A formalisation of the sign in HoTT

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Introduction

This short file aims to explain the content of the agda files written during my internship at Chalmers University during spring 2023. The purpose of this internship is to give a new definition of the sign of a permutation using univalence. As cubical Agda gives a computational meaning to univalence, this construction should be able to compute directly the sign of a permutation. Unfortunately, for the moment, the idea seems to work but the computation is too heavy to give back any result.

This file is made of three parts corresponding to each folder made in the agda part, in chronological order, so we will cover in this order:

- The definition and properties of the type of torsors for ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$.
- The definition and properties of the type of abelian groups, and the construction of the biproduct of a finite family of abelian groups.
- The definition and properties of the orientation of a finite set, with its first consequence: the sign of a permutation.

The convention we will use here are from cubical Agda, as it makes the files easier to read. The universes won't be specified more precisely than by using Type. For a dependant product, however, we will write $(a : A) \times B(a)$ for what the Book writes as $\sum_{a:A} B(a)$ and cubical Agda writes as $\Sigma[a : A]B(a)$. Being close to the syntax of Agda also means that the definition equality will br written as $=$ and the propositional equality will be $\equiv$. The hierarchy of n-types will start at 0, so a set is a 2-type and a groupoid is a 3-type.

Only a few results will be proven here, as the file already contains all the proofs. The detailed proofs here are the one giving an intuition on the situation. The construction roughly goes as follows: we define the type of sets of two elements which is a group and a finite sum of such sets by taking the codiagonal application on the finite coproduct. This allows us to construct the orientation of a finite set: the sum of every part of two elements in it. Finally the action of this function ori on a permutation $\sigma : \mathbb{N}_n \equiv \mathbb{N}_n$ is either the identity or the function switching elements, and the corresponding boolean to this function is the sign of σ.

1 The type of 2-Torsors

The first part of this work is the construction of $K({}^{\mathbb{Z}}/_{2\mathbb{Z}}, 1)$, the Eilenberg-MacLane of ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$. As $({}^{\mathbb{Z}}/_{2\mathbb{Z}}, +)$ is the same as $(\mathrm{Bool}, \oplus)$ we will just speak about Bool. The main two definitions of $K(G, 1)$ for a group G, in HoTT, are a HIT and the type of torsors, which will be the one used here. In the case of $G = {}^{\mathbb{Z}}/_{2\mathbb{Z}}$ this type is particularly simple to describe: this will be the type 2Tors with the following definition.

Definition 1.1 (2Tors)**.** *We define the type* $K(\mathrm{Bool}, 1) = \mathrm{2Tors}$ *as follows:*

$$\mathrm{2Tors} = (X : \mathrm{Type}) \times \|X \simeq \mathrm{Bool}\|_1$$

It is thus the type of sets with only two elements.

Now, we want to show that 2Tors is a group in a sense to be precised latter, but we can start by showing it is a connected pointed groupoid.

Proposition 1.1. 2Tors *is a connected pointed groupoid, its point being*

$$\bullet_t = \mathrm{Bool}, |\mathrm{id}_{\simeq}|_1$$

We can define the obvious action of Bool on any 2-torsor, by saying that true switches both elements and false is the identity.

Definition 1.2 ($+_t$)**.** *For all* $(X, e) : \text{2Tors}$ *there is an action of* Bool *on* X*, which we will write*

$$\_ +_t \_ : \text{Bool} \to X \simeq X$$

The following properties hold:

Proposition 1.2.

- *The action on* $\bullet_t$ *is* $\oplus : \text{Bool} \to \text{Bool} \to \text{Bool}$*:*

$$(b_0\ b_1 : \text{Bool}) \to b_0 +_t b_1 \equiv b_0 \oplus b_1$$

- false *is neutral:*

$$((X, e) : \text{2Tors}) \to (x : X) \to \text{false} +_t x \equiv x$$

- *The action is simply transitive:*

$$((X, e) : \text{2Tors}) \to (x : X) \to \text{isEquiv}(\lambda(b : \text{Bool}).b +_t x)$$

Now, we want to define a binary operation, which we will call the sum of torsors. This will give an analogous to a group operation on 2Tors, except that 2Tors is a groupoid. For this, we will use a universal property, *i.e.* we will define a type defining uniquely our sum in 2Tors.

Definition 1.3. *First, for* $(X, e)\ (Y, e')\ (Z, e'') : \text{2Tors}$ *we define the property* bilin *describing that a function* $+_Z : X \to Y \to Z$ *commutes with the action of* Bool*:*

$$\text{bilin } +_Z = (x : X)\ (y : Y)\ (b : \text{Bool}) \to ((b +_t x) +_Z \equiv b +_t (x +_Z y)) \times (x +_Z (b +_t y) \equiv b +_t (x +_Z y))$$

This type is a property in the hType sense.
Now the type of sums of $X : \text{2Tors}$ *and* $Y : \text{2Tors}$ *are torsors equiped with such a bilinear function:*

$$\text{SumTors } (X, e)\ (Y, e') = ((Z, e'') : \text{2Tors}) \times (+_Z : X \to Y \to Z) \times (\text{bilin } +_Z)$$

Theorem 1.1. *For all* $X\ Y : \text{2Tors}$*, the type* sumTors $X\ Y$ *is contractible.*

Proof. First, as isContr is a proposition, and each torsor is merely equal to Bool (by univalence) it is enough to prove that SumTors $\bullet_t\ \bullet_t$ is contractible. In this case we can take as the center of the fiber the pair

$$(\bullet_t, \oplus)$$

and the obvious proofs that this is bilinear.

Now suppose that there is $((Z, e''), +_Z)$ with $+_Z : \text{Bool} \to \text{Bool} \to Z$ a bilinear map. Then making an equality between this type and our center of contraction can be done by giving an equivalence between Z and Bool which commutes with $\oplus$ and $+_Z$:

$$\begin{tikzcd} & \text{Bool} \arrow[d, "f_\simeq"] \\ \text{Bool} \times \text{Bool} \arrow[ur, "\oplus"] \arrow[r, "+_Z"'] & Z \end{tikzcd}$$

We can define this function $f_\simeq$ by $\lambda(b : \text{Bool}).b +_Z \text{false}$ which is an equivalence because it can also be written as the expression $\lambda(b : \text{Bool}).b +_t (\text{false} +_Z \text{false})$ and the simple transitivity of $+_t$ means that this function is an equivalence. The fact that this function makes the above diagramm commute is direct. □

Definition 1.4. *We define the function* $\_ +^\top \_ : \text{2Tors} \to \text{2Tors} \to \text{2Tors}$ *as the center of contraction of* SumTors $X\ Y$.

The main properties of this sum are the following:

Proposition 1.3.

- *It is commutative:*
$$(X\ Y : \mathrm{2Tors}) \to X +^{\top} Y \equiv Y +^{\top} X$$
- *It is associative:*
$$(X\ Y\ Z : \mathrm{2Tors}) \to X +^{\top} (Y +^{\top} Z) \equiv (X +^{\top} Y) +^{\top} Z$$
- $\bullet_t$ *is neutral:*
$$(X : \mathrm{2Tors}) \to X +^{\top} \bullet_t \equiv X$$
- *Each element is its own inverse:*
$$(X : \mathrm{2Tors}) \to X +^{\top} X \equiv \bullet_t$$

This leads to the final result of this part:

Theorem 1.2 (Homogeneity of 2Tors)**.** *The type* 2Tors *is homogeneous, which means that for any* $X : \mathrm{2Tors}$ *there is a path*
$$h_X : (\mathrm{2Tors}, \bullet_t) \equiv (\mathrm{2Tors}, X)$$
and such that
$$h_{\bullet_t} \equiv \mathrm{refl}_{(\mathrm{2Tors}, \bullet_t)}$$

Proof. Let $X : \mathrm{2Tors}$, we define the function
$$h_{\simeq, X}\ Y = Y +^{\top} X$$
which is an involution because
$$\begin{aligned}(Y +^{\top} X) +^{\top} X &\equiv Y +^{\top} (X +^{\top} X)\\ &\equiv Y +^{\top} \bullet_t\\ &\equiv Y\end{aligned}$$

Moreover, $h_{\simeq, X}\ \bullet_t \equiv X$ by neutrality of $\bullet_t$. Thus, by univalence, we can create from $h_{\simeq, X}$ a path
$$h_X : (\mathrm{2Tors}, \bullet_t) \equiv (\mathrm{2Tors}, X)$$

Finally, this path for $X = \bullet_t$ is $X \mapsto \bullet_t +^{\top} X$ which is equal to X, so this is the same as the identity: this means that $h_{\bullet_t} \equiv \mathrm{refl}_{(\mathrm{2Tors}, \bullet_t)}$. □

2 The category of Abelian groups

In this section, we will present the main results about the category $\mathbb{Ab}$ of abelian groups. An important point is that this is not the type of abstract group (as named in the summetry book) but a type of spaces, more precisely the type of homogeneous pointed connected groupoids.

Definition 2.1 ($\mathbb{Ab}$)**.** *We define the type* $\mathbb{Ab}$ *as the type of pointed connected groupoids equiped with an homogeneity which is constant at the point associated to the space:*

$$\begin{aligned}
&\mathrm{isConnected}\ X = (x_1\ x_2 : X) \to \|x_1 \equiv x_2\|_1\\
&\mathrm{isGroupoid}\ X = (x_1\ x_2 : X) \to \mathrm{isSet}\ (x_1 \equiv x_2)\\
&\mathrm{isHomo}\ X\ \bullet_X = (x : X) \to (X, \bullet_X) \equiv (X, x)\\
&\mathrm{isHomoConst}\ X\ \ \bullet_X\ \ h = h_{\bullet_X} \equiv \mathrm{refl}_{(X, \bullet_X)}\\
&\mathrm{isAbGrp}\ X\ \bullet_X = (\mathrm{isConnected}\ X) \times (\mathrm{isGroupoid}\ X) \times ((h : \mathrm{isHomo}\ X\ \bullet_X) \times \mathrm{isHomoConst}\ X\ \ \bullet_X\ \ h)\\
&\mathbb{Ab} = (X : \mathrm{Type}) \times (\bullet_X : X) \times (\mathrm{isAbGrp}\ X\ \bullet_X)
\end{aligned}$$

For a group $X : \mathbb{Ab}$*, we will write* $\lceil X \rfloor$ *for its underlying type,* $\bullet_X$ *for its point,* h_X *for its homogeneity and* $h_{X,\equiv}$ *for the proof that this homogeneity is constant at* $\bullet_X$*.*

We will prove that being an abelian group, for a pointed type, is a proposition, but for this we need a first lemma:

Lemma 2.1. *Let A : Type be a connected type, and $B : A \to$ Type be a family of n-types. Let $a_0 : A$ and $b_0 : B(a)$. Then the type*

$$F = (f : (a : A) \to B(a)) \times (f(a_0) \equiv b_0)$$

is a $n-1$-type.

Proof. We will prove this result by induction on $n \geq 1$:

- If $n = 1$ then let's prove that F is contractible. As B is a family of propositions, we only need to give an element of F. Let $a : A$, by hypothesis of A being connected we know that we have $|p| : \|a_0 \equiv a\|_1$ but as B is a family of propositions, this means that we can take $p : a_0 \equiv a$ from our previous $|p|$, and thus write subst$Bpb_0 : B(a)$. Now we check that this expression is b_0 on a_0, but (up to changing p to $p \cdot q^{-1}$ with $|q| : \|a_0 \equiv a_0\|_1$ given by the connectedness) it is refl on a_0 so transportRefl we have that this expression is b_0 on a_0. So F is inhabited, so as it is obviously a proposition, it is contractible.

- Suppose that B is a family of n-types. We will prove that F is a $n-1$-type. For this, let's take $f\ g : F$ and let's prove that $f \equiv g$ is a $n-2$-type. Let's write $f = f_0, f_\equiv$ and $g = g_0, g_\equiv$, the type of $f \equiv g$ is equivalent to the type of pairs (p, q) where $p : f_0 \equiv g_0$ and q : PathP $(\lambda i.p\ i\ a_0 \equiv b_0)$. p can also be seen as an homotopy *i.e.* a function of equalities $p : \forall x \to f_0\ x \equiv g_0\ x$ and for such a p, a q can be seen as a path from $f_\equiv$ to $g_\equiv$ above $p\ a_0$ or equivalently as a proof that the following diagramm commutes:

$$\begin{tikzcd} f(a_0) \arrow[rr, "p\ a_0"] \arrow[dr, "f_\equiv"'] & & g(a_0) \arrow[dl, "g_\equiv"] \\ & b_0 & \end{tikzcd}$$

  So $f \equiv g$ is equivalent to the following type :

$$(p : (a : A) \to f_0\ a \equiv g_0\ a) \times (p\ a_0 \equiv f_\equiv \cdot g_\equiv^{-1})$$

  but, as B is a family of n-types, $f_0\ a \equiv g_0\ a$ is a $n-1$-type, so by induction hypothesis with $b'_0 = f_\equiv \cdot g_\equiv^{-1}$ we deduce that $f = g$ is a $n-2$-type.

□

Theorem 2.1. *Let $(X, \bullet_X)$: Type$_\bullet$, then* isAbGrp $X\ \bullet_X$ *is a proposition.*

Proof. Being connected and being a groupoid are propositions. For the homogeneity, we can first see that

$$(X, \bullet_X) \equiv (X, x)$$

for some $x : X$ is equivalent to the type

$$((e, e_\simeq) : X \simeq X) \times (e\ \bullet_X \equiv x)$$

and with the previous lemma, as X is a groupoid and is connected, and because $X \simeq X$ is at the same level as $X \to X$, this means that $(X, \bullet_X) \equiv (X, x)$ is a set. Thus $(x : X) \to (X, \bullet_X) \equiv (X, x)$ is also a set, and thus

$$(h : (x : X) \to (X, \bullet_X) \equiv (X, x)) \times (h\ \bullet_X \equiv \text{refl})$$

is a proposition by the previous lemma. □

A useful consequence is that for $X\ Y : \mathbb{Ab}$, the two groups are equal exactly when they are equal as pointed types, equivalently when there is an equivalence between the two which sends the point of X to the point of Y.

We can define the type of homomorphisms.

Definition 2.2. *Let $X\ Y : \mathbb{Ab}$. We define the type*

$$X \to_{\mathrm{a}} Y = (f : \lceil X \rfloor \to \lceil Y \rfloor) \times (f\ \bullet_X \equiv \bullet_Y)$$

Remark 2.1. By the previous lemma, $X \to_{\mathrm{a}} Y$ is a set. It also has a null morphism $0_{X \to_{\mathrm{a}} Y}$.

We will just write about the function to describe an element of $X \to_{\mathrm{a}} Y$ because the pointed part is often direct and because, as types in $\mathbb{Ab}$ are homogeneous, the equality of elements of $X \to_{\mathrm{a}} Y$ just needs an equality of the underlying functions (this is a result proved in the Cubical library).

As a category, $\mathbb{Ab}$ can be seen as a full subcategory of Type$_\bullet$.

2.1 Addition in an abelian group

Another important notion in an abelian group is its operation. Like $+_t$ in 2Tors, or the multiplication on the circle – which is $K(\mathbb{S}^1, 1)$ – any classifying space of an abelian group has a group operation.

Definition 2.3 ($+_\mathrm{a}$). *Let $X : \mathbb{A}\mathrm{b}$, we define the following:*

$$\_ +_\mathrm{a} \_ : \lceil X \rfloor \to \lceil X \rfloor \to \lceil X \rfloor$$
$$x_0 +_\mathrm{a} x_1 = \mathrm{transport}\ (h\ x_0)\ x_1$$

Proposition 2.1. *For $X : \mathbb{A}\mathrm{b}$, the operation $+_\mathrm{a}$ has the group operation properties in the following sense:*

- *it is coherently associative:*
$$(\lambda x_0\ x_1\ x_2 \to x_0 +_\mathrm{a} (x_1 +_\mathrm{a} x_2)) \equiv (\lambda x_0\ x_1\ x_2 \to (x_0 +_\mathrm{a} x_1) +_\mathrm{a} x_2)$$
- *it is coherently commutative:*
$$(\lambda x_0\ x_1 \to x_0 +_\mathrm{a} x_1) \equiv (\lambda x_0\ x_1 \to x_1 +_\mathrm{a} x_0)$$
- $\bullet_X$ *is neutral:*
$$(\lambda x \to x +_\mathrm{a} \bullet_X) \equiv \mathrm{id}$$
- *we can define an inversion function:*
$$x^{-1} = \mathrm{transport}\ (h\ x)^{-1}\ \bullet_X$$
- *this inversion function gives a symmetrical element:*
$$(\lambda x \to x +_\mathrm{a} x^{-1}) \equiv (\lambda \_ \to \bullet_x)$$

Remark 2.2. The properties are presented with equalities of functions because equalities inside $\lceil X \rfloor$ are not proposition, while equalities in $X \to_\mathrm{a} X$ are propositions (this is also why we call the properties coherent).

All the proofs of those properties rely on the same argument, which we will show in the proof of the next proposition.

Proposition 2.2. *Let $X\ Y : \mathbb{A}\mathrm{b}$ and $(f, f_\bullet) : X \to_\mathrm{a} Y$, then we have*

$$(\lambda x_0\ x_1 \to f\ (x_0 +_\mathrm{a} x_1)) \equiv (\lambda x_0\ x_1 \to f\ (x_0) +_\mathrm{a} f\ (x_1))$$

Proof. First we define a change of base operation: given $X : \mathbb{A}\mathrm{b}$ and $x : \lceil X \rfloor$ we can construct the new group $X_{(\bullet = x)}$ with the same underling type, the new point being x and the same propositions.

The idea is to find a defining property for both functions and express it in a way which allows us to use lemma 2.1. This property is that both are functions of two arguments which are equal to f when their first argument is $\bullet_X$. Thus we define the type

$$A = (g : (x : \lceil X \rfloor) \to X \to_\mathrm{a} Y_{(\bullet = f\ x)}) \times (g\ \bullet_X \equiv (f, \mathrm{refl}))$$

The fact that $\bullet_X$ is neutral in X and $\bullet_Y$ is neutral in Y, and that $f\ \bullet_X \equiv \bullet_Y$ means that both functions are A (when adding the necessary equalities).

From lemma 2.1, and because $X \to_\mathrm{a} Y_{(\bullet = f\ x)}$ is a set, A is a proposition, thus the two functions are equal. □

2.2 Finite Biproduct

The main goal here is, given $X : \mathbb{A}\mathrm{b}$ and a finite set F, to define a function

$$\Sigma_X^F : (F \to \lceil X \rfloor) \to \lceil X \rfloor$$

which behaves like $+_\mathrm{a}$. For this, we use a categorical approach: if we construct the finite coproduct in $\mathbb{A}\mathrm{b}$ and take $\coprod_{\_ : F} X$ then the codiagonal $[\mathrm{id}, \ldots, \mathrm{id}]$ plays the role of our finite sum. But as we will proceed by universal property, we won't have explicit access to the carrying type of a coproduct $\coprod_{i : F} X_i$ (even if it should clearly be $(i : F) \to X_i$). To avoid that we construct the type of biproduct instead and compose the codiagonal with the pairing of each projection $((i : F) \to X_i) \to_\mathrm{a} X_i$. The construction will be done in two times: first we construct the binary biproduct and then we generalize it to a finite biproduct. As the type of biproduct will be shown to be contractible we can proceed by induction on the fact that F is finite, *i.e.* we can just show it for the type Fin n for each $n : \mathbb{N}$. We will only show the details of the binary biproduct because its generalization is direct.

Definition 2.4. *Let $X\ Y : \mathbb{Ab}$. We define the type of preproducts and precoproducts as follows:*

$$\begin{aligned}\mathrm{Pre}\times\ X\ Y &= (Z : \mathbb{Ab}) \times (\lceil Z \rfloor \to_{\mathrm{a}} \lceil X \rfloor) \times (\lceil Z \rfloor \to_{\mathrm{a}} \lceil Y \rfloor)\\ \mathrm{Pre}+\ X\ Y &= (Z : \mathbb{Ab}) \times (X \to_{\mathrm{a}} \lceil Z \rfloor) \times (Y \to_{\mathrm{a}} \lceil Z \rfloor)\end{aligned}$$

We will call π_1, π_2 the projections and κ_1, κ_2 the coprojections. A preproduct (resp precoproduct) naturally induces a function $(A \to_{\mathrm{a}} X) \times (A \to_{\mathrm{a}} Y) \to (A \to_{\mathrm{a}} Z)$ (resp $(X \to_{\mathrm{a}} A) \times (Y \to_{\mathrm{a}} A) \to (Z \to_{\mathrm{a}} A)$) for all $A : \mathbb{Ab}$. A product (resp coproduct) is a preproduct (resp precoproduct) such that the induced function is an equivalence for all $A : \mathbb{Ab}$.

We write $X \sqcap Y$ for the type of products and $X \amalg Y$ for the type of coproducts, and $\langle f, g\rangle_Z$ for the pairing of f and g, $[f, g]_Z$ for their copairing with the product Z.

We define isProd Z *and* isCoprod Z *for a pair of projections (resp coprojections) and the proof that they are universal.*

Then a biproduct is a type which is both a product and a coproduct, but we also need a compatibility condition. We chose it to be the fact that $[\langle \mathrm{id}, 0_{X\to_{\mathrm{a}}Y}\rangle, \langle 0_{Y\to_{\mathrm{a}}X}, \mathrm{id}\rangle] \equiv \mathrm{id}$ (which is a proposition).

Definition 2.5. *Let $X\ Y : \mathbb{Ab}$. We define the type $X \oplus_{\mathrm{a}} Y$ as follows:*

$$X \oplus_{\mathrm{a}} Y = (Z : \mathbb{Ab}) \times (\mathrm{isProd}\ Z) \times (\mathrm{isCoprod}\ Z) \times [\langle \mathrm{id}_X, 0_{X\to_{\mathrm{a}}Y}\rangle, \langle 0_{Y\to_{\mathrm{a}}X}, \mathrm{id}_Y\rangle] \equiv \mathrm{id}_Z$$

Theorem 2.2. *For all $X\ Y : \mathbb{Ab}$, the type $X \oplus_{\mathrm{a}} Y$ is a proposition.*

Proof. Let $Z, \pi_1, \pi_2, \kappa_1, \kappa_2$ and $Z', \pi'_1, \pi'_2, \kappa'_1, \kappa'_2$ be two biproducts of X and Y. First the universality of the projections and coprojections, and the compatibility condition are propositions so we just need to construct an equality between Z and Z' and equalities between π_i and π'_i, and κ_i and κ'_i above this equality. This amount to constructing an equlivalence (by univalence) $Z \simeq Z'$ such that the following diagramm commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "\kappa_1"] \arrow[dr, bend left, "\kappa'_1"] & \\
Z \arrow[ur, bend left, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, "\simeq"] & & Z' \arrow[ul, "\pi'_1"'] \arrow[dl, bend left, "\pi'_2"] \\
 & Y \arrow[ul, bend left, "\kappa_2"] \arrow[ur, "\kappa'_2"'] &
\end{tikzcd}$$

We will make two equivalences, the first one for the product and the second one for the coproduct:

- We define the equivalence $p : Z \simeq Z'$ by
$$p = \langle \pi_1, \pi_2\rangle_{Z'}$$
which has the pseudo-inverse $\langle \pi'_1, \pi'_2\rangle_Z$. Furthermore, we can see that
$$\begin{aligned}\pi'_1 \circ p &\equiv \pi'_1 \circ \langle \pi_1, \pi_2\rangle_{Z'}\\ &\equiv \pi_1\\ \pi'_2 \circ p &\equiv \pi'_2 \circ \langle \pi_1, \pi_2\rangle_{Z'}\\ &\equiv \pi_2\end{aligned}$$
So p makes an equality $(Z, \pi_1, \pi_2) \equiv (Z', \pi'_1, \pi'_2)$.
- We define the equivalence $p' : Z \simeq Z'$ by
$$p = [\kappa'_1, \kappa'_2]_Z$$
which has the pseudo-inverse $[\kappa_1, \kappa_2]_{Z'}$. It also commutes with the coprojections:
$$\begin{aligned}p \circ \kappa_1 &\equiv [\kappa'_1, \kappa'_2]_Z \circ \kappa_1\\ &\equiv \kappa'_1\\ p \circ \kappa_2 &\equiv [\kappa'_1, \kappa'_2]_Z \circ \kappa_2\\ &\equiv \kappa'_2\end{aligned}$$
So p' makes an equality $(Z, \kappa_1, \kappa_2) \equiv (Z', \kappa'_1, \kappa'_2)$.

The two equivalences p and p' are in fact equal. For this we only need to show that $p \circ \kappa_i \equiv \kappa'_i$ by definition of $[\_,\_]_Z$. We will show it only for $i = 1$:

$$\begin{aligned}
\langle \pi_1, \pi_2 \rangle_{Z'} \circ \kappa_1 &\equiv \langle \pi_1, \pi_2 \rangle_{Z'} \circ [\langle \mathrm{id}_X, 0_{X \to_{\mathrm{a}} Y} \rangle_Z, \langle 0_{Y \to_{\mathrm{a}} X}, \mathrm{id}_Y \rangle_Z]_Z \circ \kappa_1 \\
&\equiv \langle \pi_1, \pi_2 \rangle_{Z'} \circ \langle \mathrm{id}, 0_{(X \to_{\mathrm{a}} Y)} \rangle_Z \\
&\equiv \langle \pi_1 \circ \langle \mathrm{id}, 0 \rangle_Z, \pi_2 \circ \langle \mathrm{id}, 0 \rangle_Z \rangle_{Z'} \\
&\equiv \langle \mathrm{id}, 0 \rangle_{Z'} \\
&\equiv [\langle \mathrm{id}_X, 0_{X \to_{\mathrm{a}} Y} \rangle_{Z'}, \langle 0_{Y \to_{\mathrm{a}} X}, \mathrm{id}_Y \rangle_{Z'}]_{Z'} \circ \kappa'_1 \\
&\equiv \kappa'_1
\end{aligned}$$

Thus $p \equiv p'$ which means that we can transport along this equality the fact that p' commutes with the coprojection to show that p commutes with the coprojections. So

$$(Z, \pi_1, \pi_2, \kappa_1, \kappa_2) \equiv (Z', \pi'_1, \pi'_2, \kappa'_1, \kappa'_2)$$

□

To show that $X \oplus_{\mathrm{a}} Y$ is contractible we must construct an inhabitant of this type.

Theorem 2.3. *Let $X\ Y : \mathbb{Ab}$ then the type $X \oplus_{\mathrm{a}} Y$ is inhabited.*

Proof. The carrier is simply the type $\lceil X \rfloor \times \lceil Y \rfloor$, which is also a connected pointed groupoid with its point being $(\bullet_X, \bullet_Y)$. The homogeneity is simply the pointwise homogeneity as a path in $A \times B$ is equivalent to a pair of paths, the first in A and the second in B. We will call Z this group for the time of this proof.

The projections are the obvious projection functions:

$$\pi_1\ (x, y) = x \qquad \pi_2\ (x, y) = y$$

and the fact that these projections are universal is direct.

The coprojections are the following functions:

$$\kappa_1\ x = (x, \bullet_Y) \qquad \kappa_2\ y = (\bullet_X, y)$$

these coprojection are indeed universal. Take $f : X \to_{\mathrm{a}} A$ and $g : Y \to_{\mathrm{a}} A$ for $A : \mathbb{Ab}$. We construct the function

$$\begin{aligned}
&[f, g] : Z \to_{\mathrm{a}} A \\
&[f, g]\ (x, y) = f(x) +_{\mathrm{a}} g(y)
\end{aligned}$$

And we show that it is a pseudo-inverse of the previously defined function. Let $h : Z \to_{\mathrm{a}} A$, let's show that $[h \circ \kappa_1, h \circ \kappa_2] \equiv h$ by function extensionality:

$$\begin{aligned}
[h \circ \kappa_1, h \circ \kappa_2]\ (x, y) &= h(x, \bullet_Y) +_{\mathrm{a}} h(\bullet_X, y) \\
&\equiv h((x, \bullet_Y) +_{\mathrm{a}} (\bullet_X, y)) && \text{because } h \text{ commutes with } +_{\mathrm{a}} \\
&\equiv h(x +_{\mathrm{a}} \bullet_X, \bullet_Y +_{\mathrm{a}} y) && \text{because the homogeneity is applied pointwise} \\
&\equiv h(x, y)
\end{aligned}$$

Let $f : X \to_{\mathrm{a}} A$ and $g : Y \to_{\mathrm{a}} A$, let's show that $[f, g] \circ \kappa_1 \equiv f$ by function extensionality (the same works with κ_2):

$$\begin{aligned}
([f, g] \circ \kappa_1)\ x &= [f, g](x, \bullet_Y) \\
&\equiv f(x) +_{\mathrm{a}} g(\bullet_Y) \\
&\equiv f(x) +_{\mathrm{a}} \bullet_Z && \text{because } g \text{ is a morphism} \\
&\equiv f(x)
\end{aligned}$$

Finally we want to prove the compatibility condition:

$$\begin{aligned}
[\langle \mathrm{id}, 0 \rangle, \langle 0, \mathrm{id} \rangle]\ (x, y) &= \langle \mathrm{id}, 0 \rangle\ (x, y) +_{\mathrm{a}} \langle 0, \mathrm{id} \rangle\ (x, y) \\
&= (x, \bullet_Y) +_{\mathrm{a}} (\bullet_X, y) \\
&\equiv (x, y)
\end{aligned}$$

□

So $X \oplus_{\mathrm{a}} Y$ is contractible.

As we said the generalization to a finite family of abelian groups is straightforward, the fact that it is a property having almost the same proof. For the construction we simply iterate the case $n = 2$.

Definition 2.6. *Let* $(F, \mathrm{isFin}) : \mathrm{FinSet}$ *and* $X : F \to \mathbb{Ab}$ *a finite family of abelian groups. We define* $\bigoplus_{\mathrm{a}} X$ *as the center of contraction of the type of finite biproducts for the family* X*. It is equiped with a pairing function which for* $(f_i : A \to_{\mathrm{a}} X_i)_{i:F}$ *gives the unique function* $\langle f_i \rangle : A \to_{\mathrm{a}} \bigoplus_{\mathrm{a}} X$ *such that for all* $i : F$ *the following diagramm commutes:*

$$\begin{tikzcd} A \arrow[rr, "f_i"] \arrow[dr, "\langle f_i \rangle"'] & & X_i \\ & \bigoplus_{\mathrm{a}} X \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

Similarly for $(g_i : X_i \to_{\mathrm{a}} A)_{i:F}$ *we can define the copairing* $[g_i] : \bigoplus_{\mathrm{a}} \to_{\mathrm{a}} A$ *as the unique function such that the following diagramm commutes for all* $i : F$*:*

$$\begin{tikzcd} X_i \arrow[rr, "g_i"] \arrow[dr, "\kappa_i"'] & & A \\ & \bigoplus_{\mathrm{a}} X \arrow[ur, "{[g_i]}"'] & \end{tikzcd}$$

Definition 2.7. *Let* $(F, \mathrm{isFin}) : \mathrm{FinSet}$ *and* $X : F \to \mathbb{Ab}$*, we define* $\prod_F X$ *with the underlying type* $(i : F) \to X_i$ *and the point* $\lambda i \to \bullet_{X_i}$.

We won't show that it is indeed a group, the proof is straightforward except for the connectedness which uses a proof on only Fin n and then extends it to any finite set. Finally this allows us that construct the group for the constant family and take its codiagonal.

Definition 2.8. *Let* $X : \mathbb{Ab}$ *and* $(F, \mathrm{isFin}) : \mathrm{FinSet}$*. We define* $\prod_F X = \prod_F (\lambda\_ \to X)$*. Let*

$$\nabla : \bigoplus_{\mathrm{a}} (\lambda\_ \to X) \to_{\mathrm{a}} X$$
$$\nabla = [\lambda\_ \to \mathrm{id}_X]$$

and $\pi_{p,i} : \prod_F X \to_{\mathrm{a}} X$ *defined by* $\pi_{p,i}\ f = f\ i$*, then we define the following:*

$$\Sigma_X^F : \prod_F X \to_{\mathrm{a}} X$$
$$\Sigma_X^F = \nabla \circ \langle \pi_{p,i} \rangle$$

3 Orientation and Sign

Now that we can make a finite sum in a group, and knowing that 2Tors is a group, we can sum sets with two elements. This is the idea behind the orientation of a set which is simply the sum of all the parts of two elements in a finite set. Using the previous work of section 1, we have a term

$$\mathrm{isAbTors} : \mathrm{isAbGrp}\ \mathrm{2Tors}\ \bullet_t$$

Definition 3.1. *Let* $F : \mathrm{Type}$*, then the define the decidable powerset of* F *as*

$$\mathbb{P}^{\mathrm{d}}\ F = F \to \mathrm{Bool}$$

and the set of size i *parts of* F *as*

$$\mathbb{P}^{\mathrm{d}}_i\ F = (P : \mathbb{P}^{\mathrm{d}}\ F) \times \|\mathrm{fiber}\ P\ \mathrm{true} \simeq \mathrm{Fin}\ i\|$$

Theorem 3.1. *For* $(F, \mathrm{isFin}) : \mathrm{FinSet}$ *and all* $i : \mathbb{N}$*, the type* $\mathbb{P}^{\mathrm{d}}_i\ F$ *is finite and of cardinal* $\binom{\mathrm{card}(F)}{2}$.

We know that Fin 2 $\simeq$ Bool so we can easily define a function $\mathbb{P}^{\mathrm{d}}_2 \to \mathrm{2Tors}$. This means that we can now directly apply our function Σ_X^F.

Definition 3.2. *We define the function of orientation of a finite set as follows:*

$$\mathrm{ori} : \mathrm{FinSet} \rightarrow \mathrm{2Tors}$$
$$\mathrm{ori}\ (F, \mathrm{isFin}) = \Sigma^{(\mathrm{2Tors}, \mathrm{isAbTors})}_{(\mathbb{P}^{\mathrm{d}}_{2}\ F)} (\mathbb{P}^{\mathrm{d}}_{2} \rightarrow \mathrm{2Tors})$$

Definition 3.3. *We define* $\mathfrak{S}_n = \mathrm{Fin}\ n \equiv \mathrm{Fin}\ n$ *and the sign function:*

$$\varepsilon : \mathfrak{S}_n \rightarrow \mathrm{Bool}$$
$$\varepsilon\ \sigma = \mathrm{actionToBool}\ (\mathrm{cong}\ \mathrm{ori}\ \sigma)$$

Where actionToBool is the function which sends an automorphism of a twotorsor to Bool*.*