K1 - MAT211 Kalkulus II
Bentuk Taktentu

TBK
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Masih ingatkah dengan bentuk-bentuk berikut:

$$
\begin{aligned}
\frac{0}{1} &= 0, \\
\frac{1}{0} &= \text{tak-terdefinisi}, \\
\lim_{x \to 0} \frac{1}{x^2} &= +\infty \text{ (tidak ada)}, \\
\frac{2}{0} &= \text{tak-terdefinisi}, \\
\frac{1}{\infty} &\Rightarrow \lim_{x \to \infty} \frac{1}{x} = 0, \\
\frac{2}{\infty} &\Rightarrow \lim_{x \to \infty} \frac{2}{x} = 0.
\end{aligned}
$$

Di PPKU Anda sudah bertemu dengan limit-limit berikut:

$$
\begin{aligned}
\lim_{x \to 0^+} \frac{1}{x} &= +\infty, \lim_{x \to 0^-} \frac{1}{x} = -\infty, \text{ sehingga } \lim_{x \to 0} \frac{1}{x} = \text{tidak ada.} \\
\lim_{x \to 0} \frac{1}{x^2} &= +\infty \\
\lim_{x \to \infty} \frac{1}{x} &= 0 \\
\lim_{x \to -\infty} \frac{1}{x} &= 0.
\end{aligned}
$$

Pernahkah Anda mengerjakan soal limit dengan cara berikut?

$$
\begin{aligned}
\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2} &\stackrel{*}{=} \lim_{x \to 2} \frac{2x + 2}{1} = 6. \\
\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2} &= \lim_{x \to 2} \frac{(x + 4)(x - 2)}{x - 2} = 6.
\end{aligned}
$$

1 Bentuk Taktentu (*Indetermined Forms*)

Type	Example
$[0/0]$	$\lim_{x\to 0} \frac{\sin x}{x}$
$[\infty/\infty]$	$\lim_{x\to 0} \frac{\ln(1/x^2)}{\cot(x^2)}$
$[0\cdot\infty]$	$\lim_{x\to 0+} x\ln\frac{1}{x}$
$[\infty-\infty]$	$\lim_{x\to(\pi/2)-}\left(\tan x - \frac{1}{\pi-2x}\right)$
$[0^0]$	$\lim_{x\to 0+} x^x$
$[\infty^0]$	$\lim_{x\to(\pi/2)-} (\tan x)^{\cos x}$
$[1^\infty]$	$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

1.1 Bentuk Taktentu 0/0

Theorem 1 (Aturan l'Hopital / l'Hospital) *Misalkan* $\lim_{x\to u} f(x) = 0$ *dan* $\lim_{x\to u} g(x) = 0$. *Jika* $\lim_{x\to u} \frac{f'(x)}{g'(x)}$ *ada, baik bernilai terhingga* (L) *maupun takhingga* ($-\infty$ *atau* ∞), *maka*

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Notasi $x \to u$ *mewakili sembarang notasi-notasi lainnya, seperti*

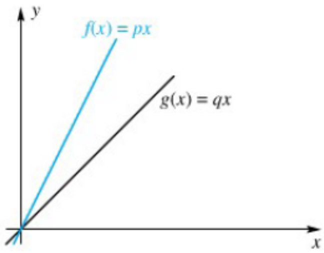
$$\begin{aligned} x &\to a \\ x &\to a^-, x \to a^+ \\ x &\to -\infty, x \to \infty. \end{aligned}$$

Proof. (Hanya akan dibuktikan untuk f dan g kontinu) Fungsi f kontinu di $x = 0$ jhj $\lim_{x\to 0} f(x) = f(0)$. Misalkan $\lim_{x\to u} f(x) = 0 = f(u)$ dan $\lim_{x\to u} g(x) = 0 = g(u)$.

$$\begin{aligned} \lim_{x\to u} \frac{f(x)}{g(x)} &= \lim_{x\to u} \frac{f(x)-0}{g(x)-0} \\ &= \lim_{x\to u} \frac{f(x)-f(u)}{g(x)-g(u)} = \lim_{x\to u} \frac{\frac{f(x)-f(u)}{x-u}}{\frac{g(x)-g(u)}{x-u}} \\ &= \frac{\lim_{x\to u} \frac{f(x)-f(u)}{x-u}}{\lim_{x\to u} \frac{g(x)-g(u)}{x-u}} = \frac{f'(u)}{g'(u)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}. \end{aligned}$$

Rumus definisi turunan:

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'(x) &= \lim_{a\to x} \frac{f(x)-f(a)}{x-a} \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \end{aligned}$$



$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{px}{qx} = \lim_{x\to 0}\frac{p}{q} = \frac{p}{q} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}.$$

Teorema nilai rata-rata untuk turunan:

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a}, \\ g'(c) &= \frac{g(b)-g(a)}{b-a}. \end{aligned}$$

Teorema nilai rata-rata untuk turunan Cauchy:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

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Example 2 *Tentukan limit-limit berikut:*

1. $\lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin x} \to$ *Bentuk* $\frac{0}{0}$

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin x} \overset{H}{=} \lim_{x\to 0}\frac{e^x + e^{-x}}{4\cos x} = \frac{1+1}{4} = \frac{1}{2}.$$

2. $\lim_{x\to 0}\frac{\ln(\cos 3x)}{2x^2} \to$ *Bentuk* $\frac{0}{0}$

$$\begin{aligned} \lim_{x\to 0}\frac{\ln(\cos 3x)}{2x^2} &\overset{H}{=} \lim_{x\to 0}\frac{\frac{1}{\cos 3x}\cdot -3\sin 3x}{4x} \\ &= \lim_{x\to 0}\frac{-3\sin 3x}{4x\cos 3x} \\ &= -\frac{3}{4}\lim_{x\to 0}\frac{\tan 3x}{x} \to \textit{Bentuk } \frac{0}{0} \\ &\overset{H}{=} -\frac{3}{4}\lim_{x\to 0}\frac{3\sec^2 3x}{1} \\ &= -\frac{9}{4}\lim_{x\to 0}\frac{1}{\cos^2 3x} \\ &= -\frac{9}{4}\cdot 1. \end{aligned}$$

3. $\lim_{x\to 0}\frac{1-\cos x}{2x^2+5x} \to$ *Bentuk* $\frac{0}{0}$

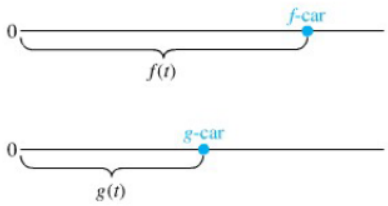
4. $\lim_{x\to 0}\frac{x^2 - x}{\sqrt[3]{x}-1} \to$ *Bentuk* $\frac{0}{0}$

1.2 Bentuk Taktentu ∞/∞, $\frac{\pm\infty}{\pm\infty}$

Theorem 3 *Misalkan* $\lim_{x\to u}|f(x)| = \infty$ *(atau* $\lim_{x\to u} f(x) = \pm\infty$*) dan* $\lim_{x\to u}|g(x)| = \infty$*. Jika* $\lim_{x\to u}\frac{f'(x)}{g'(x)}$ *ada, baik bernilai terhingga (L) maupun takhingga (*$-\infty$ *atau* ∞*), maka*

$$\lim_{x\to u}\frac{f(x)}{g(x)} = \lim_{x\to u}\frac{f'(x)}{g'(x)}.$$

Proof. Secara intuitif:



Fungsi jarak: $f(t)$ = jarak yang ditempuh mobil f pada saat t. $g(t)$ = jarak yang ditempuh mobil g pada saat t.

$$\frac{f(t)}{g(t)} \Leftrightarrow \frac{f'(t)}{g'(t)}$$

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Example 4 *Tentukan limit-limit berikut:*

1. $\lim_{x\to\infty}\frac{x^e}{e^x} \to$ *Bentuk* $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^e}{e^x} &\overset{H}{=} \lim_{x\to\infty}\frac{ex^{e-1}}{e^x} \\
&= \lim_{x\to\infty}\frac{ex^{e-1}}{e^x} \to \textit{Bentuk } \frac{\infty}{\infty} \\
\overset{H}{=} \lim_{x\to\infty}\frac{e(e-1)x^{e-2}}{e^x} &\to \textit{Bentuk } \frac{\infty}{\infty} \\
&\overset{H}{=} \lim_{x\to\infty}\frac{e(e-1)(e-2)x^{e-3}}{e^x} \\
&= \lim_{x\to\infty}\frac{e(e-1)(e-2)}{e^x x^{3-e}} \\
&= 0.
\end{aligned}$$

2. $\lim_{x\to 0^+}\frac{\cot x}{\ln x} \to$ *Bentuk* $\frac{\infty}{-\infty}$

$$\begin{aligned}
\lim_{x\to 0^+}\frac{\cot x}{\ln x} &\overset{H}{=} \ldots \\
&= \ldots \to \textit{Bentuk } \frac{0}{0} \\
&= \ldots \\
&= \ldots \\
&\overset{H}{=} \ldots \\
&= \ldots
\end{aligned}$$

1.3 Bentuk Taktentu $0 \cdot \infty$ atau $\infty - \infty$

Hint: Ubah ke bentuk $\frac{0}{0}$ atau $\frac{\infty}{\infty}$, lalu gunakan aturan l'Hopital.

Example 5 *Tentukan limit-limit berikut:*

1. $\lim\limits_{x \to 0} x^2 \csc x \to$ *Bentuk* $0 \cdot \infty$

$$\begin{aligned} \lim_{x \to 0} x^2 \csc x &= \lim_{x \to 0} \frac{x^2}{\sin x} \to \textit{Bentuk } \frac{0}{0} \\ &\overset{H}{=} \lim_{x \to 0} \frac{2x}{\cos x} \\ &= \frac{0}{1} \\ &= 0. \end{aligned}$$

2. $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) \to$ *Bentuk* $\infty - \infty$

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \to \textit{Bentuk } \frac{0}{0} \\ \overset{H}{=} \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} &\to \textit{Bentuk } \frac{0}{0} \\ \overset{H}{=} \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} &= \frac{0}{1 + 1 - 0} = 0. \end{aligned}$$

1.4 Bentuk Taktentu 0^0, ∞^0, atau 1^∞

Biasanya memiliki bentuk

$$\lim_{x \to u} (f(x))^{g(x)}.$$

Hint: tarik logaritmanya.

Example 6 *Tentukan limit-limit berikut:*

1. $\lim\limits_{x \to 0^+} (x+1)^{1/x} \to$ *Bentuk* 1^∞

$$\begin{aligned} y &= (x+1)^{1/x} \\ \ln y &= \ln (x+1)^{1/x} \\ \ln y &= \frac{\ln(x+1)}{x} \\ e^{\ln y} &= e^{\frac{\ln(x+1)}{x}} \\ y &= e^{\frac{\ln(x+1)}{x}} \\ \lim_{x \to 0^+} y &= \lim_{x \to 0^+} e^{\frac{\ln(x+1)}{x}} \end{aligned}$$

$$\lim_{x \to 0^+} y = e^{\lim\limits_{x \to 0^+} \frac{\ln(x+1)}{x}}$$

$$\begin{aligned} \lim_{x \to 0^+} \frac{\ln(x+1)}{x} &\to \textit{Bentuk } \frac{0}{0} \\ \overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x+1}}{1} &= \lim_{x \to 0^+} \frac{1}{x+1} = 1. \end{aligned}$$

$$\lim_{x\to 0^+} y = e^1 = e$$

$$\lim_{x\to 0^+} (x+1)^{1/x} = e.$$

Misal: $z = \frac{1}{x} \Leftrightarrow x = \frac{1}{z}$, *ketika* $x \to 0^+$ *maka* $z \to \infty$

$$\lim_{x\to 0^+} (x+1)^{1/x} = \lim_{z\to\infty} (\frac{1}{z}+1)^z = e.$$

2. $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^x \to$ *Bentuk* 1^∞

$$\begin{aligned} y &= \left(1+\frac{a}{x}\right)^x \\ \ln y &= x\ln\left(1+\frac{a}{x}\right) \\ y &= e^{x\ln\left(1+\frac{a}{x}\right)} \\ \lim_{x\to\infty} y &= e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)} \end{aligned}$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right) \to \textit{Bentuk } \infty\cdot 0$$

$$\begin{aligned} \lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{1/x} \to \textit{Bentuk } \frac{0}{0} \\ \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\cdot\frac{-a}{x^2}}{-1/x^2} &= \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} = a. \end{aligned}$$

$$\lim_{x\to\infty} y = e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)} = e^a.$$

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a.$$

3. $\lim\limits_{x\to 0^+} (\sin x)^x$

4. $\lim\limits_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin x}$

1.5 Bukan Bentuk Taktentu

- $\dfrac{0}{\infty} \to 0 : \lim_{x\to 1} \frac{1-x}{\cot(1-x)} = 0.$
- $\dfrac{\infty}{0} \to \infty : \lim_{x\to 1} \frac{\cot(1-x)}{1-x} = \infty.$
- $\infty + \infty \to \infty : \lim_{x\to\infty} (x+e^x) = \infty.$
- $\infty \cdot \infty \to \infty : \lim_{x\to\infty} xe^x = \infty.$
- $\infty^\infty \to \infty$

- $0^\infty \to 0$

$$\begin{aligned}
\lim_{x\to 0^+} (\sin x)^{\frac{1}{x}} &\to \text{Bentuk } 0^\infty \text{ (bukan bentuk taktentu)} \\
y &= (\sin x)^{\frac{1}{x}} \\
\ln y &= \frac{\ln \sin x}{x} \\
y &= e^{\frac{\ln \sin x}{x}} \\
\lim_{x\to 0^+} y &= e^{\lim_{x\to 0^+} \frac{\ln \sin x}{x}}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln \sin x}{x} &\to \text{Bentuk } \frac{-\infty}{0} \text{ (bukan bentuk taktentu)} \\
\lim_{x\to 0^+} \frac{\ln \sin x}{x} &= -\infty
\end{aligned}$$

$$\lim_{x\to 0^+} y = e^{\lim_{x\to 0^+} \frac{\ln \sin x}{x}} = e^{-\infty} = \frac{1}{e^\infty} = 0$$

Example 7 *Hitung*

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}.$$