A formalisation of the sign in HoTT

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1 Introduction

This is a document written in the context of an internship at Chalmers university. The purpose of this internship is to construct a new definition of the sign of a permutation in Homotopy Type Theory (HoTT). A formal proofs in Agda are quite hard too read, this document explains the main ideas behind the construction and tries to give an intuition of the results. We assume some familiarity with HoTT.

The construction is made in the following steps :

- Construction of the group 2Tors of 2-elements sets.
- Construction of a sum indexed by a finite set in a group, using the notion of biproduct in **Ab**.
- Definition of the orientation of a finite set and construction of the sign.

We will thus explain all the steps, and begin by defining the notion of group and more precisely abelian group, as they are not seen here as sets with a binary product but as spaces.

NB: in the formalised work in Agda we keep count of the universe, but for a concision purpose we will just use $\mathcal{U}$ for any universe here.

2 What is a group?

In HoTT, two definitions can be considered for a group.

2.1 A set with a convenient operation

The first one, which is the usual one in classical maths, is the definition of a set with inversible associative operation and an identity element.

Definition 1 (Abstract group). *We define the type of abstract groups as a set X with a group structure, i.e. a binary operation, a distinguished point and the usual properties of a group, described by the following types:*

$$\mathrm{assoc} : \prod_{X:\mathcal{U}} (X \to X \to X) \to \mathcal{U}$$
$$\mathrm{assoc}\ X\ (\cdot) :\equiv \prod_{x\ y\ z:X} ((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

$$\mathrm{id_G} : \prod_{X:\mathcal{U}} (X \to X \to X) \to X \to \mathcal{U}$$
$$\mathrm{id_G}\ X\ (\cdot)\ e :\equiv \prod_{x:X} (x \cdot e = x) \times (e \cdot x = x)$$

$$\mathrm{inv} : \prod_{X:\mathcal{U}} (X \to X \to X) \to X \to \mathcal{U}$$
$$\mathrm{inv}\ X\ (\cdot)\ e :\equiv \prod_{x:X} \sum_{x':X} ((x \cdot x' = e) \times (x' \cdot x = e))$$

$$\mathrm{GrpStr} : \mathcal{U} \to \mathcal{U}$$
$$\mathrm{GrpStr}\ X :\equiv \left(\sum_{e:X} \sum_{\cdot:X \to X \to X} (\mathrm{assoc}\ X\ (\cdot)) \times (\mathrm{id_G}\ X\ (\cdot)\ e) \times (\mathrm{inv}\ X\ (\cdot)\ e) \right)$$

$$\mathrm{isGrp} : \mathcal{U} \to \mathcal{U}$$
$$\mathrm{isGrp}\ X :\equiv (\mathrm{isGrp}\ X) \times (\mathrm{GrpStr}\ X)$$

Remark 1. *We could also define a function i instead of making a Σ-type for* inv*.*

We will call *abstract group* such a group, and the predicate isGrp is now renamed as isSetGrp, as this notion of group is a group structure on a set.

2.2 A connected pointed groupoid

However, another way of representing a group is to consider the Eilenberg-MacLane space of it. For example instead of considering $(\mathbb{Z}, +, 0)$ we can just consider $(\mathbb{S}^1, \mathrm{base})$. This point of view is developped in the Symmerty book, and we can prove that **Grp** is equivalent (thus equal) to **GrpSpc**, the category of $K(G, 1)$ for each $G : \mathbf{Grp}$. For this work, we will thus consider **Grp** as the type of spaces corresponding to some $K(G, 1)$. The question of characterizing those spaces is answered in the symmerty book by the notion of connected pointed groupoid.

Definition 2 (Group)**.** *A group is a space X with a distinguished point x_0 which is connected and is a* 1*-type. This can be summarized by the following types:*

$$
\begin{aligned}
&\text{connected} : \mathcal{U} \to \mathcal{U} \\
&\text{connected } X :\equiv \prod_{x\ y : X} \|x = y\|_1 \\
&\text{isGrpoid} : \mathcal{U} \to \mathcal{U} \\
&\text{isGrpoid } X :\equiv \text{is}-n-\text{type } 1\ X \\
&\text{isGrp} : \mathcal{U} \to X \to \mathcal{U} \\
&\text{isGrp } X\ x_0 :\equiv (\text{connected X}) \times (\text{isGrpoid } X)
\end{aligned}
$$

We also define

$$\boldsymbol{Grp} :\equiv \sum_{X:\mathcal{U}} \sum_{x_0:X} \text{isGrp } X\ x_0$$

For a space (X, x_0) its corresponding abstract group is $\pi_1(X, x_0) \equiv (x_0 = x_0, \text{refl}_{x_0})$.

2.3 Abelian group

Now we focus on abelian groups, also as spaces. For abstract groups, being abelian just means that for all $x, y : X$ we have $x \cdot y = y \cdot x$, but in the case of spaces the proposition we add is a structure defined uniquely.

Definition 3 (Homogeneous pointed type)**.** *A pointed type (X, x_0) is homogeneous if, for all $x : X$ we have a path $(X, x_0) = (X, x)$ which can equivalently be formulated as giving for each $x : X$ an equivalence $e : X \simeq X$ such that $e(x_0) = x$. We say that (X, x_0) is pointed-homogeneous if, moreover, there is an homogeneity h such that $h\ x_0 = \text{refl}_{(X,x_0)}$.*

$$
\begin{aligned}
&\text{Homogeneous} : \prod_{X:\mathcal{U}} X \to \mathcal{U} \\
&\text{Homogeneous } X\ x_0 :\equiv \prod_{x:X} ((X, x_0) = (X, x)) \\
&\text{Homogeneous}\bullet : \prod_{X:\mathcal{U}} \prod_{x_0:X} \text{Homogeneous } X\ x_0 \to \mathcal{U} \\
&\text{Homogeneous}\bullet\ X\ x_0\ h :\equiv h = \text{refl}_{(X,x_0)} \\
&\text{isHomo} : \prod_{X:\mathcal{U}} X \to \mathcal{U} \\
&\text{isHomo } X\ x_0 :\equiv \sum_{h:\text{Homogeneous } X\ x_0} (\text{Homogeneous}\bullet\ X\ x_0\ h)
\end{aligned}
$$

The type Homogeneous is, by itself, not a proposition. However by imposing that it is refl on the basepoint, this type becomes a proposition. The proof use an essential lemma:

Lemma 4. *If X is a n-connected type and $B : X \to \mathcal{U}$ is a family of n-types, then given $x_0 : X$ and $b_0 : B(x_0)$ the following type is a $n-1$-type:*

$$\sum_{f:\prod_{x:X} B(x)} (f(x_0) = b_0)$$

Proof. We prove this lemma by induction on n.

- If $B(x)$ is a proposition then we can prove that it is contractible using the fact that X is connected, by transporting b_0 through B, along the equalities $\|x_0 = x\|_1$ (as B is a family of propositions we can use the induction principle of propositional truncation), to give us the center of contraction. As each $B(x)$ is a proposition, this is enough to show thee contractibility.
- If we take $(f, f_\bullet, (g, g_\bullet)$ in our type supposed to be a $n-1$-type, we want to show that $(f, f_\bullet) = (g, g_\bullet)$ is a n-type. An equality in this type is equivalently a pair of a homotopy $h : \prod_{x:X}(f\ x = g\ x)$ and $f_\bullet = g_\bullet$ above h. We can also say that the following diagramm commutes :

$$\begin{tikzcd} f\ x_0 \arrow[rr, "h_{x_0}"] \arrow[dr, "f_\bullet"'] & & g\ x_0 \arrow[dl, "g_\bullet"] \\ & b_0 & \end{tikzcd}$$

  So $(f, f_\bullet) = (g, g_\bullet)$ is equivalent to the type

$$\sum_{h:\prod_{x:X} f\ x = g\ x} (h\ x_0 = f_\bullet \cdot g_\bullet)$$

  on which we can apply the induction hypothesis to conclude.

$\square$

Corrolary 4.1. *The type* isHomo $X\ x_0$ *is a proposition.*

Proof. First, for $x : X$, the type $(X, x_0) = (X, x)$ is equivalent to

$$\sum_{(f, f_\simeq):X \simeq X} (f(x_0) = x)$$

and X is a connected 1-type, so $(X, x_0) = (X, x)$ is a set. Now isHomo can be seen as the type

$$\sum_{f:\prod_{x:X} B(x)} (f(x_0) = b_0)$$

for $B(x) :\equiv (X, x_0) = (X, x)$ and $b_0 :\equiv \mathrm{refl}_{(X, x_0)}$, so this type is a -1-type, i.e. a proposition. $\square$

Definition 5 (The category **Ab**)**.** *We define the objects as the following type:*

$$\boldsymbol{Ab} :\equiv \sum_{X:\mathcal{U}} \sum_{x_0:X} (\text{isGrp } X\ x_0) \times (\text{isHomo } X\ x_0)$$

For $X : \boldsymbol{Ab}$ we will write $(\lceil X \rfloor, x_0, g_X, h_X, h_{X\bullet})$ for its different components. We also define the type of arrows between $XY : \boldsymbol{Ab}$ as follows:

$$X \to_{\mathrm{a}} Y :\equiv \sum_{f:\lceil X \rfloor \to \lceil Y \rfloor} (f\ x_0 = y_0)$$

A first result is that **Ab** is indeed a category.

Proposition 6. *The type $X \to_{\mathrm{a}} Y$ is a set.*

Proof. It is also an application of the previous lemma, taking $B(x) :\equiv \lceil Y \rfloor$ and $b_0 :\equiv y_0$. □

This category is univalent, in the sense that having an equivalence of pointed type between two groups is the same as having an equality in **Grp** (this is because the other parts of this type are propositions).

2.4 Addition in an abelian group

We conclude this section on the notion of space-addition. The homogeneity gives us a natural notion of addition.

Definition 7. *Given $X : \boldsymbol{Ab}$ we define the following operation :*

$$\_ +_X \_ : \lceil X \rfloor \to \lceil X \rfloor \to \lceil X \rfloor$$
$$x +_X y :\equiv \text{transport}^{(\lambda(X,x_0).X)}(h_X\ x, y)$$

We write it as an addition because it has the properties of a group. The difference, however, is that X is not a set. The proof of associativity, commutativity and so on relies on the fact that we can define a good type of addition as follows:

$$\prod_{f:(x:\lceil X \rfloor) \to X \to_{\mathrm{a}} X'} (f\ x_0 = \mathrm{id}_X)$$

where X' is X but with x as its basepoint. This type can be easily shown to be a proposition, and both $(x, y) \mapsto x +_X y$ and $(x, y) \mapsto y +_X x$ are in this type. In a similar way we can define a ternary sum and thus show that $+_X$ is associative. We give a final result regarding the addition:

Proposition 8. *Let $X\ Y : \boldsymbol{Ab}$ and $f : X \to_{\mathrm{a}} Y$. Then for all $x\ y : X$ we have*

$$f(x +_X y) = f(x) +_Y f(y)$$

Proof. Given $(f, f_\bullet) : X \to_{\mathrm{a}} Y$, we define the following type:

$$F :\equiv \sum_{g:\prod_{x:X} X \to_{\mathrm{a}} Y'} (g(x_0) = (f, \mathrm{refl}_{f\ x_0}))$$

where Y' is Y with basepoint set at $f\ x$.

F is a proposition. Indeed, with the previous lemma, we juste have to show that

$$\prod_{x:X} X \to_{\mathrm{a}} Y'$$

is a set, but $X \to_{\mathrm{a}} Y'$ is a set as we proved before, and a family os sets is always a set, this type is a set.

Now we can see that both $(x, y) \mapsto f(x +_X y)$ and $(x, y) \mapsto f(x) +_Y f(y)$ can be proved to belong to F, so they are equal, which is equivalent to the homotopy

$$\prod_{x\ y:X} (f(x +_X y) = f(x) +_Y f(y))$$

□

3 The $\mathbb{Z}/2\mathbb{Z}$ torsors

A way of constructing the Eilenberg-MacLane space of a group G is to study the type of G-torsors. We will not consider the G-torsors in the general case but in the case of $G = \mathbb{Z}/2\mathbb{Z}$. In this case, the type can be described easily as the type of sets with 2 elements. To be more concise, we will speak of 2-torsors instead of $\mathbb{Z}/2\mathbb{Z}$-torsors.

Definition 9. *We define the type of* 2*-torsors as such :*

$$\mathrm{2Tors} := \sum_{X:\mathcal{U}} \|X \simeq \mathbb{2}\|_1$$

Remark 2. *It is essential that we have a propositional truncation here, because otherwise it would just be a singleton of* $\mathbb{2}$*.*

3.1 2Tors is a group

Some properties easily follow from the definition.

Proposition 10. *The type* 2Tors *is pointed, connected and is a groupoid.*

Proof. The point is obviously $\mathbb{2}$:

$$(\mathbb{2}, |\mathrm{id}_{\mathbb{2}}|_1) : \mathrm{2Tors}$$

For $X\ Y$: 2Tors, we can reason by induction on $\|X \simeq \mathbb{2}\|_1$ and $\|Y \simeq \mathbb{2}\|_1$ because $\|X \simeq Y\|_1$ is also a proposition, and we use the following function:

$$\lambda(e : X \simeq \mathbb{2}).\lambda(e' : Y \simeq \mathbb{2}).|e \bullet e'^{-1}|_1 : X \simeq \mathbb{2} \to Y \simeq \mathbb{2} \to \|X \simeq Y\|_1$$

For $(X,e) : \mathrm{2Tors}$ we know that isSet X because $X \simeq \mathbb{2}$ in regard to propositions. As a family of sets is also a set, for $(X,e)\ (Y,e') : \mathrm{2Tors}$, the type $X = Y$ is also a set, and as $\|X \simeq \mathbb{2}\|_1$ is a proposition, $((X,e) = (Y,e')) \simeq X = Y$ so the equalities in 2Tors are a set. So 2Tors is a 1-type, i.e. a groupoid. ◻

Thus 2Tors : **Grp**.

Let's give an important result to describe the structure of 2Tors.

Proposition 11. $(\mathbb{2} \simeq \mathbb{2}) \simeq \mathbb{2}$

Corrolary 11.1. $\Omega(\mathrm{2Tors}, (\mathbb{2}, |\mathrm{id}_{\mathbb{2}}|_1)) \simeq (\mathbb{2}, \oplus)$

Proof. As $(\mathbb{2} = \mathbb{2}) \simeq (\mathbb{2} \simeq \mathbb{2})$ and $(\mathbb{2} \simeq \mathbb{2}) \simeq \mathbb{2}$, we conclude the result. ◻

In fact, for all $X : \mathrm{2Tors}$, we can define the action of $\mathbb{2}$ on it. The intuition is that 0 acts as the identity and 1 as the permutation of the two elements of X.

Proposition 12. *For* $(X,e) : \mathrm{2Tors}$ *we define the action* $\bullet$ *of* $\mathbb{2}$ *on* X*. It is a group action of* $(\mathbb{2}, \oplus)$ *on* X *and it is moreover transitive.*

3.2 Sum of torsors

To show that 2Tors is homogeneous, we construct an operation on the torsors, which we will call a sum. To construct it, we define the type of structures on two 2-torsors which commute with the action.

Definition 13. *Let* $X\ Y : \mathrm{2Tors}$*. We define the following types:*

$$\begin{aligned}
&\mathrm{bilinl} : \prod_{Z:\mathrm{2Tors}} (X \to Y \to Z) \to \mathcal{U} \\
&\mathrm{bilinl}\ Z\ (+_Z) :\equiv \prod_{x:X} \prod_{y:Y} \prod_{b:\mathbb{2}} ((x \bullet b) +_Z y = (x +_Z y) \bullet b) \\
&\mathrm{bilinr} : \prod_{Z:\mathrm{2Tors}} (X \to Y \to Z) \to \mathcal{U} \\
&\mathrm{bilinr}\ Z\ (+_Z) :\equiv \prod_{x:X} \prod_{y:Y} \prod_{b:\mathbb{2}} (x +_Z (y \bullet b) = (x +_Z y) \bullet b) \\
&_ +_{\mathrm{2Tors}} _ : \mathcal{U} \\
&X +_{\mathrm{2Tors}} Y :\equiv \sum_{Z:\mathrm{2Tors}} \sum_{(+_Z):X \to Y \to Z} (\mathrm{bilinl}\ Z\ (+_Z)) \times (\mathrm{bilinr}\ Z\ (+_Z))
\end{aligned}$$

Theorem 14. *For* $X\ Y : \mathrm{2Tors}$*, the type* $X +_{\mathrm{2Tors}} Y$ *is contractible.*

Proof. Showing that a type is contractible is a proposition. This means that, to prove this for each $X\ Y : \mathrm{2Tors}$, it suffices to show it for $\mathbb{2} +_{\mathrm{2Tors}} \mathbb{2}$. It is straightforward to check that the pair $(\mathbb{2}, \oplus)$ is such an element, and for any $(Z, +_Z)$ in $\mathbb{2} +_{\mathrm{2Tors}} \mathbb{2}$ we can make the equivalence $\_ +_Z 0 : \mathbb{2} \simeq Z$ because $\bullet$ is transitive and we can rewrite $b +_Z 0$ as $(0 +_Z 0) \bullet b$, it is then left to check that this equivalence commutes with the action, which it does. ◻

We will thus write $X +_{2\mathrm{Tors}} Y$ for the torsor in the center of contraction of the associated type.

This sum makes 2Tors into a monoidal symmetric category and X is its own inverse.

Proposition 15. *The following properties hold for $X\ Y : 2\mathrm{Tors}$:*

- $X +_{2\mathrm{Tors}} \mathbb{2} = X$
- $X +_{2\mathrm{Tors}} Y = Y +_{2\mathrm{Tors}} X$
- $X +_{2\mathrm{Tors}} X = \mathbb{2}$
- $X +_{2\mathrm{Tors}} (Y +_{2\mathrm{Tors}} Z) = (X +_{2\mathrm{Tors}} Y) +_{2\mathrm{Tors}} Z$

Proof. We can see that $\cdot$ is a bilniear function $X \to \mathbb{2} \to X$ so by contractibility of the type of sum, the equality follows.

We take $(Y +_{2\mathrm{Tors}} X, \lambda x.\lambda y.y + x)$ to be in the type of the sum of X and Y, hence the equality.

We associate (x, y) to 0 and $(x, y), x \neq y$ to 1 and check that this function is indeed bilinear.

We construct the type of ternary sums, in a similar way as for the binary sum, and show it is a proposition. They are both ternary sums so they are equal. □

This leads to the homogeneity.

Theorem 16. *The type* 2Tors *is homogeneous.*

Proof. Given $X : 2\mathrm{Tors}$, we define

$$\begin{aligned} h &: 2\mathrm{Tors} \to 2\mathrm{Tors} \\ h_X\ Z &:\equiv Z +_{2\mathrm{Tors}} X \end{aligned}$$

This is an equivalence because it is involutive :

$$\begin{aligned} (Z +_{2\mathrm{Tors}} X) +_{2\mathrm{Tors}} X &= Z +_{2\mathrm{Tors}} (X + 2\mathrm{Tors} X) \\ &= Z +_{2\mathrm{Tors}} \mathbb{2} \\ &= Z \end{aligned}$$

and $\mathbb{2} + X = X$ so $\mathrm{ua}(h_X) : (2\mathrm{Tors}, \mathbb{2}) = (2\mathrm{Tors}, X)$, and $h_{\mathbb{2}}$ is the function $X \mapsto X + \mathbb{2}$ which is $X \mapsto X$, so $\mathrm{ua}(h_{\mathbb{2}}) = \mathrm{refl}_{(2\mathrm{Tors}, \mathbb{2})}$. □

This means that $2\mathrm{Tors} : \mathbf{Ab}$.

4 Defining a sum indexed by a finite set

The purpose of this section is, for an abelian group A, to define an operator Σ_X^I, for a finite set J, taking a family $(x_j)_{j\in J} \in X^J$ to X. In the finite case, we could easily define it by induction by saying that $\Sigma_X^{\varnothing} x_j :\equiv 0_X$ and that $\Sigma_X^{\{1,\ldots,n+1\}} x_j :\equiv \left(\Sigma_X^{\{1,\ldots,n\}} x_j\right) + x_{n+1}$, but this is dependant on the order in which the elements come. If we want to extend this definition to any finite set, which is a type with a mere identification with some Fin n, we need to construct a sum which is a proposition. To do so, we will use the categorical language, to construct finite biproducts and then use the codiagonal.

4.1 The biproduct

4.1.1 The binary biproduct

The category **Ab**, as one expects, is abelian. This implies that we can construct a type $A \oplus_{\mathrm{a}} B$ for two groups A and B equiped with:

- two projections $\pi_1 : A \oplus_{\mathrm{a}} B \to_{\mathrm{a}} A$ and $\pi_2 : A \oplus_{\mathrm{a}} B \to_{\mathrm{a}} B$ universal in the sense that the associated function

$$(X \to_{\mathrm{a}} A \oplus_{\mathrm{a}} B) \to (X \to_{\mathrm{a}} A) \times (X \to_{\mathrm{a}} B)$$

  is an equivalence for all group X.

- two coprojections $\kappa_1 : A \to_{\mathrm{a}} A \oplus B$ and $\kappa_2 : B \to_{\mathrm{a}} A \oplus_{\mathrm{a}} B$ universal in the sens that the associated function

$$(A \oplus_{\mathrm{a}} B \to_{\mathrm{a}} X) \to (A \to_{\mathrm{a}} X) \times (B \to_{\mathrm{a}} X)$$

  is an equivalence for all group X.

- a proof that $[\langle \mathrm{id_a} A, 0_{\mathrm{a}}\ A\ B\rangle, \langle 0_{\mathrm{a}}\ B\ A, \mathrm{id_a} B\rangle] = \mathrm{id}_{A\oplus_{\mathrm{a}} B}$.

Definition 17. *Let $A\ B$: $\boldsymbol{Ab}$. We define the following terms for C : $\boldsymbol{Ab}$:*

$$\begin{aligned}
&\mathrm{Prod}_0\ C :\equiv (C \to_{\mathrm{a}} A) \times (C \to_{\mathrm{a}} B)\\
&\mathrm{distr}_\times\ C : \prod_{\pi:\mathrm{Prod}_0\ C}\ \prod_{X:\boldsymbol{Ab}} (X \to_{\mathrm{a}} C) \to (X \to_{\mathrm{a}} A) \times (X \to_{\mathrm{a}} B)\\
&\mathrm{distr}_\times\ C\ (\pi_1, \pi_2)\ X\ f :\equiv (\pi_1 \circ f, \pi_2 \circ f)\\
&\mathrm{univ}_\times\ C\ \pi :\equiv \prod_{X:\boldsymbol{Ab}} \mathrm{isEquiv}\ (\mathrm{distr}_\times C\ \pi\ X)\\
&\mathrm{Prod}\ C :\equiv \sum_{\pi:\mathrm{Prod}_0\ C} (\mathrm{univ}_\times \pi)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Coprod}_0\ C :\equiv (A \to_{\mathrm{a}} C) \times (B \to_{\mathrm{a}} C)\\
&\mathrm{distr}_+\ C : \prod_{\kappa:\mathrm{Coprod}_0\ C}\ \prod_{X:\boldsymbol{Ab}} (C \to_{\mathrm{a}} X) \to (A \to_{\mathrm{a}} X) \times (B \to_{\mathrm{a}} X)\\
&\mathrm{distr}_+\ C\ (\kappa_1, \kappa_2)\ X\ f :\equiv (f \circ \kappa_1, f \circ \kappa_2)\\
&\mathrm{univ}_+\ C\ \kappa :\equiv \prod_{X:\boldsymbol{Ab}} \mathrm{isEquiv}\ (\mathrm{distr}_+ C\ \kappa\ X)\\
&\mathrm{Coprod}\ C :\equiv \sum_{\kappa:\mathrm{Coprod}_0\ C} (\mathrm{univ}_\times C\ \kappa)\\
&\mathrm{Biprod} :\equiv \sum_{C:\boldsymbol{Ab}}\ \sum_{p:\mathrm{Prod}\ C \times \mathrm{Coprod}\ C} \mathrm{compatibility}
\end{aligned}$$

Where compatibility is the property described above showing that π and κ are compatible.

Remark 3. *As isEquiv is a proposition and equalities in $\to_{\mathrm{a}}$ are propositions (because $\to_{\mathrm{a}}$ is a set), the only real structures are the carrier C, the projections and the coprojections.*

Theorem 18. *The type* Biproduct $A\ B$ *is a proposition.*

Proof. Suppose that there are two biproducts (X, π, κ) and (Y, π', κ') of A and B. We can define by unicity of the universal property of product an equality $p_\times : X = Y$ which commutes with π and π'. Similarly we can construct $p_+ : X = Y$ which commutes with κ and κ'. Finaly by a quick computation using the compatibility condition we can find that $p_\times = p_+$ so there is an equality which makes everything commute, so the two structures are equal. □

Theorem 19. *The type* Biproduct $A\ B$ *is contractible.*

Proof. We prove that there is an inhabitant of this type. The groupe is simply $A \times B$ with the point (a_0, b_0), and all the propositions follow directly. We define:

- $\kappa_1 : a \mapsto (a, b_0)$.
- $\kappa_2 : b \mapsto (a_0, b)$.
- $\pi_1 : (a, b) \mapsto a$.
- $\pi_2 : (a, b) \mapsto b$.

The fact that π_1, π_2 are a product structure is straightforward. For the coproduct structure, we take $X : \mathbf{Ab}$ and two functions $f : A \to_{\mathrm{a}} X$, $g : B \to_{\mathrm{a}} X$ and construct the center of the fiber $[f, g] : A \oplus_{\mathrm{a}} B \to_{\mathrm{a}} X$ as $(a, b) \mapsto f(a) +_{A \oplus_{\mathrm{a}} B} g(b)$. As the homogeneity acts componentwise we can always destruct (a, b) as $(a, b_0) +_{A \oplus_{\mathrm{a}} B} (a_0, b)$ and thus for $h : A \oplus_{\mathrm{a}} B \to_{\mathrm{a}} X$ we have $[h \circ \kappa_1, h \circ \kappa_2] : (a, b) \mapsto h(a, b)$ because every function commutes with $+$.

Finally, checking that $[\langle \mathrm{id}_{\mathrm{a}} A, 0_{\mathrm{a}}\ A\ B \rangle, \langle 0_{\mathrm{a}}\ A\ B, \mathrm{id}_{\mathrm{a}}\ B \rangle] = \mathrm{id}_{A \oplus_{\mathrm{a}} B}$ is straightforward. □

4.1.2 Generalizing to a finite family

We can change slightly the definitions to make a type of biproducts for any finite family of abelian groups. The proof that the type of such biproducts is a proposition is the same as the one for the binary case.

Theorem 20. *The type* $\mathrm{Biproduct}(A_i)$ *for a finite family* (A_i) *of abelian groups is contractible.*

Proof. We want to construct an element of this type, but this type is a proposition, so as a finite set J is a set $\langle J \rangle$ with a truncated equivalent $e : \|\langle J \rangle \simeq \mathrm{Fin}\ n\|_1$ for some $n : \mathbb{N}$, we can proceed by induction on this truncated equivalence (i.e. equality by univalence) and construct an inhabitant only for the case of $J = 1, \ldots, n$. But in this case, we can take the following construction :

- The biproduct of an empty family is 0_{a}, the initial and final element of **Ab**.
- The biproduct $\displaystyle\bigoplus_{j=1}^{n+1} A_j$ is $\displaystyle\left(\bigoplus_{j=1}^{n} A_j\right) \oplus A_{n+1}$.

□

4.2 The codiagonal

Now we want to sum elements of a same group. For this, we use the structure of biproduct to take the pairing on one side, to construct a function from the family (x_i) to an element of $\bigoplus_{j \in J} X$ and the codiagonal on the other side to map this element to an element of X.

Definition 21. *Let* J *be a finite set and* (A_j) *be a finite family of abelian groups over* J*. We define the space* ΠA_j *as the type*

$$\prod_{j:J} A_j$$

of families of elements of A_j *with the point* $\lambda j.(a_j)_0$*. This space is an abelian group and we have a natural family of functions*

$$\prod_{j:J} \Pi A_j \to_{\mathrm{a}} A_j$$

which gives rise to the function

$$\langle \_ \rangle : \Pi A_j \to_{\mathrm{a}} \bigoplus_{j \in J} A_j$$

Definition 22. *Let* J *be a finite set and* $X : \boldsymbol{Ab}$*. We define*

$$\nabla_X^J : \bigoplus_{j \in J} X \to_{\mathrm{a}} X$$

by

$$\nabla_X^J :\equiv [\mathrm{id_a} X, \dots, \mathrm{id_a} X]$$

Combining the two functions, we have the expected sum:

Definition 23. *Let $X : \boldsymbol{Ab}$ and J a finite set. We define the function Σ_X^J as follows:*

$$\begin{aligned}
&\Sigma_J^X : \Pi X \to_{\mathrm{a}} X \\
&\Sigma_J^X \ (x_j)_{j \in J} :\equiv \nabla_X^J (\langle\langle (x_j)_{j \in J} \rangle\rangle)
\end{aligned}$$

5 Definition of the orientation

We now combine the results of the previous two sections.

Definition 24. *Let F be a finite set. We define the decidable powerset of F and its subset of subsets of size 2:*

$$\begin{aligned}
&\mathbb{P}^d : \mathrm{FinSet} \to \mathcal{U} \\
&\mathbb{P}^d \ F :\equiv F \to \mathbb{2} \\
&\mathbb{P}_2^d : \mathrm{F}inSet \to \mathcal{U} \\
&\mathbb{P}_2^d \ F :\equiv \sum_{P : \mathbb{P}^d \ F} \| \mathrm{fiber} \ P \ 1 \simeq \mathbb{2} \|_1
\end{aligned}$$

Both $\mathbb{P}^d$ and $\mathbb{P}_2^d$ are finite sets for F finite.

Definition 25. *The orientation of a finite set F is the following torsor :*

$$\begin{aligned}
&\mathrm{or} : \mathrm{FinSet} \to 2\mathrm{Tors} \\
&\mathrm{or} \ F :\equiv \Sigma_{2\mathrm{Tors}} \ (\mathbb{P}_2^d \ F)
\end{aligned}$$

This leads to the following definition of the sign of a permutation.

Definition 26. *We define $\mathfrak{S}_n :\equiv \mathrm{Fin} \ n = \mathrm{Fin} \ n$ and the following function :*

$$\begin{aligned}
&\varepsilon : \mathfrak{S}_n \to \mathbb{2} \\
&\varepsilon \ \sigma = \mathrm{actToBool}(\mathrm{ap_{or}} \ \sigma)
\end{aligned}$$

Where actToBool *is the function which, given an action on a torsor, returns its corresponding boolean.*