Tugas Bayes Kuliah

Raziqizzan Putrandi - G1401241040

Dik: Y: banyak bola dalam wadah (0,1,2,3,4,5,6,7)
di dapatkan 3 bola yg terambil berwarna biru.
dapat disimpulkan $P(Y=0 \mid A) = P(Y=1 \mid A) = P(Y=2 \mid A) = 0$
atau $Y=0$, $Y=1$, $Y=2$ bernilai nol (mustahil)
Y lainnya adalah:

Jawab: $\dfrac{\binom{Y}{3}\binom{7-Y}{0}}{\binom{7}{3}}$ sehingga $\dfrac{\text{prior} \times \text{likelihood}}{\sum \text{prior} \times \text{likelihood}}$

Y	prior $P(Y=y)$	likelihood	prior × likelihood	Posterior
0	1/8	0	0	0
1	1/8	0	0	0
2	1/8	0	0	0
3	1/8	1/35	1/280	1/70
4	1/8	4/35	4/280	4/70
5	1/8	10/35	10/280	10/70
6	1/8	20/35	20/280	20/70
7	1/8	1	35/280	35/70

2) ~~Poisson~~ Dik: $X \sim \text{Poisson}(\theta)$
Sebaran prior θ menyebar uniform (1,5)

Dit: a.) Sebaran posterior bagi θ
b.) Penduga bayes bagi θ
c.) Credible interval 95% bagi θ

Jwb: a.) $\theta \sim \text{uniform}(1,5) \rightarrow$ Prior: $P(\theta) = \begin{cases} 1/4 & 1 \le \theta \le 5 \\ 0 & \text{lainnya} \end{cases}$

$X \sim \text{Poisson}(\theta) \rightarrow$ Likelihood: $P(x \mid \theta) = \int_1^5 \dfrac{e^{-\theta}\theta^x}{x!}$, $x = 0,1,2,\ldots$

Sebaran Posterior

$$P(\theta \mid x) = \frac{P(\theta) \times P(x \mid \theta)}{\int P(\theta) \times P(x \mid \theta)\, d\theta} = \frac{1/4 \times ((\theta^x \cdot e^{-\theta})/x!)}{\int_1^5 1/4((\theta^x \cdot e^{-\theta})/x!)\, d\theta}$$

$$\propto = e^{-\theta}\theta^x$$

Sebaran posterior akan sesuai dgn sebaran gamma karena gamma adalah konjugat prior untuk distribusi Poisson.

$P(\theta \mid x) = \text{Gamma}(\alpha', \beta')$
$\alpha' = \alpha + \sum x_i$
$\beta' = \beta + n$

b.) $E(\theta \mid x) = \hat{\theta}$

$= \frac{\alpha'}{\beta'} = \frac{1+\Sigma x_i}{1+n}$

c.) $\theta_1 \to$ kuantil 0,025 dari distribusi gamma $\approx 1,19$

$\theta_2 \to$ kuantil 0,975 dari distribusi gamma $\approx 3,25$

credible interval 95% bagi θ adalah $1,19 \leq \theta \leq 3,25$

3.) Dik: $X \sim Beta(2, \theta)$; $\theta > 0$

D.t: a.) Tentukan Jeffrey's prior bagi θ

b.) Berdasarkan jawaban anda pada point a) tentukan penduga bayes bagi θ

Jwb: a) $I(\theta) = E\left[\frac{-d^2 \log f(x;\theta)}{d\theta^2}\right]$

$= E\left[\frac{d^2 \log(\theta^{2x-1}(1-\theta)^{2-2x})}{d\theta^2}\right]$

$= E\left[\frac{d(2x-1)}{d\theta^2}\log(\theta) + \frac{d(2-2x)}{d\theta^2}\log(1-\theta)\right]$

$= E\left[\frac{-2}{\theta^2}\log(\theta) - \frac{2}{(1-\theta^2)}\log(1-\theta)\right]$

$X \sim B(2,\theta)$

$E(x) = \frac{\alpha}{\alpha+\beta} = \frac{2}{2+\theta}$

$I(\theta) = \frac{-2}{(\frac{2}{2+\theta})^2}\log\left(\frac{2}{2+\theta}\right) - \frac{2}{(1-\frac{2}{2+\theta})^2}\log\left(1-\frac{2}{2+\theta}\right)$

$= \frac{2}{(\theta^2(1-\theta)^2)}$

Jeffrey's prior : $\pi(\theta) \propto \sqrt{\frac{2}{\theta^2(1-\theta)^2}}$

b.) $\frac{1}{c}\sqrt{\frac{2}{\theta^2(1-\theta)^2}} \times \theta^{2x-1}(1-\theta)^{2-2x}$

** c = konstanta normalisasi

Dipindai dengan CamScanner

4.) Dik: $Y \sim$ normal (μ, θ^2) μ tidak diketahui, θ^2 diketahui
Sebaran prior bagi μ adalah normal $(3, \theta^2)$
Dit: a.) Sebaran posterior bagi μ
b.) Penduga bayes bagi μ
c.) credible interval 95% bagi μ

Jwb:

a.) $\mu \sim N(3, \theta^2)$

$$f(\mu) = \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(\mu-3)^2}{2\theta^2}}$$

$$\text{Likelihood} = f(\underline{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(y_i-\mu)^2}{2\theta^2}}$$

$$\text{Posterior} = f(\underline{y}) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(y_i-\mu)^2}{2\theta^2}} \times \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{(\mu-3)^2}{2\theta^2}}$$

b.) mean: $\dfrac{\dfrac{\sum_{i=1}^{n} y_i}{\sigma^2} + \dfrac{3}{\theta^2}}{\dfrac{n}{\sigma^2} + \dfrac{1}{\theta^2}}$ var: $\dfrac{1}{\dfrac{n}{\sigma^2} + \dfrac{1}{\theta^2}}$

$$\Rightarrow \hat{\mu} = \frac{\frac{\sum y_i}{\sigma^2} + \frac{3}{\theta^2}}{\frac{n}{\sigma^2} + \frac{1}{\theta^2}}$$

c.) $\hat{\mu} - 1{,}96\sqrt{var} \le \mu \le \hat{\mu} + 1{,}96\sqrt{var}$

$$\Rightarrow \frac{\frac{\sum_{i=1}^{n} y_i}{\sigma^2} + \frac{3}{\theta^2}}{\frac{n}{\sigma^2} + \frac{1}{\theta^2}} - 1{,}96\sqrt{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\theta^2}}} \le \mu \le \frac{\frac{\sum_{i=1}^{n} y_i}{\sigma^2} + \frac{3}{\theta^2}}{\frac{n}{\sigma^2} + \frac{1}{\theta^2}} + 1{,}96\sqrt{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\theta^2}}}$$

5.) Dik: Y: Jumlah mahasiswa setuju
θ: peluang mahasiswa setuju
65 orang setuju
Dit: a.) Sebaran prior θ adalah Beta (4, 7) tentukan sebaran posterior bagi θ
b.) Tentukan penduga bayes bagi θ
c.) Lakukan pengujian hipotesis $H_0: \theta = 0{,}6$ vs $H_1: \theta = 0{,}9$ dengan
d.) — " — " — " $H_0: \theta = 0{,}6$ vs $H_1: \theta \neq 0{,}6$ pada taraf 5%. Selesaikan!

Jawab: $n = 100$

a.) $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(4)+\Gamma(7)}{\Gamma(4+7)} = \frac{3!\,6!}{10!} = \frac{1}{840}$

Prior : $P(\theta) = \frac{1}{B(\alpha,\beta)} \theta^{(4-1)}(1-\theta)^{7-1} = 840(\theta)^3(1-\theta)^6$

Likelihood : $P(y \mid \theta) = \binom{100}{65} \theta^{65}(1-\theta)^{35}$

Posterior : $P(\theta \mid y) = \frac{P(y \mid \theta)\, P(\theta)}{P(y)} \propto \binom{100}{65} \theta^{65}(1-\theta)^{35}(840)(\theta)^3(1-\theta)^6$

$\propto \theta^{69-1}(1-\theta)^{42-1}$

Sebaran posteriornya adalah Beta (69, 42)

b.) $\hat{\theta} = \frac{\alpha \text{ posterior}}{\alpha \text{ posterior} + \beta \text{ posterior}} = \frac{69}{69+42} = \frac{69}{111} = 0{,}6216$

c.) ⊕ H_0 :

$P(\theta = 0{,}6 \mid y) = \frac{\text{Beta}(69,42)(0{,}6)}{\text{Beta}(69,42)(0{,}6) + \text{Beta}(69,42)(0{,}4)}$

$= \frac{0{,}6^{(69-1)}\ 0{,}4^{(42-1)}}{0{,}6^{(69-1)}\ 0{,}4^{(42-1)} + 0{,}4^{(69-1)}\ 0{,}6^{(42-1)}}$

$= 0{,}999982$

⊕ H_1 :

$P(\theta = 0{,}4 \mid y) = \frac{\text{Beta}(69,42)(0{,}4)}{\text{Beta}(69,42)(0{,}4) + \text{Beta}(69,42)(0{,}6)}$

$= 0{,}0000176$

$\Rightarrow P(\theta = 0{,}6 \mid y) > P(\theta = 0{,}4 \mid y) \rightarrow$ Terima H_0 $\theta = 0{,}6$

d.)
- kuantil ke – 0,025 : $P(\theta < \text{kuantil } 0{,}025) = 0{,}025 \approx 0{,}492$
- kuantil ke – 0,975 : $P(\theta < \text{kuantil } 0{,}975) = 0{,}975 \approx 0{,}792$

di dapatkan probabilitas posterior bahwa θ berada dalam interval $\approx 0{,}059 > 0{,}05$ sehingga terima H_0 $\theta = 0{,}6$