Beberapa peubah acak diskret dan kontinu

	Peubah acak	Fmp/fkp	Nilai harapan	Ragam	Fungsi pembangkit momen
1.	Bernoulli(p)	$f(x\mid p) = p^x(1-p)^{1-x}$, utk $x = 0, 1;\ 0 \le p \le 1$	$E(X) = p$	$var(X) = p(1-p)$	$M_X(t) = (1-p) + pe^t$
2.	Binomial(n,p)	$f(x) = C(n,x)p^x(1-p)^{n-x}$, utk $x = 0,1,2,\ldots,n;\ 0 \le p \le 1$	$E(X) = np$	$var(X) = np(1-p)$	$M_X(t) = [pe^t+(1-p)]^n$
3.	Uniform diskret	$f(x\mid N) = 1/N$, utk $x = 1,2,3,\ldots,N;$ $N = 1,2,3,\ldots$	$E(X) = (N+1)/2$	$var(X) = (N+1)(N-1)/12$	$M_X(t) = (\sum_{x=1}^{N} e^{tx})/N$
4.	Geometrik(p)	$f(x\mid p)) = p(1-p)^{x-1}$, utk $x = 1,2,3,\ldots;\ 0 \le p \le 1$	$E(X) = 1/p$	$var(X) = (1-p)/p^2$	$M_X(t) = pe^t/[1-(1-p)e^t]$, utk $t < -\ln(1-p)$
5.	Binomial negatif(r,p)	$f(x\mid r,p) = C(r+x-1,x)p^r(1-p)^x$, utk $x = 0,1,2,\ldots;$ $0 \le p \le 1$ $\binom{w-1}{r-1} p^r (1-p)^{w-r}$ $w = r, r+1, r+2,$	$E(X) = r(1-p)/p$	$var(X) = r(1-p)/p^2$	$M_X(t) = [pe^t/\{1-(1-p)e^t\}]^r$, utk $t < -\ln(1-p)$
6.	Poisson(λ)	$f(x\mid \lambda) = e^{-\lambda}\lambda^x/x!$, utk $x = 0,1,2,\ldots;\ 0 \le \lambda < \infty$	$E(X) = \lambda$	$var(X) = \lambda$	$M_X(t) = \exp[\lambda(e^t-1)]$
7.	Beta(α,β)	$f(x\mid \alpha,\beta) = [B(\alpha,\beta)]^{-1}x^{\alpha-1}(1-x)^{\beta-1}$, utk $0 \le x \le 1$, $\alpha > 0, \beta > 0.$ $B(\alpha,\beta) = [\Gamma(\alpha)\Gamma(\beta)]/\Gamma(\alpha+\beta)$	$E(X) = \alpha/(\alpha+\beta)$	$var(X) = \alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$	
8.	Cauchy(θ,σ)	$f(x\mid \theta,\sigma) = (\pi\sigma)^{-1}[1+\{(x-\theta)/\sigma\}^2]^{-1}$, utk $-\infty < x < \infty$, $-\infty < \theta < \infty, \sigma > 0$	Tidak ada	Tidak ada	Tidak ada
9.	Khi-kuadrat	$f(x\mid p) = [\Gamma(p/2)2^{p/2}]^{-1}x^{(p/2)-1}e^{-x/2}$, utk $0 \le x < \infty;$ $p = 1,2,3,\ldots$	$E(X) = p$	$var(X) = 2p$	$M_X(t) = (1-2t)^{-p/2}$, utk $t < 1/2$
10.	Eksponensial ganda(μ,σ)	$f(x\mid \mu,\sigma) = (2\sigma)^{-1}e^{-\lvert x-\mu\rvert/\sigma}$, utk $-\infty < x < \infty$, $-\infty < \mu < \infty, \sigma > 0$	$E(X) = \mu$	$var(X) = 2\sigma^2$	$M_X(t) = e^{\mu t}/[1-(\sigma t)^2]$, utk $\lvert t\rvert < 1/\sigma$
11.	Eksponensial (β)	$f(x\mid \beta) = (1/\beta)e^{-x/\beta}$, utk $0 \le x < \infty, \beta > 0$	$E(X) = \beta$	$var(X) = \beta^2$	$M_X(t) = 1/(1-\beta t)$, utk $t < 1/\beta$

Beberapa peubah acak diskret dan kontinu (lanjutan)

No.	Peubah acak	Fmp/fkp	Nilai harapan	Ragam	Fungsi pembangkit momen (fpm)
12.	F-Snedecor(ν,ω)	$f(x\mid \nu,\omega) = \Gamma[(\nu+\omega)/2]\,[\Gamma(\nu/2)\Gamma(\omega/2)]^{-1}$ $(\nu/\omega)^{\nu/2}\, x^{(\nu-2)/2}[1+(\nu/\omega)x]^{-(\nu+\omega)/2}$, utk $0 \le x < \infty;\ \nu, \omega = 1,2,3,\ldots$	$E(X) = \omega/(\omega-2), \omega>2$	$var(X) = 2[\omega/(\omega-2)]^2$ $[(\nu+\omega-2)][\nu(\omega-4)]^{-1}$, utk $\omega > 4$	fpm tidak ada.
13.	Gamma(α,β)	$f(x\mid \alpha,\beta) = [\Gamma(\alpha)\beta^\alpha]^{-1}x^{\alpha-1}e^{-x/\beta}$, utk $0 \le x < \infty;\ \alpha > 0, \beta > 0$	$E(X) = \alpha\beta$	$var(X) = \alpha\beta^2$	$M_X(t) = (1-\beta t)^{-\alpha}$, utk $t < 1/\beta$
14.	Logistik(μ,β)	$f(x\mid \mu,\beta) = \beta^{-1}e^{-(x-\mu)/\beta}[1+e^{-(x-\mu)/\beta}]^{-2}$, utk $-\infty < x < \infty;\ -\infty < \mu < \infty, \beta > 0$	$E(X) = \mu$	$var(X) = \pi^2\beta^2/3$	$M_X(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t)$, utk $\lvert t\rvert < 1/\beta$
15.	Lognormal(μ,σ²)	$f(x\mid \mu,\sigma^2) =$ $[\sigma\sqrt{(2\pi)}]^{-1}\exp\{-[(\ln x)-\mu]^2/(2\sigma^2)\}$, utk $0 \le x < \infty;\ -\infty < \mu < \infty, \sigma > 0$	$E(X) = \exp(\mu + \sigma^2/2)$	$var(X) = \exp[2(\mu + \sigma^2)]$ $- \exp(2\mu + \sigma^2)$	fpm tidak ada. Tetapi momen ke-n adalah $E(X^n) = \exp[n\mu + (n^2\sigma^2)/2]$
16.	Normal(μ,σ²)	$f(x\mid \mu,\sigma^2)= [\sigma\sqrt{(2\pi)}]^{-1}\exp[-(x-\mu)^2/(2\sigma^2)]$, utk $-\infty < x < \infty;\ -\infty < \mu < \infty, \sigma > 0$	$E(X) = \mu$	$Var(X) = \sigma^2$	$M_X(t) = \exp[\mu t + (\sigma^2t^2)/2]$
17.	Pareto(α,β)	$f(x\mid \alpha,\beta) = \beta\alpha^\beta/(x^{\beta+1})$, utk $\alpha < x < \infty;$ $\alpha > 0, \beta > 0$	$E(X) = \beta\alpha/(\beta-1)$, utk $\beta > 1$	$var(X) = \beta\alpha^2/[(\beta-1)^2(\beta-2)]$, utk $\beta > 2$	fpm tidak ada.
18.	t-Student(ν)	$f(x\mid \nu) = \{\Gamma[(\nu+1)/2]/[\Gamma(\nu/2)]\}[(\nu\pi)^{-1/2}]$ $[1+x^2/\nu]^{-(\nu+1)/2}$, utk $-\infty<x<\infty;\ \nu=1,2,\ldots$	$E(X) = 0$, utk $\nu > 1$	$var(X) = \nu/(\nu-2)$, utk $\nu > 2$	fpm tidak ada. Momen ke-n ada (lihat di buku).
19.	Uniform(a,b)	$f(x\mid a,b) = 1/(b-a)$, utk $a \le x \le b$	$E(X) = (b + a)/2$	$var(X) = (b-a)^2/12$	$M_X(t) = (e^{bt}-e^{at})/[(b-a)t]$
20.	Weibull(γ,β)	$f(x\mid \gamma,\beta) = (\gamma/\beta)x^{\gamma-1}\exp[-x^\gamma/\beta]$, utk $0 < x < \infty;\ \gamma > 0, \beta > 0$	$E(X) = \beta^{1/\gamma}\,\Gamma(1 + 1/\gamma)$	$var(X) = \beta^{2/\gamma}\{\Gamma(1 + 2/\gamma)$ $- [\Gamma(1 + 1/\gamma)]^2\}$	fpm ada hanya untuk $\gamma \ge 1$. Momen ke-n adalah $E(X^n) = \beta^{n/\gamma}(1 + n/\gamma)$

Ringkasan Beberapa Conjugate Prior

$f(x\|\theta)$	$\pi(\theta)$	$\pi(\theta\|x)$
Normal $\mathcal{N}(\theta, \sigma^2)$	Normal $\mathcal{N}(\mu, \tau^2)$	$\mathcal{N}(\varrho(\sigma^2\mu + \tau^2 x), \varrho\sigma^2\tau^2)$ $\varrho^{-1} = \sigma^2 + \tau^2$
Poisson $\mathcal{P}(\theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	$\mathcal{G}(\alpha + x, \beta + 1)$
Gamma $\mathcal{G}(\nu, \theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	$\mathcal{G}(\alpha + \nu, \beta + x)$
Binomial $\mathcal{B}(n, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	$\mathcal{B}e(\alpha + x, \beta + n - x)$
Negative Binomial $\mathcal{N}eg(m, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	$\mathcal{B}e(\alpha + m, \beta + x)$
Multinomial $\mathcal{M}_k(\theta_1, \ldots, \theta_k)$	Dirichlet $\mathcal{D}(\alpha_1, \ldots, \alpha_k)$	$\mathcal{D}(\alpha_1 + x_1, \ldots, \alpha_k + x_k)$
Normal $\mathcal{N}(\mu, 1/\theta)$	Gamma $\mathcal{G}a(\alpha, \beta)$	$\mathcal{G}(\alpha + 0.5, \beta + (\mu - x)^2/2)$

(1). Conjugate Prior: Beta-Binomial

(Binomial Bayes estimation) Let $X_1, \ldots, X_n$ be iid Bernoulli(p). Then $Y = \sum X_i$ is binomial(n, p). We assume the prior distribution on p is beta(α, β). The joint distribution of Y and p is

$$f(y,p) = \left[\binom{n}{y} p^y (1-p)^{n-y}\right]\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\right] \quad \left(\begin{matrix}\text{conditional} \times \text{marginal} \\ f(y|p) \times \pi(p)\end{matrix}\right)$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{y+\alpha-1}(1-p)^{n-y+\beta-1}.$$

The marginal pdf of Y is

$$f(y) = \int_0^1 f(y,p)\,dp = \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)},$$

posterior distribution, the distribution of p given y, is

$$f(p|y) = \frac{f(y,p)}{f(y)} = \frac{\Gamma(n+\alpha+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)} p^{y+\alpha-1}(1-p)^{n-y+\beta-1},$$

which is beta($y+\alpha, n-y+\beta$). (Remember that p is the variable and y is treated as fixed.) A natural estimate for p is the mean of the posterior distribution, which would give us as the Bayes estimator of p,

$$\hat{p}_B = \frac{y+\alpha}{\alpha+\beta+n}.$$

Catatan: $X \sim beta(\alpha, \beta) \to E(X) = \frac{\alpha}{\alpha+\beta}$

(2). Conjugate Prior: Poisson-Gamma

The number of diseased trees per acre can be modeled by a Poisson distribution with mean θ. Since θ changes from area to area, the forester believes that $\theta \sim \text{Exp}(\lambda)$. Thus,

$$p(\theta) = (1/\lambda)e^{-\theta/\lambda}, \quad \text{if } \theta > 0, \text{and 0 elsewhere}$$

The forester takes a random sample of size n from n different one-acre plots.

Note: $\text{Exp}(\lambda) = \text{Gamma}(1, 1/\lambda)$

$$X \sim \Gamma(\alpha, \beta) \equiv \text{Gamma}(\alpha, \beta)$$

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^\alpha}{\Gamma(\alpha)} \quad \text{for } x > 0 \quad \alpha, \beta > 0,$$

The number of diseased trees per acre can be modeled by a Poisson distribution with mean θ. Since θ changes from area to area, the forester believes that $\theta \sim \text{Exp}(\lambda)$. Thus,

Let $y = (y_1, \ldots y_n)$ be a sample from Poi(θ). Then the likelihood is

$$p(y|\theta) = \prod_{i=1}^{n} \frac{\theta^{y_i}e^{-\theta}}{y_i!} \propto \theta^{\sum y_i}e^{-n\theta}.$$

prior $\rightarrow$ $p(\theta) = (1/\lambda)e^{-\theta/\lambda}, \quad \text{if } \theta > 0, \text{and 0 elsewhere}$

The likelihood function is

$$p(y|\theta) = \prod_{i=1}^{n} \frac{\theta^{y_i}}{y_i!}e^{-\theta} = \frac{\theta^{\sum_{i=1}^n y_i}}{\prod y_i!}e^{-n\theta}.$$

Consequently, the posterior distribution is

$$p(\theta|y) = \frac{\theta^{\sum_{i=1}^n y_i}e^{-\theta(n+1/\lambda)}}{\int_0^\infty \theta^{\sum_{i=1}^n y_i}e^{-\theta(n+1/\lambda)}}.$$

We see that this is a Gamma-distribution with parameters $\alpha = \sum_{i=1}^n y_i + 1$ and $\beta = n + 1/\lambda$. Thus,

posterior $\rightarrow$ $p(\theta|y) = \frac{(n+1/\lambda)^{\sum_{i=1}^n y_i + 1}}{\Gamma(\sum_{i=1}^n y_i + 1)}\theta^{\sum_{i=1}^n y_i}e^{-\theta(n+1/\lambda)}.$