Answers to Last Year's Tests

Limits Test

1. C
2. E
3. B
4. B
5. E
6. B
7. A
8. E
9. D
10. B
11. C
12. A
13. C
14. C
15. C

1. a. Since

$$f(x) = \frac{|x|(x-3)}{9-x^2} = \frac{|x|(x-3)}{(3-x)(3+x)} = \frac{-|x|}{3+x},$$

we have that both 3 and -3 are not in the domain; hence, $D = \{x|x \neq \pm 3\}$. The zeros are clearly 0 and 3, but 3 is not in the domain; hence, the only zero is 0.

b.

$$\lim_{x\to 3} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to 3} \frac{-|x|}{3+x} = -\frac{1}{2}.$$

c. Clearly, $x = -3$ is the only vertical asymptote since -3 makes the denominator zero. To find the horizontal asymptotes, simply find the limits at positive and negative infinity:

$$\lim_{x\to\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to\infty} \frac{-|x|}{3+x} = -1$$

$$\lim_{x\to-\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to-\infty} \frac{-|x|}{3+x} = 1$$

So there are two horizontal asymptotes: $y = 1$ and $y = -1$.

d. Based on the previous parts, it should be easy to see that $x = -3$ is an infinite discontinuity, and therefore is not removable. (Note that $x = 3$ is a hole and so is removable.)

2. a. We have the following values:

x	1	0.5	0.4	0.3
x^x	1	0.707	0.693	0.697

x	0.2	0.1	0.01
x^x	0.725	0.794	0.955

b. Judging from the data in the table, it appears as if both limits are 1. This is confirmed by the graphing calculator.

c. Any answer between 0.697 and 0.707 is fine as long as you justify it using values in the table.

d. The average rate of change is

$$\frac{g(0.4)-g(0.1)}{0.4-0.1} = \frac{0.693-0.794}{0.3} = -0.337.$$

3. a. This question becomes much simpler if you rewrite F as

$$(a^{-1}-x^{-1})^{-1} = \left(\frac{1}{a}-\frac{1}{x}\right)^{-1} = \left(\frac{x-a}{ax}\right)^{-1} = \frac{ax}{x-a}.$$

Then we can easily see that the domain $D = \{x|x \neq 0, x \neq a\}$ and that there are no zeros.

b. Since $x = a$ is not in the domain, $x = a$ is the vertical asymptote. Since the degree of the numerator is equal to the degree of denominator, we have $y = a$ as the horizontal asymptote. The discontinuities are the infinite discontinuity at $x = a$ and the removable discontinuity at $x = 0$.

c. $\lim_{x\to 0} F(x) = 0$; $\lim_{x\to\infty} F(x) = a$; and $\lim_{x\to a} F(x)$ does not exist.

d. Solve $\frac{6a}{6-a} = 12$ to get $a = 4$.

Derivatives Test

1. D	**6.** C	**11.** D
2. D	**7.** D	**12.** D
3. A	**8.** B	**13.** D
4. C	**9.** C	**14.** D
5. A	**10.** D	**15.** E

1. a. Taking the derivative implicitly, we have

$$\begin{aligned} y' - y' \sin y &= 1 \\ y'(1 - \sin y) &= 1 \\ y' &= \frac{1}{1 - \sin y} \end{aligned}$$

b. Vertical tangents have an undefined slope. Hence, we set the denominator of y' equal to zero and solve to get $\sin y = 1$, or $y = \pi/2$. Now we find the x value when $y = \pi/2$:

$$\begin{aligned} \frac{\pi}{2} + \cos\frac{\pi}{2} &= x + 1 \\ \frac{\pi}{2} &= x + 1 \\ x &= \frac{\pi}{2} - 1 \end{aligned}$$

Hence, the vertical tangent is $x = \frac{\pi}{2} - 1$.

c. We find the second derivative implicitly.

$$y'' = -\frac{y' \cos y}{(1 - \sin y)^2}$$

Now plug in the expression for y'.

$$\begin{aligned} y'' &= -\frac{\frac{1}{(1-\sin y)} \cos y}{(1 - \sin y)^2} \\ &= -\frac{\cos y}{(1 - \sin y)^3} \end{aligned}$$

2. a. The volume is $V = Bh$, where B is the area of the triangular base. Hence, $V = \left(\frac{1}{2}(3)(2)\right)(5) = 15$.

b. By similar triangles, we have

$$\frac{\text{base of triangle}}{\text{height of triangle}} = \frac{2}{3},$$

or $b = \frac{2}{3}h$; so that

$$V = \frac{1}{2}\left(\frac{2}{3}(h)(h)\right)(5) = \frac{5}{3}h^2.$$

When the trough is $\frac{1}{4}$ full by volume, we have $\frac{15}{4} = \frac{5}{3}h^2$, so $h = \frac{3}{2}$ at this instant. Now, we find the implicit derivative with respect to t:

$$\frac{dV}{dt} = \frac{10}{3}h\frac{dh}{dt}$$

and plug in our value of h:

$$\begin{aligned} -2 &= \frac{10}{3} \cdot \frac{3}{2} \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{2}{5} \end{aligned}$$

c. The area of the surface is $A = 5b = 5 \cdot \frac{2}{3}h = \frac{10}{3}h$. Finding the implicit derivative and using the value of dh/dt from part (b), we have

$$\begin{aligned} \frac{dA}{dt} &= \frac{10}{3} \cdot \frac{dh}{dt} \\ &= \frac{10}{3} \cdot \frac{-2}{5} = -\frac{4}{3} \end{aligned}$$

3. a. The domain is whatever makes $x^4 - 16x^2 \geq 0$, or $x^2(x^2 - 16) \geq 0$; thus, we find have either $x = 0$ or $x^2 \geq 16$. The domain is therefore $(-\infty,\ -4) \cup \{0\} \cup (4,\ \infty)$.

b. We have

$$\begin{aligned} f(-x) &= \sqrt{(-x)^4 - 16(-x)^2} \\ &= \sqrt{x^4 - 16x^2} = f(x) \end{aligned}$$

so f is even.

c. Observe:

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^4 - 16x^3)^{-1/2}(4x^3 - 32x) \\ &= \frac{2x^3 - 16x}{\sqrt{x^4 - 16x^3}} = \frac{2x(x^2 - 8)}{|x|\sqrt{x^2 - 16}} \end{aligned}$$

d. From part (c), we have

$$f'(5) = \frac{10(25 - 8)}{5\sqrt{25 - 16}} = \frac{34}{3}$$

so the slope of the normal is $-\frac{3}{34}$.

Applications of Derivatives Test

1. D
2. D
3. D
4. C
5. D
6. A
7. B
8. E
9. B
10. A
11. B
12. D
13. D
14. C
15. A

1. a. We have

$$\begin{aligned} v(t) &= x'(t) = 2\pi - 2\pi \sin 2\pi t \\ &= 2\pi(1 - \sin 2\pi t) \end{aligned}$$

b. We have

$$a(t) = v'(t) = x''(t) = -4\pi^2 \cos 2\pi t$$

c. The particle is at rest when $v(t) = 0$:

$$\begin{aligned} 2\pi(1 - \sin 2\pi t) &= 0 \\ \sin 2\pi t &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4}, \frac{9}{4} \end{aligned}$$

d. We find the critical points of $v(t)$ by setting $a(t) = 0$:

$$\begin{aligned} -4\pi^2 \cos 2\pi t &= 0 \\ \cos \pi t &= 0 \\ \pi t &= \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \\ t &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \end{aligned}$$

However, $v\left(\frac{1}{4}\right) = 0$ and $v\left(\frac{3}{4}\right) = 4\pi$ are the only possible maximum values (since all other odd multiples of $\frac{1}{4}$ give values equivalent to these two). Thus, the maximum velocity is 4π.

2. a. The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$. The absolute minimum must occur at $x = 1$ or at an endpoint. However, f is decreasing on the interval $[-1, 3]$; therefore, the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

b. There is an inflection point at $x = 1$ because the graph changes from concave up to concave down (or f'' changes from positive to negative) there.

3. a. We first find critical points:

$$\begin{aligned} f'(x) = 3x^2 - 10x + 3 &= 0 \\ (3x - 1)(x - 3) &= 0 \\ x &= \frac{1}{3} \text{ and } 3 \end{aligned}$$

Since f' is positive for $x < \frac{1}{3}$ and for $x > 3$, the increasing intervals are $\left(-\infty,\ \frac{1}{3}\right)$ and $(3,\ \infty)$.

b. Since $f''(x) = 6x - 10$, the inflection point is $x = \frac{5}{3}$. Thus, since f'' is negative for $x < \frac{5}{3}$, the graph of f is concave down on $\left(-\infty,\ \frac{5}{3}\right)$.

c. From part (a), we know that $x = 3$ gives the minimum value. Hence, we must have $f(3) = 11$:

$$\begin{aligned} f(3) = 3^3 - 5(3^2) + 3(3) + k &= 11 \\ -9 + k &= 11 \\ k &= 20 \end{aligned}$$

Integrals Test

1. E	**6.** D	**11.** C
2. B	**7.** A	**12.** B
3. E	**8.** B	**13.** E
4. B	**9.** C	**14.** D
5. C	**10.** B	**15.** A

1. a. We have $T(0) = -15$ and $T(12) = 5$. This gives the system of equations

$$\begin{aligned} -A - B &= -15 \\ -A + B &= 5 \end{aligned}$$

Hence, $A = 5$ and $B = 10$.

b.

$$\frac{1}{10}\int_0^{10}\left(-5 - 10\cos\left(\frac{\pi h}{12}\right)\right)\,dh = -6.910$$

c.

$$\begin{aligned} \int_6^{10} T(h)\,dh &= \frac{1}{2}[T(6) + 2T(7) + 2T(8) \\ &\quad + 2T(9) + T(10)] \\ &= \frac{1}{2}[-5 + 2(-2.412) + 2(0) \\ &\quad + 2(2.071) + 3.66] \\ &= -1.011 \end{aligned}$$

This integral represents the average temperature in degrees Fahrenheit from 6 AM to 10 AM.

d. Since $T(h) = -5 - 10\cos\left(\frac{\pi h}{12}\right)$, we have

$$T'(h) = -\frac{5\pi}{6}\sin\left(\frac{\pi h}{12}\right).$$

2. Differentiating the expression in 1) gives $f''(x) = 2ax + b$. From 2) we have that $f'(1) = 2a + b = 6$ and $f''(1) = 2a + b = 18$. Thus we have a system of equations in a and b that we can easily solve to get $a = 12$ and $b = -6$. Therefore

$$f'(x) = 12x^2 - 6x,$$

and

$$f(x) = \int (12x^2 - 6x)\,dx = 4x^3 - 3x^2 + C.$$

Using 3) we can solve for C:

$$\begin{aligned} 18 = \int_1^2 f(x)\,dx &= x^4 - x^3 + Cx\Big|_1^2 \\ &= 16 - 8 + 2C - (1 - 1 + C) = 8 + C \end{aligned}$$

thus, $C = 10$, and $f(x) = 4x^3 - 3x^2 + 10$.

3. a.

$$a(t) = v'(t) = -2\pi\cos(2\pi t)$$

b. Set $v(t) = 0$ and solve.

$$\begin{aligned} 1 - \sin(2\pi t) &= 0 \\ \sin(2\pi t) &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4} \end{aligned}$$

c.

$$\begin{aligned} x(t) &= \int v(t)\,dt \\ &= \int (1 - \sin(2\pi t))\,dt \\ &= t + \frac{1}{2\pi}\cos(2\pi t) + C \end{aligned}$$

Since $x(0) = 0$, we have

$$\begin{aligned} 0 &= 0 + \frac{\cos 0}{2\pi} + C \\ 0 &= \frac{1}{2\pi} + C \\ C &= -\frac{1}{2\pi} \end{aligned}$$

Thus, $x(t) = t + \frac{1}{2\pi}\cos(2\pi t) - \frac{1}{2\pi}$.

Applications of Integrals Test

1. A	**6.** D	**11.** B
2. B	**7.** A	**12.** B
3. E	**8.** A	**13.** D
4. A	**9.** C	**14.** B
5. E	**10.** D	**15.** C

1. a. First, we find the x-coordinates of the intersection points of the two graphs. Set them equal and solve, using your calculator:

$$4e^{-x} = \tan\left(\frac{x}{2}\right)$$
$$x = 1.4786108$$

Let $a = 1.4786108$. Thus, the area A is

$$A = \int_0^a \left(4e^{-x} - \tan\left(\frac{x}{2}\right)\right)\, dx = 2.483$$

b. The volume V is

$$V = \pi \int_0^a \left[\left(4e^{-x}\right)^2 - \left(\tan\left(\frac{x}{2}\right)\right)^2\right]\, dx$$
$$= 7.239\pi = 22.743$$

c. Since the diameter is in R, the length of the radius is $\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]$. The area of a semicircle with radius r is $A = \frac{1}{2}\pi r^2$. Hence,

$$A = \frac{\pi}{2}\left(\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]\right)^2$$
$$= \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2.$$

Therefore, the volume V is

$$V = \int_0^a \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2\, dx$$
$$= 0.755\pi = 2.373$$

2. a. You should have segments of zero slope at the three points where $x = 0$. You should have negative slopes with increasing steepness bottom to top at the points where $x = -1$. Finally, you should have positive slopes with increasing steepness from bottom to top and from left to right at the points where $x = 1$ and $x = 2$.

b. You should draw a graph that is concave up, decreasing for $x < 0$, increasing for $x > 0$, and that passes through the point $(0, 2)$.

c. To solve, we separate and integrate:

$$\frac{dy}{dx} = \frac{xy}{2}$$
$$\int \frac{dy}{y} = \int \frac{x}{2}\, dx$$
$$\ln y = \frac{1}{4}x^2 + C$$
$$y = Ce^{x^2/4}$$

With the initial condition, we find that $C = 2$, so the equation is $y = 2e^{x^2/4}$. Therefore, $y(2) = 2e^{4/4} = 2e = 5.4365$.

3. a. Since $v(1.5) = 1.167 > 0$ the particle is moving up the y-axis.

b. The acceleration is

$$a(t) = v'(t) = \sin(t^2) + 2t^2\cos(t^2)$$

so that $a(1.5) = -2.049 < 0$, which indicates the velocity is decreasing.

c. We have

$$y(t) = \int v(t)\, dt = -\frac{\cos(t^2)}{2} + C$$

and using the initial condition $y(0) = 3$, we find $C = \frac{7}{2}$. Hence,

$$y(t) = \frac{7 - \cos(t^2)}{2}.$$

Therefore, $y(2) = \frac{7-\cos 4}{2} = 3.827$.

d. The total distance is given by

$$\int_0^2 |v(t)|\, dt = 1.173,$$

or

$$\int_0^{\sqrt{\pi}} v(t)\, dt - \int_{\sqrt{\pi}}^2 v(t)\, dt = 1.173.$$

Techniques of Integration Test

1. C	**6.** D	**11.** B
2. B	**7.** C	**12.** C
3. C	**8.** C	**13.** C
4. C	**9.** C	**14.** C
5. E	**10.** D	**15.** C

1. a. The average value of f from 0 to 3 is

$$\frac{1}{3}\int_0^3 f(x)\ dx = \frac{5-1}{2},$$

and solving for the integral gives

$$\int_0^3 f(x)\ dx = 6.$$

b. Again, the average value of f from 0 to x is

$$\frac{1}{x}\int_0^x f(t)\ dt = \frac{5+f(x)}{2},$$

or

$$\int_0^x f(t)\ dt = \frac{5x + xf(x)}{2}.$$

Using the Fundamental Theorem to differentiate both sides, we have

$$\begin{aligned} f(x) &= \frac{5}{2} + \frac{1}{2}f(x) + \frac{1}{2}xf'(x) \\ 2f(x) &= 5 + f(x) + xf'(x) \\ f'(x) &= \frac{f(x)-5}{x}. \end{aligned}$$

c. From part (b), we have a differential equation that can be solved.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y-5}{x} \\ \int \frac{dy}{y-5} &= \int \frac{dx}{x} \\ \ln(y-5) &= \ln x + C \\ y - 5 &= Cx \\ y &= Cx + 5 \end{aligned}$$

and since $f(3) = -1$, we get that $C = -2$; hence, $y = f(x) = 5 - 2x$.

2. a.

$$\begin{aligned} R &= \int_1^3 \ln x\ dx = (x\ln x - x)\big|_1^3 \\ &= 3\ln 3 - 2 = 1.296. \end{aligned}$$

b.

$$V = \pi\int_1^3 (\ln x)^2\ dx = 1.029\pi = 3.233$$

c. We solve $y = \ln x$ for x to get $x = e^y$. When $x = 1$, $y = 0$, and when $x = 3$, $y = \ln 3$. Thus,

$$V = \pi\int_0^{\ln 3} (3 - e^y)\ dy$$

3. a.

$$\begin{aligned} \frac{dy}{dx} &= \frac{-xy}{\ln y} \\ \int \frac{\ln y}{y}\ dy &= \int -x\ dx \\ \frac{(\ln y)^2}{2} &= \frac{-x^2}{2} + C \\ (\ln y)^2 &= -x^2 + C \\ \ln y &= \pm\sqrt{C - x^2} \\ y &= e^{\pm\sqrt{C-x^2}} \end{aligned}$$

b. We find C.

$$\begin{aligned} y &= e^{\pm\sqrt{C-x^2}} \\ e^2 &= e^{\pm\sqrt{C}} \\ 2 &= \pm\sqrt{C} \\ C &= 4 \end{aligned}$$

so that $y = e^{\pm\sqrt{4-x^2}}$.

c. If $x = 2$, then $y = 1$ and $\ln y = 0$. This causes the derivative $\dfrac{-xy}{\ln y}$ to be undefined.

Series, Vectors, Parametric, and Polar Test

1. E	**6.** C	**11.** D
2. D	**7.** D	**12.** C
3. B	**8.** D	**13.** E
4. A	**9.** E	**14.** D
5. A	**10.** C	**15.** A

1. a.

$$\begin{aligned}\mathbf{v}(t) &= \left\langle -\frac{3\pi}{4}\sin\frac{\pi t}{4},\ \frac{5\pi}{4}\cos\frac{\pi t}{4}\right\rangle \\ \mathbf{v}(3) &= \left\langle -\frac{3\pi\sqrt{2}}{8},\ -\frac{5\pi\sqrt{2}}{8}\right\rangle \\ ||\mathbf{v}(3)|| &= \sqrt{\frac{18\pi^2}{64}+\frac{50\pi^2}{64}} \\ &= \frac{\pi\sqrt{17}}{4} = 1.031\pi = 3.238\end{aligned}$$

b.

$$\begin{aligned}\mathbf{a}(t) &= \left\langle -\frac{3\pi^2}{16}\cos\frac{\pi t}{4},\ -\frac{5\pi^2}{16}\sin\frac{\pi t}{4}\right\rangle \\ \mathbf{a}(3) &= \left\langle \frac{3\pi^2\sqrt{2}}{32},\ -\frac{5\pi^2\sqrt{2}}{32}\right\rangle \\ &= \langle 0.133\pi^2,\ -0.221\pi^2\rangle \\ &= \langle 1.309,\ -2.181\rangle\end{aligned}$$

c. Since

$$\sin^2\theta + \cos^2\theta = 1,$$

we have, upon solving $x(t)$ and $y(t)$ for the trigonometric terms,

$$\frac{x^2}{3}+\frac{y^2}{5}=1.$$

2. a. This curve is a 3-petal rose with petal tips at Cartesian coordinates $(\sqrt{3},1)$, $(-\sqrt{3},1)$, and $(0,-2)$.

b.

$$\frac{1}{2}\int_0^{\pi}(2\sin 3\theta)^2\,d\theta = \pi = 3.142$$

c.

$$\begin{aligned}\frac{dy}{dx} &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &= \frac{6\cos 3\theta\sin\theta + 2\sin 3\theta\cos\theta}{6\cos 3\theta\cos\theta - 2\sin 3\theta\sin\theta} \\ \left.\frac{dy}{dx}\right|_{\theta=\pi/4} &= \frac{1}{2}\end{aligned}$$

3. a.

$$f(x)\approx 5-3x+\frac{x^2}{2}+\frac{4x^3}{6}$$

b.

$$g(x)\approx 5-3x^2+\frac{x^4}{2}$$

c.

$$h(x)\approx 5x-\frac{3x^2}{2}+\frac{x^3}{6}$$

d. $h(1)=\int_0^1 f(t)\,dt$, but the exact value cannot be determined since $f(t)$ is only known at $t=0$ and $t=1$.