The Gram-Schmidt (Orthogonalization) Process

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For linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, there exist mutually perpendicular vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ with the same linear span. These may be constructed by setting:

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{x}_1 \\
\mathbf{u}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2'\mathbf{u}_1}{\mathbf{u}_1'\mathbf{u}_1}\mathbf{u}_1 \\
\vdots \quad & \quad \vdots \qquad \qquad \vdots \\
\mathbf{u}_k &= \mathbf{x}_k - \frac{\mathbf{x}_2'\mathbf{u}_1}{\mathbf{u}_1'\mathbf{u}_1}\mathbf{u}_1\mathbf{x}_1 - \cdots - \frac{\mathbf{x}_k'\mathbf{u}_{k-1}}{\mathbf{u}_{k-1}'\mathbf{u}_{k-1}}\mathbf{u}_{k-1}
\end{aligned}
$$

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We can normalize (convert to vectors **z** of unit length) the vectors **u** by setting

$$\mathbf{z}_j = \frac{\mathbf{u}_j}{\sqrt{\mathbf{u}_j'\mathbf{u}_j}}$$

Finally, note that we can project a vector $\mathbf{x}_k$ onto the linear span of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}$:

$$\sum_{j=1}^{k-1} \left(\mathbf{x}_k'\mathbf{z}_j\right)$$

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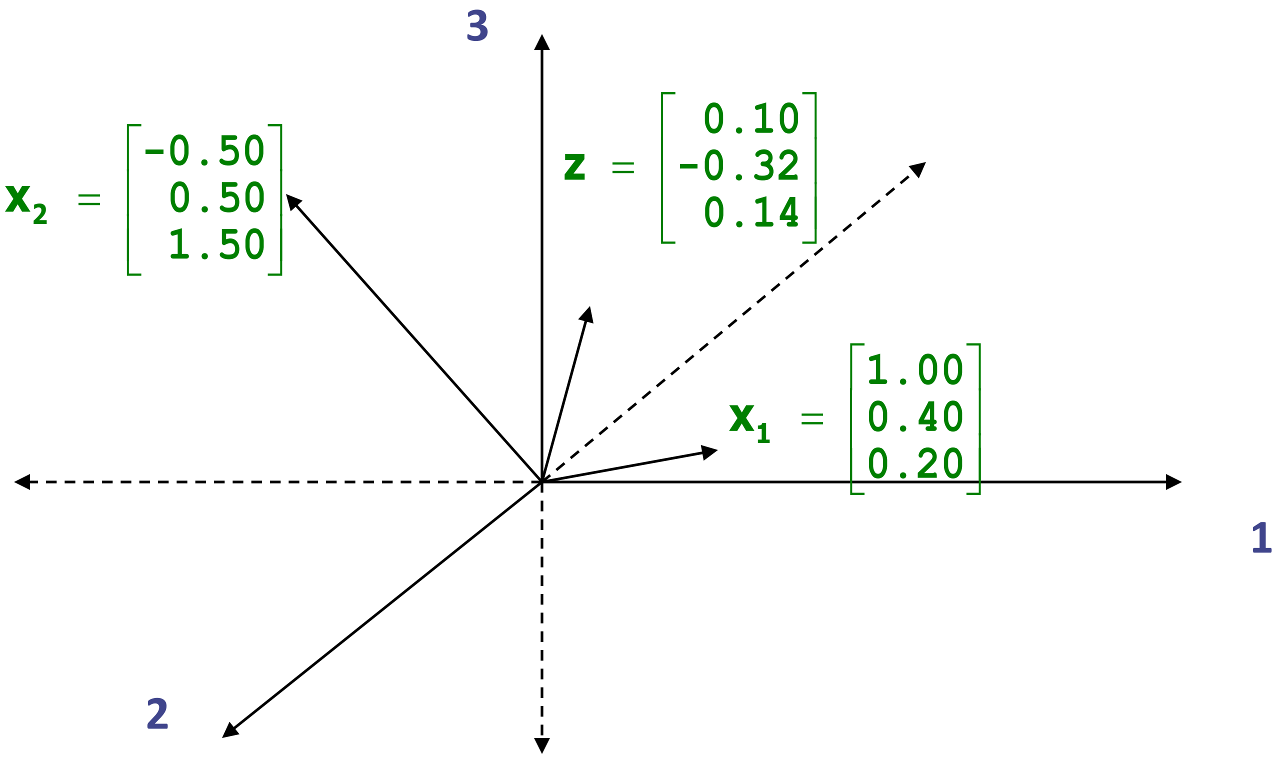
Here are vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{z}$ from our previous problem:



The Gram-Schmidt (Orthogonalization) Process

Let's construct mutually perpendicular vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ with the same linear span – we'll arbitrarily select the first axis as $\mathbf{u}_1$:

$$\mathbf{u}_1 = \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}$$

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Now we construct a vector $\mathbf{u}_2$ perpendicular with vector $\mathbf{u}_1$ (and in the linear span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{z}$):

$$\mathbf{u_2} = \begin{bmatrix} -0.50 \\ 0.50 \\ 1.50 \end{bmatrix} - \frac{\begin{bmatrix} -0.50 & 0.50 & 1.50 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}}{\begin{bmatrix} 1.00 & 0.00 & 0.00 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}} \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}$$

$$= \begin{bmatrix} -0.50 \\ 0.50 \\ 1.50 \end{bmatrix} - (-0.50) \begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.50 \\ 1.50 \end{bmatrix}$$

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Finally, we construct a vector $\mathbf{u}_3$ perpendicular with vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (and in the linear span of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{z}$):

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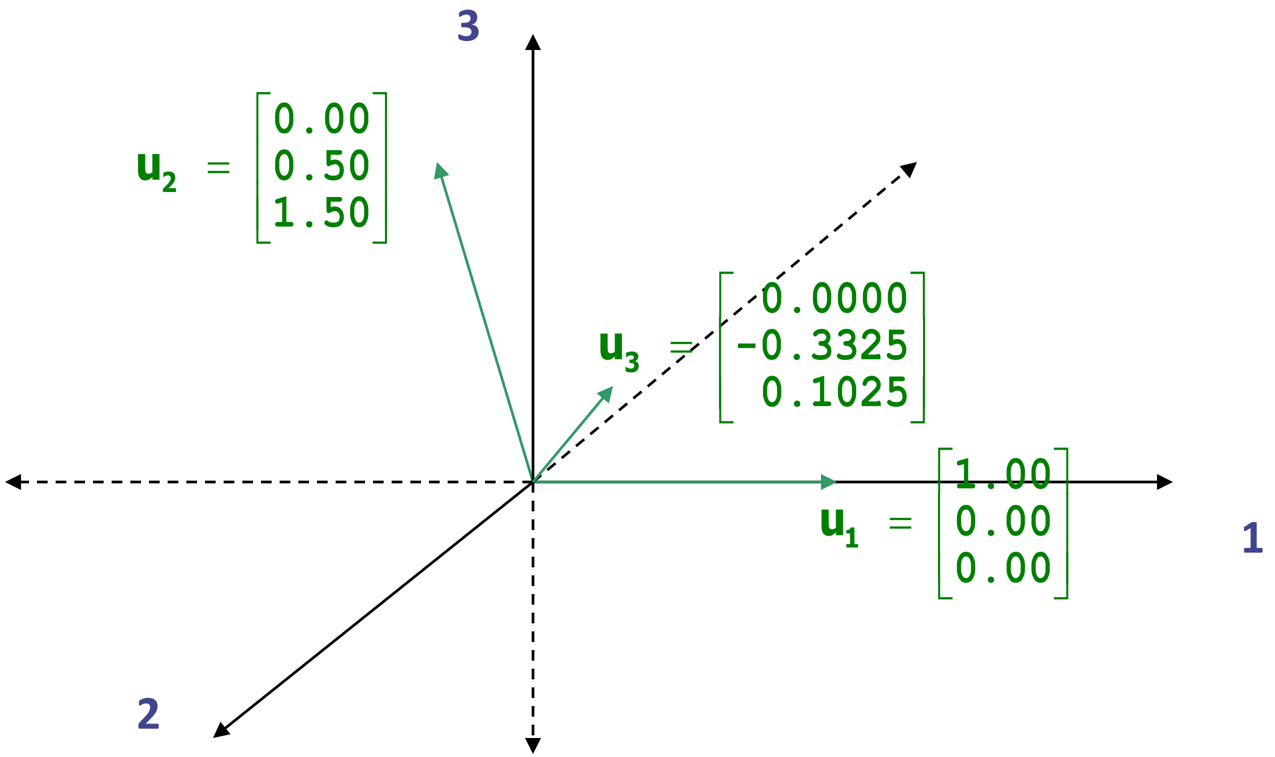
$$\mathbf{u}_3 = \mathbf{z} - \frac{\mathbf{z}'\mathbf{u}_1}{\mathbf{u}_1'\mathbf{u}_1}\mathbf{u}_1\mathbf{x}_1 - \frac{\mathbf{z}'\mathbf{u}_2}{\mathbf{u}_2'\mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 0.10 \\ -0.32 \\ 0.14 \end{bmatrix} - \frac{\begin{bmatrix} 0.10 & -0.32 & 0.14 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}}{\begin{bmatrix} 1.00 & 0.00 & 0.00 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}}\begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix}$$

$$- \frac{\begin{bmatrix} 0.10 & -0.32 & 0.14 \end{bmatrix}\begin{bmatrix} 0.00 \\ 0.50 \\ 1.50 \end{bmatrix}}{\begin{bmatrix} 0.00 & 0.50 & 1.50 \end{bmatrix}\begin{bmatrix} 0.00 \\ 0.50 \\ 1.50 \end{bmatrix}}\begin{bmatrix} 0.00 \\ 0.50 \\ 1.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.10 \\ -0.32 \\ 0.14 \end{bmatrix} - 0.10\begin{bmatrix} 1.00 \\ 0.00 \\ 0.00 \end{bmatrix} - 0.25\begin{bmatrix} 0.00 \\ 0.50 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ -0.3325 \\ 0.1025 \end{bmatrix}$$

The Gram-Schmidt (Orthogonalization) Process

Here are our orthogonal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$:



The Gram-Schmidt (Orthogonalization) Process

If we normalized our vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, we get:

$$\mathbf{z}_1 = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1'\mathbf{u}_1}} = \frac{\begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}}{\sqrt{\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \end{bmatrix}\begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}}}$$

$$= \frac{1.0}{\sqrt{1.0}}\begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

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and:

$$\mathbf{z}_2 = \frac{\mathbf{u}_2}{\sqrt{\mathbf{u}_2'\mathbf{u}_2}} = \frac{\begin{bmatrix} 0.0000 \\ 0.5000 \\ 1.5000 \end{bmatrix}}{\sqrt{\begin{bmatrix} 0.0000 & 0.5000 & 1.5000 \end{bmatrix}\begin{bmatrix} 0.0000 \\ 0.5000 \\ 1.5000 \end{bmatrix}}}$$

$$= \frac{1.0}{\sqrt{2.5}}\begin{bmatrix} 0.0000 \\ 0.5000 \\ 1.5000 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 0.3162 \\ 0.9487 \end{bmatrix}$$

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and:

$$\mathbf{z}_3 = \frac{\mathbf{u}_3}{\sqrt{\mathbf{u}_3'\mathbf{u}_3}} = \frac{\begin{bmatrix} 0.0000 \\ -0.3325 \\ 0.1025 \end{bmatrix}}{\sqrt{\begin{bmatrix} 0.0000 & -0.3325 & 0.1025 \end{bmatrix}\begin{bmatrix} 0.0000 \\ -0.3325 \\ 0.1025 \end{bmatrix}}}$$

$$= \frac{1.0}{\sqrt{0.1211}}\begin{bmatrix} 0.0000 \\ -0.3325 \\ 0.1025 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ -0.9556 \\ 0.2946 \end{bmatrix}$$

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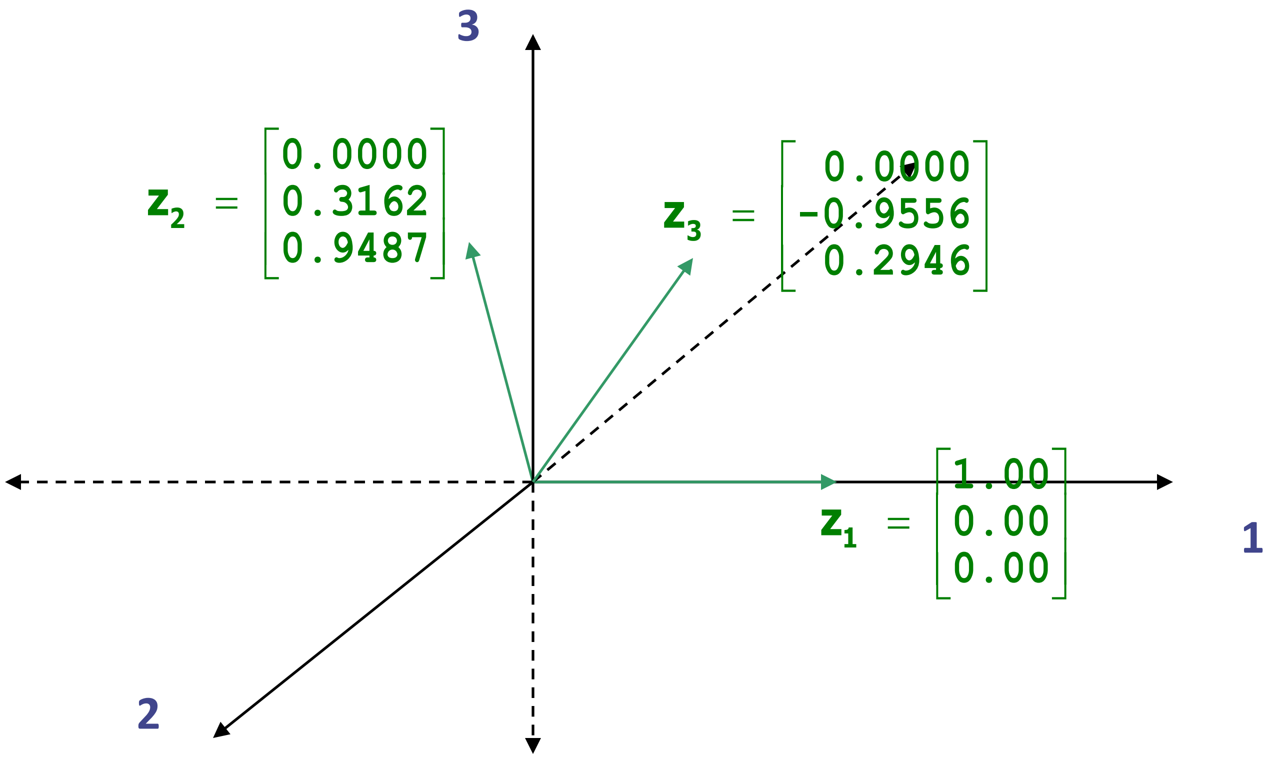
The normalized vectors $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ look like this:



LATIHAN SOAL

Kerjakan kembali contoh di atas, dengan mengambil $\underline{\mathbf{u}}_1$=$\underline{\mathbf{x}}_1$. Carilah vector-vector orthogonal x2 dan z. Lakukan proses orthogonalisasi Gram-Schmidt