Departemen Matematika FMIPA IPB

UJIAN TENGAH SEMESTER GANJIL 2015/2016

Kode - Nama MK	**: MAT211 - Kalkulus II**
Hari/Tanggal	**: Jumat/6 November 2015**
Waktu	**: 2 Jam**
Sifat Ujian	**: Catatan Tertutup**

Selesaikan ke-8 soal berikut **secara berurutan**. Bekerjalah dengan **jujur**, **teliti**, dan **sepenuh kemampuan**. Segala bentuk kecurangan bersanksi akademik.

1. **(13 poin)** Tentukan

$$\int (\cos x \sin x)^2 \cos x dx.$$

Jawab

$$\begin{aligned}\int (\cos x \sin x)^2 \cos x dx &= \int \cos^2 x \sin^2 x \cos x dx \\ &= \int \left(1-\sin^2 x\right) \sin^2 x \cos x dx \\ &= \int \left(\sin^2 x - \sin^4 x\right) \cos x dx\end{aligned}$$

Misalkan $u = \sin x$, maka $du = \cos x dx$

$$\begin{aligned}\int (\cos x \sin x)^2 \cos x dx &= \int \left(u^2 - u^4\right) du \\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \\ &= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c\end{aligned}$$

2. **(12 poin)** Tentukan

$$\lim_{x \to 0^+} x^x.$$

Jawab

$$\begin{aligned}
& \lim_{x \to 0^+} x^x \qquad (\text{bentuk } 0^0) \\
= \; & \exp\left(\lim_{x \to 0^+} x \ln x\right) \qquad (\text{bentuk } 0 \times \infty) \\
= \; & \exp\left(\lim_{x \to 0^+} \frac{\ln x}{1/x}\right) \qquad (\text{bentuk } \frac{\infty}{\infty}) \\
& \stackrel{L}{=} \exp\left(\lim_{x \to 0^+} \frac{1/x}{-1/x^2}\right) \qquad (\text{bentuk } \frac{\infty}{\infty}) \\
= \; & \exp\left(\lim_{x \to 0^+} -x\right) \\
= \; & \exp(0) \\
= \; & 1
\end{aligned}$$

3. **(13 poin)** Tentukan

$$\int \frac{1}{\sqrt{x^2-b^2}}dx$$

dengan b suatu konstanta real positif. (Diketahui: $\int \sec x dx = \ln|\sec x + \tan x| + c$)

$\boxed{\textbf{Jawab}}$

Misalkan $x = b\sec\theta$, maka $dx = b\sec\theta\tan\theta d\theta$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-b^2}}dx &= \int \frac{1}{\sqrt{(b\sec\theta)^2-b^2}}b\sec\theta\tan\theta d\theta \\
&= \int \frac{1}{\sqrt{b^2\sec^2\theta-b^2}}b\sec\theta\tan\theta d\theta \\
&= \int \frac{1}{\sqrt{b^2\left(\sec^2\theta-1\right)}}b\sec\theta\tan\theta d\theta \\
&= \int \frac{1}{\sqrt{b^2\tan^2\theta}}b\sec\theta\tan\theta d\theta \\
&= \int \frac{1}{b\tan\theta}b\sec\theta\tan\theta d\theta \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta+\tan\theta| + c
\end{aligned}$$

Karena $\sec\theta = \frac{x}{b}$, maka $\tan\theta = \frac{\sqrt{x^2-b^2}}{b}$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-b^2}}dx &= \ln\left|\frac{x}{b}+\frac{\sqrt{x^2-b^2}}{b}\right| + c \\
&= \ln\left|\frac{x+\sqrt{x^2-b^2}}{b}\right| + c
\end{aligned}$$

4. **(12 poin)** Tentukan

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

$\boxed{\textbf{Jawab}}$

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^0 \\
&= \lim_{a\to-\infty} \left(0-\tan^{-1} a\right) \\
&= -\left(-\frac{\pi}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b \\
&= \lim_{b\to\infty} \left(\tan^{-1} b-0\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \frac{\pi}{2}+\frac{\pi}{2} \\
&= \pi
\end{aligned}$$

5. **(12 poin)** Diberikan kurva dalam persamaan parametrik berikut:

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Buktikan bahwa

$$\frac{d^2y}{dx^2} = -\frac{1}{(1-\cos t)^2}.$$

Jawab

Karena $\dfrac{dx}{dt} = 1 - \cos t$ dan $\dfrac{dy}{dt} = \sin t$ maka

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\sin t}{1-\cos t}.$$

Selanjutnya,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d\left(\frac{dy}{dx}\right)}{dx} \\
&= \frac{d\left(\frac{dy}{dx}\right)}{dt}\frac{dt}{dx} \\
&= \frac{\cos t(1-\cos t) - \sin t(\sin t)}{(1-\cos t)^2}\frac{1}{1-\cos t} \\
&= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^3} \\
&= \frac{-(1-\cos t)}{(1-\cos t)^3} \\
&= -\frac{1}{(1-\cos t)^2}.
\end{aligned}$$

6. **(13 poin)** Diberikan persamaan kurva

$$y^2 - 4x^2 = 4.$$

(a) Tuliskan dalam bentuk baku persamaan hiperbola.
(b) Tentukan kedua titik fokus.
(c) Tentukan kedua titik puncak.
(d) Tentukan persamaan asimtot.
(e) Buat sketsa kurva.

Jawab

(a) Bentuk baku persamaan hiperbola:

$$y^2 - 4x^2 = 4 \Leftrightarrow \frac{y^2}{4} - \frac{x^2}{1} = 1.$$

$$\begin{aligned} a^2 &= 4 \\ a &= 2 \end{aligned}$$

$$\begin{aligned} b^2 &= 1 \\ b &= 1 \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 4 + 1 \\ &= 5 \\ c &= \sqrt{5} \end{aligned}$$

(b) Titik fokus:

$$(0, \pm c)$$
$$\left(0, -\sqrt{5}\right) \text{ dan } (0, \sqrt{5}).$$

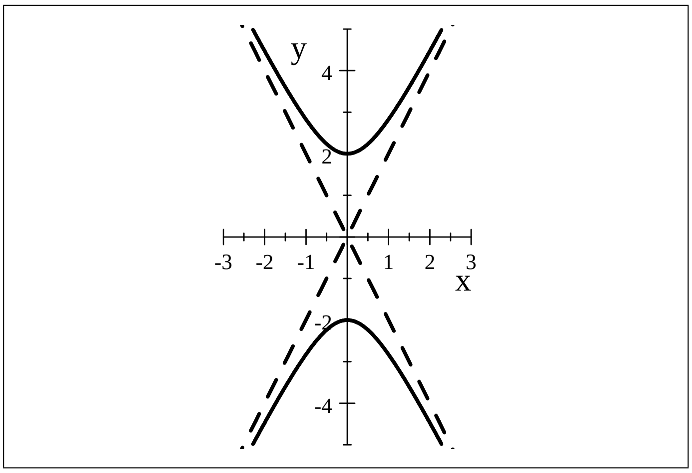
(c) Titik puncak:

$$(0, \pm a)$$
$$(0, -2) \text{ dan } (0, 2).$$

(d) Persamaan asimtot:

$$\begin{aligned} y &= \pm\frac{a}{b}x \\ y &= -2x \text{ dan } y = 2x. \end{aligned}$$

(e) Sketsa kurva:



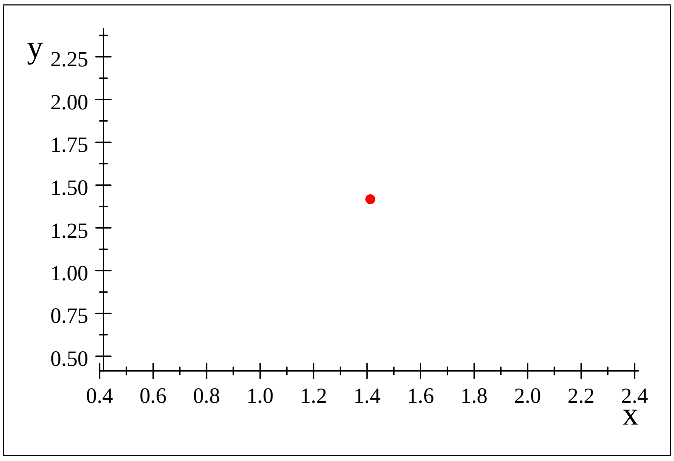
7. **(12 poin)** Plot titik-titik dengan koordinat polar berikut dan tentukan koordinat Cartesiusnya.

(a) $\left(2, \frac{\pi}{4}\right)$,

(b) $\left(-3, -\frac{5\pi}{4}\right)$.

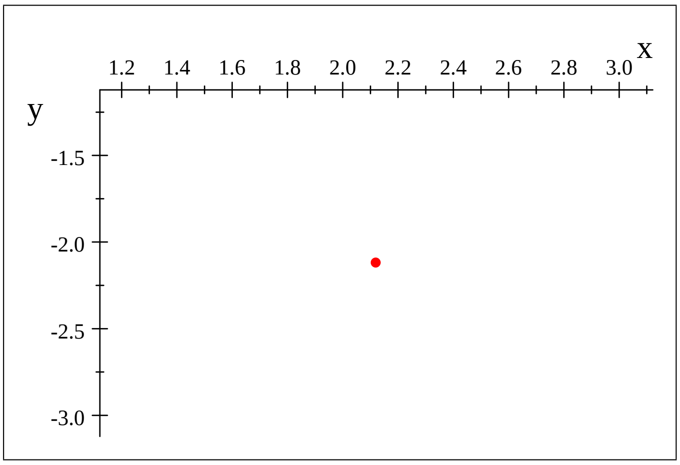
$\boxed{\textbf{Jawab}}$

(a) $\left(2, \frac{\pi}{4}\right)$



$$
\begin{aligned}
x &= 2\cos\frac{\pi}{4} = 2\frac{1}{2}\sqrt{2} = \sqrt{2} \\
y &= 2\sin\frac{\pi}{4} = 2\frac{1}{2}\sqrt{2} = \sqrt{2} \\
(x, y) &= \left(\sqrt{2}, \sqrt{2}\right)
\end{aligned}
$$

(b) $\left(-3, -\frac{5\pi}{4}\right)$



$$
\begin{aligned}
x &= -3\cos\left(-\frac{5\pi}{4}\right) = -3\left(-\frac{1}{2}\sqrt{2}\right) = \frac{3}{2}\sqrt{2} \\
y &= -3\sin\left(-\frac{5\pi}{4}\right) = -3\left(\frac{1}{2}\sqrt{2}\right) = -\frac{3}{2}\sqrt{2} \\
(x, y) &= \left(\frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)
\end{aligned}
$$

8. **(13 poin)** Misalkan daerah D terletak di bawah kurva parametrik

$$x = 4\cos t, \quad y = 3\sin t, \quad 0 \le t \le \pi$$

dan di atas sumbu-x.

(a) Gambarkan daerah D.

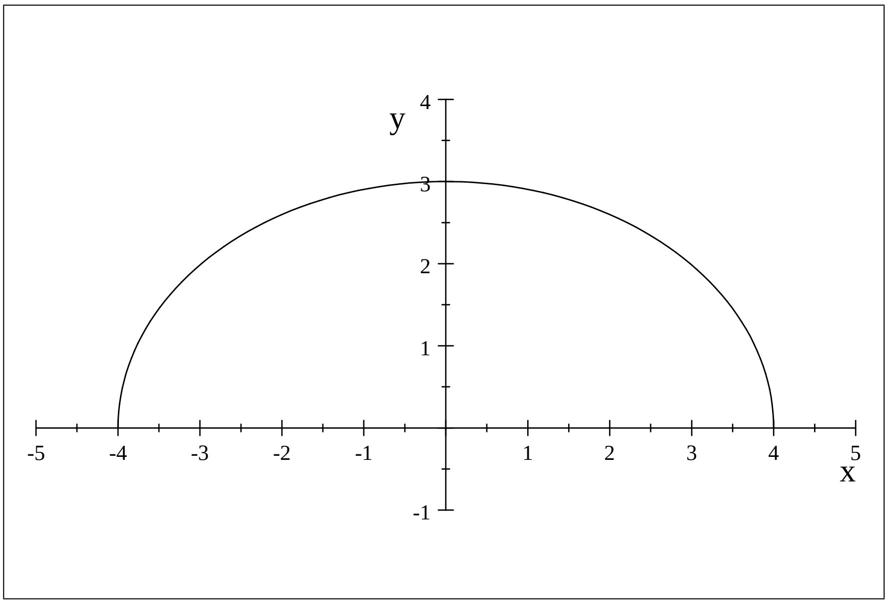
(b) Tentukan luas daerah D.

$\boxed{\textbf{Jawab}}$

(a)

t	x	y
0	4	0
$\frac{\pi}{4}$	$2\sqrt{2}$	$\frac{3}{2}\sqrt{2}$
$\frac{\pi}{2}$	0	3
$\frac{3\pi}{4}$	$-2\sqrt{2}$	$\frac{3}{2}\sqrt{2}$
π	-4	0

Daerah D



(b) Luas daerah D:

$$\begin{aligned}
L &= \int_{\pi}^{0} (3\sin t)\,(-4\sin t)\,dt \\
&= 12\int_0^{\pi} \sin^2 t dt \\
&= 12\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt \\
&= 12\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi} \\
&= 12\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right] \\
&= 6\pi
\end{aligned}$$