MAT211 Kalkulus II
K3 - Barisan Takhingga

TBK (AKT)
IPB University

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1 Pengertian

Definition 1 *Suatu* ***barisan takhingga*** $a_1, a_2, a_3, ...$ *(notasi:* $\{a_n\}_{n=1}^{\infty}$ *atau cukup* $\{a_n\}$ *saja) adalah susunan bilangan terurut sesuai dengan urutan bilangan asli sebagai indeksnya.*

Definition 2 *Suatu barisan takhingga merupakan fungsi* f *dengan* $D_f = \mathbb{N}$*, ditulis* $f_n,\ n \in \mathbb{N}$.

Cara menyatakan barisan takhingga:

1. Dengan memberikan suku awal yang cukup untuk membentuk suatu pola. Misalnya barisan: $2, 5, 8, 11, 14, ...$ (Pola: lompat tiga, dimulai dari 2).

2. Dengan rumus eksplisit:

$$\begin{aligned} a_n &= 3n - 1,\ n \in \mathbb{N} \\ \{a_n\} &= 2, 5, 8, 11, 14, ..., 89, ... \end{aligned}$$

3. Dengan rumus rekursif:

$$\begin{aligned} a_1 &= 2, \\ a_n &= a_{n-1} + 3,\ n \geq 2, \\ \text{atau } a_{n+1} &= a_n + 3,\ n \geq 1. \end{aligned}$$

2 Kekonvergenan

Perhatikan empat barisan berikut:

1. $\{a_n\}$ dengan $a_n = 1 - \frac{1}{n} = \frac{n-1}{n}$:

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \frac{2020}{2021}, ... \to 1.$$

2. $\{b_n\}$ dengan $b_n = 1 + (-1)^n \frac{1}{n} = \frac{n+(-1)^n}{n}$:

$$0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, ..., \frac{2020}{2021}, ... \to 1.$$

3. $\{c_n\}$ dengan $c_n = (-1)^n + \frac{1}{n} = \frac{(-1)^n n+1}{n}$:

$$0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, ..., -\frac{2020}{2021}, ... \nrightarrow 1$$

4. $\{d_n\}$ dengan $d_n = 0.999$.

$$0.999, 0.999, 0.999, ... \to 0.999.$$

Semua barisan di atas memiliki suku-suku yang dekat dengan 1. Apakah semuanya konvergen ke 1?

Konvergen ke 1 berarti suku-suku barisan mendekati 1 dan **tetap dekat** ke 1 untuk n yang makin membesar.

Definition 3 *Barisan $\{a_n\}$ disebut **konvergen** ke L dan ditulis*

$$\lim_{n\to\infty} a_n = L$$

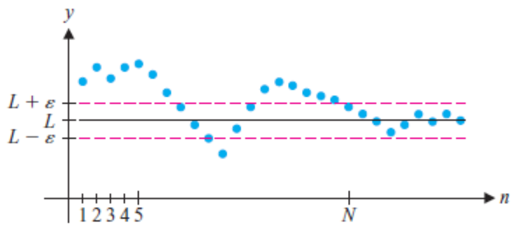
jika untuk setiap bilangan positif ε terdapat bilangan positif N sedemikian sehingga

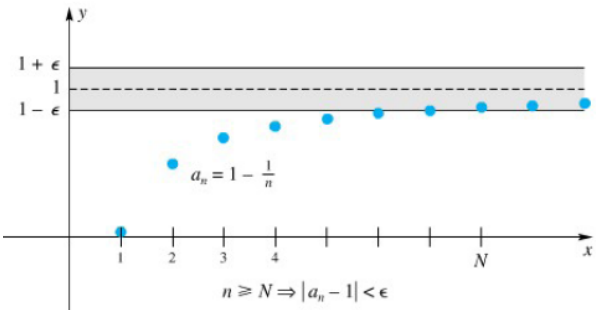
$$n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

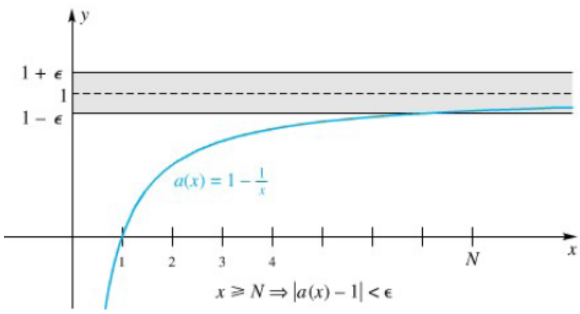
*Selain itu disebut **divergen**.*

Ilustrasi:

$$|a_n - L| < \varepsilon \Leftrightarrow -\varepsilon \leq a_n - L \leq \varepsilon \Leftrightarrow L - \varepsilon \leq a_n \leq L + \varepsilon.$$







Example 4 *Diberikan barisan $\{a_n\}$ dengan $a_n = \frac{1}{n^2}$. Tunjukkan bahwa $\{a_n\}$ konvergen ke* 0.

- *Diketahui $L = 0$*
- *Akan dibuktikan untuk setiap bilangan positif ε terdapat bilangan positif N sedemikian sehingga*

$$n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

- *Misalkan* $\varepsilon > 0$ *sembarang diberikan.*
- *Pilih* $N = \sqrt{1/\varepsilon}$.
- *Untuk* $n \geq N > 0$ *diperoleh*

$$\begin{aligned} n &\geq N \\ n^2 &\geq N^2 \\ \frac{1}{n^2} &\leq \frac{1}{N^2}. \end{aligned}$$

$$|a_n - L| = \left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} = \frac{1}{(\sqrt{1/\varepsilon})^2} = \varepsilon.$$

Example 5 *Buktikan* $\{a_n\}$ *dengan* $a_n = (2n+3)/n$ *konvergen ke* 2 *untuk* $n \geq 1$.

- *Diketahui* $L = 2$
- *Akan dibuktikan untuk setiap bilangan positif* ε *terdapat bilangan positif* N *sedemikian sehingga*
$$n \geq N \Rightarrow |a_n - L| < \varepsilon.$$
- *Misalkan* $\varepsilon > 0$ *sembarang diberikan.*
- *Pilih* $N = \frac{3}{\varepsilon}$.
- *Analisis pendahuluan: untuk* $n \geq N > 0$ *diperoleh*
$$|a_n - L| = \left|\frac{2n+3}{n} - 2\right| = \left|\frac{3}{n}\right| = \frac{3}{n} \leq \frac{3}{N} = \frac{3}{3/\varepsilon} = \varepsilon.$$

Untuk selanjutnya, memeriksa kekonvergenan barisan $\{a_n\}$ cukup dilakukan dengan menghitung $\lim_{n\to\infty} a_n$.

Example 6 *Periksa kekonvergenan barisan-barisan berikut:*

1. $a_n = \frac{2n+3}{n}$,
$$\lim_{n\to\infty} \frac{2n+3}{n} = 2.$$
Barisan $\{a_n\}$ *konvergen ke* 2.

2. $b_n = \frac{4n^2+5}{2n^2+7n}$,
$$\lim_{n\to\infty} \frac{4n^2+5}{2n^2+7n} = 2.$$
Barisan $\{b_n\}$ *konvergen ke* 2.

3. $c_n = \frac{n^2}{e^n}$,
$$\lim_{n\to\infty} \frac{n^2}{e^n} =_H \lim_{n\to\infty} \frac{2n}{e^n} =_H \lim_{n\to\infty} \frac{2}{e^n} = 0.$$
Barisan $\{c_n\}$ *konvergen ke* 0.

4. $d_n = 8 - 2n$.
$$\lim_{n\to\infty} (8 - 2n) = -\infty.$$
Barisan $\{d_n\}$ *divergen.*

Example 7 *Sebuah barisan dinyatakan secara rekursif sebagai*

$$\begin{aligned} y_1 &= 1, \\ y_{n+1} &= \frac{1}{2}\left(y_n + \frac{2}{y_n}\right), \quad n = 1, 2, 3, ... \end{aligned}$$

Dengan mengasumsikan barisan di atas konvergen, tunjukkan limitnya $\sqrt{2}$.

Misalkan barisan di atas konvergen ke L*, artinya*

$$\begin{aligned} y_n &\to L \text{ ketika } n \to \infty, \\ y_{n+1} &\to L \text{ ketika } n \to \infty, \end{aligned}$$

sehingga ketika $n \to \infty$ *diperoleh*

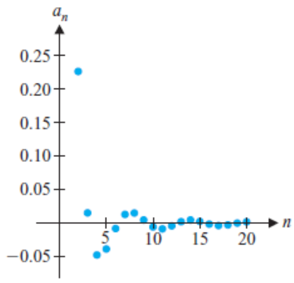
$$\begin{aligned} y_{n+1} = \frac{1}{2}\left(y_n + \frac{2}{y_n}\right) &\Rightarrow L = \frac{1}{2}\left(L + \frac{2}{L}\right) \\ &\Leftrightarrow 2L = L + \frac{2}{L} \\ &\Leftrightarrow L = \frac{2}{L} \\ &\Leftrightarrow L^2 = 2 \\ &\Leftrightarrow L = +\sqrt{2} \text{ (karena suku-suku positif).} \end{aligned}$$

3 Menghitung Limit

Theorem 8 (Teorema Apit) *Jika* $\{a_n\}$ *dan* $\{c_n\}$ *adalah barisan-barisan yang konvergen ke* L *dan* $a_n < b_n < c_n$ *untuk semua* $n > K$*, dengan* $K \in \mathbb{N}$ *konstanta, maka* $\{b_n\}$ *konvergen ke* L.

Example 9 *Tentukan kekonvergenan barisan-barisan berikut:*

1. $\{a_n\}$ *dengan* $a_n = \frac{\sin n}{n^2}$,



$$-1 \leq \sin n \leq 1 \Leftrightarrow -\frac{1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}.$$

Karena $\lim_{n\to\infty}(-\frac{1}{n^2}) = 0$ *dan* $\lim_{n\to\infty}\frac{1}{n^2} = 0$ *maka berdasarkan Teorema Apit diperoleh* $\lim_{n\to\infty}\frac{\sin n}{n^2} = 0$*, sehingga* $\{\frac{\sin n}{n^2}\}$ *konvergen ke* 0.

2. $\{b_n\}$ *dengan* $b_n = \frac{n+\sin^2 n}{2n+3}$,

$$\begin{aligned} -1 \leq \sin n \leq 1 &\Leftrightarrow 0 \leq \sin^2 n \leq 1 \\ &\Leftrightarrow n \leq n + \sin^2 n \leq n + 1 \\ &\Leftrightarrow \frac{n}{2n+3} \leq \frac{n + \sin^2 n}{2n+3} \leq \frac{n+1}{2n+3}. \end{aligned}$$

Karena

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{2n+3} &= \frac{1}{2},\\ \lim_{n\to\infty}\frac{n+1}{2n+3} &= \frac{1}{2},\end{aligned}$$

maka menurut Teorema Apit

$$\lim_{n\to\infty}\frac{n+\sin^2 n}{2n+3}=\frac{1}{2},$$

sehingga $\{\frac{n+\sin^2 n}{2n+3}\}$ *konvergen ke* $\frac{1}{2}$.

3. $\{c_n\}$ *dengan* $c_n=\frac{n!}{n^n}=1,\frac{1}{2},\frac{2}{9}$

$$\begin{aligned}\frac{n!}{n^n} &= \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdot\cdots\cdot n}\\ &= \frac{1}{n}\left(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n}\right)\\ &= \frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdot\cdots\cdot 1\right)\\ &\leq \frac{1}{n}(1)\\ \frac{n!}{n^n} &\leq \frac{1}{n}\\ \frac{n!}{n^{n-1}} &\leq 1.\end{aligned}$$

$$\begin{aligned}2\cdot 3\cdot\cdots\cdot n &\leq \underbrace{n\cdot n\cdot\cdots\cdot n}_{n-1}\\ \frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n} &\leq 1\end{aligned}$$

sehingga

$$0\leq\frac{n!}{n^n}\leq\frac{1}{n}.$$

Menurut Teorema Apit

$$\frac{n!}{n^n}\to 0.$$

Teorema berikut seringkali bermanfaat dalam penghitungan limit barisan dengan suku-suku berayun positif-negatif. Dinyatakan, jika $\{|a_n|\}$ konvergen ke 0, maka $\{a_n\}$ konvergen juga ke 0.

Theorem 10 *Jika* $\lim_{n\to\infty}|a_n|=0$ *maka* $\lim_{n\to\infty}a_n=0$.

Proof.

$$|a_n|\leq|a_n|\Leftrightarrow-|a_n|\leq a_n\leq|a_n|\,.$$

Karena diketahui $\lim_{n\to\infty}|a_n|=0$ maka

$$\begin{aligned}\lim_{n\to\infty}(-|a_n|) &= -\lim_{n\to\infty}|a_n|=0,\\ \lim_{n\to\infty}|a_n| &= 0,\end{aligned}$$

sehingga menurut Teorema Apit

$$a_n \to 0.$$

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Example 11 *Periksa kekonvergenan barisan berikut:*

$$a_n = (-1)^n \frac{1}{2^n}.$$

Diperoleh

$$\begin{aligned}
\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|(-1)^n \frac{1}{2^n}\right| \\
&= \lim_{n\to\infty} |(-1)^n| \left|\frac{1}{2^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{1}{2^n}\right| \\
&= \lim_{n\to\infty} \frac{1}{2^n} \\
&= 0,
\end{aligned}$$

sehingga

$$\lim_{n\to\infty} (-1)^n \frac{1}{2^n} = 0.$$

Example 12 *Periksa kekonvergenan barisan berikut:*

$$b_n = (-1)^n \frac{\ln n^2}{n}.$$

Diperoleh

$$\begin{aligned}
\lim_{n\to\infty} |b_n| &= \lim_{n\to\infty} \left|(-1)^n \frac{\ln n^2}{n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\ln n^2}{n}\right| \\
&= \lim_{n\to\infty} \frac{\ln n^2}{n} \\
&= \lim_{n\to\infty} \frac{2\ln n}{n} \\
&=_H \lim_{n\to\infty} \frac{2}{n} \\
&= 0,
\end{aligned}$$

sehingga

$$\lim_{n\to\infty} (-1)^n \frac{\ln n^2}{n} = 0.$$

Example 13 $b_n = 1 + (-1)^n \frac{1}{n}$

$$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \left|1 + (-1)^n \frac{1}{n}\right| = 1 + \lim_{n\to\infty} \left|(-1)^n \frac{1}{n}\right| = 1 + \lim_{n\to\infty} \frac{1}{n} = 1.$$

Teorema tidak dapat digunakan.

4 Barisan Monoton

Definition 14 *Barisan* $\{a_n\}$ *disebut*

- *naik (strictly increasing) jika* $a_1 < a_2 < a_3 < \cdots < a_n < \cdots : a_n < a_{n+1}$
- *takturun (increasing) jika* $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots : a_n \leq a_{n+1}$
- *turun (strictly decreasing) jika* $a_1 > a_2 > a_3 > \cdots > a_n > \cdots : a_n > a_{n+1}$
- *taknaik (decreasing) jika* $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots : a_n \geq a_{n+1}$

Barisan di atas disebut ***barisan monoton.***
Cara memeriksa kemonotonan:

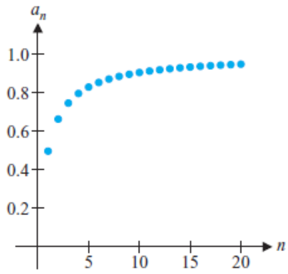
$$\begin{aligned} a_n < a_{n+1} &\Leftrightarrow a_n - a_{n+1} < 0, \\ a_n < a_{n+1} &\Leftrightarrow \frac{a_n}{a_{n+1}} < 1 \text{ atau } \frac{a_{n+1}}{a_n} > 1 \text{ (asalkan } a_{n+1} \text{ atau } a_n \text{ positif)} \\ a_n < a_{n+1} &\Leftrightarrow a'(x) > 0. \end{aligned}$$

Example 15 *Tentukan kemonotonan barisan-barisan berikut:*

1. $a_n = n : 1, 2, 3, 4, \ldots$*(barisan naik, strictly increasing)*
$$a_n - a_{n+1} = n - (n+1) = -1 < 0 \Leftrightarrow a_n < a_{n+1}.$$
$$\frac{a_n}{a_{n+1}} = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} < 1 \Leftrightarrow a_n < a_{n+1}.$$
2. $b_n = \frac{1}{n} : 1, \frac{1}{2}, \frac{1}{3}, \ldots$ *(barisan turun, strictly decreasing)*
$$a_n - a_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0 \Leftrightarrow a_n > a_{n+1}.$$
$$\frac{a_n}{a_{n+1}} = \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \Leftrightarrow a_n > a_{n+1}.$$
3. $1, 1, 2, 2, 3, 3, ... : a_n \leq a_{n+1}$ *(barisan takturun, increasing)*
4. $1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, ... : a_n \geq a_{n+1}$ *(barisan taknaik, decreasing)*
5. $c_n = (-1)^n \frac{1}{n} = -1, \frac{1}{2}, -\frac{1}{3}, ...$ *(bukan barisan monoton)*

Example 16 $a_n = \frac{n}{n+1}$

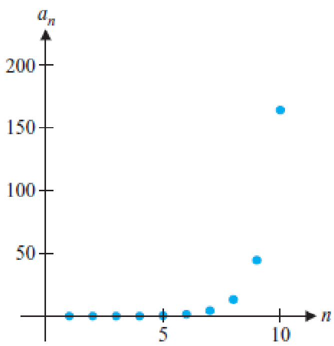
$$\begin{aligned} \frac{a_n}{a_{n+1}} &= \frac{\frac{n}{n+1}}{\frac{n+1}{n+2}} \\ &= \frac{n(n+2)}{(n+1)^2} \\ &= \frac{n^2+2n}{n^2+2n+1} \\ &= 1 - \frac{1}{n^2+2n+1} \\ &< 1 \\ a_n &< a_{n+1}. \end{aligned}$$



Example 17 $a_n = \frac{n!}{e^n} = \{\frac{1}{e}, \frac{2}{e^2}, \frac{6}{e^3}, ...\} = \{0.36788, 0.27067, 0.29872, ...\}$

$$\frac{a_n}{a_{n+1}} = \frac{n!}{e^n} \cdot \frac{e^{n+1}}{(n+1)!} = \frac{e}{n+1} < 1, n \geq 2.$$

Jadi $\{a_n\}$ *merupakan barisan naik untuk* $n \geq N$*, dengan* $N = 2$.



Example 18 $a_n = \frac{10^n}{n!}$

5 Barisan Terbatas

Definition 19 *Barisan* $\{a_n\}$ *disebut* ***terbatas*** *(bounded) jika ada bilangan* $M > 0$ *sedemikian sehingga untuk semua* n *berlaku*

$$|a_n| \leq M \Leftrightarrow -M \leq a_n \leq M.$$

Terbatas di atas: $a_n \leq U$.
Terbatas di bawah: $L \leq a_n$.

Theorem 20 *Jika* $\{a_n\}$ *monoton dan terbatas, maka* $\{a_n\}$ *konvergen.*

Theorem 21 *Jika* $\{a_n\}$ *barisan yang memenuhi:*

- *takturun:* $a_n \leq a_{n+1}$*, untuk* $n \geq N$
- *terbatas di atas:* $a_n \leq U$

maka $\{a_n\}$ *konvergen ke* $A \leq U$.

Theorem 22 *Jika* $\{b_n\}$ *barisan yang memenuhi:*

- *taknaik:* $b_n \geq b_{n+1}$*, untuk* $n \geq N$

- *terbatas di bawah:* $L \le b_n$

maka $\{b_n\}$ *konvergen ke* $B \ge L$.

Example 23 $a_n = \frac{n^2}{2^n} = \{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \frac{9}{16}, \frac{49}{128}, ...\}$

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}} \\ &= \frac{n^2}{2^n} \cdot \frac{2^{n+1}}{(n+1)^2} \\ &= \frac{2n^2}{n^2+2n+1}.\end{aligned}$$

Klaim: $a_n > a_{n+1}$ *untuk* $n \ge 3$:

$$\begin{aligned}\frac{a_n}{a_{n+1}} > 1 &\Leftrightarrow \frac{2n^2}{n^2+2n+1} > 1 \\ &\Leftrightarrow 2n^2 > n^2+2n+1 \\ &\Leftrightarrow n^2-2n > 1 \\ &\Leftrightarrow n(n-2) > 1.\end{aligned}$$

Terlihat bahwa pertaksamaan di atas benar untuk $n \ge 3$, *sehingga* $\{a_n\}$ *turun untuk* $n \ge 3$. *Selain itu karena* $a_n = \frac{n^2}{2^n} > 0$ *maka* $\{a_n\}$ *terbatas di bawah. Jadi karena* $\{a_n\}$ *monoton dan terbatas di bawah,* $\{a_n\}$ *konvergen.*

Example 24 $b_n = \frac{2^n}{n!}$.
Memeriksa kemonotonan:

$$\begin{aligned}\frac{b_n}{b_{n+1}} &= \frac{\frac{2^n}{n!}}{\frac{2^{n+1}}{(n+1)!}} \\ &= \frac{n+1}{2} \\ &\ge 1, \ n \ge 1.\end{aligned}$$

Karena $b_n \ge b_{n+1}$ *untuk* $n \ge 1$, *maka* $\{b_n\}$ *taknaik dengan batas atas* b_1, *yaitu berlaku*

$$b_1 \ge b_2 \ge b_3 \ge \cdots$$

$$b_n \le b_1 \Leftrightarrow b_n \le \frac{2^1}{1!} \Leftrightarrow b_n \le 2.$$

Karena $\{b_n\}$ *monoton dan terbatas di atas, maka* $\{b_n\}$ *konvergen.*

Example 25 $1, 1, 2, 2, 3, 3, ...$

$$a_n \le a_{n+1}.$$