Metode Momen

Definisi

Misalkan $X_1, X_2, \dots, X_n$ adalah contoh acak dari populasi dengan fungsi peluang $f(x|\theta_1, \theta_2, \dots, \theta_k)$. Momen contoh ke k didefinisikan sebagai:

$$m_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$$

Penduga metode momen dari μ_k diperoleh dengan menyelesaikan persamaan:

$$\mu_k(\theta_1, \dots, \theta_k) = m_k$$

dengan $\mu_k(\theta_1, \dots, \theta_k) = E(X^k)$ adalah momen populasi ke k

Momen Populasi / Momen Contoh

$$M_1 = E(X) \;\&\; m_1 = \frac{1}{n}\sum x_i$$

$$M_2 = E(X^2) \;\&\; m_2 = \frac{1}{n}\sum x_i^2$$

$$\vdots$$

$$M_k = E(X^k) \;\&\; m_k = \frac{1}{n}\sum x_i^k$$

Integral Parsial

$$\int u\, dv = uv - \int v\, du$$

Sifat-Sifat Logaritma

$$a^c = b \Leftrightarrow {}^a\log b = c$$

1. ${}^a\log 1 = 0$
2. ${}^a\log a = 1$
3. ${}^a\log bc = {}^a\log b + {}^a\log c$
4. ${}^a\log \frac{b}{c} = {}^a\log b - {}^a\log c$
5. ${}^a\log b^m = m\, {}^a\log b$
6. ${}^a\log b^{\frac{1}{n}} = \frac{1}{n}\, {}^a\log b$
7. ${}^{a^n}\log b^m = \frac{m}{n}\, {}^a\log b$
8. ${}^a\log b = \frac{{}^p\log b}{{}^p\log a}$
9. ${}^a\log b = \frac{1}{{}^b\log a}$
10. $a^{{}^a\log b} = b$
11. ${}^a\log d = {}^a\log b \cdot {}^b\log c \cdot {}^c\log d$

Misalkan $X_1, X_2, \dots, X_n$ contoh acak dari peubah acak yang memiliki fungsi peluang sebagai berikut:

$$f(x|\theta) = \begin{cases} \theta^x(1-\theta)^{1-x} & \text{untuk } x = 0,1 \\ 0 & x \text{ lainnya} \end{cases}$$

Tentukan penduga momen bagi θ!

- Fungsi peluang tersebut adalah fungsi massa peluang dari sebaran Bernoulli
- Momen pertama sebaran Bernoulli: $\mu_1 = E(X) = \theta$
- Momen pertama contoh adalah $m_1 = \frac{1}{n}\sum X_i$

Penduga momen bagi θ:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\theta = \frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

Sehingga Penduga momen bagi θ adalah $\bar{X}$

Sebelumnya telah diperoleh penduga momen bagi θ adalah $\bar{X}$

Suatu penduga dikatakan tidak berbias jika $E(\hat{\theta}) = \theta$

$$E(\hat{\theta}) = E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$X_i \sim Bernoulli(\theta)$ maka $E(X_i) = \theta$ sehingga:

$$E(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}(n\theta) = \theta$$

Maka $\hat{\theta}_{MM} = \bar{X}$ adalah penduga yang tak bias terhadap θ

Metode Kemungkinan Maksimum

Ilustrasi 1:

Misal $Y \sim Poisson\ (\lambda)$. Carilah penduga maksimum likelihood bagi λ!

$$f(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!}, \quad y = 0,1,2, \dots$$

- Fungsi log-likelihood:

$$\mathcal{L}(\lambda|\underline{y}) = \prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!}$$

$$l(\lambda|\underline{y}) = \log \mathcal{L}(\lambda|\underline{y})$$

$$= -n\lambda + (\log \lambda)\sum_{i=1}^{n} y_i - \log\left(\prod_{i=1}^{n} y_i!\right)$$

- Turunan pertama fungsi log-likelihood

$$\frac{dl(\lambda|\underline{y})}{d\lambda} = -n + \frac{\sum_{i=1}^n y_i}{\lambda}$$

$$-n + \frac{\sum_{i=1}^n y_i}{\lambda} = 0$$

$$\hat{\lambda} = \frac{\sum_{i=1}^n y_i}{n} = \bar{y}$$

Memeriksa turunan kedua:

$$\frac{d^2 l(\lambda|\underline{y})}{d\lambda^2} = -\frac{\sum_{i=1}^n y_i}{\lambda^2}$$

$$\left.\frac{d^2 l(\lambda|\underline{y})}{d\lambda^2}\right|_{(\lambda=\bar{y})} = -\frac{\sum_{i=1}^n y_i}{\bar{y}^2}$$

Karena $\sum_{i=1}^n y_i > 0$ dan $\bar{y}^2 > 0$, maka $\frac{d^2 l(\lambda|\underline{y})}{d\lambda^2} < 0$ sehingga terbukti bahwa $\hat{\lambda} = \bar{y}$ memaksimumkan fungsi likelihood

Metode Bayes

Definisi

Misalkan $(X_1, X_2, \dots, X_n)$ adalah peubah acak dengan fungsi peluang bersama $f(x_1, x_2, \dots, x_n|\theta)$ untuk $-\infty < x < \infty$ dan $\theta \in \Omega$.

Parameter θ merupakan suatu peubah acak dengan fungsi peluang $\pi(\theta)$ atau disebut sebagai **sebaran awal** (*prior distribution*) dari θ.

Fungsi peluang bersama dari peubah acak $(X_1, X_2, \dots, X_n, \theta)$ adalah:

$$g(x_1, x_2, \dots, x_n, \theta) = f(x_1, x_2, \dots, x_n|\theta)\pi(\theta)$$

Fungsi peluang marjinal dari peubah acak $(X_1, X_2, \dots, X_n)$ selanjutnya adalah:

$$m(x_1, x_2, \dots, x_n) = \begin{cases} \int_{\theta\in\Omega} g(x_1, x_2, \dots, x_n, \theta)\, d\theta & \text{jika } \theta \text{ kontinu} \\ \sum_{\theta\in\Omega} g(x_1, x_2, \dots, x_n, \theta) & \text{jika } \theta \text{ diskret} \end{cases}$$

Metode Bayes

Fungsi peluang bersyarat dari peubah acak θ pada nilai $(X_1, X_2, \dots, X_n)$ tertentu adalah:

$$\pi(\theta|x_1, x_2, \dots, x_n) = \frac{g(x_1, x_2, \dots, x_n, \theta)}{m(x_1, x_2, \dots, x_n)} \quad \text{dengan } m(x_1, x_2, \dots, x_n) > 0$$

$\pi(\theta|x_1, x_2, \dots, x_n)$ disebut sebagai **sebaran posterior** dari peubah acak θ untuk $(X_1, X_2, \dots, X_n)$ tertentu.

Selanjutnya **Penduga Bayes** bagi θ adalah:

$$\hat{\theta} = E(\theta|x_1, x_2, \dots, x_n) == \begin{cases} \int_{\theta\in\Omega} \theta\, \pi(\theta|x_1, x_2, \dots, x_n)\, d\theta & \text{jika } \theta \text{ kontinu} \\ \sum_{\theta\in\Omega} \theta\, \pi(\theta|x_1, x_2, \dots, x_n) & \text{jika } \theta \text{ diskret} \end{cases}$$

Jika A saling bebas dengan B, maka

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

Dari kedua persamaan di atas diperoleh

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Maka dari definisi peluang bersyarat, diperoleh **rumus bayes** dapat ditulis

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

$X \sim Exp(\theta) \rightarrow \Sigma X \sim Gamma\ (n, \theta)$

$X \sim Pois(\theta) \rightarrow \Sigma X \sim Pois(n\theta)$

$X \sim Bernoulli(\theta) \rightarrow \Sigma X \sim Binom\ (n, \theta)$

$X \sim Gamma\ (\alpha, \beta) \rightarrow \Sigma X \sim Gamma\ (n\alpha, \beta)$

$X \sim Chisquare\ (v) \rightarrow \Sigma X \sim Chisquare\ (nv)$

$X \sim Normal\ (\mu, \sigma^2) \rightarrow \bar{X} = \frac{\Sigma X}{n} \sim Normal\ (\mu, \frac{\sigma^2}{n})$

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1. Cari:
   a) $f(\underline{x}|\theta) = \prod_{i=1}^n f(x_i|\theta)$
   b) $g(\underline{x}, \theta) = f(\underline{x}|\theta)\pi(\theta)$
   c) $m(\underline{x}) = \int_\theta g(\underline{x}, \theta)\, d\theta$ atau $\sum_\theta g(\underline{x}, \theta)$
   d) $\pi(\theta|\underline{x}) = \frac{g(\underline{x}, \theta)}{m(\underline{x})}$
2. Penduga bayes:

$$\hat{\theta} = E(\theta|\underline{x})$$

Misal $X_1, X_2, \dots, X_n$ adalah contoh acak dari populasi Bernoulli (θ), sedangkan $0 < \theta < 1$. Diketahui sebaran awal dari θ adalah Seragam $(0,1)$. Tentukan penduga Bayes bagi θ berdasarkan contoh acak $X_1, X_2, \dots, X_n$!

- $X_i \sim Bernoulli(\theta)$ maka $f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$, $x_i = 0,1$
- $\theta \sim Seragam\ (0,1)$ maka $\pi(\theta) = 1$, $0 < \theta < 1$
- Mencari fungsi peluang bersama $f(\underline{x}|\theta)$

$$f(\underline{x}|\theta) = \prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}$$

- Mencari fungsi peluang bersama $g(\underline{x}, \theta)$

$$g(\underline{x}, \theta) = f(\underline{x}|\theta)\pi(\theta) = \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}$$

- Mencari peluang marjinal

$$m(\underline{x}) = \int_0^1 g(\underline{x}, \theta)\, d\theta = \int_0^1 \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}\, d\theta$$

$$= B\left(\sum_{i=1}^{n} x_i + 1, n - \sum_{i=1}^{n} x_i + 1\right)$$

Catatan:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\Gamma(a+1) = a\Gamma(a)$$

Perhatikan bahwa bentuk integral di samping menyerupai fungsi peluang dari sebaran Beta

- Mencari fungsi peluang bersyarat $\pi(\theta|\underline{x})$

$$\pi(\theta|\underline{x}) = \frac{g(\underline{x}, \theta)}{m(\underline{x})} = \frac{\theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}}{B(\sum_{i=1}^n x_i + 1, n - \sum_{i=1}^n x_i + 1)} \rightarrow \theta \sim Beta\left(\sum_{i=1}^{n} x_i + 1, n - \sum_{i=1}^{n} x_i + 1\right)$$

- Mencari penduga bayes bagi θ

$$\hat{\theta} = E(\theta|\underline{x}) = \int_0^1 \theta\, \frac{\theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}}{B(\sum_{i=1}^n x_i + 1, n - \sum_{i=1}^n x_i + 1)}\, d\theta$$

$$= \frac{1}{B(\sum_{i=1}^n x_i + 1, n - \sum_{i=1}^n x_i + 1)}\int_0^1 \theta^{\sum_{i=1}^n x_i + 1}(1-\theta)^{n-\sum_{i=1}^n x_i}\, d\theta$$

$$= \frac{1}{B(\sum_{i=1}^n x_i + 1, n - \sum_{i=1}^n x_i + 1)} \times B\left(\sum_{i=1}^{n} x_i + 2, n - \sum_{i=1}^{n} x_i + 1\right)$$

$$= \frac{\Gamma(\sum_{i=1}^n x_i + 2)\Gamma(n - \sum_{i=1}^n x_i + 1)}{\Gamma(\sum_{i=1}^n x_i + 2 + n - \sum_{i=1}^n x_i + 1)} \times \frac{\Gamma(\sum_{i=1}^n x_i + 1 + n - \sum_{i=1}^n x_i + 1)}{\Gamma(\sum_{i=1}^n x_i + 1)\Gamma(n - \sum_{i=1}^n x_i + 1)} = \frac{\sum_{i=1}^n x_i + 1}{n+2}$$

Misalkan $X_1, X_2, \dots, X_n \sim iid\ Bernoulli(p)$. Selanjutnya didefinisikan peubah acak $Y = \sum_{i=1}^n X_i$. Jika diketahui p memiliki sebaran awal $Beta(\alpha, \beta)$, tentukan penduga Bayes bagi p berdasarkan Y

- $Y_1, Y_2, \dots, Y_n \sim iid\ Bernoulli(p)$ maka $Y = \sum_{i=1}^n X_i \sim Binom(n, p)$ sehingga

$$f(y|p) = \binom{n}{y}p^y(1-p)^{n-y} \quad y = 0,1,2, \dots, n$$

- $p \sim Beta(\alpha, \beta)$ maka $\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$, $0 < p < 1$
- Mencari fungsi peluang bersama $g(y, p)$

$$g(y,p) = f(y|p)\,\pi(p) = \binom{n}{y}p^y(1-p)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

- Mencari peluang marjinal

$$m(y) = \int_0^1 g(y,p)\, dp = \int_0^1 \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}\, dp$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{y+\alpha-1}(1-p)^{n-y+\beta-1}\, dp = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}$$

- Mencari fungsi peluang bersyarat $\pi(p|y)$

$$\pi(p|y) = \frac{g(y,p)}{m(y)} = \frac{\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}}{\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}}$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

$p \sim Beta(y+\alpha, n-y+\beta)$

- Mencari penduga bayes bagi p

$$\hat{p} = E(p|y) = \int_0^1 p\, \frac{\Gamma(\alpha+\beta+n)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}\, dp$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\int_0^1 p^{y+\alpha}(1-p)^{n-y+\beta-1}\, dp$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)} \times \frac{\Gamma(y+\alpha+1)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n+1)} = \frac{y+\alpha}{\alpha+\beta+n}$$

Karena $p \sim Beta(y+\alpha, n-y+\beta)$ maka

$$E(p) = \frac{\alpha}{\alpha+\beta} = \frac{y+\alpha}{(y+\alpha)+(n-y+\beta)} = \frac{y+\alpha}{\alpha+\beta+n}$$

Sehingga, $\hat{p} = E(p|y) = \frac{y+\alpha}{\alpha+\beta+n}$

...ggunaan Bayes' Box:

a posterior probability.

$$P(H|D) = \frac{P(H)P(D|H)}{P(D)}$$

D: The ball that was removed from the bag was black.

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A|B) \times P(B) + P(A|\tilde{B}) \times P(\tilde{B})}$$

Likelihood = Probability(Data | Hypothesis)

n bola terambil hitam

Hypotheses	Possible Data	Probability
H_1: BB	Black Ball	1
	White Ball	0
H_2: BW	Black Ball	0.5
	White Ball	0.5

Hypotheses	prior	likelihood	h = prior × likelihood	posterior
BB	0.5	1	0.5	0.667
BW	0.5	0.5	0.25	0.333
Totals:	1		0.75	1

Hypotesis	Description	Prior	Likelihood	prd	Posterior
H_1	P = W & N = C	0.4	1	0.4	0.662252
H_2	P = W & N = ~C	0.4	0.01	0.004	0.006622
H_3	P = B & N = C	0.1	1	0.1	0.165563
H_4	P = B & N = ~C	0.1	1	0.1	0.165563
			marginal likelihood :	0.604	total 1

Solusi Ilustrasi 1

1. Peluang bahwa siswa yang terpilih adalah perempuan (F).
   - Peluang memilih KP dan KT

$$P(KP) = P(KT) = \frac{1}{2}$$

   - Peluang perempuan terpilih pada KP dan KT

$$P(F|KP) = \frac{12}{30}$$

$$P(F|KT) = \frac{20}{35}$$

$$P(F) = P(F|KP)P(KP) + P(F|KT)P(KT) = \frac{12}{30} \times \frac{1}{2} + \frac{20}{35} \times \frac{1}{2} = \frac{17}{35}$$

2. $P(KP|F)$ menggunakan Aturan Bayes

$$P(KP|F) = \frac{P(F|KP)P(KP)}{P(F)}$$

$$P(KP|F) = \frac{\frac{12}{30} \times \frac{1}{2}}{\frac{17}{35}}$$

$$= \frac{12/60}{17/35} = \frac{7}{17}$$

Solusi Ilustrasi 3

Diketahui:

- X adalah banyaknya bola merah di dalam kotak
- Likelihood: kita mengamati dua kali pengambilan dari kotak (tanpa pengembalian), pengambilan pertama berwarna merah, hasil pengambilan kedua berwarna hijau

Ditanya: distribusi posterior dari X dengan mengisi tabel bayes box

- Berdasarkan table bayes box kita perlu mendefinisikan prior dan likelihood untuk menghitung posterior
- Misal Madalah pengambilan pertama berwarna merah
- Misal H adalah pengambilan kedua berwarna hijau
- Likelihood tersebut dapat dihitung dengan

$$P(M, H|X = x) = \left(\frac{x}{9}\right)\left(\frac{9-x}{8}\right), x = 0,1,2,3, \dots, 9$$

Agar mudah misal kita mendefinisikan prior yang sama untuk setiap $X = x$, sedemikian sehingga

$$P(X = x) = \frac{1}{9}, x = 0,1,2,3, \dots, 9$$

X	prior	likelihood	priorxlikelihood	posterior
0	1/9	0	0	0
1	1/9	1/9	1/81	1/15
2	1/9	7/36	7/324	7/60
3	1/9	1/4	1/36	3/20
4				1/6
5				1/6
6				3/20
7	1/9	7/36	7/324	7/60
8	1/9	1/9	1/81	1/15
9	1/9	0	0	0
Total			5/27	

We often write Bayes' theorem in its proportional form as

$$posterior \propto prior \times likelihood$$

- The chance of a certain medical test being positive is 90%, if a patient has disease D. Known 1% of the population have the disease, and the test records a false positive 5% of the time.
- If you receive a positive test, what is your probability of having D?

false positive = orang yang tidak sakit dinyatakan sakit

- We are told: $P(+|D) = 0.90$, $P(D) = 0.01$, $P(+|\bar{D}) = 0.05$

$P(\bar{D}) = 0.99$

- We want to know: $P(D|+)$

posterior likelihood

- Bayes' Theorem: $P(D|+) = \frac{P(+|D)\,P(D)}{P(+)} = \frac{P(+|D)\,P(D)}{[P(+|D)\,P(D)] + [P(+|\bar{D})\,P(\bar{D})]}$
- Substituting in the data: $P(D|+) = \frac{0.9 \times 0.01}{(0.9 \times 0.01) + (0.05 \times 0.99)} = 0.15$

$Y|\pi \sim Binom\ (n = 4, \pi)$

$\pi = 0.4, 0.5, 0.6$

$y = 3$

$$P(Y=3|\pi = 0.4) = \binom{4}{3}(0.4)^3(0.6)^1$$

$$P(Y=3|\pi = \pi_i) = \binom{4}{3}(\pi_i)^3(1-\pi_i)^1$$

π	$p(\pi_i)$ prior	$p(Y=3\|\pi_i)$ likelihood	prior × likelihood	$p(\pi_i\|y=3)$ posterior
0.4	1/3	0.1536	0.0512	0.205
0.5	1/3	0.2500	0.0833	0.334
0.6	1/3	0.3456	0.1152	0.461
Marginal $P(Y=3)$			0.2497	1

$Y|\mu \sim Poisson\ (\mu)$

$\mu = 1, 1.5, 2, 2.5$

1.5 & 2 → 2x value 1, 2.5

$y = 2$

$$f(y|\mu) = \frac{\mu^y e^{-\mu}}{y!}$$

$a + b + b + a = 1$ $\quad b = 2a$

$2a + 2b = 1$ $\quad b = 2\left(\frac{1}{6}\right) = \frac{1}{3}$

$2a + 4a = 1$

$6a = 1$

$a = \frac{1}{6}$

μ	prior	likelihood	prior × likelihood	posterior
1.0	$\frac{1}{6}$	$\frac{1.0^2 e^{-1.0}}{2!} = .1839$	.0307	$\frac{.0307}{.2473}$ = .124
1.5	$\frac{1}{3}$	$\frac{1.5^2 e^{-1.5}}{2!} = .2510$	.0837	$\frac{.0837}{.2473}$ = .338
2.0	$\frac{1}{3}$	$\frac{2.0^2 e^{-2.0}}{2!} = .2707$	.0902	$\frac{.0902}{.2473}$ = .365
2.5	$\frac{1}{6}$	$\frac{2.5^2 e^{-2.5}}{2!} = .2565$	.0428	$\frac{.0428}{.2473}$ = .173
marginal $P(Y=2)$			.2473	1.000

Soal 1 pak kusman

sensitivity = .95 → $p(T|D) = 0.95$

specificity = .90 → $p(\tilde{T}|D^c) = 0.90$ → $p(T|D^c) = 1 - 0.90 = 0.10$

false positive rate → $p(T|D^c) = 0.10$

1% disease → $p(D) = 0.01$ → $p(D^c) = 1 - 0.01 = 0.99$

$$p(D|T) = \frac{p(T|D)\,p(D)}{p(T)} = \frac{0.95(0.01)}{0.1085} = 0.087$$

$$p(T) = p(T|D)p(D) + p(T|D^c)\,p(D^c)$$

$$= 0.95\,(0.01) + 0.10\,(0.99)$$

$$= 0.1085$$

Soal 2 pak kusman

sensitivity = 0.90 → $p(A|B) = 0.90$ $\quad$ B → cancer

spesifisity = 0.95 → $p(A^c|B^c) = 0.95$ $\quad$ A → test result

$p(B) = 0.001$

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)} = \frac{0.90\,(0.001)}{0.05085} = 0.01769$$

tak efektif karena peluangnya sangat kecil.

$$p(A) = p(A|B)\,p(B) + p(A|B^c)\,p(B^c)$$

$$= 0.90(0.001) + (1 - 0.95)(1 - 0.001)$$

	Disorder	No Disorder
Positive Test Result	True Positive (TP)	False Positive (FP)
Negative Test Result	False Negative (FN)	True Negative (TN)

Sensitivity = TP/(TP+FN)

Specificity = TN/(TN+FP)

PPV = TP/(TP+FP)

NPV = TN/(FN+TN)

soal 4 pak kusman

$n=10$

Sukses / gagal $\to$ binom

$p(\pi)=\frac{1}{4}$, dgn $\pi = 0.20, 0.40, 0.60, 0.80$

$y_i = 7$

$Y \sim Binom(n=10, \pi_i)$

$\pi_i = 0.20, 0.40, 0.60, 0.80$

$p(Y=7|\pi_i) = \underbrace{\binom{10}{7}}_{120} (\pi_i)^7 (1-\pi_i)^3$

π	$p(\pi_i)$ prior	$p(Y=7\mid\pi_i)$ likelihood	prior x likelihood	$p(\pi_i\mid Y=7)$ posterior
0.20	1/4	0.000786432	0.000196608	0.00171123
0.40	1/4	0.042467328	0.010616832	0.09240642
0.60	1/4	0.214990848	0.0537477712	0.467807149
0.80	1/4	0.201326592	0.050331648	0.4380748663
marginal $P(Y=7)$			0.1148928	1

soal 14

$Y|n \sim Binomial(n, \theta)$

$n \sim Poisson(5)$

$Y \in \{0, 5, 10\}$ $\theta \in \{0.2, 0.5\}$

ditanya: $n|y = ?$

prior: $f(n) = \frac{e^{-5}5^n}{n!}$, $n = 0, 1, \dots, 5$

$Y=0$ $\theta=0.2$ $\binom{n}{0}(0.2)^0(0.8)^n$

$Y \sim Binom(n, \theta \mid p(y=3))$

$Y \sim Binom(n=20, \theta=0.2 \mid P(Y=0))$

n	prior	likelihood	prior x likelihood	posterior
0	0.006738	1	0.006737947	0.00886
1	0.03369	0.8		
2	0.084224	0.64		
3	0.140374	0.512		
4	0.175467	0.4096		
5	0.175467	0.32768		
			0.65112836	

Soal 15

$n=10$

$Y \sim Binom(n=10, \theta) \to y=2$

$\theta \sim Uniform(0,1) \to \frac{1}{b-a} = \frac{1}{1-0} = 1$

prior	likelihood	prior x likelihood	posterior
1	$C^{10}_2 (1)^2(0)^8 = 0$	0	0
$p(\theta)$	$p(y\mid\theta)$		$p(\theta\mid y=2)$

Sebaran posterior bagi θ: $\pi(\theta|\boldsymbol{x})$

$$\pi(\theta|\boldsymbol{x}) = \frac{f(x,\theta)}{m(x)} = \frac{\{f(x|\theta)\}\{\pi(\theta)\}}{\int_{-\infty}^{\infty} f(x,\theta)d\theta} = \frac{\{f(x|\theta)\}\{\pi(\theta)\}}{\int_{-\infty}^{\infty}\{f(x|\theta)\}\{\pi(\theta)\}d\theta}$$

Inferensi didasarkan pada sebaran poterior $\pi(\theta|\boldsymbol{x})$ misalnya penduga Bayes bagi θ:

$$\hat{\theta}_B = E(\theta|\boldsymbol{x}) = \int_{-\infty}^{\infty} \theta\{\pi(\theta|\boldsymbol{x})\}\, d\theta$$

$$p(\theta|\mathbf{Y}) = \frac{f(\mathbf{Y}|\theta)\pi(\theta)}{\int f(\mathbf{Y}|\theta)\pi(\theta)d\theta} \propto f(\mathbf{Y}|\theta)\pi(\theta).$$

Contoh kasus 1

$\theta \in [0,1]$

$Y \sim Binom(n, \theta)$

$\hat{\theta} = Y/n$

$\theta \sim Unif(0.05, 0.15)$

posterior:

$$p(\theta|y) = \frac{p(\theta)\, p(y|\theta)}{\int p(\theta)\, p(y|\theta)\, d\theta} = \frac{\cancel{10}\binom{n}{y}\theta^y(1-\theta)^{n-y}}{\int \cancel{10}\binom{n}{y}\theta^y(1-\theta)^{n-y}d\theta} = \frac{\theta^y(1-\theta)^{n-y}}{\int \theta^y(1-\theta)^{n-y}d\theta}$$

prior:

$$p(\theta) = \begin{cases} \frac{1}{0.15-0.05} = \frac{1}{0.1} = 10 & , 0.05 \le \theta \le 0.15 \\ 0 & , \text{selainnya} \end{cases}$$

likelihood:

$p(y|\theta) = \binom{n}{y}(\theta)^y(1-\theta)^{n-y}$

$$p(\theta|y) = \begin{cases} \frac{\theta^y(1-\theta)^{n-y}}{\int_{0.05}^{0.15}\theta^y(1-\theta)^{n-y}d\theta} & , 0.05 \le \theta \le 0.15 \\ 0 & , \text{selainnya} \end{cases}$$

Contoh kasus 2 (Beta - Binomial)

$X_1, \dots, X_n \overset{iid}{\sim} Bernoulli(p)$

$Y = \sum X_i \sim Binomial(n, p)$

prior $\sim Beta(\alpha, \beta)$

$$f(y,p) = f(y|p)\pi(p) = \left[\binom{n}{y}p^y(1-p)^{n-y}\right]\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}\right]$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

$$f(y) = \int_0^1 f(y,p)\,dp = \int_0^1 \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}dp$$

$\alpha - 1 = y+\alpha-1$ $\quad \beta-1 = n-y+\beta-1$

$$B(\alpha,\beta) = \int_0^1 p^{\alpha-1}(1-p)^{\beta-1}dp \qquad 1 = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}dp$$

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$f(y) = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}$$

posterior:

$$f(p|y) = \frac{f(y,p)}{f(y)}$$

$$= \frac{\cancel{\binom{n}{y}}\cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}}{\cancel{\binom{n}{y}}\cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}}$$

$$= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

posterior $f(p|y) \sim Beta(y+\alpha, n-y+\beta)$

$$\hat{p}_{Bayes} = E(p|y) = \int p\underbrace{f(p|y)}_{post.}dp = \frac{y+\alpha}{\alpha+\beta+n}$$

$X \sim Beta(\alpha,\beta)$ $\quad E(X) = \frac{\alpha}{\alpha+\beta}$

a) $E(a) = a$ and $Var(a) = 0$

b) $E(bX) = b\mu_X$ and $Var(bX) = b^2\sigma^2_X$

c) Misalkan $Y = a + bX$
- Mean peubah acak Y adalah: $\mu_Y = E(a+bX) = a + b\mu_X$
- ragam peubah acak Y adalah: $\sigma^2_Y = Var(a+bX) = b^2\sigma^2_X$

Ringkasan Beberapa Conjugate Prior

$f(x\mid\theta)$	$\pi(\theta)$	$\pi(\theta\mid x)$
Normal $\mathcal{N}(\theta,\sigma^2)$	Normal $\mathcal{N}(\mu,\tau^2)$	$\mathcal{N}(\varrho(\sigma^2\mu+\tau^2x), \varrho\sigma^2\tau^2)$ $\varrho^{-1} = \sigma^2+\tau^2$
Poisson $\mathcal{P}(\theta)$	Gamma $\mathcal{G}(\alpha,\beta)$	$\mathcal{G}(\alpha+x, \beta+1)$
Gamma $\mathcal{G}(\nu,\theta)$	Gamma $\mathcal{G}(\alpha,\beta)$	$\mathcal{G}(\alpha+\nu, \beta+x)$
Binomial $\mathcal{B}(n,\theta)$	Beta $\mathcal{B}e(\alpha,\beta)$	$\mathcal{B}e(\alpha+x, \beta+n-x)$
Negative Binomial $\mathcal{N}eg(m,\theta)$	Beta $\mathcal{B}e(\alpha,\beta)$	$\mathcal{B}e(\alpha+m, \beta+x)$
Multinomial $\mathcal{M}_k(\theta_1,\ldots,\theta_k)$	Dirichlet $\mathcal{D}(\alpha_1,\ldots,\alpha_k)$	$\mathcal{D}(\alpha_1+x_1,\ldots,\alpha_k+x_k)$
Normal $\mathcal{N}(\mu,1/\theta)$	Gamma $\mathcal{G}a(\alpha,\beta)$	$\mathcal{G}(\alpha+0.5, \beta+(\mu-x)^2/2)$

Poisson - Gamma

$Y \sim Poisson(\theta)$

$\theta \sim Exp(\lambda)$

$Exp(\lambda) = Gamma(1, \frac{1}{\lambda})$

$X \sim \Gamma(\alpha,\beta) \equiv Gamma(\alpha,\beta)$

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^\alpha}{\Gamma(\alpha)}, \quad x>0 \quad \alpha,\beta>0$$

prior:

$$p(\theta) = \begin{cases} \frac{1}{\lambda}e^{-\theta/\lambda} & , \theta>0 \\ 0 & , \text{selainnya} \end{cases}$$

likelihood:

Misal $y = (y_1, \dots, y_n)$

$$p(y|\theta) = \prod_{i=1}^{n}\frac{\theta^{y_i}e^{-\theta}}{y_i!} = \frac{\theta^{\sum y_i}}{\prod y_i!}e^{-n\theta} \propto \theta^{\sum y_i}e^{-n\theta}$$

$f(y,\theta) = p(y|\theta)p(\theta)$

$$-\frac{n\theta\lambda}{\lambda}-\frac{\theta}{\lambda} = -\theta\left(\frac{n\lambda}{\lambda}+\frac{1}{\lambda}\right) = -\theta\left(n+\frac{1}{\lambda}\right)$$

$$= \left[\frac{\theta^{\sum y_i}e^{-n\theta}}{\prod y_i!}\right]\left[\frac{1}{\lambda}e^{-\theta/\lambda}\right] = \frac{\theta^{\sum y_i}}{\prod y_i!}e^{-n\theta-\frac{\theta}{\lambda}} = \frac{\theta^{\sum y_i}}{\prod y_i!}e^{-\theta(n+1/\lambda)}$$

$$f(y) = \int_0^\infty f(y,\theta)\,d\theta = \int_0^\infty \frac{\theta^{\sum y_i}}{\prod y_i!}e^{-\theta(n+\frac{1}{\lambda})}d\theta$$

posterior:

$$p(\theta|y) = \frac{f(y,\theta)}{f(y)} = \frac{\theta^{\sum y_i}e^{-\theta(n+\frac{1}{\lambda})}}{\int_0^\infty \theta^{\sum y_i}e^{-\theta(n+\frac{1}{\lambda})}d\theta} = \frac{\theta^{\sum y_i}e^{-\theta(n+\frac{1}{\lambda})}}{\frac{\Gamma(\sum y_i+1)}{(n+\frac{1}{\lambda})^{\sum y_i+1}}}$$

$$\int_0^\infty \frac{\theta^{\alpha-1}e^{-\beta\theta}\beta^\alpha}{\Gamma(\alpha)}d\theta = 1$$

$$\int_0^\infty \theta^{\alpha-1}e^{-\beta\theta}d\theta = \frac{\Gamma(\alpha)}{\beta^\alpha} \qquad = \frac{(n+\frac{1}{\lambda})^{\sum y_i+1}}{\Gamma(\sum y_i+1)}\theta^{\sum y_i}e^{-\theta(n+1/\lambda)}$$

$\alpha - 1 = \sum y_i$ $\quad -\beta\theta = -(n+\frac{1}{\lambda})\theta$

$\alpha = \sum y_i + 1$ $\quad \beta = n+\frac{1}{\lambda}$

posterior $p(\theta|y) \sim Gamma(\sum y_i+1, n+\frac{1}{\lambda})$

Laplace's Prior

- *Laplace's prior* dikenal juga sebagai sebaran eksponensial ganda (*double exponential distibution*), yaitu:

$$\pi(\theta) = \frac{1}{2\sigma}exp\left(-\frac{|\theta-\mu|}{\sigma}\right); \quad -\infty<\mu<\infty; \quad \sigma>0$$

Jeffreys' Prior

The univariate JP for θ is

$$\pi(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ is the expected Fisher information defined as

$$I(\theta) = -\mathrm{E}\left(\frac{d^2\log f(\mathbf{Y}|\theta)}{d\theta^2}\right)$$

and the expectation is with respect to $\mathbf{Y}|\theta$.

Jeffreys' Prior (Multivariate)

JP can also be applied in multivariate problems. The multivariate JP for $\theta = (\theta_1, \ldots, \theta_p)$ is

$$\pi(\theta) \propto \sqrt{|\mathbf{I}(\theta)|}$$

where $\mathbf{I}(\theta)$ is the $p \times p$ expected Fisher information matrix with (i,j) element

$$-\mathrm{E}\left(\frac{\partial^2\log f(\mathbf{Y}|\theta)}{\partial\theta_i\partial\theta_j}\right).$$

$Y \sim Binom(n, \theta)$

$f(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

$$W = \frac{d^2\log f(y|\theta)}{d\theta^2} = \frac{d^2}{d\theta^2}\left[\log\binom{n}{y} + y\log\theta + (n-y)\log(1-\theta)\right]$$

$$= \frac{d}{d\theta}\left[\frac{y}{\theta} - \frac{n-y}{1-\theta}\right]$$

$$= -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}$$

$$I(\theta) = -E(W) = \frac{E(y)}{\theta^2} + \frac{n-E(y)}{(1-\theta)^2} \qquad E(X) \text{ dari Binom} = n\theta$$

$$= \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n(1-\theta)}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n(1-\theta)+n\theta}{\theta(1-\theta)}$$

$$= \frac{n(1-\cancel{\theta}+\cancel{\theta})}{\theta(1-\theta)} = \frac{n}{\theta(1-\theta)}$$

JP $\pi(\theta) \propto \sqrt{I(\theta)}$

$$\propto \sqrt{\frac{n}{\theta(1-\theta)}} \propto \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} \propto \theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1} \propto Beta(\tfrac{1}{2}, \tfrac{1}{2})$$

$\theta \sim Beta(\frac{1}{2}, \frac{1}{2})$

Rataan atau nilai harapan dari X didefinisikan sebagai:

$\mu_X = E(X) = \sum_{\forall x} x\,P(X=x)$ jika X p.a. diskret

dan

$\mu_X = E(X) = \int_{-\infty}^{\infty} x\,f(x)dx$ jika X p.a. kontinu

Ragam peubah acak diskret X

$\sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 P(X)$

Ragam peubah acak kontinu X

$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

- Alternatif formula dalam perhitungan nilai ragam suatu sebaran

$\sigma^2 = E(X^2) - (\mu)^2$

- Simpangan baku peubah acak diskret/kontinu

$\sigma = \sqrt{\sigma^2}$

Sophie, the editor of the student newspaper, is going to conduct a survey of students to determine the level of support for the current president of the students' association. She needs to determine her prior distribution for π, the proportion of students who support the president. She decides her prior mean is .5, and her prior standard deviation is .15.

(a) Determine the $beta(a, b)$ prior that matches her prior belief.

(b) What is the equivalent sample size of her prior?

(c) Out of the 68 students that she polls, $y = 21$ support the current president. Determine her posterior distribution.

Ilustrasi 1

prior mean 0.5 $\rightarrow \mu = \frac{1}{2}$

standard deviation 0.15 $\rightarrow s = \frac{15}{100} = \frac{3}{20}$

$\pi \sim beta(a, b)$

(a) $\mu = E(\pi) = \frac{\alpha}{\alpha+\beta}$

$s = \sqrt{Var(\pi)}$

$\frac{3}{20} = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$

$var(\pi) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$\frac{9}{400} = \frac{\alpha^2}{4\alpha^2(2\alpha+1)}$

$\frac{\alpha}{\alpha+\beta} = \frac{1}{2}$

$\frac{9}{400} = \frac{1}{8\alpha+4}$

$2\alpha = \alpha+\beta$

$\alpha = \beta$

$72\alpha + 36 = 400$

$72\alpha = 364$

$\alpha = \frac{364}{72} = \frac{91}{18}$

$\alpha = \frac{91}{18}$, $\beta = \frac{91}{18}$

(b) $\pi \sim Beta(\frac{91}{18}, \frac{91}{18})$

$n = a+b+1$

$= \frac{91}{18} + \frac{91}{18} + 1$

$= \frac{182+18}{18} = \frac{200}{18} = 11{,}11 \approx 11$

(c) $n = 68$ student

$y = 21$ support the current president $\rightarrow$ peubah acak x,y

n 1,0,1,0 } likelihood: Binom(n, π)

Binom $\rightarrow$ likelihood

peubah acaknya, hal yg mau diamati & didapatkan datanya

prior: Beta$(\frac{91}{18}, \frac{91}{18})$

posterior $\propto$ prior $\times$ likelihood

$f(x|\theta)$: Binom(n, θ)

$\pi(\theta)$: Beta(α, β)

conjugate prior sbg:

$\pi(\theta|x)$: Beta$(\alpha+x, \beta+n-x)$ $\rightarrow$ size

nilai p.a.

posterior $\sim$ Beta$(\frac{91}{18}+21, \frac{91}{18}+68-21)$

$\sim$ Beta$(21\frac{91}{18}, 47\frac{91}{18})$

Ilustrasi 2

$Y|\lambda \sim$ Poisson(λ)

(a) Derive and plot the Jeffreys' prior for λ

(b) Is this prior proper?

(c) Derive the posterior and give conditions on Y to ensure it is proper

$I(\lambda) = -E\left[\frac{\partial^2 \log f(y|\lambda)}{\partial\lambda^2}\right]$

$I(\lambda) = -E\left[-\frac{\Sigma y_i}{\lambda^2}\right]$

$f(y) = \frac{e^{-\lambda}\lambda^y}{y!}$

$f(y|\lambda) = \Pi f(y_i) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{\Pi y_i!}$

$= \frac{\Sigma E(y_i)}{\lambda^2}$ $\leftarrow$ $E(y) = \lambda$

$\log f(y|\lambda) = -n\lambda + \Sigma y_i \log\lambda - \Sigma \log y_i!$

$= \frac{\Sigma\lambda}{\lambda^2} = \frac{n\cancel{\lambda}}{\lambda^{\cancel{2}}}$

$\frac{d}{d\lambda} \log f(y|\lambda) = -n + \frac{\Sigma y_i}{\lambda}$

$= \frac{n}{\lambda}$

$\frac{d^2}{d\lambda^2} \log f(y|\lambda) = -\frac{\Sigma y_i}{\lambda^2}$

proportional

JP: $g(\lambda) \propto \sqrt{I(\lambda)}$

$\propto \sqrt{\frac{n}{\lambda}}$ $\rightarrow$ n diabaikan misalkan:

$\sqrt{\frac{2}{\lambda^2(1-\lambda)}}$ $\rightarrow$ hapus konstanta tdk menempel parameter

$\propto \sqrt{\frac{1}{\lambda}}$

$\propto \sqrt{\frac{1}{\lambda^2(1-\lambda)}}$ ...

$\propto \lambda^{-\frac{1}{2}}$

$\propto \lambda^{\frac{1}{2}-1}$

Beta: $x^{\alpha-1}(1-x)^{\beta-1}$

Gamma: $x^{\alpha-1}e^{-x\beta}$

(b) prior $\sim$ Beta$(\frac{1}{2}, 1)$ $\quad \lambda^{\frac{1}{2}-1}(1-\lambda)^{1-1}$ $\quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ $\quad$ PPr: $\frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(1)}$ $\quad \frac{\Gamma(1)}{\sqrt{\pi}}$

$\lambda^{-\frac{1}{2}}$, $0 < \lambda < 1$

likelihood poisson $y = 0, 1, 2, \ldots$ $0 < \lambda < \infty$

fkp/fmp $\rightarrow$ integral / sum : CDF $(F(\lambda))$

jmlh terbatas max 1

$\int_0^\infty \lambda^{-\frac{1}{2}} d\lambda = 2\lambda^{\frac{1}{2}}\Big]_0^\infty = \infty$

Is the prior proper? $\rightarrow$ tdk proper sebab nilai

Fnya harusnya terbatas max 1.

(c) posterior $\propto$ prior $\times$ likelihood

$\frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{\Pi y_i!} \times \lambda^{-\frac{1}{2}}$

$\frac{e^{-n\lambda}\lambda^{\Sigma y_i - \frac{1}{2}}}{\Pi y_i!}$ $\quad \Sigma y_i - \frac{1}{2} = \alpha - 1$ $\quad -n\lambda = -\beta\lambda$

$\alpha = \Sigma y_i - \frac{1}{2} + 1$

$\propto e^{-n\lambda}\lambda^{\Sigma y_i - \frac{1}{2}} \sim$ Gamma$(\Sigma y_i + \frac{1}{2}, n)$

$\int_0^\infty e^{-n\lambda}\lambda^{\Sigma y_i - \frac{1}{2}} d\lambda$

harus finite

$\lambda^{\Sigma y_i - \frac{1}{2}} = 1$

$\Sigma y_i - \frac{1}{2} = 0 \nearrow \lambda^0$

$\Sigma y_i = \frac{1}{2}$

Fungsi Gamma

- Definisi: $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$
- $\Gamma(\alpha+1) = \int_0^\infty y^{\alpha}e^{-y}\,dy = \alpha\Gamma(\alpha)$
- jika α bilangan bulat, maka $\Gamma(\alpha) = (\alpha-1)!$
- $\int_0^\infty y^{\alpha-1}e^{-\frac{y}{\beta}}\,dy = \beta^\alpha\Gamma(\alpha)$

Ilustrasi 3

prior Uniform$(0,1)$

$\hookrightarrow g(\theta) = 1$ $\quad \theta^{1-1}e^{\theta(0)}$

Gamma$(1, 0)$

Ilustrasi 4

Suppose we start with a Jeffreys' prior for the Poisson parameter μ

$g(\mu) = \mu^{-\frac{1}{2}}$

(a) What gamma (r, v) prior will give this form?

(b) Find the posterior distribution using the macro PoisGamP in Minitab or the function poisgamp in R.

(c) Find the posterior mean and median.

$g(\mu) = \mu^{-\frac{1}{2}} \rightsquigarrow$ Gamma$(\frac{1}{2}, 0)$

Gamma : $x^{\alpha-1}e^{-\frac{x}{\beta}}$

$Exp(\lambda)$: $\lambda e^{-x\lambda} \rightarrow x^{\alpha-1}e^{-x\beta}$

$Exp(\beta)$: $\frac{1}{\beta}e^{-x/\beta}$

$\lambda^{-\frac{1}{2}} \rightarrow \lambda^{\frac{1}{2}-1}e^{-\lambda(0)}$

On the other hand, the expectation of the loss function over the sampling distribution of y is called *risk function*:

$$R_{\hat\theta}(\theta) = \mathsf{E}[L(\theta, \hat\theta)|\theta] = \int L(\theta, \hat\theta)p(y|\theta)dy.$$

Further, the expectation of the risk function over the prior distribution of θ,

$$\mathsf{E}[R_{\hat\theta}(\theta)] = \int R_{\hat\theta}(\theta)p(\theta)d\theta,$$

is called *Bayes risk*.

in summary $\quad \theta \sim$ posterior $\quad p(\theta|x)$

$\hat\theta_B = E(\theta|x) = \int \theta\, p(\theta|x)\, d\theta$

Credible Intervals

$$P(a \le \theta \le b|x) = \alpha$$

Pengujian Hipotesis

$$P(H_0|Y=y) = \frac{f_Y(y|H_0)P(H_0)}{f_Y(y)},$$

$$P(H_1|Y=y) = \frac{f_Y(y|H_1)P(H_1)}{f_Y(y)}.$$

To be more specific, according to the MAP test, we choose H_0 if and only if

$$P(H_0|Y=y) > P(H_1|Y=y).$$ $\rightarrow$ Terima H_0

In other words, we choose H_0 if and only if

$$f_Y(y|H_0)P(H_0) > f_Y(y|H_1)P(H_1).$$

Langkah Pengujian Hipotesis (Bayesian)

- $\boldsymbol{X} \mid \theta \sim f(\boldsymbol{x} \mid \theta)$.
- To test: $H_0 : \theta \in \Theta_0 \quad vs \quad H_1 : \theta \in \Theta_1$.
- Prior distribution:
  - Prior probabilities $\Pr(H_0)$ and $\Pr(H_1)$ of the hypotheses.
  - Proper prior densities $\pi_0(\theta)$ and $\pi_1(\theta)$ on Θ_0 and Θ_1.
- Posterior distribution $\pi(\theta \mid \boldsymbol{x})$:
$$\Pr(H_0 \mid \boldsymbol{x}) = 1 - \Pr(H_1 \mid \boldsymbol{x})$$
- Based on the posterior odds. By default, H_0 accepted if $\Pr(H_0 \mid \boldsymbol{x}) > \Pr(H_1 \mid \boldsymbol{x})$ $\rightarrow$ posterior dari H_1 benar

$\hookrightarrow$ posterior dari H_0 benar

Suppose that the random variable X is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + W,$$

where $W \sim N(0, \sigma^2)$ is independent of X. Suppose that $X = 1$ with probability p, and $X = -1$ with probability $1-p$. The goal is to decide between $X = 1$ and $X = -1$ by observing the random variable Y. Find the MAP test for this problem.

Studi Kasus 2

H_0: $x = 1$

H_1: $x = -1$

$Y = X + W \quad W \sim N(0, \sigma^2)$

H_0 benar: $Y = 1 + W$, $Y|H_0 \sim N(1, \sigma^2)$

$f_Y(y|H_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2\sigma^2}}$

H_1 benar: $Y = -1 + W$, $Y|H_1 \sim N(-1, \sigma^2)$

$f_Y(y|H_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2\sigma^2}}$

H_0 diterima jika dan hanya jika:

$f_Y(y|H_0)P(H_0) > f_Y(y|H_1)P(H_1)$

$\cancel{\frac{1}{\sigma\sqrt{2\pi}}} e^{-\frac{(y-1)^2}{2\sigma^2}} P(H_0) > \cancel{\frac{1}{\sigma\sqrt{2\pi}}} e^{-\frac{(y+1)^2}{2\sigma^2}} P(H_1)$

$e^{-\frac{(y-1)^2}{2\sigma^2}} (p) > e^{-\frac{(y+1)^2}{2\sigma^2}} (1-p)$

$e^{-\frac{(y-1)^2}{2\sigma^2} + \frac{(y+1)^2}{2\sigma^2}} > \frac{1-p}{p}$ $\quad -(y^2 - 2y + 1) + (y^2 + 2y + 1)$ $\quad \cancel{-y^2} + 2y \cancel{-1} \cancel{+y^2} + 2y \cancel{+1}$ $\quad 4y$

$e^{\frac{2y}{\sigma^2}} > \frac{1-p}{p}$

$\exp\left(\frac{2y}{\sigma^2}\right) > \frac{1-p}{p}$

$\frac{2y}{\sigma^2} > \ln\left(\frac{1-p}{p}\right)$

$y > \frac{\sigma^2}{2}\ln\left(\frac{1-p}{p}\right)$ $\rightarrow$ terima H_0

Pengujian Hipotesis Dapat Menggunakan *Credible Interval*
(Khusus Uji Hipotesis Dua Arah)

$$H_0 : \pi = \pi_0 \qquad H_1 : \pi \neq \pi_0$$

Compute a $(1-\alpha) \times 100\%$ credible interval for π. If π_0 lies inside the credible interval, accept (do not reject) the null hypothesis $H_0 : \pi = \pi_0$; and if π_0 lies outside the credible interval, then reject the null hypothesis.

1

Suppose that the random variable X is transmitted over a communication channel. Assume that the received signal is given by

$$Y = 2X + W,$$

where $W \sim N(0, \sigma^2)$ is independent of X. Suppose that $X = 1$ with probability p, and $X = -1$ with probability $1-p$. The goal is to decide between $X = -1$ and $X = 1$ by observing the random variable Y. Find the MAP test for this problem.

2

In the study of water quality in New Zealand streams, documented in McBride et al. (2002), a high level of *Campylobacter* was defined as a level greater than 100 per 100 ml of stream water. $n = 116$ samples were taken from streams having a high environmental impact from birds. Out of these, $y = 11$ had a high *Campylobacter* level. Let π be the true probability that a sample of water from this type of stream has a high *Campylobacter* level.

(a) Find the frequentist estimator for π.

(b) Use a $beta(1, 10)$ prior for π. Calculate the posterior distribution $g(\pi|y)$.

(c) Find the posterior mean and variance. What is the Bayesian estimator for π?

(d) Find a 95% credible interval for π.

(e) Test the hypothesis

$$H_0 : \pi = .10 \quad \text{versus} \quad H_1 : \pi \neq .10$$

at the 5% level of significance.

3

The number of defects per 10 meters of cloth produced by a weaving machine has the *Poisson* distribution with mean μ. You examine 100 meters of cloth produced by the machine and observe 71 defects.

(a) Your prior belief about μ is that it has mean 6 and standard deviation 2. Find a $gamma(r, v)$ prior that matches your prior belief.

(b) Find the posterior distribution of μ given that you observed 71 defects in 100 meters of cloth.

(c) Calculate a 95% Bayesian credible interval for μ.

---

1. $Y = 2X + W \qquad W \sim N(0, \sigma^2)$

$X = 1$ prob : p

$X = -1$ prob : $1-p$

maximum a posteriori (MAP) test

$H_0 : X = 1 \qquad W \sim N(0, \sigma^2) \qquad Y \sim N(\mu, \sigma^2)$

$E(W) = 0 \quad Var(W) = \sigma^2$

$H_1 : X = -1$

H_0 benar : $E(Y|X=1) = E(2X+W|X) = E(2(1)+W) = E(2+W) = 2$

$Var(Y|X=1) = Var(2X+W) = Var(2+W) = Var(2) + Var(W) = \sigma^2$

$$f_Y(y|H_0) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right)$$

H_1 benar : $E(Y|X=-1) = E(2(-1)+W) = -2 + E(W) = -2 + 0 = -2$

$Var(Y|X=-1) = Var(2(-1)+W) = Var(W) = \sigma^2$

$$f_Y(y|H_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y+2)^2}{2\sigma^2}\right)$$

posterior $\propto$ likelihood $\times$ prior

H_0 diterima (gagal tolak H_0) jika dan hanya jika:

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right) p(H_0) > \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y+2)^2}{2\sigma^2}\right) p(H_1)$$

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right)(p) > \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y+2)^2}{2\sigma^2}\right)(1-p)$$

$\ln(ab) = \ln a + \ln b$

$\ln(\frac{a}{b}) = \ln a - \ln b$

$$\cancel{\ln \frac{1}{\sigma\sqrt{2\pi}}} - \frac{(y-2)^2}{2\sigma^2} + \ln p > \cancel{\ln \frac{1}{\sigma\sqrt{2\pi}}} - \frac{(y+2)^2}{2\sigma^2} + \ln(1-p)$$

$$-\frac{(\cancel{y^2} - 4y + \cancel{4})}{2\sigma^2} + \frac{\cancel{y^2} + 4y + \cancel{4}}{2\sigma^2} > \ln(1-p) - \ln p$$

$$\frac{8y}{2\sigma^2} > \ln\left(\frac{1-p}{p}\right)$$

$$\frac{4y}{\sigma^2} > \ln\left(\frac{1-p}{p}\right)$$

MAP

$$y > \frac{\sigma^2}{4} \ln\left(\frac{1-p}{p}\right)$$

---

1. Diketahui 7 bola dlm wadah dgn ukuran sama.

- merah & biru (jmlh bola tak diketahui)
- dilakukan percobaan pengambilan 3 bola diperoleh hasilnya biru semua
- Y menyatakan jmlh bola biru dlm wadah

a) prior → mengasumsikan peluang jmlh bola biru yg terambil dlm wadah sama yakni 1/8 untuk $Y = 0, 1, 2, \ldots, 7$

tak ditemukan informasi probabilitas awal terkait peluang jmlh bola biru yg terambil

b) likelihood : $\left(\frac{7-Y}{7}\right)^3$, catatan: bahwa pengambilan bola scr pengembalian

↳ probabilitas mengambil 3 bola biru dari 7 bola dlm wadah

c) Bayes' box

y	prior	likelihood	prior×likelihood	posterior
0	1/8	$(\frac{7-0}{7})^3 = 1$	1/8	
1	1/8	$(\frac{7-1}{7})^3 = (\frac{6}{7})^3$	$(1/8)(6/7)^3$	
2	1/8	$(\frac{7-2}{7})^3 = (\frac{5}{7})^3$	$(1/8)(5/7)^3$	
3	1/8	$(\frac{7-3}{7})^3 = (\frac{4}{7})^3$	$(1/8)(4/7)^3$	
4	1/8	$= (3/7)^3$	$(1/8)(3/7)^3$	
5	1/8	$= (2/7)^3$	$(1/8)(2/7)^3$	
6	1/8	$= (1/7)^3$	$(1/8)(1/7)^3$	
7	1/8	$(\frac{7-7}{7})^3 = 0$	0	
			Hitung Total	pastikan jmlh = 1

---

2) Diketahui

$X_1, X_2, \ldots, X_n \sim$ poisson $(\theta) \to f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}; x = 0, 1, \ldots$

$\theta \sim$ Uniform $(1,5) \to f(\theta) \sim \frac{1}{4}, 1 < \theta < 5$

a) Sebaran posterior bagi θ

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$f(\underset{\sim}{x}, \theta) = f(\underset{\sim}{x}|\theta) f(\theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \cdot \frac{1}{4} = \frac{1}{4} \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$m(\underset{\sim}{x}) = \int_1^5 f(\underset{\sim}{x}, \theta) = \int_1^5 \frac{1}{4} \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod x_i!} d\theta = \frac{1}{4} \cdot \frac{1}{\prod_{i=1}^{n} x_i!} \int_1^5 e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i} d\theta$$

Selesaikan $\int_1^5 e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i} d\theta$ dengan metode substitusi $u = \theta^{\sum x_i + 1} \to du = (\sum x_i + 1)\theta^{\sum x_i} d\theta$ gunakan $\theta = u^{\frac{1}{\sum x_i + 1}}$

$$\int_1^5 e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i} d\theta = \frac{1}{\sum x_i + 1} \int_1^5 e^{-n \cdot u^{\frac{1}{\sum x_i + 1}}} du$$

Selanjutnya selesaikan lagi $\int_1^5 e^{-n \cdot u^{\frac{1}{\sum x_i + 1}}} du$ dengan metode substitusi, misalkan

$v = n^{\sum x_i + 1} u \to dv = n^{\sum x_i + 1} du$

$$\int_1^5 e^{-n \cdot u^{\frac{1}{\sum x_i + 1}}} du = n^{-\sum x_i - 1} \int_1^5 e^{-v^{\frac{1}{\sum x_i + 1}}} dv$$

Selanjutnya selesaikan $\int_1 e^{-v^{\frac{1}{\sum x_i + 1}}} dv$ dengan menggunakan fungsi gamma tak lengkap

$$\int_1^5 e^{-v^{\frac{1}{\sum x_i + 1}}} = -\frac{\Gamma\left(\frac{1}{\frac{1}{\sum x_i + 1}}, v^{\frac{1}{\sum x_i + 1}}\right)}{\frac{1}{\sum x_i + 1}} = -(\sum x_i + 1)\Gamma(\sum x_i + 1, v^{\frac{1}{\sum x_i + 1}})$$

sehingga

$$\int_1^5 e^{-n \cdot u^{\frac{1}{\sum x_i + 1}}} du = -n^{-\sum x_i - 1}(\sum x_i + 1)\Gamma(\sum x_i + 1, v^{\frac{1}{\sum x_i + 1}})$$

Substitusikan kembali $v = n^{\sum x_i + 1} u$, maka diperoleh

$$\int_1^5 e^{-n \cdot u^{\frac{1}{\sum x_i + 1}}} du = -(\sum x_i + 1) n^{-\sum x_i - 1}\Gamma(\sum x_i + 1, n u^{\frac{1}{\sum x_i + 1}})$$

Sehingga

$$\int_1^5 e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i} d\theta = -n^{-\sum x_i - 1}\Gamma(\sum x_i + 1, n u^{\frac{1}{\sum x_i + 1}})$$

Substitusikan kembali $u = \theta^{\sum x_i + 1}$ maka diperoleh

$$\int_1^5 e^{-n\theta}\theta^{\sum x_i} d\theta = -n^{-\sum x_i - 1}\Gamma(\sum x_i + 1, n\theta) + C$$

Karena batasnya θ, 1 sampai 5 maka

$$\int_1^5 e^{-n\theta}\theta^{\sum x_i} d\theta = -n^{-\sum x_i - 1}\left(\Gamma(\sum x_i + 1, 5\theta) - \Gamma(\sum x_i + 1, n)\right)$$

Sehingga

$$m(\underset{\sim}{x}) = \frac{1}{4} \frac{1}{\prod x_i} \int_1^5 e^{-n\theta}\theta^{\sum x_i} d\theta = -\frac{1}{4\prod_{i=1}^{n} x_i} \cdot \frac{n}{\sum x_i + 1}\left(\Gamma(\sum x_i + 1, 5\theta) - \Gamma(\sum x_i + 1, n)\right)$$

atau

$$m(\underset{\sim}{x}) = \frac{1}{4\prod_{i=1}^{n} x_i} \cdot \frac{n}{\sum_{i=1}^{n} x_i + 1}\left(\Gamma(\sum x_i + 1, n) - \Gamma(\sum x_i + 1)\right)$$

$$f(\theta|\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}, \theta)}{m(\underset{\sim}{x})} = \frac{\frac{1}{4\prod x_i} e^{-n\theta}\theta^{\sum x_i}}{\frac{1}{4\prod x_i} \cdot \frac{n}{\sum x_i + 1}\left(\Gamma(\sum x_i + 1, n) - \Gamma(\sum x_i + 1, 5n)\right)}$$

$$= \frac{\sum_{i=1}^{n} x_i + 1}{n\left(\Gamma(\sum x_i + 1, n) - \Gamma(\sum x_i + 1, 5n)\right)} \cdot e^{-n\theta}\theta^{\sum x_i}$$

Jika ~~[illegible]~~ diproporsionalkan

$$f(\theta|\underset{\sim}{x}) \propto e^{-n\theta}\theta^{\sum x_i}$$

Sehingga posterior distribusinya adalah sebaran gamma $(\sum x_i + 1, n\theta)$

sehingga penduga bagi θ adalah

$$E(\theta) = \frac{\sum x_i + 1}{n\theta}$$

---

2. (a) $\hat{\pi} = \frac{11}{116} = 0.0948 \approx 0.1$ ($y = 11$, $n = 116$)

(b) beta(1,10) $\alpha = 1 \quad \beta = 10$

prior : $\pi \sim$ beta(1,10)

posterior $\propto$ likelihood $\times$ prior

posteriornya: $Y \sim bin(n, \pi)$, $\pi \sim beta(\alpha, \beta)$

beta-binomial

$Y \sim beta(\alpha, \beta)$ $\quad E(Y) = \frac{\alpha}{\alpha+\beta}$ $\quad Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

posteriornya $\sim$ beta$(Y+\alpha, n-Y+\beta)$

$\sim$ beta$(11+1, 116-11+10)$

$\sim$ beta$(12, 115)$

$$\hat{\pi} = \frac{12}{12+115} = \frac{12}{127}$$

($Y+\alpha$; $\alpha+\beta+n$)

---

3. (a) $\mu \sim$ Gamma(r, v) = Gamma$(r, \frac{1}{v})$

$E(\mu) = \frac{r}{v} \quad \to \quad rv = r \cdot \frac{1}{v}$

$Var(\mu) = \frac{r}{v^2} \quad \to \quad rv^2 = r(\frac{1}{v})^2$

mean = 6	standard deviation = 2
$\frac{r}{v} = 6$	$\frac{r}{v^2} = 2^2 = 4$
$r = 6v$	$\frac{6v}{v^2} = 4$
$r = 6(\frac{3}{2})$	$\frac{6}{v} = 4$
$r = 9$	$v = 3/2$

prior: Gamma$(9, 3/2)$

(b) posterior Gamma-Poisson

Poisson $P(\theta)$, gamma $G(a,b)$ → $G(a+x, \beta+1)$

→ Gamma$(r+y, v+1)$

$$f(\mu) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \mu^{\alpha-1} \exp\left(-\frac{\mu}{\beta}\right) \qquad E(\mu) = \alpha\beta \quad Var(\mu) = \alpha\beta^2$$

$$E(\mu) = \int_0^\infty \mu f(\mu)$$

$$f(\mu) = \frac{\beta^\alpha}{\Gamma(\alpha)} \mu^{\alpha-1} \exp(-\beta\mu) \qquad E(\mu) = \alpha/\beta \quad Var(\mu) = \alpha/\beta^2$$

3) Misalkan

$x_1, x_2, \ldots x_n \sim Beta(2, \theta)$, $\theta > 0$

$$f(x) = \frac{\Gamma(2+\theta)}{\Gamma(2)\Gamma(\theta)} x (1-x)^{\theta-1}$$

Karena ~~...~~ $\Gamma(n+1) = n\Gamma(n)$ maka $\Gamma(2) = 1$ dan $\Gamma(2+\theta) = \theta(\theta+1)\Gamma(\theta)$, diperoleh

$$f(x) = \frac{\theta(\theta+1)\Gamma(\theta)}{\Gamma(\theta)} x(1-x)^{\theta-1}$$

$$f(x) = \theta(\theta+1) x (1-x)^{\theta-1}, \quad \theta > 0$$

a) Jeffreys prior

likelihood

$$f(\underline{x}|\theta) = \prod_{i=1}^{n} \theta(\theta+1) x_i (1-x_i)^{\theta-1}$$

$$= \theta^n(\theta+1)^n \prod_{i=1}^{n} x_i \left[\prod_{i=1}^{n}(1-x_i)\right]^{\theta-1}$$

Log likelihood

$$\log f(\underline{x}|\theta) = n\log\theta + n\log(\theta+1) + \sum_{i=1}^{n}\log x_i + (\theta-1)\sum_{i=1}^{n}\log(1-x_i)$$

$$\frac{\partial \log f(\underline{x}|\theta)}{\partial\theta} = \frac{n}{\theta} + \frac{n}{\theta+1} + \sum_{i=1}^{n}\log(1-x_i)$$

$$\frac{\partial^2 \log f(\underline{x}|\theta)}{\partial\theta^2} = -\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2}$$

Informasi fisher $I(\theta)$ diperoleh

$$I(\theta) = -E\left(-\frac{n}{\theta^2} - \frac{n}{(\theta+1)^2}\right) = n\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)$$

b) penduga bagi θ

perlu menghitung posterior distribution

$$f(\underline{x}, \theta) = f(\underline{x}|\theta)\,\pi(\theta)$$

$$= \theta^n(\theta+1)^n \prod_{i=1}^{n} x_i \left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1} \cdot \left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)^{1/2}$$

$$= \theta^n(\theta+1)^n \prod_{i=1}^{n} x_i \left(\prod(1-x_i)\right)^{\theta-1} \cdot \frac{1}{\theta(\theta+1)}(\theta^2+(\theta+1)^2)^{1/2}$$

$$= \frac{(\theta(\theta+1))^n}{\theta(\theta+1)}(\theta^2+(\theta+1)^2)^{1/2}\prod_{i=1}^{n} x_i \left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1}$$

$$= [\theta(\theta+1)]^{n-1}[\theta^2+(\theta+1)^2]^{1/2}\prod_{i=1}^{n} x_i \left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1}$$

jika diproporsionalkan

$$f(\theta|\underline{x}) \propto [\theta(\theta+1)]^{n-1}[\theta^2+(\theta+1)^2]^{1/2}\left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1}$$

untuk mendapatkan penduga bagi θ maka

$$E(\theta) = \int \theta \cdot [\theta(\theta+1)]^{n-1}[\theta^2+(\theta+1)^2]^{1/2}\left(\prod_{i=1}^{n}(1-x_i)\right)^{\theta-1}$$

Kerjakan dengan komputasi

4) Diketahui

$y_1, y_2, \ldots y_n \sim N(\mu, \sigma^2)$

$\pi(\mu) \sim N(3, \theta^2)$ dimana σ^2 diketahui

a) Sebaran posterior bagi μ

likelihood

$$f(\underline{y}|\mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2\mu y_i + \mu^2)}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right)}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\mu n\bar{y} + n\mu^2\right)}$$

jika diproporsionalkan diperoleh

$$f(\underline{y}|\mu) \propto e^{-\frac{1}{2\sigma^2}(-2\mu n\bar{y} + n\mu^2)}$$

prior

$$\pi(\mu) = \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{1}{2\theta^2}(\mu-3)^2}$$

$$= \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{1}{2\theta^2}(\mu^2 - 6\mu + 9)}$$

jika di proporsionalkan diperoleh

$$\pi(\mu) \propto e^{-\frac{1}{2\theta^2}(\mu^2 - 6\mu)}$$

dengan menggunakan Teorema Bayes

$$f(\mu|\underline{y}) \propto \text{likelihood} \times \text{prior}$$

$$= e^{-\frac{1}{2\sigma^2}(-2\mu n\bar{y} + n\mu^2)} \cdot e^{-\frac{1}{2\theta^2}(\mu^2 - 6\mu)}$$

$$= e^{-\frac{1}{2\sigma^2}(-2\mu n\bar{y} + n\mu^2) - \frac{1}{2\theta^2}(\mu^2 - 6\mu)}$$

$$= e^{-\frac{n}{2\sigma^2}\mu^2 - \frac{1}{2\theta^2}\mu^2 + \frac{n\bar{y}}{\sigma^2}\mu + \frac{3}{\theta^2}\mu}$$

$$= e^{\left[-\frac{1}{2}\left(\frac{1}{\theta^2} + \frac{n}{\sigma^2}\right)\mu^2 + \left(\frac{3}{\theta^2} + \frac{n\bar{y}}{\sigma^2}\right)\mu\right]}$$

~~Misalkan~~

$$= e^{\left[-\frac{1}{2}\left(\frac{\sigma^2 + n\theta^2}{\theta^2\sigma^2}\right)\mu^2 + \left(\frac{3\sigma^2 + n\theta^2\bar{y}}{\theta^2\sigma^2}\right)\mu\right]}$$

Misalkan

$$\sigma^2_{new} = \frac{\theta^2\sigma^2}{\sigma^2 + n\theta^2} \quad \text{dan} \quad \mu_{new} = \left(\frac{3}{\theta^2} + \frac{n\bar{y}}{\sigma^2}\right)\sigma^2_{new}$$

maka di peroleh

$$f(\mu|\underline{y}) = e^{\left[-\frac{1}{2\sigma^2_{new}}\mu^2 + \frac{\mu_{new}}{\sigma^2_{new}}\mu\right]}$$

$$= e^{\left[-\frac{1}{2\sigma^2_{new}}(\mu^2 - 2\mu_{new}\mu)\right]}$$

$$f(\mu|\underline{y}) \propto e^{\left[-\frac{1}{2\sigma^2_{new}}(\mu^2 - 2\mu_{new}\mu + \mu^2_{new})\right]}$$

$$= e^{\left[-\frac{1}{2\sigma^2_{new}}(\mu - \mu_{new})^2\right]}$$

Sehingga posterior distribution adalah

$$f(\mu|y) \sim N(\mu_{new}, \sigma^2_{new})$$

~~dimana~~

$$f(\mu|y) \sim N\left(\left(\frac{\theta^2\sigma^2}{\sigma^2+n\theta^2}\right)\left(\frac{3\sigma^2+n\theta^2\bar{y}}{\theta^2\sigma^2}\right), \frac{\theta^2\sigma^2}{\sigma^2+n\theta^2}\right)$$

b) Penduga bayes bagi μ

$$E(\mu) = \mu_{new}$$

$$= \left(\frac{\theta^2\sigma^2}{\sigma^2+n\theta^2}\right)\left(\frac{3\sigma^2+n\theta^2\bar{y}}{\theta^2\sigma^2}\right)$$

//

c) Credible interval of 2 bagi μ

$$P\left(-Z_{\alpha/2} < \frac{\mu - \mu_{new}}{\sqrt{\sigma^2_{new}}} < Z_{\alpha/2}\right) = 1-\alpha$$

//

5) Misalkan y menyatakan jumlah mahasiswa setuju

θ adalah peluang mahasiswa setuju

diketahui:

$n = 100$ dan $y = 65$

a) Sebaran posterior jika prior θ adalah $Beta(4,7)$

Karena y mengikuti sebaran binomial maka sebaran posteriornya adalah Beta dimana

$$g(\theta|y) \sim Beta(\alpha+y, \beta+n-y)$$

dimana $\alpha = 4$ dan $\beta = 7$

sehingga diperoleh sebaran posterior

$$g(\theta|y) \sim Beta(4+65, 7+100-65)$$

$$g(\theta|y) \sim Beta(69, 42)$$

catatan:

untuk membuktikan sebaran posterior dari likelihood binomial dan prior beta bisa di lihat kembali materi dari pak kusman

b) penduga bayes bagi θ

Karena sebaran posterior adalah Beta maka penduga bagi θ diperoleh dari

$$E(\theta) = \mu_{posterior} = \frac{\alpha+y}{\alpha+\beta+n}$$

$$\mu_{posterior} = \frac{4+65}{4+7+100} = \frac{69}{111} = 0{,}621$$

c. $H_0: \theta = 0.6$

$H_1: \theta = 0.4$

$$H_0: P(\theta=0.6|y) = \frac{Beta(69,42)\ (0,6)}{Beta(69,42)(0,6) + Beta(69,42)(0,4)}$$

$$= \frac{0{,}6^{(69-1)}\ 0{,}4^{(42-1)}}{0{,}6^{(69-1)}\ 0{,}4^{(42-1)} + 0{,}4^{(69-1)}\ 0{,}6^{(42-1)}}$$

$$= 0{,}9999982$$

H_1:

$$P(\theta=0.4|y) = \frac{Beta(69,42)\ (0,4)}{Beta(69,42)(0,4) + Beta(69,42)(0,6)}$$

$$= 0{,}0000176$$

$P(\theta=0{,}6|y) > P(\theta=0.4|y) \longrightarrow$ terima H_0 $\theta = 0.6$

d. pengujian hipotesis $H_0: \theta = 0.6$ vs $H_1: \theta \neq 0.6$ dgn trf nyata 0.05

probabilitas posterior bahwa θ berada di luar interval $= (\theta_0 - \delta)$ hingga $(\theta_0 + \delta)$ dgn kuantil -0.025 dan ke -0.975 dari $B(69,42)$

- kuantil ke -0.025: $P(\theta <$ kuantil $0.025) = 0.025 \approx 0.497$
- $-$ll$-$ 0.975: $P(\theta <$ kuantil $0.975) = 0.975 \approx 0.748$

probabilitas posterior θ berada diluar interval $\approx 0.259 > 0.05$

maka terima $H_0: \theta = 0.6$

Secara **analitik** tentukan penduga Bayes bagi θ. Bandingkan dengan hasil yang diperoleh melalui MCMC. Apa kesimpulannya?

posterior: $g(\theta|y) = e^{-\theta}$; $\theta > 0$

$\hat{\theta}_B$ — analitik $\rightarrow \hat{\theta}_B = 1$; numerik $\rightarrow \hat{\theta}_B = 0.9985$

$$\hat{\theta}_B = E(\theta|y) = \int_0^\infty \theta g(\theta|y) = \int_0^\infty \underbrace{\theta}_{u}\underbrace{e^{-\theta}d\theta}_{dv}$$

- $u = \theta$ $\quad du = d\theta$

$dv = e^{-\theta}d\theta$ $\quad v = \int e^{-\theta}d\theta = -e^{-\theta}$

$= -\theta e^{-\theta}\Big|_0^\infty - \int_0^\infty -e^{-\theta}d\theta$

$= -\theta e^{-\theta}\Big|_0^\infty + \int_0^\infty e^{-\theta}d\theta$

$= (-\infty e^{-\infty} + 0e^{0}) + (-e^{-\theta})\Big|_0^\infty$

$= (0+0) + (-e^{-\infty} + e^{0})$

$= 0 + 0 + 1 = 1$

Misalkan diketahui sebaran posterior bagi θ adalah $g(\theta|y) = 6\theta(1-\theta)$ dengan $0 < \theta < 1$.

$$\hat{\theta} = E(\hat{\theta}) = \int_0^1 \theta g(\theta|y)\, d\theta$$

$$= \int_0^1 \theta\, 6\theta(1-\theta)\, d\theta$$

$$= \int_0^1 6\theta^2 - 6\theta^3\, d\theta$$

$$= 2\theta^3 - \frac{3}{2}\theta^4\Big|_0^1$$

$$\hat{\theta} = \left[2 - \frac{3}{2}\right] = \frac{1}{2} = 0.5$$

	Posterior Mean	Posterior Std. Deviation
Intercept:	11.94	0.099036
Slope:	2.46	0.084019 ✓

Lakukan pengujian hipotesis untuk H_0: $\beta_1 = 0$ vs H_1: $\beta_1 \neq 0$ pada taraf uji α = 0.05.

$n = 100$ $\quad$ Stbl $t_{(\frac{\alpha}{2};\, db=98)} \approx Z_{(\frac{\alpha}{2})} = Z_{0{,}025} = 1{\cdot}96$

$$Z_{hit} = \frac{\hat{\beta}_1 - \beta_1}{St.dev(\hat{\beta}_1)} = \frac{2{\cdot}46 - 0}{0.084019} = 29.28$$

Karena $29.28 > 1{\cdot}96$ maka TOLAK H_0

2] $X_1, X_2, .., X_n \sim$ Poisson (θ)

dgn $\theta \sim$ Gamma (α, β)

Perlihatkan bahwa penduga Bayes bagi θ adalah

$$\hat{\theta}_B = \left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\frac{\beta}{1+n\beta}\right)$$

Penyelesaian

* fkp bersama $X_1, X_2, .., X_n$ dgn syarat θ :

$$f_X(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x!}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!} \quad ; \quad x_i = 0, 1, 2, \dots; \quad i = 1, 2, \dots, n$$

→ Sebaran prior bagi θ :

$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{\alpha-1} e^{-\theta/\beta} \quad ; \quad \alpha, \beta, \theta > 0$$

→ fkp bersama antara $x_1, x_2, \dots, x_n$ dan θ :

$$f(x, \theta) = f_X(x|\theta) \cdot \pi(\theta) = \left(\frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}\right)\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{\alpha-1} e^{-\theta/\beta}\right)$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta(n + 1/\beta)}}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} \quad ; \quad x_i = 0, 1, 2, \dots; \quad i = 1, 2, 3, \dots, n; \quad \alpha, \beta, \theta > 0$$

* fkp marjinal bagi $x_1, x_2, \dots, x_n$, $f(x)$:

$$f(x) = \int_0^{\infty} \frac{\theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta(n + 1/\beta)}}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} d\theta$$

Misal

$u = \theta(n + 1/\beta)$

$du = (n + 1/\beta)\, d\theta$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} \int_0^{\infty} \theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta(n + 1/\beta)} d\theta$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} \int_0^{\infty} \left(\frac{u}{n + 1/\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-u} \, du/(n + 1/\beta)$$

$$\Leftrightarrow \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i! \, (n + 1/\beta)^{\sum_{i=1}^{n} x_i + \alpha}} \int_0^{\infty} u^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-u} \, du$$

$$\Leftrightarrow \frac{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i! \, (n + 1/\beta)^{\sum_{i=1}^{n} x_i + \alpha}} \quad ; \quad x_i = 0, 1, 2, \dots; \quad i = 1, 2, \dots, n; \quad \alpha, \beta > 0$$

Sebaran Posterior bagi θ :

$$\pi(\theta|x) = \frac{f(x, \theta)}{f(x)} = \frac{\cancel{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!}\, \theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta(n + 1/\beta)} (n + 1/\beta)^{\sum_{i=1}^{n} x_i + \alpha}}{\cancel{\Gamma(\alpha)\beta^{\alpha}}\, \Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right) \cancel{\prod_{i=1}^{n} x_i!}}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta(n + 1/\beta)} (n + 1/\beta)^{\sum_{i=1}^{n} x_i + \alpha}}{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\theta/\left(\frac{\beta}{n\beta+1}\right)}}{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)\left(\frac{\beta}{n\beta+1}\right)^{\sum_{i=1}^{n} x_i + \alpha}} \quad ; \quad x_i = 0, 1, 2, \dots; \quad i = 1, 2, \dots, n; \quad \alpha, \beta, \theta > 0$$

∴ Penduga Posterior bagi $\hat{\alpha} = \sum_{i=1}^{n} x_i + \alpha$

$$\hat{\beta} = \frac{\beta}{n\beta + 1}$$

diketahui bahwa $\theta \sim$ Gamma (α, β)

$$\hat{\theta}_B = E(\theta|x) = \hat{\alpha}\hat{\beta}$$

$$= \left(\sum_{i=1}^{n} x_i + \alpha\right)\left(\frac{\beta}{n\beta+1}\right)$$

$$= \frac{\sum_{i=1}^{n} x_i + \alpha}{n + 1/\beta} = \left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\frac{\beta}{n\beta+1}\right)$$

Sehingga terbukti bahwa penduga bayes bagi θ adalah

$$\hat{\theta}_B = \left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\frac{\beta}{1+n\beta}\right)$$

Revisi Nomor 1

AHMAD RIFAI NASU
G14180045

1] Dik : $f(x|\theta) = \frac{1}{2\theta} x^{-1/2} \exp\left(\frac{-x^{1/2}}{\theta}\right)$; $x > 0$

fkp diatas adalah bentuk fkp sebaran weibull (α, β) ; dengan $\alpha = \theta$; $\beta = \frac{1}{2}$

Sehingga $X \sim$ Weibull $(\theta, \beta = 1/2)$

a) Tentukan Penduga momen bagi θ, $\hat{\theta}_m$

Penduga Momen

$\mu_1 = m_1$, dgn $\mu_1 = E(x)$

$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

Maka. $\mu_1 = m_1$

$E(x) = \bar{x}$

$2\theta^2 = \bar{x}$

$\theta^2 = \frac{\bar{x}}{2}$

$\hat{\theta}_m = \left(\frac{\bar{x}}{2}\right)^{1/2}$

Sehingga Penduga momen bagi θ, adalah

$$\hat{\theta}_m = \left(\frac{\bar{x}}{2}\right)^{1/2}$$

Catatan :

karena $X \sim$ Weibull $(\theta, \beta = 1/2)$

maka $E(x) = \int_0^{\infty} x \frac{1}{2\theta} x^{-1/2} \exp\left(\frac{-x^{1/2}}{\theta}\right) dx$

$= \frac{1}{2\theta} \int_0^{\infty} x^{1/2} \exp\left(\frac{-x^{1/2}}{\theta}\right) dx$

$= \frac{1}{2\theta}\left[-2\theta\left(x + 2\theta\sqrt{x} + 2\theta^2\right) e^{-\sqrt{x}/\theta}\right]_0^{\infty}$

$= \frac{1}{2\theta}\left[4\theta^3\right] = 2\theta^2$

atau $E(x) = \alpha^{1/\beta}\, \Gamma(1 + 1/\beta)$

$= (\theta)^2\, \Gamma(1 + 2)$

$\boxed{= 2!\,\theta^2 = 2\theta^2}$

b) Tentukan MLE bagi θ, $\hat{\theta}_L$

. fungsi likelihood :

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} x_i^{-1/2} \exp\left(-\frac{x_i^{1/2}}{\theta}\right) \quad ; \quad x_i > 0 \quad ; \quad i = 1, 2, \dots, n$$

$$= \frac{1}{2^n\theta^n} \prod_{i=1}^{n} x_i^{-1/2} e^{-\sum_{i=1}^{n} x_i^{1/2}/\theta} \quad ; \quad x_i > 0 \quad ; \quad i = 1, 2, \dots, n$$

Fungsi logaritma likelihood :

$$\ell(\theta|x) = \log\left[\frac{1}{2^n\theta^n} \prod_{i=1}^{n} x_i^{-1/2} e^{-\sum_{i=1}^{n} x_i^{1/2}/\theta}\right]$$

$$= -n\log 2 - n\log\theta + \log\left[\prod_{i=1}^{n} x_i^{-1/2}\right] - \sum_{i=1}^{n} x_i^{1/2}/\theta$$

Untuk mendapatkan Penduga MLE, maka fungsi Likelihood harus maksimum :

$$\frac{\partial \ell(\theta|x)}{\partial\theta} = 0$$

$$\frac{-n}{\theta} + \frac{\sum_{i=1}^{n} x_i^{1/2}}{\theta^2} = 0$$

$$\frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i^{1/2}}{\theta^2}$$

$$\hat{\theta}_L = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$$

$$\frac{\partial^2 \log\left[f_X(x|\theta)\right]}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2\sum_{i=1}^{n} x_i^{1/2}}{\theta^3} < 0$$

∴ Penduga MLE Untuk θ adalah

$$\hat{\theta}_L = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$$

c) Perlihatkan bahwa $\hat{\theta}_L$ adalah Penduga tak bias bagi θ

Adb. $E(\hat{\theta}_L) = \theta$, maka $\hat{\theta}_L$ adalah Penduga tak bias

$$E(\hat{\theta}_L) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}\right) = \frac{1}{\cancel{n}} \cancel{n} E(x^{1/2})$$

$$E(x^{1/2}) = \int_0^{\infty} x^{1/2} \cdot \frac{1}{2\theta} x^{-1/2} e^{-x^{1/2}/\theta} dx$$

$$= \frac{1}{2\theta} \int_0^{\infty} e^{-x^{1/2}/\theta} dx$$

$$= \frac{1}{2\theta}\left[2\theta\left(x^{1/2} + \theta\right) e^{-x^{1/2}/\theta}\right]_0^{\infty}$$

$$= \frac{1}{\cancel{2\theta}} \cdot \cancel{2}\theta^{\cancel{2}} = \theta$$

∴ $E(\hat{\theta}_L) = \theta$, terbukti bahwa $\hat{\theta}_L = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$ adalah Penduga tak bias

d) Perlihatkan bahwa $\hat{\theta}_L$ adalah Penduga yang efisien bagi θ

Jika diketahui fkp bersama $Y_1, Y_2, \dots, Y_n$:

$$f_X(x|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} x_i^{-1/2} \exp\left(\frac{-x_i^{1/2}}{\theta}\right) \quad ; \quad x_i > 0 \quad ; \quad i = 1, 2, \dots, n$$

$$= \frac{1}{2^n\theta^n} \prod_{i=1}^{n} x_i^{-1/2} \exp\left(\frac{-\sum_{i=1}^{n} x_i^{1/2}}{\theta}\right) \quad ; \quad x_i > 0 \quad ; \quad i = 1, 2, \dots, n$$

Dengan faktorisasi Neymon-fisher diperoleh bahwa :

$$f_X(x|\theta) = g\left[T(x_1, x_2, \dots, x_n)\right] h(x_1, x_2, \dots, x_n)$$

$$= \left(\frac{1}{2^n\theta^n} \exp\left[-\frac{\sum_{i=1}^{n} x_i^{1/2}}{\theta}\right]\right)\left(\prod_{i=1}^{n} x_i^{-1/2}\right)$$

maka $T = \sum_{i=1}^{n} x_i^{1/2}$ adalah Statistik cukup bagi θ

agar tak bias maka $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$.

∴ Sehingga terbukti bahwa $\hat{\theta}_L$ adalah Penduga yang efisien bagi θ

Apakah $\hat{\theta}_L$ adalah Penduga tak bias terbaik ?

Misal $W = \hat{\theta}_L = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$ (Penduga tak bias bagi θ)

$T = \sum_{i=1}^{n} x_i^{1/2}$ (Statistik cukup bagi θ)

* Berdasarkan teorema Rao-Blackwell

$\phi T = E(W|T)$

$= E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1/2} \middle| T\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} x_i^{1/2} \middle| T\right)$

$= \frac{1}{n} E(T|T)$

$= \frac{1}{n} \cdot T$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2} = W$

Jadi $W = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$ adalah Penduga tak bias terbaik bagi θ

Let $X_1, X_2, \dots, X_n$ be Poisson random variables with parameter λ. Assume that λ has a Gamma (α, β) prior, α and β are known. Obtain the Bayes estimate of λ.

Penyelesaian:

$X_1, X_2, \dots, X_n \sim$ Poisson (λ)

$E(X) = \lambda$ dan $V(X) = \lambda$

1. Tentukan fkp/fmp bersama antara $X_1, X_2, \dots, X_n$ dengan syarat λ, $f_{\underline{X}}(\underline{x}|\lambda)$.

$$f_{\underline{X}}(\underline{x}|\lambda) = \prod_{i=1}^{n} f_x(x_i|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \times \frac{\lambda^{x_2} e^{-\lambda}}{x_2!} \times \dots \times \frac{\lambda^{x_n} e^{-\lambda}}{x_n!}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}, x_i = 0, 1, 2, \dots; \ i = 1, 2, \dots, n$$

2. Tentukan fkp/fmp dari sebaran prior bagi λ, $\pi(\lambda)$.

$$\pi(\lambda) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{-\lambda/\beta}, \qquad \alpha, \beta, \lambda > 0$$

3. Tentukan fkp/fmp bersama antara $X_1, X_2, \dots, X_n$ dan λ, $f(\underline{x}, \lambda)$.

$$f(\underline{x}, \lambda) = f_{\underline{X}}(\underline{x}|\lambda)\pi(\lambda) = \left(\frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}\right)\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}\right) = \frac{\lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)}}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!}$$

dimana $x_i = 0, 1, 2, \dots, i = 1, 2, \dots, n$, dan $\alpha, \beta, \lambda > 0$

4. Tentukan fkp/fmp marginal bagi $X_1, X_2, \dots, X_n$, $f(\underline{x})$.

$$f(\underline{x}) = \int_{\forall\lambda} f(\underline{x}, \lambda)\, d\lambda$$

Misal: $t = \lambda\left(n + \frac{1}{\beta}\right)$

$dt = \left(n + \frac{1}{\beta}\right) d\lambda$

$$= \int_0^{\infty} \frac{\lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)}}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} d\lambda$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} \int_0^{\infty} \lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)} d\lambda$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!} \int_0^{\infty} \left(\frac{t}{n + \frac{1}{\beta}}\right)^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-t} \frac{dt}{\left(n + \frac{1}{\beta}\right)}$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i! \left(n + \frac{1}{\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha}} \int_0^{\infty} t^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-t} dt$$

$$= \frac{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i! \left(n + \frac{1}{\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha}}$$

$$\boxed{\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt}$$

dimana $x_i = 0, 1, 2, \dots, i = 1, 2, \dots, n$, dan $\alpha, \beta > 0$

5. Tentukan sebaran posterior bagi λ, $\pi(\lambda|\underline{x})$.

$$\pi(\lambda|\underline{x}) = \frac{f(\underline{x}, \lambda)}{f(\underline{x})} = \frac{\dfrac{\lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)}}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!}}{\dfrac{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)}{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i! \left(n + \frac{1}{\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha}}}$$

$$= \frac{\Gamma(\alpha)\beta^{\alpha} \prod_{i=1}^{n} x_i!\, \lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)} \left(n + \frac{1}{\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha}}{\Gamma(\alpha)\beta^{\alpha} \Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right) \prod_{i=1}^{n} x_i!}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)} \left(n + \frac{1}{\beta}\right)^{\sum_{i=1}^{n} x_i + \alpha}}{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right)}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i + \alpha - 1} e^{-\lambda/\left(\frac{\beta}{n\beta+1}\right)}}{\Gamma\left(\sum_{i=1}^{n} x_i + \alpha\right) \left(\frac{\beta}{n\beta + 1}\right)^{\sum_{i=1}^{n} x_i + \alpha}}$$

dimana $x_i = 0, 1, 2, \dots, i = 1, 2, \dots, n$, dan $\alpha, \beta, \lambda > 0$. Penduga sebaran posterior bagi α dan β adalah $\hat{\alpha} = \sum_{i=1}^{n} x_i + \alpha$ dan $\hat{\beta} = \frac{\beta}{n\beta+1}$.