Kalkulus II

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(1) (a) Rumus eksplisit barisan dan tentukan kekonvergenannya :

$$\cos \pi, \frac{\cos 2\pi}{4}, \frac{\cos 3\pi}{9}, \frac{\cos 4\pi}{16}, \ldots$$

rumus eksplisit : $\frac{\cos n\pi}{n^2}$

kekonvergenan : $-1 \leq \cos n\pi \leq 1$

$$\frac{-1}{n^2} \leq \frac{\cos n\pi}{n^2} \leq \frac{1}{n^2}$$

$$\frac{-1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{-1}{n^2} = 0$$

$$\frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{\cos n\pi}{n^2} = 0$$ $\Rightarrow$ konvergen ke 0.

(b.) $\{a_n\}$ konvergen ke A $\Rightarrow \lim_{n\to\infty} a_n = A$, maka setiap $\varepsilon_1 > 0$ terdapat $N_1 > 0$ sehingga $n > N_1$, berlaku :

$$|a_n - A| < \frac{1}{2}\varepsilon$$

$\{b_n\}$ konvergen ke B $\Rightarrow \lim_{n\to\infty} b_n = B$, maka setiap $\varepsilon_2 > 0$ terdapat $N_2 > 0$ sehingga $n > N_2$, berlaku:

$$|b_n - B| < \frac{1}{2}\varepsilon$$

$N = \max\{N_1, N_2\}$. Diperoleh :

$$|a_n + b_n - (A+B)| = |(a_n - A) + (b_n - B)|$$
$$\leq |a_n - A| + |b_n - B|$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$
$$= \varepsilon$$

maka dapat dibuktikan $\lim_{n\to\infty} (a_n + b_n) = A + B$

(c.) Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan berikut :

$$a_n = \frac{\sin n\pi}{4}$$

$\Rightarrow$ Kemonotonan : $a_n = a_{n+1} = \frac{\sin n\pi}{4} - \frac{\sin (n+1)\pi}{4} = \frac{\sin n\pi}{4} - \left(\frac{\sin \pi}{4} - \frac{\sin n\pi}{4}\right) = -\frac{\sin \pi}{4}$

•) Limit : $-1 \leq \sin n\pi \leq 1$

$$\frac{-1}{4} \leq \frac{\sin n\pi}{4} \leq \frac{1}{4} \quad \text{(divergen)}$$

•) Keterbatasan : memiliki batas $[-1, 1]$.

(2.) (a.) Rumus eksplisit dan tentukan kekonvergenannya

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$$

rumus eksplisit

$$a_n = \frac{1}{-n(-1^n)}$$

kekonvergenan : $\lim_{n \to \infty} \frac{1}{-n(-1^n)}$ (divergen)

(b.) Dengan definisi limit, buktikan barisan $\{a_n\}$ berikut konvergennya :

$$a_n = \frac{3 - 8 \cdot 2^n}{5 + 4 \cdot 2^n}$$

•) Limit : $\lim_{n \to \infty} \frac{3 - 8 \cdot 2^n}{5 + 4 \cdot 2^n} = \lim_{n \to \infty} \frac{\frac{3}{2^n} - 8}{\frac{5}{2^n} + 4} = \frac{0 - 8}{0 + 4} = -2$

•) Pembuktian barisan :

$L = -2$

$$|a_n - L| = \left| \frac{3 - 8 \cdot 2^n}{5 + 4 \cdot 2^n} + 2 \right| = \frac{3 - 8 \cdot 2^n + 10 \cdot 8 + 2^n}{5 + 4 \cdot 2^n} = \frac{13}{5 + 4 \cdot 2^n}$$

$$< \frac{13}{5 + 4 \cdot 2^N}$$

$$= \frac{13}{5 + 4\left(\frac{\frac{13}{\varepsilon} - 5}{4}\right)} = \varepsilon$$

•) Konvergen ke -2.

(c.) Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan berikut :

$$a_n = \frac{\ln n}{n}$$

•) Kemonotonan

$$a'(n) = \frac{1/n \cdot n - \ln n}{n^2} = \frac{1 - \ln(n)}{n^2}$$

$= a'(n) > 0$

$= \frac{1 - \ln(n)}{n^2} > 0$

$= \ln(e) - \ln(n) > 0$

$= \ln(e) > \ln(n)$

$= e > n$ (naik pada $0, e$)

$= a'(n) < 0$

$= \frac{1 - \ln(n)}{n^2} < 0$

$= 1 - \ln(n) < 0$

$= \ln(e) < \ln(n)$

$= e < n$ (turun pada e, ∞)

•> Keterbatasan : Melakukan pengecekan pada a_1, a_2, a_3.

$$a_1 = \frac{\ln(1)}{1} = 0$$

$$a_2 = \frac{\ln(2)}{2} \approx 0,34657$$

$$a_3 = \frac{\ln(3)}{3} \approx 0,3662$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{n} = 0$$

maka $\{a_n\}$ terbatas di bawah oleh 0 dan terbatas di atas oleh a_3.

(3) (a) Rumus eksplisit dan tentukan kekonvergenannya :

0,9, 0,99, 0,999, 0,9999, ...

rumus eksplisit = $a_n = \left(1 - \frac{1}{10^n}\right)$

konvergen = $\lim_{n\to\infty} 1 - \frac{1}{10^n} = 1 - \frac{1}{10^\infty} = 1 - 0 = 1$

(b) Dengan definisi limit, buktikan barisan $\{a_n\}$ konvergen :

$$a_n = \frac{n+3}{3n-2} = \lim_{n\to\infty} \frac{n+3}{3n-2} = \frac{1}{3} \text{ (pangkat terbesar)}$$

• misal $\varepsilon > 0$

$$|a_n - L| = \left|\frac{n+3}{3n-2} - \frac{1}{3}\right| = \left|\frac{3n+9-(3n-2)}{9n-6}\right| = \frac{11}{9n-6}$$

• untuk $n > N$

$$\frac{11}{9n-6} < \frac{11}{9N-6} = \varepsilon, \frac{11}{9N-6} = \varepsilon \Leftrightarrow 9N-6 = \frac{11}{\varepsilon} \Leftrightarrow N = \frac{\frac{11}{\varepsilon}+6}{9}$$

• maka $|a_n - L| = \left|\frac{n+3}{3n-2} - \frac{1}{3}\right| = \frac{11}{9n-6} < \frac{11}{9N-6}$

$$= \frac{11}{9N-6}$$

$$= \frac{11}{9\left(\frac{11/\varepsilon+6}{9}\right)-6} = \frac{11}{(11/\varepsilon+6)-6} = \frac{11}{11/\varepsilon} = \varepsilon$$

(c) Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan :

$$a_n = \frac{n!}{10^n}$$

→ Kemonotonan : $a_n = \frac{n!}{10^n} \Leftrightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{n+1}{10}$

maka : $\frac{a_{n+1}}{a_n} \leq 1$, dengan $\{a_n\}$ tak naik pada $n = 1, 2, 3, \ldots, 9$

$\frac{a_{n+1}}{a_n} > 1$, dengan $\{a_n\}$ naik pada $n = 10, 11, 12, \ldots$

→ Keterbatasan :

$\{a_n\}$ tak naik pada $n = 1, 2, 3, \ldots, 9$, yakni : $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 9$

maka : $a_9 = \frac{9!}{10^9} \approx 3.6288 \times 10^{-4}$ (batas bawah)

$a_{10} = \frac{10!}{10^{10}} = \frac{10 \cdot 9!}{10 \cdot 10^9} = \frac{9!}{10^9}$ (batas bawah duga)

$\{a_n\}$

$$\lim_{n \to \infty} \frac{n!}{10^n} = \lim_{n \to \infty} \frac{n(n-1)(n-2) \ldots 3 . 2 . 1}{10 . 10 . 10 \ldots . 10}$$

$$= \lim_{n \to \infty} \frac{n}{10} \cdot \frac{n-1}{10} \cdot \frac{n-2}{10} \ldots \frac{3}{10} \cdot \frac{2}{10} \cdot \frac{1}{10}$$

$$= \frac{\infty}{10} \cdot \frac{\infty - 1}{10} \cdot \frac{\infty - 2}{10} \ldots \frac{3}{10} \cdot \frac{2}{10} \cdot \frac{1}{10}$$

$= \infty$ (batas atas tak hingga)