Lecture notes on **Topological vector spaces**
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Abstract

These are lecture notes based on the course “Topological vector spaces” taught by Alexei Yu. Pirkovskii at the Faculty of Mathematics at HSE in the Fall Semester 2023.

Contents

Lecture 1 (2023.09.08)

Topological Vector Spaces

We will work on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers.

Definition. A topological vector space (tvs) is (X, τ), $X =$ a vector space over $\mathbb{K}$, $\tau =$ a topology on X s.t.

$$\left.\begin{array}{r} X \times X \to X, (x, y) \mapsto x + y \\ \mathbb{K} \times X \to X, (\lambda, x) \mapsto \lambda x \end{array}\right\} \text{are continuous.}$$

Topological vector spaces form a category and natural morphisms are continuous maps. So we will use the following notation.

Notation. $X, Y =$ tvs's.
$\mathscr{L}(X, Y) = \{$continuous linear maps $X \to Y\}$.
$X' = \mathscr{L}(X, \mathbb{K})$ is the dual of X.

Proposition. $X, Y =$ tvs's.

(1) A linear map $\varphi\colon X \to Y$ is continuous $\iff$ φ is continuous at 0.

(2) $\mathscr{L}(X, Y)$ is a vector subspace of $\operatorname{Hom}_{\mathbb{K}}(X, Y)$.

Proof. Exercise. □

Definition. $X =$ vector space. A seminorm on X is $p\colon X \to [0, +\infty)$ s.t.

(1) $p(x + y) \le p(x) + p(y)$ $(x, y \in X)$.

(2) $p(\lambda x) = |\lambda| p(x)$ $(x \in X, \lambda \in \mathbb{K})$.

$(X, p) =$ a seminormed space.

Exercise. A seminormed space is a tvs (in a standard way).

Definition. A polynormed space is (X, P), where X is a vector space, P is a family of seminorms on X.

Notation. $X =$ vector space. $p =$ a seminorm on X, $x \in X$, $\varepsilon > 0$.

$$U_{p,\varepsilon}(x) = \{y \in X : p(y - x) < \varepsilon\} \quad \text{the "open" ball}$$

$$\overline{U}_{p,\varepsilon}(x) = \{y \in X : p(y - x) \le \varepsilon\} \quad \text{the "closed" ball}$$

$$U_p = U_{p,1}(0), \quad \overline{U}_p = \overline{U}_{p,1}(0).$$

Definition. If (X, P) is a polynormed space, then the topology on X generated by P is the topology $\tau(P)$, whose subbase is

$$\{U_{p,\varepsilon} : p \in P, x \in X, \varepsilon > 0\}.$$

Exercise. $\overline{U}_{p,\varepsilon}(x)$ is the closure of $U_{p,\varepsilon}(x)$ in $\tau(P)$ $(x \in X, p \in P, \varepsilon > 0)$.

Proposition. (X, P) is a polynormed space $\implies$ $(X, \tau(P))$ is a tvs.

Proof. Exercise. □

Hint: reduce to seminormed space; $\forall p \in P, X_p = (X, p), i_p\colon X \to X_p, i_p(x) = x$. If Y is a topological space, then $f\colon Y \to X$ is continuous $\iff$ $i_p \circ f\colon Y \to X_p$ is continuous for every p.

Proposition. $(X, \tau(P))$ is Hausdorff $\iff$ $\forall 0 \ne x \in X, \exists p \in P$ s.t. $p(x) > 0$.

Proof. ($\implies$) otherwise $x \in$ every neighborhood of 0.
($\impliedby$) $x, y \in X, x \ne y, \exists p \in P, p(x - y) = d > 0$.

$$U_{p,\frac{d}{2}}(x) \cap U_{p,\frac{d}{2}}(y) = \varnothing.$$ □

Proposition. $x_n \to x$ in $\tau(P)$ $\iff$ $p(x_n - x) \to 0, \forall p \in P$.

Proof. Exercise. □

Example 1. $\mathbb{K}^X = \{\text{functions } X \to \mathbb{K}\}$ (X = a set). $\forall x \in X$, define $\|\cdot\|_x$ on $\mathbb{K}^X$ by $\|f\|_x = |f(x)|$. The family $\{\|\cdot\|_x : x \in X\}$ generates the topology of pointwise convergence on $\mathbb{K}^X$.

$$f_n \to f \text{ in } \mathbb{K}^X \iff f_n(x) \to f(x) \quad \forall x.$$

Exercise. This topology = the product topology on $\mathbb{K}^X$.

Example 2. $C(X)$ (X = a topological space). $\forall$ compact set $K \subset X$, define $\|\cdot\|_K$ on $C(X)$ by $\|f\|_K = \sup_{x\in K} |f(x)|$. The family $\{\|\cdot\|_K : K \subset X \text{ compact}\}$ generates the topology of compact convergence (a special case of the compact-open topology).

$$f_n \to f \text{ in } C(X) \iff f_n \underset{K}{\rightrightarrows} f \quad \forall \text{ compact } K \subset X.$$

Example 3. $C^\infty[a,b]$. $\forall n \in \mathbb{Z}_{\geq 0}$, define $\|\cdot\|_n$ on $C^\infty[a,b]$ by $\|f\|_n = \sup_{t\in[a,b]} |f^{(n)}(t)|$. The standard topology on $C^\infty[a,b]$ is generated by $\{\|\cdot\|_n : n \in \mathbb{Z}_{\geq 0}\}$.

$$f_k \to f \text{ in } C^\infty[a,b] \iff f_k^{(n)} \underset{[a,b]}{\rightrightarrows} f^{(n)} \quad \forall n.$$

Example 4. $C^\infty(U)$ ($U \subset \mathbb{R}^n$ open). $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}, f \in C^\infty(U), \mathrm{D}^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ ($|\alpha| = \alpha_1 + \cdots + \alpha_n$). $\forall$ compact $K \subset U$, $\alpha \in \mathbb{Z}^n_{\geq 0}$ define $\|\cdot\|_{K,\alpha}$ on $C^\infty(U)$ by $\|f\|_{K,\alpha} = \sup_{x\in K} |\mathrm{D}^\alpha f(x)|$. The standard topology on $C^\infty(U)$ is generated by $\left\{\|\cdot\|_{K,\alpha} : K \subset \text{ compact}, \alpha \in \mathbb{Z}^n_{\geq 0}\right\}$.

Exercise. Define a standard topology on $C^\infty(M)$, where M is a smooth manifold.

Example 5. The space of holomorphic functions $\mathcal{O}(U)$ ($U \subset \mathbb{C}^n$ open). The standard topology on $\mathcal{O}(U)$ is the topology of compact convergence induced from $C(U)$.

Exercise. This topology on $\mathcal{O}(U)$ = the topology induced from $C^\infty(U)$.

Example 6. The Schwartz space $\mathscr{S}(\mathbb{R}^n) = \left\{\varphi \in C^\infty(\mathbb{R}^n) : x^\alpha \mathrm{D}^\beta \varphi(x) \text{ is bounded } \forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0}\right\}$ (here $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). $\forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ define $\|\cdot\|_{\alpha,\beta}$ on $\mathscr{S}(\mathbb{R}^n)$ by $\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \mathrm{D}^\beta \varphi(x)|$. The standard topology on $\mathscr{S}(\mathbb{R}^n)$ is generated by $\left\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \in \mathbb{Z}^n_{\geq 0}\right\}$.

Example 7. The space of rapidly decreasing sequences

$$s = \left\{x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \text{the sequence } (x_n n^k)_{n\in\mathbb{N}} \text{ is bounded } \forall k \in \mathbb{Z}_{\geq 0}\right\}.$$

$\forall k \in \mathbb{Z}_{\geq 0}$ define $\|\cdot\|_k$ on s by $\|x\|_k = \sup_{n\in\mathbb{N}} |x_n| n^k$. The standard topology on s is given by $\{\|\cdot\|_k : k \in \mathbb{Z}_{\geq 0}\}$.

Example 8. X = tvs. $\forall f \in X'$ define $\|\cdot\|_f$ on X $\|x\|_f = |f(x)|$. The weak topology on X is generated by $\{\|\cdot\|_f : f \in X'\}$.

Example 9. X = tvs. $\forall x \in X$ define $\|\cdot\|_x$ on X' by $\|f\|_x = |f(x)|$. The weak* topology on X' is generated by $\{\|\cdot\|_x : x \in X\}$.

Remark. It is induced from $\mathbb{K}^X$.

Some notions related to convexity

X = vector space.

Definition. A subset $S \subset X$ is

(1) convex if $\forall x, y \in S$ we have $[x,y] \subset S$ where $[x,y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$.

(2) circled (balanced) if $\lambda S \subset S$ $\forall \lambda \in \mathbb{K}$ s.t. $|\lambda| \leq 1$.

(3) absolutely convex if S is convex and circled.

Definition. The convex hull of $S \subset X$ is

$$\operatorname{conv}(S) = \bigcap \{T \subset X \colon T \text{ is convex}, T \supset S\}.$$

This is the smallest convex set containing S.

Similarly, we may define the circled hull $\operatorname{circ}(S)$ and the absolutely convex hull $\Gamma(S)$.

Exercise.

(1) $\operatorname{conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \colon x_i \in S, \lambda_i \geq 0, \sum \lambda_i = 1 \right\}$.

(2) $\operatorname{circ}(S) = \{\lambda x \colon x \in S, |\lambda| \leq 1\}$.

(3) $\Gamma(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \colon x_i \in S, \lambda_i \in \mathbb{K}, \sum |\lambda_i| \leq 1 \right\}$.

Corollary.

(1) S is circled $\implies \operatorname{conv}(S)$ is circled.

(2) $\Gamma(S) = \operatorname{conv}(\operatorname{circ}(S))$.

Definition. $S, T \subset X$. S is absorbed by T (T absorbs S) if $\exists C \geq 0$ s.t. $S \subset \lambda T \ \forall |\lambda| \geq C$ ($S \prec T$).

Exercise. If T is circled, then $S \prec T \iff \exists \lambda \geq 0$ s.t. $S \subset \lambda T$.

Definition. A subset $S \subset X$ is absorbing if $\{x\} \prec S \ \forall x \in X$.

Example. A neighborhood of 0 in a tvs X is absorbing:

$$x \in \lambda U \iff \lambda^{-1} x \in U.$$

This is for large λ because $\mathbb{K} \times X \to X$ is continuous.

Definition. X = vector space, $S \subset X$ absorbing.
The Minkowski functional is $p_S \colon X \to [0, +\infty), p_S(x) = \inf \{\lambda \geq 0 \colon x \in \lambda S\}$.

Proposition.

(1) $p_S(\lambda x) = \lambda p_S(x) \ \forall \lambda \geq 0$.

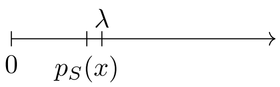
(2) If S is convex, then $p_S(x + y) \leq p_S(x) + p_S(y) \ \forall x, y$.

(3) If S is circled, then $p_S(\lambda x) = |\lambda| p_S(x) \ \forall \lambda \in \mathbb{K}, \forall x \in X$.

(4) If S is absolutely convex, then p_S is a seminorm. Moreover, $U_{p_S} \subset S \subset \overline{U}_{p_S}$.

Proof. (1) Exercise.

(2) Let's look at the picture:



$$\forall \varepsilon > 0 \quad \begin{cases} \exists \lambda \text{ s.t. } x \in \lambda S, \lambda < p_S(x) + \varepsilon, \\ \exists \mu \text{ s.t. } y \in \mu S, \mu < p_S(y) + \varepsilon. \end{cases}$$

$\implies x + y \in \lambda S + \mu S = (\lambda + \mu) S$ (by convexity). $\implies p_S(x+y) \leq \lambda + \mu < p_S(x) + p_S(y) + 2\varepsilon$ ($\forall \varepsilon > 0$).

(3) Exercise.

(4) $S \subset \overline{U}_{p_S}$ clear. If $x \in U_{p_S}$, then $\exists \lambda \in (0,1)$ s.t. $x \in \lambda S$, that is, $\lambda^{-1} x \in S \implies x \in S$. $\square$

Example. q = a seminorm. $p_{U_q} = p_{\overline{U}_q} = q$.

Lecture 2 (2023.09.15)

Locally convex spaces

Definition. $X =$ topological space, $x \in X$.

(1) A neighborhood of x is a set $U \subset X$ s.t. $x \in \operatorname{Int} U$ (i.e., $\exists$ an open $V \subset X$ s.t. $x \in V \subset U$).

(2) A family β_x of neighborhoods of x is a base of neighborhoods of x (a base at x) if $\forall$ neighborhood $U \ni x$ $\exists V \in \beta_x$ s.t. $V \subset U$.

(3) A family σ_x of neighborhoods of x is a subbase at x if $\{U_1 \cap \cdots \cap U_n \colon U_i \in \sigma_x, n \in \mathbb{N}\}$ is a base at x.

Notation. (X, P) polynormed. $x \in X, p_1, \ldots, p_n \in P, \varepsilon > 0$.

$$U_{p_1,\ldots,p_n,\varepsilon}(x) = \bigcap_{i=1}^{n} U_{p_i,\varepsilon}(x) = \{y \in X \colon p_i(y - x) < \varepsilon, \forall i = 1, \ldots, n\}.$$

Proposition.

(1) $\{U_{p,\varepsilon}(x) \colon p \in P, \varepsilon > 0\}$ is a subbase of neighborhoods of x.

(2) $\{U_{p_1,\ldots,p_n,\varepsilon}(x) \colon p_1, \ldots, p_n \in P, \varepsilon > 0\}$ is a base at x.

Proof. Exercise. $\square$

Observation. $U_{p_1,\ldots,p_n,\varepsilon}(x)$ is convex. If $x = 0$, then it is absolutely convex.

Definition. A tvs X is locally convex if X has a base of neighborhoods of 0 consisting of convex sets.

(lcs) locally convex space $=$ locally convex tvs.

Exercise. $X =$ tvs, $S \subset X$ convex $\implies \operatorname{Int} S$ is convex.

Corollary. X is locally convex $\iff$ X has a base of open neighborhoods of 0 consisting of convex sets.

Example. (X, P) polynormed $\implies (X, \tau(P))$ is a lcs.

Theorem. A tvs (X, τ) is locally convex $\iff \exists$ a family P of seminorms on X s.t. $\tau = \tau(P)$.

Definition. Any such P is a defining family of seminorms on X.

Lemma.

(1) $X =$ tvs $\implies$ each neighborhood $U \subset X$ of 0 contains a circled neighborhood of 0.

(2) $X =$ lcs $\implies$ each neighborhood of 0 contains an absolutely convex neighborhood of 0.

Proof.

(1)
$$\left.\begin{aligned} \mathbb{K} \times X \to X &\text{ is continuous} \\ (\lambda, x) &\mapsto \lambda x \\ \overline{\mathbb{D}} = \{z \in \mathbb{K} &\colon |z| \le 1\} \end{aligned}\right\} \implies \exists \text{ a neighborhood } V \subset X \text{ of } 0 \text{ s.t. } \overline{\mathbb{D}} \cdot V \subset U.$$

Let $W = \overline{\mathbb{D}} \cdot V$. W is circled, $W \subset U$.

(2) We may assume that U is convex. $\exists$ a circled neighborhood W of 0, $W \subset U$. Let $W' = \operatorname{conv}(W) \implies W'$ is absolutely convex, $W' \subset U$. $\square$

Corollary. A tvs X is locally convex $\iff$ X has a base of neighborhood of 0 consisting of absolutely convex sets.

Exercise. We can replace neighborhoods by open neighborhoods, we still get the equivalent definition.

Proof of Theorem. (X,τ) = lcs. β = a base of neighborhoods of 0 consisting of absolutely convex sets. $P=\{p_V : V\in\beta\}$. We know: $\forall V\in\beta, U_{p_V}\subset V\subset U_{p_V,2}$.

$$\left.\begin{array}{c} U_{p_V}\subset V \implies (X,\tau(P))\to(X,\tau) \text{ is continuous} \\ x\mapsto x \\ \dfrac{\varepsilon}{2}V\subset U_{p_V,\varepsilon} \implies (X,\tau)\to(X,\tau(P)) \text{ is continuous} \\ x\mapsto x \end{array}\right\} \implies \tau=\tau(P).$$

□

A continuity criterion for linear operators between lcs's

Motivation. X,Y normed, $T\colon X\to Y$ linear. T is continuous $\iff \exists C\geq 0$ s.t. $\forall x\in X, \|Tx\|\leq C\|x\|$.

X = vector space; p,q = seminorms on X.

Definition. q is dominated by p ($q\prec p$) $\iff \exists C\geq 0$ s.t. $q(x)\leq Cp(x)\ \forall x$.

Proposition.

(1) $q\leq p \iff U_p\subset U_q$.

(2) $q\prec p \iff U_p\prec U_q$.

Proof. Exercise. □

Notation. $\max\{p,q\}\colon X\to[0,+\infty), x\mapsto \max\{p(x),q(x)\}$.

Proposition.

(1) $\max\{p,q\}$ is a seminorm.

(2) $U_{\max\{p,q\}}=U_p\cap U_q$.

Theorem 1. X = tvs, q = a seminorm on X. Then the following conditions are equivalent (TFAE):

(1) q is continuous.

(2) q is continuous at 0.

(3) U_q is open (in τ).

(4) U_q is a neighborhood of 0 (in τ).
Moreover, if X is locally convex, and P is a defining family of seminorms on X, then (1)-(4) are equivalent to

(5) $\exists p_1,\dots,p_n\in P$ s.t. $q\prec \max\limits_{1\leq i\leq n} p_i$.

Proof. (1) $\implies$ (3) $\implies$ (4) clear.
(4) $\implies$ (2) $q(U_q)\subset[0,1) \implies q(\varepsilon U_q)\subset[0,\varepsilon), \varepsilon U_q$ is a neighborhood of 0 $\implies$ q is continuous at 0.
(2) $\implies$ (1) $\forall\varepsilon>0\ \exists$ a neighborhood $U\subset X$ of 0 s.t. $q(U)\subset[0,\varepsilon)$.
$\forall x\in X, x+U$ is a neighborhood of x. $\forall y\in x+U, |q(y)-q(x)|\leq q(x-y)<\varepsilon \implies q$ is continuous at x.
Assume that P is a defining family of seminorms.

$$\begin{aligned}(4) &\iff \exists\varepsilon>0,\exists p_1,\dots,p_n\in P \text{ s.t. } U_q\supset U_{p_1,\dots,p_n,\varepsilon}(0)=U_{p,\varepsilon}(0)=U_{\frac{p}{\varepsilon}}\ (p=\max_{1\leq i\leq n}p_i)\\ &\iff q\leq\frac{p}{\varepsilon}\\ &\iff (5)\end{aligned}$$

□

Notation. $\mathrm{SN}(X)=\{$continuous seminorms on $X\}$.

Corollary. $(X,\tau(P))$ = lcs.

(1) $P\subset\mathrm{SN}(X)$.

(2) $\mathrm{SN}(X)$ is the largest defining family of seminorms on X.

Theorem 2. $(X, \tau(P)), (Y, \tau(Q)) =$ lcs's. $T\colon X \to Y$ linear. TFAE:

(1) T is continuous.

(2) $\forall q \in Q, q \circ T\colon X \to [0, +\infty)$ is continuous.

(3) $\forall q \in Q, \exists C \geq 0, \exists p_1, \ldots, p_n \in P$ s.t. $\forall x \in X, q(Tx) \leq C \max\limits_{1 \leq i \leq n} p_i(x)$.

Proof. (2) $\iff$ (3): apply the previous theorem to $q \circ T$.
(1) $\implies$ (2) clear.
(2) $\implies$ (1)

$$T^{-1}(U_q) = U_{q \circ T} \text{ is open} \implies \forall \varepsilon > 0, T^{-1}(U_{q,\varepsilon}) \text{ is open} \implies T \text{ is continuous at } 0 \implies T \text{ is continuous.}$$ ☐

Observation. $(X, \tau(P)) =$ lcs; $X_0 \subset X$ vector subspace $\implies$ the topology on X_0 induced from X is generated by $\{p|_{X_0} \colon p \in P\}$.

Theorem 3. $X =$ lcs, $X_0 \subset X$ vector subspace $\implies \forall f_0 \in X_0' \ \exists f \in X'$ s.t. $f|_{X_0} = f_0$.

Corollary (from Theorem 2)**.** $(X, \tau(P)) =$ lcs, $f\colon X \to \mathbb{K}$ linear. TFAE:

(1) f is continuous.

(2) $|f|$ is continuous.

(3) $\exists p_1, \ldots, p_n \in P$ s.t. $|f| \prec \max\limits_{1 \leq i \leq n} p_i$.

Proof of Theorem 3. $\exists$ a continuous seminorm p on X s.t. $|f_0(x)| \leq p(x) \ \forall x \in X_0$.
Hahn-Banach $\implies \exists$ linear $f\colon X \to \mathbb{K}$ s.t. $f|_{X_0} = f_0$ and $|f(x)| \leq p(x) \ \forall x \in X$. Corollary $\implies f$ is continuous. ☐

Corollary 1. $X =$ Hausdorff lcs $\implies \forall 0 \neq x \in X \ \exists f \in X'$ s.t. $f(x) \neq 0$.

Proof. $\exists$ a continuous seminorm p on X s.t. $p(x) > 0$. Define $f_0\colon \mathbb{K}X \to \mathbb{K}, f_0(x) = p(x)$. $\implies \forall y \in \mathbb{K}X, |f_0(y)| = p(y)$. $\implies f_0$ is continuous. It remains to apply Theorem 3. ☐

Corollary 2. $X =$ Hausdorff lcs $\implies \forall x, y \in X, x \neq y, \exists f \in X'$ s.t. $f(x) \neq f(y)$.

Example. $(X, \mu) =$ measure space; For $0 < p < 1$, consider $L^p(X, \mu)$. $\forall f \in L^p(X, \mu)$ let $|f|_p = \int_X |f|^p d\mu$.

Exercise.

(1) $\rho(f, g) = |f - g|_p$ is a metric on L^p.

(2) $L^p(X, \mu)$ is a tvs for the topology determined by ρ.

(3) $(L^p[0,1])' = 0$.

Lecture 3 (2023.09.22)

Recall:

Theorem. $(X, \tau(P)), (Y, \tau(Q)) =$ lcs's. $T\colon X \to Y$ linear. TFAE:

(1) T is continuous.

(2) $\forall q \in Q, q \circ T\colon X \to [0, +\infty)$ is continuous.

(3) $\forall q \in Q, \exists C \geq 0, \exists p_1, \ldots, p_n \in P$ s.t. $\forall x \in X, q(Tx) \leq C \max\limits_{1 \leq i \leq n} p_i(x)$.

Example/Exercise 1 (linear differential operator on $C^\infty(U)$; $U \subset \mathbb{R}^n$ open)**.**

$$T = \sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ |\alpha| \leq N}} a_\alpha \mathrm{D}^\alpha \colon C^\infty(U) \to C^\infty(U),$$

where $a_\alpha \in C^\infty(U)$, is a continuous linear operator.

Example/Exercise 2. Let $U \subset \mathbb{C}$ open.

$$T = \sum_{k=0}^{N} a_k \frac{\mathrm{d}}{\mathrm{d}z^k} \colon \mathcal{O}(U) \to \mathcal{O}(U), a_k \in \mathcal{O}(U),$$

is a continuous linear operator.

Example/Exercise 3. Consider

$$s(\mathbb{Z}) = \left\{ x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \colon \forall k \in \mathbb{N}, \|x\|_k = \sup_{n \in \mathbb{Z}} |x_n||n|^k < \infty \right\}, \mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}.$$

The Fourier transform $\mathscr{F}\colon C^\infty(\mathbb{T}) \to s(\mathbb{Z}), f \mapsto \hat{f}$,

$$\hat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} \mathrm{d}\mu(z) \quad \left(\mu = \text{ normalized length measure on } \mathbb{T} = \frac{\text{length}}{2\pi}\right).$$

Prove: $\mathscr{F}$ takes $C^\infty(\mathbb{T})$ to $s(\mathbb{Z})$, and it is a topological isomorphism.

Definition. $X, Y =$ tvs, $T\colon X \to Y$ linear. T is

(1) a topological isomorphism if T is a homeomorphism of X onto Y.

(2) topological injective if T is a homeomorphism of X onto $T(X)$.

(3) open if $\forall$ open $U \subset X$ $T(U)$ is open in Y.

Exercise 1. open $\implies$ surjective.

Exercise 2. Characterize topological injective and open linear maps between locally convex spaces in terms of seminorms.

Definition. $X =$ vector space, P, Q are the families of seminorms on X.
Q is dominated by P ($Q \prec P$) if $\tau(Q) \subset \tau(P)$.
Q and P are equivalent ($Q \sim P$) if $\tau(Q) = \tau(P)$.

Corollary 1. TFAE:

(1) $Q \prec P$;

(2) every $q \in Q$ is $\tau(P)$-continuous

(3) $\forall q \in Q$ $\exists p_1, \ldots, p_n \in P$ s.t. $q \prec \max\limits_{1 \leq i \leq n} p_i$.

Proof. Consider $(X, \tau(P)) \xrightarrow{I} (X, \tau(Q)), x \mapsto x$. (1) $\iff$ I is continuous $\iff$ (2) $\iff$ (3). $\square$

Example/Exercise (2 equivalent families of seminorms on $\mathcal{O}(\mathbb{D}_R)$, where $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$).
Consider $P = \{\|\cdot\|_K : K \subset \mathbb{D}_R \text{ compact}\}$ on $\mathcal{O}(\mathbb{D}_R)$, where $\|f\|_K = \sup_{z \in K} |f(z)|$.

$\forall n \in \mathbb{Z}_{\geq 0}$ let $c_n(f) = \frac{f^{(n)}(0)}{n!}$. $\forall r \in (0, R)$ define $\|\cdot\|_r$ on $\mathcal{O}(\mathbb{D}_R)$ by $\|f\|_r = \sum_{n=0}^{\infty} |c_n| r^n$.

Consider $Q = \{\|\cdot\|_r : r \in (0, R)\}$.
Prove: $Q \sim P$.
Hint:

$$\|\cdot\|_K \leq \|\cdot\|_r, \quad \text{where } K \subset \overline{\mathbb{D}}_r, r < R.$$

$$\|\cdot\|_r \prec \|\cdot\|_K, \quad \text{where } K = \overline{\mathbb{D}}_\rho, r < \rho < R.$$

Just use the Cauchy inequalities.

Corollary 2. X = a vector space.

(1) $\exists$ the strongest locally convex topology on X. It is given by the family of all seminorms on X. It is Hausdorff.

(2) $\exists$ the weakest locally convex topology on X. It is generated by $\{0\}$ (= antidiscrete topology). It is not Hausdorff unless $X = 0$.

(3) $\forall$ lcs Y

$$\mathrm{Hom}_{\mathbb{K}}(X, Y) = \mathscr{L}(X_{\text{strongest}}, Y)$$
$$\mathrm{Hom}_{\mathbb{K}}(Y, X) = \mathscr{L}(Y, X_{\text{weakest}}).$$

In other words, from the viewpoint of category theory, above two qualities imply exactly that the forgetful functor LCS $\to$ Vect from the category of locally convex spaces to the category of vector spaces has a left joint and right joint. The first one is the left joint and the second one is the right joint.

Definition. P = a family of seminorms on a vector space X. P is directed if $\forall p, q \in P \; \exists r \in P$ s.t. $p \prec r, q \prec r$.

Notation. $(X, \tau(P))$ = lcs.
$\forall x \in X, \sigma_x = \{U_{p,\varepsilon}(x) : p \in P, \varepsilon > 0\}$.

Proposition.

$$\begin{aligned} P \text{ is directed} &\iff \sigma_x \text{ is a base at } x \\ &\iff \sigma_0 \text{ is a base at } 0. \end{aligned}$$

Observation. Every family of seminorms P is equivalent to a directed family.
Indeed: let

$$P_\infty = \left\{ \max_{1 \leq i \leq n} p_i : p_i \in P, n \in \mathbb{N} \right\}$$
$$P_1 = \left\{ \sum_{i=1}^{n} p_i : p_i \in P, n \in \mathbb{N} \right\}$$

We have $P \subset P_\infty \subset \mathrm{SN}(X, \tau(P))$ and $P \subset P_1 \subset \mathrm{SN}(X, \tau(P)) \implies P \sim P_1 \sim P_\infty \sim \mathrm{SN}(X)$. On the other hand, $P_1, P_\infty, \mathrm{SN}(X)$ are directed.

Finite-dim, normable, metrizable lcs's

Recall: any two norms on a finite-dim vector space X are equivalent, and they generate the standard topology τ_{std} on X.

Proposition 1. X = finite-dim vector space.
Then τ_{std} is the only Hausdorff locally convex topology on X.

Proof. Let $e_1, \dots, e_n$ = a basis in X.

$$\|x\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^n |x_i| \quad \text{if } x = \sum_{i=1}^n x_i e_i.$$

$\forall$ seminorm p on X we have $p \prec \|\cdot\|_1$. $\implies$ $\forall$ family P of seminorms on X, $\tau(P) \subset \tau_{\text{std}}$.

Assume $\tau(P)$ is Hausdorff.

Claim. $\exists$ a $\tau(P)$-continuous norm on X.

Induction over $\dim X$. Choose any $p_1 \in P, p_1 \neq 0, X_1 = p_1^{-1}(0)$. $\dim X_1 < \dim X \implies \exists \tau(P|_{X_1})$-continuous norm on $X_1 \implies \exists p_2 \in P$ s.t. $p_2|_{X_1}$ is a norm.

$p = \max\{p_1, p_2\}$ is a $\tau(P)$-continuous norm on X. $\implies \tau_{\text{std}} \supset \tau(P) \supset \tau(\{p\}) = \tau_{\text{std}}$. ☐

Exercise.

(1) A Hausdorff tvs is finite-dim $\iff$ it is locally compact $\iff$ it has a compact neighborhood of 0.

(2) Extend Proposition 1 to Hausdorff vector space topologies.

Proposition 2. $(X, \tau(P))$ = lcs.

X is seminormable $\iff$ $\exists$ a finite $Q \subset P, Q \sim P$.

Proof. ($\Longrightarrow$) p = a defining seminorm. $\underset{\text{Corollary 1}}{\Longrightarrow}$ $\exists$ finite $Q \subset P$ s.t. $P \prec Q$. Q is what we need.

($\Longleftarrow$) $Q = \{p_1, \dots, p_n\} \sim P$. $p = \max_{1 \le i \le n} p_i$ is a defining seminorm. ☐

Definition. X = tvs.

$B \subset X$ is bounded if $\forall$ neighborhood $U \subset X$ of 0 we have $B \prec U$.

Observation. $(X, \tau(P))$ = lcs.

$$\begin{aligned} B \subset X \text{ is bounded} &\iff B \prec U_{p,\varepsilon} (\forall p \in P, \forall \varepsilon > 0) \\ &\iff B \prec U_p (\forall p \in P) \\ &\iff \sup_{x \in B} p(x) < \infty \quad \forall p \in P. \end{aligned}$$

Proposition. (obvious)

(1) $\forall x \in X, \{x\}$ is bounded.

(2) $B_1, B_2 \subset X$ bounded $\implies B_1 \cup B_2$ bounded.

(3) $B_1 \subset B_2 \subset X, B_2$ is bounded $\implies B_1$ is bounded.

In particular, (1)&(2) $\implies$ all finite sets are bounded.

Exercise. compact $\implies$ bounded and the converse direction is not true.

Proposition (Kolmogorov's criterion)**.** X = lcs.

X is seminormable $\iff$ X has a bounded neighborhood of 0.

Proof. ($\Longrightarrow$) clear.

($\Longleftarrow$) $B \subset X$ an absolutely convex bounded neighborhood of 0.

$$p_B = \text{ the Minkowski functional of } B.$$

Claim. $\tau(\{p_B\}) = \tau$ (where τ is the original topology on X).

B is a neighborhood of 0 $\implies p_B$ is continuous $\implies \tau(\{p_B\}) \subset \tau$.

B is bounded $\implies \forall$ continuous seminorm p on X, $B \subset U_p \implies p \prec p_B \implies \tau \subset \tau(\{p_B\})$. ☐

Definition. X = vector space, $p \colon X \to [0, +\infty)$.

p is an F-seminorm (named after Fréchet) if

(1) $p(x+y) \leq p(x) + p(y) \quad (x, y \in X)$.

(2) $p(\lambda x) \leq p(x) \quad (x \in X, |\lambda| \leq 1)$.

(3) $\forall$ sequence (λ_n) in $\mathbb{K}$ s.t. $\lambda_n \to 0$, $\forall x$ we have $p(\lambda_n x) \to 0$.

If $p(x) > 0$ for all $x \neq 0$, then p is an F-norm.

Example/Exercise. (X, μ) = measure space. For $0 < p < 1$, consider $L^p(X, \mu)$.

$$|f|_p \stackrel{\text{def}}{=} \int_X |f|^p \mathrm{d}\mu \text{ is an F-norm on } L^p(X, \mu).$$

Observation. If p is an F-(semi)norm, then $\rho(x, y) = p(x - y)$ is a (semi)metric.

Exercise. (X, τ_ρ) is a tvs.

Definition. X = topological space.
X is 1st countable if $\forall x \in X$ $\exists$ an at most countable base at x.

Example. (semi)metrizable $\iff$ 1st countable.

Exercise. X, Y = topological space, X is 1st countable.
$f\colon X \to Y$ is continuous at $x \in X$ $\iff$ it is sequentially continuous at x, that is, if (x_n) is a sequence in X, $x_n \to x$, then $f(x_n) \to f(x)$.

Proposition. $(X, \tau(P))$ = lcs. TFAE:

(1) X is 1st countable.

(2) X is semimetrizable.

(3) The topology on X can be generated by a shift invariant semimetric.

(4) X is F-seminormable.

(5) The topology on X can be generated by countably many seminorms.

(6) $\exists$ an at most countable $Q \subset P, Q \sim P$.

Proof. (4) $\implies$ (3) $\implies$ (2) $\implies$ (1) clear.
(1) $\implies$ (5) almost clear.
(5) $\implies$ (6) Suppose $Q' \sim P, Q'$ is at most countable.
$\forall q \in Q'$ $\exists$ finite $P_q \subset P$ s.t. $q \prec P_q$. $\implies Q = \bigcup\limits_{q \in Q} P_q$ is what we need.
(6) $\implies$ (5) clear.
(5) $\implies$ (4) $\{\|\cdot\|_n \colon n \in \mathbb{N}\}$ = a defining family.
Choose $\varphi\colon [0, +\infty) \to [0, +\infty)$ s.t. $\varphi \nearrow, \varphi(0) = 0, \varphi$ is bounded, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$ $\forall s, t$ and φ is a homeomorphism between $[0, \delta]$ and $[0, \varepsilon]$ for some $\delta, \varepsilon > 0$. For example: $\varphi(t) = \frac{t}{1+t}$ or $\varphi(t) = \min\{t, 1\}$.
Define $|x| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \varphi(\|x\|_n)$.

Exercise. $|\cdot|$ is an F-seminorm.

Claim. $|\cdot|$ determines the same topology.

It suffices to show: $x_k \underset{\tau}{\to} 0 \iff x_k \underset{\tau_F}{\to} 0$ (τ = original topology, τ_F is generated by $|\cdot|$).
($\Longrightarrow$) $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $\sum\limits_{n > N} \frac{1}{2^n} < \frac{\varepsilon}{2}$.
$|x_k| \leq \sum\limits_{n=1}^{N} \frac{1}{2^n} \varphi(\|x_k\|_n) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $(k \geq k_0)$.
($\Longleftarrow$) $x_k \underset{\tau_F}{\to} 0 \implies \forall n\ \|x_k\|_n \to 0 \implies x_k \underset{\tau}{\to} 0$. □

Example/Exercise. Let $X \subset \mathbb{R}^n$ open.
$C(X)$ is metrizable, but $\nexists$ continuous norm on $C(X)$ ($\implies$ it is not normable).

Lecture 4 (2023.09.29)

Quotients

$X = $ tvs, $X_0 \subset X$ vector subspace.

$$Q\colon X \to X/X_0, x \mapsto x + X_0.$$

Definition. The quotient topology on X/X_0 is:

$$U \subset X/X_0 \text{ is open} \overset{\text{def}}{\iff} Q^{-1}(U) \text{ is open in } X.$$

Proposition.

(1) X/X_0 is a tvs.

(2) $Q\colon X \to X/X_0$ is continuous and open.

(3) If β is a base at $0 \in X$, then $\{Q(U)\colon U \in \beta\}$ is a base at $0 \in X/X_0$.

(4) X/X_0 is Hausdorff $\iff$ X_0 is closed in X.

(5) X is locally convex $\implies$ so is X/X_0.

Proof. Exercise. □

Definition. $X = $ vector space, $X_0 \subset X$ vector subspace, $p = $ a seminorm on X.

The quotient seminorm of p is

$$\hat{p}\colon X/X_0 \to [0, +\infty), \hat{p}(x + X_0) = \inf \{p(x + y)\colon y \in X_0\}.$$

Exercise. $\hat{p}$ is a seminorm.

Observation. Let $\rho_p(x, y) = p(x - y)$. Then $\hat{p}(x + X_0) = \inf\limits_{y \in X_0} \rho_p(x, y) = \operatorname{dist}_{\rho_p}(x, X_0)$.

Proposition. $(X, \tau(P)) = $ lcs, P is a directed defining family of seminorms on X. Then the quotient topology on X/X_0 is generated by $\{\hat{p}\colon p \in P\}$.

Proof. $Q(U_{p,\varepsilon}) = U_{\hat{p},\varepsilon}$ (exercise). □

Theorem 1 (universal property of quotients)**.** $X = $ tvs, $X_0 \subset X$ vector subspace.

Then $\forall$ tvs $Y, \forall$ continuous linear $T\colon X \to Y$ s.t. $T(X_0) = 0$ $\exists$ a unique continuous linear $\widehat{T}$ s.t. the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "Q"'] & Y \\
X/X_0 \arrow[ur, "\widehat{T}"'] &
\end{tikzcd}$$

Proof. The existence and the uniqueness of such linear map $\widehat{T}$ are known from the algebra course. To show $\widehat{T}$ is continuous, we just observe the following $Q^{-1}(\widehat{T}^{-1}(U)) = T^{-1}(U)$. □

Theorem 1′ (equivalent to Theorem 1)**.** $\forall$ tvs Y the map

$$\begin{gathered}\mathscr{L}(X/X_0, Y) \to \{T \in \mathscr{L}(X, Y)\colon T(X_0) = 0\} \\ S \mapsto S \circ Q\end{gathered}$$

is bijective.

Theorem 2. Let $X, X_0, Y, T, \widehat{T}$ be as in Theorem 1. Then

(1) $\widehat{T}$ is open $\iff$ T is open;

(2) $\widehat{T}$ is a topological isomorphism $\iff$ T is open, and $X_0 = \operatorname{Ker} T$.

Proof. Exercise. □

Remark. $X/\operatorname{Ker} T \not\cong T(X) \subset Y$!

This is true iff T is an open map $X \to T(X)$.

Kernels and cokernels

$\mathscr{A}$ = a category with 0; $\varphi\colon X \to Y$ morphism in $\mathscr{A}$.

Definition. The kernel of φ is $(K,k), K \in \operatorname{Ob}\mathscr{A}, k\colon K \to X$ s.t.

(1) $\varphi k = 0$

(2) $\forall Z, \forall \psi\colon Z \to X$ s.t. $\varphi\psi = 0$ $\exists$ a unique $\chi\colon Z \to K$ s.t. the following diagram commutes.

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "\varphi"] & Y \\ Z \arrow[u, "\chi"] \arrow[ur, "\psi"'] & & \end{tikzcd}$$

Notation. $K = \operatorname{Ker}\varphi, k = \ker\varphi$.

Example. $\mathscr{A} \in \{\mathrm{Ab}, \mathrm{Vect}, \mathrm{Set}_*\}$, where Set_* = the category of sets with a basepoint.
Every $\varphi\colon X \to Y$ has a kernel, $\operatorname{Ker}\varphi = \varphi^{-1}(0), k\colon K \hookrightarrow X$ inclusion map.

Definition′. The kernel of $\varphi\colon X \to Y$ in $\mathscr{A}$ is $\operatorname{Ker}\varphi \in \operatorname{Ob}\mathscr{A}$ together with a natural isomorphism

$$\operatorname{Hom}(Z, \operatorname{Ker}\varphi) \cong \operatorname{Ker}(\operatorname{Hom}(Z,X) \to \operatorname{Hom}(Z,Y))$$

Exercise. Definition $\iff$ Definition′.

Exercise. Every morphism $\varphi\colon X \to Y$ in the category TVS of topological vector spaces has a kernel, and $\operatorname{Ker}\varphi = (\varphi^{-1}(0);\ \text{subspace topology}), \ker\varphi\colon\ \operatorname{Ker}\varphi \hookrightarrow X$ inclusion.
This same holds for the following full subcategories of TVS: the category HTVS of Hausdorff lcs's; the category LCS of locally compact spaces (l.c.s's); the category HLCS of Hausdorff lcs's.

Definition. $\mathscr{A}$ = a category with 0; $\varphi\colon X \to Y$ morphism in $\mathscr{A}$.
The cokernel of φ is (C,c) where $C \in \operatorname{Ob}\mathscr{A}, c\colon Y \to C$ s.t. (C,c) is the kernel of φ in $\mathscr{A}^{\mathrm{op}}$.

Notation. $C = \operatorname{Cokernel}\varphi, c = \operatorname{cokernel}\varphi$.

Definition′. The cokernel of φ is $\operatorname{Cokernel}\varphi \in \operatorname{Ob}\mathscr{A}$ together with a natural isomorphism

$$\operatorname{Hom}(C,Z) = \operatorname{Ker}(\operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z))$$

Exercise. Definition $\iff$ Definition′.

Example/Exercise.

(1)

$$\begin{aligned} \mathscr{A} \in \{\mathrm{Ab}, \mathrm{Vect}\} \implies & \operatorname{Cokernel}\varphi = Y/\varphi(X); \\ & \operatorname{cokernel}\varphi\colon y \mapsto y + \varphi(X). \end{aligned}$$

(2) $\mathscr{A} \in \{\mathrm{TVS}, \mathrm{LCS}\} \implies \operatorname{Cokernel}\varphi = (Y/\varphi(X), \text{quotient topology})$.

(3) $\mathscr{A} \in \{\mathrm{HTVS}, \mathrm{HLCS}\} \implies \operatorname{Cokernel}\varphi = (Y/\overline{\varphi(X)}, \text{quotient topology})$.

Projective and inductive locally convex topologies

X = vector space; $(X_i)_{i\in I}$ = a family of lcs's, $(\varphi_i\colon X \to X_i)$ a family of linear maps.

Definition. The projective locally convex topology on X generated by (φ_i) is the locally convex topology on X generated by

$$P_{\mathrm{proj}} = \{p \circ \varphi_i\colon i \in I, p \in \mathrm{SN}(X_i)\}\,. \qquad (*)$$

Proposition.

(1) We can replace $\mathrm{SN}(X_i)$ in $(*)$ by any defining family of seminorms on X_i.

(2) $\tau(P_{\text{proj}})$ is the weakest locally convex topology on X making all φ_i continuous.

(3) $\tau(P_{\text{proj}})$ is the weakest topology on X making all φ_i continuous.

(4) $\tau(P_{\text{proj}})$ is the only locally convex topology on X with the following property: if Y is a lcs, $\varphi\colon Y \to X$ linear, then

$$\varphi \text{ is continuous } \iff \varphi_i \circ \varphi \text{ is continuous } \forall i \in I.$$

$$Y \xrightarrow{\varphi} X \xrightarrow{\varphi_i} X_i$$

(5) If σ_i is a subbase at $0 \in X_i (i \in I)$, then

$$\left\{\varphi_i^{-1}(U)\colon i \in I, U \in \sigma_i\right\}$$

is a subbase at $0 \in X$.

Proof. Exercise. □

Example 1. The weakest locally convex topology (anti-discrete) on X = the projective locally convex topology generated by $X \to 0$.

Example 2. $X = \text{lcs}, X_0 \subset X$ vector subspace.
The topology on X_0 induced from X = projective locally convex topology generated by $X_0 \hookrightarrow X$.

Example 3. $X = \text{tvs}$.
The weak topology on X = the projective locally convex topology generated by $\{f\colon X \to \mathbb{K}\colon f \in X'\}$.

Notation. p = a seminorm on a vector space X.
Observe: $p^{-1}(0) \subset X$ is a vector subspace.
$X_p = X/p^{-1}(0)$ is a normed space with respect to the quotient norm $\hat{p}$ of p; $\hat{p}(x+p^{-1}(0)) = p(x)$. (exercise)

Example 4. $(X, \tau(P)) = \text{lcs}$.
$\tau(P)$ = the projective locally convex topology generated by

$$\{\pi_p\colon X \to X_p\colon p \in P\}, \quad \pi_p(x) = x + p^{-1}(0).$$

Moreover, we can replace X_p by any normed space $Y_p \supset X_p$.

Special case. X = topological space; $K \subset X$ compact. $r_K\colon C(X) \to C(K)$ restriction map. $C(K)$ is a Banach space (with respect to the sup norm).
The topology of compact convergence on $C(X)$ = the projective locally convex topology generated by $\{r_K\colon C(X) \to C(K)\colon K \subset X \text{ compact}\}$.

X = vector space; $(X_i)_{i\in I}$ a family of lcs's, $(\varphi_i\colon X_i \to X)_{i\in I}$ a family of linear maps.

Definition. The inductive locally convex topology on X generated by (φ_i) is the locally convex topology generated by

$$P_{\text{ind}} = \{p\colon p = \text{a seminorm on } X \text{ s.t. } p \circ \varphi_i \in \mathrm{SN}(X_i), \forall i\}.$$

Proposition.

(1) $\tau(P_{\text{ind}})$ is the strongest locally convex topology on X making all φ_i continuous.

(2) $\tau(P_{\text{ind}})$ is the only locally convex topology on X with the following property: if Y is a lcs and $\varphi\colon X \to Y$ linear, then φ is continuous $\iff$ $\varphi \circ \varphi_i$ is continuous $\forall i \in I$.

$$X_i \underset{\varphi_i}{\to} X \underset{\varphi}{\to} Y$$

(3) The following family is a base at $0 \in X$:

$$\left\{U \subset X \colon U \text{ is absolutely convex, absorbing, } \varphi_i^{-1}(U) \text{ is a neighborhood of } 0\ \forall i \in I\right\}$$

Example 1. The strongest locally convex topology on X = the inductive locally convex topology generated by $0 \to X$.

Example/Exercise 2. $X = \text{lcs}, X_0 \subset X$ vector subspace.
The quotient topology on X/X_0 = the inductive locally convex topology generated by $Q \colon X \to X/X_0$.

Example 3. X = topological space.

$$C_c(X) = \{f \in C(X) \colon \operatorname{supp} f \text{ is compact}\}\,.$$

$\forall$ compact $K \subset X$ let

$$C_K(X) = \{f \in C(X) \colon \operatorname{supp} f \subset K\}\,.$$

Equip $C_K(X)$ with the sup norm.
The canonical topology on $C_c(X)$ is the inductive locally convex topology generated by

$$\{C_K(X) \hookrightarrow C_c(X) \colon K \subset X \text{ compact}\}.$$

Exercise. Suppose X is locally compact, σ-compact, noncompact.
Prove:

(1) $C_c(X)$ is not metrizable.

(2) The canonical topology on $C_c(X)$ is strictly stronger than the topology induced from $C(X)$.

Example 4. Let $U \subset \mathbb{R}^n$ open.

$$C_c^\infty(U) = \{f \in C^\infty(U) \colon \operatorname{supp} f \text{ is compact}\}\,.$$

$\forall$ compact $K \subset U$ let

$$C_K^\infty(U) = \{f \in C^\infty(U) \colon \operatorname{supp} f \subset K\}\,.$$

Equip $C_K^\infty(U)$ with the topology induced from $C^\infty(U)$.
The canonical topology on $C_c^\infty(U)$ is the inductive locally convex topology generated by

$$\{C_K^\infty(U) \hookrightarrow C_c^\infty(U) \colon K \subset U \text{ compact}\}\,.$$

Exercise.

(1) The canonical topology on $C_c^\infty(U)$ is not metrizable.

(2) This topology is strictly stronger than the topology induced from $C^\infty(U)$.

Example 5 (The space $\mathcal{O}(K)$ of germs of holomorphic functions on a compact set $K \subset \mathbb{C}^n$)**.** The canonical topology on $\mathcal{O}(K)$ is the inductive locally convex topology generated by

$$\left\{\mathcal{O}(U) \xrightarrow{\text{restriction}} \mathcal{O}(K) \colon U \supset K \text{ open}\right\}.$$

Lecture 5 (2023.10.06)

Products

$\mathscr{A}$ = category; $(X_i)_{i\in I}$ objects.

Definition. The product of (X_i) is $X \in \operatorname{Ob}\mathscr{A}$ together with $\{\pi_i \colon X \to X_i | i \in I\}$ s.t. $\forall Y \in \operatorname{Ob}\mathscr{A}$, $\forall \{\varphi_i \colon Y \to X_i | i \in I\}$ $\exists$ a unique $\chi \colon Y \to X$ s.t. the following diagram commutes.

$$\begin{tikzcd} Y \arrow[rr, "\chi"] \arrow[dr, "\varphi_i"'] & & X \arrow[dl, "\pi_i"] \\ & X_i & \end{tikzcd}$$

Notation. $X = \prod\limits_{i\in I} X_i$.

Example. $\mathscr{A}$ = Sets.

The product of (X_i) $\exists$; $\prod X_i$ is the Cartesian product. $\pi_i \colon X \to X_i$ is the canonical projection.

Definition′. The product of (X_i) in $\mathscr{A}$ is $\prod X_i \in \operatorname{Ob}\mathscr{A}$ together with a natural isomorphism

$$\operatorname{Hom}\left(Y, \prod_{i\in I} X_i\right) \cong \prod_{i\in I} \operatorname{Hom}(Y, X_i) \quad (\forall Y \in \operatorname{Ob}\mathscr{A}).$$

Exercise. Definition $\iff$ Definition′.

$\mathscr{A}$ = LCS. (X_i) = a family of lcs's.

Exercise.

$$\left(\prod_{i\in I} X_i \text{ in Sets; projective locally convex topology generated by } \{\pi_i\}\right)$$

is the product of (X_i) in LCS.

Example. S = a set.

$$\mathbb{K}^S = \prod_{s\in S} \mathbb{K}.$$

Product topology = topology of pointwise convergence.

Observation. X = vector space; $(\varphi_i \colon X \to X_i)_{i\in I}$ a family of linear maps to lcs's X_i.

Define $\varphi \colon X \to \prod\limits_{i\in I} X_i, \pi_i \circ \varphi = \varphi_i, \forall i \in I$. Then the projective locally convex topology on X generated by (φ_i) = projective locally convex topology generated by φ.

In particular, if $\bigcap \operatorname{Ker}\varphi_i = 0$, that is, $\operatorname{Ker}\varphi = 0$,

$$(X; \text{projective locally convex topology}) \cong \text{subspace of } \prod X_i.$$

Exercise. Suppose $X_i \neq 0$ $\forall i$. Then

(1) $\prod X_i$ is normable $\iff$ all X_i are normable, and I is finite.

(2) $\prod X_i$ is metrizable $\iff$ all X_i are metrizable, and I is at most countable.

Coproducts

$\mathscr{A}$ = category; (X_i) a family of objects.

Definition. The coproduct of (X_i) is $X \in \operatorname{Ob}\mathscr{A}$ together with $(\kappa_i\colon X_i \to X)_{i\in I}$ s.t. $(X, \{\kappa_i\})$ is the product of (X_i) in $\mathscr{A}^{\mathrm{op}}$.

Notation. $X = \coprod_{i\in I} X_i$.

Definition′. The coproduct of (X_i) is $\coprod X_i \in \operatorname{Ob}\mathscr{A}$ together with a natural isomorphism

$$\operatorname{Hom}\left(\coprod_{i\in I} X_i, Y\right) \cong \prod_{i\in I} \operatorname{Hom}(X_i, Y) \quad (Y \in \operatorname{Ob}\mathscr{A}).$$

Exercise. Definition $\iff$ Definition′.

Example/Exercise.

(1) $\mathscr{A} = \mathrm{Sets} \implies \coprod X_i$ is the disjoint union. $\kappa_i\colon X_i \hookrightarrow \coprod_{j\in I} X_j$ canonical inclusion.

(2) $\mathscr{A} = \mathrm{Vect}$ or $\mathrm{Ab} \implies \coprod X_i = \bigoplus_{i\in I} X_i; \kappa_i\colon X_i \hookrightarrow \bigoplus_{j\in I} X_j$ canonical inclusion.

If we work in the category Vect, the difference between a direct sum and product is that

$$\bigoplus_{i\in I} X_i \hookrightarrow \prod_{i\in I} X_i.$$

If I is finite, then they are the same. If I is infinite, then they are not the same.

$\mathscr{A} = \mathrm{LCS}, (X_i)$ = a family of lcs's.

Definition. The locally convex direct sum of (X_i) is $\bigoplus_{i\in I} X_i$ equipped with the inductive locally convex topology generated by $\left\{\kappa_i\colon X_i \hookrightarrow \bigoplus_{j\in I} X_j\right\}$.

Exercise. Coproducts in LCS = locally convex direct sums.

Exercise. If I is infinite, and $X_i \neq 0\ \forall i$, then the topology of locally convex direct sum on $\bigoplus X_i$ is strictly stronger than the topology induced from $\prod X_i$.

Example. X = vector space; $(e_i)_{i\in I}$ basis of X.

$$X \cong \bigoplus_{i\in I} \mathbb{K}e_i \cong \bigoplus_{i\in I} \mathbb{K} \quad \text{(algebraically)}.$$

The the topology of locally convex direct sum on X = the strongest locally convex topology.

Observation. X = vector space; $(\varphi_i\colon X_i \to X)$ family of linear maps from lcs's X_i to X.

Define $\varphi\colon \bigoplus_{i\in I} X_i \to X$ s.t. $\varphi \circ \kappa_i = \varphi_i (i \in I)$.

The inductive locally convex topology on X generated by (φ_i) = the inductive locally convex topology generated by φ.

In particular, if $\sum \varphi_i(X_i) = X$, then φ is onto, and

$$(X; \text{inductive locally convex topology}) \cong \text{quotient of } \bigoplus X_i.$$

Exercise. Suppose $X_i \neq 0\ \forall i$.

(1) $\bigoplus X_i$ is normable $\iff$ all X_i are normable, and I is finite.

(2) $\bigoplus X_i$ is metrizable $\iff$ all X_i are metrizable, and I is finite.

In particular: the strongest locally convex topology on an infinite-dim space is not metrizable.

Limits

$I =$ a small category.

For example. We start with a poset $(I, \leq)$.

Make I into a category, $I = \mathrm{Ob}(I); \mathrm{Hom}_I(i,j) = \begin{cases} \{*\} & \text{if } i \leq j \\ \varnothing & \text{otherwise.} \end{cases}$

$\mathscr{A} =$ a category; $F \colon I \to \mathscr{A}$ covariant functor.

Definition. A cone over F is $X \in \mathrm{Ob}\,\mathscr{A}$ together with $(\varphi_i \colon X \to F(i))_{i \in I}$ s.t. $\forall \alpha \colon i \to j$ in I the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ F(i) \arrow[rr, "F(\alpha)"'] & & F(j) \end{tikzcd}$$

Definition. The limit of F is a cone $(X; (\varphi_i))$ over F s.t. $\forall$ cone $(Y, (\psi_i))$ over F $\exists$ a unique $\chi \colon Y \to X$ s.t. the following diagram commutes for all $i \in I$.

$$\begin{tikzcd} Y \arrow[rr, "\chi"] \arrow[dr, "\psi_i"'] & & X \arrow[dl, "\varphi_i"] \\ & F(i) & \end{tikzcd}$$

(The limit is the terminal object in the category of cones over F)

Notation. $X = \lim F$.

Notation. $Y \in \mathrm{Ob}\,\mathscr{A}$.
The constant functor $\Delta_Y \colon I \to \mathscr{A}, \Delta_Y(i) = Y \ \forall i; \ \Delta_Y(\alpha) = 1_Y \ \forall \alpha \colon i \to j$.

Definition′. The limit of F is $\lim F \in \mathrm{Ob}\,\mathscr{A}$ together with a natural isomorphism

$$\mathrm{Hom}(Y, \lim F) \cong \mathrm{Hom}(\Delta_Y, F) \quad (Y \in \mathrm{Ob}\,\mathscr{A}).$$

Example. $\mathscr{A} =$ Sets $F \colon I \to$ Sets.

$$X = \left\{ x = (x_i) \in \prod_{i \in I} F(i) \,\middle|\, x_j = F(\alpha)(x_i) \quad \forall \alpha \colon i \to j \text{ in } I \right\}.$$

Exercise. $X = \lim F$.
$\varphi_i \colon \lim F \to F(i)$ is the restriction of the canonical projection.

Definition″. The limit of $F \colon I \to \mathscr{A}$ is $\lim F \in \mathrm{Ob}\,\mathscr{A}$ together with a natural isomorphism

$$\mathrm{Hom}(Y, \lim F) \cong \lim \mathrm{Hom}(Y, X_i) \quad (Y \in \mathrm{Ob}\,\mathscr{A}).$$

Exercise. Definition $\iff$ Definition′ $\iff$ Definition″.

Example 1 (products $\subset$ limits)**.** $(X_i)_{i \in I}$ family of objects of $\mathscr{A}$.
I can be viewed as a discrete category:

$$\mathrm{Hom}(i,j) = \begin{cases} \varnothing & \text{if } i \neq j \\ \{1\} & \text{if } i = j. \end{cases}$$

Then a functor $I \xrightarrow{F} \mathscr{A} =$ a family of objects $(X_i)_{i \in I}$ of $\mathscr{A}$,

$$\lim F \cong \prod_{i \in I} X_i.$$

Example/Exercise 2 (Kernels $\subset$ limits).

Concrete examples.

(1) $\mathscr{A} = \mathrm{Vect}$ $F\colon I \to \mathrm{Vect}$.

Observe: $\lim F$ taken in Sets is a vector subspace of $\prod_{i \in I} F(i)$.

Exercise. $\lim F$ is the limit of F in Vect.

(2) $\mathscr{A} = \mathrm{LCS}$ $F\colon I \to \mathrm{LCS}$.

Equip $\lim F$ taken in Vect with the projective locally convex topology generated by $(\lim F \to F(i))_{i \in I}$ (equivalently, with the topology induced from $\prod F(i)$).

Exercise. the resulting space is the limit of F in LCS.

Special case. $(I, \leq)$ poset.

Definition. I is directed if $\forall i, j \in I, \exists k \in I$ s.t. $i \leq k, j \leq k$.

$\mathscr{A}$ = a category.

Definition. An inverse system (projective system) in $\mathscr{A}$ is a covariant functor $F\colon I^{\mathrm{op}} \to \mathscr{A}$.
Explicitly: The projective system is

$$\{(X_i)_{i \in I}; (\varphi_{ij}\colon X_j \to X_i | i \leq j)\} \quad i \leq j \leq k \quad \varphi_{ik} = \varphi_{ij}\varphi_{jk} \quad \varphi_{ii} = 1.$$

φ_{ij} are called the connecting maps.

Definition. The limit of $F\colon I^{\mathrm{op}} \to \mathscr{A}$ is called the inverse limit (projective limit).

Notation. $\varprojlim F$ or $\varprojlim X_i$.

Here is a picture for the projective limit:

$$\begin{tikzcd}
& \varprojlim X_i \arrow[dl] \arrow[dr] & \\
X_i & & X_j \arrow[ll, "\varphi_{ij}"'] \\
& Y \arrow[ul] \arrow[ur] \arrow[uu, bend left=80, "\exists!"] &
\end{tikzcd} \qquad i \leq j$$

Example 1. X = lcs.

$$X_1 \supset X_2 \supset X_3 \supset \cdots \text{subspaces}$$

This is a projective system indexed by $\mathbb{N}$. The connecting maps = inclusions.
$\implies \varprojlim X_i = \bigcap_{i \in \mathbb{N}} X_i$.

Example 2.

$$\begin{array}{ccccccc}
\mathbb{K} & \longleftarrow & \mathbb{K}^2 & \longleftarrow & \mathbb{K}^3 & \longleftarrow & \cdots \\
x_1 & \mapsfrom & (x_1, x_2) & \mapsfrom & (x_1, x_2, x_3) & \mapsfrom & \cdots
\end{array}$$

The projective limit is $\varprojlim \mathbb{K}^n \cong \mathbb{K}^{\mathbb{N}}$ (exercise).

Lecture 6 (2023.10.13)

Let's briefly recall the definition.

$\mathscr{A}$ = category; I = small category, $F\colon I \to \mathscr{A}$ covariant

The limit of F is

(1) $\lim F \in \operatorname{Ob} \mathscr{A}$

(2) a natural isomorphism

$$\operatorname{Hom}_{\mathscr{A}}(Y, \lim F) \cong \operatorname{Hom}_{\operatorname{Fun}(I,\mathscr{A})}(\Delta_Y, F),$$

Δ_Y = the constant functor

$$\Delta_Y(i) = Y \quad (i \in I); \quad \Delta_Y(i \to j) = 1_Y.$$

Remark. The elements of $\operatorname{Hom}(\Delta_Y, F)$ are cones over F with vertex Y:

$$\begin{tikzcd} & Y \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ F(i) \arrow[rr, "F(\alpha)"'] & & F(j) \end{tikzcd} \qquad \forall \alpha\colon i \to j \text{ in } I.$$

The limit of F is the terminal object in the category of cones over F.

Limits in Sets. $\mathscr{A}$ = Sets.

$$X = \left\{ x = (x_i) \in \prod_{i \in I} F(i) \,\middle|\, \forall \alpha\colon i \to j \quad x_j = F(\alpha)(x_i) \right\}.$$

$\varphi_i\colon X \to F(i)$ is restriction of the canonical projection.
$\implies (X, \{\varphi_i\})$ is the limit of F.

Limits in LCS. $\mathscr{A}$ = LCS.

$X = \lim F$ taken in Sets. X is a vector subspace of $\prod F(i)$. Equip X with the topology induced from $\prod F(i) \implies (X, \{\varphi_i\})$ is the limit of F in LCS.

Special case. I = a directed poset.

Definition. A projective system (inverse system) in $\mathscr{A}$ indexed by I is a covariant functor $I^{\mathrm{op}} \xrightarrow{F} \mathscr{A}$.
Explicitly: $(X_i)_{i \in I}$ = a family of objects.

$$(\varphi_{ij}\colon X_j \to X_i | i \le j) \text{ s.t.} \quad \varphi_{ik} = \varphi_{ij}\varphi_{jk} \quad (i \le j \le k) \quad \varphi_{ii} = 1.$$

The limit of F is called the projective limit (the inverse limit)

Notation. $\varprojlim F$ or $\varprojlim X_i$.

A special case. $I = \mathbb{N}$.
A projective sequence: $X_1 \xleftarrow{\varphi_{12}} X_2 \xleftarrow{\varphi_{23}} X_3 \xleftarrow{\varphi_{34}} \cdots$.

$$\varphi_{ij} = \varphi_{i,i+1} \circ \cdots \circ \varphi_{j-1,j} \quad (i \le j).$$

Let's write the following picture once again

$$
\begin{tikzcd}
& \varprojlim F \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\
X_i & & X_j \arrow[ll, "\varphi_{ij}"'] \\
& Y \arrow[ul, "\psi_i"] \arrow[ur, "\psi_j"'] \arrow[uu, bend left=60] &
\end{tikzcd}
\qquad (i \le j)
$$

Example 1.

$$X = \text{lcs}; \quad X_1 \supset X_2 \supset X_3 \supset \cdots \text{subspaces}$$

$\varprojlim X_i = \bigcap\limits_{n=1}^{\infty} X_n$.

Example 2.

$$
\begin{array}{ccccccc}
\mathbb{K} & \longleftarrow & \mathbb{K}^2 & \longleftarrow & \mathbb{K}^3 & \longleftarrow & \cdots \\
x_1 & \mapsfrom & (x_1, x_2) & \mapsfrom & (x_1, x_2, x_3) & \mapsfrom & \cdots
\end{array}
$$

$$\varprojlim_{n \in \mathbb{N}} \mathbb{K}^n = \mathbb{K}^{\mathbb{N}}.$$

Example 3. $(X_i)_{i \in I}$ = a family of lcs's.

$$\operatorname{Fin}(I) = \{J \subset I \colon J \text{ is finite}\}.$$

$(\operatorname{Fin}(I), \subset)$ is a directed poset.

$\forall J \in \operatorname{Fin}(I)$ let $X_J = \prod\limits_{i \in J} X_i$. If $K \subset J$, then we have the projection $X_J \to X_K$. So we see that X_J form a projective system indexed by $\operatorname{Fin}(I)$.

Exercise. $\varprojlim\limits_{J \in \operatorname{Fin}(I)} X_J \cong \prod\limits_{i \in I} X_i$.

Example 4. Consider the following projective sequence of spaces of continuous functions with sup norm

$$C[-1,1] \xleftarrow[\text{restriction}]{} C[-2,2] \xleftarrow[\text{restriction}]{} C[-3,3] \longleftarrow \cdots .$$

$\varprojlim C[-n,n] \cong C(\mathbb{R})$ (exercise).

To prove that such isomorphism exists, we first of all should make $C(\mathbb{R})$ into a cone over the Banach spaces of continuous functions on the intervals. Just take the restriction maps and they are compatible in the obvious sense. If we have another locally convex space Y together with a compatible family of maps from Y to $C[-n.n]$, then we must show there is a unique map from Y to $C(\mathbb{R})$ s.t. the following diagram commutes.

$$
\begin{tikzcd}
Y \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi_n"'] & & C(\mathbb{R}) \arrow[dl, "\text{restriction}"] \\
& C[-n,n] &
\end{tikzcd}
$$

Define $\chi\colon Y \to C(\mathbb{R})$ s.t. $[\chi(y)](t) = [\varphi_n(y)](t)$ if $t \in [-n, n]$. Now, to complete the proof we must prove that χ is well defined (it doesn't depend on n if we replace n by $n+1$ then we get the same number), continuous, and that it is a unique map making the diagram commute. This is more or less straightforward.

Example 5. X = locally compact Hausdorff topological space.

$$\text{Prove: } C(X) \cong \varprojlim \{C(K)\colon K \subset X \text{ compact}\}.$$

If X is 2nd countable (that is, the topology on X has a countable base), then $\exists$ a compact exhaustion of X, that is, a sequence (K_j) of compact subsets of X s.t. $X = \bigcup K_j$, $K_j \subset \operatorname{Int} K_{j+1}$ $\forall j$. In this case, $C(X) \cong \varprojlim C(K_j)$.

Example 6.

$$C^1[-1,1] \xleftarrow[\text{restriction}]{} C^2[-2,2] \xleftarrow[\text{restriction}]{} C^3[-3,3] \longleftarrow \cdots$$

Prove: $\varprojlim C^k[-k,k] \cong C^\infty(\mathbb{R})$.

Exercise. Generalize this to $C^\infty(M)$, M = a manifold.

Example 7. Let $K \subset \mathbb{C}^n$ compact.

$$\mathscr{A}(K) = \{f \in C(K)\colon f \text{ is holomorphic on } \operatorname{Int} K\} \subset C(K) \text{ closed.}$$

Let $U \subset \mathbb{C}^n$ open; (K_j) = a compact exhaustion of U.
Prove: $\mathcal{O}(U) \cong \varprojlim_{j\in\mathbb{N}} \mathscr{A}(K_j)$.

Example 8. Consider the following projective sequence

$$\ell^\infty \xleftarrow[\varphi]{} \ell^\infty \xleftarrow[\varphi]{} \ell^\infty \xleftarrow[\varphi]{} \cdots.$$

$$\varphi(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right).$$

Exercise.

(1) $\varprojlim(\ell^\infty, \varphi) \cong s$.

(2) ℓ^∞ can be replaced by $\ell^p (1 \le p \le \infty)$ or by c_0.

Colimits

I = a small category; $\mathscr{A}$ = a category; $F\colon I \to \mathscr{A}$ covariant.

Definition. The colimit of F is the limit of $F\colon I^{\mathrm{op}} \to \mathscr{A}^{\mathrm{op}}$.

Definition′. The colimit of F is

(1) $\operatorname{colim} F \in \operatorname{Ob} \mathscr{A}$;

(2) a natural isomorphism
$$\operatorname{Hom}_{\mathscr{A}}(\operatorname{colim} F, Y) \cong \operatorname{Hom}_{\mathrm{Fun}(I,\mathscr{A})}(F, \Delta_Y).$$

Observation. The elements of $\operatorname{Hom}(F, \Delta_Y)$ are the cocones over F with vertex Y:

$$\begin{tikzcd} & Y & \\ F(i) \arrow[ur, "\varphi_i"] \arrow[rr, "F(\alpha)"'] & & F(j) \arrow[ul, "\varphi_j"'] \end{tikzcd} \qquad \forall \alpha\colon i \to j.$$

Exercise. The colimit of F is the initial object in the category of all cocones over F.

Exercise. Coproducts $\subset$ colimits. Cokernels $\subset$ colimits.

Exercise.

(1) $\mathscr{A} = \mathrm{Vect}$ $F\colon I \to \mathrm{Vect}$.

$$X = \left(\bigoplus_{i\in I} F(i) \right) \Big/ \operatorname{span} \left\{ x - F(\alpha)(x) \middle| x \in F(i) \quad \alpha\colon i \to j \right\}.$$

$$\begin{tikzcd} F(j) \arrow[r, hook] \arrow[rr, bend right, "\varphi_j"'] & \bigoplus\limits_{i\in I} F(i) \arrow[r, "\text{quotient}"] & X \end{tikzcd}$$

Prove: $(X, \{\varphi_i\})$ is the colimit of F.

(2) $\mathscr{A} = \mathrm{LCS}$ $F\colon I \to \mathrm{LCS}$.

$X =$ the colimit of F in Vect. Equip X with the quotient topology.
Prove: $(X, \{\varphi_i\})$ is the colimit of F in LCS.

A special case. $I =$ a directed poset.

Definition. An inductive system (a direct system) in $\mathscr{A}$ is a covariant $F\colon I \to \mathscr{A}$.
Explicitly: a family $(X_i)_{i\in I}$ of objects; a family $(\varphi_{ij}\colon X_i \to X_j | i \le j)$ of morphisms s.t. $\varphi_{ik} = \varphi_{ij}\varphi_{jk}$ $(i \le j \le k)$.

Definition. The colimit of F is called the inductive limit (the direct limit).

Notation. $\varinjlim F$ or $\varinjlim X_i$.

A very special case. inductive sequences ($I = \mathbb{N}$)

$$X_1 \xrightarrow[\varphi_{12}]{} X_2 \xrightarrow[\varphi_{23}]{} X_3 \to \cdots \qquad \varphi_{ij} = \varphi_{i,i+1} \circ \cdots \circ \varphi_{j-1,j}.$$

$$\begin{tikzcd} & \varinjlim X_i \arrow[dd, bend right=80] & \\ X_i \arrow[ur, "\varphi_i"] \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] & & X_j \arrow[ul, "\varphi_j"'] \arrow[dl, "\psi_j"] \\ & Y & \end{tikzcd} \qquad (i \le j)$$

Proposition. X = vector space; $(X_i)_{i\in I}$ a family of vector subspaces indexed by a directed poset I. Assume that

(1) $\forall i \leq j\ X_i \subset X_j$.

(2) Every X_i is equipped a locally convex topology s.t. $X_i \hookrightarrow X_j$ are continuous $\forall i \leq j$.

(3) $X = \bigcup_{i\in I} X_i$.

Then $\varinjlim_{i\in I} X_i = (X;$ inductive locally convex topology generated by the inclusion maps $X_i \hookrightarrow X)$.

Proof. Exercise. □

Example/Exercise 1.

$$\begin{array}{ccccc} \mathbb{K} & \hookrightarrow & \mathbb{K}^2 & \hookrightarrow & \mathbb{K}^3 \hookrightarrow \\ x_1 & \longmapsto & (x_1, 0) & & \\ & & (x_1, x_2) & \longmapsto & (x_1, x_2, 0) \end{array}$$

$$\varinjlim_{n\in\mathbb{N}} \mathbb{K}^n = \bigoplus_{n\in\mathbb{N}} \mathbb{K}$$

(with the strongest locally convex topology).

Example/Exercise 2. $(X_i)_{i\in I}$ = a family of lcs's.

$$\bigoplus_{i\in I} X_i \cong \varinjlim \left\{ \bigoplus_{j\in J} X_j \colon J \in \operatorname{Fin}(I) \right\}.$$

Example/Exercise 3. X = a topological space.

$$C_c(X) \cong \varinjlim \{C_K(X) \cong K \subset X \text{ compact}\}.$$

Example/Exercise 4. Let $U \subset \mathbb{R}^n$ open; (K_j) = a compact exhaustion of U.

$$C_c^\infty(U) \cong \varinjlim C_{K_j}^\infty(U).$$

Example/Exercise 5. Let $K \subset \mathbb{C}^n$ compact, $\mathscr{U}$ = a base of open neighborhoods of K. $(\mathscr{U}, \supseteq)$ is a directed poset.

$$\mathcal{O}(K) \cong \varinjlim_{U\in\mathscr{U}} \mathcal{O}(U) \cong \varinjlim_{U\in\mathscr{U}} \mathscr{A}(\overline{U}) \quad \text{(if all } U \in \mathscr{U} \text{ are bounded, then the 2nd isomorphism exists)}.$$

Example/Exercise 6. $\mathscr{C}_0$ = the space of germs of continuous functions at $0 \in \mathbb{R}$.

$$\mathscr{C}_0 \cong \varinjlim_{n\in\mathbb{N}} C\left(-\frac{1}{n}, \frac{1}{n}\right) \cong \varinjlim_{n\in\mathbb{N}} C\left[-\frac{1}{n}, \frac{1}{n}\right].$$

Equip $\mathscr{C}_0$ with the inductive topology.

$$\mathscr{C}_0 = \mathbb{K}1 \bigoplus E \quad E = \{f \in \mathscr{C}_0 \colon f(0) = 0\}.$$

Exercise. The topology on E induced from $\mathscr{C}_0$ is anti-discrete.

Lecture 7 (2023.10.20)

Strict inductive limits

Definition. An inductive system $(X_i, \varphi_{ij})_{i\in I}$ of lcs's is strict if $\varphi_{ij}\colon X_i \to X_j$ are topological injective ($i \le j$). A strict inductive limit is the inductive limit of a strict inductive system.

Remark. As rule, $I = (\mathbb{N}, \le)$.

Example 1. A vector space X of countable dimension equipped with the strongest locally convex topology is isomorphic to $\varinjlim_{n\in\mathbb{N}} \mathbb{K}^n$; this is a strict inductive limit.

Example 2. $X =$ locally compact 2nd countable Hausdorff topological space; $(K_j)_{j\in\mathbb{N}} =$ a compact exhaustion of X. $\implies C_c(X) \cong \varinjlim_{n\in\mathbb{N}} C_{K_n}(X)$ is a strict inductive limit.

Example 3. Let $U \subset \mathbb{R}^n$ open; $(K_j)_{j\in\mathbb{N}}$ compact exhaustion of U $\implies$ $C_c^\infty(U) \cong \varinjlim C^\infty_{K_j}(U)$ is a strict inductive limit.

Nonexample/Exercise. Fix $z_0 \in \mathbb{C}$.
$\mathscr{U}$ = a base of open neighborhoods of z_0; $\mathscr{U} = \{U_n\}, U_{n+1} \subset U_n \ \forall n$.

$$\mathcal{O}_{z_0} = \{\text{germs of holomorphic functions at } z_0\}.$$

Then

$$\mathcal{O}_{z_0} \cong \varinjlim \mathcal{O}(U_j) \quad \mathcal{O}_{z_0} \cong \varinjlim \mathscr{A}(\overline{U}_j)$$

are not strict inductive limits.

Theorem. Suppose $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow$ is a strict inductive sequence of lcs's, suppose X_n is closed in X_{n+1} $\forall n$; $X = \varinjlim X_n$.
Then

(1) X is Hausdorff $\iff$ all the X_n's are Hausdorff.

(2) $\forall n \in \mathbb{N}, X_n \to X$ is topological injective, and X_n is closed in X $\forall n$.

(3) A sequence (x_i) in X converges in X $\iff$ (x_i) contained in X_n for some n, and (x_i) converges in X_n.

(4) X is not metrizable unless (X_n) stabilizes.

Lemma 1. $X = $ lcs; $Y \subset X$ vector subspace.

(1) $\forall$ absolutely convex neighborhood $V \subset Y$ of 0 $\exists$ an absolutely convex neighborhood $U \subset X$ of 0 s.t. $V = U \cap Y$.

(2) $\forall q \in \mathrm{SN}(Y)\ \exists p \in \mathrm{SN}(X)$ s.t. $q = p|_Y$.

Proof.

(1) $\exists$ an absolutely convex neighborhood $W \subset X$ of 0 s.t. $W \cap Y \subset V$. Let $U = \operatorname{conv}(V \cup W)$.

Exercise. U is what we need.

(2) follows from (1). □

Lemma 2. $X = $ lcs, $Y \subset X$ closed vector subspace. Then $\forall x \in X \setminus Y\ \exists f \in X'$ s.t. $f|_Y = 0, f(x) \neq 0$.

Proof. $Q\colon X \to X/Y$ quotient map. X/Y is a Hausdorff lcs, $Q(x) \neq 0$.
$\implies \exists g \in (X/Y)'$ s.t. $g(Q(x)) \neq 0$. Let $f = g \circ Q$. □

Lemma 3. $X =$ metrizable tvs; $(x_n) =$ a sequence in X. $\implies$ $\exists$ a sequence (λ_n) in $\mathbb{R}$, $\lambda_n > 0$, s.t. $\lambda_n x_n \to 0$.

Proof. $(U_n)_{n\in\mathbb{N}} =$ a base at 0; $U_{n+1} \subset U_n\ \forall n$. $\forall n\ \exists \lambda_n > 0$ s.t. $\lambda_n x_n \in U_n$ (because U_n is absorbing) $\implies \lambda_n x_n \to 0$. □

Proof of Theorem.

(2) Let $n \in \mathbb{N}, U_n \subset X_n$ an absolutely convex neighborhood of 0. By Lemma 1, $\exists$ an absolutely convex neighborhood $U_{n+1} \subset X_{n+1}$ of 0 s.t. $U_n = U_{n+1} \cap X_n$. Constantly use Lemma 1, we get a chain $U_n \subset U_{n+1} \subset \cdots, U_i \subset X_i$ an absolutely convex neighborhood of 0, $U_{i+1} \cap X_i = U_i (i \in \mathbb{N})$.

Let $U = \bigcup\limits_{i=n}^{\infty} U_i \subset X$. U is absolutely convex absorbing set; $U \cap X_i$ is a neighborhood of 0. $\implies$ U is a neighborhood of 0 in X; $U \cap X_n = U_n$. $\implies$ $X_n \to X$ is topological injective.

Exercise. X_n is closed in X.

(1) ($\Longrightarrow$) Just note that we have a continuous injection of X_n to X.

($\Longleftarrow$) Suppose all X_n are Hausdorff.

Let $x \in X, x \neq 0$. $\exists n \in \mathbb{N}$ s.t. $x \in X_n$. $\exists f_0 \in X_n'$ s.t. $f_0(x) \neq 0$. Hahn-Banach&(2) $\implies$ $\exists f \in X'$ s.t. $f|_{X_n} = f_0; f(x) \neq 0 \implies X$ is Hausdorff.

(3) ($\Longleftarrow$) Because the map $X_n \to X$ is continuous.

($\Longrightarrow$) Assume there is no such n. We may assume that $x_n \notin X_n$ $\forall n \in \mathbb{N}; x_n \to 0$. By Lemma 2, $\forall n \, \exists f_n \in X'$ s.t. $f_n|_{X_n} = 0$, but $f_n(x_n) = a_n > 0$. Let $p(x) = \sum\limits_{j=1}^{\infty} \frac{|f_j(x)|}{a_j} (x \in X), p(x) \in \mathrm{SN}(X)$. $p(x_n) \geq \frac{|f_n(x_n)|}{a_n} = 1 \implies x_n \not\to 0$, a contradiction.

(4) $\forall n$ choose $x_n \in X_n \setminus X_{n-1}$. $\forall$ sequence (λ_n) of positive numbers, $\lambda_n x_n \in X_n \setminus X_{n-1} \overset{(3)}{\implies} \lambda_n x_n \not\to 0$ $\xRightarrow[\text{Lemma 3}]{}$ X is not metrizable. $\square$

Example. Let $U \subset \mathbb{R}^n$ open. $C_c^\infty(U)$ is Hausdorff and nonmetrizable.
A sequence (f_j) in $C_c^\infty(U)$ converges to f $\iff$ $\exists$ a compact set $K \subset U$ s.t. $\operatorname{supp} f_j \subset K$ $\forall j, \operatorname{supp} f \subset K$, and

$$\mathrm{D}^\alpha f_j \underset{K}{\rightrightarrows} \mathrm{D}^\alpha f \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^n.$$

Topologies on $\mathscr{L}(X,Y)$

$X, Y =$ lcs's; $\mathscr{B} =$ a family of bounded subsets of X.
$\forall p \in \mathrm{SN}(Y), \forall B \in \mathscr{B}$ define a seminorm p^B on $\mathscr{L}(X,Y)$ by

$$p^B(T) = \sup_{x \in B} p(Tx).$$

Exercise. $T(B)$ is bounded.

Definition. The topology on $\mathscr{L}(X,Y)$ generated by

$$\left\{p^B \colon B \in \mathscr{B}, p \in \mathrm{SN}(Y)\right\}$$

is called the topology of uniform convergence on elements of $\mathscr{B}$.

Notation. $\mathscr{T}_{\mathscr{B}}$; $\mathscr{L}_{\mathscr{B}}(X,Y) = (\mathscr{L}(X,Y), \mathscr{T}_{\mathscr{B}})$.

Exercise.

(1) A sequence (T_n) in $\mathscr{L}_{\mathscr{B}}(X,Y)$ converges to T $\iff$ $T_n \underset{B}{\rightrightarrows} T \quad \forall B \in \mathscr{B}$, that is, $\forall$ neighborhood $U \subset Y$ of 0, $\forall B \in \mathscr{B} \; \exists N \in \mathbb{N}$ s.t. $\forall n > N \; \forall x \in B \; T_n(x) - T(x) \in U$.

(2) If P is a defining family of seminorms on Y, then $\mathscr{T}_{\mathscr{B}}$ is generated by $\left\{p^B \colon B \in \mathscr{B}; p \in P\right\}$.

Notation. $B \subset X, U \subset Y$.

$$M(B,U) = \left\{T \in \mathscr{T}(X,Y) \colon T(B) \subset U\right\}.$$

Exercise. $\mathscr{U}$ = a subbase at $0 \in Y$, then

$$\{M(B,U)\colon B \in \mathscr{B}, U \in \mathscr{U}\}$$

is a subbase at $0 \in \mathscr{T}_{\mathscr{B}}(X,Y)$.

Special cases.

(1) $\mathscr{B} = \{\text{all bounded subsets of } X\}$.

$$\mathscr{L}_{\mathscr{B}}(X,Y) = \mathscr{L}_b(X,Y); \quad \mathscr{L}_b(X,\mathbb{K}) = X'_\beta(\text{the } \underline{\text{strong dual}} \text{ of } X).$$

Observation. If X and Y are normed spaces, then

$$\mathscr{L}_b(X,Y) = (\mathscr{L}_b(X,Y); \text{operator norm topology})\,.$$

(2) $\mathscr{B} = \{\text{finite subsets of } X\}$.

$$\mathscr{L}_{\mathscr{B}}(X,Y) = \mathscr{L}_s(X,Y); \quad \mathscr{L}_s(X,\mathbb{K}) = X'_\sigma(\text{the } \underline{\text{weak dual}} \text{ of } X).$$

(3) $\mathscr{B} = \{\text{compact subsets of } X\} \cdots\cdots$

Lecture 8 (2023.10.27)

Nets in topological spaces

Definition. A partially pre-ordered set is $(\Lambda, \le)$. Λ = a set, $\le$ = a binary relation on Λ s.t.

(1) $\lambda \le \lambda (\lambda \in \Lambda)$

(2) $\lambda \le \mu, \mu \le \nu \implies \lambda \le \nu$.

$(\Lambda, \le)$ is directed if $\forall \lambda, \mu \in \Lambda\ \exists \nu \in \Lambda$ s.t. $\lambda \le \nu, \mu \le \nu$.
Directed set = a directed partially pre-ordered set.

X = topological space.

Definition. A net in X is (Λ, x): Λ = directed set, $x \colon \Lambda \to X$.

Notation. $x_\lambda = x(\lambda), x = (x_\lambda)_{\lambda \in \Lambda}$. If $\Lambda = \mathbb{N}$, then x is a sequence.

Definition. (x_λ) converges to x $(x_\lambda \to x)$ if $\forall$ neighborhood $U \ni x\ \exists \lambda_0 \in \Lambda$ s.t. $x_\lambda \in U\ \forall \lambda \ge \lambda_0$.

Example. $x \in X; \beta$ = a base at x. $(\beta, \supset)$ is a directed set. $\forall U \in \beta$ choose $x_U \in U \implies x_U \to x$.

Example. $\Lambda = \{(y, U) \colon U \in \beta, y \in U\}$.

$$(y_1, U_1) \le (y_2, U_2) \overset{\text{def}}{\iff} U_2 \subset U_1.$$

$\forall \lambda = (y, U) \in \Lambda$ let $x_\lambda = y \implies x_\lambda \to x$.

Exercise. $Y \subset X; x \in X$. Then

$$x \in \overline{Y} \iff \exists \text{ a net } (y_\lambda) \text{ in } Y \text{ s.t. } y_\lambda \to x.$$

Exercise. $f \colon X \to Y$ is continuous at $x \in X \iff \forall$ net (x_λ) in X s.t. $x_\lambda \to x$ we have $f(x_\lambda) \to f(x)$.

Definition. $x \in X$ is an accumulation point of a net (x_λ) if $\forall$ neighborhood $U \ni x\ \forall \lambda_0 \in \Lambda\ \exists \lambda \ge \lambda_0$ s.t. $x_\lambda \in U$.

Exercise. X is compact $\iff$ every net in X has an accumulation point.

Completeness

X = tvs.

Definition. A net (x_λ) in X is a Cauchy net if $\forall$ neighborhood $U \ni 0\ \exists \lambda_0 \in \Lambda$ s.t. $x_\lambda - x_\mu \in U\ \forall \lambda, \mu \ge \lambda_0$.

Proposition.

(1) (x_λ) converges in $X \implies (x_\lambda)$ is Cauchy.

(2) (x_λ) is a Cauchy net which has an accumulation point $x \in X$, then $x_\lambda \to x$.

(3) $\varphi \in \mathscr{L}(X, Y), (x_\lambda)$ Cauchy in $X \implies (\varphi(x_\lambda))$ is Cauchy in Y.

(4) $(X, \tau(P))$ = lcs.

$$(x_\lambda) \text{ is Cauchy} \iff \forall p \in P\ \forall \varepsilon > 0\ \exists \lambda_0 \in \Lambda \text{ s.t. } p(x_\lambda - x_\mu) < \varepsilon\ \forall \lambda, \mu \ge \lambda_0.$$

Proof. Exercise. ☐

X = Hausdorff tvs, $S \subset X$ subset.

Definition. S is complete if every Cauchy net in S has a limit in S.

Observation.

(1) $S \subset X$, S is complete $\implies S$ is closed in X.

(2) $S \subset T \subset X$, T is complete, S is closed in $T \implies S$ is complete.

(3) $K \subset X$ is compact $\implies K$ is complete.

Theorem (extension by continuity)**.** X = tvs, $X_0 \subset X$ dense vector subspace; Y = complete tvs. Then $\forall \varphi_0 \in \mathscr{L}(X_0, Y)$ $\exists$ a unique $\varphi \in \mathscr{L}(X, Y)$ s.t. $\varphi|_{X_0} = \varphi_0$.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \\ X_0 \arrow[u, hook] \arrow[ur, "\varphi_0"'] & \end{tikzcd}$$

Proof. Exercise (Similar to the case of normed spaces). ☐

Theorem. X = metrizable tvs; ρ = a shift-invariant metric on X which generates the topology on X. TFAE:

(1) X is complete

(2) X is sequentially complete (that is, every Cauchy sequence converges)

(3) (X, ρ) is a complete metric space.

Definition. A Fréchet space is a complete metrizable locally convex space.

Proposition 1. $(X_i)_{i \in I}$ a family of lcs's.

$$\prod X_i \text{ is complete } \iff \text{ all } X_i \text{ are complete.}$$

Proof. Exercise. ☐

Corollary. $\mathbb{K}^S$ (S = a set) is complete.

Proposition 2. (X_i, φ_{ij}) = a projective system of complete lcs's
$\implies \varprojlim X_i$ is a closed subspace of $\prod X_i$
$\implies \varprojlim X_i$ is complete.

Proof. Exercise. ☐

Trick of proving completeness

X = lcs. We want to show that X is complete.
Suppose Y is a complete lcs, $X \hookrightarrow Y$ continuous linear injection.
(x_λ) = Cauchy in $X \implies x_\lambda \to y \in Y$.
We have to prove:

(1) $y \in X$

(2) $x_\lambda \to y$ for the topology of X.

Illustration. The standard proof of the completeness of ℓ^p

$$\ell^p \hookrightarrow \mathbb{K}^{\mathbb{N}}$$

Proposition 3. $(X_i)_{i \in I}$ a family of lcs's. Then $\bigoplus_{i \in I} X_i$ is complete $\iff$ all X_i are complete.

Proof. Exercise (Hint: $\bigoplus X_i \hookrightarrow \prod X_i$ and use the Trick). ☐

Proposition 4. X = a Fréchet space, $X_0 \subset X$ closed vector subspace $\implies X/X_0$ is a Fréchet space.

Proof. (a sketch) $\|\cdot\|_1 \le \|\cdot\|_2 \le \|\cdot\|_3 \le \cdots$ a defining sequence of seminorms on X. (u_k) = Cauchy sequence in X/X_0. We may assume that the quotient seminorm on X/X_0 satisfies $\|u_{k+1} - u_k\|_k < \frac{1}{2^k}$. Choose any $x_1 \in u_1$. Induction: choose $x_{k+1} \in u_{k+1}$ s.t. $\|x_{k+1} - x_k\|_k < \frac{1}{2^k} \implies (x_k)$ is Cauchy in $X \implies x_k \to x \in X \implies u_k \to x + X_0$ in X/X_0. ☐

Warning. X = a complete lcs, $X_0 \subset X$ closed vector subspace $\not\Longrightarrow X/X_0$ is complete (Examples: see Exercise sheets).

Fact: The strict inductive limit of a sequence of complete lcs's is complete.

Proof. Later. ☐

Examples of complete lcs's

1. X = a locally compact Hausdorff topological space. Then

$$C(X) \cong \varprojlim C(K)(K \subset X \text{ compact}) \implies C(X) \text{ is complete.}$$

If, moreover, X is 2nd countable, then $C(X)$ is a Fréchet space.

2. $U \subset \mathbb{C}^n$(open) $\underset{\text{(exercise)}}{\Longrightarrow}$ $\mathcal{O}(U)$ is closed in $C(U) \implies \mathcal{O}(U)$ is a Fréchet space.

3. Let $U \subset \mathbb{R}^n$ open.
$(M_p)_{p\in\mathbb{N}}$ = a compact exhaustion of U s.t. each M_p is a smooth manifold with boundary. Then

$$C^\infty(U) \cong \varprojlim_{p\in\mathbb{N}} C^p(M_p) \implies C^\infty(U) \text{ is a Fréchet space.}$$

4. $s; \mathscr{S}(\mathbb{R}^n); \mathcal{O}(U)$ are countable projective limits of Banach spaces (exercise) $\implies$ they are Fréchet spaces.

5. $C_c(X)$ (X = 2nd countable, locally compact) and $C_c^\infty(U)$ ($U \subset \mathbb{R}^n$ open) are strict inductive limits of sequence of Fréchet spaces $\implies$ they are complete.

6. $\mathcal{O}_{z_0}(z_0 \in \mathbb{C}); C_c(X); C_c^\infty(U)$ are (uncountable) projective limits of Banach spaces (exercise) $\implies$ they are complete.
<u>Hint</u>: For $C_c(X); C_c^\infty(U)$: use "concrete" families of seminorms (Exercise sheet 7).
For $\mathcal{O}_{z_0}$: a power series $\sum c_n z^n$ has a positive radii of convergence $\iff$ $\sum |c_n| p_n < \infty$ $\forall$ sequence $(p_n), p_n > 0$ s.t. $p_n = o(\varepsilon^n)$ $\forall \varepsilon > 0$.

7. (Nonexample/Exercise) S = uncountable set.

$$X = \left\{ f \in \mathbb{K}^S : f(s) = 0 \text{ for all but countably many } s \in S \right\}.$$

Then X is sequentially complete, but is not complete.

Completions

X = lcs.

<u>Definition</u>. The <u>completion</u> of X is $(\widetilde{X}, J)$, where $\widetilde{X}$ is a complete lcs, $J\colon X \hookrightarrow \widetilde{X}$ continuous linear map s.t. $\forall$ complete lcs Y, $\forall \varphi \in \mathscr{L}(X,Y)$ $\exists$ a unique $\widetilde{\varphi} \in \mathscr{L}(\widetilde{X}, Y)$ s.t. the following diagram commutes.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\varphi}"] & Y \\ X \arrow[u, "J"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

<u>Notation</u>. CLCS $\subset$ LCS full subcategory consisting of complete spaces.

<u>Definition′</u>. The <u>completion</u> of X is a complete lcs $\widetilde{X}$ together with a natural isomorphism

$$\mathscr{L}(X,Y) \cong \mathscr{L}(\widetilde{X}, Y) \quad (Y \in \mathrm{Ob(CLCS)}).$$

<u>Exercise</u>. Definition $\iff$ Definition′.

Remark (uniqueness of the completion)**.** If we have two completions of X: $(\widetilde{X}, \widetilde{J})$ and $(\widehat{X}, \widehat{J})$, then there exists a unique topological isomorphism between $\widetilde{X}$ and $\widehat{X}$ which makes the following diagram commute.

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "\exists!"] & & \widehat{X} \\
& X \arrow[ul, "\widetilde{J}"] \arrow[ur, "\widehat{J}"'] &
\end{tikzcd}$$

This is immediately from our definition. Firstly, take a map from $\widetilde{X}$ to $\widehat{X}$ by definition. Then get a map $\widehat{X} \to \widetilde{X}$ in the opposite direction by definition again. The uniqueness implies that the composite map is identity map in both directions.

Observation. $X = \text{lcs}; X_h = X/\overline{\{0\}}$. X_h is a Hausdorff lcs.

Definition. X_h is the Hausdorff lcs associated to X.

Observation.

(1) $\mathscr{L}(X_h, Y) \cong \mathscr{L}(X, Y) \quad (Y \in \text{Ob(HLCS)})$

(2) Suppose $\widetilde{X}$ is a complete lcs, $J \in \mathscr{L}(X_h, \widetilde{X})$. Then: $(\widetilde{X}, J)$ is a completion of $X_h \iff (\widetilde{X}, J \circ Q_h)$ is a completion of X (where $Q_h \colon X \to X_h$ is the quotient map).

$$\begin{tikzcd}
\widetilde{X} \arrow[r] & Y \\
X_h \arrow[u] \arrow[ur] & \\
X \arrow[u] \arrow[uur] &
\end{tikzcd}$$

Lecture 9 (2023.11.03)

Proposition. X = Hausdorff lcs, $\widetilde{X}$ = complete lcs, $J\colon X \to \widetilde{X}$ topological injective s.t. $\overline{J(X)} = \widetilde{X}$, then $(\widetilde{X}, J)$ is a completion of X.

Proof. See "Extension by continuity" Theorem. $\square$

Corollary. X = normed; $(\widetilde{X}, J)$ = a completion of X as a normed space $\implies$ $(\widetilde{X}, J)$ is a completion of X as a lcs.

Construction of the completion

X = Hausdorff lcs; P = directed defining family of seminorms.
$\forall p \in P, X_p = \big(X/p^{-1}(0); \widehat{p}\big)$ is a normed space.
$\widetilde{X}_p$ = the completion of X_p.

$$\begin{tikzcd}
X \arrow[r, "\pi_p"] \arrow[rr, bend right, "\varphi_p\colon x \mapsto x + p^{-1}(0)"'] & X_p \arrow[r, hook] & \widetilde{X}_p
\end{tikzcd}$$

If $p, q \in P, p \prec q \implies q^{-1}(0) \subset p^{-1}(0) \implies \exists \pi_{pq}\colon X_q \to X_p, x + q^{-1}(0) \mapsto x + p^{-1}(0)$; π_{pq} is continuous.

$$\begin{tikzcd}
& X \arrow[dl, "\pi_p"'] \arrow[dr, "\pi_q"] \arrow[ddl, bend right=60, "\varphi_p"'] \arrow[ddr, bend left=60, "\varphi_q"] & \\
X_p \arrow[d, hook] & & X_q \arrow[ll, "\pi_{pq}"] \arrow[d, hook] \\
\widetilde{X}_p & & \widetilde{X}_q \arrow[ll, "\varphi_{pq}"]
\end{tikzcd}$$

If $p \prec q \prec r$, then $\varphi_{pq}\varphi_{qr} = \varphi_{pr} \implies \mathscr{X} = (\widetilde{X}_p, \varphi_{pq})$ is a projective system. Moreover, (X, φ_p) is a cone over $\mathscr{X}$.

Let $\widetilde{X} = \varprojlim \mathscr{X}$; $\psi_p\colon \widetilde{X} \to \widetilde{X}_p$ canonical maps. Then $\exists!\ J \in \mathscr{L}(X, \widetilde{X})$ s.t. the following diagram commute $\forall p$.

$$\begin{tikzcd}
X \arrow[rr, "J"] \arrow[dr, "\varphi_p"'] & & \widetilde{X} \arrow[dl, "\psi_p"] \\
& \widetilde{X}_p &
\end{tikzcd}$$

Theorem. $(\widetilde{X}, J)$ is a completion of X.

Lemma. (Z_i, π_{ij}) = projective system of lcs's; $Z = \varprojlim Z_i, \tau_i\colon Z \to Z_i$ canonical maps. Suppose $Y \subset Z$ vector subspace, $\tau_i(Y) = Z_i\ \forall i. \implies \overline{Y} = Z$.

Proof. Exercise. $\square$

Proof of Theorem. $\widetilde{X}$ is complete.

Lemma $\implies$ $\overline{J(X)} = \widetilde{X}$; $\operatorname{Ker} J = \bigcap_{p\in P} p^{-1}(0) = 0$, since X is Hausdorff.

(1) The topology on X = the projective locally convex topology generated by $\{\varphi_p\}_{p\in P}$.

(2) The topology on $J(X)$ = the projective locally convex topology generated by $\left\{\psi_p|_{J(X)}\right\}_{p\in P}$.

(3)

$$\begin{tikzcd} X \arrow[rr, "J"] \arrow[dr] & & J(X) \arrow[dl] \\ & \widetilde{X}_p & \end{tikzcd}$$

commutes.

(1), (2), (3) $\implies$ $J\colon X \to J(X)$ is a topological isomorphism. □

Corollary 1. Every lcs X has a completion $(\widetilde{X}, J)$. Moreover, $\overline{J(X)} = \widetilde{X}, \operatorname{Ker} J = \overline{\{0\}}$, and $J\colon X \to J(X)$ is open. In particular, if X is Hausdorff, then J is topological injective.

Corollary 2. X = complete lcs $\implies$ $X \cong \varprojlim \widetilde{X}_p$.

Exercise 1. Describe this isomorphism explicitly for the following spaces:

(1) $\mathbb{K}^S$ (S = a set)

(2) $C(T)$ (T = locally compact topological space)

(3) $\mathscr{O}(U)$ ($U \subset \mathbb{C}^n$ open)

(4) $C^\infty(U)$ ($U \subset \mathbb{R}^n$ open)

(5) s

(6) $\mathscr{S}(\mathbb{R}^n)$

(7)* $C_c(T)$ (T = 2nd countable locally compact space)

(8)* $C_c^\infty(U)$ ($U \subset \mathbb{R}^n$)

(9)* $\mathscr{O}_{z_0}$ ($z_0 \in \mathbb{C}$)

Exercise 2. $X \mapsto \widetilde{X}$ is a functor LCS $\to$ CLCS, left adjoint to CLCS $\hookrightarrow$ LCS.

Theorem. The strict inductive limit of a sequence of complete lcs's is complete.

Proof. Let $X = \varinjlim X_n$ (strict); X_n are complete.

$$X_1 \subset X_2 \subset X_3 \subset \cdots \subset X. \quad \mathscr{U} = \{\text{absolutely convex neighborhoods of } 0 \in X\}.$$

Let $a \in \widetilde{X}$.

$$\mathscr{F} = \left\{ V \cap X : V = \text{an absolutely convex neighborhood of } a \in \widetilde{X} \right\}.$$

Lemma. $\exists n \in \mathbb{N}$ s.t. $\forall F \in \mathscr{F}\ \forall U \in \mathscr{U}\ \ (F + U) \cap X_n \neq \varnothing$.

Assume Lemma is already proved.

Let W be a neighborhood of a in $\widetilde{X}$. $\exists F \in \mathscr{F}\ \exists U \in \mathscr{U}$ s.t. $F + U \subset W$. $\implies$ $X_n \cap W \neq \varnothing$ $\implies$ $a \in$ (closure of X_n in $\widetilde{X}$). But X_n is complete $\implies$ closed in $\widetilde{X}$ $\implies$ $a \in X_n \subset X$. □

Proof of Lemma. Suppose $\forall n\ \exists F_n \in \mathscr{F}\ \exists U_n \in \mathscr{U}$ s.t. $(F_n+U_n)\cap X_n = \varnothing$. We may assume that $U_{n+1} \subset \frac{1}{2}U_n$.

Let $U = \sum\limits_{n=1}^{\infty}(U_n \cap X_n) = \left\{\sum\limits_{i=1}^{m} u_i \colon u_i \in U_i \cap X_i, m \in \mathbb{N}\right\}$, U is a neighborhood of 0 in X. $\implies \exists F \in \mathscr{F}$ s.t. $F - F \subset U$ (exercise).

Choose $x \in F$; choose $n \in \mathbb{N}$ s.t. $x \in X_n$. Let's show: $x \in F_n + U_n$.

Take any $y \in F \cap F_n \implies x = y + (x-y) \in F_n + (F-F) \subset F_n + U \implies x = a_n + \sum\limits_{i=1}^{m} u_i$, $a_n \in F_n$, $u_i \cap X_i \implies X_n \ni x - \sum\limits_{i=1}^{n} u_i = a_n + \sum\limits_{i>n} u_i \in a_n + U_{n+1} + U_{n+2} + \cdots + U_{n+k} \subset a_n + \frac{1}{2}U_n + \frac{1}{4}U_n + \cdots + \frac{1}{2^k}U_n \subset a_n + U_n \subset F_n + U_n \implies x - \sum\limits_{i\le n} u_i \in X_n \cap (F_n + U_n)$, a contradiction. $\square$

Corollary. $C_c(T)$ (T = 2nd countable, locally compact) and $C_c^\infty(U)$ ($U \subset \mathbb{R}^n$ open) are complete.

Tensor products of seminormed spaces

X, Y, Z = lcs's.

Notation. $\mathscr{L}^{(2)}(X \times Y, Z) = \{$jointly continuous bilinear maps $X \times Y \to Z\}$.

Proposition 1. $(X, \tau(P)), (Y, \tau(Q)), (Z, \tau(R))$ lcs's.

Suppose P, Q are directed. A bilinear map $\Phi \colon X \times Y \to Z$ is continuous $\iff \forall r \in R\ \exists C > 0\ \exists p \in P\ \exists q \in Q$ s.t. $r(\Phi(x,y)) \le C p(x) q(y)$ $(\forall x, y)$.

Proof. Exercise. $\square$

X, Y, Z = seminormed spaces. $\Phi \colon X \times Y \to Z$ bilinear.

Definition. The operator seminorm of Φ is

$$\|\Phi\| = \sup\{\|\Phi(x,y)\| \colon x \in X, y \in Y, \|x\| \le 1, \|y\| \le 1\} \in [0, +\infty].$$

Proposition 2.

(1) $\|\Phi\| < \infty \iff \Phi$ is continuous

(2) $\mathscr{L}^{(2)}(X \times Y, Z)$ is a seminormed space

(3) It is a normed space $\iff Z$ is a normed space

In particular, $\mathscr{L}^{(2)}(X \times Y, \mathbb{K})$ is a normed space.

Proof. Exercise. $\square$

Definition. Φ is a contraction if $\|\Phi\| \le 1$.

Notation. $U_X = \{x \in X \colon \|x\| < 1\}$.

Proposition 3. $\|\Phi\| \le 1 \iff \Phi(U_X \times U_Y) \subset U_Z \iff \Phi(\overline{U}_X \times \overline{U}_Y) \subset \overline{U}_Z \iff \Phi(U_X \times U_Y) \subset \overline{U}_Z$.

Exercise.

(1) Prove Proposition 3.

(2) Prove analogs of Propositions 2,3 for linear maps.

Definition. X, Y = seminormed.

The projective tensor product of X, Y is $(X \otimes_\pi Y, \theta)$, where $X \otimes_\pi Y$ is a seminormed space, $\theta \colon X \times Y \to X \otimes_\pi Y$ is a bilinear contraction s.t. $\forall$ seminormed Z $\forall$ bilinear contraction $\Phi \colon X \times Y \to Z$ $\exists!$ linear contraction $\varphi \colon X \otimes_\pi Y \to Z$ s.t. the following diagram commutes.

$$\begin{tikzcd}
X \times Y \arrow[r, "\Phi"] \arrow[d, "\theta"'] & Z \\
X \otimes_\pi Y \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

(φ is the linearization of Φ.)

Category SNorm_1: objects = seminormed spaces, morphisms = linear contractions.

Observation. Isomorphism in SNorm_1 = an isometric isomorphism.

Definition′. The projective tensor product of X, Y is a seminormed space $X \otimes_\pi Y$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathrm{SNorm}_1}(X \otimes_\pi Y, Z) \cong \{\text{Bilinear contraction } X \times Y \to Z\} \quad (Z \in \mathrm{Ob}(\mathrm{SNorm}_1)).$$

Exercise. Definition $\iff$ Definition′.

Uniqueness:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\theta"'] \arrow[dr, "\theta'"] & \\ X \otimes_\pi Y \arrow[rr, "\exists! \text{ isometric isomorphism}"'] & & X \otimes'_\pi Y \end{tikzcd}$$

Construction of $X \otimes_\pi Y$

The underlying vector space of $X \otimes_\pi Y$ is $X \times Y$.

$$\theta \colon X \times Y \to X \otimes Y, \theta(x, y) = x \otimes y.$$

Definition. The projective tensor seminorm on $X \otimes Y$ is

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \colon u = \sum_{i=1}^n x_i \otimes y_i, x_i \in X, y_i \in Y \right\}.$$

Exercise. $\| \cdot \|_\pi$ is a seminorm on $X \otimes Y$.

Notation. $S \subset X, T \subset Y$.

$$S \odot T = \{x \otimes y \colon x \in S, y \in T\} \subset X \otimes Y.$$

Proposition. $U_{X \otimes Y, \|\cdot\|_\pi} = \Gamma(U_X \odot U_Y) = \mathrm{conv}(U_X \odot U_Y)$. In particular, $\| \cdot \|_\pi$ = the Minkowski functional of $\Gamma(U_X \odot U_Y)$.

Notation. $X \otimes_\pi Y = (X \otimes Y, \| \cdot \|_\pi)$.

Proposition. $(X \otimes_\pi Y, \theta)$ is the projective tensor product of X and Y.

Proof. $\theta(U_X \times U_Y) = U_X \odot U_Y \subset \Gamma(U_X \odot U_Y) \implies \theta$ is a contraction.

$$\begin{tikzcd} X \times Y \arrow[r, "\Phi"] \arrow[d, "\theta"'] & Z \\ X \otimes_\pi Y \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Φ = a bilinear contraction
φ = the linearization of Φ. (it exists, it is unique).

$\Phi(U_X \times U_Y) \subset U_Z \implies \varphi(U_X \odot U_Y) \subset U_Z \implies \varphi(\Gamma(U_X \odot U_Y)) \subset U_Z \implies \varphi$ is a contraction. $\square$

Lecture 10 (2023.11.10)

Projective tensor product of seminormed spaces

$$\begin{tikzcd} X \times Y \arrow[r, "\text{contraction } \Phi"'] \arrow[d, "\theta"'] & Z \\ X \otimes_\pi Y \arrow[ur, "\exists! \text{ linear contraction } \varphi"'] & \end{tikzcd} \qquad \|\theta\| \le 1 (\text{contraction}) \qquad (\dagger)$$

$$X \otimes Y \quad \theta \colon X \times Y \to X \otimes Y$$
$$(x, y) \mapsto x \otimes y.$$

Projective tensor seminorm: $\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \colon u = \sum_{i=1}^n x_i \otimes y_i, x_i \in X, y_i \in Y \right\}$

$$U_{X \otimes_\pi Y} = \Gamma(U_X \odot U_Y) = \operatorname{conv}(U_X \odot U_Y), \quad U_X \odot U_Y = \{x \otimes y \colon x \in U_X, y \in U_Y\}.$$

Proposition. $\forall$ seminormed Z, $\forall \Phi \in \mathscr{L}^{(2)}(X \times Y, Z)$ $\exists$ a unique linear $\varphi \in \mathscr{L}(X \otimes_\pi Y, Z)$ s.t. ($\dagger$) commutes; moreover, $\|\varphi\| = \|\Phi\|$.

Proof. Choose any $C > \|\Phi\|$; $\Phi' = \Phi/C$ is a bilinear contraction; $\varphi' \colon X \otimes_\pi Y \to Z$ linearizes Φ' $\implies$ $\varphi = C\varphi'$ linearizes Φ; $\|\varphi\| \le C \implies \|\varphi\| \le \|\Phi\|$. $\Phi = \varphi \circ \theta \implies \|\Phi\| \le \|\varphi\| \|\theta\| \le \|\varphi\| \implies \|\varphi\| = \|\Phi\|$. $\square$

Algebraic notation

$f_i \colon X_i \to Y_i (i = 1, 2)$ linear maps.

$f_1 \otimes f_2 \colon X_1 \otimes X_2 \to Y_1 \otimes Y_2$ the linear maps uniquely defined by $x_1 \otimes x_2 \mapsto f_1(x_1) \otimes f_2(x_2) (x_i \in X_i)$.

If $Y_1 = Y_2 = \mathbb{K}$, then

$$\begin{tikzcd} X_1 \otimes X_2 \arrow[r, "f_1 \otimes f_2"] \arrow[rr, bend right, "f_1 \otimes f_2"'] & \mathbb{K} \otimes \mathbb{K} \cong & \mathbb{K} \end{tikzcd}$$

$$C[0,1] \widehat{\otimes}_\pi C[0,1] \subsetneqq C([0,1]^2).$$

Definition. X, Y = seminormed; α = a seminorm on $X \otimes Y$. α is a reasonable cross-seminorm if

(C1) $\alpha(x \otimes y) \le \|x\| \|y\| (x \in X, y \in Y)$

(C2) If $f \in X', g \in Y' \implies f \otimes g \in (X \otimes Y, \alpha)'$, and $\|f \otimes g\| \le \|f\| \|g\|$.

Observation. (C2) $\iff$ (C2$'$): If $f \in X', g \in Y', \|f\| \le 1, \|g\| \le 1 \implies f \otimes g \in (X \otimes Y, \alpha)'$ and $\|f \otimes g\| \le 1$.

Proposition. If α is a reasonable cross-seminorm, then we have equalities in (C1), (C2).

Proof.

(C2)

$$\begin{aligned} \|f\| \|g\| &= \sup \{ |(f \otimes g)(x \otimes y)| \colon \|x\| \le 1, \|y\| \le 1 \} \\ &\le \sup \{ |(f \otimes g)(x \otimes y)| \colon \alpha(x \otimes y) \le 1 \} \\ &\le \|f \otimes g\| \implies \text{equalities in (C2).} \end{aligned}$$

(C1) Let $x \in X, y \in Y$. Hahn-Banach $\implies$ $\exists f \in X', g \in Y'$ s.t. $\|f\| = \|g\| \le 1, f(x) = \|x\|, g(y) = \|y\| \implies \|x\| \|y\| = |(f \otimes g)(x \otimes y)| \le \alpha(x \otimes y) \implies$ equality in (C1). $\square$

Remark. A seminorm α on $X \otimes Y$ satisfying

$$\alpha(x \otimes y) = \|x\| \|y\|$$

is a cross-seminorm.

Proposition. $\|\cdot\|_\pi$ is the greatest reasonable cross-seminorm on $X \otimes Y$.

Proof. (C1) $\iff \|\theta\| \le 1$; this is true for $\|\cdot\|_\pi$.
(C2$'$)

$$\begin{tikzcd} X \times Y \arrow[r, "\Phi"] \arrow[d] & \mathbb{K} \\ X \otimes_\pi Y \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\Phi(x,y) = f(x)g(y)$ bilinear contraction $\implies \varphi$ is a contraction; $\varphi = f \otimes g$.

Suppose α is a reasonable cross-seminorm

$$\begin{tikzcd} & X \times Y \arrow[dl, "\theta"'] \arrow[dr, "\theta_\alpha"] & \\ X \otimes_\pi Y \arrow[rr, "I"'] & & (X \otimes Y, \alpha) \end{tikzcd}$$

$\|\theta_\alpha\| \le 1$ by (C1) for α
$\implies \|I\| \le 1$, that is, $I(u) = u \,\forall u$. $\quad \alpha \le \|\cdot\|_\pi$. ☐

Definition. The injective tensor seminorm on $X \otimes Y$ is

$$\|u\|_\varepsilon = \sup\left\{|(f \otimes g)(u)| \colon f \in X', g \in Y', \|f\| \le 1, \|g\| \le 1\right\}.$$

A construction

$\forall x \in X, y \in Y$ $\Phi_{x,y} \colon X' \times Y' \to \mathbb{K}, \Phi_{x,y}(f,g) = f(x)g(y)$ bilinear. $\|\Phi_{x,y}\| \le \|x\|\|y\|$.

$$\begin{tikzcd} X \times Y \arrow[r, "\Phi"] \arrow[d] & \mathscr{L}^{(2)}(X' \times Y', \mathbb{K}) \\ X \otimes Y \arrow[ur, "\varphi \text{ linear}"'] & \end{tikzcd}$$

$(x,y) \mapsto \Phi_{x,y}$ bilinear.

Observation.

(1) $\varphi(u)(f,g) = (f \otimes g)(u) \quad \forall u \in X \otimes Y$.

(2) $\|u\|_\varepsilon = \|\varphi(u)\|$.

In particular, $\|u\|_\varepsilon < \infty$, and $\|\cdot\|_\varepsilon$ is a seminorm.

Definition. $(X \otimes Y, \|\cdot\|_\varepsilon) = X \otimes_\varepsilon Y$ the injective tensor product of X, Y.

Observation. (2) $\iff$ we have an isometry $X \otimes_\varepsilon Y \to (X' \otimes_\pi Y')'$.

Proposition. $\|\cdot\|_\varepsilon$ is the smallest reasonable cross-seminorm on $X \otimes Y$.

Proof. $\|\cdot\|_\varepsilon$ satisfies (C2$'$) and it is the smallest seminorm satisfying (C2$'$) (see Definition).

$$\|x \otimes y\|_\varepsilon = \|x\|\|y\| \implies \text{(C1) for } \|\cdot\|_\varepsilon.$$ ☐

Corollary/Exercise. A seminorm α on $X \otimes Y$ is a reasonable cross-seminorm $\iff \|\cdot\|_\varepsilon \le \alpha \le \|\cdot\|_\pi$.

Tensor products of lcs's

Definition. The projective tensor product of lcs's X, Y is $(X \otimes_\pi, \theta)$ where $X \otimes_\pi Y$ is a lcs, $\theta \in \mathscr{L}^{(2)}(X \times Y, X \otimes_\pi Y)$ s.t. $\forall$ lcs Z

$$\begin{tikzcd} X \times Y \arrow[r, "\Phi"] \arrow[d, "\theta"'] & Z \\ X \otimes_\pi Y \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\forall \Phi \in \mathscr{L}^{(2)}(X \times Y, Z), \exists$ a unique $\varphi \in \mathscr{L}(X \otimes_\pi Y, Z)$ s.t. the diagram commutes.

The complete projective tensor product of X, Y is $(X \widehat{\otimes}_\pi Y, \widehat{\theta})$ where $X \widehat{\otimes}_\pi Y$ is a complete lcs, $\widehat{\theta} \in \mathscr{L}(X \times Y, X \widehat{\otimes}_\pi Y)$ satisfying the same condition for complete Z.

Construction

$\forall p \in \mathrm{SN}(X), q \in \mathrm{SN}(Y)$, $p \otimes_\pi q$ = the respective projective tensor seminorm on $X \otimes Y$. P, Q = directed defining families of seminorms on X, Y.

$$X \otimes_\pi Y = (X \otimes Y; \tau\{p \otimes_\pi q \colon p \in P, q \in Q\}).$$

$$\theta \colon X \times Y \to X \otimes_\pi Y, (x, y) \mapsto x \otimes y.$$

$X \widehat{\otimes}_\pi Y = (X \otimes_\pi Y)^\sim$ (complete).

$$\begin{tikzcd} X \times Y \arrow[r, "\theta"] \arrow[rr, bend right, "\widehat{\theta}"'] & X \otimes_\pi Y \arrow[r, "\text{canonical}"] & X \widehat{\otimes}_\pi Y \end{tikzcd}$$

Proposition.

(1) $(X \otimes_\pi Y, \theta)$ is a projective tensor product of X, Y.

(2) $(X \widehat{\otimes}_\pi Y, \widehat{\theta})$ is a complete projective tensor product of X, Y.

Notation. $X = \mathrm{lcs}, p \in \mathrm{SN}(X)$. $X^p = (X, p)$ a seminormed space.

Proof.

(1) Choose a defining family R of seminorms on Z. Φ is continuous $\implies$ $r \in R, \exists p \in P, q \in Q$ s.t. $\Phi \colon X^p \times Y^q \to Z^r$ is continuous. φ = the linearization of Φ. $\implies$ $(X \otimes Y)^{p \otimes_\pi q} = X^p \otimes_\pi Y^q \to Z^r$ is continuous $\implies$ $\varphi \colon X \otimes_\pi Y \to Z$ is continuous.

(2) Exercise (see Definition). $\square$

Exercise.

(1) $\mathscr{U}$ = a base of absolutely convex neighborhoods of 0 in X. $\mathscr{V}$ = a base of absolutely convex neighborhoods of 0 in Y. $\implies$ $\{\Gamma(U \odot V) U \in \mathscr{U}, V \in \mathscr{V}\}$ is a base of neighborhoods of 0 in $X \otimes_\pi Y$.

(2) The topology on $X \otimes_\pi Y$ is the strongest locally convex topology on $X \otimes Y$ making θ continuous.

Proposition.

(1) $\varphi_i \in \mathscr{L}(X_i, Y_i) (i = 1, 2)$ $\implies$ $\exists$ a unique $\varphi_1 \otimes \varphi_2 \in \mathscr{L}(X_1 \otimes_\pi X_2, Y_1 \otimes_\pi Y_2)$ s.t. $(\varphi_1 \otimes \varphi_2)(x_1 \otimes x_2) = \varphi_1(x_1) \otimes \varphi_2(x_2)$.

(2) If X_i, Y_i are seminormed spaces, then

$$\|\varphi_1 \otimes \varphi_2\| = \|\varphi_1\| \|\varphi_2\|.$$

Proof. (1)

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "\psi_1 \times \psi_2"] \arrow[d, "\theta_1"'] & Y_1 \times Y_2 \arrow[d, "\theta_2"] \\ X_1 \otimes_\pi X_2 \arrow[r, "\varphi_1 \otimes \varphi_2"] & Y_1 \otimes_\pi Y_2 \end{tikzcd}$$

$\theta_2 \circ (\varphi_1 \times \varphi_2)$ is a continuous bilinear map $\implies$ $\varphi_1 \otimes \varphi_2$ is continuous.

(2) $\|\varphi_1 \otimes \varphi_2\| = \|\theta_2 \circ (\varphi_1 \times \varphi_2)\| \overset{\text{exercise}}{=\!=\!=} \|\varphi_1\| \|\varphi_2\|$. $\square$

Corollary. We have functors

$$\otimes_\pi \colon \mathrm{LCS} \times \mathrm{LCS} \to \mathrm{LCS}$$

$$\widehat{\otimes}_\pi \colon \mathrm{LCS} \times \mathrm{LCS} \to \mathrm{CLCS}$$

and natural isomorphisms

$$\mathscr{L}(X \otimes_\pi Y, Z) \cong \mathscr{L}^{(2)}(X \times Y, Z) \quad (Z \in \mathrm{LCS})$$

$$\mathscr{L}(X \widehat{\otimes}_\pi Y, Z) \cong \mathscr{L}^{(2)}(X \times Y, Z) \quad (Z \in \mathrm{CLCS}).$$

Lecture 11 (2023.11.17)

The injective tensor product of lcs's

$X, Y =$ lcs's, $p \in \mathrm{SN}(X), q \in \mathrm{SN}(Y)$.

Notation. $p \otimes_\varepsilon q =$ the injective tensor seminorm on $X \otimes Y$ associated to p, q.

Recall: $(p \otimes_\varepsilon q)(u) = \sup\{|(f \otimes g)(u)| \colon f \in (X,p)', \|f\| \le 1; g \in (X,q)', \|g\| \le 1\}$.

Definition. The injective tensor product of X and Y is

$$X \otimes_\varepsilon Y = (X \otimes Y, \tau\{p \otimes_\varepsilon q \colon p \in \mathrm{SN}(X), q \in \mathrm{SN}(Y)\}).$$

The complete injective tensor product is

$$X \widehat{\otimes}_\varepsilon Y = \text{the completion of } X \otimes_\varepsilon Y.$$

Exercise. $P, Q =$ defining directed families of seminorms on X, Y, respectively. We can replace $\mathrm{SN}(X)$ by P and $\mathrm{SN}(Y)$ by Q.

Observation $(\varepsilon \subset \pi)$**.**

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "\theta_\pi"'] \arrow[dr, "\theta_\varepsilon"] \arrow[ddl, bend right, "\widehat{\theta}_\pi"'] \arrow[ddr, bend left, "\widehat{\theta}_\varepsilon"] & \\
X \otimes_\pi Y \arrow[rr, "I"] \arrow[d] & & X \otimes_\varepsilon Y \arrow[d] \\
X \widehat{\otimes}_\pi Y \arrow[rr, "\widehat{I}"] & & X \widehat{\otimes}_\varepsilon Y
\end{tikzcd}$$

$I(u) = u \quad (u \in X \otimes Y)$.

$p \otimes_\varepsilon q \le p \otimes_\pi q \implies I$ is continuous, that is, $\varepsilon \subset \pi$

$\implies \theta_\varepsilon$ is continuous.

Warning In general, $\widehat{I}$ is neither surjective, nor injective.

Proposition.

(1) $\varphi_i \in \mathscr{L}(X_i, Y_i) (i = 1, 2) \implies \exists$ a unique continuous $\varphi_1 \otimes_\varepsilon \varphi_2 \colon X_1 \otimes_\varepsilon X_2 \to Y_1 \otimes_\varepsilon Y_2$ s.t. $(\varphi_1 \otimes \varphi_2)(x_1 \otimes x_2) = \varphi_1(x_1) \otimes \varphi_2(x_2) (\forall x_1, x_2)$.

(2) If X_i, Y_i are seminormed, then $\|\varphi_1 \otimes_\varepsilon \varphi_2\| = \|\varphi_1\| \|\varphi_2\|$.

(3) Similar results hold for $\widehat{\otimes}_\varepsilon$.

Proof. (2)

$$\begin{tikzcd}
X_1 \otimes_\varepsilon X_2 \arrow[r, "\varphi_1 \otimes_\varepsilon \varphi_2"] \arrow[d, "\text{isometry}"] & Y_1 \otimes_\varepsilon Y_2 \arrow[d, "\text{isometry}"] \\
(X_1' \otimes_\pi X_2')' \arrow[r, "(\varphi_1' \otimes_\pi \varphi_2')'"] & (Y_1' \otimes_\pi Y_2')'
\end{tikzcd}$$

commutes (exercise).

$\implies \|\varphi_1 \otimes_\varepsilon \varphi_2\| \le \|(\varphi_1' \otimes_\pi \varphi_2')'\| = \|\varphi_1' \otimes_\pi \varphi_2'\| = \|\varphi_1'\| \|\varphi_2'\| = \|\varphi_1\| \|\varphi_2\|$.

Exercise. $\|\varphi_1\| \|\varphi_2\| \le \|\varphi_1 \otimes_\varepsilon \varphi_2\|$.

(1) Take $q_i \in \mathrm{SN}(Y_i) (i = 1, 2)$; let $p_i = q_i \circ \varphi_i \implies p_i \in \mathrm{SN}(X_i)$, and $\varphi_i \colon X_i^{p_i} \to Y_i^{q_i}$ is continuous.

$$\implies \begin{array}{ccc} X_1^{p_1} \otimes_\varepsilon X_2^{p_2} & \longrightarrow & Y_1^{q_1} \otimes_\varepsilon Y_2^{q_2} \\ \| & & \| \\ (X_1 \otimes_\varepsilon X_2)^{p_1 \otimes_\varepsilon p_2} & & (Y_1 \otimes_\varepsilon Y_2)^{q_1 \otimes_\varepsilon q_2} \end{array} \quad \text{is continuous by (2).}$$

$\implies \varphi_1 \otimes_\varepsilon \varphi_2 \colon X_1 \otimes_\varepsilon X_2 \to Y_1 \otimes_\varepsilon Y_2$ is continuous.

(3) Follows from (1), (2) and the universal property of completions. □

Corollary. We have functors

$$\otimes_\varepsilon\colon \text{ LCS} \times \text{LCS} \to \text{LCS}$$
$$\widehat{\otimes}_\varepsilon\colon \text{ LCS} \times \text{LCS} \to \text{CLCS}$$

Proposition. $X, Y =$ lcs's. TFAE:

(1) $X \otimes_\pi Y$ is Hausdorff;

(2) $X \otimes_\varepsilon Y$ is Hausdorff;

(3) X and Y are Hausdorff.

Proof. (2) $\implies$ (1) clear ($\varepsilon \subset \pi$).
(1) $\implies$ (3) exercise.
(3) $\implies$ (2) Let $u \in X \otimes_\varepsilon Y, u \neq 0$. $\implies$ $u = \sum_{i=1}^{n} x_i \otimes y_i$; $x_1, \ldots, x_n$ are linear independent; $y_1 \neq 0$. Hahn-Banach $\implies \exists f \in X'$ s.t. $f(x_1) \neq 0, f(x_2) = \cdots = f(x_n) = 0$. $\exists g \in Y'$ s.t. $g(y_1) \neq 0$.

$$f \otimes g \in (X \otimes_\varepsilon Y)', (f \otimes g)(u) = f(x_1)g(y_1) \neq 0 \implies X \otimes_\varepsilon Y \text{ is Hausdorff.}$$ □

Corollary. $X, Y =$ normed spaces $\implies$ $X \otimes_\pi Y, X \otimes_\varepsilon Y$ are normed spaces; $X \widehat{\otimes}_\pi Y, X \widehat{\otimes}_\varepsilon Y$ are Banach spaces.

Exercise.

(1) Prove the commutativity and associativity for $\otimes_\pi, \otimes_\varepsilon, \widehat{\otimes}_\pi, \widehat{\otimes}_\varepsilon$.

(2) Prove: $(X \oplus Y) \otimes_\pi Z \cong (X \otimes_\pi Z) \oplus (Y \otimes_\pi Z)$, and similarly for $\otimes_\varepsilon, \widehat{\otimes}_\pi, \widehat{\otimes}_\varepsilon$.

(3) $X, Y, Z =$ seminormed space.
Construct natural isometric isomorphisms

$$\mathscr{L}(X \otimes_\pi Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y, Z))$$

and

$$\mathscr{L}(X \widehat{\otimes}_\pi Y, Z) \cong \mathscr{L}(X, \mathscr{L}(Y, Z)) \quad \text{(if } Z \text{ is a Banach space).}$$

(Such isomorphisms are called the adjoint associativity or exponential law).

Examples of topological tensor products

Notation. $X =$ seminormed.

$$X_1^n = (X^n, \|\cdot\|_1) \quad \|(x_1, \ldots, x_n)\|_1 = \sum \|x_i\|.$$
$$X_\infty^n (X^n, \|\cdot\|_\infty) \quad \|(x_1, \ldots, x_n)\|_\infty = \max_{1 \le i \le n} \|x_i\|.$$

Proposition. $\exists$ isometric isomorphisms

(1) $\mathbb{K}_1^n \otimes_\pi X \cong X_1^n$

(2) $\mathbb{K}_\infty^n \otimes_\varepsilon X \cong X_\infty^n$

Proof.

(1) Let $e_i = (0, \ldots, 1, 0, \ldots)$ be the usual basis of $\mathbb{K}^n$.
We have the algebraic isomorphism

$$e_i \otimes x \mapsto (0, \ldots, 0, x, 0, \ldots)$$

where x in the nth slot and 0 elsewhere.

Now the only point is to show that it is isometric. So to show that it is isometric, we use the universal property of tensor product which implies that this map is a contraction and the fact that the opposite map is a contraction follows immediately from the triangle inequality. Hence this map is actually isometric.

(2)

$$\begin{tikzcd}
\mathbb{K}^n_\infty \otimes_\varepsilon X \arrow[r, "\text{isometry}"] \arrow[drrr, "\text{(algebraic isomorphism)}"'] & (\mathbb{K}^n_1 \otimes_\pi X')' \arrow[r, phantom, "\cong"] & ((X')^n_1)' \arrow[r, phantom, "\cong"] & (X'')^n_\infty \\
 & & & X^n_\infty \arrow[u, "\text{isometry}"']
\end{tikzcd} \implies \text{this is an isometric isomorphism.}$$

$\square$

Example/Exercise. $M_n(\mathbb{K}) = n \times n$-matrices over $\mathbb{K}$.

$$\mathbb{K}^n_1 \otimes_\pi \mathbb{K}^n_\infty \xrightarrow{\sim} M_n(\mathbb{K}) \xleftarrow{\sim} \mathbb{K}^n_1 \otimes_\varepsilon \mathbb{K}^n_\infty$$

$$e_i \otimes e_j \longmapsto e_{ij}$$

Find an explicit formula for $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$ on $M_n(\mathbb{K})$ and show that $\|\cdot\|_\pi \neq \|\cdot\|_\varepsilon$ (if $n \geq 2$).

$I =$ a set.

$$\ell^1(I) = \left\{ x = (x_i) \in \mathbb{K}^I : \|x\|_1 = \sum_{i \in I} |x_i| < \infty \right\} \text{is a Banach space.}$$

Proposition 1. $\exists$ an isometric isomorphism

$$\ell^1(I) \widehat{\otimes}_\pi \ell^1(J) \xrightarrow{\sim} \ell^1(I \times J), e_i \otimes e_j \mapsto e_{(i,j)}.$$

Here $e_i \in \ell^1(I), (e_i)_j = \delta_{ij} (i, j \in I)$

Proof.

$$\begin{tikzcd}
\ell^1(I) \times \ell^1(J) \arrow[r, "\Phi"] \arrow[d] & \ell^1(I \times J) \\
\ell^1(I) \widehat{\otimes}_\pi \ell^1(J) \arrow[ur, "\varphi"'] &
\end{tikzcd} \qquad \Phi(x, y)_{ij} = x_i y_j.$$

Φ is bilinear, $\|\Phi\| \leq 1$. φ is linear, $\|\varphi\| \leq 1$. $\varphi(e_i \otimes e_j) = e_{(i,j)}$.

Define $\psi \colon \ell^1(I \times J) \to \ell^1(I) \widehat{\otimes}_\pi \ell^1(J), \psi((u_{ij})) = \sum_{i,j} u_{ij} e_i \otimes e_j$.

$\|\psi\| \leq 1, \varphi\psi = 1, \psi\varphi = 1 \implies \varphi$ is an isometric isomorphism. $\square$

Notation. $E =$ Banach space.

$$\ell^1(I, E) = \left\{ x = (x_i) \in E^I : \|x\| = \sum_{i \in I} \|x_i\| < \infty \right\} \text{is a Banach space.}$$

Exercise. Construct an isometric isomorphism

$$\ell^1(I) \widehat{\otimes}_\pi E \cong \ell^1(I, E).$$

Proposition 2. $(X, \mu), (Y, \nu) =$ measure spaces (σ-finite).

Then $\exists$ an isometric isomorphism

$$L^1(X, \mu) \widehat{\otimes}_\pi L^1(Y, \nu) \underset{\varphi}{\xrightarrow{\sim}} L^1(X \times Y, \mu \times \nu), f \times g \mapsto h,$$

where $h(x, y) = f(x) g(y)$.

Proof. (a sketch) (assume μ, ν are finite).

The existence of φ and $\|\varphi\| \leq 1$ follow from the universal property of $\widehat{\otimes}_\pi$.

$$X = X_1 \bigsqcup \cdots \bigsqcup X_n, \quad Y = Y_1 \bigsqcup \cdots \bigsqcup Y_m \text{ measurable partitions.}$$

$$E = \operatorname{span} \left\{ \chi_{X_i} \otimes \chi_{Y_j} : 1 \leq i \leq n, 1 \leq j \leq m \right\}.$$

$$F = \operatorname{span} \left\{ \chi_{X_i \times Y_j} : 1 \leq i \leq n, 1 \leq j \leq m \right\}.$$

Exercise. $\varphi|_E \colon E \to F$ is an isometric isomorphism.

$$\bigcup_{\text{(partitions)}} E \text{ is dense in } L^1(X,\mu)\widehat{\otimes}_\pi L^1(Y,\nu)$$

$$\bigcup_{\text{(partitions)}} F \text{ is dense in } L^1(X\times Y,\mu\times\nu).$$ □

Köthe spaces

$I =$ a set. $P \subset [0,+\infty)^I, p \in P, p = (p_i)_{i\in I}$.

Definition. P is a Köthe set if

(1) $\forall i \in I, \exists p \in P$ s.t. $p_i > 0$.

(2) $\forall p,q \in P, \exists C > 0, \exists r \in P$ s.t. $\max\{p_i,q_i\} \le Cr_i (\forall i)$.

Definition. The Köthe space $\lambda^1(I,P)$ is

$$\lambda^1(I,P) = \left\{x = (x_i) \in \mathbb{K}^I \colon \forall p \in P \quad \|x\|_p = \|(x_ip_i)\|_{\ell^1(I)} < \infty\right\}.$$

Similarly we can define $\lambda^\nu(I,P)$ for $\nu \in [1,+\infty]$.

Exercise. $\lambda^\nu(I,P)$ is a vector subspace of $\mathbb{K}^I$ and is a complete lcs for $\tau(\{\|\cdot\|_p \colon p \in P\})$.

Example 1. $I = \mathbb{N}, P = \left\{p^{(1)}, p^{(2)}, \dots\right\}, p^{(m)} = (\underbrace{1,\dots,1}_{m}, 0, \dots) \implies \lambda^\nu(\mathbb{N},P) = \mathbb{K}^{\mathbb{N}}\ \forall\nu$ (exercise).

Example 2. $I = \mathbb{N}, P = \{(1,1,1,\dots)\} \implies \lambda^\nu(I,P) = \ell^\nu(I)$.

Example 3. $I = \mathbb{N}, P = \left\{p^{(1)}, p^{(2)}, \dots\right\}; p_k^{(m)} = k^m \implies \lambda^\infty(\mathbb{N},P) = s \cong \lambda^\nu(\mathbb{N},P)\ \forall\nu$ (exercise).

Example 4. $I = \mathbb{Z}_{\ge 0}, P = \left\{p^{(1)}, p^{(2)}, \dots\right\}, p_k^{(m)} = m^k \implies \lambda^\nu(\mathbb{Z}_{\ge 0},P) \cong \mathcal{O}(\mathbb{C})\ \forall\nu$ (exercise).
Just consider

$$f \mapsto \text{Taylor coefficients at } 0.$$

Example 5. $I = \mathbb{Z}^n, P = \left\{p^{(1)}, p^{(2)}, \dots\right\}, p_k^{(m)} = (|k|+1)^m (k = (k_1,\dots,k_n), |k| = k_1+\dots+k_n) \implies \lambda^\nu(\mathbb{Z},P) \cong s(\mathbb{Z}^n) \underset{\text{Fourier}}{\cong} C^\infty(\mathbb{T}^n) \quad \forall\nu$.

Example 6. $R = (R_1,\dots,R_n) \in (0,+\infty]^n, \mathbb{D}_R^n = \{z = (z_1,\dots,z_n) \in \mathbb{C}^n \colon |z_i| < R_i\ \forall i\}$ is the polydisc of polyradius R.

$I = \mathbb{Z}_{\ge 0}^n; P = \left\{(r_1^{k_1}\cdots r_n^{k_n})_{k=(k_1,\dots,k_n)\in\mathbb{Z}_{\ge 0}^n} \colon 0 < r_i < R_i\right\} \implies \lambda^\nu(\mathbb{Z}_{\ge 0}^n,P) \cong \mathcal{O}(\mathbb{D}_R^n)\ \forall\nu$ (exercise).

Example 7. $\mathcal{O}_0 = \{\text{germs of holomorphic functions at } 0 \in \mathbb{C}\}$.
$I = \mathbb{Z}_{\ge 0};\ P = \left\{p = (p_n) \in (0,+\infty)^{\mathbb{Z}_+} \colon p_n = o(\varepsilon^n)\,\forall\varepsilon > 0 (n \to \infty)\right\}$.

Exercise*. $\lambda^\nu(\mathbb{Z}_{\ge 0},P) \cong \mathcal{O}_0\ \forall\nu$. As a corollary, $\mathcal{O}_0$ is complete.

Lecture 12 (2023.11.24)

Recall:

Proposition 1.

$$\ell^1(I)\widehat{\otimes}_\pi \ell^1(J) \cong \ell^1(I\times J)$$
$$e_i \otimes e_j \mapsto e_{(i,j)} \quad (e_i)_j = \delta_{ij}.$$

Exercise. $\ell^1(I)\widehat{\otimes}_\pi E \cong \ell^1(I,E)$.

Proposition 2.

$$L^1(X,\mu)\widehat{\otimes}_\pi L^1(Y,\nu) \cong L^1(X\times Y, \mu\times\nu)$$
$$f\otimes g \mapsto h; \quad h(x,y) = f(x)g(y).$$

Köthe spaces

$P \subset [0,+\infty)^I$ (I = a set)

(1) $\forall i \in I\ \exists p \in P$ s.t. $p_i > 0$.

(2) $\forall p,q \in P\ \exists C>0\ \exists r \in P$ s.t. $\max\{p_i, q_i\} \le C r_i\ \forall i$.

P is a Köthe set if (1), (2) are satisfied.
The Köthe space is

$$\lambda^1(I,P) = \left\{ x = (x_i) \in \mathbb{K}^I : \forall p \in P \quad \|x\|_p = \|(x_i p_i)\|_{\ell^1(I)} < \infty \right\}.$$

Special cases. $\ell^1(I); \mathbb{K}^I; s; s(\mathbb{Z}^n); \mathcal{O}(\mathbb{C}); \mathcal{O}(\mathbb{D}^n_R)$.

Notation. $P \subset [0,+\infty)^I$; $Q \subset [0,+\infty)^J$ Köthe sets.
Define $P \odot Q \subset [0,+\infty)^{I\times J}$,

$$P \odot Q = \left\{ (p_i q_j)_{(i,j)\in I\times J} : p \in P, q \in Q \right\}$$

is a Köthe set on $I \times J$.

Proposition 3. $\exists$ a topological isomorphism

$$\lambda^1(I,P)\widehat{\otimes}_\pi \lambda^1(J,Q) \xrightarrow{\sim} \lambda^1(I\times J; P\odot Q)$$
$$e_i \otimes e_j \mapsto e_{(i,j)}$$

Proof. Exercise (similar to Proposition 1). ☐

Corollary 1. $\exists$ a topological isomorphism

$$C^\infty(\mathbb{T}^m)\widehat{\otimes}_\pi C^\infty(\mathbb{T}^n) \xrightarrow{\sim} C^\infty(\mathbb{T}^{m+n})$$
$$f\otimes g \mapsto h; \quad h(x,y) = f(x)g(y)$$

Proof. (a sketch)

$$\begin{tikzcd}
s(\mathbb{Z}^m)\widehat{\otimes}_\pi s(\mathbb{Z}^n) \arrow[r, "\sim"] & s(\mathbb{Z}^{m+n}) \\
C^\infty(\mathbb{T}^m)\widehat{\otimes}_\pi C^\infty(\mathbb{T}^n) \arrow[r, "\sim"] \arrow[u, "\sim"] & C^\infty(\mathbb{T}^{m+n}) \arrow[u, "\sim(\text{Fourier})"']
\end{tikzcd}$$ (Proposition 3) ☐

Corollary 2. $\exists$ a topological isomorphism

$$\mathcal{O}(\mathbb{D}^n_R)\widehat{\otimes}_\pi \mathcal{O}(\mathbb{D}^n_S) \xrightarrow{\sim} \mathcal{O}(\mathbb{D}^{m+n}_{(R,S)})$$
$$f\otimes g \mapsto h, \quad h(x,y) = f(x)g(y)$$

Proof. Similar to Corollary 1. ☐

E = seminormed.

Definition. A set $S \subset \overline{U_{E'}}$ is norming if $\forall x \in E$

$$\|x\| = \sup_{f \in S} |f(x)|.$$

Example.

(1) $\overline{U}_{E'}$ is norming (Hahn-Banach).

(2) Any dense subset of $\overline{U}_{E'}$ is norming.

(3) $E = C(X)$ X = a compact topological space.
$\forall x \in X, \mathrm{ev}_x \colon C(X) \to \mathbb{K}, \mathrm{ev}_x(f) = f(x) \implies \{\mathrm{ev}_x \colon x \in X\}$ is a norming set.

Lemma. E, F = seminormed, $S \subset \overline{U}_E, T \subset \overline{U}_F$, norming sets $\implies \forall u \in E \otimes F$,

$$\|u\|_\varepsilon = \sup \{|(f \otimes g)(u)| \colon f \in S, g \in T\}.$$

Notation. E, F = vector spaces, $f \colon E \to \mathbb{K}, g \colon F \to \mathbb{K}$ linear.

$$\begin{tikzcd} E \otimes F \arrow[r, "f \otimes 1_F"] \arrow[rr, bend right, "f \otimes 1"'] & \mathbb{K} \otimes F \arrow[r, phantom, "\cong"] & F \end{tikzcd}$$

$$\begin{tikzcd} E \otimes F \arrow[r, "1_E \otimes g"] \arrow[rr, bend right, "1 \otimes g"'] & E \otimes \mathbb{K} \arrow[r, phantom, "\cong"] & E \end{tikzcd}$$

Observation. $f \otimes g = f \circ (1 \otimes g) = g \circ (f \otimes 1)$.

Proof of Lemma.

$$\begin{aligned} \|u\|_\varepsilon &= \sup_{\|g\| \le 1} \sup_{\|f\| \le 1} |(f \otimes g)(u)| = \sup_{\|g\| \le 1} \underbrace{\sup_{\|f\| \le 1} |f((1 \otimes g)(u))|}_{\|(1 \otimes g)(u)\|} \\ &= \sup_{\|g\| \le 1} \sup_{f \in S} |f((1 \otimes g)(u))| = \sup_{f \in S} \underbrace{\sup_{\|g\| \le 1} |g((f \otimes 1)(u))|}_{\|(1 \otimes g)(u)\|} \\ &= \sup_{f \in S} \sup_{g \in T} |(f \otimes g)(u)|. \end{aligned}$$

□

Proposition 4. X, Y = compact Hausdorff topological spaces $\implies \exists$ an isometric isomorphism

$$C(X) \widehat{\otimes}_\varepsilon C(Y) \xrightarrow{\sim} C(X \times Y)$$
$$f \otimes g \mapsto h, \quad h(x, y) = f(x) g(y).$$

Proof.

$$\begin{tikzcd} C(X) \times C(Y) \arrow[r, "\Phi"] \arrow[d] & C(X \times Y) \\ C(X) \otimes C(Y) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\Phi(f, g)(x, y) = f(x) g(y)$ bilinear.

$\varphi(u)(x, y) = (\mathrm{ev}_x \otimes \mathrm{ev}_y)(u) \implies \|\varphi(u)\| = \sup_{x \in X, y \in Y} |(\mathrm{ev}_x \otimes \mathrm{ev}_y)(u)| \overset{\text{Lemma}}{=\!=\!=} \|u\|_\varepsilon$. $\forall u \in C(X) \otimes C(Y) \implies \varphi$ is an isometry (with respect to $\| \cdot \|_\varepsilon$). Stone-Weierstrass $\implies \varphi$ has dense range $\implies \varphi$ extends to an isometric isomorphism

$$C(X) \widehat{\otimes}_\varepsilon C(Y) \cong C(X \times Y).$$

□

Exercise. E = Banach space; X = compact Hausdorff topological space. Construct an isometric isomorphism

$$C(X)\widehat{\otimes}_\varepsilon E \xrightarrow{\sim} C(X,E).$$

Notation. I = a set, E = Banach space

$$c_0(I,E)=\Big\{x=(x_i)\in E^I\colon \lim_{i\to\infty}\|x_i\|=0\Big\}\subset \ell^\infty(I,E)$$

closed subspace $\implies$ Banach space; $\|x\|=\sup\limits_{i\in I}\|x_i\|$.

Exercise. Construct an isometric isomorphism

$$c_0(I)\widehat{\otimes}_\varepsilon E\cong c_0(I,E).$$

Recall: E,F = lcs's

$$\begin{aligned}&E\widehat{\otimes}_\pi F\to E\widehat{\otimes}_\varepsilon F \text{ canonical map}\\ &x\otimes y\mapsto x\otimes y.\end{aligned}$$

Exercise.

$$\begin{array}{ccc}\ell^1\widehat{\otimes}_\pi c_0 & \longrightarrow & \ell^1\widehat{\otimes}_\varepsilon c_0\\ \| & & \| \\ \ell^1(\mathbb{N},c_0) & & c_0(\mathbb{N},\ell^1)\end{array}\quad \text{is neither topological injective nor surjective.}$$

Fact ($\approx$ Dvoretzkl-Rogers) E = an infinite-dim Banach space.

$$\ell^1\widehat{\otimes}_\pi E\to \ell^1\widehat{\otimes}_\varepsilon E \text{ is neither topological injective nor surjective.}$$

Proposition 5. X,Y = locally compact Hausdorff topological spaces. Then $\exists$ a topological isomorphism

$$\begin{aligned}&C(X)\widehat{\otimes}_\varepsilon C(Y)\xrightarrow{\sim} C(X\times Y)\\ &f\otimes g\mapsto h,\quad h(x,y)=f(x)g(y).\end{aligned}$$

Proof. (a sketch) $K_1\subset X, K_2\subset Y$ compact.

$$\begin{tikzcd} C(X)\otimes_\varepsilon C(Y) \arrow[r, "\alpha"] \arrow[d, "r_1\otimes r_2"] & C(X\times Y) \arrow[d, "r"] \\ C(K_1)\otimes_\varepsilon C(K_2) \arrow[r, "\text{isometry with dense range}"] & C(K_1\times K_2) \end{tikzcd}$$

where r_1,r_2,r = restriction maps. $\implies$ α is topological injective and has dense range (the density follows from $C(X\times Y)=\varprojlim C(K_1\times K_2)$). $\square$

Exercise. $U\subset\mathbb{R}^m, V\subset\mathbb{R}^n$ open $\implies$ $\exists$ a topological isomorphism

$$\begin{aligned}&C^\infty(U)\widehat{\otimes}_\varepsilon C^\infty(V)\xrightarrow{\sim} C(U\times V)\\ &f\otimes g\mapsto h,\quad h(x,y)=f(x)g(y).\end{aligned}$$

Exercise. $\mathscr{S}(\mathbb{R}^m)\widehat{\otimes}_\varepsilon\mathscr{S}(\mathbb{R}^n)\cong\mathscr{S}(\mathbb{R}^{m+n})$.

Some functorial properties of $\otimes_\pi$ and $\otimes_\varepsilon$

$\mathscr{A}$ = a category with 0. $X\xrightarrow{\varphi}Y\xrightarrow{\psi}Z, \psi\varphi=0$.

Definition. (Z,ψ) is a cokernel of φ if $\forall W\in\operatorname{Ob}\mathscr{A}$ the sequence

$$0\to\operatorname{Hom}(Z,W)\to\operatorname{Hom}(Y,W)\to\operatorname{Hom}(X,W) \text{ is exact.}$$

Exercise. This is equivalent to our earlier definition.

Definition. $X, Y = $ lcs's; $\varphi \in \mathscr{L}(X,Y)$. φ is <u>strict</u> if $\varphi\colon X \to \varphi(X)$ is open.

Exercise. (Z, ψ) is cokernel of φ iff

(a) in LCS: $\varphi(X) = \psi^{-1}(0)$, and ψ is open $\iff$ $\exists$ a topological isomorphism

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr] & \\ Z & \cong & Y/\varphi(X) \end{tikzcd}$$

(b) in HCLS, Norm, Ban, Fr: $\overline{\varphi(X)} = \psi^{-1}(0)$, and ψ is open $\iff$ $\exists$ a topological isomorphism

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr] & \\ Z & \cong & Y/\overline{\varphi(X)} \end{tikzcd}$$

(c) in $\mathrm{CLCS}, \mathrm{Fr}, \mathrm{Ban}$: $\overline{\varphi(X)} = \psi^{-1}(0)$, and ψ is strict with dense range $\iff$ $\exists$ a topological isomorphism

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr] & \\ Z & \cong & (Y/\overline{\varphi(X)})^{\sim} \end{tikzcd}$$

(d) in SNorm_1: $\varphi(X) = \psi^{-1}(0)$, and ψ is a coisometry(i.e. $\psi(U_Y) = U_Z$) $\iff$ $\exists$ an isometric isomorphism as in (a).

Proposition.

(1) $\mathscr{A} \in \{\mathrm{CLCS}, \mathrm{Fr}, \mathrm{Ban}, \mathrm{Ban}_1\}, Y \in \mathrm{Ob}\,\mathscr{A}$. $\psi = \operatorname{cokernel}\varphi$ in $\mathscr{A}$ $\implies$ $\psi \widehat{\otimes}_\pi 1_Y = \operatorname{cokernel}(\varphi \widehat{\otimes}_\pi 1_Y)$ in $\mathscr{A}$.

(2) A similar result holds for $\otimes_\pi$ and $\mathscr{A} = \{\mathrm{LCS}, \mathrm{HLCS}, \mathrm{Norm}_{(1)}, \mathrm{SNorm}_{(1)}\}$.

Proof. (for $\mathscr{A} = \mathrm{CLCS}$).

$$x_1 \overset{\varphi}{\to} X_2 \overset{\psi}{\to} X_3 \quad X_1 \widehat{\otimes}_\pi Y \underset{\varphi\otimes 1}{\to} X_2 \widehat{\otimes}_\pi Y \underset{\psi\otimes 1}{\to} X_3 \widehat{\otimes}_\pi Y.$$

We want: $\forall$ complete lcs Z the sequence

$$0 \to \mathscr{L}(X_3 \widehat{\otimes}_\pi Y, Z) \to \mathscr{L}(X_2 \widehat{\otimes}_\pi Y, Z) \to \mathscr{L}(X_1 \widehat{\otimes}_\pi Y, Z) \text{ is exact.}$$

By universal property of complete injective tensor product we have $\mathscr{L}^{(2)}(X_i \times Y, Z) \cong \mathscr{L}(X_i \widehat{\otimes}_\pi Y, Z)$ $\forall i$.

Let's consider

$$0 \to \mathscr{L}^{(2)}(X_3 \times Y, Z) \underset{\alpha}{\to} \mathscr{L}^{(2)}(X_2 \times Y, Z) \underset{\beta}{\to} \mathscr{L}^{(2)}(X_1 \times Y, Z)$$

$\overline{\psi(X_2)} = X_3 \implies \operatorname{Ker}\alpha = 0$.

Let $F \in \operatorname{Ker}\beta \implies F(x,y) = 0, \forall x \in \varphi(X_1) \implies F(x,y) = 0, \forall x \in \operatorname{Ker}\psi \implies \exists$ a well-defined bilinear

$$G_0\colon \psi(X_2) \times Y \to Z \text{ s.t. } G_0(\psi(x), y) = F(x,y).$$

ψ is strict $\implies$ G_0 is continuous.

But $\overline{\psi(X_2)} = X_3 \implies G_0$ uniquely extends to a continuous bilinear $G\colon X_3 \times Y \to Z; \alpha(G) = F \implies$ the sequence is exact. ☐

Corollary. $\varphi_i \in \mathscr{L}(X_i, Y_i) (i = 1, 2)$

(1) If φ_1, φ_2 are open $\implies$ so is $\varphi_1 \otimes_\pi \varphi_2$

(2) If φ_1, φ_2 are strict with dense ranges $\implies$ so is $\varphi_1 \widehat{\otimes}_\pi \varphi_2$

(3) If X_i, Y_i are seminormed, φ_1, φ_2 are continuous $\implies$ so are $\varphi_1 \otimes_\pi \varphi_2, \varphi_1 \widehat{\otimes}_\pi \varphi_2$.

Proof. $\varphi_1 \otimes_\pi \varphi_2 = (\varphi_1 \otimes_\pi 1)(1 \otimes_\pi \varphi_2)$. ☐

Lecture 13 (2023.12.01)

Theorem (Banach; open mapping theorem). $X, Y =$ Fréchet spaces; $\varphi \in \mathscr{L}(X,Y)$ surjective $\implies$ φ is open.

Proof. Exercise. $\square$

Corollary. $X_i, Y_i (i=1,2)$ Fréchet spaces, $\varphi_i \in \mathscr{L}(X_i, Y_i)$ surjective $\implies$ $\varphi_1 \widehat{\otimes}_\pi \varphi_2$ is surjective.

Warning. φ_1, φ_2 topological injective $\not\Longrightarrow$ $\varphi_1 \otimes_\pi \varphi_2, (\varphi_1 \widehat{\otimes}_\pi \varphi_2)$ is topological injective.

Theorem. $X_i, Y_i (i=1,2), \varphi_i \in \mathscr{L}(X_i, Y_i)$

(1) If φ_1, φ_2 are topological injective $\implies$ so are $\varphi_1 \otimes_\varepsilon \varphi_2, \varphi_1 \widehat{\otimes}_\varepsilon \varphi_2$.

(2) If X_i, Y_i are seminormed, φ_1, φ_2 are isometries $\implies$ so are $\varphi_1 \otimes_\varepsilon \varphi_2, \varphi_1 \widehat{\otimes}_\varepsilon \varphi_2$.

Lemma. X, Y seminormed, $\varphi \in \mathscr{L}(X,Y)$.

(1) φ is an isometry $\implies$ φ' is a coisometry.

(2) φ is a coisometry $\implies$ φ' is an isometry.

Proof. Exercise. $\square$

Proof of Theorem. (2)

$$\begin{tikzcd}
(X_1' \otimes_\pi X_2')' \arrow[r, "(\varphi_1' \otimes_\pi \varphi_2')'"'] & (Y_1' \otimes_\pi Y_2')' \\
X_1 \otimes_\varepsilon X_2 \arrow[u, "\text{isometry}"'] \arrow[r, "\varphi_1 \otimes_\varepsilon \varphi_2"'] & Y_1 \otimes_\varepsilon Y_2 \arrow[u, "\text{isometry}"']
\end{tikzcd}$$

(1) follows from (2) (exercise). $\square$

Exercise. $\psi = \ker \varphi$ in LCS $\implies$ $\forall \operatorname{lcs} Y, \psi \otimes_\varepsilon 1_Y = \ker(\varphi \otimes_\varepsilon 1_Y)$.

Warning. $\widehat{\otimes}_\varepsilon$ does not preserve kernels.

Theorem. $X, Y =$ metrizable lcs's $\implies$ every $u \in X \widehat{\otimes}_\pi Y$ has the form $u = \sum_{i=1}^\infty \lambda_i x_i \otimes y_i$, where $\sum_{i=1}^\infty |\lambda_i| < \infty$, $x_i \to 0, y_i \to 0$.

Proof. $\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots$ a defining sequence of seminorms on X (and on Y).

$\|\cdot\|_n = \|\cdot\|_n \otimes_\pi \|\cdot\|_n$ on $X \widehat{\otimes}_\pi Y$.

Induction: $\exists u_1, u_2, \dots \in X \otimes Y$ s.t. $\left\| u - \sum_{i=1}^n u_i \right\|_{n+1} < \frac{1}{(n+1)^2 2^{n+1}} (*)$ $\implies$ $\|u_n\|_n \le \frac{1}{(n+1)^2 2^{n+1}} + \frac{1}{n^2 2^n} < \frac{1}{n^2 2^{n-1}}$ $\implies$ $u_n = \sum_{i=k_n+1}^{k_{n+1}} \lambda_i x_i \otimes y_i, \sum |\lambda_i| \|x_i\|_n \|y_i\|_n < \frac{1}{n^2 2^{n-1}}$.

We may assume that $\|x_i\|_n < \frac{1}{n}, \|y_i\|_n < \frac{1}{n}, \sum_{i=k_n+1}^{k_{n+1}} |\lambda_i| < \frac{1}{2^{n-1}}$.

Now $(*)$ $\implies$ $u = \sum_{i=1}^\infty \lambda_i x_i \otimes y_i$ is what we need. $\square$

Corollary. X, Y normed $\implies$ every $u \in X \widehat{\otimes}_\pi Y$ has the form $u = \sum_{i=1}^\infty x_i \otimes y_i, \sum \|x_i\| \|y_i\| < \infty$.

Nuclear operators

$X, Y =$ Banach spaces.

$$\begin{tikzcd}
Y \times X' \arrow[r] \arrow[d] & \mathscr{L}(X,Y) \\
Y \widehat{\otimes}_\pi X' \arrow[ur, "\gamma_{X,Y} = \gamma"'] &
\end{tikzcd}
\qquad (y,f) \mapsto f(\cdot)y \quad \text{rank-1 map} \quad \|f(\cdot)y\| = \|f\| \|y\|.$$

Definition. $\varphi \in \mathscr{L}(X,Y)$ is nuclear if $\varphi \in \gamma(Y \widehat{\otimes}_\pi X')$.

$$\mathscr{N}(X,Y) = \gamma(Y \widehat{\otimes}_\pi X') \text{ is the space of nuclear operators.}$$

Exercise.

(1) $\gamma(Y \otimes X') = \{\varphi \in \mathscr{L}(X,Y) \colon \dim \varphi(X) < \infty\} \subset \mathscr{N}(X,Y)$.

(2) $\gamma|_{Y \otimes X'}$ is injective.

Warning. γ is not always injective.

$\mathscr{N}(X,Y) \cong (Y \widehat{\otimes}_\pi X')/\operatorname{Ker}\gamma$.

Definition. The nuclear norm on $\mathscr{N}(X,Y)$ is the quotient norm of $\|\cdot\|_\pi$ on $Y \widehat{\otimes}_\pi X'$ ($\|\cdot\|_{\mathscr{N}}$).

Proposition. $X, Y =$ Banach, $\varphi \in \mathscr{L}(X,Y)$. TFAE:

(1) φ is nuclear

(2) $\varphi = \sum\limits_{n=1}^{\infty} \varphi_n, \varphi_n =$ a rank-1 open map, $\sum \|\varphi_n\| < \infty$

(3) $\varphi = \sum\limits_{n=1}^{\infty} \lambda_n f_n(\cdot) y_n, \sum |\lambda_n| < \infty$, (f_n) bounded in X', (y_n) bounded in Y

(4) $\varphi = \sum\limits_{n=1}^{\infty} \lambda_n f_n(\cdot) y_n, \sum |\lambda_n| < \infty$, $f_n \to 0, y_n \to 0$.

Proof. (1) $\implies$ (4) follows from the series decomposition of elements of $Y \widehat{\otimes}_\pi X'$.
(4) $\implies$ (3) $\implies$ (2) clear.
(2) $\implies$ (1) $\varphi_n = f_n(\cdot) y_n, f_n \in X', y_n \in Y, \|\varphi_n\| = \|f_n\| \|y_n\| \implies \varphi = \gamma(u)$, where $u = \sum y_n \otimes f_n \in Y \widehat{\otimes}_\pi X'$. $\square$

Observation. nuclear $\implies$ compact.

Remark. compact $\not\Longrightarrow$ nuclear.

Proposition. $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$, φ or ψ is nuclear $\implies$ $\psi\varphi$ is nuclear.

Proof. Exercise. $\square$

Example/Exercise 1. $1 \le p \le +\infty$; $X = \ell^p$ or $X = c_0$.
$\alpha = (\alpha_n) \in \ell^\infty$; $M_\alpha \colon X \to X$, $M_\alpha(x) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$ (the diagonal operator).
M_α is bounded, $\|M_\alpha\| = \|\alpha\|_\infty$.
M_α is compact $\iff \alpha_n \to 0$.
Prove: M_α is nuclear $\iff \sum |\alpha_n| < \infty$.

Hint/subexercise. $\frac{1}{p} + \frac{1}{q} = 1$

$$\mathcal{D} = \overline{\operatorname{span}}\,\{e_n \otimes e_n \colon n \in \mathbb{N}\} \subset \ell^p \widehat{\otimes}_\pi \ell^q \quad e_n = (0, \dots, 0, 1, 0, \dots), \text{ where 1 is in the } n\text{th slot.}$$

Prove: $\mathcal{D} \cong \ell^1$ isometrically.

Example/Exercise 2. $I = [a,b] \subset \mathbb{R}, K \in C(I \times I)$.

$$\varphi \colon C(I) \to C(I) \quad \varphi(f)(x) = \int_a^b K(x,y) f(y) \mathrm{d}y.$$

Prove φ is nuclear; find $\|\varphi\|_{\mathscr{N}}$.

$X, Y = $ lcs's.

Definition. $\varphi \in \mathscr{L}(X,Y)$ is nuclear if $\exists$ a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & Y \\ E \arrow[r, "\psi"'] & F \arrow[u] \end{tikzcd}$$

where E, F are Banach spaces, $\psi \in \mathscr{N}(E,F)$, $\uparrow, \downarrow$ are continuous linear maps.

Observation. This is consistent with our earlier definition if X, Y are Banach spaces.

Notation. $\mathscr{N}(X,Y) = \{\text{nuclear operators } X \to Y\}$.

Two constructions of Banach spaces

$X = $ lcs.

1. $p \in \mathrm{SN}(X), X_p = (X/p^{-1}(0); \hat{p})$ a normed space; $\widetilde{X}_p = $ the completion of X_p.

$$\pi_p \colon X \to X_p, x \mapsto x + p^{-1}(0) \text{ is continuous.}$$

2. $B \subset X$ absolutely convex, bounded.
   $X_B = \mathrm{span}(B) = \bigcup_{n \in \mathbb{N}} nB$. (X_B, p_B) is a seminormed space. $j_B \colon X_B \to X$ inclusion map.

   Exercise.

   (1) j_B is continuous.
   (2) If X Hausdorff, then X_B is a normed space.

Definition. B is a Banach disk if X_B is a Banach space.

Example/Exercise. If X is Hausdorff, B is absolutely convex, bounded and complete $\implies$ B is a Banach disk.

Exercise. $\varphi \in \mathscr{L}(X,Y), B \subset X$ Banach disk $\implies \varphi(B) \subset Y$ is a Banach disk. ($Y_{\varphi(B)}$ is a quotient of X_B)

Example/Exercise. $X = \text{lcs}, p \in \mathrm{SN}(X)$

$$B_p = \{f \in X' \colon |f(x)| \le p(x) \, \forall x \in X\}.$$

$\implies B_p$ is a Banach disk in X', and

$$(X')_{B_p} \cong (X_p)' \cong (\widetilde{X}_p)' \text{ isometrically.}$$

Definition. $X, Y = $ lcs's; $S \subset \mathscr{L}(X,Y)$.
S is equicontinuous if $\forall$ neighborhood $V \subset Y$ of 0, $\exists$ a neighborhood $U \subset X$ of 0 s.t. $\forall \varphi \in S, \varphi(U) \subset V$.

Example/Exercise. If X, Y are normed, then S is equicontinuous $\iff$ S is bounded with respect to the operator norm in $\mathscr{L}(X,Y)$.

Example/Exercise. ($Y = \mathbb{K}$)
$S \subset X'$ is equicontinuous $\iff \exists p \in \mathrm{SN}(X)$ s.t. $\forall f \in S, |f(x)| \le p(x), \forall x$.

Proposition. $X, Y = $ lcs's, $\varphi \in \mathscr{L}(X,Y)$. TFAE:

(1) φ is nuclear

(2) $\varphi = \sum_{n=1}^{\infty} \lambda_n f_n(\cdot) y_n, \sum |\lambda_n| < \infty, (f_n)$ is equicontinuous in X', $(y_n) \subset B \subset Y$ for some Banach disk $B \subset Y$

(3) $\exists p \in \mathrm{SN}(X), \exists$ a Banach disk $B \subset Y, \exists \psi \in \mathscr{N}(\widetilde{X}_p, Y_B)$ s.t. $\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi_p"'] & Y \\ \widetilde{X}_p \arrow[r, "\psi"] & Y_B \arrow[u, "j_B"'] \end{tikzcd}$ commutes.

Proof. (1) $\implies$ (2)

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & Y \\ E \arrow[r, "\psi"] & F \arrow[u, "\beta"'] \end{tikzcd}$$

E, F Banach spaces; $\psi \in \mathscr{N}(E, F)$, $\psi = \sum_n \lambda_n g_n(\cdot) z_n, \sum |\lambda_n| < \infty$, $\|g_n\| \leq 1$, $\|z_n\| \leq 1$, $g_n \in E'$, $z_n \in F \implies \varphi = \sum_n \lambda_n f_n(\cdot) y_n$, where $y_n = \beta(z_n) \in \beta(\overline{U}_F)$, $\beta(\overline{U}_F)$ is a Banach disk, $f_n = g_n \circ \alpha$, (f_n) is equicontinuous in X'.

(2) $\implies$ (3) $\varphi = \sum \lambda_n f_n(\cdot) y_n$ (as in (2))

$$\begin{aligned} (f_n) \text{ is equicontinuous } &\implies \exists p \in \mathrm{SN}(X) \text{ s.t. } |f_n(x)| \leq p(x) \, \forall x, \forall n. \\ &\implies f_n = g_n \circ \widetilde{\pi}_p, g_n \in (\widetilde{X}_p)', \|g_n\| \leq 1 \end{aligned}$$

$\psi = \sum \lambda_n g_n(\cdot) y_n \in \mathscr{N}(\widetilde{X}_p, Y_B)$.

(3) $\implies$ (1) clear. □

Lecture 14 (2023.12.08)

E, F Banach

$$F \widehat{\otimes}_\pi E' \underset{\gamma}{\to} \mathscr{L}(E, F) \quad y \otimes f \mapsto f(\cdot)y.$$

$\varphi \in \mathscr{L}(E, F)$ is <u>nuclear</u> if $\varphi \in \gamma(F \widehat{\otimes}_\pi E')$.

X, Y lcs's

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & Y \\ E \arrow[r, "\psi"] & F \arrow[u] \end{tikzcd}$$

φ is <u>nuclear</u> if $\exists$ Banach E, F and $\psi \in \mathscr{N}(E, F)$ s.t. the diagram commutes.

Proposition. TFAE:

(1) φ is nuclear.

(2) $\varphi = \sum\limits_{n=1}^{\infty} \lambda_n f_n(\cdot) y_n$, where $\sum |\lambda_n| < \infty$, $(f_n) \subset X'$ equicontinuous, $(y_n) \subset B \subset Y$, where $B \subset Y$ is a Banach disk.

(3) $\exists p \in \mathrm{SN}(X), \exists$ a Banach disk $B \subset Y, \exists \psi \in \mathscr{N}(\widetilde{X}_p, Y_B)$ s.t. $\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi_p"'] & Y \\ \widetilde{X}_p \arrow[r, "\psi"] & Y_B \arrow[u, "j_B"'] \end{tikzcd}$ commutes.

Recall: $X_p = (X/p^{-1}(0), \widehat{p})$ normed, $\widehat{X}_p =$ the completion of X_p. $\pi_p \colon X \to \widehat{X}_p, x \mapsto x + p^{-1}(0)$. $Y_B = \operatorname{span}(B), j_B \colon Y_B \hookrightarrow Y, Y_B = (Y_B, p_B)$.

<u>Some properties of nuclear operators</u>

Proposition. $X, Y =$ lcs's, $\varphi \in \mathscr{N}(X, Y) \implies \exists$ a Hilbert space H and $\alpha \in \mathscr{L}(X, H), \beta \in \mathscr{L}(H, Y)$ s.t. $\varphi = \beta\alpha$.

Proof. We may assume that X, Y are Banach spaces.

$$\varphi = \sum_{n=1}^{\infty} \lambda_n f_n(\cdot) y_n, (f_n) \subset X' \text{ bounded}, (y_n) \subset Y \text{ bounded}, \sum |\lambda_n| < \infty, \lambda)n \geq 0.$$

$H = \ell^2$;

$$\alpha \colon X \to \ell^2, \alpha(x) = (\sqrt{\lambda_n} f_n(x))_{n \in \mathbb{N}}$$

$$\beta \colon \ell^2 \to Y, \beta((z_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} z_n y_n.$$

$\implies \varphi = \beta\alpha$, β and α are bounded. □

Proposition.

(1) $\varphi_k \in \mathscr{N}(X_k, Y_k) (k = 1, 2) \implies \varphi_1 \otimes_\pi \varphi_2, \varphi_1 \widehat{\otimes}_\pi \varphi_2$ are nuclear.

(2) $\varphi \in \mathscr{N}(X, Y) \implies \varphi' \in \mathscr{N}(Y'_\beta, X'_\beta)$

Proof. Exercise (reduce to Banach spaces). □

Warning.

(1) $\varphi \in \mathscr{N}(X, Y), \varphi'$ is nuclear $\not\Longleftrightarrow \varphi$ is nuclear (Fiegel, Johnson 1973).

(2) $\varphi \in \mathscr{N}(X, Y), Z \subset Y$ closed subspace, $\varphi(X) \subset Z \not\Longleftrightarrow \varphi \in \mathscr{N}(X, Z)$.

Proposition. X_1, X_2 Hilbert, $\varphi \in \mathscr{N}(X_1, X_2)$; $Y_1 \subset X_1, Y_2 \subset X_2$ closed subspaces; $\varphi(Y_1) \subset Y_2$.

$$\begin{tikzcd}
Y_1 \arrow[r, "\varphi_1"] \arrow[d, hook, "i_1"'] & Y_2 \arrow[d, hook, "i_2"'] \\
X_1 \arrow[r, "\varphi"] \arrow[d, "q_1"] & X_2 \arrow[d, "q_2"] \arrow[u, bend right, "p"'] \\
X_1/Y_1 \arrow[r, "\varphi_2"] \arrow[u, bend left, "j"] & X_2/Y_2
\end{tikzcd}$$

Then φ_1 and φ_2 are nuclear.

Proof. $\exists p$ s.t $pi_2 = 1$ (orthogonal projection) $\implies \varphi_1 = pi_2\varphi_1 = p\varphi i_1$ is nuclear.
$\exists j$ s.t. $q_1 j = 1 \implies \varphi_2 = \varphi_2 q_1 j = q_2 \varphi j$ is nuclear. $\square$

Remark. The proof shows that: if X_1, X_2 are lcs's
$Y_2 \subset X_2$ complemented $\implies \varphi_1$ is nuclear
$Y_1 \subset X_1$ complemented $\implies \varphi_2$ is nuclear.

Nuclear spaces

Definition. A lcs X is nuclear if $\forall$ Banach space E,

$$\mathscr{L}(X, E) = \mathscr{N}(X, E).$$

Example.

(1) $\dim X < \infty \implies X$ is nuclear.

(2) An infinite-dim normed space is not nuclear ($X \hookrightarrow \widehat{X}$ is not compact $\implies$ is not nuclear.)

Recall: $X = \text{lcs}, p, q \in \text{SN}(X), p \prec q$

$$\begin{tikzcd}
& X \arrow[dl, "\pi_p"'] \arrow[dr, "\pi_q"] & \\
\widetilde{X}_p & & \widetilde{X}_q \arrow[ll, "\pi_{pq}"]
\end{tikzcd} \qquad \pi_{pq} \colon x + q^{-1}(0) \mapsto x + p^{-1}(0).$$

Proposition. $X = \text{lcs}$; $P =$ a directed defining family of seminorms on X. TFAE:

(1) X is nuclear;

(2) $\forall p \in P, \pi_p \colon X \to \widetilde{X}_p$ is nuclear;

(3) $\forall p \in P, \exists q \in P$ s.t. $p \prec q$ and $\pi_{pq} \colon \widetilde{X}_q \to \widetilde{X}_p$ is nuclear.

Lemma. $X, p =$ as in Proposition. $E =$ Banach space, $\varphi \in \mathscr{L}(X, E) \implies \exists p \in P, \exists \psi \in \mathscr{L}(\widetilde{X}_p, E)$ s.t.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\pi_p"'] & E \\
\widetilde{X}_p \arrow[ur, "\psi"'] &
\end{tikzcd} \qquad \text{commutes.}$$

Proof. $\|\varphi(\cdot)\| \in \text{SN}(X) \implies \exists p \in P$ s.t. $\|\varphi(\cdot)\| \le p$. $\psi \colon X_p \to E, \psi \colon x + p^{-1}(0) \mapsto \varphi(x)$ is well defined and bounded. Extend ψ to $\widetilde{X}_p$. $\square$

Proof of Proposition. (1) $\implies$ (3)

$$\begin{tikzcd}
& X \arrow[dl, "\pi_p"'] \arrow[d] \arrow[dr, "\pi_q"] & \\
\widetilde{X}_p & E \arrow[l, "\text{nuclear}"] & X_q \arrow[l] \arrow[ll, bend left, "\pi_{pq}"]
\end{tikzcd} \implies \pi_{pq} \text{ is nuclear.}$$

(3) $\implies$ (2) $\pi_p = \pi_{pq}\pi_q$ is nuclear.
(2) $\implies$ (1) see Lemma. □

Corollary. X is nuclear $\iff$ $\widetilde{X}$ is nuclear.

Example 1. $\mathbb{K}^S$ (S = a set) is nuclear (because $(\mathbb{K}^S)_p$ is finite-dim $\forall p$).

Proposition (The Grothendieck-Pietsch criterion)**.** I = a set; P = a Köthe set on I; $\nu \in [1, +\infty]$.
The Köthe space $\lambda^\nu(I; P)$ is nuclear $\iff$ $\forall p \in P, \exists q \in P, \exists \lambda \in \ell^1(I)_{\geq 0}$ s.t $p_i \leq \lambda_i q_i, \forall i \in I$.

Proof. (a sketch)
Assume that $I = \mathbb{N}$ and $p_i > 0, \forall p \in P, \forall i \in \mathbb{N}, X = \lambda^\nu(I, P)$.

$$\begin{tikzcd}
& & X \arrow[dl, "\pi_p"'] \arrow[dr, "\pi_q"] & \\
(x_i) \arrow[d, mapsto] & \widetilde{X}_p \arrow[d, equal] & & \widetilde{X}_q \arrow[ll, "\pi_{pq}"'] \arrow[d, equal] \\
(x_i p_i) & \ell^\nu & & \ell^\nu \arrow[ll, mapsto, "M_{p/q}"']
\end{tikzcd}$$

Exercise. Construct the isometric isomorphisms and show that the diagram commutes.

$M_{p/q}$ is nuclear $\iff$ $\frac{p}{q} \in \ell^1$. □

Example 2. $s; s(\mathbb{Z}^n)$ are nuclear.

Example 3. $C^\infty(\mathbb{T}^n)$ is nuclear (it is $\cong s(\mathbb{Z}^n)$).

Example 4. $\mathcal{O}(\mathbb{D}_R^n)$ is nuclear.

Example 5. $\mathcal{O}_z (z \in \mathbb{C})$ is nuclear.

Theorem.

(1) X nuclear, $Y \subset X$ subspace $\implies$ $Y, X/Y$ are nuclear.

(2) $(X_n)_{n\in\mathbb{N}}$ are nuclear $\implies$ $\bigoplus_{n\in\mathbb{N}} X_n$ is nuclear.

(3) $(X_i)_{i\in I}$ are nuclear $\implies$ $\prod X_i$ is nuclear.

(4) X = lcs equipped with the inductive locally topology generated by $\{\varphi_n : X_n \to X\}_{n\in\mathbb{N}}$, where all X_n are nuclear and $\sum \varphi_n(X_n) = X$, then X is nuclear.

(5) X = lcs equipped with the projective locally convex topology generated by $\{\varphi_i : X \to X_i\}_{i\in I}$, where all X_i are nuclear and $\bigcap \operatorname{Ker} \varphi_i = 0$, then X is nuclear.

(6) A countable $\varinjlim$ and an arbitrary $\varprojlim$ of nuclear spaces are nuclear.

(7) X, Y are nuclear $\implies$ $X \otimes_\pi Y, X \widehat{\otimes}_\pi Y$ are nuclear.

Lemma. $X =$ nuclear lcs $\implies \exists$ a directed defining family P of seminorms on X s.t. $\widetilde{X}_p$ are Hilbert spaces $(p \in P)$.

Proof. $Q =$ a directed defining family of seminorms.

$$\begin{tikzcd}
& X \arrow[dl, "\pi_q"'] \arrow[dr, "\alpha"] & \\
\widetilde{X}_q & & H \arrow[ll, "\beta"]
\end{tikzcd}$$

$\forall q \in Q, \exists$ a Hilbert space H and α, β s.t. $\pi_q = \beta\alpha$.
We may assume that $\overline{\alpha(X)} = H$. Let $p_q(x) = \|\alpha(x)\|$.

Exercise. $\widetilde{X}_q \cong H$ isometrically; $\{p_q \colon q \in Q\}$ is a directed defining family of seminorms. $\square$

Proof of Theorem.

(1) $p \in \mathrm{SN}(X), \dot{p} = p|_\gamma, \widehat{p} =$ the quotient seminorm of p on $Z = X/Y$.

Exercise. $\widetilde{Y}_{\dot{p}} \hookrightarrow \widetilde{X}_p$ isometrically, and $\exists$ an isometric isomorphism $\widetilde{Z}_{\widehat{p}} = \widetilde{X}_p / \widetilde{Y}_{\dot{p}}$.

Choose $q \in \mathrm{SN}(X)$ s.t. $p \prec q$ s.t. π_{pq} is nuclear.

$$\begin{tikzcd}
\widetilde{Y}_{\dot{q}} \arrow[r, "\pi_{\dot{p}\dot{q}}"] \arrow[d, hook] & \widetilde{Y}_{\dot{p}} \arrow[d, hook] \\
\widetilde{X}_q \arrow[r, "\pi_{pq}"'] \arrow[d] & \widetilde{X}_p \arrow[d] \\
\widetilde{Z}_{\widehat{q}} \arrow[r, "\pi_{\widehat{p}\widehat{q}}"'] & \widetilde{Z}_{\widehat{p}}
\end{tikzcd}$$

By Lemma, we may assume that $\widetilde{X}_p, \widetilde{X}_q$ are Hilbert space $\implies \pi_{\dot{p}\dot{q}}$ and $\pi_{\widehat{p}\widehat{q}}$ are nuclear $\implies Y, X/Y$ are nuclear.

(2) $E =$ Banach space, $\varphi \colon X = \bigoplus X_n \to E$ continuous linear.
$\varphi_n = \varphi|_{X_n} \colon X_n \to E$ is nuclear $\implies \varphi_n = \sum\limits_k \lambda_{nk} f_{nk}(\cdot) y_{nk}, \|y_{nk}\| \le 1, (f_{nk})_{k \in \mathbb{N}} \subset X_n'$ equicontinuous, $\sum\limits_k |\lambda_{nk}| \le \frac{1}{2^n}$. Extend f_{nk} to X: $f_{nk}|_{X_m} = 0, \forall m \neq n$.

Exercise. $(f_{nk})_{n,k \in \mathbb{N}}$ is equicontinuous.

$\varphi = \sum\limits_{n,k} \lambda_{nk} f_{nk}(\cdot) y_{nk}, \sum\limits_{n,k} |\lambda_{nk}| < \infty \implies \varphi$ is nuclear.

(3) $E =$ Banach space.

Exercise. $\exists$ finite $J \subset I$ and a commutative diagram.

$$\begin{tikzcd}
\prod\limits_{i \in I} X_i \arrow[r] \arrow[d] & E \\
\prod\limits_{i \in J} X_i \arrow[ur] &
\end{tikzcd}$$

$\prod_{i \in I} X_i$ is nuclear by (2).

(4) X is a quotient of $\bigoplus X_n$.

(5) X is subspace of $\prod X_i$.

(6) follows from (4), (5).

(7) follows from the next lemma. $\square$

Lemma. X, Y lcs's, $p \in \mathrm{SN}(X), q \in \mathrm{SN}(Y)$. $\exists$ an isometric isomorphism s.t. the diagram commutes.

$$\begin{tikzcd}
(X \otimes_\pi Y)^{\sim}_{p \otimes_\pi q} \arrow[rr, "\cong"] & & \widetilde{X}_p \widehat{\otimes}_\pi \widetilde{Y}_q \\
& X \otimes_\pi Y \arrow[ul, "\pi_{p \otimes q}"] \arrow[ur, "\pi_p \otimes \pi_q"'] &
\end{tikzcd}$$

This is almost immediately from the construction if we recall the construction of the associated Banach spaces, then we will easily see that this is indeed the case. But now if you want to show that $X \otimes_\pi Y$ is nuclear, it's enough to show that all $\pi_{p \otimes q}$ are nuclear. we already know that the projective tensor product of two nuclear operators is a nuclear operator, so $\pi_p \otimes \pi_q$ is nuclear. This completes the proof.

Lecture 15 (2023.12.15)

The class of nuclear spaces is stable under: subspaces; quotients; countable $\bigoplus$; arbitrary $\prod$; completions; $\otimes_\pi, \widehat{\otimes}_\pi$; arbitrary $\varprojlim$; countable $\varinjlim$.

Exercise. $\bigoplus\limits_{i\in I}\mathbb{K}$ (I is uncountable) is not nuclear.

More examples of nuclear spaces

Example 1. $[0,1] \hookrightarrow \mathbb{T}, [0,1] \mapsto \left\{ \mathrm{e}^{\sqrt{-1}\theta} \colon \theta \in [\pi, 2\pi] \right\}$.

$$C^\infty(\mathbb{T}) \xrightarrow[\text{restriction}]{} C^\infty[0,1] \text{ is surjective (exercise).}$$

By open mapping theorem, the restriction is open. So $C^\infty[0,1] \cong$ quotient of $C^\infty(\mathbb{T})$. Hence $C^\infty[0,1]$ is nuclear.

Similarly: $C^\infty([0,1]^n)$ is nuclear (exercise).

Example 2. $C^\infty(\mathbb{R}^n) \cong \varprojlim\limits_k C^\infty([-k,k]^n)$ is nuclear.

Example 3. M = a smooth real manifold.

$$M = \bigcup_{i\in\mathbb{N}} U_i \quad U_i \cong \mathbb{R}^n.$$

$$\left.\begin{array}{r} C^\infty(M) \hookrightarrow \prod\limits_{i\in\mathbb{N}} C^\infty(U_i) \text{ topological injective} \\ f \mapsto (f|_{U_i})_{i\in\mathbb{N}} \end{array}\right\} \implies C^\infty(M) \text{ is nuclear.}$$

Example 4. $\mathbb{R}^n \hookrightarrow S^n \cong \mathbb{R}^n \cup \{\infty\}$.

Exercise. $\mathscr{S}(\mathbb{R}^n) \cong \{f \in C^\infty(S^n) \colon f \text{ vanishes at } \infty \text{ together with all partial derivatives}\}$.

$\implies \mathscr{S}(\mathbb{R}^n)$ is nuclear.

Example 5. M = smooth real manifold. $(K_n)_{n\in\mathbb{N}}$ compact exhaustion of M.

$$C_c^\infty(M) = \varinjlim_n C_{K_n}^\infty(M); \quad C_{K_n}^\infty(M) \subset C^\infty(M) \text{ is nuclear} \implies C_c^\infty(M) \text{ is nuclear.}$$

Example 6. M = complex manifold.

$$M = \bigcup_{i\in\mathbb{N}} U_i \quad U_i \cong \mathbb{D}_R^n.$$

$$\left.\begin{array}{r} \mathcal{O}(M) \hookrightarrow \prod\limits_{i\in\mathbb{N}} \mathcal{O}(U_i) \text{ topological injective} \\ f \mapsto (f|_{U_i})_{i\in\mathbb{N}} \end{array}\right\} \implies \mathcal{O}(M) \text{ is nuclear.}$$

Example 7. $K \subset \mathbb{C}^n$ compact. $(U_i)_{i\in\mathbb{N}}$ a base of open neighborhoods of K.

$$\mathcal{O}(K) = \{\text{germs of holomorphic functions at } K\} \cong \varinjlim \mathcal{O}(U_i) \implies \mathcal{O}(K) \text{ is nuclear.}$$

Tensor products & nuclear spaces

Theorem. X, Y = lcs, X is nuclear. Then the canonical maps

$$X \otimes_\pi Y \to X \otimes_\varepsilon Y \quad \text{and} \quad X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\varepsilon Y$$

are topological isomorphisms.

Lemma 1. X, Y = seminormed. The map

$$\alpha\colon X \otimes_\varepsilon Y \to \mathscr{L}(X', Y), x \otimes y \mapsto (f \mapsto f(x)y)$$

is an isometry.

Proof. Observe: $\forall u \in X \otimes Y, \alpha(u)(f) = (f \otimes 1)(u)$.

$$\begin{aligned}\|u\|_\varepsilon &= \sup_{\substack{\|f\|\le 1\\ \|g\|\le 1}} |(f \otimes g)(u)| = \sup_{\|f\|\le 1} \sup_{\|g\|\le 1} |g(f \otimes 1)(u)| \\ &= \sup_{\|f\|\le 1} \|(f \otimes 1)(u)\| = \|\alpha(u)\|.\end{aligned}$$

□

Lemma 2. X, Y, Z Banach spaces, $\varphi \in \mathscr{N}(X, Y)$. The map

$$\varphi \otimes 1\colon X \otimes_\varepsilon Z \to Y \otimes_\pi Z$$

is continuous.

Proof. $Y \widehat{\otimes}_\pi X' \xrightarrow{\gamma} \mathscr{L}(X, Y), \gamma(y \otimes f) = f(\cdot)y$. $\exists u \in Y \widehat{\otimes}_\pi X'$ s.t. $\varphi = \gamma(u)$.

$$\begin{tikzcd} X \otimes_\varepsilon Z \arrow[rr, "\varphi\otimes 1"] \arrow[dr, "\alpha"'] & & Y \widehat{\otimes}_\pi Z \\ & \mathscr{L}(X', Z) \arrow[ur, "\beta"'] & \end{tikzcd} \qquad \beta(\psi) = (1_Y \widehat{\otimes}_\pi \psi)(u) \quad (\psi \in \mathscr{L}(X', Z)).$$

Exercise. β is bounded, the diagram commutes. □

Proof of Theorem. $p \in \mathrm{SN}(X), r \in \mathrm{SN}(Y)$.

It suffices to show: $\exists q \in \mathrm{SN}(X)$ s.t. $p \otimes_\pi r \prec q \otimes_\varepsilon r$. $\exists q \in \mathrm{SN}(X)$ s.t. $\pi_{pq}\colon \widetilde{X}_q \to \widetilde{X}_p$ is nuclear.

$$\begin{tikzcd} & X \otimes Y \arrow[dl, "\pi_p\otimes\pi_r"'] \arrow[dr, "\pi_q\otimes\pi_r"] & \\ \widetilde{X}_p \otimes_\pi \widetilde{Y}_r & & \widetilde{X}_q \otimes_\varepsilon \widetilde{Y}_r \arrow[ll, "\pi_{pq}\otimes 1"] \end{tikzcd}$$

$$\begin{aligned}(p \otimes_\pi r)(u) &= \|(\pi_p \otimes \pi_q)(u)\|_\pi \text{ (because } \exists \text{ an isometric isomorphism } \widetilde{X}_p \widehat{\otimes}_\pi \widetilde{Y}_q = (X \widehat{\otimes}_\pi Y)^{\sim}_{p\otimes_\pi q}) \\ &= \|(\pi_{pq} \otimes 1)(\pi_q \otimes \pi_r)(u)\|_\pi \\ &\le C\|(\pi_q \otimes \pi_r)(u)\|_\varepsilon \\ &= C(q \otimes_\varepsilon r)(u).\end{aligned}$$

□

Theorem. X = lcs. TFAE:

(1) X is nuclear;

(2) $\forall\, \mathrm{lcs}\, Y, X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\varepsilon Y$ is a topological isomorphism;

(3) $X \widehat{\otimes}_\pi \ell^1 \to X \widehat{\otimes}_\varepsilon \ell^1$ is a topological isomorphism.

Conjecture (Grothendieck 1952). X = lcs. If $\exists$ a non-nuclear lcs Y s.t. $X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\varepsilon Y$ is a topological isomorphism, then X is nuclear.

Counterexample (Pisier 1983). $\exists$ an infinite-dim Banach space P s.t. $P \widehat{\otimes}_\pi P \to P \widehat{\otimes}_\varepsilon P$ is a topological isomorphism.

Theorem (K. John 1982). X = lcs s.t. $X \widehat{\otimes}_\pi X \widehat{\otimes}_\pi X \to X \widehat{\otimes}_\varepsilon X \widehat{\otimes}_\varepsilon X$ is a topological isomorphism $\implies$ X is nuclear.

Tensor product & projective limits

(X_i, φ_{ij}) = a projective system of lcs's; $X = \varprojlim X_i$, $\varphi_i \colon X \to X_i$ canonical map.

Definition. (X_i, φ_{ij}) is reduced if $\overline{\varphi_i(X)} = X_i, \forall i$. X is a reduced projective limit.

Example. X = complete lcs; P = a directed defining family of seminorms on $X \implies X \cong \varprojlim \widetilde{X}_p$ is reduced ($\widetilde{X}_p$ = the completion of $(X/p^{-1}(0), \widehat{p})$).

Theorem. (X_i, φ_{ij}) and (Y_k, ψ_{kl}) reduced projective systems of lcs's; $\varphi_i \colon X \to X_i, \psi_k \colon Y \to Y_k$ canonical maps. $X = \varprojlim X_i, Y = \varprojlim Y_k$. Then $\exists$ a unique topological isomorphism χ s.t.

$$\begin{tikzcd}
X \widehat{\otimes} Y \arrow[rr, "\chi"'] \arrow[dr, "\varphi_i \widehat{\otimes} \psi_k"'] & & \varprojlim X_i \widehat{\otimes} Y_k \arrow[dl, "\text{canonical } \tau_{ik}"] \\
& X_i \widehat{\otimes} Y_k &
\end{tikzcd}$$

commutes $\forall i, \forall k, \widehat{\otimes} \in \{\widehat{\otimes}_\pi, \widehat{\otimes}_\varepsilon\}$.

Lemma. $X_k, Y_k (k = 1, 2)$ seminormed, $\varphi_k \in \mathscr{L}(X_k, Y_k)$ isometries with dense ranges. Then $\varphi_1 \otimes_\pi \varphi_2, \varphi_1 \otimes_\varepsilon \varphi_2$ are isometries.

Proof. trivial for $\otimes_\varepsilon$; exercise for $\otimes_\pi$. **Hint**: φ_k extends to an isometric isomorphism $\widetilde{X}_k \to \widetilde{Y}_k$. □

Proof of Theorem. $\exists$ a unique $\chi \in \mathscr{L}(X \widehat{\otimes} Y, \varprojlim X_i \widehat{\otimes} Y_k)$ s.t. the diagram commutes; χ has dense range.

Let's show the topological injectivity of χ.
Take $p \in \mathrm{SN}(X_i), q \in \mathrm{SN}(Y_k)$.

$$\begin{tikzcd}
(X \otimes Y, p\varphi_i \otimes q\psi_k) \arrow[rr, "\chi"] \arrow[dr, "\text{isometry } \varphi_i \otimes \psi_k \text{ (Lemma)}"'] & & (\varprojlim (X_i \otimes Y_k), (p \otimes q) \circ \tau_{ik}) \arrow[dl, "\text{isometry } \tau_{ik}"] \\
& (X_i \otimes Y_k, p \otimes q) &
\end{tikzcd}$$

$\implies \chi$ is an isometry with respect to $p\varphi_i \otimes q\psi_k$ and $(p \otimes q)\tau_{ik}$. This is true $\forall p, \forall q, \forall i, \forall k \implies \chi$ is topological injective. □

Corollary.

$$\left(\prod_{i \in I} X_i \right) \widehat{\otimes} \left(\prod_{k \in K} Y_k \right) \cong \prod_{i,k} (X_i \widehat{\otimes} Y_k) \qquad \widehat{\otimes} \in \{\widehat{\otimes}_\pi, \widehat{\otimes}_\varepsilon\}.$$

Proof. $\prod X_i \cong \varprojlim \left\{ \prod_{i \in J} X_i \colon J \in \mathrm{Fin}(I) \right\}$ reduced. □

Theorem. $X_k, Y_k (k = 1, 2)$ complete lcs's. $\varphi_k \in \mathscr{L}(X_k, Y_k)$ injective $\implies \varphi_1 \widehat{\otimes}_\varepsilon \varphi_2$ is injective.

Recall: X normed, Y = lcs.

$$\mathscr{L}_b(X, Y) = (\mathscr{L}(X, Y); \text{uniform convergence on the closed unit ball of } X).$$

Lemma 1. X = normed, (Y_i, φ_{ij}) projective system of lcs's $\implies \exists$ topological isomorphism $\mathscr{L}_b \left(X, \varprojlim Y_i \right) \cong \varprojlim \mathscr{L}_b(X, Y_i)$.

Proof. Exercise. □

Corollary. X = normed, Y = complete lcs $\implies \mathscr{L}_b(X, Y)$ is complete.

Proof. $Y =\cong \varprojlim Y_i$, Y_i Banach $\implies \mathscr{L}_b(X,Y) = \varprojlim \underbrace{\mathscr{L}_b(X,Y_i)}_{\text{Banach}}$ is complete. □

Lemma 2. X = normed, Y = Hausdorff lcs.

(1) The map $\alpha\colon X \otimes_\varepsilon Y \to \mathscr{L}_b(X',Y), x\otimes y \mapsto (f \mapsto f(x)y)$ is topological injective.

(2) If Y is complete, then α extends to a topological injective map $X \widehat{\otimes}_\varepsilon Y \hookrightarrow \mathscr{L}_b(X',Y)$.

Proof.

(1) reduce to seminormed space (exercise).

(2) follows from (1) and Corollary. □

Proof of Theorem. $\varphi_1 \widehat{\otimes}_\varepsilon \varphi_2 = (\varphi_1 \widehat{\otimes}_\varepsilon 1_{Y_2})(1_{X_1} \widehat{\otimes}_\varepsilon \varphi_2)$.
It suffices to show: if X, Y, Z are complete lcs's, $\varphi \in \mathscr{L}(X,Y)$ injective $\implies 1_Z \widehat{\otimes}_\varepsilon \varphi$ is injective.
Case 1: Z is a Banach space.

$$\begin{tikzcd} Z\widehat{\otimes}_\varepsilon X \arrow[r, "1\otimes\varphi"] \arrow[d, hook, "\alpha"] & Z\widehat{\otimes}_\varepsilon Y \arrow[d, hook, "\alpha"] \\ \mathscr{L}_b(Z',X) \arrow[r, "\psi\mapsto\varphi\circ\psi \text{ is injective}"] & \mathscr{L}_b(Z',Y) \end{tikzcd}$$

commutes (exercise).
General case: see the next lecture. □

Lecture 16 (2023.12.22)

Proposition. X, Y, Z complete lcs's, $\varphi \in \mathscr{L}(X, Y)$ injective $\implies 1_Z \widehat{\otimes}_\varepsilon \varphi \colon Z \widehat{\otimes}_\varepsilon X \to Z \widehat{\otimes}_\varepsilon Y$ is injective.

Proof. **Case 1**: Z is a Banach space (see previous lecture).
General case: $Z = \varprojlim Z_i$ reduced; Z_i = Banach space.

$$\begin{tikzcd}
Z\widehat{\otimes}_\varepsilon X \arrow[r] \arrow[d, equal] & Z\widehat{\otimes}_\varepsilon Y \arrow[d, equal] \\
\varprojlim (Z_i \widehat{\otimes}_\varepsilon X) \arrow[r, "\text{injective}"] & \varprojlim (Z_i \widehat{\otimes}_\varepsilon Y)
\end{tikzcd}$$

because $\lim(Z_i \widehat{\otimes}(-)) \hookrightarrow \prod (Z_i \widehat{\otimes}_\varepsilon(-))$. $\square$

The exactness of $\widehat{\otimes}_\pi$ for nuclear Fréchet spaces

Proposition. $0 \to X_1 \xrightarrow{\varphi} X_2 \xrightarrow{\psi} X_3 \to 0$ exact sequence of Fréchet spaces. Y = a Fréchet space. Suppose either X_2 or Y is nuclear. Then

$$0 \to X_1 \widehat{\otimes}_\pi Y \xrightarrow[\varphi\widehat{\otimes}_\pi 1]{} X_2 \widehat{\otimes}_\pi Y \xrightarrow[\psi\widehat{\otimes}_\pi 1]{} X_3 \widehat{\otimes}_\pi Y \to 0 \tag{$*$}$$

is exact.

Proof. Open mapping theorem $\implies \psi$ is open, φ is topological injective $\implies \widehat{\otimes}_\pi = \widehat{\otimes}_\varepsilon$ in $(*)$.
We know $\psi\widehat{\otimes}_\pi 1$ is surjective, $\overline{(\varphi\widehat{\otimes}1)(X_1\widehat{\otimes}Y)} = \operatorname{Ker}(\psi\widehat{\otimes}_\pi 1)$. Also, $\varphi\widehat{\otimes}_\pi 1 = \varphi\widehat{\otimes}_\varepsilon 1$ is topological injective $\implies$ the exactness at $X_1\widehat{\otimes}Y$ and $X_2\widehat{\otimes}Y$. $\square$

Exercise. The result holds if only X_1 is nuclear.

Theorem. $X_1 \xrightarrow{\varphi} X_2 \xrightarrow{\psi} X_3$ an exact sequence of Fréchet spaces. Y = a Fréchet space. Suppose either X_2 or Y is nuclear. Then

$$X_1 \widehat{\otimes}_\pi Y \xrightarrow[\varphi\widehat{\otimes}_\pi 1]{} X_2 \widehat{\otimes}_\pi Y \xrightarrow[\psi\widehat{\otimes}_\pi 1]{} X_3 \widehat{\otimes}_\pi Y$$

is exact.

Proof. Let $K = \operatorname{Ker}\psi, C = X_2/K$. $i\colon K \hookrightarrow X_2, q\colon X_2 \to C$ quotient.
Proposition $\implies$ the following sequence is exact:

$$\begin{tikzcd}
K\widehat{\otimes}_\pi Y \arrow[r, "i\widehat{\otimes}1"] & X_2\widehat{\otimes}_\pi Y \arrow[r, "q\widehat{\otimes}1"] \arrow[dr, "\psi\widehat{\otimes}1"'] & C\widehat{\otimes}_\pi Y \arrow[r, "\sim"] \arrow[d] & C\widehat{\otimes}_\varepsilon Y \arrow[d, "\text{injective}"] \\
X_1\widehat{\otimes}_\pi Y \arrow[u, "\text{surjective}"] \arrow[ur, "\varphi\widehat{\otimes}1"'] & & X_3\widehat{\otimes}_\pi Y \arrow[r] & X_3\widehat{\otimes}_\varepsilon Y
\end{tikzcd}$$

$\square$

Corollary. If $\varphi = \operatorname{Ker}\psi$, and the conditions of Theorem are satisfied, then $\varphi\widehat{\otimes}_\pi 1 = \operatorname{Ker}(\psi\widehat{\otimes}_\pi 1)$.

Proof. Consider $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$. We apply our theorem to $0 \to X \xrightarrow{\varphi} Y$ and then apply the same theorem to $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$. $\square$

Proposition. M, N = complex manifolds. Then $\exists$ a topological isomorphism

$$\begin{aligned}
\mathcal{O}(M)\widehat{\otimes}_\pi\mathcal{O}(N) &\xrightarrow{\sim} \mathcal{O}(M \times N) \\
f \otimes g &\mapsto ((x, y) \mapsto f(x)g(y)).
\end{aligned}$$

Proof. **Case 1**: true if M, N are polydisks.

Case 2: M = a polydisk, N is arbitrary.

$$N = \bigcup_{i \in I} U_i, \quad U_i \cong \text{a polydisk}$$

$$0 \to \mathcal{O}(N) \xrightarrow{\varphi} \prod_{i \in I} \mathcal{O}(U_i) \xrightarrow{\psi} \prod_{i,j} \mathcal{O}(U_i \cap U_j) \text{ exact.}$$

$$\varphi(f)_i = f|_{U_i}; \quad \psi((f_i)_{i \in I})_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}.$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(M)\widehat{\otimes}_\pi \mathcal{O}(N) \arrow[r] \arrow[d] & \prod_i \mathcal{O}(M)\widehat{\otimes}\mathcal{O}(U_i) \arrow[r] \arrow[d, "\sim"] & \prod_{i,j} \mathcal{O}(M)\widehat{\otimes}\mathcal{O}(U_i \cap U_j) \arrow[d, "\sim"] \\
0 \arrow[r] & \mathcal{O}(M \times N) \arrow[r] & \prod_i \mathcal{O}(M \times U_i) \arrow[r] & \prod_{i,j} \mathcal{O}((M \times U_i) \cap (M \times U_j))
\end{tikzcd}$$

The rows are exact $\implies \mathcal{O}(M)\widehat{\otimes}\mathcal{O}(N) \to \mathcal{O}(M \times N)$ is a topological isomorphism (open mapping theorem).

Case 3: M, N are arbitrary.

$$M = \bigcup_{j \in J} V_j, \quad V_j \cong \text{a polydisk.}$$

Apply the same argument as in Case 2. $\square$

Proposition. M, N = smooth real manifolds. Then $\exists$ a topological isomorphism

$$C^\infty(M)\widehat{\otimes}_\pi C^\infty(N) \xrightarrow{\sim} C^\infty(M \times N)$$
$$f \otimes g \mapsto ((x,y) \mapsto f(x)g(y)).$$

Proof. Exercise (use the topological isomorphism $C^\infty(\mathbb{R}^m)\widehat{\otimes}_\varepsilon C^\infty(\mathbb{R}^n) \xrightarrow{\sim} C^\infty(\mathbb{R}^{m+n})$). $\square$

The Cartan-Serre finiteness theorem

1 The Čech complex

X = topological space, $\mathscr{F}$ = a sheaf of abelian groups on X. $\mathscr{U} = (U_i)_{i \in I}$ open cover of X.

Notation. $\alpha = (i_0, \dots, i_n) \in I^{n+1}, U_\alpha = U_{i_0} \cap \dots \cap U_{i_n}$.

$$C^n(\mathscr{U}, \mathscr{F}) = \prod_{\alpha \in I^{n+1}} \mathscr{F}(U_\alpha) \text{ the group of Čech } n\text{-cochains.}$$

$$\mathrm{d}^n \colon C^n(\mathscr{U}, \mathscr{F}) \to C^{n+1}(\mathscr{U}, \mathscr{F}), (\mathrm{d}^n f)_{i_0 \dots i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k f_{i_0 \dots \widehat{i}_k \dots i_{n+1}}|_{U_{i_0 \dots i_{n+1}}}.$$

Exercise. $\mathrm{d}^{n+1}\mathrm{d}^n = 0$.

Definition. $C(\mathscr{U}, \mathscr{F}) = (C^n(\mathscr{U}, \mathscr{F}), \mathrm{d}^n)_{n \geq 0}$ is the Čech complex for $\mathscr{F}$ and $\mathscr{U}$.

Observation. $H^0(C(\mathscr{U}, \mathscr{F})) = \mathscr{F}(X) = H^0(X, \mathscr{F})$.

Definition. $U \subset X$ open.

$\mathscr{F}$ is acyclic on U if $H^i(U, \mathscr{F}) = 0, \forall i \geq 1$.

$\mathscr{U}$ is an acyclic cover (a Leray cover) for $\mathscr{F}$ if $\mathscr{F}$ is acyclic on $U_\alpha, \forall \alpha \in I^{n+1}, \forall n \geq 0$.

Theorem 1 (Leray)**.** If $\mathscr{U}$ is an acyclic cover for $\mathscr{F}$, then $H^n(C(\mathscr{U}, \mathscr{F})) \cong H^n(X, \mathscr{F}), \forall n \geq 0$.

Definition. C, D cochain complexes (of abelian groups). $\varphi \colon C \to D$ morphism. φ is a quasiisomorphism (qis) if $H^n(C) \xrightarrow[H^n(\varphi)]{\sim} H^n(C)$ for every $n \in \mathbb{Z}$.

$\mathscr{U}, \mathscr{V}$ = open covers of X; $\mathscr{U} = (U_i)_{i \in I}, \mathscr{V} = (V_j)_{j \in J}$.

Definition. $\mathscr{V}$ is a refinement of $\mathscr{U}$ if every element of $\mathscr{V}$ is contained in an element of $\mathscr{U}$. Equivalently, $\exists \varphi \colon J \to I$ s.t. $\forall j \in J, V_j \subset U_{\varphi(j)}$.

Define $\varphi^* \colon C(\mathscr{U}, \mathscr{F}) \to C(\mathscr{V}, \mathscr{F}), (\varphi^*)^n(f)_{j_0 \dots j_n} = f_{\varphi(j_0) \dots \varphi(j_n)}|_{V_{j_0 \dots j_n}}$. φ^* is a morphism of complexes.

Theorem 2 (Leray)**.** $\mathscr{U}, \mathscr{V}$ acyclic covers for $\mathscr{F}$, $\mathscr{V}$ is a refinement of $\mathscr{U}$ $\implies$ $C(\mathscr{U}, \mathscr{F}) \xrightarrow{\varphi^*} C(\mathscr{V}, \mathscr{F})$ is a qis.

(2) Coherent analytic sheaves

X = a complex manifold. $\mathcal{O} = \mathcal{O}_X$ structure sheaf of X.

Definition. An analytic sheaf on X is a sheaf of $\mathcal{O}$-modules.

Definition. $\mathscr{F}$ = an analytic sheaf on X.
$\mathscr{F}$ is coherent if $\forall \alpha \in X, \exists$ a neighborhood $U \ni x$ and an exact sequence $\mathcal{O}^m \to \mathcal{O}^n \to \mathscr{F} \to 0$ $(m, n \geq 0)$.

Example. $\mathcal{O}$; $\mathcal{O}^n$; a locally free sheaf = sheaf of sections of a holomorphic vector bundle; $\cdots$.

Fact: $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ short exact sequence of analytic sheaves. Then $\mathscr{G}$ is coherent $\iff$ $\mathscr{F}$ and $\mathscr{H}$ are coherent.

Definition. X = complex manifold. $K \subset X$ compact.
The holomorphically convex hull of K is

$$\widehat{K} = \left\{ x \in X : \forall f \in \mathcal{O}(X), |f(x)| \leq \sup_{y \in K} |f(y)| \right\}.$$

Definition. X is a Stein manifold if

(1) $\mathcal{O}(X)$ separates the points of X

(2) $\forall$ compact set $K \subset X$, $\widehat{K}$ is compact.

Example. $\mathbb{C}^n$; a convex open subset $U \subset \mathbb{C}^n$; any open $U \subset \mathbb{C}$.

Nonexample.

(1) a compact manifold X of positive dim (because $\mathcal{O}(X) = \mathbb{C}$ if X is connected)

(2) $\mathbb{C}^n \setminus \{0\}$ $(n \geq 2)$

"Theorem B" (H. Cartan 1951)**.** X = a Stein manifold $\implies$ every coherent analytic sheaf on X is acyclic on X.

Theorem (H. Cartan, J.-P. Serre 1953)**.** X = a compact complex manifold, $\mathscr{F}$ = a coherent analytic sheaf on X $\implies$ $\dim_{\mathbb{C}} H^i(X, \mathscr{F}) < \infty, \forall i = 0, 1, 2, \dots$.

(3) Topologizing the section spaces of coherent sheaves

X = complex manifold, $\mathscr{F}$ = a coherent analytic sheaf on X. Suppose $U \subset X$ is a Stein open, $\exists$ an exact sequence

$$0 \to \mathscr{K} \to \mathcal{O}^n \to \mathscr{F} \to 0 \text{ over } U \tag{$*$}$$

We know: $H^1(U, \mathscr{K}) = 0 \implies \mathcal{O}(U)^n \to \mathscr{F}(U)$ is onto. Equip $\mathscr{F}(U)$ with the quotient topology.
Fact: This topology is Hausdorff.

Exercise. This topology does not depend on $(*)$.

Now let $U \subset X$ be an arbitrary open set. $\mathscr{U} = (U_i)$ open cover of U s.t. $\mathscr{F}(U_i)$ are already topologized. Equip $\mathscr{F}(U)$ with the projective topology generated by $\{\mathscr{F}(U) \to \mathscr{F}(U_i)\}_{i \in I}$.

Exercise.

(1) This topology does not depend on $\mathscr{U}$.

(2) $\mathscr{F}(U)$ is a nuclear Fréchet space.

(4) The Schwartz-type theorem on nuclear perturbations

Theorem. $X, Y =$ Fréchet spaces, $f, u \in \mathscr{L}(X,Y)$, f is onto, u is nuclear $\implies$ Cokernel$(f-u)$ is finite-dim.

Lemma 1. $X =$ Fréchet space, $u \in \mathscr{N}(X,Y) \implies$ Cokernel$(1-u)$ is finite-dim.

Proof. True if X is a Banach space (follows from Riesz' theorem)

General case:

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "u_1"'] & & X \\ & E \arrow[ur, "u_2"'] & \end{tikzcd}$$

$E =$ Banach space, $u = u_2 u_1$, u_1 is nuclear. $v = u_1 u_2 \in \mathscr{N}(E,E)$.

$$\begin{tikzcd} X \arrow[r, "1-u"] \arrow[d, "u_1"'] \arrow[dd, bend left, "u"] & X \arrow[r] \arrow[d, "u_1"'] \arrow[dd, bend left, "u"] & \mathrm{Cokernel}(1-u) \arrow[d, "\overline{u}_1"] \arrow[dd, bend left, "\overline{u}=1"] \\ E \arrow[r, "1-v"] \arrow[d, "u_2"'] & E \arrow[r] \arrow[d, "u_2"'] & \mathrm{Cokernel}(1-v) \arrow[d, "\overline{u}_2"] \\ X \arrow[r, "1-u"] & X \arrow[r] & \mathrm{Cokernel}(1-u) \end{tikzcd}$$

$\implies \overline{u}_2 \overline{u}_1 = 1$
$\implies \overline{u}_1$ is injective
$\implies$ Cokernel$(1-u)$ is finite-dim. $\square$

Lemma 2. $X, Y =$ Fréchet spaces, $f \in \mathscr{L}(X,Y)$ surjective. $(y_n) =$ a sequence in Y, $y_n \to 0 \implies \exists$ a sequence (x_n) in X, s.t. $x_n \to 0$ and $f(x_n) = y_n$.

Proof. Exercise (see the proof of the completeness of X/X_0). $\square$

Lemma 3. X, Y, Z Fréchet spaces, $f \in \mathscr{L}(X,Y)$ surjective, $u \in \mathscr{N}(Z,Y)$. Then $\exists v \in \mathscr{N}(Z,X)$ s.t. the diagram commutes.

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[ur, "v"] \arrow[r, "u"'] & Y \end{tikzcd}$$

Proof. $u = \sum\limits_n \lambda_n f_n(\cdot) y_n, (f_n) =$ an equicontinuous sequence in Z', $\sum |\lambda_n| < \infty, y_n \to 0$ in Y.

Lemma 2 $\implies \exists$ a sequence (x_n) in X, $x_n \to 0$, $f(x_n) = y_n$.
Let $v \colon Z \to X, v = \sum\limits_n \lambda_n f_n(\cdot) x_n$.

Let $B = \overline{\Gamma\{x_n : n \in \mathbb{N}\}}$, B is bounded, absolutely convex, complete $\implies$ B is a Banach disk $\implies$ v is nuclear. $\square$

Proof of the Schwartz-type theorem. Lemma 3 $\implies \exists$ a nuclear v s.t. the diagram
$$\begin{tikzcd} & X \arrow[d, "f"] \\ X \arrow[ur, "v"] \arrow[r, "u"'] & Y \end{tikzcd}$$
commutes.

$$\begin{tikzcd} X \arrow[r, "1-v"] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "f"] & \mathrm{Cokernel}(1-v) \arrow[d, "\text{surjective}"] \\ X \arrow[r, "f-u"'] & X \arrow[r] & \mathrm{Cokernel}(f-u) \end{tikzcd}$$

$\implies$ Cokernel$(f-u)$ is finite-dim. $\square$

5 Binuclear qis's and the proof of the Cartan-Serre

Definition. $X, Y =$ lcs's, $\varphi \in \mathscr{L}(X, Y)$.
φ is binuclear if $\varphi = \varphi_2\varphi_1$, where $\varphi_1 \in \mathscr{N}(X, Z), \varphi_2 \in \mathscr{N}(Z, Y)$.

Example/Exercise. $X =$ complex manifold, $\mathscr{F} =$ a coherent sheaf on X, $U, V \subset X$ open, s.t. $\overline{V}$ is compact, $\overline{V} \subset U$. Then the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is binuclear.
Hint: This map factorizes through a Banach space.

Lemma. $X, Y =$ lcs's, $\varphi \in \mathscr{L}(X, Y)$ binuclear, $Y_1 \subset Y$ closed subspace, $\varphi(X) \subset Y_1$.
Consider $\varphi_0 \colon X \to Y_1, \varphi_0(x) = \varphi(x), \forall x \in X$. Then φ_0 is nuclear.

Proof. $\varphi = \varphi_2\varphi_1$, φ_1, φ_2 are nuclear.

$$\begin{tikzcd}
X \arrow[r, "\varphi_1"'] \arrow[drr, "\gamma"'] & Z \arrow[r, "\alpha"'] \arrow[rr, bend left, "\varphi_2"] & H \arrow[r, "\beta"'] & Y \\
 & & H_1 \arrow[u, "j"'] \arrow[r, "\beta_1"'] & Y_1 \arrow[u, "i"']
\end{tikzcd} \quad H_1 = \beta^{-1}(Y_1), \beta_1 = \beta|_{H_1}, (\alpha\varphi_1)(X) \subset H_1.$$

$\implies \exists\gamma$ s.t. $j\gamma = \alpha\varphi_1$. We have $\varphi_0 = \beta_1\gamma$. $p =$ the orthogonal projection of H onto H_1. $\gamma = pj\gamma = p\alpha\varphi_1$ is nuclear $\implies$ φ_0 is nuclear. ☐

Theorem. $C, D =$ cochain complexes of Fréchet spaces, $\varphi \colon C \to D$ binuclear qis $\implies$ $H^n(C)(\cong H^n(D))$ is finite-dim.

Proof.

$$\begin{tikzcd}
C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & Z^n(C) \arrow[d, "\overline{\varphi}^n"] & \subset & C^n \arrow[d, "\varphi^n"] \\
D^{n-1} \arrow[r, "\overline{d}_D^{n-1}"'] & Z^n(D) & \subset & D^n
\end{tikzcd}$$

φ^n is binuclear $\xRightarrow{\text{Lemma}}$ $\overline{\varphi}^n$ is nuclear.

$$f \colon D^{n-1} \bigoplus Z^n(C) \to Z^n(D), f = \overline{d}_D^{n-1} \bigoplus \overline{\varphi}^n$$

$$(x, y) \mapsto \overline{d}_D^{n-1}(x) + \overline{\varphi}^n(y) \quad f \text{ is surjective.}$$

$0 \bigoplus \overline{\varphi}^n$ is nuclear $\xRightarrow[\text{(Schwartz)}]{}$ Cokernel$(f - 0 \bigoplus \overline{\varphi}^n)$ $\Big(\cong$ Cokernel$(\overline{d}_D^{n-1}) \cong H^n(D)\Big)$ is finite-dim. ☐

Exercise. $X =$ a complex manifold, $U, V \subset X$ Stein open sets $\implies$ $U \cap V$ is Stein.

Proof of the Cartan-Serre theorem. Let $\mathscr{U}, \mathscr{V}$ be finite open covers of X s.t. $\forall V \in \mathscr{V}, \exists U \in \mathscr{U}$ s.t. $\overline{V} \subset U$.

$$\begin{tikzcd}
C^n(\mathscr{U}, \mathscr{F}) \arrow[r, "(\varphi^*)^n"'] \arrow[d] & C^n(\mathscr{V}, \mathscr{F}) \arrow[d] \\
\mathscr{F}(U_{i_0 \ldots i_n}) \arrow[r, "\text{binuclear restriction}"] & \mathscr{F}(V_{j_0 \ldots j_n})
\end{tikzcd} \quad i_k = \varphi(j_k), \overline{V}_j \subset U_{\varphi(j)}.$$

$\xRightarrow[\text{(exercise)}]{}$ $(\varphi^*)^n$ is binuclear (follows from the restriction maps are binuclear and from the fact that C^n is finite product of these section spaces).
Leray 2 $\implies$ φ^* is a qis (because $\mathscr{U}, \mathscr{V}$ are acyclic for $\mathscr{F}$ by Cartan's Theorem B and by Exercise).
Theorem $\implies$ $H^n(C(\mathscr{U}, \mathscr{F}))$ is finite-dim.
Leray 1 $\implies$ $H^n(C(\mathscr{U}, \mathscr{F})) \cong H^n(X, \mathscr{F})$. ☐