Harmonic Analysis and Banach Algebras

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Abstract

These are lecture notes based on the course "Harmonic Analysis and Banach Algebras" taught by Alexei Yu. Pirkovskii at the Faculty of Mathematics at HSE in the Fall Semester 2024.

Contents

Lecture 1 (2024.09.06)

Harmonic analysis of finite abelian groups

Convention. Everything is over $\mathbb{C}$.

Notation. $\mathbb{C}^\times = (\mathbb{C}\backslash\{0\}, \cdot)$, $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$.

G = a group.

Definition. A character of G is a group homomorphism $\chi \colon G \to \mathbb{C}^\times$. χ is unitary if $\chi(G) \subset \mathbb{T}$.

Exercise. G is finite $\implies$ all characters of G are unitary.

Observe. $\mathrm{Hom}(G, \mathbb{T})$ is an abelian group.

$$(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x), \quad \chi^{-1}(x) = \frac{1}{\chi(x)} = \overline{\chi(x)}.$$

Assume. G is finite and abelian.

Definition. $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ is the dual of G.

Example. $G = \langle x_0 \rangle_n$, $\mathbb{U}_n = \{z \in \mathbb{C} \colon z^n = 1\}$.

Exercise. The map $\widehat{G} \to \mathbb{U}_n$, $\chi \mapsto \chi(x_0)$ is an isomorphism.

$\implies$ $G \cong \widehat{G}$ (not canonically!).

Exercise. $\widehat{G_1 \times G_2} \cong \widehat{G}_1 \times \widehat{G}_2$, $\chi \mapsto (\chi|_{G_1}, \chi|_{G_2})$.

Proposition. G = a finite abelian group $\implies$ $G \cong \widehat{G}$ (not canonically!).

Proof. $\widehat{G} \cong \widehat{\langle x_1 \rangle}_{n_1} \times \cdots \times \widehat{\langle x_k \rangle}_{n_k} \cong \mathbb{U}_{n_1} \times \cdots \times \mathbb{U}_{n_k} \cong G$. □

Let $\widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T})$. For any $x \in G$, consider the evaluation map $\varepsilon_x \colon \widehat{G} \to \mathbb{T}$, $\varepsilon_x(\chi) = \chi(x)$, $\varepsilon_x \in \widehat{\widehat{G}}$. Then the map $i_G \colon G \to \widehat{\widehat{G}}$, $G \ni x \mapsto \varepsilon_x$ is a group homomorphism.

Theorem (**"Pontryagin duality"**). G = a finite abelian group $\implies$ $i_G \colon G \to \widehat{\widehat{G}}$ is an isomorphism.

Lemma. $x \in G$, $x \neq e$ $\implies$ $\exists \chi \in \widehat{G}$ s.t. $\chi(x) \neq 1$.

Proof.

(1) $G = \langle x_0 \rangle_n$. Then there exists an injective character χ, $\chi(x_0^k) = e^{\frac{2\pi i k}{n}}$.

(2) General case:

$$G \cong G_1 \times \cdots \times G_m, \ G_i \text{ cyclic.}$$ □

Proof of Theorem. Lemma $\iff$ $\mathrm{Ker}\, i_G = \{e\}$.

$$|\widehat{\widehat{G}}| = |G| \implies i_G \text{ is an isomorphism.}$$ □

Notation. $\mathrm{Fun}(G) = \mathbb{C}^G$, $f \in \mathrm{Fun}(G)$.

Definition. The Fourier transform of f is

$$\widehat{f} \colon \widehat{G} \to \mathbb{C}, \quad \widehat{f}(\chi) = \sum_{x \in G} f(x)\chi(x).$$

Consider $\mathscr{F} = \mathscr{F}_G \colon \mathrm{Fun}(G) \to \mathrm{Fun}(\widehat{G})$, $\mathscr{F}(f) = \widehat{f}$. $\mathscr{F}$ is the Fourier transform of G.

Another definition. $\widehat{f}(\chi) = \sum_{x \in G} f(x)\overline{\chi(x)} = \sum_{x \in G} f(x)\chi^{-1}(x)$.

Example. $x \in G$, $\delta_x \in \mathrm{Fun}(G)$, $\delta_x(y) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases}$ $\widehat{\delta}_x(\chi) = \sum_{y \in G} \delta_x(y)\chi(y) = \chi(x) = \varepsilon_x(\chi) \implies \widehat{\delta}_x = \varepsilon_x$.

Lemma. $\chi \in \widehat{G}$, $\chi \neq 1 \implies \sum_{x \in G} \chi(x) = 0$.

Proof. Take $y \in G$ s.t. $\chi(y) \neq 1$.

$$\chi(y) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(yx) = \sum_{x \in G} \chi(x) \implies \sum_{x \in G} \chi(x) = 0. \qquad \square$$

Inner product on $\mathrm{Fun}(G)$: $\langle f | g \rangle = \sum_{x \in G} f(x)\overline{g(x)}$.

Proposition. $\widehat{G}$ is an orthogonal basis in $\mathrm{Fun}(G)$; $\forall \chi \in \widehat{G}$, $\|\chi\| = \sqrt{\langle \chi | \chi \rangle} = \sqrt{n}$, where $n = |G|$.

Proof. $\chi_1, \chi_2 \in \widehat{G}$, $\chi_1 \neq \chi_2$.

$$\langle \chi_1 | \chi_2 \rangle = \sum_x \chi_1(x)\overline{\chi_2(x)} = \sum_x \left(\chi_1 \chi_2^{-1}\right)(x) \overset{\text{Lemma}}{=\!=\!=} 0.$$

$$\|\chi\|^2 = \langle \chi | \chi \rangle = \sum_{x \in G} |\chi(x)|^2 = n.$$

$|\widehat{G}| = n = \dim \mathrm{Fun}(G) \implies \widehat{G}$ is a basis of $\mathrm{Fun}(G)$. $\square$

Example. $\chi \in \widehat{G}$, $\widehat{\chi}(\varphi) = \sum_{x \in G} \varphi(x)\chi(x) = \langle \varphi | \overline{\chi} \rangle = \langle \varphi | \chi^{-1} \rangle = \begin{cases} 0 & \text{if } \varphi \neq \chi^{-1}, \\ n & \text{if } \varphi = \chi^{-1}. \end{cases} \implies \widehat{\chi} = n\delta_{\chi^{-1}}$.

Inner product on $\mathrm{Fun}(\widehat{G})$: $\langle f | g \rangle = \frac{1}{n} \sum_{\chi \in \widehat{G}} f(\chi)\overline{g(\chi)}$.

Theorem (**"Plancherel theorem"**). $\mathscr{F} \colon \mathrm{Fun}(G) \to \mathrm{Fun}(\widehat{G})$ is a unitary isomorphism.

Proof. $\left\{ \frac{\chi}{\sqrt{n}} \colon \chi \in \widehat{G} \right\}$ is an ONB (orthonormal basis) in $\mathrm{Fun}(G)$; $\left\{ \sqrt{n}\delta_\chi \colon \chi \in \widehat{G} \right\}$ is an ONB in $\mathrm{Fun}(\widehat{G})$.

$$\mathscr{F} \colon \frac{\chi}{\sqrt{n}} \mapsto \sqrt{n}\delta_{\chi^{-1}}. \qquad \square$$

Remark. $\widehat{f}(\chi) = \int_G f(x)\chi(x)\,\mathrm{d}\mu(x)$, $\mu =$ counting measure, $\mu(\{x\}) = 1$, $\forall x$.

$$\mathrm{Fun}(G) = L^2(G, \mu), \quad \mathrm{Fun}(\widehat{G}) = L^2(\widehat{G}, \widehat{\mu}), \quad \widehat{\mu} = \frac{\text{counting}}{n} \text{ is the dual of } \mu.$$

Another alternative approach: $\mu = \frac{\text{counting}}{\sqrt{n}} \implies \widehat{\mu} = \frac{\text{counting}}{\sqrt{n}}$.

Identify G and $\widehat{G}$ (canonically)

$$\begin{tikzcd} \mathrm{Fun}(G) \arrow[r, bend left, "\mathscr{F}_G"] & \mathrm{Fun}(\widehat{G}) \arrow[l, bend left, "\mathscr{F}_{\widehat{G}}"] \end{tikzcd}$$

Definition.

$$\mathscr{F}_{\widehat{G}} \colon \mathrm{Fun}(\widehat{G}) \to \mathrm{Fun}(G),$$
$$g \mapsto \widehat{g}, \quad \widehat{g}(x) = \frac{1}{n} \sum_{\chi \in \widehat{G}} g(\chi)\chi(x).$$

Notation. $S_G \colon \mathrm{Fun}(G) \to \mathrm{Fun}(G)$, $(S_G f)(x) = f(x^{-1})$.

Theorem (**"Inversion formula"**). $\mathscr{F}_{\widehat{G}} \circ \mathscr{F}_G = S_G$. That is, $\forall f \in \mathrm{Fun}(G)$,

$$f(x) = \frac{1}{n} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\overline{\chi(x)}.$$

Proof. $\delta_x \xmapsto[\mathscr{F}_G]{} \varepsilon_x \xmapsto[\mathscr{F}_{\widehat{G}}]{} \delta_{\varepsilon_x^{-1}} = \delta_{x^{-1}}$ $\square$

Definition. Algebra = associative $\mathbb{C}$-algebra = associative ring together with a vector space structure s.t. the multiplication $A \times A \to A$ is $\mathbb{C}$-bilinear. A is unital if there exists $1 = 1_A \in A$ s.t. $a \cdot 1 = 1 \cdot a = a$, $\forall a$.

Definition. An algebra homomorphism $\varphi\colon A \to B$ is a ring homomorphism which is $\mathbb{C}$-linear. If A, B are unital, then φ is unital if $\varphi(1_A) = 1_B$.

G = a group, $\mathbb{C}G$ = a vector space s.t. G is a basis of $\mathbb{C}G$.

$$\mathbb{C}G = \left\{ \sum_{x \in G} \alpha_x \cdot x \colon \alpha_x \in \mathbb{C}, \alpha_x = 0 \text{ for all but finitely many } x \right\}.$$

Multiplication on $\mathbb{C}G$: $(u, v) \mapsto u * v$ is uniquely determined by $x * y = xy$ (x, $y \in G$). $(\mathbb{C}G, *)$ is the group algebra of G. Assume G is finite, then we have a vector isomorphism $\alpha\colon \mathbb{C}G \to \mathrm{Fun}(G)$, $G \ni x \mapsto \delta_x$.

Definition. The convolution of f, $g \in \mathrm{Fun}(G)$ is

$$f * g = \alpha(\alpha^{-1}(f) * \alpha^{-1}(g)).$$

$\mathrm{Fun}_*(G) = (\mathrm{Fun}(G), *)$ is the convolution algebra of G.

Exercise. $(f * g)(x) = \sum\limits_{y \in G} f(y) g(y^{-1}x)$.

Theorem. G = a finite abelian group. Then

$$\mathscr{F}\colon\ \mathrm{Fun}_*(G) \to \mathrm{Fun}(\widehat{G}).$$

(Here, $\mathrm{Fun}(\widehat{G})$ is equipped with the pointwise product.) That is, $\widehat{f * g} = \widehat{f}\widehat{g}$.

Proof. $\widehat{\delta_x * \delta_y} = \widehat{\delta}_{xy} = \varepsilon_{xy} = \varepsilon_x \varepsilon_y = \widehat{\delta}_x \widehat{\delta}_y$. $\square$

Lecture 2 (2024.09.13)

The Pontryagin duality for $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{R}$

Notation. $\widehat{\mathbb{Z}} = \mathrm{Hom}_{\mathsf{cont}}(\mathbb{Z}, \mathbb{T})$.

$\forall z \in \mathbb{T}$, $\chi_z \colon \mathbb{Z} \to \mathbb{T}$, $\chi_z(n) = z^{-n}$. Consider the map

$$\mathbb{T} \to \widehat{\mathbb{Z}}, \quad z \mapsto \chi_z \tag{1}$$

Exercise 1. (1) is an isomorphism.

Notation. $\widehat{\mathbb{T}} = \mathrm{Hom}_{\mathsf{cont}}(\mathbb{T}, \mathbb{T})$.

$\forall n \in \mathbb{Z}$, $\chi_n \colon \mathbb{T} \to \mathbb{T}$, $\chi_n(z) = z^{-n}$. Consider the map

$$\mathbb{Z} \to \widehat{\mathbb{T}}, \quad n \mapsto \chi_n \tag{2}$$

Exercise 2. (2) is an isomorphism.

Notation. $\widehat{\mathbb{R}} = \mathrm{Hom}_{\mathsf{cont}}(\mathbb{R}, \mathbb{T})$.

$\forall \lambda \in \mathbb{R}$, $\chi_\lambda \colon \mathbb{R} \to \mathbb{T}$, $\chi_\lambda(t) = e^{-2\pi \mathrm{i} \lambda t}$. Consider the map

$$\mathbb{R} \to \widehat{\mathbb{R}}, \quad \lambda \mapsto \chi_\lambda \tag{3}$$

Exercise 3. (3) is an isomorphism.

Hint to Exercise 2, 3: Surjectivity of (3): $\chi \in \widehat{\mathbb{R}}$, $\chi(0) = 1$. $\exists \delta > 0$ s.t. $\forall t \in (-\delta, \delta)$, $\mathrm{Re}\,\chi(t) > 0$. Let $a \in (0, \delta)$, $b = \chi(a)$, then there exists a unique λ s.t. $b = e^{-2\pi \mathrm{i} \lambda a}$ and $|2\pi\lambda a| < \frac{\pi}{2}$.
Claim: $\chi = \chi_\lambda$ (prove it).
Exercise 2 follows from Exercise 3: $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. ■
Equip $\widehat{\mathbb{Z}}$, $\widehat{\mathbb{T}}$, $\widehat{\mathbb{R}}$ with the topology induced from $C(\mathbb{Z})$, $C(\mathbb{T})$, $C(\mathbb{R})$.

Proposition. Isomorphisms (1), (2), (3) $\widehat{\mathbb{T}} \cong \mathbb{Z}$, $\widehat{\mathbb{Z}} \cong \mathbb{T}$, $\widehat{\mathbb{R}} \cong \mathbb{R}$ are topological isomorphisms.

Proof.

(1) $\mathbb{T} \to \widehat{\mathbb{Z}}$, $z \mapsto \chi_z$.

$$\chi_{z_n} \to \chi_z \text{ in } \widehat{\mathbb{Z}} \iff \chi_{z_n} \to \chi_z \text{ pointwise} \iff z_n = \chi_{z_n}(-1) \to \chi_z(-1) = z.$$

(2) $\mathbb{Z} \to \widehat{\mathbb{T}}$, $n \mapsto \chi_n$.

$$\begin{aligned}
\chi_{n_k} \to \chi_n &\iff \chi_{n_k} \underset{\mathbb{T}}{\rightrightarrows} \chi_n \\
&\iff \sup_{z \in \mathbb{T}} |z^{n_k} - z^n| \to 0 \\
&\iff \sup_{z \in \mathbb{T}} |z^{m_k} - 1| \to 0 \quad (m_k k = n_k - n) \\
&\iff \exists \ell \text{ s.t. } m_k = 0 \; \forall k \geqslant \ell \\
&\iff n_k \to n \text{ in } \mathbb{Z}.
\end{aligned}$$

(3) $\mathbb{R} \to \widehat{\mathbb{R}}$, $\lambda \mapsto \chi_\lambda$.

$$\chi_{\lambda_k} \to \chi_\lambda \iff \forall [a,b] \subset \mathbb{R}, \sup_{t \in [a,b]} \left| e^{-2\pi \mathrm{i}(\lambda_k - \lambda)t} \right| \to 0 \underset{\text{(exercise)}}{\iff} \lambda_k \to \lambda \text{ in } \mathbb{R}.$$ □

Theorem (Pontryagin duality for $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{R}$)**.** Let $G \in \{\mathbb{Z}, \mathbb{T}, \mathbb{R}\}$, then $i_G \colon G \to \widehat{\widehat{G}}$ is a topological isomorphism.

Proof. We have the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "i_{\mathbb{Z}}"] \arrow[dr, "\alpha_{\mathbb{Z}}"'] & & \widehat{\widehat{\mathbb{Z}}} \arrow[dl, "\widehat{\alpha_{\mathbb{T}}}"] \\ & \widehat{\mathbb{T}} & \end{tikzcd}$$

where $\alpha_{\mathbb{T}} \colon \mathbb{T} \to \widehat{\mathbb{Z}}$ as in (1), $\alpha_{\mathbb{Z}} \colon \mathbb{Z} \to \widehat{\mathbb{T}}$ as in (2) and $\widehat{\alpha_{\mathbb{T}}} \colon \widehat{\widehat{\mathbb{Z}}} \to \widehat{\mathbb{T}}$, $\chi \mapsto \chi \circ \alpha_{\mathbb{T}}$. This implies that $i_{\mathbb{Z}}$ is a topological isomorphism. Similarity for $i_{\mathbb{T}}$, $i_{\mathbb{R}}$. $\square$

Harmonic analysis on $\mathbb{Z}$ and $\mathbb{T}$

Quasi-definition. $f \colon \mathbb{Z} \to \mathbb{C}$, $\widehat{f} \colon \widehat{\mathbb{Z}} \to \mathbb{C}$,

$$\widehat{f}(\chi) = \sum_{n \in \mathbb{Z}} f(n)\chi(n) \quad \widehat{\mathbb{Z}} \cong \mathbb{T}.$$

$$\widehat{f} \colon \mathbb{T} \to \mathbb{C} \quad \widehat{f}(z) = \sum_{n \in \mathbb{Z}} f(n) z^{-n}.$$

Notation. $\ell^1(\mathbb{Z}) = \left\{ f \colon \mathbb{Z} \to \mathbb{C} \colon \sum_{n \in \mathbb{Z}} |f(n)| < \infty \right\}$ is a Banach space w.r.t the norm $\|f\|_1 = \sum |f(n)|$.

Definition. The Fourier transform of $f \in \ell^1(\mathbb{Z})$ is $\widehat{f} \colon \mathbb{T} \to \mathbb{C}$, $\widehat{f}(z) = \sum_{n \in \mathbb{Z}} f(n) z^{-n}$.

Observe. The series converges absolutely and uniformly on $\mathbb{T}$, $\widehat{f} \in C(\mathbb{T})$, $\|\widehat{f}\|_\infty \leqslant \|f\|_1$ (where $\|\cdot\|_\infty$ is the sup norm).

Notation. $\mathscr{F}_{\mathbb{Z}} = \mathscr{F} \colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$, $f \mapsto \widehat{f}$ is the Fourier transform on $\mathbb{Z}$. $\mathscr{F}_{\mathbb{Z}}$ is a bounded linear map.

Example. $n \in \mathbb{Z}$, $\widehat{\delta}_n = \chi_n$.

Quasi-definition. $f \colon \mathbb{T} \to \mathbb{C}$, $\widehat{f} \colon \widehat{\mathbb{T}} \to \mathbb{C}$,

$$\widehat{f}(\chi) = \text{“}\sum_{z \in \mathbb{T}}\text{”} f(z)\chi(z) \quad \widehat{\mathbb{T}} \cong \mathbb{Z}.$$

$$\widehat{f} \colon \mathbb{Z} \to \mathbb{C} \quad \widehat{f}(n) = \text{“}\sum_{z \in \mathbb{T}}\text{”} f(z) z^{-n}.$$

$$\text{“}\sum\text{”} \rightsquigarrow \int.$$

Notation. $L^1(\mathbb{T}) = L^1(\mathbb{T}, \mu)$, $\mu = \frac{\text{length measure}}{2\pi}$.

Definition. The Fourier transform of $f \in L^1(\mathbb{T})$ is $\widehat{f} \colon \mathbb{Z} \to \mathbb{C}$, $\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} \, \mathrm{d}\mu(z)$.

Notation. $\ell^\infty(\mathbb{Z}) = \{ f \colon \mathbb{Z} \to \mathbb{C} \colon \sup |f(n)| < \infty \}$ is a Banach space w.r.t. the norm $\|f\|_\infty = \sup |f(n)|$.

Observe. $\forall f \in L^1(\mathbb{T})$, $\widehat{f} \in \ell^\infty(\mathbb{Z})$.

Notation. $\mathscr{F}_{\mathbb{T}} = \mathscr{F} \colon L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$, $f \mapsto \widehat{f}$ is the Fourier transform on $\mathbb{T}$.

Lemma. $\{\chi_n : n \in \mathbb{Z}\}$ is an ON family.

Proof. Exercise. $\square$

Exercise. $\widehat{\chi}_n(k) = \langle \chi_n | \chi_{-k} \rangle = \delta_{n,-k} \implies \widehat{\chi}_n = \delta_{-n}$.

Theorem (**Stone-Weierstrass theorem**). X = compact Hausdorff topological space; $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A = a $\mathbb{K}$-subalgebra of $C(X) = C(X, \mathbb{K})$. Suppose:

(1) $1 \in A$

(2) A separates the points of X ($\forall x, y \in X$, $x \neq y$, $\exists f \in A$ s.t. $f(x) \neq f(y)$)

(3) (for $\mathbb{K} = \mathbb{C}$) $f \in A \implies \overline{f} \in A$.

Then A is dense in $C(X)$ (w.r.t. $\|\cdot\|_\infty$).

Corollary 1 (**Weierstrass**). $\mathbb{K}[t] \hookrightarrow C[a, b]$ is a dense subalgebra.

Notation. $\mathscr{R}(\mathbb{T}) = \operatorname{span}\{\chi_n : n \in \mathbb{Z}\} \subset C(\mathbb{T})$ (subalgebra of trigonometric polynomials) ($\cong$ Laurent polynomials).

Corollary 2. $\mathscr{R}(\mathbb{T})$ is dense in $C(\mathbb{T})$ ($\mathbb{K} = \mathbb{C}$).

Notation.

$$c_0(\mathbb{Z}) = \left\{ f : \mathbb{Z} \to \mathbb{C} : \lim_{n \to \infty} f(n) = 0 \right\}.$$

$c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ is a closed vector space, hence it is a Banach space.

Corollary 3 (**Riemann-Lebesgue lemma for $\mathscr{F}_\mathbb{T}$**). $\mathscr{F}_\mathbb{T}(L^1(\mathbb{T})) \subset c_0(\mathbb{Z})$.

Proof. Note that $\mathscr{R}(\mathbb{T})$ dense in $L^1(\mathbb{T})$ and $\mathscr{F}_\mathbb{T}(\mathscr{R}(\mathbb{T})) \subset \operatorname{span}\{\delta_n : n \in \mathbb{Z}\} \subset c_0(\mathbb{Z})$, so $\mathscr{F}_\mathbb{T}(L^1(\mathbb{T})) \subset c_0(\mathbb{Z})$. $\square$

We have $\mathscr{F}_\mathbb{T} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$.

Corollary 4. $\{\chi_n : n \in \mathbb{Z}\}$ is an ON basis of $L^2(\mathbb{T})$.

Lecture 3 (2024.09.20)

Recall:

$$\mathbb{T} \cong \widehat{\mathbb{Z}}, \quad z \in \mathbb{T} \mapsto \chi_z \in \widehat{\mathbb{Z}}, \quad \chi_z(n) = z^{-n}.$$

$$\mathbb{Z} \cong \widehat{\mathbb{T}}, \quad n \in \mathbb{Z} \mapsto \chi_n \in \widehat{\mathbb{T}}, \quad \chi_n(z) = z^{-n}.$$

$$\mathbb{R} \cong \widehat{\mathbb{R}}, \quad \lambda \in \mathbb{R} \mapsto \chi_\lambda \in \widehat{\mathbb{R}}, \quad \chi_\lambda(t) = e^{-2\pi \mathrm{i} \lambda t}.$$

Fourier transform on $\mathbb{Z}$: If $f \in \ell^1(\mathbb{Z})$, then $\widehat{f}\colon \mathbb{T} \to \mathbb{C}$ is given by

$$\widehat{f}(z) = \sum_{n\in\mathbb{Z}} f(n) z^{-n}.$$

It is easy to check that

$$\widehat{f} \in C(\mathbb{T}); \quad \|\widehat{f}\|_\infty \leqslant \|f\|_1.$$

$$\mathscr{F}_{\mathbb{Z}}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T}), \quad \widehat{\delta}_n = \chi_n.$$

Fourier transform on $\mathbb{T}$: If $f \in L^1(\mathbb{T})$, then $\widehat{f}\colon \mathbb{Z} \to \mathbb{C}$ is given by

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} \,\mathrm{d}\mu(z).$$

It is easy to check that

$$\widehat{f} \in c_0(\mathbb{Z}); \quad \|\widehat{f}\|_\infty \leqslant \|f\|_1.$$

$$\mathscr{F}_{\mathbb{T}}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z}), \quad \widehat{\chi}_n = \delta_{-n}.$$

Definition. $f, g\colon \mathbb{Z} \to \mathbb{C}$. The convolution of f, g is

$$(f * g)(n) = \sum_{k\in\mathbb{Z}} f(k) g(n-k). \tag{4}$$

Convention. $f * g$ is defined at those $n \in \mathbb{Z}$ for which (4) converges.

Exercise. Let $f, g \in \ell^1(\mathbb{Z})$. Then

(1) $f * g$ is defined everywhere on $\mathbb{Z}$;

(2) $f * g \in \ell^1(\mathbb{Z})$;

(3) $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$;

(4) $(\ell^1(\mathbb{Z}), *)$ is a commutative algebra;

(5) δ_0 is an identity of $\ell^1(\mathbb{Z})$; $\mathbb{C}\mathbb{Z}$ is isomorphic to a dense subalgebra of $\ell^1(\mathbb{Z})$, $n \in \mathbb{Z} \mapsto \delta_n$;

(6) $\mathscr{F}_{\mathbb{Z}}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is an algebra homomorphism.

Definition. $f, g \in \mathbb{T} \to \mathbb{C}$ measurable. The convolution of f, g is

$$(f * g)(z) = \int_{\mathbb{T}} f(\zeta) g(\zeta^{-1} z) \,\mathrm{d}\mu(\zeta). \tag{5}$$

Convention. $f * g$ is defined at those $z \in \mathbb{T}$ for which (5) exists.

Exercise. Let $f, g \in L^1(\mathbb{T})$. Then

(1) $f * g$ is defined a.e. on $\mathbb{T}$;

(2) $f * g \in L^1(\mathbb{T})$;

(3) $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$;

(4) $(L^1(\mathbb{T}), *)$ is a commutative algebra;

(5) $(L^1(\mathbb{T}), *)$ is not unital;

(6) $\mathscr{F}_{\mathbb{T}}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is an algebra homomorphism.

Before state Plancherel theorem, let's draw a picture:

$$\begin{tikzcd}
\ell^1(\mathbb{Z}) \arrow[rr, "\mathscr{F}_{\mathbb{Z}}"] \arrow[d, hook] & & C(\mathbb{T}) \arrow[d] \\
\ell^2(\mathbb{Z}) \arrow[loop left, "S"] \arrow[rr, bend left, "\mathscr{F}_{\mathbb{Z}}"] \arrow[d, hook] & & L^2(\mathbb{T}) \arrow[loop right, "S"] \arrow[ll, bend left, "\sim"] \arrow[d] \\
c_0(\mathbb{Z}) & & L^1(\mathbb{T}) \arrow[ll, "\mathscr{F}_{\mathbb{T}}"]
\end{tikzcd}$$

Theorem (**Plancherel theorem for $\mathbb{Z}$ and $\mathbb{T}$**).

(1) $\mathscr{F}_{\mathbb{T}}|_{L^2(\mathbb{T})}$ is a unitary isomorphism of $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$.

(2) $\mathscr{F}_{\mathbb{Z}}$ uniquely extends to a unitary isomorphism

$$\mathscr{F}_{\mathbb{Z}}\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}), \quad \mathscr{F}_{\mathbb{Z}} g = \sum_{n\in\mathbb{Z}} g(n)\chi_n.$$

(3) $\mathscr{F}_{\mathbb{Z}}\mathscr{F}_{\mathbb{T}} = S_{\mathbb{T}}$ on $L^2(\mathbb{T})$; $\mathscr{F}_{\mathbb{T}}\mathscr{F}_{\mathbb{Z}} = S_{\mathbb{Z}}$ on $\ell^2(\mathbb{Z})$.

Proof. Recall that $\{\chi_n : n \in \mathbb{Z}\}$ is an ON basis of $L^2(\mathbb{T})$.

(1) If $f \in L^2(\mathbb{T})$, then $(\mathscr{F}_{\mathbb{Z}} f)(n) = \langle f|\chi_{-n}\rangle$. Hence by Riesz-Fischer theorem, (1) follows.

(2) Riesz-Fischer theorem implies that there is a unitary isomorphism

$$U\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}), \quad Ug = \sum_{n\in\mathbb{Z}} g(n)\chi_n.$$

We have $U|_{\ell^1(\mathbb{Z})} = \mathscr{F}_{\mathbb{Z}}$.

(3) $\delta_n \mapsto \chi_n \mapsto \delta_{-n}$; $\chi_n \mapsto \delta_{-n} \mapsto \chi_{-n}$. □

Corollary.

(1) (Uniqueness for $\mathscr{F}_{\mathbb{Z}}$). $\mathscr{F}_{\mathbb{Z}}\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is injective.

(2) (Density theorem for $\mathscr{F}_{\mathbb{Z}}$). $\mathscr{F}_{\mathbb{Z}}(\ell^1(\mathbb{Z}))$ is dense in $C(\mathbb{T})$.

(3) (The Fourier inversion formula). $\mathscr{F}_{\mathbb{T}}\mathscr{F}_{\mathbb{Z}} = S_{\mathbb{Z}}$ on $\ell^1(\mathbb{Z})$. That is, $\forall f \in \ell^1(\mathbb{Z})$,

$$f(n) = \int_{\mathbb{T}} \widehat{f}(z) z^n \,\mathrm{d}\mu(z) \quad (n \in \mathbb{Z}).$$

Theorem.

(1) (Uniqueness theorem for $\mathscr{F}_{\mathbb{T}}$). $\mathscr{F}_{\mathbb{T}}\colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is injective.

(2) (Density theorem). $\mathscr{F}_{\mathbb{T}}(L^1(\mathbb{T}))$ is dense in $c_0(\mathbb{Z})$.

Proof.

(1) Let $f \in L^1(\mathbb{T})$. Suppose $\widehat{f} = 0$. Define $F\colon C(\mathbb{T}) \to \mathbb{C}$, $F(g) = \int_{\mathbb{T}} fg \,\mathrm{d}\mu$. F is bounded linear functional, and $\|F\| \leqslant \|f\|_1$. $\widehat{f} = 0 \implies F(\chi_n) = 0$, $\forall n$. Since $\operatorname{span}\{\chi_n : n \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$, $F = 0$.
Take an interval $I \subset \mathbb{T}$, choose a sequence $\{g_n\}$ in $C(\mathbb{T})$ s.t. $g_n \mapsto \chi_I$ pointwise, and $0 \leqslant g_n \leqslant 1$.

$$\int_I f \,\mathrm{d}\mu = \int_{\mathbb{T}} f\chi_I \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{T}} f g_n \,\mathrm{d}\mu = \lim_{n\to\infty} F(g_n) = 0 \implies f = 0 \text{ a.e.}$$

(2) $\forall n$, $\delta_n \in \mathscr{F}_{\mathbb{T}}(L^1(\mathbb{T}))$. $\operatorname{span}\{\delta_n : n \in \mathbb{Z}\}$ is dense in $c_0(\mathbb{Z})$. □

Exercise. $\mathscr{F}_{\mathbb{Z}} \colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ is not surjective.

Notation. $A(\mathbb{T}) = \mathscr{F}_{\mathbb{Z}}(\ell^1(\mathbb{Z}))$ is the Fourier algebra on $\mathbb{T}$ (Wiener algebra).
$A(\mathbb{T})$ is a proper dense subalgebra of $C(\mathbb{T})$.

Exercise. $\mathscr{F}_{\mathbb{T}} \colon L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is not surjective.

Notation. $A(\mathbb{Z}) = \mathscr{F}_{\mathbb{T}}(L^1(\mathbb{T}))$ is the Fourier algebra on $\mathbb{Z}$.
$A(\mathbb{Z})$ is a proper dense subalgebra of $c_0(\mathbb{Z})$.

Theorem (Fourier inversion formula for $\mathscr{F}_{\mathbb{T}}$). Let $f \in L^1(\mathbb{T})$. TFAE:

(1) $\widehat{f} \in \ell^1(\mathbb{Z})$

(2) there exists $f_0 \in A(\mathbb{T})$ (necessarily unique) s.t. $f = f_0$ a.e.

If (1) or (2) holds, then $f_0 = S\widehat{\widehat{f}}$. That is,

$$f(z) \underset{\text{a.e.}}{=\!=} f_0(z) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) z^n.$$

Exercise. $S_{\mathbb{T}}\mathscr{F}_{\mathbb{Z}} = \mathscr{F}_{\mathbb{Z}}S_{\mathbb{Z}}$, and $S_{\mathbb{Z}}\mathscr{F}_{\mathbb{T}} = \mathscr{F}_{\mathbb{T}}S_{\mathbb{T}}$.

Proof. (2) $\implies$ (1). $f_0 = \widehat{g}$, $g \in \ell^1(\mathbb{Z}) \implies \widehat{f} = \widehat{f_0} = \widehat{\widehat{g}} = Sg \in \ell^1(\mathbb{Z})$.
(1) $\implies$ (2). Let $f_0 = S\widehat{f}$, then $f_0 \in A(\mathbb{T})$ (because $A(\mathbb{T})$ is S-invariant, see Exercise.).
We want: $f \underset{\text{a.e.}}{=\!=} f_0$. It suffices to show that $\widehat{f} = \widehat{f_0}$:

$$\widehat{f_0} = (S\widehat{\widehat{f}})\widehat{\ } \underset{\text{(Exercise)}}{=\!=\!=\!=} S((\widehat{f})\widehat{\ }\widehat{\ }) = S(S(\widehat{f})) = \widehat{f}.$$

□

Harmonic analysis on $\mathbb{R}$ (a survey)

Quasi-definition. $f \colon \mathbb{R} \to \mathbb{C}$, $\widehat{f} \colon \widehat{\mathbb{R}} \to \mathbb{C}$.

$$\widehat{f}(\chi) = \text{“}\sum_{t\in\mathbb{R}}\text{”} f(t)\chi(t), \quad \widehat{\mathbb{R}} \cong \mathbb{R}.$$

$$\widehat{f} \colon \mathbb{R} \to \mathbb{C}, \quad \widehat{f}(\lambda) = \text{“}\sum_{t\in\mathbb{R}}\text{”} f(t)e^{-2\pi i\lambda t}.$$

Definition. $f \in L^1(\mathbb{R})$. The Fourier transform of f is $\widehat{f} \colon \mathbb{R} \to \mathbb{R}$,

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t)e^{-2\pi i\lambda t}\, dt.$$

Observe. $\widehat{f} \in \ell^\infty(\mathbb{R})$, $\|\widehat{f}\|_\infty \leqslant \|f\|_1$.

Notation. $\mathscr{F}_{\mathbb{R}} \colon L^1(\mathbb{R}) \to \ell^\infty(\mathbb{R})$, $f \mapsto \widehat{f}$ is the Fourier transform on $\mathbb{R}$. This is a bounded linear map.

Notation.

$$C_0(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) \colon \lim_{t\to\infty} f(t) = 0 \right\}.$$

$C_0(\mathbb{R})$ is a closed vector subspace of $\ell^\infty(\mathbb{R})$.

Proposition (Riemann-Lebesgue lemma). $\mathscr{F}_{\mathbb{R}}(L^1(\mathbb{R})) \subset C_0(\mathbb{R})$.

Proof. $(\chi_{[a,b]})\widehat{\ } \in C_0(\mathbb{R})$ (exercise). □

We have $\mathscr{F}_{\mathbb{R}} \colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$.

Exercise. Define the convolution of $f, g \colon \mathbb{R} \to \mathbb{C}$ and prove its basic properties (like for $\mathbb{T}$). In particular, $(L^1(\mathbb{R}), *)$ is a non-unital commutative algebra, and $\mathscr{F}_{\mathbb{R}} \colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is an algebra homomorphism.

Theorem.

(1) (Uniqueness theorem). $\mathscr{F}_{\mathbb{R}}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is injective.

(2) (Density theorem). $\mathscr{F}_{\mathbb{R}}(L^1(\mathbb{R}))$ is dense in $C_0(\mathbb{R})$.

(3) (Plancherel theorem). $\mathscr{F}_{\mathbb{R}}|_{(L^1 \cap L^2)(\mathbb{R})}$ uniquely extends to a unitary isomorphism $\mathscr{F}^\bullet\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Moreover, $(\mathscr{F}^\bullet)^2 = S$ on $L^2(\mathbb{R})$.

Exercise. $\mathscr{F}_{\mathbb{R}}(L^1(\mathbb{R})) \neq C_0(\mathbb{R})$.

Notation. $A(\mathbb{R}) = \mathscr{F}_{\mathbb{R}}(L^1(\mathbb{R}))$ is the Fourier algebra on $\mathbb{R}$. $A(\mathbb{R})$ is a proper dense subalgebra of $C_0(\mathbb{R})$.

Theorem (**Fourier inversion formula**). Let $f \in L^1(\mathbb{R})$. TFAE:

(1) $\widehat{f} \in L^1(\mathbb{R})$

(2) there exists $f_0 \in A(\mathbb{R})$ (unique) s.t. $f = f_0$ a.e.

If (1) or (2) holds, then $f_0 = S\widehat{\widehat{f}}$. That is,

$$f(t) \overset{}{\underset{\text{a.e.}}{=\!=}} f_0(t) = \int_{\mathbb{R}} \widehat{f}(\lambda) e^{2\pi i \lambda t}\, d\lambda.$$

Lecture 4 (2024.09.27)

Recall: If $f \in L^1(\mathbb{R})$, then $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ is given by $\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t)e^{-2\pi\mathrm{i}\lambda t}\,\mathrm{d}t$ and $\widehat{f} \in C_0(\mathbb{R})$. The Fourier transform $\mathscr{F}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is given by $f \mapsto \widehat{f}$, $\|\widehat{f}\|_\infty \leqslant \|f\|_1$, and $\widehat{f * g} = \widehat{f}\widehat{g}$. $\mathscr{F}(L^1(\mathbb{R})) = A(\mathbb{R}) \subseteq C_0(\mathbb{R})$ is the Fourier algebra.

Theorem.

(1) (Uniqueness theorem) $\mathscr{F}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is injective.

(2) (Density theorem) $A(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

(3) (Plancherel theorem) $\mathscr{F}((L^1 \cap L^2)(\mathbb{R})) \subset L^2(\mathbb{R})$, and $\mathscr{F}|_{L^1\cap L^2}$ uniquely extends to a unitary isomorphism $\mathscr{F}^\bullet\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Moreover, $(\mathscr{F}^\bullet)^2 = S$ $(f \mapsto (t \mapsto f(-t)))$.

(4) (Inversion formula) Let $f \in L^1(\mathbb{R})$. Then:

$$\widehat{f} \in L^1(\mathbb{R}) \iff \exists f_0 \in A(\mathbb{R}) \text{ s.t. } f \xlongequal{\text{a.e.}} f_0.$$

If they hold, then $f_0 = S\widehat{\widehat{f}}$. That is,

$$f(t) \xlongequal{\text{a.e.}} f_0(t) = \int_{\mathbb{R}} \widehat{f}(\lambda)e^{2\pi\mathrm{i}\lambda t}\,\mathrm{d}\lambda.$$

Ingredients of the proof

Lemma.

(1) $f \in C^1(\mathbb{R})$, $f, f' \in L^1(\mathbb{R}) \implies \widehat{f'}(\lambda) = 2\pi\mathrm{i}\lambda\widehat{f}(\lambda)$.

(2) $f \in C^p(\mathbb{R})$, $f, \dots, f^{(p)} \in L^1(\mathbb{R}) \implies \widehat{f}(\lambda) = o(|\lambda|^{-p})$ $(\lambda \to \infty)$.

(3) $f, tf \in L^1(\mathbb{R})$ (where $t = \mathrm{id}_{\mathbb{R}}$) $\implies \widehat{f} \in C^1(\mathbb{R})$, and $\widehat{f}\,'(\lambda) = -2\pi\mathrm{i}\,\widehat{tf}(\lambda)$.

(4) $f, tf, \dots, t^p f \in L^1(\mathbb{R}) \implies \widehat{f} \in C^p(\mathbb{R})$.

Definition. The Schwartz space is

$$\mathscr{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R})\colon \forall k, \ell \in \mathbb{Z}_{\geqslant 0}, t^k f^{(\ell)} \text{ is bounded} \right\}.$$

The topology on $\mathscr{S}(\mathbb{R})$ is generated by the family $\{\|\cdot\|_{k,\ell}\colon k, \ell \in \mathbb{Z}_{\geqslant 0}\}$ of seminorms on $\mathscr{S}(\mathbb{R})$, where

$$\|f\|_{k,\ell} = \sup_{t\in\mathbb{R}} |t^k f^{(\ell)}(t)|.$$

Theorem. $\mathscr{F}(\mathscr{S}(\mathbb{R})) = \mathscr{S}(\mathbb{R})$, and $\mathscr{F}\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is a topological isomorphism. Moreover, $\mathscr{F}^2 = S$ on $\mathscr{S}(\mathbb{R})$.

The proof of the theorem will be divided into the following easy parts (all left as exercises):

Lemma/Exercise 0. E, F = vector spaces; $P = \{\|\cdot\|_i\colon i \in I\}, Q = \{\|\cdot\|_j\colon j \in J\}$ families of seminorms on E, F respectively. $T\colon E \to F$ is linear. Then T is continuous $\iff \forall j \in J, \exists C > 0, \exists i_1, \dots, i_n \in I$ s.t. $\forall v \in E$, $\|Tv\|_j \leqslant C \max\limits_{1\leqslant k\leqslant n} \|v\|_{i_k}$.

Lemma/Exercise 1. Let $\widehat{\mathscr{F}} = S\mathscr{F} = \mathscr{F}S$. Then $\mathscr{F}(\mathscr{S}(\mathbb{R})) \subset \mathscr{S}(\mathbb{R})$, and $\mathscr{F}, \widehat{\mathscr{F}}\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ are continuous.

Lemma/Exercise 2. Define $M, D\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$, $Mf = tf$, $D = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$. Then $\mathscr{F}D = M\mathscr{F}$, $\mathscr{F}M = -D\mathscr{F}$.

Lemma/Exercise 3. Let $T = \widehat{\mathscr{F}}\mathscr{F}$. Then $T = \mathscr{F}\widehat{\mathscr{F}}$, and $TM = MT$, $TD = DT$.

Lemma/Exercise 4. Suppose $T\colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is a linear map s.t. $TM = MT$, $TD = DT \implies T = c1$ for some $c \in \mathbb{C}$.

Hint. $\forall a \in \mathbb{R}$, consider $m_a = \{f \in \mathscr{S}(\mathbb{R})\colon f(a) = 0\}$. Then $TM = MT \implies T(m_a) \subset m_a, \forall a \in \mathbb{R} \implies \exists c \in C^\infty(\mathbb{R})$ s.t. $Tf = cf$, $\forall f \in \mathscr{S}(\mathbb{R})$. $TD = DT \implies c = \text{const}$.

Lemma/Exercise 5. $f(t) = e^{-\pi t^2} \implies \widehat{f} = f$.

Hint. $f' + 2\pi tf = 0 \implies \widehat{f}\,' + 2\pi t\widehat{f} = 0 \implies \widehat{f} = cf$ $(c \in \mathbb{C})$; $f(0) = 1 = \widehat{f}(0) = \int_{\mathbb{R}} e^{-\pi t^2}\,\mathrm{d}t \implies c = 1$.

Notation.

$$\mathscr{S}'(\mathbb{R}) = \text{ the topological dual of } \mathscr{S}(\mathbb{R}) = \{\text{continuous linear functionals } \mathscr{S}(\mathbb{R}) \to \mathbb{C}\}$$

(The space of tempered distributions).

Exercise. Let $p \in [1, +\infty]$, then

$$L^p(\mathbb{R}) \hookrightarrow \mathscr{S}'(\mathbb{R}), \quad f \mapsto \left(\varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)\,\mathrm{d}t\right).$$

Notation.

$$\mathscr{F}' : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R}), \quad \mathscr{F}'g = g \circ \mathscr{F}$$

(that is, $\mathscr{F}'$ is dual to $\mathscr{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$).

$\mathscr{F}'$ is the Fourier transform on $\mathscr{S}'(\mathbb{R})$. $\mathscr{F}' : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is an isomorphism.

Exercise.

(1)

$$\begin{tikzcd}
L^1(\mathbb{R}) \arrow[r, "\mathscr{F}"] \arrow[d, hook] & C_0(\mathbb{R}) \arrow[d, hook] \\
\mathscr{S}'(\mathbb{R}) \arrow[r, "\mathscr{F}'"] & \mathscr{S}'(\mathbb{R})
\end{tikzcd} \quad \text{commutes} \implies \text{uniqueness theorem.}$$

(2)

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[r, "\mathscr{F}"] \arrow[d, hook] & \mathscr{S}(\mathbb{R}) \arrow[d, hook] \\
L^1(\mathbb{R}) \arrow[r, "\mathscr{F}"] & C_0(\mathbb{R})
\end{tikzcd} \quad \mathscr{S}(\mathbb{R}) \text{ is dense in } C_0(\mathbb{R}) \implies \mathscr{F}(L^1(\mathbb{R})) \text{ is dense in } C_0(\mathbb{R}).$$

(3)

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[r, "\text{unitary } \mathscr{F}"] \arrow[d] & \mathscr{S}(\mathbb{R}) \arrow[d] \\
L^2(\mathbb{R}) \arrow[r, "\text{unitary } \mathscr{F}^\bullet"] \arrow[d] & L^2(\mathbb{R}) \arrow[d] \\
\mathscr{S}'(\mathbb{R}) \arrow[r, "\mathscr{F}'"'] & \mathscr{S}'(\mathbb{R})
\end{tikzcd}$$

$\mathscr{F}'$ extends both $\mathscr{F}^\bullet : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \mathscr{F}_{L^1(\mathbb{R})} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$

$\implies \mathscr{F}^\bullet|_{(L^1 \cap L^2)(\mathbb{R})} = \mathscr{F}|_{(L^1 \cap L^2)(\mathbb{R})}$

$\implies$ Plancherel theorem.

Locally compact spaces. Radon measures

Definition. A topological space X is locally compact if $\forall x \in X$, there is a neighborhood $U \ni x$ s.t. $\overline{U}$ is compact.

Examples. (1) compact; (2) discrete; (3) $\mathbb{R}^n$; (4) any C^0-manifold.

Nonexamples. (1) $\mathbb{Q}$; (2) an infinite-dim normed space; (3) (exercise) an infinite product of noncompact spaces.

Exercise. The product of finitely many locally compact spaces is locally compact.

Theorem (Urysohn's lemma)**.** $X =$ a locally compact Hausdorff space, $K, F \subset X$, $K \cap F = \varnothing$, K is compact, F is closed $\implies$ there exists a continuous $\varphi\colon X \to [0,1]$ s.t. $\varphi|_K = 1$, $\varphi|_F = 0$, $\operatorname{supp}\varphi$ is compact.

Let X be a Hausdorff locally compact topological space.

Notation. $\mathrm{Bor}(X)$= Borel σ-algebra on $X =$ the smallest σ-subalgebra of 2^X containing open sets.

Definition. A (positive) Borel measure on X is a σ-additive measure $\mu\colon \mathrm{Bor}(X) \to [0, +\infty]$.

Definition. $\mu =$ a Borel measure on X, $B \subset X$ is a Borel set. μ is

(1) outer regular on B if $\mu(B) = \inf\{\mu(U) \colon U \supset B, U \text{ open}\}$.

(2) inner regular on B if $\mu(B) = \sup\{\mu(K) \colon K \subset B \text{ compact}\}$.

Definition. μ is an outer Radon measure if

(1) $\forall$ compact set $K \subset X$, $\mu(K) < \infty$.

(2) μ is outer regular on all Borel sets.

(3) μ is inner regular on all open sets.

Example.

(1) The Lebesgue measure on $\mathbb{R}^n$.

(2) X is discrete. $\mu(A) = \begin{cases} \mathrm{Card}\, A, & \text{if } A \text{ is finite,} \\ +\infty, & \text{if } A \text{ is infinite} \end{cases}$ (counting measure).

Facts/Exercise.

(1) Suppose X is σ-compact (i.e., $X = \bigcup_{n\in\mathbb{N}} X_n$, X_n is compact). Then each Radon measure on X is inner regular on all Borel sets.

(2) Suppose X is 2nd countable, $\mu =$ a Borel measure on X s.t. $\mu(K) < \infty$ for each compact set $K \subset X$. Then μ is inner regular and outer regular on all Borel sets.

Notation.

$$C_c(X) = \{f \in C(X) \colon \operatorname{supp} f \text{ is compact}\}.$$

Let $f, g \in C_c(X)$, then $f \geqslant 0 \underset{\text{def}}{\iff} \forall x \in X, f(x) \geqslant 0$; $f \leqslant g \underset{\text{def}}{\iff} g - f \geqslant 0$.

$$C_c^+(X) = \{f \in C_c(X) \colon f \geqslant 0\}.$$

Definition. A linear functional $I\colon C_c(X) \to \mathbb{C}$ is positive ($I \geqslant 0$) if $I(f) \geqslant 0$ for all $f \geqslant 0$.

Example. $\mu =$ a positive Radon measure on X.

$$I_\mu\colon C_c(X) \to \mathbb{C}, \quad I_\mu(f) = \int_X f \,\mathrm{d}\mu \implies I_\mu \geqslant 0.$$

Theorem (Riesz, Markov, Kakutani)**.** There exists a bijection

$$\{\text{Positive Radon measures on } X\} \to \{\text{Positive linear functionals on } C_c(X)\}$$
$$\mu \mapsto I_\mu.$$

Locally compact groups

Definition. A topological group is a group G equipped with a topology such that

$$\left.\begin{aligned} G \times G \to G, (x, y) \mapsto xy \\ G \to G, x \mapsto x^{-1} \end{aligned}\right\} \text{are continuous.}$$

Observe.

(1) $\forall x \in G$ the maps $y \mapsto xy$ and $y \mapsto yx$ are homeomorphisms $G \to G$.

(2) $x \mapsto x^{-1}$ is a homeomorphism $G \to G$.

Notation. $S, T \subset G$.

$$ST = \{xy \colon x \in S, y \in T\}, \quad S^{-1} = \{x^{-1} \colon x \in S\}.$$

S is symmetric if $S = S^{-1}$.

Observe. Every neighborhood $U \ni e$ contains a symmetric neighborhood of e (namely $U \cap U^{-1}$).

Definition. A locally compact group is a locally compact Hausdorff topological group.

Examples.

(1) Discrete groups.

(2) $\mathbb{Z}, \mathbb{R}, \mathbb{R}^\times, \mathbb{C}, \mathbb{C}^\times, \mathbb{T}, \mathbb{Q}_p, \mathbb{Q}_p^\times, \mathbb{Z}_p$.

(3) $\mathrm{GL}_n(\mathbb{K}), \mathrm{SL}_n(\mathbb{K})$ $(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\})$, $\mathrm{U}_n, \mathrm{SU}_n, \mathrm{O}_n, \mathrm{SO}_n, \ldots$.

(4) Any Lie group.

Lecture 5 (2024.10.04)

Let G be a topological group.

Definition. G = topological group, $f\colon G \to \mathbb{C}$. f is left (resp. right) uniformly continuous if $\forall \varepsilon > 0$, there exists a neighborhood $U \ni e$ s.t. $\forall x \in G$, $\forall u \in U$ we have $|f(x) - f(xu)| < \varepsilon$ (resp. $|f(x) - f(ux)| < \varepsilon$).

Remark. For $G = \mathbb{R}$ we get the "usual" uniform continuity.

Equivalently: f is left (resp. right) uniformly continuous iff $\forall \varepsilon > 0$, there exists a neighborhood $U \ni e$ s.t. $\forall x, y \in G$ satisfying $x^{-1}y \in U$ (resp. $yx^{-1} \in U$) we have $|f(x) - f(y)| < \varepsilon$.

Proposition. G = locally compact group, $f \in C_c(G)$. Then f is left and right uniformly continuous.

Lemma. X, Y, Z topological spaces; $F\colon X \times Y \to Z$ continuous; $Z_0 \subset Z$ open, $Y_0 \subset Y$ compact. Let $X_0 = \{x \in X\colon F(x, y) \in Z_0\ \forall y \in Y_0\}$. Then X_0 is open.

Proof. Exercise. □

Proof of Proposition. Note that: f is left uniformly continuous $\iff$ Sf is right uniformly continuous ($(Sf)(x) = f(x^{-1})$). Let's show that f is left uniformly continuous.
Let $F = \operatorname{supp} f$; $V \ni e$ a relative compact symmetric neighborhood of e ($V^{-1} = V$). Let $K = F \cdot \overline{V}$. K is compact. Let $\varepsilon > 0$; let $W = \{y \in G\colon \forall x \in K, |f(x) - f(xy)| < \varepsilon\}$. Lemma $\implies$ W is open; $e \in W$. Let $U = V \cap W$. If $x \in K, y \in U$ $\implies$ $|f(x) - f(xy)| < \varepsilon$. Suppose $x \in G \backslash K, y \in U$. Then $f(x) = 0$. **Claim**: $f(xy) = 0$. If not, then $xy \in F$ $\implies$ $x \in F \cdot y^{-1} \subset F \cdot V \subset K$, a contradiction. This implies that $|f(x) - f(xy)| < \varepsilon, \forall x \in G, \forall y \in U$. □

The Haar measure

G = locally compact group, μ = a Radon measure on G. (positive)

Definition. μ is left (resp. right) invariant if for any $x \in G$ and any Borel set $B \subset G$, we have $\mu(xB) = \mu(B)$ (resp. $\mu(Bx) = \mu(B)$). If, moreover, $\mu \neq 0$, then μ is a left (resp. right) Haar measure.

Observe. If μ is left invariant, then $\nu(B) = \mu(B^{-1})$ is right invariant.

$$\{\text{left invariant}\} \rightleftarrows \{\text{right invariant}\}.$$

Convention. Haar measure = left Haar measure.

Examples.

(1) The counting measure on a discrete group.

(2) The Lebesgue measure on $\mathbb{R}^n$.

(3) The normalize measure on $\mathbb{T}$: $\frac{\text{length measure on } \mathbb{T}}{2\pi}$.

Theorem (A. Haar, J. von Neumann, A. Weil)**.** G = locally compact group.

(1) There exists a Haar measure on G.

(2) If μ, ν are Haar measures, then there exists a constant $c > 0$ s.t. $\nu = c\mu$.

Haar measure on Lie groups

G = real Lie group, $n = \dim G$. Choose $\omega_e \in \Lambda^n(T_e^* G), \omega_e \neq 0$. $\forall x \in G$, $\ell_x\colon G \to G$, $\ell_x(y) = xy$.

$$\mathrm{d}\ell_{x^{-1}} = \ell_{x^{-1},*}\colon T_x G \to T_e G \quad \ell_{x^{-1}}^*\colon \Lambda(T_e^* G) \to \Lambda(T_x^* G).$$

Let $\omega_x = \ell_{x^{-1}}^* \omega_e \in \Lambda^n(T_x^* G)$. $\omega \in \Omega^n(G)$. $\omega_x \neq 0$. In particular, G is orientable.
Choose an orientation on G such that ω is positive. For any Borel set $B \subset G$, define $\mu(B) = \int_B \omega$.
Claim: μ is a Haar measure.
Indeed: $\ell_x^* \omega = \omega, \forall x \in G$ by construction. μ is a Radon measure (because $\mu(K) < \infty$ for any compact set $K \subset G$ and G is 2nd countable).

$$\mu(xB) = \int_{xB} \omega \xlongequal{\ell_x \text{ is orientation-preserving}} \int_B \ell_x^* \omega = \int_B \omega = \mu(B) \implies \mu \text{ is left invariant.}$$

Coordinate form of ω

$y^1, \ldots, y^n$ = coordinates in a neighborhood of e; $\omega_e = \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$. $\forall p \in G$, $x^1, \ldots, x^n$ = coordinates in a neighborhood of p; $\omega(p) = \det(\ell_{p^{-1},*}(p))\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \det\left(\frac{\partial(y^i \circ \ell_{p^{-1}})}{\partial x^j}(p)\right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Example 1. $G = \mathbb{R}^\times = (\mathbb{R}\backslash\{0\}, \cdot)$, $p \in G$, $\omega(p) = \frac{\mathrm{d}x}{p}$. x = the global coordinate on $\mathbb{R}$ ($x = \mathrm{id}_{\mathbb{R}}$).
The orientation of $\mathbb{R}^\times$ compatible with ω is

- the standard orientation on $\mathbb{R}_{>0}$.
- the opposite orientation on $\mathbb{R}_{<0}$.

$\forall f \in C_c(\mathbb{R}^\times)$, $\int_{\mathbb{R}^\times} f\,\mathrm{d}\mu = \int_{\mathbb{R}^\times} \frac{f(x)}{|x|}\,\mathrm{d}x$, i.e., $\mu = \frac{\lambda}{|x|}$ (λ = Lebesgue measure).

Example/Exercise 2. $G = \mathrm{GL}_n(\mathbb{R})$. Prove: $\mu_{\text{left}} = \mu_{\text{right}} = \frac{\lambda}{|\det|^n}$.

Example/Exercise 3. $\forall a, b \in \mathbb{R}$ ($a \neq 0$). $L_{a,b}\colon \mathbb{R} \to \mathbb{R}$, $L_{a,b}(x) = ax + b$.

$$G = \left\{L_{a,b}\colon a \in \mathbb{R}^\times, b \in \mathbb{R}\right\} = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R}\right\}.$$

Find explicitly (in terms of a, b) μ_{left} and μ_{right}; show that $\mu_{\text{left}} \neq \mu_{\text{right}}$.

The existence of a Haar measure

G = locally compact group.
A rough idea Let $U \subset G$ be a neighborhood of e. For any Borel set $B \subset G$, let

$$(B : U) = \min\left\{n\colon B \subset x_1 U \cup \cdots \cup x_n U \text{ for some } x_1, \ldots, x_n \in G\right\}.$$

Intuitively, $(B : U) \approx$ "$\frac{\text{area of } B}{\text{area of } U}$". $U \to \{e\} \implies (B : U) \to \infty$.
Choose $K \subset G$ compact, $\operatorname{Int} K \neq \varnothing$.

$$\lim_{U \to \{e\}} \frac{(B : U)}{(K : U)} = \mu(B).$$

Notation.

(1) μ = a Radon measure on G, $x \in G$. Define Radon measure $L_x\mu, R_x\mu$:

$$(L_x\mu)(B) = \mu(x^{-1}B), \quad (R_x\mu)(B) = \mu(Bx).$$

We have

$$L_{xy} = L_x L_y; \quad R_{xy} = R_x R_y \tag{6}$$

μ is left invariant $\iff L_x\mu = \mu, \forall x \in G$.

(2) $f \in \mathrm{Fun}(G) = \mathbb{C}^G$. Define $L_x f, R_x f \in \mathrm{Fun}(G)$:

$$(L_x f)(y) = f(x^{-1}y), \quad (R_x f)(y) = f(yx).$$

(6) holds.

(3) Let $I\colon C_c(G) \to \mathbb{C}$ be a linear functional. Define functionals $L_x I, R_x I\colon C_c(G) \to \mathbb{C}$.

$$L_x I = I \circ L_{x^{-1}}, \quad R_x I = I \circ R_{x^{-1}}.$$

(6) holds (exercise).

Proposition. μ = a Radon measure on G, $x \in G$, $I_\mu(f) = \int f\,\mathrm{d}\mu$ ($f \in C_c(G)$). Then $L_x I_\mu = I_{L_x\mu}$.

Proof. $(L_x I_\mu)(\chi_B) = (I_{L_x\mu})(\chi_B)$ holds for any Borel set $B \subset G$ (exercise) $\implies$ for any bounded Borel function f, $(L_x I_\mu)(f) = (I_{L_x\mu})(f)$. This result concentrated on a set of finite measures. □

Corollary. μ is left invariant $\iff$ I_μ is left invariant.

Theorem. $G =$ locally compact group.

(1) There exists a left invariant positive linear functional I on $C_c(G)$, $I \neq 0$.

(2) If I, J are such functionals, then there exists a constant $c > 0$ s.t. $J = cI$.

Lemma 1. $f, g \in C_c^+(G), g \neq 0 \implies \exists C > 0, \exists x_1, \dots, x_n \in G$ s.t. $f \leqslant C \sum_{i=1}^n L_{x_i} g$.

Proof. $\exists \varepsilon > 0$, $\exists$ open $U \subset G, U \neq \varnothing$ s.t. $g(x) > \varepsilon, \forall x \in U$. $\operatorname{supp}(f) \subset \bigcup_{i=1}^n x_i U$ for some $x_1, \dots, x_n \in G$. □

Notation. $f, g \in C_c^+(G), g \neq 0$, define

$$(f : g) = \inf \left\{ \sum_{i=1}^n c_i \colon f \leqslant \sum_{i=1}^n c_i L_{x_i} g \text{ for some } x_1, \dots, x_n \in G \right\}.$$

Geometric idea: $(f : g) \approx \frac{\int f \, \mathrm{d}x}{\int g \, \mathrm{d}x}$ (on $\mathbb{R}$).

Lemma 2.

(1) $(cf : g) = c(f : g), \forall c \geqslant 0$.

(2) $(f_1 + f_2 : g) \leqslant (f_1 : g) + (f_2 : g)$.

(3) $(L_x f : g) = (f : g), \forall x \in G$.

(4) $(f : g) \geqslant \frac{\|f\|_\infty}{\|g\|_\infty}$ $(f, g \neq 0)$.

(5) $(f : g) \leqslant (f : h)(h : g)$ $(h, g \neq 0)$.

Proof. Exercise. □

Remark. (4) $\implies$ $(f : g) > 0$ if $f, g \neq 0$.

Notation. Choose $f_0 \in C_c^+(G), f_0 \neq 0$.
$\forall \varphi \in C_c^+(G) \backslash \{0\}$, define $I_\varphi \colon C_c^+(G) \to [0, +\infty)$,

$$I_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)} \quad \text{“approximate integral”}.$$

Lemma 3.

(1) $I_\varphi(cf) = cI_\varphi(f), \forall c \geqslant 0$.

(2) $I_\varphi(f_1 + f_2) \leqslant I_\varphi(f_1) + I_\varphi(f_2)$.

(3) $I_\varphi(L_x f) = I_\varphi(f), \forall x \in G$.

(4) $\frac{1}{(f_0 : f)} \leqslant I_\varphi(f) \leqslant (f : f_0)$ if $f, \varphi \neq 0$.

Lecture 6 (2024.10.11)

Theorem 1. $G=$ locally compact group. Then there exists a positive linear functional $I\colon C_c(G)\to\mathbb{C}$, I is left invariant, $I\neq 0$.

Lemma 1. $f,g\in C_c^+(G), g\neq 0 \implies \exists C>0, \exists x_1,\dots,x_n\in G$ s.t. $f\leqslant C\sum\limits_{i=1}^n L_{x_i}g$.

Notation.

$$(f:g)=\inf\left\{\sum_{i=1}^n c_i\colon f\leqslant \sum_{i=1}^n c_iL_{x_i}g \text{ for some } x_1,\dots,x_n\in G, c_i\geqslant 0\right\}.$$

it is "a relative approximate integral" of f relative to g: $(f:g)\approx\frac{\int f\,\mathrm{d}x}{\int g\,\mathrm{d}x}$.

Lemma 2.

(1) $(cf:g)=c(f:g),\forall c\geqslant 0$.

(2) $(f_1+f_2:g)\leqslant (f_1:g)+(f_2:g)$.

(3) $(L_xf:g)=(f:g),\forall x\in G$.

(4) $(f:g)\geqslant\frac{\|f\|_\infty}{\|g\|_\infty}$ $(f,g\neq 0)$.

(5) $(f:g)\leqslant (f:h)(h:g)$ $(h,g\neq 0)$.

Notation. Choose $f_0\in C_c^+(G), f_0\neq 0$.
$\forall\varphi\in C_c^+(G)\backslash\{0\}$, define $I_\varphi\colon C_c^+(G)\to[0,+\infty)$,

$$I_\varphi(f)=\frac{(f:\varphi)}{(f_0:\varphi)}\quad\text{"an approximate integral" of } f.$$

Lemma 3.

(1) $I_\varphi(cf)=cI_\varphi(f),\forall c\geqslant 0$.

(2) $I_\varphi(f_1+f_2)\leqslant I_\varphi(f_1)+I_\varphi(f_2)$.

(3) $I_\varphi(L_xf)=I_\varphi(f),\forall x\in G$.

(4) $\frac{1}{(f_0:f)}\leqslant I_\varphi(f)\leqslant (f:f_0)$ if $f,\varphi\neq 0$.

Proof of (4). $\frac{(f:\varphi)}{(f_0:\varphi)}\leqslant (f:f_0)$ (see Lemma 2. (5)).

$$\frac{(f:\varphi)}{(f_0:\varphi)}\geqslant\frac{1}{(f_0:f)}\iff\frac{(f_0:\varphi)}{(f:\varphi)}\leqslant (f_0:f)\text{ True by Lemma 2. (5).}\qquad\square$$

Lemma 4. Let $f_1,f_2\in C_c^+(G)$. Then for any $\varepsilon>0$, there exists a neighborhood $U\ni e$ s.t. $\forall\varphi\in C_c^+(G)\backslash\{0\}$ with $\operatorname{supp}\varphi\subset U$, we have

$$I_\varphi(f_1)+I_\varphi(f_2)\leqslant I_\varphi(f_1+f_2)+\varepsilon.$$

Proof of Theorem 1. Let $P=C_c^+(G)\backslash\{0\}$. $\forall\varphi\in P, I_\varphi\in(0,+\infty)^P$. $\forall f\in P$, let $S_f=\left[\frac{1}{(f_0:f)}(f:f_0)\right]$. Lemma 3. $\implies I_\varphi\in\prod\limits_{f\in P}S_f=:S$[1], S is compact. For any neighborhood $U\ni e$, let $K_U=\overline{\{I_\varphi\colon\varphi\in P,\operatorname{supp}\varphi\subset U\}}\subset S$, $K_U\neq\varnothing$. $U\subset V\implies K_U\subset K_V$. Hence $K_{U_1}\cap\cdots\cap K_{U_n}\supset K_{U_1\cap\cdots\cap U_n}\neq\varnothing$. Hence $\{K_U\colon U\ni e\}$ has the finite intersection property, by the compactness we have $\bigcap\limits_{U\ni e}K_U\neq\varnothing$. Let $I\in\bigcap\limits_{U\ni e}K_U\subset S$. $I\colon P\to(0,+\infty)$.

Claim: I is positive homogeneous, additive, and left invariant.
$(*)$: $\forall U\ni e,\forall\varepsilon>0,\forall f_1,\dots,f_n\in P,\exists\varphi\in P$ with $\operatorname{supp}\varphi\subset U$ s.t. $|I(f_j)-I_\varphi(f_j)|<\varepsilon,\forall j=1,\dots,n$.
$(*)$ & Lemma 3. $\implies I$ is positive homogeneous, subadditive, left invariant. $(*)$ & Lemma 4. $\implies I$ is additive, that is, $I(f_1+f_2)=I(f_1)+I(f_2),\forall f_1,f_2\in P$.
Let $I(0)=0$. $\forall f\in C_c(G), f=(f_1-f_2)+\mathrm{i}(f_3-f_4)$, where $f_k\in C_c^+(G)$ $(k=1,\dots,4)$. Let $I(f)=I(f_1)-I(f_2)+\mathrm{i}(I(f_3)-I(f_4))$.

[1] $(0,+\infty)^P=\{a_f\colon f\in P\}\supset S=\prod S_f=\{a_f\colon f\in P, a_f\in S_f\}$.

Exercise. $I\colon C_c(G)\to\mathbb{C}$ is well defined, linear, $I\neq 0$, and left invariant. ☐

Proof of Lemma 4. Let $f = f_1+f_2+\delta u$, where $\delta>0$, and $u\in C_c^+(G)$ s.t. $u(x)=1, \forall x\in \operatorname{supp}(f_1+f_2)$. $f_k = fh_k$ $(k=1,2)$, where $h_k\in C_c^+(G)$. (exercise)
Suppose $f\leqslant \sum\limits_{i=1}^n c_iL_{x_i}\varphi$ $(c_i\geqslant 0, x_i\in G)$, then $f_k\leqslant \sum c_ih_kL_{x_i}\varphi$ $(k=1,2)$, that is,

$$f_k(x)\leqslant \sum c_ih_k(x)\varphi(x_i^{-1}x). \tag{$*$}$$

For any $\varepsilon>0$, there exists a neighborhood $U\ni e$ s.t. $\forall x,y\in G$ satisfying $x^{-1}y\in U$ we have $|h_k(x)-h_k(y)|<\delta$ $(k=1,2)$.
Suppose $\operatorname{supp}\varphi\subset U$. If $x_i^{-1}x\notin U$, then the RHS of $(*)$ is 0.

$$\begin{aligned}
&\text{If } x_i^{-1}x\in U, \text{then } |h_k(x)-h_k(x_i)|<\delta\\
&\implies f_k(x)\leqslant c_i(h_k(x_i)+\delta)\varphi(x_i^{-1}x), \forall x\in G\ (k=1,2)\\
&\implies (f_k:\varphi)\leqslant \sum c_i(h_k(x_i)+\delta)\ (k=1,2)\\
&\implies (f_1:\varphi)+(f_2:\varphi)\leqslant \sum c_i(1+\delta)(\text{because } h_1+h_2\leqslant 1)\\
&\implies (f_1:\varphi)+(f_2:\varphi)\leqslant (1+\delta)(f:\varphi)\\
&\implies \left(\begin{aligned}
I_\varphi(f_1)+I_\varphi(f_2)&\leqslant (1+\delta)I_\varphi(f)\\
&\leqslant (1+\delta)(I_\varphi(f_1+f_2+\delta u))\\
&\leqslant (1+\delta)(I_\varphi(f_1+f_2)+\delta I_\varphi(u))\\
&\leqslant I_\varphi(f_1+f_2)+\delta(f_1+f_2:f_0)+\delta(1+\delta)(u:f_0).
\end{aligned}\right)
\end{aligned}$$

Note that $\delta(f_1+f_2:f_0)+\delta(1+\delta)(u:f_0)<\varepsilon$ if δ is small enough. This completes the proof. ☐

The uniqueness of the Haar measure

Lemma 1. G = locally compact group, μ = a Haar measure on G. Then

(1) $\forall\varnothing\neq U\subset G$ open we have $\mu(U)>0$.

(2) If $f\in C(G)$ is μ-integrable, $f\geqslant 0$, $\int f\,\mathrm{d}\mu=0 \implies f=0$.

Proof.

(1) Suppose $\mu(U)=0$. For any compact set $K\subset G$, $K\subset x_1U\cup\cdots\cup x_nU$ for some $x_1,\dots,x_n \implies \mu(K)=0 \implies \mu=0$ on open sets (by inner regularity) $\implies \mu=0$ on all Borel sets (by outer regularity), a contradiction.

(2) $f=0$ μ-a.e., that is, $\mu(\underbrace{f^{-1}(0,+\infty)}_{\text{open}})=0 \underset{(1)}{\implies} f=0$. ☐

Lemma 2. G = locally compact group, μ = a Radon measure on G. Let $f\in C_c(G)$; define $g(x)=I_\mu(R_xf), h(x)=I_\mu(L_xf)$. Then g,h are continuous.

Proof. (continuity of g at e).

$$|g(x)-g(e)|\leqslant \int_G |f(yx)-f(y)|\,\mathrm{d}\mu(y).$$

For any $\varepsilon>0$, there exists a neighborhood $U\ni e$ s.t. $|f(yx)-f(y)|<\varepsilon$. $\forall y\in G,\forall x\in U$. Let $F=\operatorname{supp} f$, choose a relative compact, symmetric neighborhood $V\ni e$; let $K=F\cdot\overline{V}$. K is compact.
Claim: if $y\notin K$, then $f(y)=f(yx)=0$ $(x\in V)$.
Indeed, if $f(yx)\neq 0$, then $yx\in F \implies y\in F\cdot x^{-1}\subset F\cdot V\subset K$ (a contradiction).
Claim $\implies |g(x)-g(e)|\leqslant \int_K |f(yx)-f(y)|\,\mathrm{d}\mu(y)<\varepsilon\mu(K)$ $(x\in V\cap U) \implies g$ is continuous at e.

Exercise. Complete the proof. ☐

Theorem 2. G = locally compact group, μ,ν = (left) Haar measures on $G \implies \exists c>0$ s.t. $\nu=c\mu$.

Proof. $\forall f \in C_c(G)\backslash \operatorname{Ker} I_\mu$, define $D_f\colon G \to \mathbb{C}$,

$$D_f(x) = \frac{I_\nu(R_x f)}{I_\mu(f)}. \text{ Lemma 2.} \implies D_f \text{ is continuous.}$$

<u>**Claim**</u>: D_f does not depend on f. $(*)$
If $(*)$ is true, then $I_\nu(f) = D(e)I_\mu(f), \forall f \notin \operatorname{Ker} I_\mu \underset{\text{exercise}}{\implies} I_\nu = D(e)I_\mu$ everywhere on $C_c(G) \implies \nu = c\mu$, where $c = D(e)$.
<u>Let's prove $(*)$</u>

$$\begin{aligned}
I_\mu(f)I_\nu(g) &= \iint f(x)g(y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \iint f(x)g(x^{-1}y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) \\
&\overset{\text{(Fubini)}}{=\!=\!=} \iint f(x)g(x^{-1}y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \iint f(yx)g(x^{-1})\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \\
&\overset{\text{(Fubini)}}{=\!=\!=} \iint f(yx)g(x^{-1})\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \iint I_\nu(R_x f)g(x^{-1})\,\mathrm{d}\mu(x) \\
&\implies I_\nu(g) = \int D_f(x)g(x^{-1})\,\mathrm{d}\mu(x).
\end{aligned}$$

Suppose $f, f' \in C_c(G)\backslash \operatorname{Ker} I_\mu$, then $\int (D_f - D_{f'})g\,\mathrm{d}\mu = 0, \forall g \in C_c(G)$. Replace g by $|D_f - D_{f'}||g|^2 \implies \int |(D_f - D_{f'})g|^2\,\mathrm{d}\mu = 0 \underset{\text{Lemma 1.}}{\implies} (D_f - D_{f'})g = 0, \forall g \in C_c(G) \implies D_f = D_{f'} \implies (*)$. □

Lecture 7 (2024.10.18)

Some operators on measures

X = locally compact Hausdorff topological space, $C^+(X) = \{f \in C(X) : f \geqslant 0\}$.

1. Multiplication by a function

Notation.

(1) For any linear $I : C_c(X) \to \mathbb{C}, \forall f \in C(X)$, define $f \cdot I : C_c(X) \to \mathbb{C}$ by $(f \cdot I)(g) = I(fg)$.

Observe. If $I, f \geqslant 0$, then $f \cdot I \geqslant 0$.

(2) For each Radon measure μ on X, $\forall f \in C^+(X)$ define a Radon measure $f \cdot \mu$ on X by $I_{f\cdot\mu} = f \cdot I_\mu$.

Exercise.

(1) If X is σ-compact, then

$$(f \cdot \mu)(B) = \int_B f \, \mathrm{d}\mu \text{ for any Borel set } B \subset X.$$

(2) The same is true if $\int_X f \, \mathrm{d}\mu < \infty$.

Exercise. G = locally compact group, $f \in C^+(G)$, μ = a Radon measure on $G \implies L_x(f \cdot \mu) = L_x f \cdot L_x \mu$, $R_x(f \cdot \mu) = R_x f \cdot R_x \mu$ $(x \in G)$.

2. Reflection

G = locally compact group.

Notation.

(1) $I : C_c(G) \to \mathbb{C}$ linear. Define $S(I) : C_c(G) \to \mathbb{C}$ by $S(I) = I \circ S$ (where $(Sf)(x) = f(x^{-1})$, $\forall x \in G$).

(2) For each Radon measure μ on G define a Radon measure $S\mu$ on G by $I_{S\mu} = S(I_\mu)$.

Exercise. $(S\mu)(B) = \mu(B^{-1})$ for any Borel set $B \subset G$.

Exercise. $S(f \cdot \mu) = Sf \cdot S\mu$, $\forall f \in C^+(G)$.

The modular character (modular function)

G = locally compact group, μ = a (left) Haar measure on G.

Observe. $\forall x \in G$, $R_x\mu$ is a Haar measure. Indeed: $L_y(R_x\mu) = R_x L_y \mu = R_x \mu$.

Hence there exists $\Delta(x) > 0$ s.t. $R_x\mu = \Delta(x)\mu$. $(*)$

Definition. The function $\Delta : G \to \mathbb{R}_{>0}$ given by $(*)$ is called the modular character of G.

Proposition 1. $R_x I_\mu = \Delta(x) I_\mu$, $\forall x \in G$.[2] That is,

$$\int_G f(yx) \, \mathrm{d}\mu(y) = \Delta(x^{-1}) \int_G f \, \mathrm{d}\mu.$$

Proof. $R_x I_\mu = I_{R_x\mu} = I_{\Delta(x)\mu} = \Delta(x) I_\mu$. □

Proposition 2. $\Delta : G \to \mathbb{R}_{>0}$ is a continuous homomorphism.

Proof. $\Delta(xy)\mu = R_{xy}\mu = R_x R_y \mu = \Delta(y) R_x \mu = \Delta(x)\Delta(y)\mu \implies \Delta$ is a homomorphism. Choose $f \in C_C(G)$ s.t. $I_\mu f = 1$, then $\Delta(x) = R_x I_\mu f = I_\mu(R_{x^{-1}} f)$ is continuous (see previous lecture). □

[2] Recall: $R_x I_\mu = I_\mu \circ R_{x^{-1}}$ by definition.

Recall: μ = Haar measure $\implies S\mu$ is a right Haar measure.

Proposition 3. $S\mu = \Delta^{-1} \cdot \mu$. That is, $\forall f \in C_c(G)$,

$$\int_G f(x^{-1})\,\mathrm{d}\mu(x) = \int_G \Delta(x)^{-1} f(x)\,\mathrm{d}\mu(x).$$

Proof. Let $\nu = \Delta^{-1} \cdot \mu$. **Claim**: ν is right invariant.

Observe. For any homomorphism $\varphi\colon G \to \mathbb{C}^\times$, $R_x\varphi = \varphi(x)\varphi, S\varphi = \varphi^{-1}$.

$R_x\nu = R_x(\Delta^{-1} \cdot \mu) = R_x(\Delta^{-1}) \cdot R_x\mu = \Delta(x)^{-1}\Delta^{-1} \cdot \Delta(x)\mu = \nu \implies \nu$ is right invariant $\implies \exists c > 0$ s.t.

$$S\mu = c \cdot \Delta^{-1}\mu. \tag{7}$$

(7) $\implies c \cdot \mu = \Delta \cdot S\mu$ and (7) $\implies \mu = cS(\Delta^{-1} \cdot \mu) = c \cdot S(\Delta^{-1}) \cdot S\mu = c \cdot \Delta \cdot S\mu = c^2\mu \implies c = 1.$ $\square$

Definition. G is unimodular if $\Delta \equiv 1$.

$\Delta \equiv 1 \iff$ a left Haar measure is right invariant $\iff$ a right Haar measure is left invariant.

Example 1. Abelian $\implies$ unimodular.

Example 2. Compact $\implies$ unimodular. Indeed, $\Delta(G)$ is a compact subgroup of $R_{>0} \implies \Delta(G) = \{1\}$.

Example/Exercise 3.

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\} \text{ is not unimodular.}$$

Exercise. G = Lie group; $\mathfrak{g} = T_eG$. $\forall x \in G$, define $i_x\colon G \to G, i_x(y) = xyx^{-1}$. $\mathrm{Ad}_x = (\mathrm{d}i_x)(e)\colon \mathfrak{g} \to \mathfrak{g}$. $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism (the adjoint representation of G). **Prove**: $\Delta(x) = |\det \mathrm{Ad}_{x^{-1}}|$.

Banach algebras

Definition. A normed algebra is an algebra A equipped with a norm such that $\|ab\| \leqslant \|a\|\|b\|$, $\forall a, b \in A$ ($\|\cdot\|$ is submultiplicative). If A is unital, then we require that $\|1_A\| = 1$.

Exercise. $A \times A \to A, (a, b) \mapsto ab$ is continuous.

Definition. Banach algebra = complete nomred algebra.

Example 0. $0, \mathbb{C}$ are Banach algebras.

Example 1. X = a set. $\ell^\infty(X)$ is a Banach algebra under pointwise multiplication.

Example 2. X = topological space. $C_b(X) = C(X) \cap \ell^\infty(X)$ is a closed subalgebra in $\ell^\infty(X)$. Hence $C_b(X)$ is a Banach algebra.

Definition. A continuous function $f\colon X \to \mathbb{C}$ vanishes at ∞ if for any $\varepsilon > 0$, there exists a compact set $K \subset X$ s.t. $|f(x)| < \varepsilon$ for all $x \in X \backslash K$.

Example 3. $C_0(X) = \{f \in C(X)\colon f \text{ vanishes at } \infty\}$ is a closed ideal in $C_b(X)$, hence $C_0(X)$ is a Banach algebra. If X is compact, then $C_0(X) = C_b(X) = C(X)$.

Example 4. (X, μ) = measure space. $L^\infty(X, \mu)$ is a Banach algebra under pointwise multiplication (exercise).

Example/Exercise 5. $C^n[a, b]$ is a Banach algebra with respect to the norm $\|f\|_{C^n} = \sum\limits_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}$ (equivalent to $\|f\| = \max\left\{\|f^{(k)}\|_\infty\colon 0 \leqslant k \leqslant n\right\}$).

Example 6. $K \subset \mathbb{C}$ compact set.

$$\mathscr{A}(K) = \{f \in C(K)\colon f \text{ is holomorphic on } \operatorname{Int} K\}$$

is a closed subalgebra, hence $\mathscr{A}(K)$ is a Banach algebra. Consider $\overline{\mathbb{D}} = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$, then $\mathscr{A}(\overline{\mathbb{D}})$ is the disc algebra.

Example 7. E = Banach space.

$$\mathscr{B}(E) = \{\text{bounded linear operators } E \to E\} \text{ is a Banach algebra.}$$

Example 8.

$$\mathscr{K}(E) = \{T \in \mathscr{B}(E) \colon T \text{ is compact}\} \text{ is a closed 2-sided ideal of } \mathscr{B}(E) \implies \mathscr{K}(E) \text{ is a Banach algebra.}$$

Definition. A = an algebra. An involution on A is a map $A \to A, a \in A \mapsto a^* \in A$, such that

(1) $a^{**} = a$ $(a \in A)$.

(2) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ $(a, b \in A,\ \lambda, \mu \in \mathbb{C})$.

(3) $(ab)^* = b^* a^*$.

$(A, *)$ is a $*$-algebra.

Definition. A Banach $*$-algebra is a Banach algebra A equipped with an involution such that $\|a^*\| = \|a\|$, $\forall a \in A$.

Definition. A Banach $*$-algebra A is a C^*-algebra if $\|a^* a\| = \|a\|^2$ $(a \in A)$ (C^*-axiom).

Definition. A, B = $*$-algebra.
A algebra homomorphism $\varphi \colon A \to B$ is a $*$-homomorphism if $\varphi(a^*) = \varphi(a)^*$ $(a \in A)$.

Definition. A = $*$-algebra. $S \subset A$ is a $*$-subset if $\forall a \in S$ we have $a^* \in S$ (that is, $S^* = S$).

Example. $0, \mathbb{C}$ are C^*-algebras; $\lambda^* = \overline{\lambda}$ $(\lambda \in \mathbb{C})$.

Example/Exercise. $\underbrace{\ell^\infty(X), C_b(X), C_0(X), L^\infty(X, \mu)}_{C^*\text{-algebras}}, C^n[a, b]$ are Banach $*$-algebra w.r.t. $f^*(x) = \overline{f(x)}$.

Exercise. $C^n[a, b]$ is not a C^*-algebra if $n \geqslant 1$.

Example/Exercise. $\mathscr{A}(\overline{\mathbb{D}})$ is a Banach $*$-algebra w.r.t. $f^*(z) = \overline{f(\overline{z})}$ but is not a C^*-algebra.

Example. H = Hilbert space. $\mathscr{B}(H)$ is a C^*-algebra; $\langle T^* x | y \rangle = \langle x | T y \rangle$.

$$\mathscr{K}(H) \text{ is a closed } *\text{-ideal in } \mathscr{B}(H) \implies \mathscr{K}(H) \text{ is a } C^*\text{-algebra.}$$

The algebra $L^1(G)$

Proposition 1. X, Y = 2nd countable Hausdorff locally compact spaces. Then

(1) $\mathrm{Bor}(X \times Y)$ is generated by $\{B_1 \times B_2 \colon B_1 \in \mathrm{Bor}(X), B_2 \in \mathrm{Bor}(Y)\}$.

(2) μ = a Radon measure on X, ν = a Radon measure on $Y \implies \mu \otimes \nu$ is a Radon measure on $X \times Y$.

Proof. Exercise. □

Proposition 2. G_1, G_2 = locally compact groups, 2nd countable. μ_1, μ_2 = Haar measure on G_1, G_2, resp. Then $\mu_1 \otimes \mu_2$ is a Haar measure on $G_1 \times G_2$.

Proof. Exercise. □

G = locally compact group, μ = Haar measure on G.

$$\mathfrak{m}_\mu = \{A \subset G \colon A \text{ is } \mu\text{-measurable}\}.$$

Recall: $\mathfrak{m}_\mu$ is a σ-algebra; $\mathfrak{m}_\mu = \{B \cup N \colon B \subset G \text{ is Borel}, N \subset G \text{ is a } \mu\text{-null set}\}$ (the completion of $\mathrm{Bor}(X)$).

Lemma. $f, g \colon G \to \mathbb{C}$ $\mathfrak{m}_\mu$-measurable. Let

$$F \colon G \times G \to G, \quad F(y, x) = f(y) g(y^{-1} x).$$

Then F is $\mathfrak{m}_{\mu \otimes \nu}$-measurable.

Lecture 8 (2024.10.25)

<u>The algebra $L^1(G)$</u>

$G =$ 2nd countable locally compact group (i.e., there exists a countable base for the topology on G). $\mu =$ Haar measure.
<u>Recall</u>: $\mu \otimes \mu$ is a Haar measure on $G \times G$.
$\mathfrak{m}_\mu = \{A \subset G \colon A \text{ is } \mu\text{-measurable}\}$. $\mathfrak{m}_\mu = \{B \cup N \colon B \subset G \text{ Borel}, N = \text{a } \mu\text{-null set}\}$ (i.e., $N \subset C$ for some Borel C, $\mu(C) = 0$). $\mathfrak{m}_\mu$ is the <u>completion</u> of $\mathrm{Bor}(X)$.

<u>Lemma</u>. $f, g \colon G \to \mathbb{C}$ $\mathfrak{m}_\mu$-measurable. Define

$$F \colon G \times G \to G, \quad F(y,x) = f(y)g(y^{-1}x).$$

Then F is $\mathfrak{m}_{\mu\otimes\mu}$-measurable.

Proof.

$$\begin{tikzcd}
G \times G \arrow[r, "\alpha"] \arrow[rrr, bend left, "F"] & G \times G \arrow[r, "f \times g"] & \mathbb{C} \times \mathbb{C} \arrow[r, "\text{multiplication}"] & \mathbb{C}
\end{tikzcd}$$

$$(y,x) \longmapsto (y, y^{-1}x)$$

Multiplication is continuous $\implies$ it is a Borel map. $f \times g$ is $\mathfrak{m}_{\mu\otimes\mu}$-measurable. α is a homeomorphism $\implies$ it is a Borel map.
<u>Claim</u>: α is $\mathfrak{m}_{\mu\otimes\mu}$-$\mathfrak{m}_{\mu\otimes\mu}$-measurable.
Indeed: $\forall \varphi \in C_c(G \times G)$

$$\begin{aligned}
\int_{G\times G} \varphi(y, y^{-1}x) \, \mathrm{d}(\mu \otimes \mu)(y,x) &= \int_G \int_G \varphi(y, y^{-1}x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\
&= \int_G \int_G \varphi(y,x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) = \int_{G\times G} \varphi \, \mathrm{d}(\mu \otimes \mu).
\end{aligned}$$

Hence α is measure-preserving, $\forall \mu \otimes \mu$-null set $N \subset G \times G$, $\alpha^{-1}(N)$ is a null set. It implies that α is $\mathfrak{m}_{\mu\otimes\mu}$-$\mathfrak{m}_{\mu\otimes\mu}$-measurable. Therefore F is $\mathfrak{m}_{\mu\otimes\mu}$-measurable. $\square$

<u>Definition</u>. $f, g \colon G \to \mathbb{C}$ measurable. The <u>convolution</u> of f and g is

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) \, \mathrm{d}\mu(y). \tag{8}$$

<u>Convention</u>. $f * g$ is defined at those $x \in G$ where (8) exists.

<u>Theorem</u>.

(1) f, g are integrable $\implies$ $f * g$ is defined a.e., $f * g$ is integrable, and $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$.

(2) $(L^1(G), *)$ is a Banach $*$-algebra w.r.t.

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}).$$

Proof.

(1) $F(y,x) = f(y)g(y^{-1}x)$ is measurable. It follows from

$$\begin{aligned}
\int_{G\times G} |F| \, \mathrm{d}(\mu \otimes \mu) &\overset{\text{(Tonelli)}}{=\!=} \int_G \left(\int_G |f(y) g(y^{-1}x)| \, \mathrm{d}\mu(x) \right) \mathrm{d}\mu(y) \\
&= \int_G \int_G |f(y) g(x)| \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) = \|f\|_1 \|g\|_1 < \infty
\end{aligned}$$

that $|F|$ is integrable, so is F. By Fubini theorem, $F(\cdot, x)$ is integrable for almost all $x \in G$ (that is, $f * g$ is defined a.e.), and moreover, the function $x \mapsto \int_G F(y,x)\,\mathrm{d}\mu(y)$ is integrable (that is, $f*g$ is integrable).

$$\begin{aligned}\|f * g\|_1 &= \int_G \left| \int_G f(y) g(y^{-1}x)\,\mathrm{d}\mu(y) \right| \mathrm{d}\mu(x) \\ &\leqslant \int_{G\times G} |F|\,\mathrm{d}(\mu \otimes \mu) = \|f\|_1 \|g\|_1.\end{aligned}$$

(2) Exercise. □

Exercise.

(1) $L^1(G)$ is commutative $\iff$ G is commutative.

(2) $L^1(G)$ is unital $\iff$ G is discrete. If G is discrete, then $L^1(G) = \ell^1(G)$ and δ_e is the identity of $\ell^1(G)$.

(3) $L^1(G)$ is not a C^*-algebra unless $G = \{e\}$.

Complex measures

$X =$ a set, $\mathscr{A} \subset 2^X$ is a σ-algebra.

Definition. A complex measure on $\mathscr{A}$ is $\mu\colon \mathscr{A} \to \mathbb{C}$ s.t. $\forall A_1, A_2, \dots \in \mathscr{A}$ s.t. $A_i \cap A_j = \varnothing$ $(i \neq j)$ we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{9}$$

Remark.

(1) (9) converges absolutely.

(2) $\mu(A) \neq +\infty$!

Definition. The variation of a complex measure μ is

$$|\mu|\colon \mathscr{A} \to [0,+\infty], \quad |\mu|(A) = \sup\left\{ \sum_{i=1}^{n} |\mu(A_i)| \colon A = \bigsqcup_{i=1}^{n} A_i,\ A_i \in \mathscr{A} \right\}.$$

Facts.

(1) $|\mu|$ is a positive measure (σ-additive).

(2) $|\mu|(A) < +\infty$, $\forall A$.

Observe. If ν is a positive measure on $\mathscr{A}$ s.t.

$$\forall A \in \mathscr{A} \quad |\mu(A)| \leqslant \nu(A) \implies |\mu| \leqslant \nu.$$

Fact (Jordan decomposition)**.** $\mu\colon \mathscr{A} \to \mathbb{R}$ is a measure. Then there exists a unique pair (μ_+, μ_-) of positive measures on $\mathscr{A}$ s.t. $\mu = \mu_+ - \mu_-$ and $\mu_+ \perp \mu_-$. Moreover, $|\mu| = \mu_+ + \mu_-$.

Definition. Suppose $f\colon X \to \mathbb{C}$ is $\mathscr{A}$-measurable, $\mu\colon \mathscr{A} \to \mathbb{R}$ is a measure. f is μ-integrable if f is μ_+-integrable and μ_--integrable. Define

$$\int f\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu_+ - \int f\,\mathrm{d}\mu_-.$$

Definition. $\mu\colon \mathscr{A} \to \mathbb{C}$ is a measure; $\mu = \mu_1 + \mathrm{i}\mu_2$, where μ_1, μ_2 are real. f is μ-integrable if f is μ_1-integrable and μ_2-integrable. Define

$$\int f\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu_1 + \mathrm{i}\int f\,\mathrm{d}\mu_2.$$

Exercise. f is μ-integrable $\iff$ f is $|\mu|$-integrable, and $\left|\int f\,\mathrm{d}\mu\right| \leqslant \int |f|\,\mathrm{d}|\mu|$.

Let X be a locally compact Hausdorff topological space.

Definition. A complex Borel measure μ on X is a Radon measure if $|\mu|$ is a Radon measure.

Notation.

$$M(X) = \{\text{complex Radon measures on } X\}.$$

Exercise. $M(X)$ is a Banach space w.r.t.

$$\|\mu\| = |\mu|(X).$$

Notation.

$$B(X) = \{\text{bounded Borel function } X \to \mathbb{C}\}.$$

This is a Banach space under the uniform norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

Observe. We have a linear map

$$M(X) \to B(X)^*,$$
$$\mu \mapsto I_\mu, \quad \text{where } I_\mu(f) = \int_X f \, \mathrm{d}\mu.$$

Theorem (Riesz, Markov, Kakutani)**.**

$$\begin{array}{c} M(X) \to C_0(X)^* \\ \mu \mapsto I_\mu \end{array} \text{ is an isometric isomorphism.}$$

Definition. X, Y = locally compact Hausdorff spaces, $\mu \in M(X), \nu \in M(Y)$.

(1) If μ, ν are real, then

$$\mu \otimes \nu = (\mu_+ - \mu_-) \otimes (\nu_+ - \nu_-) \overset{\text{def}}{=\!=} \mu_+ \otimes \nu_+ - \mu_+ \otimes \nu_- - \mu_- \otimes \nu_+ + \mu_- \otimes \nu_-.$$

(2) General case:

$$\mu \otimes \nu = (\mu_1 + \mathrm{i}\mu_2) \otimes (\nu_1 + \mathrm{i}\nu_2) \overset{\text{def}}{=\!=} (\mu_1 \otimes \nu_1 - \mu_2 \otimes \nu_2) + \mathrm{i}(\mu_2 \otimes \nu_1 + \mu_1 \otimes \nu_2) \; (\mu_1, \mu_2, \nu_1, \nu_2 \text{ real}).$$

The measure algebra

G = 2nd countable locally compact group.

Notation.

$$\Delta \colon C_0(G) \to C_b(G \times G),$$
$$(\Delta f)(x, y) = f(xy).$$

Let's call it the comultiplication on $C_0(G)$.

Definition. $\mu, \nu \in M(G)$. The convolution of μ and ν is $\mu * \nu \in M(G)$:

$$\langle \mu * \nu, f \rangle = \langle \mu \otimes \nu, \Delta f \rangle \quad (f \in C_0(G)),$$

where $\langle \mu, g \rangle = \int g \, \mathrm{d}\mu$.

Proposition.

(1) $(M(G), *)$ is a Banach $*$-algebra w.r.t.

$$\mu^*(B) = \overline{\mu(B^{-1})}.$$

It is the measure algebra of G.

(2) $\langle \mu^*, f \rangle = \overline{\langle \mu, \overline{Sf} \rangle}$, $f \in C_0(G)$.

Proof. Exercise. $\square$

Remark.

$$\int f \, \mathrm{d}(\mu * \nu) = \iint\limits_{G\times G} f(xy) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y).$$

Exercise. $M(G)$ is commutative $\iff$ G is commutative.

Proposition. For any $x \in G$, let δ_x denote the Dirac measure concentrated at x.

(1) Then $\forall \mu \in M(G)$,

$$\delta_x * \mu = L_x\mu, \ \mu * \delta_x = R_{x^{-1}}\mu; \ \delta_x * \delta_y = \delta_{xy}.$$

(2) $\mathbb{C}G \xrightarrow{\alpha} M(G), \ x \in G \mapsto \delta_x$ is an injective homomorphism.

(3) α is an isomorphism $\iff$ G is finite.

Proof. Exercise. $\square$

Exercise.

(1) α extends to an isometric homomorphism

$$\beta \colon \ell^1(G) \to M(G), \ x \in G \mapsto \delta_x.$$

(2) β is an isomorphism $\iff$ G is discrete.

Notation. μ = Haar measure on G. $\forall f \in L^1(G)$, define $f \cdot \mu \in M(G)$ by

$$\langle f \cdot \mu, g\rangle = \langle \mu, fg\rangle \quad (g \in C_0(G)).$$

Exercise. $(f \cdot \mu)(B) = \int_B f \, \mathrm{d}\mu$, $\forall$ Borel $B \subset G$.

Proposition.

(1) The map

$$i \colon L^1(G) \to M(G), \ i(f) = f \cdot \mu$$

is an isometric $*$-homomorphism.

(2) Identify $L^1(G)$ with $i(L^1(G)) \subset M(G)$. Then $L^1(G)$ is a closed 2-sided ideal of $M(G)$, and $\forall f \in L^1(G)$, $\forall \nu \in M(G)$

$$(\nu * f)(x) = \int_G f(y^{-1}x) \, \mathrm{d}\nu(y);$$
$$(f * \nu)(x) = \int_G f(xy^{-1})\Delta(y^{-1}) \, \mathrm{d}\nu(y).$$

Proof. Exercise. $\square$

Lecture 9 (2024.11.01)

Approximate identities

Let $(\Lambda, \leqslant)$ be a poset.

Definition. $(\Lambda, \leqslant)$ is directed if $\forall \lambda, \mu \in \Lambda$, $\exists \nu \in \Lambda$ s.t. $\lambda \leqslant \nu, \mu \leqslant \nu$.

Examples.

(1) $(\mathbb{N}, \leqslant)$.

(2) $X =$ topological space, $x \in X$.

$$\Lambda = \{\text{neighborhoods of } x\}. \quad (\Lambda, \supset) \text{ is a directed poset.}$$

$X =$ topological space.

Definition. A net in X is a map $x \colon \Lambda \to X$, where Λ is a directed poset.

Notation. $x_\lambda = x(\lambda)$, $x = (x_\lambda)_{\lambda \in \Lambda}$.

Definition. (x_λ) converges to $x \in X$ ($x_\lambda \underset{\Lambda}{\to} x$; $\lim\limits_{\Lambda} x_\lambda = x$) if for any neighborhood $U \ni x$, there exists $\lambda_0 \in \Lambda$ s.t. $\forall \lambda \geqslant \lambda_0$, $x_\lambda \in U$.

Example. $\Lambda =$ the poset from Examples (2). $\forall U \in \Lambda$, choose $x_U \in U$, then $x_U \to x$.

$A =$ normed algebra.

Definition. An approximate identity (a.i.) in A is a net (e_λ) in A s.t. $\forall a \in A$, $ae_\lambda \to a, e_\lambda a \to a$.

Definition.

(1) An a.i. $(e_\lambda)_{\lambda \in \Lambda}$ is sequential if $\Lambda = \mathbb{N}$ with the standard order.

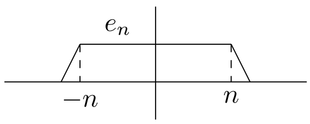
(2) $(e_\lambda)_{\lambda \in \Lambda}$ is a bounded a.i. if $\exists C > 0$ s.t. $\|e_\lambda\| \leqslant C$, $\forall \lambda \in \Lambda$. (b.a.i. = bounded a.i.)

Example 1. $A = c_0 = C_0(\mathbb{N}) = \left\{x = (x_n) \in \mathbb{C}^{\mathbb{N}} \colon \lim\limits_{n \to \infty} x_n = 0\right\}$. $\forall n \in \mathbb{N}$, $e_n = (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots) \in A$, $\|e_n\| = 1$. $\forall a \in A$

$$\|a - ae_n\| = \sup_{k > n} |a_k| \to 0 \implies (e_n) \text{ is a b.a.i. in } A.$$

Example 2. $A = \ell^1$ with pointwise multiplication $\implies (e_n)_{n \in \mathbb{N}}$ is an unbounded a.i.

Exercise. ℓ^1 does not have a b.a.i.

Example 3. $A = C_0(\mathbb{R})$. Define e_n by setting 

(e_n) is a b.a.i. in $C_0(\mathbb{R})$.

Example 4. $A = C_0(X)$ ($X =$ locally compact Hausdorff space).

$$\Lambda = \{K \subset X \colon K \text{ is compact}\}. \quad (\Lambda, \subset) \text{ is a directed poset.}$$

$\forall K \in \Lambda$, choose $e_K \in C_0(X)$ s.t. $e_K|_K = 1$, $\|e_K\| \leqslant 1$.

Exercise. $(e_K)_{K \in \Lambda}$ is a b.a.i. in $C_0(X)$.

Exercise. $C_0(X)$ has a sequential b.a.i. $\iff X$ is σ-compact.

Example 5. $A = \mathscr{K}(H)$ ($H =$ Hilbert space).

$$\Lambda = \{L \subset H \colon L \text{ is a finite-dimensional vector subspace}\}.$$

$(\Lambda, \subset)$ is a directed poset. $\forall L \in \Lambda$, let $P_L =$ the orthogonal projection onto L.

Exercise. $(P_L)_{L\in\Lambda}$ is a b.a.i. in $\mathscr{K}(H)$.

Exercise. $\mathscr{K}(H)$ has a sequential b.a.i. $\iff$ H is separable.

Example 6.

(1) $(A, \text{zero multiplication})$ does not have an a.i.

(2) $A = \{f \in C^1[0,1] \colon f(0) = 0\}$ does not have an a.i.

Proposition/Exercise. A = normed algebra, (e_λ) is a bounded net in A. Suppose $S \subset A$ generates a dense subalgebra of A and $e_\lambda a \to a, ae_\lambda \to a$, $\forall a \in S$. Then (e_λ) is a b.a.i. in A.

G = locally compact group (2nd countable), μ = Haar measure, β = a base of relative compact symmetric neighborhoods of $e \in G$. $\forall V \in \beta$ choose $u_V \in L^1(G)$ s.t.

(1) $u_V \geqslant 0$;

(2) $u_V|_{G\setminus V} = 0$;

(3) $\int_G u_V \, \mathrm{d}\mu = 1$.

Definition. A net $(u_V)_{V\in\beta}$ satisfying (1)-(3) is a Dirac net in $L^1(G)$.

Example. $u_V = \frac{\chi_V}{\|\chi_V\|_1}$.

Remark. There exists a Dirac net in $C_c(G)$. (Urysohn's lemma)

Proposition. Any Dirac net in $L^1(G)$ is a b.a.i. for $L^1(G)$.

Proof. $C_c(G)$ is dense in $L^1(G)$ (Urysohn's lemma). Hence it suffices to show that $u_V * f \to f$ and $f * u_V \to f$, $\forall f \in C_c(G)$. We may assume that $\exists V_0 \in \beta$ s.t. $V \subset V_0$, $\forall V \in \beta$

$$
\begin{aligned}
(u_V * f - f)(x) &= \int_V u_V(y)(f(y^{-1}x) - f(x)) \, \mathrm{d}\mu(y). \\
\|u_V * f - f\|_1 &= \int_G \left| \int_V u_V(y)(f(y^{-1}x) - f(x)) \, \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x) \\
&\leqslant \int_V \int_G |u_V(y)||f(y^{-1}x) - f(x)| \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\
&= \int_V u_V(y) \|L_y f - f\|_1 \, \mathrm{d}\mu(y) \\
&\leqslant \sup_{y\in V} \|L_y f - f\|_1.
\end{aligned}
$$

Exercise. $\exists C > 0$ s.t. $\forall y \in V_0$, $\|L_y f - f\|_1 \leqslant C\|L_y f - f\|_\infty$.

Hence $\|u_V * f - f\|_1 \leqslant C \sup_{y\in V} \|L_y f - f\|_\infty \to 0$ by the uniform continuity of f.

$f * u_V \to f$: exercise. □

Spectral theory in Banach algebras (a survey)

A = unital algebra, $A^\times = \{a \in A \colon a \text{ is invertible}\}$ (multiplication group of A).

Definition. The spectrum of $a \in A$ is

$$\sigma_A(a) = \sigma(a) = \left\{\lambda \in \mathbb{C} \colon a - \lambda 1 \notin A^\times\right\}.$$

Example 1. $A = \mathbb{C}$, $\sigma_{\mathbb{C}}(\lambda) = \{\lambda\}$.

Example 2. $A = \mathrm{End}_{\mathbb{C}}(E)$, $\dim E < \infty$. $\forall T \in A$, $\sigma_A(T) = \{\text{eigenvalues of } T\}$.

Example 3. $A = \mathbb{C}^X$ (X = a set). $\sigma_A(f) = f(X)$. The same is true for $A = C(X)$ (X = topological space).

Example 4. $A = \ell^\infty(X)$ (X = a set). $\sigma_A(f) = \overline{f(X)}$. The same is true, for example, for $A = C_b(X)$ (X = a topological space).

Example 5. $A = \mathbb{C}G$ (G = a finite abelian group). $\sigma_A(f) = \widehat{f}(\widehat{G})$.

Proposition. $\varphi\colon A \to B$ is a unital algebra homomorphism. Then

(1) $\varphi(A^\times) \subset B^\times$.

(2) $\sigma_B(\varphi(a)) \subset \sigma_A(a)$, $\forall a \in A$.

(3) $\forall a \in A$, $\sigma_B(\varphi(a)) = \sigma_A(a) \iff \varphi(A \backslash A^\times) \subset B \backslash B^\times$.

Corollary. A = unital algebra, $B \subset A$ subalgebra, $1_A \in B$. Then $\forall b \in B$, $\sigma_A(b) \subset \sigma_B(b)$.

Definition. B is spectrally invariant in A if $\forall b \in B$

$$\begin{aligned} \sigma_B(b) = \sigma_A(b) &\iff B\backslash B^\times \subset A \backslash A^\times \\ &\iff B \cap A^\times = B^\times. \end{aligned}$$

Examples.

(1) $C(X) \subset \mathbb{C}^X$ is spectrally invariant (X = a topological space).

(2) $\ell^\infty(X) \subset \mathbb{C}^X$ is not spectrally invariant (X = an infinite set).

(3) $\mathscr{B}(E) \subset \mathrm{End}_{\mathbb{C}}(E)$ is spectrally invariant (E = a Banach space).

Proposition (Polynomial spectral mapping theorem)**.** A = unital algebra. $a \in A$, $f \in \mathbb{C}[t]$. Then

$$\boxed{\sigma_A(f(a)) = f(\sigma_A(a))}$$

unless $\sigma_A(a) = \varnothing$ and $f \in \mathbb{C}1$.

Proposition. If $a \in A^\times$, then $\sigma(a^{-1}) = \{\lambda^{-1} \colon \lambda \in \sigma(a)\}$.

Theorem. A = unital Banach algebra. Then

(1) $A^\times$ is open in A. Moreover: $\forall a \in A$ s.t. $\|a\| < 1$, we have $1 - a \in A^\times$, and

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

(2) The map $A^\times \to A^\times, a \mapsto a^{-1}$, is continuous.

Definition. A = an algebra. A character of A is an algebra homomorphism $\chi\colon A \to \mathbb{C}$.

Observe. If A is unital and $\chi \neq 0$, then $\chi(1) = 1$.

Corollary. A = unital Banach algebra, $\chi\colon A \to \mathbb{C}$ a character $\implies$ χ is continuous, and $\|\chi\| \leqslant 1$.

Proof. If χ is unbounded or $\|\chi\| > 1$, then there exists $a \in A$ s.t. $|\chi(a)| > \|a\| \implies \exists b \in A$ s.t. $\|b\| < 1$, $\chi(b) = 1$ ($b = \frac{a}{\chi(a)}$) $\implies 1 - b \in A^\times \implies 0 = 1 - \chi(b) = \chi(1-b) \neq 0$, a contradiction. ☐

Theorem (Gelfand)**.** A = unital Banach algebra, $a \in A$. Then

(1) $\forall \lambda \in \sigma_A(a)$, $|\lambda| \leqslant \|a\|$.

(2) $\sigma_A(a)$ is compact.

(3) $\sigma_A(a) \neq \varnothing$ (if $A \neq 0$).

Theorem (Gelfand-Mazur theorem)**.** A = a Banach division algebra (that is, $A \neq 0$ and all $a \in A\backslash\{0\}$ are invertible). Then $A \cong \mathbb{C}$.

Proof. $\forall a \in A$, $\exists \lambda \in \mathbb{C}$ s.t. $a - \lambda 1 = 0$, that is, $a = \lambda 1 \implies A = \mathbb{C}1 \cong \mathbb{C}$. ☐

Lecture 10 (2024.11.08)

Spectral radius

$A =$ unital Banach algebra, $a \in A$ ($A \neq 0$).

Definition. The spectral radius of a is

$$r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$$

Gelfand's theorem $\implies r_A(a) \leqslant \|a\|$.

Example. $A = \ell^\infty(X) \implies r_A(a) = \|a\|$. The same holds for $C_b(X)$, $X =$ topological space.

Example. $A = \mathscr{B}(H)$, $H =$ finite-dimensional Hilbert space. $a \in A$, $a = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ w.r.t. an orthonormal basis $\implies r_A(a) = \max\limits_{1 \leqslant i \leqslant n} |\lambda_i| = \|a\|$.

Example. $A = \mathscr{B}(\mathbb{C}^2)$, $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies r_A(a) = 0$, but $\|a\| > 0$.

Exercise. $a \in A$ is nilpotent $\implies \sigma_A(a) = \{0\} \implies r_A(a) = 0$.

Theorem (Beurling, Gelfand)**.** $A =$ unital Banach algebra, $a \in A \implies r(a) = \lim\limits_{n\to\infty} \|a^n\|^{1/n} = \inf\limits_{n \geqslant 1} \|a^n\|^{1/n}$.

Idea. $\leqslant$ is trivial; since $|\lambda^n| \leqslant \|a^n\|$. For $\geqslant$, consider $f \in A^*$ and the map $\lambda \mapsto f((1-\lambda a)^{-1})$.

Corollary 1. $r(a) = 0 \iff \lim\limits_{n\to\infty} \|a^n\|^{1/n} = 0 \iff \forall \varepsilon > 0, \|a^n\| = o(\varepsilon^n)$ $(n \to \infty)$.

Definition. Such elements are called quasinilpotent.

Example/Exercise. The Volterra integral operator

$$V_K \colon L^2[a,b] \to L^2[a,b], \quad (V_K f)(x) = \int_a^x K(x,y) f(y)\, \mathrm{d}y$$

is quasinilpotent for any bounded measurable K on $[a,b] \times [a,b]$.

Corollary 2. $A =$ unital Banach algebra, $B \subset A$ closed subalgebra, $1_A \in B \implies \forall b \in B, r_B(b) = r_A(b)$.

The maximal spectrum and the Gelfand transform

$A =$ commutative unital algebra.

Definition. An ideal $I \subsetneq A$ is maximal $\iff \nexists$ ideal J s.t. $I \subsetneq J \subsetneq A$.

Exercise. I is maximal $\iff A/I$ is a field.

Definition. The maximal spectrum of A is

$$\mathrm{Max}(A) = \{\text{maximal ideals of } A\}.$$

Example/Exercise.

$$\begin{aligned} \mathrm{Max}\,\mathbb{C}[t] &\overset{1-1}{\rightleftarrows} \mathbb{C} \\ p \in \mathbb{C} &\mapsto m_p = \{f \in \mathbb{C}[t] : f(p) = 0\}. \end{aligned}$$

Proposition. Each proper ideal of A is contained in a maximal ideal.

Proof. $I \subsetneq A$ ideal.

$$M = \{J : J \subsetneq A \text{ is an ideal}, I \subset J\}.$$

Claim. $(M, \subset)$ satisfies the conditions of Zorn's lemma.

Indeed: Suppose $C \subset M$ is a chain. Let $K = \bigcup\{J : J \in C\}$, K is an ideal, $I \subset K$. $\forall J \in C$, $1 \notin J \implies 1 \notin K \implies K \neq A$. K is an upper bound for $C \implies M$ has a maximal element. ∎

Definition. The character space of A is

$$\widehat{A} = \{\chi\colon A \to \mathbb{C}\colon \chi \text{ is a character}, \chi \neq 0\}.$$

Observe. $\forall \chi \in \widehat{A}$, $\operatorname{Ker}\chi \in \operatorname{Max}(A)$.

Proposition. The map $\widehat{A} \to \operatorname{Max}(A), \chi \mapsto \operatorname{Ker}\chi$, is injective.

Proof. $\chi_1, \chi_2 \in \widehat{A}$; $\operatorname{Ker}\chi_1 = \operatorname{Ker}\chi_2 \implies \chi_1 = \lambda\chi_2$ $(\lambda \in \mathbb{C})$; $1 = \chi_1(1) = \lambda\chi_2(1) = \lambda \implies \chi_1 = \chi_2$. □

Example/Exercise.

(1) $A = \mathbb{C}[t] \implies \widehat{A} \to \operatorname{Max}(A)$ is a bijection.

(2) $A \supsetneq \mathbb{C}$ is a field $\implies \widehat{A} = \varnothing$, but $\operatorname{Max}(A) = \{0\}$.

Lemma. A = commutative unital Banach algebra $\implies$ each maximal ideal of A is closed in A.

Proof. Let $I \in \operatorname{Max}(A) \implies \bar{I}$ is an ideal. Suppose $I \neq \bar{I} \implies \bar{I} = A \implies I \cap A^\times \neq \varnothing$ (because $A^\times$ is open in A) $\implies I = A$, a contradiction. □

Corollary. A commutative unital Banach algebra does not have dense proper ideals.

Theorem. A = commutative unital Banach algebra $\implies$ the map $\widehat{A} \to \operatorname{Max}(A), \chi \mapsto \operatorname{Ker}\chi$, is a bijection.

Observation. A = Banach algebra, $I \subset A$ closed 2-sided ideal of $A \implies A/I$ is a Banach algebra w.r.t. $\|a + I\| = \inf\{\|a + b\|\colon b \in I\}$.

Proof of Theorem. Let $I \in \operatorname{Max}(A) \implies A/I$ is a Banach field $\implies A/I \cong \mathbb{C}$.

$$\begin{tikzcd} A \arrow[r, "\text{quotient}"] \arrow[rr, bend right, "\chi"'] & A/I \arrow[r, "\cong"] & \mathbb{C} \end{tikzcd} \quad I = \operatorname{Ker}\chi.$$

□

Corollary. A = commutative unital Banach algebra, $a \in A$. $a \in A^\times \iff \forall \chi \in \widehat{A}, \chi(a) \neq 0$.

Proof. ($\Longrightarrow$) clear. ($\Longleftarrow$). Suppose $a \notin A^\times \implies Aa \subsetneq A \implies \exists I \in \operatorname{Max}(A)$ s.t. $I \supset Aa$; but $I = \operatorname{Ker}\chi$ $(\chi \in \widehat{A}) \implies \chi(a) = 0$. □

Convention. Identify $\widehat{A}$ with $\operatorname{Max}(A)$.

Some facts on the weak* topology

E = normed space. $\forall v \in E$, define a seminorm $\|\cdot\|_v$ on E^* by $\|f\|_v = |f(v)|$.

Definition. The weak* topology on E^* is the locally convex topology generated by $\{\|\cdot\|_v : v \in E\}$.
Explicitly: $\forall f \in E^*$ the standard subbase of neighborhoods of f (for weak*) is

$$\sigma_f = \{U_{v,\varepsilon}(f)\colon v \in E, \varepsilon > 0\},$$

where $U_{v,\varepsilon}(f) = \{g \in E^*\colon |g(v) - f(v)| < \varepsilon\}$.

Facts.

(0) wk* is Hausdorff.

(1) $(E^*, \text{wk*}) \subset \mathbb{C}^E$. wk* = the restriction to E^* of the product (Tychonoff) topology on $\mathbb{C}^E$.

(2) $f_n \to f$ w.r.t. wk* $\iff f_n(v) \to f(v)$, $\forall v \in E$.

(3) $\forall v \in E$, let $\varepsilon_v\colon E^* \to \mathbb{C}, \varepsilon_v(f) = f(v)$. wk* = the weakest topology on E^* that makes all ε_v continuous.

(4) X = topological space. A map $\varphi\colon X \to (E^*, \text{wk*})$ is continuous $\iff \forall v \in E$, $\varepsilon_v \circ \varphi\colon X \to \mathbb{C}$ is continuous.

(5) $i_E\colon E \to E^{**}$ cannonical embedding ($v \mapsto \varepsilon_v$).

$$\operatorname{Im} i_E = \{\alpha \in E^{**} \colon \alpha \text{ is wk*-continuous}\}.$$

(6) E, F normed. A linear operator $T\colon (F^*, \text{wk*}) \to (E^*, \text{wk*})$ is continuous $\iff$ $\exists$ a bounded linear operator $S\colon E \to F$ s.t. $S^* = T$.

(7) (Banach-Alaoglu Theorem)

$$\mathbb{B}_{E*} = \{f \in E^* \colon \|f\| \leqslant 1\} \text{ is wk*-compact.}$$

The maximal spectrum and the Gelfand transform (continuation)

A = commutative unital Banach algebra.

Definition. The Gelfand topology on $\operatorname{Max}(A) \cong \widehat{A}$ is the restriction to $\widehat{A}$ of the weak* topology on A^*. ($\operatorname{Max}(A) \cong \widehat{A} \subset A^*$)

Theorem. $\operatorname{Max}(A)$ is compact and Hausdorff.

Proof. $(A^*, \text{wk*})$ is Hausdorff $\implies$ so is $\widehat{A}$. $\widehat{A} \subset \mathbb{B}_{A*}$, $(\mathbb{B}_{A*}, \text{wk*})$ is compact. We have to show that $\widehat{A} \subset \mathbb{B}_{A*}$ is closed. Let $a, b \in A$.

Observe. the maps

$$\begin{aligned} A^* &\to \mathbb{C} \\ f \in A^* &\mapsto f(ab) - f(a)f(b) \\ f \in A^* &\mapsto f(1) \end{aligned}$$

are continuous w.r.t. wk*.

$$\widehat{A} = \left\{ f \in A^* \colon \begin{matrix} f(ab) - f(a)f(b) = 0 \quad \forall a, b \in A; \\ f(1) = 1 \end{matrix} \right\} \implies \widehat{A} \text{ is closed in } \mathbb{B}_{A*}. \qquad \square$$

Definition. The Gelfand transform of $a \in A$ is

$$\widehat{a}\colon \operatorname{Max}(A) \to \mathbb{C}, \quad \widehat{a}(x) = x(a).$$

Proposition. $\widehat{a}$ is continuous.

Proof. $\widehat{a} = i_A(a)|_{\widehat{A}}$; $i_A(a)$ is wk*-continuous on A^*. $\square$

Definition. The Gelfand transform of A is

$$\Gamma_A\colon A \to C(\operatorname{Max}(A)), \quad a \in A \mapsto \widehat{a}.$$

Theorem (properties of Γ_A)**.** A = commutative unital Banach algebra.

(1) Γ_A is a unital algebra homomorphism.

(2) $\|\Gamma_A\| = 1$ (if $A \neq 0$).

(3) $\forall a \in A$, $\|\widehat{a}\|_\infty = r_A(a)$.

(4) $\forall a \in A$, $\sigma_A(a) = \widehat{a}(\operatorname{Max}(A))$.

(5) $\operatorname{Ker} \Gamma_A = \bigcap \{I \colon I \in \operatorname{Max}(A)\} = \{a \in A \colon a \text{ is quasinilpotent}\}$.

Proof.

(1) exercise.

(4) We know: $\widehat{a}(\mathrm{Max}(A)) = \sigma_{C(\mathrm{Max}(A))}(\widehat{a}) \implies$ it suffices to show that Γ(noninvertible) $\subset$ noninvertible. Suppose $a \notin A^\times \implies \exists \chi \in \widehat{A}$ s.t. $\chi(a) = 0$, that is, $\widehat{a}(\chi) = 0 \implies \widehat{a}$ is noninvertible in $C(\mathrm{Max}(A))$.

(3) follows from (4).

(2) $\forall a \in A$, $\|\widehat{a}\|_\infty = r(a) \leqslant \|a\| \implies \|\Gamma_A\| \leqslant 1$; $\Gamma_A(1) = 1 \implies \|\Gamma_A\| = 1$.

(5) $\operatorname{Ker}\Gamma_A = \bigcap \left\{\operatorname{Ker}\chi \colon \chi \in \widehat{A}\right\} = \bigcap \{I \colon I \in \mathrm{Max}(A)\} \underset{(3)}{=} \{\text{quasinilpotents}\}$. □

Definition. A = unital commutative algebra. The Jacobson radical of A is

$$J(A) = \bigcap \{I \colon I \in \mathrm{Max}(A)\}.$$

A is Jacobson semisimple $\iff J(A) = 0$.

Corollary. $\operatorname{Im}\Gamma_A$ is spectrally invariant in $C(\mathrm{Max}(A))$.

Proof. $\Gamma(a) \in C(\mathrm{Max}(A))^\times \implies a \in A^\times \implies \Gamma(a) \in (\operatorname{Im}\Gamma_A)^\times$. □

Examples: subalgebras of $C(X)$

X = compact Hausdorff topological space. $\forall x \in X, \varepsilon_x \colon C(X) \to \mathbb{C}, \varepsilon_x(f) = f(x)$; $m_x = \operatorname{Ker}\varepsilon_x$.

Lemma. For any ideal $I \subsetneq C(X)$, there exists $x \in X$ s.t. $I \subset m_x$.

Proof. Suppose $\forall x \in X$, $\exists f_x \in I$ s.t. $f_x(x) = 0$. $\exists$ a neighborhood $U_x \ni x$ s.t. $\forall y \in U_x$, $f_x(y) \neq 0$. $X = U_{x_1} \cup \cdots \cup U_{x_n}$ (by compactness). Let $f = \sum\limits_{i=1}^n |f_{x_i}|^2 = \sum\limits_{i=1}^n \overline{f}_{x_i} f_{x_i} \in I$; $f(y) > 0$, $\forall y \in X \implies f$ is invertible in $C(X) \implies I = C(X)$, a contradiction. □

Corollary. The map
$$\begin{tikzcd} \varepsilon\colon X \arrow[r] \arrow[dr] & \mathrm{Max}(C(X)), x \mapsto m_x \arrow[d, equal] \\ & \widehat{C(X)}, x \mapsto \varepsilon_x \end{tikzcd}$$
is a bijection.

Notation. X, Y compacts, Hausdorff. $f \colon X \to Y$ continuous. $f^\bullet \colon C(Y) \to C(X), f^\bullet(\varphi) = \varphi \circ f$.
Properties of $f^\bullet$:

(1) $f^\bullet$ is a unital algebra homomorphism, and $\|f^\bullet\| = 1$.

(2) $(1_X)^\bullet = 1_{C(X)}$.

(3) $X \xrightarrow{f} Y \xrightarrow{g} Z \implies (g \circ f)^\bullet = f^\bullet \circ g^\bullet$.

Observe. (1)-(3) $\implies$ if f is a homomorphism, then $f^\bullet$ is an isometric isomorphism.

Theorem. X = compact Hausdorff topological space. $A \subset C(X)$ subalgebra, $1_{C(X)} \in A$. Suppose

(1) A is a Banach algebra w.r.t. a norm that dominates the sup norm.

(2) A separates the points of X.

(3) $\forall \chi \in \widehat{A}, \exists x \in X$ s.t. $\chi = \varepsilon_x$.

Then the map $\varepsilon \colon X \to \widehat{A}, x \mapsto \varepsilon_x$, is a homeomorphism. Moreover, the following diagram commutes:

$$\begin{tikzcd} & C(X) \\ A \arrow[ur, hook] \arrow[dr, "\Gamma_A"'] & \\ & C(\mathrm{Max}(A)) \arrow[uu, "\varepsilon^\bullet"', "\sim"] \end{tikzcd}$$

Proof. (2)&(3) $\implies \varepsilon$ is a bijection. ε is continuous $\iff \forall a \in A$ the map $x \mapsto \varepsilon(x)(a) = a(x)$ is continuous. $a \in C(X) \implies \varepsilon$ is continuous $\implies \varepsilon$ is a homeomorphism. $(\varepsilon^\bullet\Gamma)(a)(x) = \Gamma(a)(\varepsilon_x) = \varepsilon_x(a) = a(x) \implies$ the diagram commutes. □

Corollary. If $A = C(X)$ (X compact, Hausdorff) $\implies \Gamma_A$ is an isometric isomorphism, and $\Gamma_A^{-1} = \varepsilon^\bullet$.

Functorial properties of Γ

Category Comp. Objects: compact Hausdorff topological spaces. Morphisms: continuous maps.
Category CUBA. Objects: commutative unital Banach algebra. Morphisms: continuous unital homomorphisms.

2 contravariant functors

$$\begin{aligned}
&C\colon \mathsf{Comp} \to \mathsf{CUBA}, \quad X \mapsto C(X);\\
&(f\colon X \to Y) \mapsto (f^\bullet\colon C(Y) \to C(X), f^\bullet(\varphi) = \varphi \circ f).\\
&\mathrm{Max}\colon \mathsf{CUBA} \to \mathsf{Comp}, \quad A \mapsto \mathrm{Max}(A);\\
&(\varphi\colon A \to B) \mapsto (\varphi^*\colon \mathrm{Max}(B) \to \mathrm{Max}(A), \varphi^*(\chi) = \chi \circ \varphi).
\end{aligned}$$

φ^* is the restriction of $\varphi^*\colon B^* \to A^*$ (dual of φ), which is wk*-continuous $\implies \varphi^*\colon \mathrm{Max}(B) \to \mathrm{Max}(A)$ is continuous.

Exercise.

(1)
$$\{\varepsilon_x\colon X \to \mathrm{Max}(C(X))\colon X \in \mathsf{Comp}\}$$
is a natural isomorphism between 1_{Comp} and $\mathrm{Max} \circ C$.

(2)
$$\{\Gamma_A\colon A \to C(\mathrm{Max}(A))\colon A \in \mathsf{CUBA}\}$$
is a natural transformation from 1_{CUBA} to $C \circ \mathrm{Max}$.

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\Gamma_A"'] & B \arrow[d, "\Gamma_B"] \\
C(\mathrm{Max}(A)) \arrow[r] & C(\mathrm{Max}(B))
\end{tikzcd}$$

(3) $\exists$ 1-1 correspondence

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{CUBA}}(A, C(X)) \cong \mathrm{Hom}_{\mathsf{Comp}}(X, \mathrm{Max}(A)) &\cong \mathrm{Hom}_{\mathsf{Comp}^{op}}(\mathrm{Max}(A), X)\\
\varphi &\mapsto \varphi^* \circ \varepsilon_x\\
f^\bullet \circ \Gamma_A &\leftarrow f
\end{aligned}$$

Hence (Max, C) is an adjoint pair of functors.

Lecture 11 (2024.11.15)

Unitization

A = algebra. $A_+ = A \oplus \mathbb{C}1_+$ (a vector space direct sum). Multiplication on A_+:

$$(a + \lambda 1_+)(b + \mu 1_+) = ab + \lambda b + \mu a + \lambda\mu 1_+.$$

A_+ becomes a unital algebra.

Definition. A_+ is the unitization of A.

Proposition/Exercise 1. A = algebra, B = unital algebra; $\varphi\colon A \to B$.

(1) Define $\varphi_+\colon A_+ \to B$ by $\varphi_+(a + \lambda 1_+) = \varphi(a) + \lambda 1_B$. Then φ_+ is a unital algebra homomorphism.

(2) $\exists$ a natural bijection

$$\begin{gathered}\mathrm{Hom}_{\mathsf{Alg}}(A, B) \rightleftarrows \mathrm{Hom}_{\mathsf{Un.Alg}}(A_+, B)\\ \varphi \mapsto \varphi_+\\ \psi|_A \leftarrow\!\shortmid \psi\end{gathered}$$

Proposition/Exercise 2. A = Banach algebra. Then

(1) A_+ is a Banach algebra w.r.t. $\|a + \lambda 1_+\| = \|a\| + |\lambda|$.

(2) Proposition 1 holds for Banach algebras with "Hom" = continuous algebra homomorphism.

Corollary. A = Banach algebra, $\chi\colon A \to \mathbb{C}$ character $\implies$ χ is continuous, and $\|\chi\| \leqslant 1$.

Example. X = locally compact Hausdorff topological space. X_+ = the 1-point compactification of X. $X_+ = X \sqcup \{\infty\}$. Topology on X_+: $\{U \subset X : U \text{ is open}\} \cup \{X_+ \backslash K : K \text{ is compact in } X\}$.

Facts.

(1) X_+ is compact and Hausdorff.

(2) Y = compact Hausdorff topological space; $X = Y \backslash \{y_0\}$, then X is locally compact, and there is a homeomorphism $X_+ \xrightarrow{\sim} Y, x \in X \mapsto x \in X, \infty \mapsto y_0$.

Exercise.

(1) $C_0(X) = \{f|_X : f \in C(X_+), f(\infty) = 0\}$.

(2) $\exists$ a topological algebra isomorphism

$$C_0(X)_+ \xrightarrow{\sim} C(X_+), \quad f + \lambda 1_+ \mapsto f + \lambda \quad (f(\infty) = 0).$$

A = algebra, $a \in A$.

Definition. The nonunital spectrum of a is

$$\sigma'_A(a) = \sigma_{A_+}(a).$$

Observe. $A \subset A_+$ is a 2-sided ideal $\implies$ $a \in A$ is not invertible in A_+ $\implies$ $0 \in \sigma'_A(a)$.

Exercise.

(1) A_1, A_2 = unital algebras, $a = (a_1, a_2) \in A_1 \oplus A_2 = A$ $\implies$ $\sigma_A(a) = \sigma_{A_1}(a_1) \cup \sigma_{A_2}(a_2)$.

(2) A = unital algebra $\implies$ $\exists$ an algebra isomorphism

$$A \oplus \mathbb{C} \xrightarrow{\sim} A_+, \quad (a, \lambda) \mapsto a + \lambda(1_+ - 1_A)$$

(3) A = unital algebra, $a \in A$ $\implies$ $\sigma'_A(a) = \sigma_A(a) \cup \{0\}$.

A = Banach algebra, $a \in A$.

Definition. The spectral radius of a is

$$r(a) = \sup\left\{|\lambda| : \lambda \in \sigma'_A(a)\right\}.$$

Theorem. $r(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf\limits_{n\geqslant 1} \|a^n\|^{\frac{1}{n}}$.

Max and Γ for nonunital commutative Banach algebras

A = commutative algebra.

Definition. An ideal $I \subset A$ is modular (regular) if A/I is unital ($\iff \exists u \in A$ s.t. $\forall a \in A, a - au \in I$. u is a modular identity for I).

Observation.

(1) $0 \subset A$ is modular $\iff$ A is unital $\iff$ all ideals of A are modular.

(2) $I \subset J \subset A$ ideals, I is modular $\implies$ J is modular.

(3) $\chi\colon A \to \mathbb{C}$ character $\implies$ $\operatorname{Ker}\chi$ is a modular ideal.

(4) Let $A^2 = \operatorname{span}\{ab\colon a, b \in A\}$. Suppose $A^2 \neq A$. Then each vector subspace I s.t. $A^2 \subset I \subsetneq A$ is a non-modular ideal of A. For example, $A = t\mathbb{C}[t]$, $I = A^2 = t^2\mathbb{C}[t]$.

Definition. The maximal spectrum of A is

$$\operatorname{Max}(A) = \{\text{maximal modular ideals of } A\}.$$

Theorem. Each proper modular ideal of A is contained in a maximal modular ideal.

Proof. Exercise. □

Exercise. Fails for non-modular ideals.

Definition. The character space of A is

$$\widehat{A} = \{\chi\colon A \to \mathbb{C}\colon \chi \text{ is character}, \chi \neq 0\}.$$

Exercise. The map $\widehat{A} \to \operatorname{Max}(A), \chi \mapsto \operatorname{Ker}\chi$, is injective.

Notation.

$$\widehat{A}_+ = \{\text{all characters } A \to \mathbb{C}\} = \widehat{A} \cup \{0\colon A \to \mathbb{C}\}, \quad \operatorname{Max}_+(A) = \operatorname{Max}(A) \cup \{A\}.$$

Proposition.

$$\begin{tikzcd}
\widehat{A_+} \arrow[r, "\chi\mapsto\chi|_A"] \arrow[d] \arrow[dr, phantom, "\text{D}"] & \widehat{A}_+ \arrow[d] & \chi \arrow[d, mapsto] \\
\operatorname{Max}(A_+) \arrow[r, "I\mapsto I\cap A"'] & \operatorname{Max}_+(A) & \operatorname{Ker}\chi
\end{tikzcd}$$

The diagram commutes, and the horizontal arrows are bijections.

Proof. Exercise. *Hint*: $I \subset A$ modular ideal, $u \in A$ a modular identity for I. Define $J = I \oplus \mathbb{C}(1_+ - u)$. Then J is an ideal of A_+, and $A_+/J \cong A/I$. □

Corollary. A = commutative Banach algebra. Then

(1) All arrows in D are bijections.

(2) All maximal modular ideals of A are closed in A.

(3) The map $\widehat{A} \to \operatorname{Max}(A), \chi \mapsto \operatorname{Ker}\chi$, is bijective.

Definition. The Gelfand topology on $\operatorname{Max}(A) \cong \widehat{A}$ and $\operatorname{Max}_+(A) \cong \widehat{A}_+$ is the restriction of the weak* topology on A^*.

Proposition. $\mathrm{Max}(A)$ and $\mathrm{Max}_+(A)$ are Hausdorff. $\mathrm{Max}_+(A)$ is compact and $\mathrm{Max}_+(A) \underset{\text{homeo}}{\cong} \mathrm{Max}(A_+)$. $\mathrm{Max}(A)$ is locally compact, and $\mathrm{Max}_+(A)$ is the 1-point compactification of $\mathrm{Max}(A)$.

A = commutative Banach algebra.

Definition. The Gelfand transform of $a \in A$ is $\widehat{a} \colon \widehat{A} = \mathrm{Max}(A) \to \mathbb{C}, \widehat{a}(\chi) = \chi(a)$ $(\chi \in \widehat{A})$.

Proposition. $\widehat{a} \in C_0(\mathrm{Max}(A))$.

Proof. Extend $\widehat{a}$ to $\widehat{a} \colon \mathrm{Max}_+(A) \cong \widehat{A}_+ \to \mathbb{C}, \widehat{a}(\chi) = \chi(a)$. $\widehat{a}$ is continuous on $\widehat{A}_+$ (see the unital case). $\widehat{a}(0) = 0 \implies \widehat{a} \in C_0(\widehat{A})$. $\square$

Definition. The Gelfand transform of A is

$$\Gamma_A \colon A \to C_0(\mathrm{Max}(A)), \quad a \mapsto \widehat{a}.$$

Observe. The following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "\Gamma_A"] \arrow[d, hook] & & C_0(\mathrm{Max}(A)) \arrow[d, hook] \\
A_+ \arrow[r, "\Gamma_{A_+}"'] & C(\mathrm{Max}(A_+)) \arrow[r, "\cong"] & C(\mathrm{Max}_+(A))
\end{tikzcd}$$

Theorem.

(1) Γ_A is an algebra homomorphism.

(2) $\|\Gamma_A\| \leqslant 1$.

(3) $\forall a \in A$, $\|\widehat{a}\|_\infty = r(a)$.

(4) $\forall a \in A$, $\sigma'_A(a) = \widehat{a}(\mathrm{Max}(A)) \cup \{0\}$.

(5) $\mathrm{Ker}\,\Gamma_A = \bigcap\{\text{maximal modular ideals of } A\} = \{\text{quasinilpotents of } A\}$.

Products and unitizations of C^*-algebras

1. Products

Observe.

(1) A, B = Banach $*$-algebras $\implies$ so is $A \oplus B$ in the following natural way:

$$(a, b)^* = (a^*, b^*);$$
$$\|(a, b)\| = \max\{\|a\|, \|b\|\}.$$

(2) $A, B = C^*$-algebra $\implies$ so is $A \oplus B$.

2. Unitizations

Observe. If A is a Banach $*$-algebra, then so is A_+:

$$(a + \lambda 1_+)^* = a^* + \bar{\lambda} 1_+ \quad (a \in A, \lambda \in \mathbb{C}).$$
$$\|a + \lambda 1_+\| = \|a\| + |\lambda|. \tag{10}$$

Exercise. If $A \neq 0$ is a C^*-algebra, then norm (10) does not satisfy the C^*-axiom.

Suppose A is a unital C^*-algebra.

$$\underset{\text{(algebra isomorphism)}}{A_+ \cong A \oplus \mathbb{C}} \qquad (a, \lambda) \in A \oplus \mathbb{C} \mapsto a + \lambda(1_+ - 1_A).$$

Hence A_+ becomes a C^*-algebra w.r.t.

$$\|a + \lambda(1_+ - 1_A)\| = \max\{\|a\|, |\lambda|\},$$

or equivalently, $\|a + \lambda 1_+\| = \max\{\|a + \lambda 1_A\|, |\lambda|\}$.

Proposition. $A =$ (strictly) nonunital C^*-algebra. $\forall a \in A_+$, let $L_a \colon A \to A, L_a(b) = ab$. Define $\|a\|_+ = \|L_a\| = \sup\{\|ab\| \colon \|b\| \leqslant 1, b \in A\}$. Then

(1) $\|\cdot\|_+$ is a norm on A_+.

(2) $\forall a \in A$, $\|a\|_+ = \|a\|$.

(3) $(A_+, \|\cdot\|_+)$ is a C^*-algebra.

Proof.

(2) $\forall b \in A$, $\|ab\| \leqslant \|a\|\|b\| \implies \|a\|_+ \leqslant \|a\|$. $\|aa^*\| = \|a\|^2 = \|a\|\|a^*\| \implies \|a\|_+ = \|a\|$.

(1) Clearly, $\|\cdot\|_+$ is a seminorm. Suppose $a \in A_+, a \neq 0, \|a\|_+ = 0$. Let $a = b + \lambda 1_+$. By (2), $\lambda \neq 0$. $\forall c \in A$, $0 = ac = bc + \lambda c \implies (-\lambda^{-1}b)c = c$, that is, $e = -\lambda^{-1}b$ is a left identity in $A \implies e^*$ is a right identity in $A \implies A$ is unital, which is a contradiction.

(3)

Lemma/Exercise 1. $E =$ normed space, $E_0 \subset E$ vector subspace of codimension 1. If E_0 is complete, then so is E.

Lemma/Exercise 2. $A =$ Banach algebra equipped with an involution s.t. $\forall a \in A$, $\|a\|^2 \leqslant \|a^*a\| \implies A$ is a C^*-algebra.

By Lemma 1, A_+ is a Banach algebra. $\forall a \in A_+, b \in A$,

$$\|ab\|^2 = \|(ab)^*ab\| = \|b^*a^*ab\| \leqslant \|b^*\|\|a^*ab\| \leqslant \|b^*\|\|a^*a\|_+\|b\| = \|a^*a\|_+\|b\|^2 \implies \|a\|_+^2 \leqslant \|a^*a\|_+.$$

By Lemma 2, A_+ is a C^*-algebra. □

Lecture 12 (2024.11.22)

Spectral properties of C^*-algebras

$A = *$-algebra, $a \in A$.

Definition. $a \in A$ is selfadjoint (Hermitian) $\iff a^* = a$. a is normal $\iff aa^* = a^*a$. If A is unital, then $u \in A$ is unitary $\iff u \in A^\times$ and $u^{-1} = u^*$.

Observe.

(1) selfadjoint $\implies$ normal, unitary $\implies$ normal.

(2) $\forall a \in A$, a^*a is selfadjoint.

Notation. $A_{\mathrm{sa}} = \{a \in A \colon a = a^*\}$.

Example 1. $A = \mathbb{C}^X$ or $A = \ell^\infty(X)$ or $A = C_b(X)$.

(1) $f \in A$ is selfadjoint $\iff f(x) \in \mathbb{R}$, $\forall x \in X$.

(2) $f \in A$ is unitary $\iff |f(x)| = 1$, $\forall x \in X$.

Example/Exercise 2. $A = \mathscr{B}(H)$ (H = Hilbert space).

(1) $T \in \mathscr{B}(H)$ is selfadjoint $\iff \langle Tx|x\rangle \in \mathbb{R}$, $\forall x \in H$.

(2) $U \in \mathscr{B}(H)$ is unitary $\iff U$ is bijective and $\langle Ux|Uy\rangle = \langle x|y\rangle$ $(x, y \in H)$.

Proposition. $\forall a \in A$, $\exists$ a unique pair (b, c) of selfadjoint s.t. $a = b + \mathrm{i}c$.

Proof. We may take $\begin{cases} b = \frac{a+a^*}{2}, \\ c = \frac{a-a^*}{2\mathrm{i}}. \end{cases}$ And actually $\begin{cases} a = b + \mathrm{i}c, \\ a^* = b - \mathrm{i}c \end{cases}$ gives b, c. $\square$

Theorem 1. $A = C^*$-algebra, $a \in A$ normal $\implies r(a) = \|a\|$.

Proof. If $b \in A_{\mathrm{sa}}$, then $\|b^2\| = \|b\|^2$. Suppose $a \in A$ is normal.

$$\|a\|^4 = \|a^*a\|^2 = \|(a^*a)^2\| = \|a^*aa^*a\| = \|(a^*)^2a^2\| = \|(a^2)^*a^2\| = \|a^2\|^2 \implies \|a\|^2 = \|a^2\|.$$

Induction $\implies \|a^{2^n}\| = \|a\|^{2^n}$.

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|. \qquad \square$$

Corollary 1. $A = C^*$-algebra $\implies \forall a \in A$, $\|a\| = \sqrt{r(a^*a)}$.

Corollary 2. If A is a $*$-algebra, then there exists at most one norm on A, i.e. $\|a\| = \sqrt{r(a^*a)}$, making A into a C^*-algebra. Equivalently, every $*$-isomorphism between C^*-algebras is isometric.

Corollary 3. A = Banach $*$-algebra, $B = C^*$-algebra. Then every $*$-homomorphism $\varphi\colon A \to B$ is continuous, and $\|\varphi\| \leqslant 1$.

Proof. $\forall a \in A_{\mathrm{sa}}$, $\varphi(a) \in A_{\mathrm{sa}} \implies \|\varphi(a)\| = r(\varphi(a)) \leqslant r(a) \leqslant \|a\|$. $\forall a \in A$,

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(\underbrace{a^*a}_{\text{in } A_{\mathrm{sa}}})\| \leqslant \|a^*a\| \leqslant \|a\|^2. \qquad \square$$

Theorem 2. $A = C^*$-algebra, $a \in A_{\mathrm{sa}} \implies \sigma'_A(a) \subset \mathbb{R}$.

Proof. We may assume that A is unital (otherwise we consider the unitization). Let $\lambda \in \sigma(a)$, $\lambda = \alpha + \mathrm{i}\beta$ $(\alpha, \beta \in \mathbb{R})$. $\forall t \in \mathbb{R}$, $\lambda + \mathrm{i}t \in \sigma(a + \mathrm{i}t1) \implies \alpha^2 + t^2 + 2t\beta + \beta^2 = \alpha^2 + (t+\beta)^2 = |\lambda + \mathrm{i}t|^2 \leqslant \|a + \mathrm{i}t1\|^2 = \|(a - \mathrm{i}t1)(a + \mathrm{i}t1)\| = \|a^2 + t^21\| \leqslant \|a^2\| + t^2 \implies \alpha^2 + \beta^2 + 2\beta t \leqslant \|a^2\|$, $\forall t \in \mathbb{R} \implies \beta = 0$. $\square$

Definition. A $*$-algebra A is Hermitian if $\forall a \in A_{\mathrm{sa}}$, $\sigma'_A(a) \subset \mathbb{R}$.

Examples.

(1) All C^*-algebras.

(2) Every spectrally invariant $*$-subalgebra of a C^*-algebra. For example, $C^n[a,b]$ is Hermitian.

Exercise. Is $\mathscr{A}(\overline{\mathbb{D}})$ Hermitian?

Proposition. A = Hermitian $*$-algebra $\implies$ all characters of A are $*$-characters.

Proof. $\forall a \in A_{\mathrm{sa}}$, $\sigma'_A(a) \subset \mathbb{R}$. $\chi\colon A \to \mathbb{C}$ character $\implies$ $\sigma'_{\mathbb{C}}(\chi(a)) \subset \mathbb{R}$, that is, $\chi(a) \in \mathbb{R}$. $\forall a \in A$, $a = b + \mathrm{i}c$ $(b, c \in A_{\mathrm{sa}})$.

$$\chi(a^*) = \chi(b - \mathrm{i}c) = \chi(b) - \mathrm{i}\chi(c) = \overline{\chi(b) + \mathrm{i}\chi(c)} = \overline{\chi(a)}. \qquad \square$$

Theorem 3. A = commutative Banach $*$-algebra. TFAE:

(1) A is Hermitian.

(2) All characters of A are $*$-characters.

(3) $\Gamma_A\colon A \to C_0(\mathrm{Max}(A))$ is a $*$-homomorphism.

Moreover, if A is Hermitian, then $\mathrm{Im}\,\Gamma_A$ is dense in $C_0(\mathrm{Max}(A))$.

Proof.
(1) $\implies$ (2). See the previous proposition.
(2) $\implies$ (3). $\forall a \in A$, $\forall \chi \in \widehat{A}$

$$\widehat{a^*}(\chi) = \chi(a^*) = \overline{\chi(a)} = \overline{\widehat{a}(\chi)} = \widehat{a}^*(\chi).$$

(3) $\implies$ (1). $\forall a \in A_{\mathrm{sa}}$, $\sigma'_A(a) = \widehat{a}(\mathrm{Max}(A)) \cup \{0\} \subset \mathbb{R}$. Let $B = \mathrm{Im}\,\Gamma_A \subset C_0(\mathrm{Max}(A))$. $B_+ \subset C_0(\mathrm{Max}(A))_+ \cong C((\mathrm{Max}(A))_+) \cong C(\widehat{A}_+)$ satisfy the conditions of the Stone-Weierstrass theorem $\implies$ B_+ is dense in $C(\widehat{A}_+) \underset{\text{(exer)}}{\implies} B$ is dense in $C_0(\mathrm{Max}(A))$. $\square$

Theorem (Gelfand, Naimark)**.** A = commutative C^*-algebra $\implies$ $\Gamma_A\colon A \to C_0(\mathrm{Max}(A))$ is an isometric $*$-isomorphism.

Proof. We know: Γ_A is a $*$-homomorphism, $\mathrm{Im}\,\Gamma_A$ is dense in $C_0(\mathrm{Max}(A))$. We have to show that Γ_A is isometric.
$\forall a \in A_{\mathrm{sa}}$, $\|\Gamma_A(a)\| = r(a) = \|a\|$. $\forall a \in A$, $\|\Gamma_A(a)\|^2 = \|\Gamma_A(a)^*\Gamma_A(a)\| = \|\Gamma(a^*a)\| = \|a^*a\| = \|a\|^2$. $\square$

A category-theoretic interpretation

$\mathscr{A}, \mathscr{B}$ = categories. $F\colon \mathscr{A} \to \mathscr{B}$ covariant functor.

Definition. F is an equivalence if there is a covariant functor $G\colon \mathscr{B} \to \mathscr{A}$ s.t. $G \circ F = \mathbf{1}_{\mathscr{A}}, F \circ G = \mathbf{1}_{\mathscr{B}}$. ($G$ is a quasi-inverse of F)

Notation. CUC^* = the category of commutative unital C^*-algebra.
Morphisms in CUC^* = unital $*$-homomorphisms.

Theorem.

$$\mathsf{Comp}^{op} \underset{\mathrm{Max}}{\overset{C}{\rightleftarrows}} \mathsf{CUC}^* \quad \text{are equivalences.}$$

Moreover,

$$\mathrm{Max} \circ C \underset{\varepsilon}{\cong} \mathbf{1}_{\mathsf{Comp}^{op}},\ \varepsilon_X\colon X \xrightarrow{\sim} \mathrm{Max}(C(X))$$
$$C \circ \mathrm{Max} \underset{\Gamma_A}{\cong} \mathbf{1}_{\mathsf{CUC}^*}.$$

The Fourier transform on locally compact abelian groups

$G =$ Locally compact abelian (LCA) group (2nd countable).
$\widehat{G} = \mathrm{Hom}_{\mathsf{cont}}(G, \mathbb{T})$.
$\widehat{G}$ is an abelian group under the pointwise multiplication.

Definition. $\widehat{G}$ is the dual of G.

Definition. The Pontryagin topology on $\widehat{G}$ is the restriction to $\widehat{G}$ of the compact-open topology on $C(G)$.
Explicitly: $\chi \in \widehat{G}$, $K \subset G$ compact, $\varepsilon > 0$.

$$U_{K,\varepsilon}(\chi) = \left\{ \varphi \in \widehat{G} \colon \|\varphi - \chi\|_K < \varepsilon \right\},$$

where $\|f\|_K = \sup\limits_{x \in K} |f(x)|$, $f \in C(G)$.

$\{U_{K,\varepsilon}(\chi) \colon K \subset G \text{ compact}, \varepsilon > 0\}$ is a base of open neighborhoods of $\chi \in \widehat{G}$.

$$U_{K_1,\varepsilon_1}(\chi) \cap U_{K_2,\varepsilon_2} \supset U_{K,\varepsilon}(\chi),$$

where $K = K_1 \cup K_2$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Hence this family is a base (and not only a subbase) of neighborhoods of χ.

Proposition. $\widehat{G}$ is a topological group.

Proof (sketch). $\chi_1, \chi_2 \in \widehat{G}$.

$$U_{K,\varepsilon}(\chi_1) U_{K,\varepsilon}(\chi_2) \subset U_{K,2\varepsilon}(\chi_1\chi_2) \text{ (exer)} \implies \text{ the multiplication on } \widehat{G} \text{ is continuous.}$$

$$U_{K,\varepsilon}(\chi)^{-1} = U_{K,\varepsilon}(\chi^{-1}) \text{ (exer)} \implies \chi \mapsto \chi^{-1} = \overline{\chi} \text{ is continuous.}$$ □

Definition. The Fourier transform of $\nu \in M(G)$ is $\widehat{\nu} \colon \widehat{G} \to \mathbb{C}$, $\widehat{\nu}(\chi) = \int_G \chi \, \mathrm{d}\nu = \langle \nu, \chi \rangle$.

$\mu =$ Haar measure on G,

$$L^1(G) = L^1(G, \mu) \hookrightarrow M(G), \quad f \mapsto f \cdot \mu.$$

Definition. The Fourier transform of $f \in L^1(G)$ is $\widehat{f} = \widehat{f \cdot \mu}$.
Explicitly:

$$\widehat{f}(\chi) = \int_G f \chi \, \mathrm{d}\mu.$$

Observe. $|\widehat{\nu}(\chi)| \leqslant \int_G |\chi| \, \mathrm{d}|\nu| = \|\nu\| \implies \widehat{\nu}$ is bounded, and $\|\widehat{\nu}\|_\infty \leqslant \|\nu\| = |\nu|(G)$.

Proposition. $\widehat{\nu} \in C_b(\widehat{G})$.

Proof. Let $\chi_0 \in \widehat{G}$, $\varepsilon > 0$. There exists a compact set $K \subset G$ s.t. $|\nu|(G \backslash K) < \varepsilon$.
$\forall \chi \in U_{K,\varepsilon}(\chi_0)$,

$$\begin{aligned} |\widehat{\nu}(\chi) - \widehat{\nu}(\chi_0)| &\leqslant \int_G |\chi - \chi_0| \, \mathrm{d}|\nu_0| = \int_K |\chi - \chi_0| \, \mathrm{d}|\nu_0| + \int_{G \backslash K} |\chi - \chi_0| \, \mathrm{d}|\nu_0| \\ &\leqslant \varepsilon \|\nu\| + 2\varepsilon = (\|\nu\| + 2)\varepsilon \implies \widehat{\nu} \text{ is continuous.} \end{aligned}$$ □

Notation. $\mathscr{F}_G \colon M(G) \to C_b(\widehat{G})$, $\nu \mapsto \widehat{\nu}$.

Definition. $\mathscr{F}_G$ is the Fourier transform on G.

Observe. $\mathscr{F}_G$ is a bounded linear map.

Lecture 13 (2024.11.29)

Example. δ_x = Dirac measure concentrated at $x \in G$. Then $\widehat{\delta}_x(\chi) = \chi(x)$, that is, $\widehat{\delta}_x = \varepsilon_x$ (evaluation at x). In particular, $\widehat{\delta}_e = 1$.

Proposition. $\mathscr{F}_G\colon M(G) \to C_b(G)$ is a unital $*$-algebra homomorphism.

Proof. Note that $\Delta\chi(x,y) = \chi(xy) = \chi(x)\chi(y) = (\chi \otimes \chi)(x,y)$,

$$\begin{aligned}
\widehat{\nu_1 * \nu_2}(\chi) &= \langle \nu_1 * \nu_2, \chi\rangle \\
&= \langle \nu_1 \otimes \nu_2, \Delta\chi\rangle \\
&= \langle \nu_1 \otimes \nu_2, \chi \otimes \chi\rangle \\
&= \langle \nu_1, \chi\rangle\langle \nu_2, \chi\rangle \\
&= \widehat{\nu}_1(\chi)\widehat{\nu}_2(\chi); \\
\widehat{\nu^*}(\chi) &= \langle \nu^*, \chi\rangle \\
&= \overline{\langle \nu, \overline{S\chi}\rangle} \\
&= \overline{\langle \nu, \chi\rangle} \\
&= \overline{\widehat{\nu}(\chi)}.
\end{aligned}$$

$\square$

Now we turn to the Fourier transform of $f \in L^1(G)$. Consider $\chi \in \widehat{G}$, define $\widetilde{\chi}\colon M(G) \to \mathbb{C}$, $\widetilde{\chi}(\nu) = \widehat{\nu}(\chi)$. $\widetilde{\chi}$ is a unital $*$-character of $M(G)$ ($\widetilde{\chi}(\delta_x) = \widehat{\delta}_x(\chi) = \chi(x) \implies \widetilde{\chi}(\delta_e) = \chi(e) = 1$).

Observe. $\widetilde{\chi}|_{L^1(G)} \neq 0$ (because $L^\infty \underset{\sim}{\to} (L^1)^*$).

Notation. $\gamma\colon \widehat{G} \to \widehat{L^1(G)}$, $\chi \mapsto \widetilde{\chi}|_{L^1(G)}$.

Theorem 1. γ is bijective.

Lemma 1. G = locally compact group, $f \in L^1(G)$. The map $G \to L^1(G), x \in G \mapsto L_x f$, is continuous.

Proof. True if $f \in C_c(G)$ (exer) (Hint: We "almost" proved this before).
Let $f \in L^1(G), \varepsilon > 0$. Choose $g \in C_c(G)$ s.t. $\|f - g\|_1 < \varepsilon$. $\forall x \in G$, there exists a neighborhood $U \ni x$ s.t. $y \in U \implies \|L_x f - L_y f\|_1 \leqslant \|L_x(f-g)\|_1 + \|L_x g - L_y g\|_1 + \|L_y(g-f)\|_1 < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$. $\square$

Lemma 2. A = commutative Banach algebra, $I \subset A$ is a closed ideal. Let

$$\widehat{A}_I = \left\{\chi \in \widehat{A}\colon \chi|_I \neq 0\right\}.$$

Then $\widehat{A}_I$ is an open subset of $\widehat{A}$, and $\widehat{A}_I \xrightarrow{\alpha} \widehat{I}, \chi \mapsto \chi|_I$, is a homeomorphism.

Proof. Let $\chi \in \widehat{A}_I$, then $\forall b \in I, \chi|_I = \varphi$.

$$\chi(a) = \frac{\varphi(ab)}{\varphi(b)} \quad \text{whenever } \varphi(b) \neq 0. \tag{11}$$

(11) $\implies \alpha$ is injective.
Let $\varphi \in \widehat{I}$. Choose $b \in I$ s.t. $\varphi(b) = 1$. Define $\chi\colon A \to \mathbb{C}$, $\chi(a) = \varphi(ab)$.

$$\left.\begin{aligned}
\chi(a_1)\chi(a_2) &= \varphi(a_1 b)\varphi(a_2 b) \\
&= \varphi(a_1 b a_2 b) \\
&= \varphi(a_1 a_2 b)\varphi(b) \\
&= \chi(a_1 a_2).
\end{aligned}\right\} \implies \chi \text{ is a character of } A, \text{ and } \chi|_I = \varphi. \implies \alpha \text{ is bijective.}$$

Exercise. Show that $\widehat{A}_I$ is open, α and α^{-1} are continuous (Hint: use (11)). $\square$

Let G be a LCA group, $A = M(G), I = L^1(G)$.

$$\widehat{G} \xrightarrow{\beta} \widehat{A}_I \xrightarrow{\alpha} \widehat{I}, \quad \beta(\chi) = \widetilde{\chi}, \widetilde{\chi}(\nu) = \widehat{\nu}(\chi), \alpha\beta = \gamma.$$

Lemma 3. β is bijective.

Proof. $\forall \chi \in \widehat{G}, \chi(x) = \widetilde{\chi}(\delta_x) \implies \beta$ is injective.
Take $\varphi \in \widehat{A}_I$. Define $\chi\colon G \to \mathbb{C}, \chi(x) = \varphi(\delta_x)$. $\chi(xy) = \varphi(\delta_{xy}) = \varphi(\delta_x * \delta_y) = \chi(x)\chi(y)$; $\chi(e) = 1$. $\chi(x)\chi(x^{-1}) = \chi(e) = 1 \implies \chi(x) \neq 0, \forall x \implies \chi\colon G \to \mathbb{C}^\times$ is a character.

$$|\chi(x)| \leqslant \|\delta_x\| = 1, \forall x \implies \left|\frac{1}{\chi(x)}\right| = |\chi(x^{-1})| \leqslant 1 \implies |\chi(x)| = 1.$$

Choose $h \in L^1(G)$ s.t. $\varphi(h) = 1 \implies \chi(x) = \varphi(\delta_x)\varphi(h) = \varphi(\delta_x * h) = \varphi(L_x h) \overset{\text{Lemma 1}}{\implies} \chi$ is continuous $\implies \chi \in \widehat{G}$.
We want: $\widetilde{\chi} = \varphi$. Lemma 2 $\implies$ it suffices to show that $\widetilde{\chi}(f) = \varphi(f), \forall f \in L^1(G)$. $\exists g \in L^\infty(G)$ s.t. $\varphi(f) = \int_G fg \,\mathrm{d}\mu, \forall f \in L^1(G)$.

$$\begin{aligned}\varphi(f) = \varphi(f)\varphi(h) = \varphi(f * h) &= \int (f * h) g \,\mathrm{d}\mu \\ &= \int\int f(y) h(y^{-1}x) g(x) \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(x) \\ &= \int f(y) \left(\int (L_y h)(x) g(x) \,\mathrm{d}\mu(x)\right) \mathrm{d}\mu(y) \\ &= \int f(y)\chi(y) \,\mathrm{d}\mu(y) = \widetilde{\chi}(f)\end{aligned} {}^{3} \implies \beta \text{ is bijection.}$$ □

Proof of Theorem 1. It follows from Lemma 2 & Lemma 3. □

Corollary. $L^1(G)$ is Hermitian.

Proof. $\forall \chi \in \widehat{G}, \widetilde{\chi}$ is a $*$-character of $L^1(G) \implies$ all characters of $L^1(G)$ are $*$-characters $\implies L^1(G)$ is Hermitian. □

Theorem 2. $\gamma\colon \widehat{G} \to \widehat{L^1(G)}$ is a homeomorphism.

Lemma 1. γ is continuous.

Proof. It suffices to show that $\chi \mapsto \widetilde{\chi}(f) = \widehat{f}(\chi)$ is continuous, $\forall f \in L^1(G)$. This is true as we proved before. □

Lemma 2. E = normed space, $B \subset E^*$ bounded[4]. Then $(B, \mathrm{wk}^*) \times E \to \mathbb{C}$, $(f, x) \mapsto f(x)$ is continuous.

Proof. Let $C = \sup_{f \in B} \|f\|$. Let $f, f_0 \in B$, $x, x_0 \in E$. Then

$$\begin{aligned}|f(x) - f_0(x_0)| &\leqslant |f(x - x_0)| + |f(x_0) - f_0(x_0)| \\ &\leqslant C\|x - x_0\| + \|f - f_0\|_{x_0}.\end{aligned}$$ □

Notation. $\widehat{G}_W = (\widehat{G};$ Gelfand topology induced from $\widehat{L^1(G)})$.

Lemma 3. $\widehat{G}_W \times G \to \mathbb{T}$, $(\chi, x) \mapsto \chi(x)$, is continuous.

Proof. $\forall \chi \in \widehat{G}, f \in L^1(G), x \in G, \widetilde{\chi}(L_x f) = \widetilde{\chi}(\delta_x * f) = \chi(x)\widetilde{\chi}(f) \implies \chi(x) = \frac{\widetilde{\chi}(L_x f)}{\widetilde{\chi}(f)}$ if $\widetilde{\chi}(f) \neq 0$.
Let $\chi_0 \in \widehat{G}$. Choose $f \in L^1(G)$ s.t. $\widetilde{\chi}_0(f) \neq 0$, there exists a neighborhood $U \ni \chi_0$ in $\widehat{G}_W$ s.t. $\forall \chi \in U, \widetilde{\chi}(f) \neq 0$.
By (11), it suffices to show that $U \times G \to \mathbb{C}$, $(\chi, x) \mapsto \widetilde{\chi}(L_x f)$ is continuous.

$$\begin{array}{ccccc} U \times G & \xrightarrow{\text{cont}} & \widehat{L^1(G)} \times L^1(G) & \xrightarrow[\text{cont (L 2)}]{\langle \cdot,\cdot \rangle} & \mathbb{C} \\ & (\chi, x) \mapsto (\widetilde{\chi}, L_x f), & & (\varphi, f) \mapsto \varphi(f). & \end{array}$$ □

[3] $\int (L_y h)(x) g(x) \,\mathrm{d}\mu(x) = \varphi(L_y h) = \varphi(\delta_y * h) = \varphi(\delta_y)\varphi(h) = \chi(y)$.
[4] This condition is essential.

Lemma 4. X, Y, Z topological spaces, $F\colon X \times Y \to Z$ continuous. $Z_0 \subset Z$ open, $Y_0 \subset Y$ compact. Then $\{x \in X\colon F(x,y) \in Z_0, \forall y \in Y_0\}$ is open in X.

Lemma 5. γ is open.

Proof. $\chi \in \widehat{G}, K \subset G$ compact, $\varepsilon > 0$.

$$U_{K,\varepsilon}(\chi) = \left\{\varphi \in \widehat{G}\colon |\varphi(x) - \chi(x)| < \varepsilon, \forall x \in K\right\}$$

(a basic open neighborhood of χ).
We want: $U_{K,\varepsilon}(\chi)$ is open in $\widehat{G}_W$. Lemma 3: $(\varphi, x) \mapsto |\varphi(x) - \chi(x)|$ is continuous on $\widehat{G}_W \times G$. Lemma 2 $\implies U_{K,\varepsilon}(\chi)$ is open in $\widehat{G}_W$. ☐

Proof of Theorem 2. It follows from Lemma 1 & Lemma 5. ☐

Corollary. $\widehat{G}$ is locally compact.

Theorem 3. $\mathscr{F}(L^1(G)) \subset C_0(\widehat{G})$, and the following diagram commutes:

$$\begin{tikzcd} & C_0(\widehat{G}) \\ L^1(G) \arrow[ur, "\mathscr{F}"] \arrow[dr, "\Gamma"'] & \\ & C_0(\widehat{L^1(G)}) \arrow[uu, "\sim", "\gamma^\bullet"'] \end{tikzcd}$$

Proof. $\forall f \in L^1(G)$,

$$\begin{aligned}(\gamma^\bullet(\Gamma f))(\chi) &= (\Gamma f \circ \gamma)(\chi) \\ &= (\Gamma f)(\gamma(\chi)) \\ &= (\Gamma f)(\widetilde{\chi}) \\ &= \widetilde{\chi}(f) = (\mathscr{F} f)(\chi).\end{aligned}$$

☐

Corollary (Density theorem)**.** $\mathscr{F}(L^1(G))$ is dense in $C_0(\widehat{G})$.

Proof. $L^1(G)$ is Hermitian. ☐

Lecture 14 (2024.12.06)

Proposition. G is 2nd countable $\implies$ so is $\widehat{G}$.

Proof. G is 2nd countable $\implies$ $L^1(G)$ is separable (exercise). E = separable Banach space $\implies$ $(\mathbb{B}_{E^*}, \mathrm{wk}^*)$ is compact and metrizable $\implies$ separable and metrizable $\implies$ 2nd countable $\implies$ $\widehat{G} \hookrightarrow (\mathbb{B}_{L^1(G)^*}, \mathrm{wk}^*)$ is 2nd countable. $\square$

Definition. A = $*$-algebra, H = Hilbert space. A $*$-representation of A is a $*$-homomorphism $\pi\colon A \to \mathscr{B}(H)$. π is faithful if $\operatorname{Ker}\pi = \{e\}$.

Lemma 1. A = commutative Banach $*$-algebra. Suppose A has a faithful $*$-representation on a Hilbert space. Then

$$\Gamma_A\colon A \to C_0(\operatorname{Max} A) \text{ is injective.}$$

Proof. $\pi\colon A \to \mathscr{B}(H)$ faithful $*$-representation; $B = \overline{\pi(A)} \subset \mathscr{B}(H)$, B is a commutative C^*-algebra $\implies B \cong C_0(X) \implies$ characters of B separate the points of B $\implies$ characters of A separate the points of A (because π is injective) $\iff \Gamma_A$ is injective. $\square$

Lemma 2. G = LCA group (2nd countable). Then

(1) $f \in L^1(G), g \in L^2(G) \implies f * g$ is defined a.e., $f * g \in L^2(G)$, and $\|f * g\|_2 \leqslant \|f\|_1 \|g\|_2$.

(2) $\lambda\colon L^1(G) \xrightarrow{\sim} \mathscr{B}(L^2(G)), \lambda(f)g = f * g$, is a faithful $*$-representation.

Proof.

(1) Exercise.

(2) λ is a $*$-representation (exercise). Let (e_α) be an a.i. of $L^1(G)$ contained in $C_c(G)$. Let $f \in \operatorname{Ker}\lambda$. Then:

$$0 = \lambda(f)e_\alpha = f * e_\alpha \to f \implies f = 0. \qquad \square$$

Theorem (Uniqueness theorem for $\mathscr{F}$). $\mathscr{F}\colon L^1(G) \to C_0(\widehat{G})$ is injective.

Proof. It follows from Lemma 1 & 2. $\square$

Corollary. $\widehat{G}$ separates the points of G, that is, $\forall e \neq x \in G$, $\exists \chi \in \widehat{G}$ s.t. $\chi(x) \neq 1$.

Proof. $\exists f \in C_c(G)$ s.t. $f(x^{-1}) \neq f(e) \implies L_x f \neq f \implies \exists \chi \in \widehat{G}$ s.t. $\chi(x)\widetilde{\chi}(f) = \widetilde{\chi}(\delta_x * f) = \widetilde{\chi}(L_x f) \neq \widetilde{\chi}(f) \implies \chi(x) \neq 1$. $\square$

Positive definite functions. Bochner's theorem

A = $*$-algebra, $\omega\colon A \to \mathbb{C}$ linear.

Definition. ω is positive ($\omega \geqslant 0$) if $\omega(a^*a) \geqslant 0$, $\forall a \in A$.

Notation. A = Banach $*$-algebra.

$$A^*_{\mathrm{pos}} = \{\omega \in A^* : \omega \geqslant 0\}$$

is a convex cone in A^*.

Example 1. $\chi\colon A \to \mathbb{C}$ $*$-character.

$$\chi(a^*a) = |\chi(a)|^2 \geqslant 0 \implies \chi \geqslant 0.$$

Example 2. X = locally compact Hausdorff space. There is a bijection

$$C_0(X)^*_{\mathrm{pos}} \cong \{\text{Finite positive Radon measures on } X\} = M(X)_{\mathrm{pos}}.$$

Indeed: $\mu \in M(X)$; $I_\mu \in C_0(X)^\times, I_\mu(f) = \int f \, d\mu$.

$$I_\mu \geqslant 0 \iff \int_X |f|^2 \, d\mu \geqslant 0 \quad \forall f \in C_0(X) \iff \mu \geqslant 0.$$

Example 3. H = Hilbert space, $A \subset \mathscr{B}(H)$ $*$-subalgebra. $v \in H$, $\omega_v\colon A \to \mathbb{C}, \omega_v(T) = \langle Tv|v\rangle$.

$$\omega_v(T^*T) = \|Tv\|^2 \geqslant 0 \implies \omega_v = 0.$$

Example/Exercise 4. $A = \mathbb{C}G$ or $A = \ell^1(G)$ (G = a group) or $A = (C_c(G);\text{convolution product}) \subset L^1(G)$ (G= LC group). $\omega\colon A \to \mathbb{C}, \omega(f) = f(e)$. Prove: $\omega \geqslant 0$.

Notation. A = $*$-algebra, $\omega\colon A \to \mathbb{C}$ positive linear functional. $\forall a, b \in A. \underline{\langle a|b\rangle_\omega} = \omega(b^*a)$. $\langle\cdot|\cdot\rangle_\omega$ is a sesquilinear form on A; $\langle a|a\rangle_\omega \in \mathbb{R}, \forall a \underset{\text{(exer)}}{\implies} \langle\cdot|\cdot\rangle_\omega$ is Hermitian, that is, $\langle b|a\rangle_\omega = \overline{\langle a|b\rangle_\omega}, \forall a, b$. Hence $\langle\cdot|\cdot\rangle_\omega$ is a semi-inner product on A.

Proposition (Cauchy–Bunyakovsky–Schwarz inequality)**.** $|\omega(b^*a)|^2 \leqslant \omega(a^*a)\omega(b^*b) \quad (a, b \in A)$.

G = a group; $\varphi\colon G \to \mathbb{C}$.
$\forall n \in \mathbb{N}, \forall x = (x_1, \ldots, x_n) \in G^n$, define $\Phi_x\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $\Phi_x(u, v) = \sum\limits_{i,j} \varphi(x_j^{-1}x_i)u_i\overline{v}_j$. This is a sesquilinear form on $\mathbb{C}^n$.

Definition. φ is <u>positive definite</u> if $\forall n \in \mathbb{N}, \forall x \in G^n, \Phi_x$ is positive definite (that is, $\Phi_x(u, u) \geqslant 0, \forall u \in \mathbb{C}^n$).

Observation. Suppose φ is positive definite.

(1) $\varphi(e) \geqslant 0$ (let $n = 1$).

(2) Φ_x is a semi-inner product on $\mathbb{C}^n$. In particular, $\Phi_x(v, u) = \overline{\Phi_x(u, v)}, \forall u, v$.

(3) (Cauchy–Bunyakovsky–Schwarz inequality). $|\Phi_x(u, v)|^2 \leqslant \Phi_x(u, u)\Phi_x(v, v) \quad (u, v \in \mathbb{C}^n)$.

(4) Let $n = 2, x = (e, s) \in G^2, s \in G$. $u = (1, 0), v = (0, 1)$.

$$\Phi_x(u, v) = u \begin{pmatrix} \varphi(e) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(e) \end{pmatrix} v^T.$$

$$(2) \implies \varphi(s^{-1}) = \overline{\varphi(s)};\ (3) \implies |\varphi(s)|^2 \leqslant \varphi(e)^2, \text{that is, } |\varphi(s)| \leqslant \varphi(e).$$

In fact, φ is bounded.

Examples.

(1) $\chi\colon G \to \mathbb{C}$ unitary character.

$$\sum_{i,j} \chi(x_j^{-1}x_i)u_i\overline{u}_j = \sum_{i,j} \overline{\chi(x_j)}\chi(x_i)u_i\overline{u}_j = \left|\sum \chi(x_i)u_i\right|^2 \geqslant 0 \implies \chi \text{ is positive definite.}$$

(2) H = Hilbert space; $U(H)$ = {unitary operators on H}. $\pi\colon G \to U(H)$ unitary representation (that is, a group homomorphism). $\forall v \in H, \pi_v\colon G \to \mathbb{C}, \pi_v(x) = \langle \pi(x)v|v\rangle$.

Exercise. π_v is positive definite.

Notation. $\mathcal{P}(G)$ = {positive definite functions on G}.

$\mathbb{C}G$ = group algebra of G.

$$\begin{aligned} \mathbb{C}G &\cong (\{\text{finitely supported functions } G \to \mathbb{C}\}, *) \\ &= \operatorname{span}\{\delta_x : x \in G\};\ \delta_x * \delta_y = \delta_{xy};\ \delta_x^* = \delta_{x^{-1}}. \end{aligned}$$

$\mathbb{C}G$ is a $*$-algebra.

Observe. There is a vector space isomorphism

$$\begin{aligned} \alpha\colon \operatorname{Fun}(G) &\xrightarrow{\sim} (\mathbb{C}G)^* \quad \text{(algebraic dual)} \\ \varphi &\mapsto \alpha_\varphi, \quad \alpha_\varphi(\delta_x) = \varphi(x). \end{aligned}$$

Proposition. $\alpha_\varphi \geqslant 0 \iff \varphi$ is positive definite.

Proof. $f \in \mathbb{C}G$, $f = \sum_{i=1}^{n} c_i \delta_{x_i}$, $f^* = \sum_j \bar{c}_j \delta_{x_j^{-1}}$.

$$\alpha_\varphi(f^* * f) = \alpha_\varphi\left(\sum_{i,j} c_i \bar{c}_j \delta_{x_j^{-1} x_i}\right) = \sum_{i,j} \varphi(x_j^{-1} x_i) c_i \bar{c}_j. \qquad \square$$

Remark. φ is positive definite $\iff$ Φ_x is positive definite for every n-tuple $(x_1, \ldots, x_n)$ of pairwise distinct elements of G.

Proposition. $G =$ a finite abelian group. Then

$$\mathcal{P}(G) = \left\{ \sum_{\chi \in \widehat{G}} c_\chi \chi \colon c_\chi \geqslant 0 \right\}.$$

Proof.

$$\mathbb{C}G \xrightarrow[\mathscr{F}]{\sim} \operatorname{Fun}(\widehat{G}) \qquad *\text{-isomorphism}$$

$$\begin{array}{ccc} \operatorname{Fun}(\widehat{G})^*_{\text{pos}} & \xrightarrow[\mathscr{F}^*]{\sim} & (\mathbb{C}G)^*_{\text{pos}} \\ \| & & \| \\ M(\widehat{G})_{\text{pos}} & \xrightarrow{\sim} & \mathcal{P}(G) \\ \| & & \\ \left\{\sum_{\chi \in \widehat{G}} c_\chi \delta_\chi \colon c_\chi \geqslant 0\right\} & & \end{array} \qquad \mathscr{F}^*(\delta_\chi) = \chi \text{ (exer)} \qquad \square$$

Exercise. Give a proof which avoids using $\mathscr{F}$.

$G =$ locally compact group (2nd countable); $\mu =$ Haar measure.
Recall: there is an isometric isomorphism of Banach spaces

$$\alpha \colon L^\infty(G) \xrightarrow{\sim} L^1(G)^*, \quad \varphi \mapsto \alpha_\varphi, \quad \alpha_\varphi(f) = \int_G f\varphi \, \mathrm{d}\mu.$$

Definition. $\varphi \in L^\infty(G)$ is of positive type if $\alpha_\varphi \geqslant 0$.

Notation.

$$\mathcal{P}^\infty(G) = \{\varphi \in L^\infty(G) \colon \varphi \text{ is of positive type}\},$$
$$\mathcal{P}(G) = \{\text{continuous positive definite functions on } G\}.$$

Theorem. Let $\varphi \in C_b(G)$. Then φ is of positive type $\iff$ φ is positive definite.

Lemma/Exercise 1. $\forall \varphi \in L^\infty(G)$, $f, g \in L^1(G)$. Then

$$\alpha_\varphi(g^* * f) = \iint_{G \times G} \varphi(y^{-1}x) f(x) \overline{g(y)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y), \; g \mapsto g^* \text{ involution on } L^1(G), g^*(x) = \overline{g(x^{-1})} \Delta(x^{-1}).$$

Lemma/Exercise 2. $\beta =$ a base of relative compact symmetric neighborhoods of $e \in G$. $(u_V)_{V \in \beta} =$ a Dirac net in $L^1(G)$. Then

(1) $\forall \varphi \in C_b(G), \int_G u_V \varphi \, \mathrm{d}\mu \to \varphi(e)$. In particular, $u_V \xrightarrow{\text{wk}^*} \delta_e$ in $M(G)$.

(2) $(u_V \otimes u_V)_{V \in \beta}$ is a Dirac net in $L^1(G \times G)$, $(u_V \otimes u_V) \colon (x, y) \mapsto u_V(x)_V(y)$.

Proof of Theorem. ($\Longrightarrow$) Suppose φ is of positive type.

$$x = (x_1, \dots, x_n) \in G^n \quad t = (t_1, \dots, t_n) \in \mathbb{C}^n.$$

Let $(u_V)_{V\in\beta}$ be a Dirac net in $L^1(G)$. Let $f_V = \sum t_i L_{x_i} u_V$ $(v \in \beta)$, $f_V^* = \sum\limits_j \bar{t}_j (L_{x_i} u_V)^*$.

$$\begin{aligned}
0 &\leqslant \alpha_\varphi(f_V^* * f_V) \\
&\overset{\text{Lemma 1}}{=\!=\!=} \sum_{i,j} \bar{t}_j t_i \iint\limits_{G\times G} \varphi(y^{-1}x) u_V(x_i^{-1}x) u_V(x_j^{-1}y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \sum_{i,j} \bar{t}_j t_i \iint\limits_{G\times G} \varphi\left((x_j y)^{-1}(x_i x)\right) u_V(x) u_V(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\to \sum_{i,j} \varphi(x_j^{-1}x_i) t_i \bar{t}_j \implies \varphi \text{ is positive definite.}
\end{aligned}$$

($\Longleftarrow$) Suppose φ is positive definite. It suffices to show that $\alpha_\varphi(f^* * f) \geqslant 0, \forall f \in C_c(G)$. Take $f \in C_c(G)$; $K = \operatorname{supp} f$. $F(x,y) = \varphi(y^{-1}x) f(x) \overline{f(y)}$. $F \in C_c(G \times G), \operatorname{supp} F \subset K \times K$.

<u>Exercise</u>. $\forall \varepsilon > 0$, there exists disjoint Borel sets $E_1, \dots, E_n$ and $x_i \in E_i$ $(i = 1, \dots, n)$ s.t. $K = \bigsqcup\limits_{i=1}^n E_i$ and

$$\left\| F - \sum_{i,j} F(x_i, x_j) \chi_{E_i \times E_j} \right\|_\infty < \varepsilon \text{ (Hint: uniform continuity of } F\text{).}$$

Denote $\sum\limits_{i,j} F(x_i, x_j) \chi_{E_i \times E_j}$ by G_ε, then

$$\left| \int F \,\mathrm{d}(\mu \otimes \mu) - \int G_\varepsilon \,\mathrm{d}(\mu \otimes \mu) \right| \leqslant \int_{K\times K} |F - G_\varepsilon| \,\mathrm{d}(\mu \otimes \mu) < \varepsilon \mu(K)^2.$$

$$\begin{aligned}
\int_{K\times K} G_\varepsilon \,\mathrm{d}(\mu \otimes \mu) &= \sum_{i,j} F(x_i, x_j) \mu(E_i) \mu(E_j) \\
&= \sum_{i,j} \varphi(x_j^{-1} x_i) f(x_i) \mu(E_i) \overline{f(x_j) \mu(E_j)} \geqslant \\
\implies \int F \,\mathrm{d}(\mu \otimes \mu) &\geqslant 0, \text{ that is, } \varphi \text{ is of positive type.}
\end{aligned}$$

$\square$

Lecture 15 (2024.12.13)

Recall:
$A = *$-algebra, $\omega\colon A \to \mathbb{C}$ is positive if $\omega(a^*a) \geqslant 0$, $\forall a \in A$. Question: $L^1(G)^*_{\text{pos}} = ?$
We have the following diagram

$$\begin{tikzcd} L^\infty(G) \arrow[r, "\sim", "\alpha"'] & L^1(G)^* \\ \mathcal{P}^\infty(G) \arrow[u, hook] \arrow[r, "\sim"] & L^1(G)^*_{\text{pos}} \arrow[u, hook] \end{tikzcd}$$

Theorem. If $\varphi \in C_b(G)$, then $\varphi \in \mathcal{P}^\infty(G) \iff \varphi \in \mathcal{P}(G)$ (positive definite), that is,

$$\sum_{i,j=1}^{n} \varphi(x_j^{-1}x_i)u_i\overline{u}_j \geqslant 0 \quad \forall (x_1, \ldots, x_n) \in G^n \quad \forall (u_1, \ldots, u_n) \in \mathbb{C}^n.$$

$\alpha\colon \varphi \mapsto \alpha_\varphi$, $\alpha_\varphi(f) = \int f\varphi \, \mathrm{d}\mu$.

$$\alpha_\varphi \geqslant 0 \iff \forall f \in L^1, \int (f^* * f)\varphi \, \mathrm{d}\mu \geqslant 0 \iff \int \varphi(y^{-1}x)f(x)\overline{f(y)} \, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \geqslant 0, \forall f \in L^1.$$

Bochner's theorem

$G = $ LCA group (2nd countable), $\mu = $ Haar measure on G.

Notation. $\forall x \in G, \varepsilon_x\colon \widehat{G} \to \mathbb{T}, \varepsilon_x(\chi) = \chi(x)$.

Observe. $\varepsilon_x \in \widehat{\widehat{G}}$.

Consider $i_G\colon G \to \widehat{\widehat{G}}, x \in G \mapsto \varepsilon_x$.

Proposition. i_G is continuous.

Lemma. $X, Y = $ topological spaces, $F\colon X \times Y \to \mathbb{C}$ continuous. Define $\varphi\colon X \to C(Y), \varphi(x)(y) = F(x, y)$. Then φ is continuous.

Proof. Let $x_0 \in X$; $K \subset Y$ compact; $\varepsilon > 0$. Let $U = \{x \in X : |F(x, y) - F(x_0, y)| < \varepsilon \ \forall y \in K\}$. U is open; $x_0 \in U$. $\forall x \in U$, $\|\varphi(x) - \varphi(x_0)\|_K < \varepsilon \implies \varphi$ is continuous. $\square$

Exercise. If Y is locally compact, then

$$\begin{aligned} C(X \times Y) &\to C(X, C(Y)), \text{ is a topological isomorphism} \\ F &\mapsto \varphi \end{aligned}$$

Proof of Proposition. Apply Lemma to

$$\begin{aligned} G \times \widehat{G} &\to \mathbb{C} \\ (x, \chi) &\mapsto \chi(x). \end{aligned}$$

$\square$

$$\begin{tikzcd} M(\widehat{G}) \arrow[r, "\mathscr{F}_{\widehat{G}}"] \arrow[rr, bend right, "\check{\mathscr{F}}"'] & C_b(\widehat{\widehat{G}}) \arrow[r, "i_G^\bullet", "f \mapsto f \circ i_G"'] & C_b(G) \end{tikzcd}$$

Definition. $\check{\mathscr{F}} = i_G^\bullet \circ \mathscr{F}_{\widehat{G}}\colon M(\widehat{G}) \to C_b(G)$ is the dual Fourier transform (inverse Fourier transform; Fourier cotransform).

$$\nu \in M(\widehat{G}), \quad \check{\nu} = \check{\mathscr{F}}(\nu)\colon G \to \mathbb{C} \text{ is the dual Fourier transform of } \nu.$$

Explicitly: $\check{\nu}(x) = \int_{\widehat{G}} \chi(x) \, \mathrm{d}\nu(\chi)$.

Proposition.

$$L^1(G) \xrightarrow{\quad\mathscr{F}\quad} C_0(\widehat{G})$$

$$\begin{tikzcd}
C_0(\widehat{G})^* \arrow[r, "\mathscr{F}^*"] \arrow[d, equal] & L^1(G)^* \arrow[d, equal] \\
M(\widehat{G}) \arrow[r] \arrow[dr, "\check{\mathscr{F}}"'] & L^\infty(G) \\
& C_b(G) \arrow[u, hook]
\end{tikzcd}$$

The diagram commutes.

Proof. Exercise. $\square$

Corollary. $\check{\mathscr{F}}$ is injective.

Proof. $\mathscr{F}(L^1(G)) \subset C_0(\widehat{G})$ dense $\implies$ $\mathscr{F}^*$ is injective $\implies$ so is $\check{\mathscr{F}}$. $\square$

Theorem 1 (Bochner's theorem; Weil, Raikov, Povzner)**.** $\check{\mathscr{F}}$ maps $M(\widehat{G})_{\mathrm{pos}}$ bijectively onto $\mathcal{P}^\infty(G)$.

Corollary. Every function on G of positive type is a.e. equal to a (unique) continuous positive definite function. Hence $\mathcal{P}^\infty(G) \cong \mathcal{P}(G)$.

Fact. This holds for nonabelian LC groups.

Theorem 2 (Generalized Bochner's theorem)**.** A = Hermitian commutative Banach algebra with a b.a.i. $\Gamma_A\colon A \to C_0(\widehat{A})$ is the Gelfand transform. Then Γ_A^* maps $M(\widehat{A})_{\mathrm{pos}}$ bijectively onto A^*_{pos}.

Theorem 2 $\implies$ *Theorem 1.* Let $A = L^1(G)$, $\Gamma_A = \mathscr{F}\colon L^1(G) \to C_0(\widehat{G})$, $\Gamma^* \cong \mathscr{F}^* \cong \check{\mathscr{F}}$; $M(\widehat{A})_{\mathrm{pos}} \cong M(\widehat{G})_{\mathrm{pos}}$, $A^*_{\mathrm{pos}} \cong \mathcal{P}^\infty(G)$. $\square$

Proof of Theorem 2. $\Gamma\colon A \to C_0(\widehat{A})$ is a $*$-homomorphism $\implies$ $\Gamma^*(M(\widehat{A})_{\mathrm{pos}}) \subset A^*_{\mathrm{pos}}$. $\Gamma(A)$ is dense in $C_0(\widehat{A})$ $\implies$ Γ^* is injective.
Let $\omega \in A^*_{\mathrm{pos}}$. Recall: $|\omega(b^*a)|^2 \leqslant \omega(a^*a)\omega(b^*b)$ $\forall a, b$.
Let (e_λ) be a b.a.i. in A.

$$|\omega(e_\lambda^* a)|^2 \leqslant \omega(e_\lambda^* e_\lambda)\omega(a^*a) \leqslant C\omega(a^*a) = C\omega(h),$$

where $C = \|\omega\| \sup\limits_\lambda \|e_\lambda\|^2$. We may assume that $C \geqslant 1$.

$$\text{Take } \lim_\lambda \implies \left.\begin{aligned} |\omega(a)| &\leqslant C^{1/2}\omega(h)^{1/2}; \\ \omega(h) &\leqslant C^{1/2}\omega(h^2)^{1/2} \end{aligned}\right\}$$

$$\implies \begin{cases} |\omega(a)| \leqslant C^{\frac12+\frac14}\omega(h^2)^{\frac14} \\ \qquad\quad \leqslant C^{\frac12+\frac14+\frac18}\omega(h^4)^{\frac18} \\ \qquad\quad \leqslant \cdots \\ \qquad\quad \leqslant C^{\frac12+\cdots+\frac{1}{2^{n+1}}}\omega(h^{2^n})^{\frac{1}{2^{n+1}}} \\ \qquad\quad \leqslant C\|\omega\|^{\frac{1}{2^{n+1}}}\|h^{2^n}\|^{\frac{1}{2^{n+1}}} \end{cases}$$

Take $\lim\limits_{n\to\infty}$,

$$|\omega(a)| \leqslant C \underbrace{\lim_{n\to\infty} \|h^{2^n}\|^{\frac{1}{2^n}\cdot\frac12}}_{r(h)} = Cr(a^*a)^{\frac12} = C\|\widehat{a^*a}\|_\infty^{\frac12} = C\|\widehat{a}^*\widehat{a}\|_\infty^{\frac12} = C\|\widehat{a}\|_\infty. \tag{$*$}$$

Let $B = \Gamma(A) \subset C_0(\widehat{A})$. Define $\tau\colon B \to \mathbb{C}$ by $\tau(\widehat{a}) = \omega(a)$ $(a \in A)$. $(*)$ $\implies$ τ is well defined and bounded; $\tau(\widehat{a}^*\widehat{a}) = \tau(\widehat{a^*a}) = \omega(a^*a) \geqslant 0$ $\implies$ τ is positive; B is dense in $C_0(\widehat{A})$ $\implies$ τ uniquely extends to a bounded linear functional $\tau\colon C_0(\widehat{A}) \to \mathbb{C}$; $\tau \geqslant 0$. We have $\omega = \Gamma^*(\tau)$. ☐

Remark.

(1) If A is not Hermitian, then Theorem 2 holds with $\widehat{A}$ replaced by $\widehat{A}_h = \left\{ *\text{-characters } \chi \in \widehat{A} \right\}$.

(2) What if A is not commutative?

Let A be a Banach $*$-algebra, B is a C^*-algebra and $C^*(A)$ is the universal C^*-algebra generated by A. Then we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "\theta"] \arrow[dr, "\forall"'] & & C^*(A) \arrow[dl, dashed, "\exists!"] \\ & B & \end{tikzcd}$$

If A is commutative & Hermitian, then $C^*(A) = C_0(\widehat{A}), \theta = \Gamma_A$. Commutative version of Bochner's theorem states that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "\theta"] \arrow[dr, "\text{pos } \omega"'] & & C^*(A) \arrow[dl, "\exists! \text{ pos } \widetilde{\omega}"] \\ & \mathbb{C} & \end{tikzcd}$$

Let H_ω be the quotient of A w.r.t. semi-inner-product $\langle a|b\rangle = \omega(b^*a)$. We have a natural representation $\pi_\omega\colon A \to \mathscr{B}(H_\omega), \pi_\omega(a)(b + N_\omega) = ab + N\omega$, where $N_\omega = \{a\colon \omega(a^*a) = 0\}$.

GNS representation

$\exists v \in H_\omega$, $\omega(a) = \langle \pi_\omega(a)v|v\rangle$.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\pi_\omega"'] & & C^*(A) \arrow[dl, "\widetilde{\pi}_\omega"] \\ & \mathscr{B}(H_\omega) & \end{tikzcd} \qquad \widetilde{\omega}(b) = \langle \widehat{\pi}_\omega(b)v|v\rangle.$$

G = LCA group (2nd countable).

Definition. $B(G) = \check{\mathscr{F}}(M(\widehat{G}))$ is the Fourier Stieltjes algebra of G (H is a $*$-subalgebra in $C_b(G)$). $A(G) = \check{\mathscr{F}}(L^1(\widehat{G}))$ is the Fourier algebra of G.

Remark.

$$\begin{tikzcd} A(G) \arrow[r, "\lhd", "\text{(ideal)}"'] \arrow[d, hook, "\text{announcement}"'] & B(G) \arrow[d, hook, "\text{subalgebra}"] \\ C_0(G) \arrow[r, "\lhd"] & C_b(G) \end{tikzcd}$$

Corollary (of Bochner's theorem)**.** $B(G) = \operatorname{span} \mathcal{P}(G)$.

Proof. $M(\widehat{G}) = \operatorname{span} M(\widehat{G})_{\text{pos}}$. ☐

The Fourier inversion formula

Notation. $B^1(G) = B(G) \cap L^1(G)$; $\mu = \mu_G$ Haar measure on G.

Theorem (Fourier inversion formula-I)**.**

(1) $f \in B^1(G) \implies \widehat{f} \in L^1(G)$;

(2) There exists a unique Haar measure on $\widehat{G}$, i.e., $\mu_{\widehat{G}}$ s.t.

$$\forall f \in B^1(G) \quad Sf = (\widehat{f})^\vee.$$

Explicitly, $f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(x)}\,\mathrm{d}\mu_{\widehat{G}}(\chi)$.

Definition. $\mu_{\widehat{G}}$ is the dual of μ_G (the Plancherel measure).

Observe. $\mu_G \rightsquigarrow c\mu_G \implies \mu_{\widehat{G}} \rightsquigarrow c^{-1}\mu_{\widehat{G}}$.

Example 1. G is finite, $n = |G| < \infty$.

$$\begin{aligned} \mu_G = \text{ counting } &\implies \mu_{\widehat{G}} = \frac{\text{counting}}{n}; \\ \mu_G = \frac{\text{counting}}{\sqrt{n}} &\implies \mu_{\widehat{G}} = \frac{\text{counting}}{\sqrt{n}}. \end{aligned}$$

Example 2. $G = \mathbb{Z}$, $\mu_G =$ counting.

$$\alpha \colon \mathbb{T} \xrightarrow{\sim} \mathbb{Z}, z \in \mathbb{T} \mapsto \chi_z,\ \chi_z(n) = z^{-n}.$$

α_*(normalized Lebesgue measure) $= \mu_{\widehat{\mathbb{Z}}}$.

Example 3. $G = \mathbb{T}$, $\mu_G =$ normalized Lebesgue measure.

$$\alpha \colon \mathbb{Z} \xrightarrow{\sim} \widehat{\mathbb{T}}, n \mapsto \chi_n,\ \chi_n(z) = z^{-n}.$$

α_*(counting) $= \mu_{\widehat{\mathbb{T}}}$.

Example 4. $G = \mathbb{R}$, $\mu_G = \lambda =$ Lebesgue measure.

$$\alpha \colon \mathbb{R} \xrightarrow{\sim} \widehat{\mathbb{R}}, \lambda \mapsto \chi_\lambda,\ \chi_\lambda(t) = e^{-2\pi \mathrm{i}\lambda t}.$$

$\alpha_*(\lambda) = \mu_{\widehat{\mathbb{R}}}$.

Lecture 16 (2024.12.20)

The Fourier inversion formula

$G = $ LCA group (2nd countable). $\mu = \mu_G$ Haar measure. $B(G) = \check{\mathscr{F}}(M(\widehat{G}))$, $B^1(G) = B(G) \cap L^1(G)$.

Theorem (Inversion formula)**.**

(1) $f \in B^1(G) \implies \widehat{f} \in L^1(G)$;

(2) There exists a unique Haar measure on $\widehat{G}$, i.e., $\mu_{\widehat{G}}$ s.t.

$$\forall f \in B^1(G) \quad Sf = (\widehat{f})^\vee.$$

Explicitly, $f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(x)}\,\mathrm{d}\mu_{\widehat{G}}(\chi)$.

Definition. $\mu_{\widehat{G}}$ is the dual of μ_G (or the Plancherel measure on $\widehat{G}$).

Strategy of proof.

- We want: $f(e) = \int_{\widehat{G}} \widehat{f}(\chi)\,\mathrm{d}\mu_{\widehat{G}}(\chi)$.
- But $\left\{\widehat{f}\colon f \in B^1(G)\right\} \neq C_c(\widehat{G})$.
- We want: $(f * g)(e) = \int_{\widehat{G}} \widehat{f}\widehat{g}\,\mathrm{d}\mu_{\widehat{G}}$.
- We'll see: $\exists \nu_g \in M(\widehat{G})$ s.t. $(f * g)(e) = \int \widehat{f}\,\mathrm{d}\nu_g$.
- "Define" $\mu_{\widehat{G}}$ by $\mu_{\widehat{G}} =$ "$\frac{\nu_g}{\widehat{g}}$". $\square$

Lemma/Exercise 1.

(1) $\mathscr{F}_G \circ S_G = S_{\widehat{G}} \circ \mathscr{F}_G$, and similarly for $\check{\mathscr{F}}$.

(2) $(L_x\nu)^\wedge = \varepsilon_x\widehat{\nu}$, $\nu \in M(G)$; $(L_\chi\nu)^\vee = \chi\check{\nu}$, $\nu \in M(\widehat{G}), \chi \in \widehat{G}$.

(3) $(\chi\nu)^\wedge = L_{x^{-1}}\widehat{\nu}$, $\nu \in M(G)$; $(\varepsilon_x\nu)^\vee = L_{x^{-1}}\check{\nu}$, $\nu \in M(\widehat{G}), x \in G$.

Corollary. $B(G)$ is stable under S_G, under translations, under multiplication by $\widehat{G}$.

Lemma/Exercise 2.

(1) $f \in L^1(G), g \in L^\infty(G) \implies f * g$ is defined everywhere, $f * g \in C_b(G)$, $\|f * g\|_\infty \leqslant \|f\|_1\|g\|_\infty$.

(2) $f, g \in L^2(G) \implies f * Sg$ is defined everywhere, $f * Sg \in C_0(G)$, $\|f * Sg\|_\infty \leqslant \|f\|_2\|g\|_2$.

(3) $f \in L^2(G) \implies f * \overline{Sf} \in \mathcal{P}(G)$.

Hints:

(1) $(f * g)(x) = \int_G (Sg)(L_x f)\,\mathrm{d}\mu$. Continuity of $f * g$ follows from the continuity of $x \mapsto L_x f$.

(2)

$$\begin{tikzcd} L^2 \times L^2 \arrow[r] & L^\infty \\ C_c \times C_c \arrow[u, hook, "\text{dense}"'] \arrow[r] & C_c \subset C_0 \arrow[u, hook, "\text{closed}"'] \end{tikzcd} \qquad (f, g) \mapsto f * Sg$$

Lemma/Exercise 3. $\varphi \in \mathcal{P}(G), \chi \in \widehat{G} \implies \chi\varphi \in \mathcal{P}(G)$.

Remark. Lemma 2 & 3 hold for nonabelian G as well.

Notation.

$$\mathcal{P}_c(G) = \{\varphi \in \mathcal{P}(G)\colon \operatorname{supp}\varphi \text{ is compact}\}.$$

Lemma 4. For all compact $K \subset \widehat{G}$, there exists $f \in \mathcal{P}_c(G)$ s.t. $\widehat{f} \geqslant 0$ and $\widehat{f}|_K > 0$.

Proof. Take any $h \in C_c(G)$ s.t. $\widehat{h}(e) = \int h \,\mathrm{d}\mu \neq 0$.
Let $g = h * h^*$ ($h^* = \overline{Sh}$).
Lemma 2 $\implies$ $g \in \mathcal{P}_c(G), \widehat{g} = |\widehat{h}|^2 \geqslant 0$; $\widehat{g}(e) = |\widehat{h}(e)|^2 > 0$. $\implies$ there exists a neighborhood $V \ni e, V \subset \widehat{G}$, s.t. $\widehat{g}|_V > 0$.
Let compact set $K \subset \bigcup_{i=1}^{n} x_i V$ $(x_1, \ldots, x_n \in \widehat{G})$.
Let $f = \sum_{i=1}^{n} x_i g$; $f \in \mathcal{P}_c(G)$ by Lemma 3, Lemma 1 $\implies$ $\widehat{f} = \sum_{i=1}^{n} L_{x_i} g > 0$ on K. □

Notation. $\forall f \in B^1(G)$, define $\nu_f \in M(\widehat{G})$ by $\boxed{Sf = \widecheck{\nu_f}}$.

Remark.

$$\left.\begin{aligned} \text{Goal: } Sf &= (\widehat{f})^\vee \\ \text{We have: } Sf &= \widecheck{\nu_f} \end{aligned}\right\} \implies \nu_f = \widehat{f} \cdot \mu_{\widehat{G}}.$$

Lemma 5.

(1) $\forall h \in L^1(G), \forall f \in B^1(G)$,
$$\int \widehat{h} \,\mathrm{d}\nu_f = \langle \widehat{h}, \nu_f \rangle = (h * f)(e).$$

(2) $\forall f, g \in B^1(G)$, $\widehat{f} \cdot \nu_g = \widehat{g} \cdot \nu_f$.

Proof.

(1)
$$\begin{aligned} \langle \widehat{h}, \nu_f \rangle &= \int_{\widehat{G}} \int_G h(x) \chi(x) \mu(x) \,\mathrm{d}\nu_f(\chi) = \int_G h(x) \widecheck{\nu_f}(x) \,\mathrm{d}\mu(x) \\ &= \int_G h(x) f(x^{-1}) \,\mathrm{d}\mu(x) = (h * f)(e). \end{aligned}$$

(2) $\forall h \in L^1(G)$,
$$\begin{aligned} \langle \widehat{h}, \widehat{f} \cdot \nu_g \rangle &= \langle \widehat{h}\widehat{f}, \nu_g \rangle = \langle \widehat{h * f}, \nu_g \rangle \\ &\overset{(1)}{=\!=} ((h * f) * g)(e) = ((h * g) * f)(e) = \langle \widehat{h}, \widehat{g} \cdot \nu_f \rangle. \end{aligned}$$

We know: $\overline{\mathscr{F}(L^1)} = C_0(\widehat{G}) \implies \widehat{f} \cdot \nu_g = \widehat{g} \cdot \nu_f$. □

Lemma 6. $L_\chi \nu_f = \nu_{\chi^{-1} f}$ $(f \in B^1(G), \chi \in \widehat{G})$.

Proof. $\widecheck{L_\chi \nu_f} \overset{\text{Lemma 1}}{=\!=\!=} \chi \widecheck{\nu_f} = \chi \cdot Sf = S(\chi^{-1} f) = \widecheck{\nu_{\chi^{-1}} f}$. □

Proof of Theorem. Define $I \colon C_c(\widehat{G}) \to \mathbb{C}$ as follows: $\forall \psi \in C_c(\widehat{G})$ choose $f \in \mathcal{P}_c(G)$ s.t. $\widehat{f} \geqslant 0$, $\widehat{f}|_{\operatorname{supp}(\psi)} > 0$. Let $I(\psi) = \left\langle \frac{\psi}{\widehat{f}}, \nu_f \right\rangle$. This does not depend on f: Indeed, if $g \in \mathcal{P}_c(G)$ is another such function, then

$$\left\langle \frac{\psi}{\widehat{f}}, \nu_f \right\rangle = \left\langle \frac{\psi}{\widehat{f}\widehat{g}}, \widehat{g} \cdot \nu_f \right\rangle \overset{\text{Lemma 5}}{=\!=\!=} \left\langle \frac{\psi}{\widehat{f}\widehat{g}}, \widehat{f} \cdot \nu_g \right\rangle = \left\langle \frac{\psi}{\widehat{g}}, \nu_g \right\rangle.$$

Clearly, I is linear.

$$\left.\widehat{f} \geqslant 0 f \quad \in \mathcal{P}(G) \implies \nu_f \geqslant 0\right\} \implies I \geqslant 0.$$

Claim. $\forall f \in B^1(G)$,
$$\widehat{f} \cdot I = \nu_f. \tag{12}$$

Indeed, $\forall \psi \in C_c(\widehat{G})$,

$$\begin{aligned}
(\widehat{f} \cdot I)(\psi) &= I(\widehat{f}\psi) \\
&= \left\langle \frac{\widehat{f}\psi}{\widehat{g}}, \nu_g \right\rangle \text{ (for a suitable } g \in \mathcal{P}_c(G)) \\
&= \left\langle \frac{\psi}{\widehat{g}}, \widehat{f}\nu_g \right\rangle = \left\langle \frac{\psi}{\widehat{g}}, \widehat{g}\nu_f \right\rangle = \langle \psi, \nu_f \rangle \implies (12) \text{ holds.}
\end{aligned}$$

$$(12) \implies I \neq 0.$$

$\forall \chi \in \widehat{G}, \forall \psi \in C_c(\widehat{G})$,

$$\begin{aligned}
(L_\chi I)(\psi) = I(L_{\chi^{-1}}\psi) &= \left\langle \frac{L_{\chi^{-1}}\psi}{\widehat{f}}, \nu_f \right\rangle \text{ (for a suitable } f \in \mathcal{P}_c(G)) \\
&= \left\langle L_{\chi^{-1}}\left(\frac{\psi}{L_\chi \widehat{f}}\right), \nu_f \right\rangle = \left\langle \frac{\psi}{L_\chi \widehat{f}}, L_\chi \nu_f \right\rangle \\
&\overset{\text{Lemma 1 \& 6}}{=\!=\!=} \left\langle \frac{\psi}{\widehat{\chi^{-1} f}}, \nu_{\chi^{-1} f} \right\rangle = I(\psi) \implies I \text{ is left invariant.}
\end{aligned}$$

We have a Haar measure $\mu = \mu_{\widehat{G}}$ on $\widehat{G}$ s.t. $I = I_\mu$. We have $\widehat{f} \cdot \mu_{\widehat{G}} = \nu_f, \forall f \in B^1(G) \underset{\text{(ex)}}{\implies} |\widehat{f}| \cdot \mu_{\widehat{G}} = |\nu_f| \implies \widehat{f}$ is $\mu_{\widehat{G}}$-integrable;

$$(\widehat{f})^\vee = \int \widehat{f}(\chi)\chi(x)\, \mathrm{d}\mu_{\widehat{G}}(\chi) = \int \chi \, \mathrm{d}\nu_f = \widecheck{\nu_f} = Sf. \qquad \square$$

Convention. G and $\widehat{G}$ are equipped with Haar measures $\mu_G, \mu_{\widehat{G}}$ s.t. $\mu_{\widehat{G}}$ is dual to μ_G.

Theorem (Plancherel's theorem; Weil, Raikov)**.** $\mathscr{F}((L^1 \cap L^2)(G)) \subset L^2(\widehat{G})$, and $\mathscr{F}|_{L^1 \cap L^2}$ uniquely extends to a unitary isomorphism $\mathscr{F}^\bullet \colon L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$.

Proof. $\forall f \in L^1 \cap L^2$, $f * f^* \underset{\text{Lemma 2}}{\in} L^1 \cap \mathcal{P} \subset B^1 \implies \int_G |f|^2 \, \mathrm{d}\mu_G = (f * f^*)(e) = \int \widehat{f * f^*} \, \mathrm{d}\mu_{\widehat{G}} = \int_{\widehat{G}} |\widehat{f}|^2 \, \mathrm{d}\mu_{\widehat{G}} \implies \mathscr{F}(L^1 \cap L^2) \subset L^2(\widehat{G})$, and $\mathscr{F}|_{L^1 \cap L^2}$ is an isometry w.r.t. $\|\cdot\|_2$. $\overline{L^1 \cap L^2} \subset L^2 \implies \mathscr{F}|_{L^1 \cap L^2}$ uniquely extends to an isometry $\mathscr{F}^\bullet \colon L^2(G) \to L^2(G)$.
Let $\psi \in L^2(\widehat{G}), \overline{\psi} \perp \mathscr{F}^\bullet(L^2)$. We want: $\psi = 0$ a.e.
$L^1 \cap L^2$ is stable under translate $\implies \overline{\psi} \perp \widehat{L_x f}, \forall f \in (L^1 \cap L^2)(G)$. That is, $0 = \int \widehat{L_x f} \psi \, \mathrm{d}\mu_{\widehat{G}} = \int_{\widehat{G}} \varepsilon_x \underbrace{\widehat{f}\psi}_{\text{in } L^1} \mathrm{d}\mu_{\widehat{G}} = (\widehat{f}\psi)^\vee(x) \implies \widehat{f}\psi = 0$ a.e. on $\widehat{G}$. This is true for all $f \in (L^1 \cap L^2)(G) \underset{\text{Lemma 4}}{\implies} \psi = 0$ a.e. on $\widehat{G} \implies \mathscr{F}^\bullet(L^2(G)) = L^2(\widehat{G})$. $\square$

We have a canonical map $i_G \colon G \to \widehat{\widehat{G}}, x \mapsto \varepsilon_x$. Pontryagin (or more precisely, Pontryagin and van Campen) duality claims that i_G is a topological isomorphism.

Sketch of proof.

- i_G is injective and continuous.
- i_G is a topological embedding. Here, the key role is the inversion formula. The inversion formula implies that if f sufficiently nice (positive definite and compact supported) continuous function on G, then $\widehat{f}$ is continuous.
- i_G has a closed range in $\widehat{\widehat{G}}$: $G \subset \widehat{\widehat{G}}$.
- By inversion formula, one can show that $L^1(G)$ is <u>regular</u>.[5] If we assume $\overline{G} \subsetneq \widehat{\widehat{G}}$, then we could find a function f s.t. $\widehat{f}$ separates G from $\widehat{\widehat{G}}$. $\widehat{f}|_G = 0$ but $\widehat{f} \not\equiv 0$. This contradicts the inversion formula. $\square$

[5]For any commutative Banach algebra A, it is called <u>regular</u> if for any closed set $F \subset \widehat{A}$, $\forall x \in \widehat{A} \backslash F$, $\exists a \in A$ s.t. $\widehat{a}|_F = 0$, $\widehat{a}(x) = 1$.