Departemen Matematika FMIPA IPB

UJIAN TENGAH SEMESTER GANJIL 2014/2015

Kode - Nama MK : **MAT211 - Kalkulus II**
Hari/Tanggal : **Jumat/31 Oktober 2014**
Waktu : **2 Jam**
Sifat Ujian : **Catatan Tertutup**

Selesaikan ke-10 soal berikut **secara berurutan** dengan **jujur**, **teliti**, dan **sepenuh kemampuan**. Segala bentuk kecurangan bersanksi akademik. Nilai maksimum setiap soal adalah 10.

1. Tentukan
$$\int \tan^3 x \, dx.$$

Jawab

$$
\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\
&= \int \tan x \left(\sec^2 x - 1 \right) dx \\
&= \int \tan x \sec^2 x \, dx - \int \tan x \, dx
\end{aligned}
$$

$$
\begin{aligned}
u &= \tan x \\
du &= \sec^2 x \, dx
\end{aligned}
$$

$$
\begin{aligned}
\int \tan x \sec^2 x \, dx &= \int u \, du \\
&= \frac{1}{2} u^2 + c_1 \\
&= \frac{1}{2} \tan^2 x + c_1
\end{aligned}
$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$
\begin{aligned}
v &= \cos x \\
dv &= \sin x \, dx
\end{aligned}
$$

$$
\begin{aligned}
\int \tan x \, dx &= \int \frac{1}{v} \, dv \\
&= \ln |v| + c_2 \\
&= \ln |\cos x| + c_2
\end{aligned}
$$

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + c$$

2. Tentukan

$$\int \sin^3 x \cos^{-3} x dx.$$

Jawab

$$
\begin{aligned}
\int \sin^3 x \cos^{-3} x dx &= \int \cos^{-3} x \sin x \sin^2 x dx \\
&= \int \cos^{-3} x \sin x \left(1 - \cos^2 x\right) dx \\
&= \int \cos^{-3} x \sin x dx - \int \cos^{-1} x \sin x dx
\end{aligned}
$$

$$
\begin{aligned}
u &= \cos x \\
du &= -\sin x dx \\
-du &= \sin x dx
\end{aligned}
$$

$$
\begin{aligned}
\int \sin^3 x \cos^{-3} x dx &= -\int u^{-3} du + \int u^{-1} du \\
&= \frac{1}{2} u^{-2} + \ln |u| + c \\
&= \frac{1}{2} \cos^{-2} x + \ln |\cos x| + c
\end{aligned}
$$

3. Tentukan
$$\int \frac{x^3}{\sqrt{1-x^2}}dx$$

dengan $|x| < 1$.

Jawab

Misalkan $x = \sin\theta$, maka $dx = \cos\theta d\theta$, sehingga

$$
\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}dx &= \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta \\
&= \int \frac{\sin^3\theta}{\sqrt{\cos^2\theta}}\cos\theta d\theta \\
&= \int \frac{\sin^3\theta}{\cos\theta}\cos\theta d\theta \\
&= \int \sin^3\theta d\theta \\
&= \int \sin^2\theta\sin\theta d\theta \\
&= \int \left(1-\cos^2\theta\right)\sin\theta d\theta \\
&= \int \sin\theta d\theta - \int \cos^2\theta\sin\theta d\theta
\end{aligned}
$$

$$\int \sin\theta d\theta = -\cos\theta + c_1$$

$$
\begin{aligned}
u &= \cos\theta \\
du &= -\sin\theta d\theta \\
-du &= \sin\theta d\theta
\end{aligned}
$$

$$
\begin{aligned}
\int \cos^2\theta\sin\theta d\theta &= -\int u^2 du \\
&= -\frac{1}{3}u^3 + c_2 \\
&= -\frac{1}{3}\cos^3\theta + c_2
\end{aligned}
$$

$$\int \frac{x^3}{\sqrt{1-x^2}}dx = -\cos\theta + \frac{1}{3}\cos^3\theta + c$$

Karena $x = \sin\theta$, maka $\cos\theta = \sqrt{1-x^2}$, sehingga

$$\int \frac{x^3}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + c$$

4. Hitunglah limit bentuk taktentu berikut:

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right).$$

Jawab

$$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) \text{ adalah bentuk taktentu } \infty - \infty$$

$$= \lim_{x \to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} \text{ adalah bentuk taktentu } \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\ln x + 1 - \frac{1}{x}} \text{ adalah bentuk taktentu } \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to 1} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

5. Tentukan integral takwajar berikut atau tunjukkan bahwa integral takwajar tersebut divergen.

$$\int_0^3 \left(\frac{1}{1-x}\right)^2 dx.$$

Jawab

Fungsi f dengan $f(x) = \left(\frac{1}{1-x}\right)^2$ kontinu pada $\mathbb{R} - \{1\}$.

$$\begin{aligned} u &= 1-x \\ du &= -dx \\ -du &= dx \end{aligned}$$

$$\begin{aligned} \int \left(\frac{1}{1-x}\right)^2 dx &= -\int \left(\frac{1}{u}\right)^2 du \\ &= -\int \frac{1}{u^2} du \\ &= \frac{1}{u} + c \\ &= \frac{1}{1-x} + c \end{aligned}$$

$$\begin{aligned} \int_0^1 \left(\frac{1}{1-x}\right)^2 dx &= \lim_{t \to 1^-} \int_0^t \left(\frac{1}{1-x}\right)^2 dx \\ &= \lim_{t \to 1^-} \left[\frac{1}{1-x}\right]_0^t \\ &= \lim_{t \to 1^-} \left(\frac{1}{1-t} - 1\right) \\ &= \infty \end{aligned}$$

$\int_0^1 \left(\frac{1}{1-x}\right)^2 dx$ divergen, sehingga $\int_0^3 \left(\frac{1}{1-x}\right)^2 dx$ divergen.

6. Terdapat suatu kurva dengan persamaan $x = 3\cos t$, $y = 3\sin t$ untuk $0 \leq t \leq \pi$. Tentukan $\dfrac{dy}{dx}$ dan $\dfrac{d^2y}{dx^2}$.

Jawab

$$
\begin{aligned}
\frac{dx}{dt} &= -3\sin t \\
\frac{dy}{dt} &= 3\cos t \\
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-3\sin t} = -\cot t
\end{aligned}
$$

$$
\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-\cot t\right) = \csc^2 t
$$

$$
\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\csc^2 t}{-3\sin t} = -\frac{1}{3}\csc^3 t
$$

7. Diberikan persamaan polar berikut:

$$r = 3 + 6\cos\theta, \quad 0 \le \theta \le 2\pi.$$

(a) Tentukan $\dfrac{dx}{d\theta}$, $\dfrac{dy}{d\theta}$, dan $\dfrac{dy}{dx}$.

(b) Tentukan persamaan garis singgung pada kurva saat $\theta = \dfrac{\pi}{2}$.

$\boxed{\text{Jawab}}$

(a)

$$
\begin{aligned}
x &= r\cos\theta = (3 + 6\cos\theta)\cos\theta \\
\frac{dx}{d\theta} &= (-6\sin\theta)\cos\theta + (3 + 6\cos\theta)(-\sin\theta) = -3\sin\theta - 6\sin(2\theta) \\
y &= r\sin\theta = (3 + 6\cos\theta)\sin\theta \\
\frac{dy}{d\theta} &= (-6\sin\theta)\sin\theta + (3 + 6\cos\theta)\cos\theta = 3\cos\theta + 6\cos(2\theta) \\
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta + 6\cos(2\theta)}{-3\sin\theta - 6\sin(2\theta)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
m &= \frac{0 - 6}{-3 - 0} = 2 \\
x &= 0 \\
y &= 3
\end{aligned}
$$

Persamaan garis singgung:

$$
\begin{aligned}
y - 3 &= 2(x - 0) \\
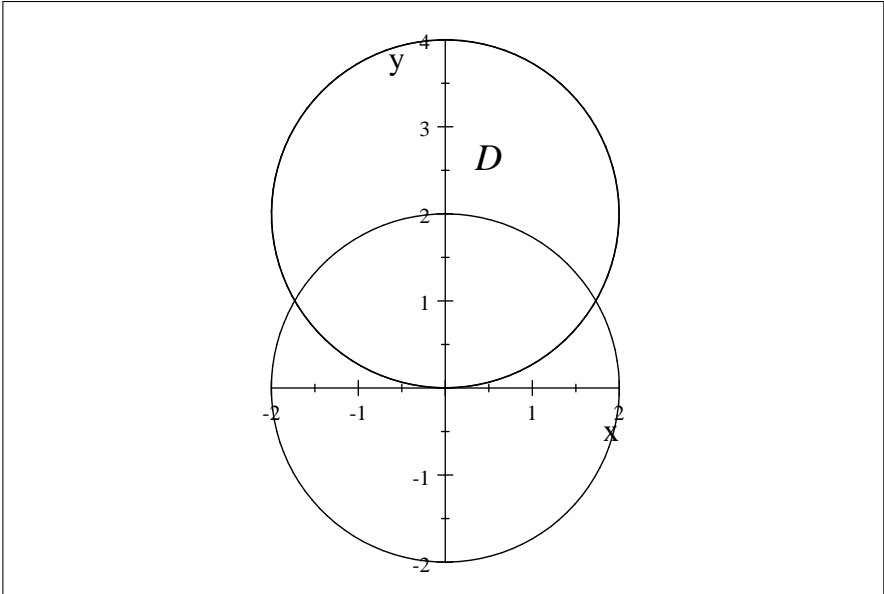y &= 2x + 3
\end{aligned}
$$

8. Misalkan daerah D terletak di dalam lingkaran $r = 4\sin\theta$ dan di luar lingkaran $r = 2$.

   (a) Gambarkan daerah D.
   (b) Tentukan luas daerah D.

Jawab

   (a)

θ	$r = 4\sin\theta$	$r = 2$
0	0	2
$\frac{\pi}{4}$	$2\sqrt{2}$	2
$\frac{\pi}{2}$	4	2
$\frac{3\pi}{4}$	$2\sqrt{2}$	2
π	0	2
$\frac{5\pi}{4}$	$-2\sqrt{2}$	2
$\frac{3\pi}{2}$	-4	2
$\frac{7\pi}{4}$	$-2\sqrt{2}$	2
2π	0	2

   (b) Titik potong

$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6} \quad \theta = \frac{5\pi}{6}$$

Luas

$$
\begin{aligned}
L &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{1}{2}(4\sin\theta)^2 - \frac{1}{2}(2)^2 \right) d\theta \\[2mm]
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 2 \right) d\theta \\[2mm]
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4 - 4\cos(2\theta) - 2 \right) d\theta \\[2mm]
&= \left[2\theta - 2\sin(2\theta) \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\[2mm]
&= \left(\frac{5\pi}{3} + \sqrt{3} \right) - \left(\frac{\pi}{3} - \sqrt{3} \right) \\[2mm]
&= \frac{4\pi}{3} + 2\sqrt{3}
\end{aligned}
$$

9. Tentukan persamaan elips yang memiliki titik-titik fokus di $(0, -1)$ dan $(0, 1)$ serta keeksentrikan $\frac{1}{2}$.

Jawab

$$c = 1$$

$$\begin{aligned} e &= \frac{1}{2} \\ \frac{c}{a} &= \frac{1}{2} \\ \frac{1}{a} &= \frac{1}{2} \\ a &= 2 \end{aligned}$$

$$b^2 = a^2 - c^2 = 4 - 1 = 3$$

Persamaan elips:

$$\begin{aligned} \frac{x^2}{b^2} + \frac{y^2}{a^2} &= 1 \\ \frac{x^2}{3} + \frac{y^2}{4} &= 1 \end{aligned}$$

10. Diberikan persamaan dalam koordinat polar berikut:

$$r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0.$$

(a) Tunjukkan bahwa dalam koordinat kartesius persamaan di atas merupakan lingkaran.

(b) Tentukan titik pusat dan jari-jari lingkaran tersebut.

Jawab

(a)

$$\begin{aligned}
x^2 + y^2 - 6x - 4y + 9 &= 0 \\
x^2 - 6x + 9 + y^2 - 4y + 4 &= 4 \\
(x-3)^2 + (y-2)^2 &= 2^2
\end{aligned}$$

(b) Titik pusat: $(3, 2)$

Jari-jari: $r = 2$