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KALKULUS II

a) Tulis rumus eksplisit barisan berikut dan tentukan kekonvergenannya:

$\cos\pi, \frac{\cos 2\pi}{4}, \frac{\cos 3\pi}{9}, \frac{\cos 4\pi}{16}, \ldots$

Rumus eksplisit $\frac{\cos n\pi}{n^2}$

kekonvergenan

$$\frac{-1}{n^2} < \frac{\cos n\pi}{n^2} < \frac{1}{n^2}$$

$\lim\limits_{n\to\infty} -\frac{1}{n^2} = 0$ $\quad \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

$\lim\limits_{n\to\infty} \frac{\cos n\pi}{n^2} = 0$ (konvergen)

b) Diketahui $\{a_n\}$ konvergen ke A dan $\{b_n\}$ konvergen ke B. Buktikan (dengan definisi limit) $\{a_n + b_n\}$ konvergen ke A + B.

Diketahui

$\lim\limits_{n\to\infty} a_n = A$ $\quad \lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

$\lim\limits_{n\to\infty} b_n = B$ $\quad \lim\limits_{n\to\infty} (a_n + b_n) = A + B$ (konvergen)

c) $\{a_n\}$ konvergen ke A, maka $\lim\limits_{n\to\infty} a_n = A$, dengan kata lain:

Untuk setiap $\varepsilon > 0$ ditemukan $N_A > 0$ sedemikian sehingga untuk $n > N_A$ berlaku

$$|a_n - A| < \tfrac{1}{2}\varepsilon$$

catatan:
$\varepsilon > 0$
$\frac{1}{2}\varepsilon > 0$

Definisi limit
$\lim\limits_{x\to a} f(x) = L \Leftrightarrow \varepsilon > 0$
Maka,
$|f(x) - L| < \varepsilon$

$\{b_n\}$ konvergen ke B, maka $\lim\limits_{n\to\infty} b_n = B$, dengan kata lain: untuk setiap $\varepsilon > 0$ ditemukan $N_B > 0$ sedemikian sehingga untuk $n > N_B$ berlaku

$$|b_n - B| < \tfrac{1}{2}\varepsilon$$

$$|(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon$$
$$|a_n + b_n - (A+B)| < \varepsilon$$

$\therefore$ Terbukti bahwa $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$

c. Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan berikut:

$$a_n = \sin \frac{n\pi}{4}$$

kemonotonan

$$a_n - a_{n+1} = \sin \frac{n\pi}{4} - \sin \frac{(n+1)\pi}{4}$$

(Bukan barisan monoton karena alternating)

Limit $\lim_{n\to\infty} \sin\left(\frac{n\pi}{4}\right)$ = Tak tentu (Divergen)

keterbatasan. - karena $\{a_n\}$ divergen, maka $\{a_n\}$ tak terbatas

2. a. Tulis rumus eksplisit dan tentukan kekonvergenannya:

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$$

rumus eksplisit $(-1)^{n+1}\left(\frac{1}{n}\right)$

kekonvergenan

$$\lim_{n\to\infty} \left|(-1)^{n+1}\left(\frac{1}{n}\right)\right| = \lim_{n\to\infty} \left|(-1)^{n+1}\right| \left|\frac{1}{n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0_{//}$$

b. Dengan definisi limit, buktikan barisan $\{a_n\}$ berikut konvergen:

$$a_n = \frac{3-8.2^n}{5+4.2^n}$$

$$\lim_{n\to\infty} \frac{3-8\,2^n}{5+4.2^n} = \frac{3/2n^2 - 8}{5/2n^2 + 4} = \frac{-8}{4} = -2_{//}$$

$$\therefore \lim_{n\to\infty} \frac{3-8.2^n}{5+4.2^n} = -2$$

Analisis Pendahuluan | mencari nilai δ sehingga

$$0 < \left|\frac{1}{5+4.2^n}\right| < \delta \Rightarrow \left|\frac{3-8\,2^n}{5+4.2^n} - (-2)\right| < \varepsilon$$

$$\left|\frac{3-8.2^n}{5+4.2^n} - (-2)\right| < \varepsilon$$

$$\left|\frac{3-8.2^n}{5+4.2^n} + 2\right| < \varepsilon$$

$$\left|\frac{3-8.2^n + 2(5+4.2^n)}{5+4.2^n}\right| < \varepsilon$$

$$\left|\frac{13}{5+4.2^n}\right| < \varepsilon$$

$$13\left|\frac{1}{5+4.2^n}\right| < \varepsilon$$

$$\left|\frac{1}{5+4.2^n}\right| < \frac{\varepsilon}{13}$$

$$\delta = \frac{\varepsilon}{13}$$

Bukti formal

Diberikan $\varepsilon > 0$, pilih $\delta = \frac{\varepsilon}{13}$ dari $0 < \left|\frac{1}{5+4.2^n}\right| < \delta$

maka,

$$\left|\frac{3-8.2^n}{5+4.2^n} - (-2)\right| = \left|\frac{3-8.2^n}{5+4.2^n} + 2\right| = \left|\frac{13}{5+4.2^n}\right|$$

$$\Leftrightarrow |13|\left|\frac{1}{5+4.2^n}\right| = 13\left|\frac{1}{5+4.2^n}\right| < 13\delta = 13\left(\frac{\varepsilon}{13}\right) = \varepsilon_{//}$$

$$\Leftrightarrow 13\left|\frac{1}{5+4.2^n}\right| < \varepsilon_{//}$$

C. Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan beri:

$$a_n = \frac{\ln n}{n}$$

kemonotonan
$$\frac{a_n}{a_{n+1}} = \frac{\frac{\ln n}{n}}{\frac{\ln(n+1)}{(n+1)}}$$
$$= \frac{\ln n\,(n+1)}{n \ln(n+1)}$$
$$= \frac{n \ln n + \ln n}{n \ln(n+1)}$$
$$= \frac{\cancel{n} \ln n}{\cancel{n} \ln(n+1)} + \frac{\ln n}{n \ln(n+1)}$$
$$= \frac{\ln n}{\ln(n+1)} + \frac{\ln n}{n \ln(n+1)}$$
(Bukan barisan monoton)
(Divergen)

keterbatasan
karena $\{a_n\}$ divergen, maka $\{a_n\}$ tak terbatas.

3a. Tulis rumus eksplisit dan kekonvergenannya
$$0{,}9\ ,\ 0{,}99\ ,\ 0{,}999,\ 0{,}9999,\ \ldots$$
rumus eksplisit
$$1 - \frac{1}{10^n}$$
kekonvergenannya
$$\lim_{n\to\infty} 1 - \frac{1}{10^n} = 1 - 0 = 1$$

b. Dengan definisi limit, buktikan barisan $\{a_n\}$ berikut konvergen:
$$a_n = \frac{n+3}{3n-2}$$
$$\lim_{n\to\infty} \frac{n+3}{3n-2} = \frac{n+3}{3n-2} \cdot \frac{1/n}{1/n} = \frac{1 + 3/n}{3 - 2/n} = \frac{1}{3}$$ → (konvergen ke $\frac{1}{3}$)

Analisis Pendahuluan
mencari nilai δ sehingga
$$0 < \frac{1}{3n-2} < \delta \Rightarrow \left|\frac{n+3}{3n-2} - \frac{1}{3}\right| < \varepsilon$$
$$\left|\frac{n+3}{3n-2} - \frac{1}{3}\right| < \varepsilon$$
$$\left|\frac{n+3-(3n-2)}{3(3n-2)}\right| < \varepsilon$$
$$\left|\frac{1}{3}\right|\left|\frac{n+3-3n+2}{3n-2}\right| < \varepsilon$$
$$\left|\frac{-2n+5}{3n-2}\right| < 3\varepsilon$$
$$|-2n+5|\left|\frac{1}{3n-2}\right| < 3\varepsilon$$
$$\left|\frac{1}{3n-2}\right| < \frac{3\varepsilon}{|-2n+5|}$$
$$\delta = \frac{3\varepsilon}{|-2n+5|}$$

Bukti Formal
Diberikan $\varepsilon > 0$, pilih $\delta = \frac{3\varepsilon}{|-2n+5|}$ dari $0 < \frac{1}{3n-2} < \delta$ maka,
$$\left|\frac{n+3}{3n-2} - \frac{1}{3}\right| = \left|\frac{n+3-(3n-2)}{3(3n-2)}\right|$$
$$\Leftrightarrow \left|\frac{1}{3}\right|\left|\frac{-2n+5}{3n-2}\right| = \left|\frac{1}{3}\right||-2n+5|\left|\frac{1}{3n-2}\right| < \frac{|-2n+5|}{3}\delta$$
$$\Leftrightarrow \frac{1}{3}|-2n+5|\left|\frac{1}{3n-2}\right| < \frac{|-2n+5|}{3} \cdot \frac{3\varepsilon}{|-2n+5|}$$
$$\Leftrightarrow \frac{1}{3}|-2n+5|\left|\frac{1}{3n-2}\right| < \varepsilon$$

c. Tentukan kemonotonan, keterbatasan, dan limit (jika ada) barisan bikt:

$$a_n = \frac{n!}{10^n}$$

kemonotonan

$$\frac{a_n}{a_{n+1}} = \frac{\frac{n!}{10^n}}{\frac{(n+1)!}{10^{(n+1)}}} = \frac{n!}{10^n} \cdot \frac{10^{(n+1)}}{(n+1)!} = \frac{10}{(n+1)} > 0 \quad \text{(monoton turun)}$$

limit

$$\lim_{n \to \infty} \frac{n!}{10^n} = \frac{1 . 2 . 3 \ldots n}{10^n \, 10^n \, 10^n \ldots 10^n} = \infty \text{, Tak tentu (Divergen)}$$

keterbatasan

karena $\{a_n\}$ divergen, maka $\{a_n\}$ tak terbatas.