MATRIX DAN VEKTOR ACAK

Random Vectors and Matrices

- Random Vector – vector whose individual elements are random variables
- Random Matrix – matrix whose individual elements are random variables

Random Vectors and Matrices

The expected value of a random vector or matrix is the matrix containing the expected values of the individual elements, i.e.,

$$E[\mathbf{X}] = \begin{bmatrix} E[x_{11}] & E[x_{12}] & \cdots & E[x_{1p}] \\ E[x_{21}] & E[x_{22}] & \cdots & E[x_{2p}] \\ \vdots & \vdots & \ddots & \vdots \\ E[x_{n1}] & E[x_{n2}] & \cdots & E[x_{np}] \end{bmatrix}$$

Random Vectors and Matrices

where

$$E\left[x_{ij}\right] = \begin{cases} \displaystyle\sum_{\text{all } x_{ij}} x_{ij} p_{ij}\left(x_{ij}\right) \\ \displaystyle\int_{-\infty}^{\infty} x_{ij} f_{ij}\left(x_{ij}\right) dx_{ij} \end{cases}$$

Random Vectors and Matrices

Note that for random matrices $\mathbf{X}$ and $\mathbf{Y}$ of the same dimension, conformable matrices of constants $\mathbf{A}$ and $\mathbf{B}$, and scalar c

- $E(c\mathbf{X}) = cE(\mathbf{X})$
- $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$
- $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$

Random Vectors and Matrices

Mean Vector – random vector whose elements are the means of the corresponding random variables, i.e.,

$$E\left[x_i\right] = \mu_i = \begin{cases} \sum\limits_{\text{all } x_i} x_i p_i\left(x_i\right) \\ \int\limits_{-\infty}^{\infty} x_i f_i\left(x_i\right) dx_i \end{cases}$$

Random Vectors and Matrices

In matrix notation we can write the mean vector as

$$E[\mathbf{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_p] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} = \boldsymbol{\mu}$$

Random Vectors and Matrices

For the bivariate probability distribution

X_1 \ X_2	-4	1	3	$p_1(x_1)$
1	0.15	0.20	0.05	0.40
2	0.10	0.25	0.25	0.60
$p_2(x_2)$	0.25	0.45	0.30	1.00

the mean vector is

$$\boldsymbol{\mu} = E[\mathbf{X}] = \begin{bmatrix} E[x_1] \\ E[x_2] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \sum_{\text{all } x_1} x_1 p_1(x_1) \\ \sum_{\text{all } x_2} x_2 p_2(x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.4(1) + 0.6(2) \\ 0.1(-4) + 0.25(1) + 0.25(3) \end{bmatrix} = \begin{bmatrix} 1.60 \\ 0.35 \end{bmatrix}$$

Random Vectors and Matrices

Covariance Matrix – random symmetric vector whose diagonal elements are variances of the corresponding random variables, i.e.,

$$E\left[\left(x_i-\mu_i\right)^2\right]=\sigma_i^2=\begin{cases}\displaystyle\sum_{\text{all } x_i}\left(x_i-\mu_i\right)^2 p_i\left(x_i\right)\\[2ex] \displaystyle\int_{-\infty}^{\infty}\left(x_i-\mu_i\right)^2 f_i\left(x_i\right)dx_i\end{cases}$$

Random Vectors and Matrices

and whose off-diagonal elements are covariances of the corresponding random variable pairs, i.e.,

$$E\left[(x_i-\mu_i)(x_k-\mu_k)\right]=\sigma_{ik}=\begin{cases}\sum_{\text{all } x_i}\sum_{\text{all } x_k}(x_i-\mu_i)(x_k-\mu_k)p_{ik}(x_i,x_k)\\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i-\mu_i)(x_k-\mu_k)f_{ik}(x_i,x_k)dx_idx_k\end{cases}$$

notice that if we this expression, when i = k, returns the variance, i.e.,

$$\sigma_{ii}=\sigma_i^2$$

Random Vectors and Matrices

In matrix notation we can write the covariance matrix as

$$E\left[(\mathbf{X}-\mu)(\mathbf{X}-\mu)'\right] = E\left[\begin{bmatrix}(x_1-\mu_1)\\(x_2-\mu_2)\\\vdots\\(x_p-\mu_p)\end{bmatrix}\begin{bmatrix}(x_1-\mu_1) & (x_2-\mu_2) & \cdots & (x_p-\mu_p)\end{bmatrix}\right]$$

$$= \begin{bmatrix} E\left[(x_1-\mu_1)^2\right] & E\left[(x_1-\mu_1)(x_2-\mu_2)\right] & \cdots & E\left[(x_1-\mu_1)(x_p-\mu_p)\right] \\ E\left[(x_2-\mu_2)(x_1-\mu_1)\right] & E\left[(x_2-\mu_2)^2\right] & \cdots & E\left[(x_2-\mu_2)(x_p-\mu_p)\right] \\ \vdots & \vdots & \ddots & \vdots \\ E\left[(x_p-\mu_p)(x_1-\mu_1)\right] & E\left[(x_p-\mu_p)(x_2-\mu_2)\right] & \cdots & E\left[(x_p-\mu_p)^2\right] \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = \Sigma$$

Random Vectors and Matrices

For the bivariate probability distribution we used earlier

X_1 \ X_2	-4	1	3	$p_1(x_1)$
1	0.15	0.20	0.05	0.40
2	0.10	0.25	0.25	0.60
$p_2(x_2)$	0.25	0.45	0.30	1.00

the covariance matrix is

$$E\left[(\mathbf{X}-\mu)(\mathbf{X}-\mu)'\right] = \begin{bmatrix} E\left[(x_1-\mu_1)^2\right] & E\left[(x_1-\mu_1)(x_2-\mu_2)\right] \\ E\left[(x_2-\mu_2)(x_1-\mu_1)\right] & E\left[(x_2-\mu_2)^2\right] \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{\text{all } x_1} (x_1-\mu_1)^2 p_1(x_1) & \sum\limits_{\text{all } x_1}\sum\limits_{\text{all } x_2} (x_1-\mu_1)(x_2-\mu_2) p_{12}(x_1,x_2) \\ \sum\limits_{\text{all } x_2}\sum\limits_{\text{all } x_1} (x_2-\mu_2)(x_1-\mu_1) p_{21}(x_2,x_1) & \sum\limits_{\text{all } x_2} (x_2-\mu_2)^2 p_2(x_2) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \Sigma$$

Random Vectors and Matrices

which can be computed as

$$E\left[(\mathbf{X}-\mu)(\mathbf{X}-\mu)'\right] = \begin{bmatrix} \sum_{\text{all } x_1} (x_1-\mu_1)^2 p_1(x_1) & \sum_{\text{all } x_1}\sum_{\text{all } x_2} (x_1-\mu_1)(x_2-\mu_2) p_{12}(x_1,x_2) \\ \sum_{\text{all } x_2}\sum_{\text{all } x_1} (x_2-\mu_2)(x_1-\mu_1) p_{21}(x_2,x_1) & \sum_{\text{all } x_2} (x_2-\mu_2)^2 p_2(x_2) \end{bmatrix}$$

$$= \begin{bmatrix} (1-1.6)^2(0.4)+(2-1.6)^2(0.6) & \begin{matrix} (1-1.6)(-4-0.35)(0.15)+(1-1.6)(1-0.35)(0.2) \\ +(1-1.6)(3-0.35)(0.05)+(2-1.6)(-4-0.35)(0.1) \\ +(2-1.6)(1-0.35)(0.25)+(2-1.6)(3-0.35)(0.25) \end{matrix} \\ \begin{matrix} (-4-0.35)(1-1.6)(0.15)+(-4-0.35)(2-1.6)(0.1) \\ +(1-0.35)(1-1.6)(0.2)+(1-0.35)(2-1.6)(0.25) \\ +(3-0.35)(1-1.6)(0.05)+(3-0.35)(2-1.6)(0.25) \end{matrix} & (-4-0.35)^2(0.25)+(1-0.35)^2(0.45)+(3-0.35)^2(0.3) \end{bmatrix}$$

$$= \begin{bmatrix} 0.24 & 0.39 \\ 0.39 & 7.0275 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \Sigma$$

Random Vectors and Matrices

Thus the population represented by the bivariate probability distribution

X_1 \ X_2	-4	1	3	$p_1(x_1)$
1	0.15	0.20	0.05	0.40
2	0.10	0.25	0.25	0.60
$p_2(x_2)$	0.25	0.45	0.30	1.00

Have population mean vector and variance-covariance matrix

$$\mu = \begin{bmatrix} 1.60 \\ 0.35 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 0.24 & 0.39 \\ 0.39 & 7.0275 \end{bmatrix}$$

Random Vectors and Matrices

We can use the population variance-covariance matrix $\boldsymbol{\Sigma}$ to calculate the population correlation matrix $\boldsymbol{\rho}$. Individual population correlation coefficients are defined as

$$\rho_{ik} = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{kk}}}$$

Random Vectors and Matrices

In matrix notation we can write the correlation matrix as

$$\boldsymbol{\rho} = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{ik} & \rho_{ik} & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & \rho_{pp} \end{bmatrix}$$

Random Vectors and Matrices

We can easily show that

$$\boldsymbol{\Sigma} = \mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2}$$

where

$$\mathbf{V}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}$$

which implies

$$\boldsymbol{\rho} = \left(\mathbf{V}^{1/2}\right)^{-1}\boldsymbol{\Sigma}\left(\mathbf{V}^{1/2}\right)^{-1}$$

Random Vectors and Matrices

For the bivariate probability distribution we used earlier

X_1 \ X_2	-4	1	3	$p_1(x_1)$
1	**0.15**	**0.20**	**0.05**	**0.40**
2	**0.10**	**0.25**	**0.25**	**0.60**
$p_2(x_2)$	**0.25**	**0.45**	**0.30**	**1.00**

the variance matrix is

$$\mathbf{V}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{bmatrix} = \begin{bmatrix} \sqrt{0.24} & 0 \\ 0 & \sqrt{7.0275} \end{bmatrix} = \begin{bmatrix} 0.48989 & 0 \\ 0 & 2.65094 \end{bmatrix}$$

Random Vectors and Matrices

so the population correlation matrix is

$$\boldsymbol{\rho} = \left(\mathbf{V}^{1/2}\right)^{-1} \Sigma \left(\mathbf{V}^{1/2}\right)^{-1}$$

$$= \begin{bmatrix} 2.04124 & 0 \\ 0 & 0.37739 \end{bmatrix} \begin{bmatrix} 0.24 & 0.39 \\ 0.39 & 7.0275 \end{bmatrix} \begin{bmatrix} 2.04124 & 0 \\ 0 & 0.37739 \end{bmatrix}$$

$$= \begin{bmatrix} 1.00 & 0.30 \\ 0.30 & 1.00 \end{bmatrix}$$

Random Vectors and Matrices

We often deal with variables that naturally fall into groups. In the simplest case, we have two groups of size q and p – q of variables.

Under such circumstances, it may be convenient to *partition the matrices and vectors*.

Random Vectors and Matrices

Here we have a mean vector and variance-covariance matrix:

$$\mu \left[\begin{array}{c} \mu_1 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{array}\right] = \left(\begin{array}{c} \mu^{(1)} \\ \hline \mu^{(2)} \end{array}\right), \quad \Sigma = \left[\begin{array}{ccc|ccc} \sigma_{11} & \cdots & \sigma_{1p} & \sigma_{1,q+1} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{q1} & \cdots & \sigma_{qq} & \sigma_{q,q+1} & \cdots & \sigma_{qp} \\ \hline \sigma_{q+1,1} & \cdots & \sigma_{q+1,q} & \sigma_{q+1,q+1} & \cdots & \sigma_{q+1,p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pq} & \sigma_{p,q+1} & \cdots & \sigma_{pp} \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$$

Random Vectors and Matrices

Rules for Mean Vectors and Covariance Matrices for Linear Combinations of Random Variables: The linear combination of real constants **c** and random variables **X**

$$\mathbf{c}'\mathbf{X} = \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = \sum_{j=1}^{p} c_j X_j$$

has mean vector

$$E\left[\mathbf{c}'\mathbf{X}\right] = \mathbf{c}'E\left[\mathbf{X}\right] = \mathbf{c}'\boldsymbol{\mu}$$

Random Vectors and Matrices

The linear combination of real constants **c** and random variables **X**

$$\mathbf{c}'\mathbf{X} = \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = \sum_{j=1}^{p} c_j X_j$$

also has variance

$$\text{Var}\left[\mathbf{c}'\mathbf{X}\right] = \mathbf{c}'\Sigma\mathbf{c}$$

Random Vectors and Matrices

Suppose, for example, we have random vector

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and we want to find the mean vector and covariance matrix for the linear combinations:

$$\begin{aligned} Z_1 &= X_1 - X_2, \\ Z_2 &= X_1 + X_2 \end{aligned} \quad \text{i.e.,}\ \mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{CX}$$