MAT211 Kalkulus II
K2 - Integral Takwajar

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1 Pendahuluan

Misal diberikan integral tentu berikut (dengan TDK-II):

$$\int_a^b f(x)\ dx = F(b) - F(a),$$

dengan F merupakan sembarang antiturunan dari f. **Integral takwajar** (*improper integral*) merupakan integral tentu dengan sifat:

- memiliki batas pengintegralan takhingga (*infinity*), yaitu $a = -\infty$ atau $b = \infty$ atau keduanya, atau
- memiliki integran, yaitu $f(x)$ tak berbatas (*unbounded*).

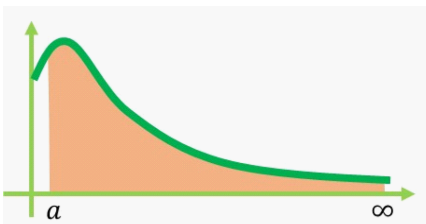
Notasi berikut dapat dipahami:

$$\int_a^\infty f(x)\ dx = F(\infty) - F(a),$$

namun ditulis dengan cara yang kurang tepat. Kenapa? Karena $x = \infty \notin D_F$ sehingga $F(\infty)$ tidak terdefinisi.

2 Integral takwajar dengan batas pengintegralan takhingga

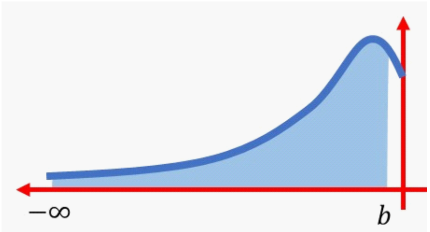
- Bentuk 1



$\lim\limits_{x\to\infty} f(x) = 0$ (bersifat asimtotik).

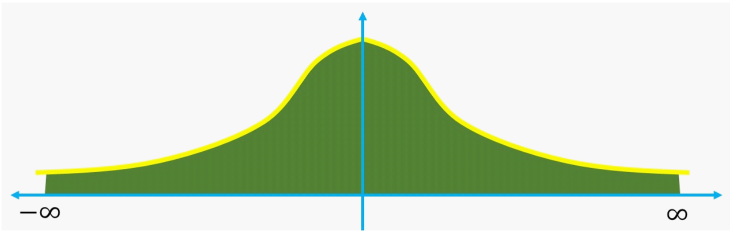
$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx = \lim_{b\to\infty} (F(b) - F(a)).$$

- Bentuk 2



$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a\to -\infty} \int_{a}^{b} f(x)\ dx = \lim_{a\to -\infty} (F(b) - F(a)).$$

- Bentuk 3



$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= \int_{-\infty}^{c} f(x)\ dx + \int_{c}^{\infty} f(x)\ dx \\ &= \lim_{a\to -\infty} \int_{a}^{c} f(x)\ dx + \lim_{b\to \infty} \int_{c}^{b} f(x)\ dx \\ &= \lim_{a\to -\infty} (F(c) - F(a)) + \lim_{b\to\infty} (F(b) - F(a)). \end{aligned}$$

Terhadap integral takwajar:

- konvergen (nilai integralnya berhingga),
- divergen (tak-konvergen, nilai integralnya takhingga atau tidak ada).

Example 1 *Tentukan integral berikut, jika ada:*

$$\int_{1}^{\infty} x^2 e^{-x^3} dx.$$

Misal: $u = -x^3$ *sehingga* $du = -3x^2\ dx$. *Dieproleh:*

$$\begin{aligned} \int_{1}^{\infty} x^2 e^{-x^3} dx &= -\frac{1}{3} \int_{-1}^{-\infty} e^u du \\ &= \frac{1}{3} \int_{-\infty}^{-1} e^u du \\ &= \frac{1}{3} \lim_{a\to -\infty} \int_{a}^{-1} e^u du \\ &= \frac{1}{3} \lim_{a\to -\infty} (e^{-1} - e^a) \\ &= \frac{1}{3e}. \end{aligned}$$

Example 2 *Tentukan integral berikut, jika ada:*

$$\begin{aligned}
\int_{-\infty}^{0} \cos x \; dx &= \lim_{a\to-\infty} \int_{a}^{0} \cos x \; dx \\
&= \lim_{a\to-\infty} (\sin 0 - \sin a) \\
&= -\lim_{a\to-\infty} \sin a \\
&= \textit{tidak ada.}
\end{aligned}$$

Example 3 *Dalam teori peluang, suatu fungsi f disebut fungsi kepekatan peluang (*probability density function*) jika dipenuhi syarat*

$$\int_{-\infty}^{\infty} f(x) \; dx = 1.$$

Fungsi kepekatan peluang sebaran Cauchy memiliki bentuk

$$f(x) = \frac{k}{x^2+1}.$$

Tentukan nilai k.

Karena f merupakan fkp maka:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \; dx = 1 \quad &\Leftrightarrow \quad \int_{-\infty}^{\infty} \frac{1}{x^2+1} \; dx = \frac{1}{k} \\
&\Leftrightarrow \quad \int_{-\infty}^{0} \frac{1}{x^2+1} \; dx + \int_{0}^{\infty} \frac{1}{x^2+1} \; dx = \frac{1}{k} \\
&\Leftrightarrow \quad \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+1} \; dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+1} \; dx = \frac{1}{k} \\
&\Leftrightarrow \quad \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) + \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{1}{k} \\
&\Leftrightarrow \quad (0 - (-\tfrac{1}{2}\pi)) + (\tfrac{1}{2}\pi - 0) = \frac{1}{k} \\
&\Leftrightarrow \quad \pi = \frac{1}{k} \\
&\Leftrightarrow \quad k = \frac{1}{\pi}.
\end{aligned}$$

Example 4 *Dalam teori peluang, fungsi kepekatan peluang sebaran **eksponensial** dengan parameter $\lambda > 0$ diberikan oleh*

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}.$$

Buktikan:

- $\int_0^\infty f(x) \; dx = 1,$
- $\int_0^\infty x f(x) \; dx = 1/\lambda.$

Karena f merupakan fungsi kepekatan peluang maka berlaku

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx = 1 \quad &\Leftrightarrow \quad \int_{-\infty}^{0} f(x)\ dx + \int_{0}^{\infty} f(x)\ dx = 1 \\
&\Leftrightarrow \quad \int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} f(x)\ dx = 1 \\
&\Leftrightarrow \quad \int_{0}^{\infty} f(x)\ dx = 1.
\end{aligned}$$

Penghitungan langsung memberikan

$$\begin{aligned}
\int_{0}^{\infty} f(x)\ dx &= \lim_{b\to\infty} \int_{0}^{b} \lambda e^{-\lambda x}\ dx \\
&= \lim_{b\to\infty} (-e^{-\lambda x}|_{0}^{b} \\
&= \lim_{b\to\infty} (-e^{-\lambda b} + 1) \\
&= \lim_{b\to\infty} \left(-\frac{1}{e^{\lambda b}} + 1\right) \\
&= 1.
\end{aligned}$$

Selanjutnya:

$$\begin{aligned}
\int_{0}^{\infty} x f(x)\ dx &= \lim_{b\to\infty} \int_{0}^{b} \lambda x e^{-\lambda x}\ dx \\
&= \lambda \lim_{b\to\infty} \int_{0}^{b} x e^{-\lambda x}\ dx \\
\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^{-\lambda x} dx \Rightarrow v = -\frac{1}{\lambda} e^{-\lambda x} \end{array} \qquad &= \lambda \lim_{b\to\infty} \left(uv|_{0}^{b} - \int_{0}^{b} v\ du \right) \\
&= \lambda \lim_{b\to\infty} \left(-\frac{1}{\lambda} x e^{-\lambda x}|_{0}^{b} + \int_{0}^{b} \frac{1}{\lambda} e^{-\lambda x}\ dx \right) \\
&= \lim_{b\to\infty} \left(-\frac{b}{e^{\lambda b}} + \int_{0}^{b} e^{-\lambda x}\ dx \right) \\
&= \lim_{b\to\infty} \left(-\frac{b}{e^{\lambda b}} - \left(\frac{1}{\lambda} e^{-\lambda x} \right|_{0}^{b} \right) \\
&= \lim_{b\to\infty} \left(-\frac{b}{e^{\lambda b}} - \frac{1}{\lambda}(e^{-\lambda b} - 1) \right) \\
&= -\lim_{b\to\infty} \frac{b}{e^{\lambda b}} - \frac{1}{\lambda} \lim_{b\to\infty} (e^{-\lambda b} - 1) \\
&= {}_{H} - \lim_{b\to\infty} \frac{1}{\lambda e^{\lambda b}} + \frac{1}{\lambda} \\
&= \frac{1}{\lambda}.
\end{aligned}$$

3 Integral takwajar dengan integran tak-berbatas

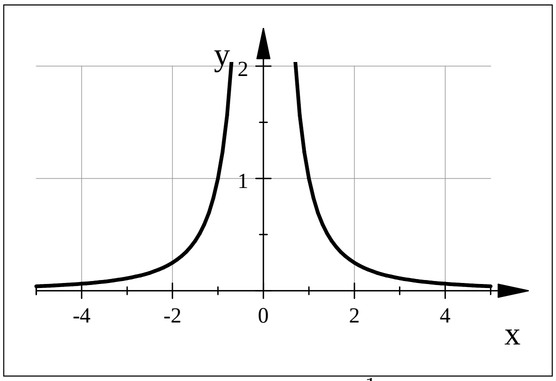
Perhatikan ilustrasi berikut:

$$\begin{aligned}\int_{-3}^{1} \frac{1}{x^2}\,dx &= \int_{-3}^{1} x^{-2}\,dx \\ &= \left(-\frac{1}{x}\right|_{-3}^{1} \\ &= -1 + \frac{1}{-3} \\ &= -\frac{4}{3}.\ \text{(SALAH)}\end{aligned}$$

Seharusnya...

$$\begin{aligned}-3 \le x \le 1 &\Leftrightarrow 0 \le x^2 \le 9 \\ &\Leftrightarrow \frac{1}{x^2} > 0 \\ &\Leftrightarrow \int_{-3}^{1} \frac{1}{x^2} dx > \int_{-3}^{1} 0 dx \\ &\Leftrightarrow \int_{-3}^{1} \frac{1}{x^2} dx > 0\end{aligned}$$

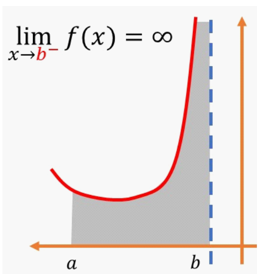
Kesalahan: Kita mengabaikan fakta bahwa nilai f di $x = 0$ membesar tanpa batas.



Grafik $f(x) = \frac{1}{x^2}$

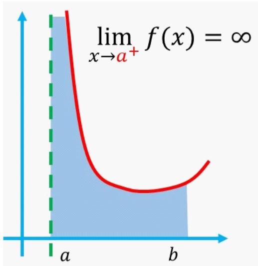
- Bentuk 1: integran tak berbatas di ujung selang kanan

  $[a, b]$: a ujung selang kiri, b ujung selang kanan



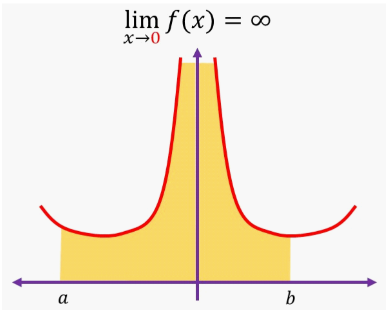
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

- Bentuk 2: integran tak berbatas di ujung selang kiri.



$$\int_a^b f(x)\ dx = \lim_{t\to a^+} \int_t^b f(x)\ dx.$$

- Bentuk 3: integran tak berbatas di titik dalam (*interior point*).



$$\begin{aligned}\int_a^b f(x)\ dx &= \int_a^0 f(x)\ dx + \int_0^b f(x)\ dx \\ &= \lim_{t\to 0^-} \int_a^t f(x)\ dx + \lim_{t\to 0^+} \int_t^b f(x)\ dx.\end{aligned}$$

Catatan: Integral harus dipecah pada titik di mana fungsi f tak berbatas.

Example 5 *Tentukan integral berikut, jika ada:*

$$\int_{-3}^{1} \frac{1}{x^2}\ dx.$$

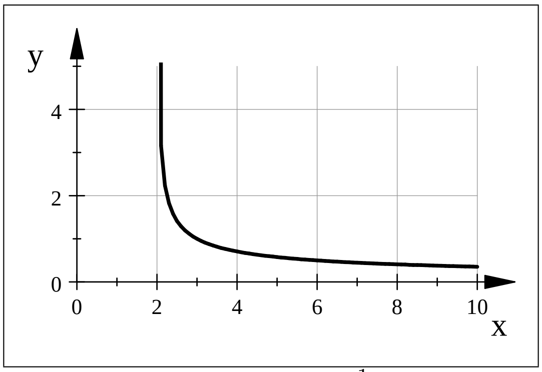
Jika $f(x) = \frac{1}{x^2}$ *maka di* $x = 0$ *nilai* f *membesar tanpa batas, sehingga*

$$\begin{aligned}\int_{-3}^{1} \frac{1}{x^2}\ dx &= \int_{-3}^{0} \frac{1}{x^2}\ dx + \int_{0}^{1} \frac{1}{x^2}\ dx \\ &= \lim_{t\to 0^-} \int_{-3}^{t} \frac{1}{x^2}\ dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2}\ dx \\ &= \lim_{t\to 0^-} \left(-\frac{1}{x}\right|_{-3}^{t} + \lim_{t\to 0^+} \left(-\frac{1}{x}\right|_{t}^{1} \\ &= \lim_{t\to 0^-} \left(-\frac{1}{t} - \frac{1}{3}\right) + \lim_{t\to 0^+} \left(-1 + \frac{1}{t}\right) \\ &= \infty.\end{aligned}$$

Example 6 *Tentukan integral berikut, jika ada:*

$$\int_2^6 \frac{1}{\sqrt{x-2}}\ dx.$$

$$f(x) = \frac{1}{\sqrt{x-2}} \Rightarrow D_f = (2, \infty).$$



Grafik $f(x) = \frac{1}{\sqrt{x-2}}$

Dengan demikian

$$\begin{aligned}
\int_2^6 \frac{1}{\sqrt{x-2}}\ dx &= \lim_{t\to 2^+} \int_t^6 \frac{1}{\sqrt{x-2}}\ dx \\
&= \lim_{t\to 2^+} \int_t^6 (x-2)^{-1/2}\ dx \\
&= \lim_{t\to 2^+} (2\sqrt{x-2}|_t^6 \\
&= \lim_{t\to 2^+} (2(2) - 2\sqrt{t-2}) \\
&= 4.
\end{aligned}$$

Example 7 *Hitung integral berikut:*

$$\int_{-1}^2 \frac{1}{x}\ dx.$$

$$\int_{-1}^1 \frac{1}{x}\ dx = tidak\ ada\ (\neq 0)$$

Diperoleh:

$$\begin{aligned}
\int_{-1}^2 \frac{1}{x}\ dx &= \int_{-1}^0 \frac{1}{x}\ dx + \int_0^2 \frac{1}{x}\ dx \\
&= \lim_{t\to 0^-} \int_{-1}^t \frac{1}{x}\ dx + \lim_{t\to 0^+} \int_t^2 \frac{1}{x}\ dx \\
&= \lim_{t\to 0^-} \ln|x|_{-1}^t + \lim_{t\to 0^+} \ln|x|_t^2 \\
&= \lim_{t\to 0^-} (\ln|t| - 0) + \lim_{t\to 0^+} (\ln 2 - \ln|t|) \\
&= (-\infty - 0) + (\ln 2 + \infty) \\
&= \ \ tidak\ ada.
\end{aligned}$$

Example 8 *Buktikan bahwa:*

$$\int_0^1 \frac{1}{x^p}\ dx$$

- *konvergen jika* $p < 1 \Leftrightarrow 0 < 1 - p$,
- *divergen jika* $p \geq 1 \Leftrightarrow 0 \geq 1 - p$.

Diperoleh

$$\begin{aligned}
\int_0^1 \frac{1}{x^p}\ dx &= \int_0^1 x^{-p}\ dx \\
&= \lim_{t \to 0^+} \int_t^1 x^{-p}\ dx \\
&= \lim_{t \to 0^+} \left(\frac{x^{1-p}}{1-p} \right|_t^1 \\
&= \lim_{t \to 0^+} \left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p} \right) \\
&= \frac{1}{1-p} - \lim_{t \to 0^+} \frac{t^{1-p}}{1-p}.
\end{aligned}$$

Aoakah bentuk di atas konvergen/divergen, bergantung pada nilai limitnya. Akan dievaluasi tiga kasus

- *Jika* $1 - p > 0$ *maka*
$$\lim_{t \to 0^+} \frac{t^{1-p}}{1-p} = 0,$$
sehingga bentuk di atas konvergen.

- *Misal* $1 - p < 0$, *yaitu* $1 - p = -q^2 < 0$, *maka diperoleh*
$$\lim_{t \to 0^+} \frac{t^{1-p}}{1-p} = \lim_{t \to 0^+} \frac{t^{-q^2}}{-q^2} = \lim_{t \to 0^+} \frac{1}{-q^2 t^{q^2}} = -\infty,$$
sehingga bentuk di atas divergen.

- *Misal* $1 - p = 0$ *atau* $p = 1$ *maka diperoleh*
$$\int_0^1 \frac{1}{x^p}\ dx = \int_0^1 \frac{1}{x}\ dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\ dx = \lim_{t \to 0^+} (\ln|x|\,|_t^1 = \infty,$$
sehingga bentuk di atas divergen.

Dengan demikian

$$\int_0^1 \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{1-p} & ; \quad p < 1 \\ -\infty & ; \quad p > 1 \\ +\infty & ; \quad p = 1 \end{cases}.$$

Example 9 *Tentukan integral berikut, jika ada:*

$$\int_1^e \frac{1}{x \ln x}\ dx.$$

Dengan metode substitusi, misalkan $u = \ln x$ *dan* $du = \frac{1}{x}dx$*, sehingga*

$$\int_1^e \frac{1}{x \ln x}\ dx = \int_0^1 \frac{1}{u}\ du.$$

Dapat dilihat bahwa integran tidak terdefinisi di $x = 0$*. Oleh karena itu*

$$\int_0^1 \frac{1}{u}\ du = \lim_{t\to 0^+} \int_t^1 \frac{1}{u}\ du = \lim_{t\to 0^+} (\ln|u|\,|_t^1 = \infty.$$

Example 10 *Tentukan integral berikut, jika ada:*

$$\int_{-8}^{27} \frac{1}{x^{2/3}}\ dx.$$

Dapat dilihat bahwa integran tidak terdefinisi di $x = 0$*. Oleh karena itu*

$$\begin{aligned}
\int_{-8}^{27} \frac{1}{x^{2/3}}\ dx &= \int_{-8}^{0} \frac{1}{x^{2/3}}\ dx + \int_{0}^{27} \frac{1}{x^{2/3}}\ dx \\
&= \lim_{t\to 0^-} \int_{-8}^{t} \frac{1}{x^{2/3}}\ dx + \lim_{t\to 0^+} \int_{t}^{27} \frac{1}{x^{2/3}}\ dx \\
&= 6 + 9 \ \textit{(mohon dicek)} \\
&= 15.
\end{aligned}$$

Example 11 *Tentukan integral berikut, jika ada:*

$$\int_0^\infty \frac{1}{\sqrt{x}}\ dx.$$

Bentuk integral di atas merupakan gabungan dua kasus integral takwajar, yaitu memiliki batas pengintegralan takhingga dan integran tak berbatas, sehingga

$$\int_0^\infty \frac{1}{\sqrt{x}}\ dx = \lim_{b\to\infty} \int_0^b \frac{1}{\sqrt{x}}\ dx = \lim_{t\to 0^+} \left(\lim_{b\to\infty} \int_t^b \frac{1}{\sqrt{x}}\ dx \right) = \infty.$$