffing system presents behavior in the frequency response that can be random and unexpected. **Figure 2** below shows the "jump phenomena" of the Duffing equation and is an example of this chaotic behavior:

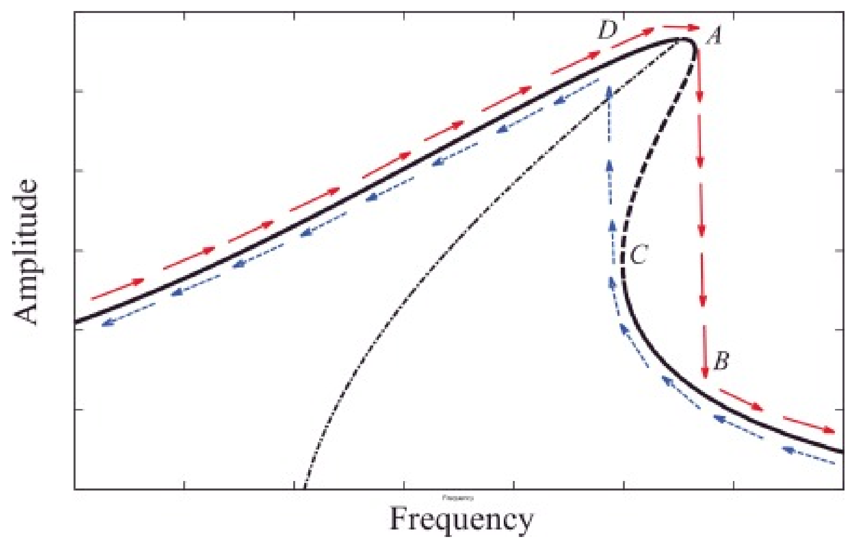

Figure 2: Jump Phenomena of the Duffing Equation

In the above figure, for a given $\boldsymbol{\beta > 0}$, as the frequency increases from *0* to *A'* ($\boldsymbol{\Omega}$ increases), the amplitude rises ($\boldsymbol{F_0}$ increases). Then, a jump occurs from *A* to *B* causing a sudden drop in the amplitude. But say instead of starting from *0*, one starts from *B*. As seen in the figure, as the frequency decreases from *B*, the amplitude will actually not jump upwards to *A* but instead continue towards *C* and then finally jump at point *D*. This "jumping" leads to a discontinuous change in the output, mainly the jumps in amplitude from *A* to *B* and from *C* to *D*. And between these vertically dashed lines as shown in the figure is an inaccessible zone of output. More specifically, this section of the graph is unstable. And will lead to random and chaotic output if the amplitude $\boldsymbol{\Omega}$ is taken from that point.

Methods

Analytic Analysis for an Undamped System:

→ Duffing equation for general motion:-

$$mx'' + \gamma x' + \alpha x + \beta x^3 = F_0 \cos \Omega t \quad ; \text{---} (1)$$

where $m=1$; γ = amount of damping ;
(Case V/I) α = linear stiffness;
β = amount of non-linearity in force;
F_0 = amplitude of driving force;
Ω = angular frequency of periodic driving force Ω.

Case 1:- $x'' + \alpha x + \beta x^3 = F_0 \cos \Omega t$; ---(2)
where $\gamma = 0$ [No damping].

Here, the relation between amplitude-frequency is given as:-

$$\boxed{\Omega^2 = \alpha + \frac{3}{4}\beta A^2 - \frac{F_0}{A}} \quad ; \text{---} (3)$$

Using the iterative scheme for (2);

$$x''_{n+1} + \alpha x_n + \beta x_n^3 = F_0 \cos \Omega t$$

$$\boxed{x''_{n+1} = F_0 \cos \Omega t - \alpha x_n - \beta x_n^3} \text{---} (4)$$

Choosing the initial iteration as $x(t) = A \cos \Omega t$

(i) For $\beta = 0$;
We know for $\beta = 0$;

$$\boxed{x'' + \alpha x = F_0 \cos \Omega t} \text{---} (5)$$

Now; $x(t) = A\cos \Omega t$

$$\boxed{\dot{x} = -A\sin(\Omega t)\,\Omega}$$

$$\ddot{x} = -A\Omega^2 \cos(\Omega t)$$

$$\boxed{\ddot{x} = -\Omega^2 x(t)}$$

Putting in (5)

$$-\Omega^2 x + \alpha\, x = F_0 \cos \Omega t$$

$$\therefore \boxed{x(t) = \frac{F_0 \cos \Omega t}{(\alpha - \Omega^2)}} \quad \text{— (6)}$$

Now; $x_0(t) = A\cos \Omega t$

Using (4); Putting $n=0$ $[x_0 = A\cos \Omega t]$

$$x''_1 = F_0 \cos \Omega t - \alpha A \cos \Omega t \quad [\beta = 0]$$

$$x''_1 = \cos \Omega t\,(F_0 - \alpha A)$$

Integrating

$$x'_1 = (F_0 - \alpha A)\frac{\sin \Omega t}{\Omega}$$

Integrating again

$$\boxed{x_1 = -\frac{(F_0 - \alpha A)}{\Omega^2} \cos \Omega t} \quad \text{— (7)}$$

Putting $n=1$ in (4)

$$x''_2 = F_0 \cos \Omega t + \frac{\alpha\,(F_0 - \alpha A)}{\Omega^2} \cos \Omega t$$

Integrate

$$x_2' = \left(F_0 + \alpha \frac{(F_0 - \alpha A)}{\Omega^2}\right) \frac{\sin \Omega t}{\Omega}$$

$$x_2' = \left[\frac{F_0}{\Omega} + \frac{\alpha(F_0 - \alpha A)}{\Omega^3}\right] \sin \Omega t$$

Integrating again

$$x_2 = -\left[\frac{F_0}{\Omega} + \frac{\alpha(F_0 - \alpha A)}{\Omega^3}\right] \frac{\cos \Omega t}{\Omega}$$

$$\boxed{x_2 = -\left[\frac{F_0}{\Omega^2} + \frac{\alpha(F_0 - \alpha A)}{\Omega^4}\right] \cos \Omega t} \quad \text{—} (8)$$

So, we can substitute $n = 1, 2, 3, \ldots n$ and by observing $n = 0 \,\&\, 1$

we can say that;

$$\boxed{x_n(t) = \left\{\left(\frac{\alpha}{\Omega^2}\right)^n A - \frac{F_0}{\Omega^2}\left[1 + \left(\frac{\alpha}{\Omega^2}\right) + \cdots + \left(\frac{\alpha}{\Omega^2}\right)^{n-1}\right]\right\} \cos \Omega t} \quad \text{—} (9)$$

→ Now checking for exact solution;

$$x_0(t) = A \cos \Omega t \; ; \; x_1(t) = \frac{(\alpha A - F_0)}{\Omega^2} \cos \Omega t$$

Comparing coefficient of $\cos \Omega t$.

$$A = \frac{\alpha A}{\Omega^2} - \frac{F_0}{\Omega^2}$$

$$\frac{\alpha A}{\Omega^2} - A = \frac{F_0}{\Omega^2}.$$

$$A\left(\frac{\alpha - \Omega^2}{\cancel{\Omega^2}}\right) = \frac{F_0}{\cancel{\Omega^2}}$$

$$\boxed{\therefore A = \frac{F_0}{\alpha - \Omega^2}}$$

which is the exact solution

$$\boxed{x(t) = \frac{F_0}{(\alpha - \Omega^2)} \cos \Omega t}$$

(ii) For $\beta \neq 0$;

$$x'' + \alpha x + \beta x^3 = F_0 \cos \Omega t$$

We know;

$$x_0(t) = A \cos \Omega t$$

By using (4), Putting $n = 0$

$$x''_1 = F_0 \cos \Omega t - \alpha x_0 - \beta x_0^3.$$

$$x''_1 = F_0 \cos \Omega t - \alpha A \cos \Omega t - \beta A^3 \cos^3 \Omega t$$

Integrating twice

$$x_1 = \frac{-1}{\Omega^2}\left[-\alpha A - \frac{3\beta A^3}{4} + F_0\right] \cos(\Omega t)$$

Comparing coefficient of $\cos \Omega t$ from $x_0(t)$ & $x_1(t)$.

$$A = \frac{-1}{\Omega^2}\left[-\alpha A - \frac{3\beta A^3}{4} + F_0\right]$$

$$\boxed{\therefore \Omega^2 = \alpha + \frac{3}{4}\beta A^2 - \frac{F_0}{A}}$$

$\therefore$ we can see for $\beta \neq 0$, the amplitude frequency relation is satisfied.

Numerical Analysis for a Damped System:

MATLAB was chosen to model the Duffing equation. First, a function to solve the Duffing equation was created called **duffing.m** which defines the given variables for the equation (***r, α, β, F_0,** *and* **Ω**) and uses the formula as shown in **Equation 1** to solve for the velocity with respect to time (x_dot). With the function defined, the main script was created called **mainduffing.m** that calls the function for various ***F_0*** (The periodic driving force) while keeping all other variables constant and graphs the chosen displacement and time (***x*** and ***t***). Shown below in **Figures 3, 4, 5, and 6** are the MATLAB scripts written to model the Duffing function and the main script to graph ***x*** and ***t***:

```
function x_dot=duffing(t,x)

global r alpha beta Fo omega

x_dot(1)=-r*x(1)+alpha^2*x(2)-beta*x(2)^3+Fo*cos(omega*t);
x_dot(2)=x(1);
x_dot=x_dot';

end
```

Figure 3: MATLAB code for duffing.m function

```
close all; clear all; clc;

global r alpha beta Fo omega

r=0.23; alpha=-1; beta=1; omega=1.2;

%% Fo = 0.20
Fo=0.20;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(1)
plot(t(2000:4000),x(2000:4000,1),'g')
axis tight
title('Time Series')
figure(2)
plot(x(5000:10000,2),x(5000:10000,1),'y')
axis tight
title('Phase Space')
%% Fo = 0.218
Fo=0.218;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(3)
plot(t(2000:4000),x(2000:4000,1),'g')
axis tight
title('Time Series')
figure(4)
plot(x(5000:10000,2),x(5000:10000,1),'y')
axis tight
title('Phase Space')
```

Figure 4: MATLAB code for mainduffing.m script

```
%% Fo = 0.22
Fo=0.22;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(5)
plot(t(2000:4000),x(2000:4000,1),'g')
axis tight
title('Time Series')
figure(6)
plot(x(2000:10000,2),x(2000:10000,1),'y')
axis tight
title('Phase Space')
%% Fo = 0.37
Fo=0.37;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(7)
plot(t(2000:4000),x(2000:4000,1),'g')
axis tight
title('Time Series')
figure(8)
plot(x(2000:10000,2),x(2000:10000,1),'y')
axis tight
title('Phase Space')
```

Figure 5: MATLAB code for mainduffing.m script

```matlab
%% Fo = 0.50
Fo=0.50;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(9)
plot(t(2000:6000),x(2000:6000,1),'g')
axis tight
title('Time Series')
figure(10)
plot(x(2000:10000,2),x(2000:10000,1),'y')
axis tight
title('Phase Space')
%% Fo = 0.65
Fo=0.65;

[t x]=ode45(@duffing,0:2*pi/omega/100:4000,[0 1]);

figure(11)
plot(t(2000:4000),x(2000:4000,1),'g')
axis tight
title('Time Series')
figure(12)
plot(x(2000:10000,2),x(2000:10000,1),'y')
axis tight
title('Phase Space')
```

Figure 6: MATLAB code for mainduffing.m script

Results

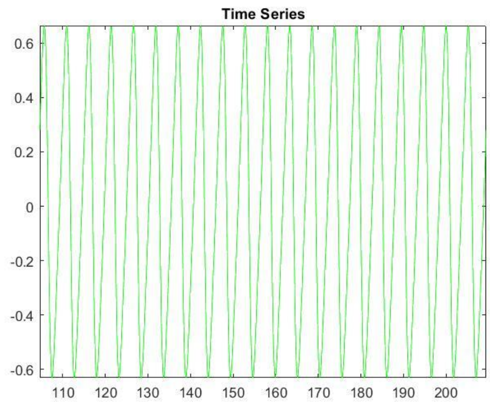

Figure 7: Time Series for Fo = 0.20

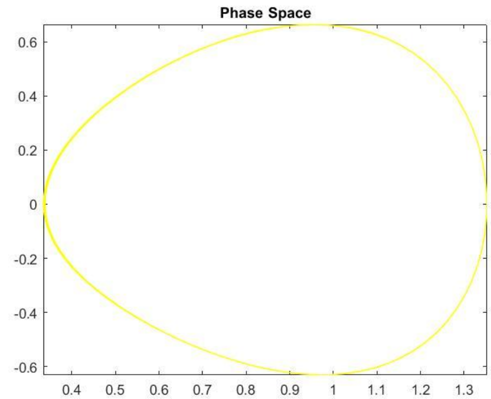

Figure 8: Phase Space for Fo = 0.20

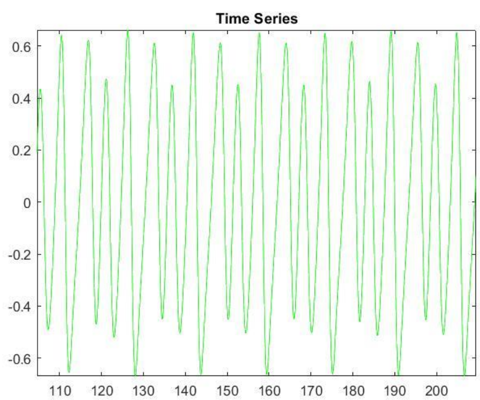

Figure 9: Time Series for Fo = 0.218

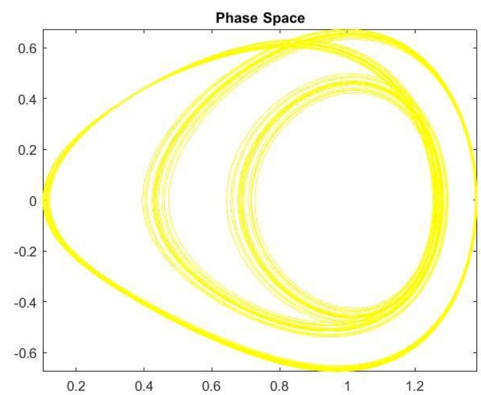

Figure 10: Phase Space for Fo = 0.218

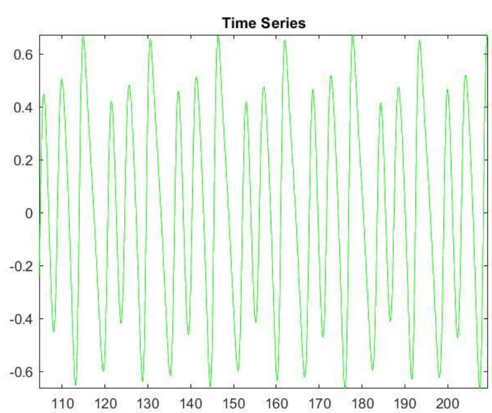

Figure 11: Time Series for Fo = 0.22

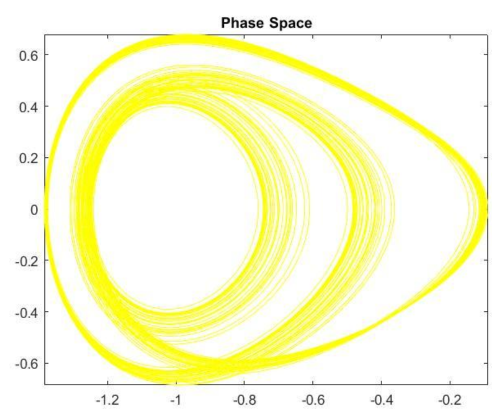

Figure 12: Phase Space for Fo = 0.22

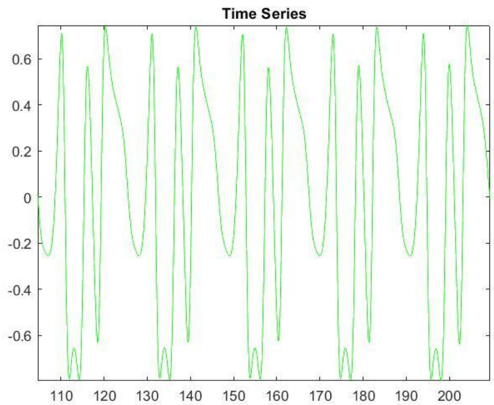

Figure 13: Time Series for Fo = 0.37

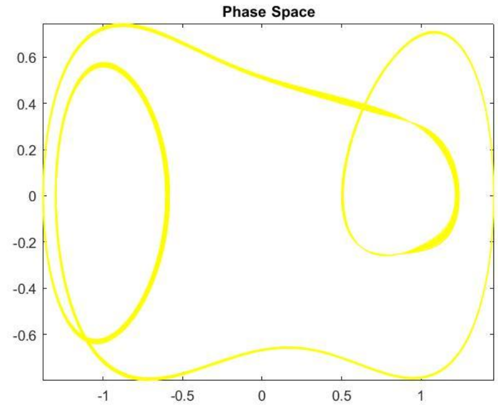

Figure 14: Phase Space for Fo = 0.37

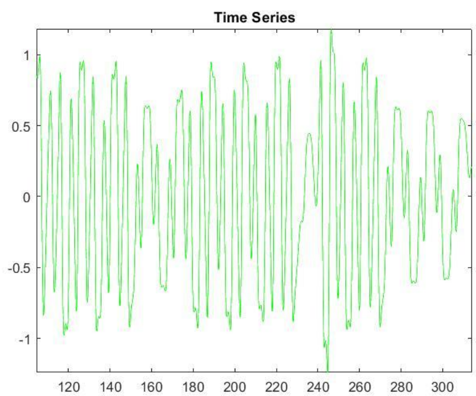

Figure 15: Time Series for Fo = 0.50

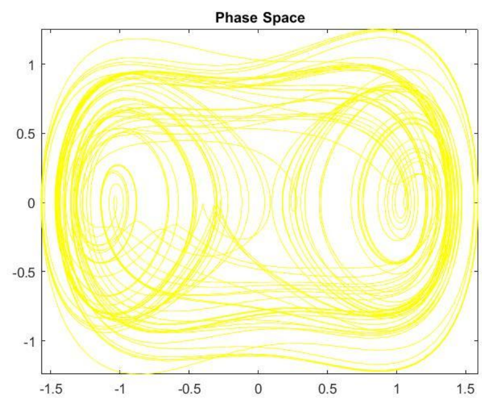

Figure 16: Phase Space for Fo = 0.50

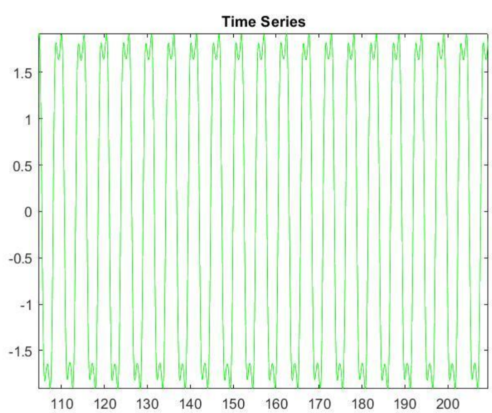

Figure 17: Time Series for Fo = 0.65

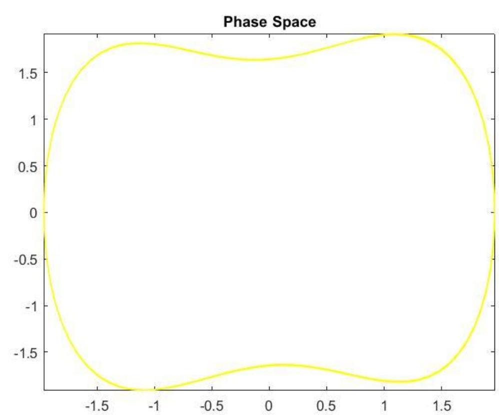

Figure 18: Phase Space for Fo = 0.65

Discussion

As seen in the numerical analysis, the duffing equation for a damped nonlinear system where ***r, α, β,*** *and* ***Ω*** are fixed and F_0 is varied, one can observe the changing amplitude vs *x* (position) and the time plot *x* vs *t;* this is done in order to observe the characteristics of the duffing equation and the chaos phenomena of the system.

As seen in **Figure 7** and **Figure 8**, when $F_0 = 0.20$, it is shown that after a brief transition period steady-state oscillation near x = 1 is reached. These oscillations are called Period-1 or Harmonic oscillations and persist for $F_0 < 0.218$.

For $F_0 = 0.218$ (refer to **Figure 9** and **Figure 10**) the solution is still periodic, but it takes more than one loop to complete one period. This is called a period-2 oscillation or subharmonic oscillation.

If one increases F_0 further to 0.22 (refer **Figure 11** and **Figure 12**), the force is sufficiently strong enough so that during the transient phase the mass is driven out of the right hand potential well and ends up in a period-4 oscillation about the left hand well.

Now increasing F_0 further to 0.37 (refer **Figure 13** and **Figure 14**), a period-5 oscillation occurs that now encompasses both stable equilibrium points.

Now the region where $0.37<F_0<0.65$ the system is found to be chaotic with random motions as a result (refer **Figure 15** and **Figure 16**,where $F_0= 0.5$).

Now as it gets out of the chaotic region that is $F_0 = 0.65$ (refer **Figure 17** and **Figure 18**), one once again obtains a periodic solution.

Therefore, one sees that the forced duffing equation admits a great variety of periodic and chaotic solutions, and these different regions correspond to different intervals on the F_0 axis.

Conclusion:

In conclusion, it is observed how the jump phenomena of the duffing equation works in a damped nonlinear system. The analytic analysis of the undamped duffing equation for both linear and non-linear systems proved a number of characteristics, and the numerical analysis of the damped nonlinear system using MATLAB observed physical properties of the duffing equation like ‘chaos’ by varying these characteristics ($\boldsymbol{r}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{F_0}$, *and* $\boldsymbol{\Omega}$). By observing all of the above properties of the duffing equation, one can observe these properties in real physical systems and predict the behavior of the systems in real life, like for instance a system using masses connected to non-linear springs, etc.

References

1. Greenberg, M.D. (1998) *Advanced Engineering Mathematics*, Pearson, NY. ISBN 13: 9780133214314 or 9788177585469 (International version)
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