STK333
Pengantar Model Linear

Selang Kepercayaan Fungsi Linear dan Selang Regional

- Selang Kepercayaan Fungsi Linier
- Selang Kepercayaan Regional

Fungsi linear koefisien $\boldsymbol{\beta}$ secara umum dapat ditulis sebagai $\mathbf{a}'\,\boldsymbol{\beta}$.
Penduga ‘tak bias’-nya (BLUE) adalah $\mathbf{a}'\mathbf{b}$.

$$E[\mathbf{a}'\mathbf{b}] = \mathbf{a}'\boldsymbol{\beta}$$

$$\begin{aligned} V[\mathbf{a}'\mathbf{b}] &= V(\mathbf{a}'(X'X)^{-1}(X'\mathbf{y})) \\ &= \mathbf{a}'(X'X)^{-1}X'V(\mathbf{y})(\mathbf{a}'(X'X)^{-1}X')' \\ &= \mathbf{a}'(X'X)^{-1}X'(\sigma^2\mathbf{I})X(X'X)^{-1}\mathbf{a} \\ &= \mathbf{a}'(X'X)^{-1}X'\,X(X'X)^{-1}\mathbf{a}(\sigma^2\mathbf{I}) \\ &= \mathbf{a}'(X'X)^{-1}X'\,X(X'X)^{-1}\mathbf{a}\sigma^2 \\ &= \mathbf{a}'(X'X)^{-1}\mathbf{a}\sigma^2 \end{aligned}$$

$$\boxed{\begin{aligned} &\text{Teorema: } \mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}\,,\ r(X) = p\,,\ \boldsymbol{\epsilon}\sim N(\mathbf{0},\sigma^2 I) \\ &\qquad\qquad \mathbf{b} \sim N(\boldsymbol{\beta},(X'X)^{-1}\sigma^2) \end{aligned}}$$

$$\boxed{[\mathbf{a}'\mathbf{b}] \sim N(\mathbf{a}'\boldsymbol{\beta},\mathbf{a}'(X'X)^{-1}\mathbf{a}\sigma^2)}$$

$$\boxed{\frac{\mathbf{a}'\mathbf{b} - \mathbf{a}'\boldsymbol{\beta}}{\sqrt{\mathbf{a}'(X'X)^{-1}\mathbf{a}\sigma^2}} \sim N(0,1)}$$

$$\frac{\mathbf{a}'\mathbf{b}-\mathbf{a}'\boldsymbol{\beta}}{\sqrt{\mathbf{a}'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{a}\sigma^2}} \sim \mathrm{N}(0,1) \qquad \text{dan} \qquad \frac{(n-p)s^2}{\sigma^2} \sim \chi^2_{n-p}$$

$$\frac{\dfrac{\mathbf{a}'\mathbf{b}-\mathbf{a}'\boldsymbol{\beta}}{\sqrt{\mathbf{a}'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{a}\sigma^2}}}{\sqrt{\dfrac{(n-p)s^2}{\sigma^2}/(n-p)}} = \frac{\mathbf{a}'\mathbf{b}-\mathbf{a}\boldsymbol{\beta}}{s\sqrt{\mathbf{a}'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{a}}} \sim \mathrm{t}_{(\mathrm{n-p})}$$

Selang kepercayaan (1-α) bagi **a'** **b** adalah:

$$\mathbf{a}'\mathbf{b} \pm t_{(n-p,\frac{\alpha}{2})} s\sqrt{\mathbf{a}'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{a}}$$

Nilai dugaan

<u>Rataan</u> nilai dugaan $\hat{y}$ untuk $\mathbf{x}_*'$ tertentu adalah $\mathbf{x}_*'\mathbf{b}$.

$$\mathrm{E}[\mathbf{x}_*'\mathbf{b}] = \mathbf{x}_*'\boldsymbol{\beta}$$

$$\begin{aligned}
\mathrm{V}[\mathbf{x}_*'\mathbf{b}] &= \mathrm{V}(\mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}(\mathrm{X}'\mathbf{y}) \\
&= \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{V}(\mathbf{y})(\mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')' \\
&= \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'(\sigma^2\mathbf{I})\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_* \\
&= \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\,\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*(\sigma^2\mathbf{I}) \\
&= \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\,\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2 \\
&= \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2
\end{aligned}$$

Nilai dugaan

Rataan nilai dugaan $\hat{y}$ untuk $\mathbf{x}_*'$ tertentu adalah $\mathbf{x}_*'\mathbf{b}$.

$$\mathrm{E}[\mathbf{x}_*'\mathbf{b}] = \mathbf{x}_*'\boldsymbol{\beta}$$

$$\mathrm{V}[\mathbf{x}_*'\mathbf{b}] = \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2$$

$$[\mathbf{x}_*'\mathbf{b}] \sim \mathbf{N}(\mathbf{x}_*'\boldsymbol{\beta}, \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2)$$

$$\frac{\mathbf{x}_*'\mathbf{b} - \mathbf{x}_*'\boldsymbol{\beta}}{\sqrt{\mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2}} \sim \mathrm{N}(0,1)$$

Selang Nilai dugaan

$$[\mathbf{x}_*'\mathbf{b}] \sim N(\mathbf{x}_*'\boldsymbol{\beta}, \mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*\sigma^2)$$

$$\frac{\mathbf{x}_*'\mathbf{b} - \mathbf{x}_*'\boldsymbol{\beta}}{\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*\sigma^2}} \sim N(0,1)$$

$$\frac{(n-p)s^2}{\sigma^2} \sim \chi^2_{n-p}$$

$$\frac{\dfrac{\mathbf{x}_*'\mathbf{b} - \mathbf{x}_*'\boldsymbol{\beta}}{\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*\sigma^2}}}{\sqrt{\dfrac{(n-p)s^2}{\sigma^2}/(n-p)}} = \frac{\mathbf{x}_*'\mathbf{b} - \mathbf{x}_*'\boldsymbol{\beta}}{s\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}} \sim t_{(n-p)}$$

Selang kepercayaan bagi (1-α) bagi $\mathbf{x}_*'\mathbf{b}$ adalah:

$$\mathbf{x}_*'\mathbf{b} \pm t_{(n-p,\frac{\alpha}{2})} s\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}$$

Nilai dugaan <u>SATU</u> amatan

$$y_* = \mathbf{x}_*'\boldsymbol{\beta} + \varepsilon$$

$$\hat{y}_* = \mathbf{x}_*'\mathbf{b}$$

$$y_* - \hat{y}_* = \mathbf{x}_*'\boldsymbol{\beta} + \varepsilon - \mathbf{x}_*'\mathbf{b}$$

$$\mathrm{E}[y_* - \hat{y}_*] = \mathrm{E}[\mathbf{x}_*'\boldsymbol{\beta} + \boldsymbol{\varepsilon} - \mathbf{x}_*'\mathbf{b}] = \mathbf{0}$$

$$\begin{aligned}\mathrm{V}[y_* - \hat{y}_*] &= \mathrm{V}[\mathbf{x}_*'\boldsymbol{\beta} + \boldsymbol{\varepsilon} - \mathbf{x}_*'\mathbf{b}] \\ &= \sigma^2 + \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*\sigma^2 \\ &= (1 + \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*)\sigma^2\end{aligned}$$

$$\boxed{[y_* - \hat{y}_*] \sim \mathrm{N}(0, (1 + \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*)\sigma^2)}$$

$$\boxed{\frac{y_* - \hat{y}_*}{\sqrt{(1 + \mathbf{x}_*'(\mathrm{X}'\mathrm{X})^{-1}\mathbf{x}_*)\sigma^2}} \sim \mathrm{N}(0,1)}$$

Nilai dugaan SATU amatan

$$\frac{\dfrac{\mathbf{x}_*'\mathbf{b}-\mathbf{x}_*'\boldsymbol{\beta}}{\sqrt{(1+\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*)\sigma^2}}}{\sqrt{\dfrac{(n-p)s^2}{\sigma^2}/(n-p)}}=\frac{\mathbf{x}_*'\mathbf{b}-\mathbf{x}_*'\boldsymbol{\beta}}{s\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}}\sim t_{(n-p)}$$

Selang kepercayaan bagi (1-α) bagi $\mathbf{x}_*'\mathbf{b}$ adalah:

$$\mathbf{x}_*'\mathbf{b}\ \pm\ t_{(n-p,\frac{\alpha}{2})}s\sqrt{1+\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}$$

Teladan: [Exp 3.3.1]

$$\boldsymbol{y} = \beta_0 + \beta_1 \mathbf{x} + \boldsymbol{\varepsilon}$$

x	y
2	1.9
3	2.7
4	4.2
5	4.8
6	4.8
7	5.1

$$\mathrm{X} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{bmatrix}$$

$$(\mathrm{X}'\mathrm{X}) = \begin{bmatrix} 6 & 27 \\ 27 & 139 \end{bmatrix} \qquad \mathrm{X}'\mathbf{y} = \begin{bmatrix} 23.5 \\ 117.2 \end{bmatrix}$$

$$('\mathrm{X}'\mathrm{X})^{-1} = \frac{1}{105}\begin{bmatrix} 139 & -27 \\ -27 & 6 \end{bmatrix}$$

$$\mathbf{b} = ('\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathbf{y} = \frac{1}{105}\begin{bmatrix} 139 & -27 \\ -27 & 6 \end{bmatrix}\begin{bmatrix} 23.5 \\ 117.2 \end{bmatrix} = \begin{bmatrix} 0.97 \\ 0.65 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 1.9 \\ 2.7 \\ 4.2 \\ 4.8 \\ 4.8 \\ 5.1 \end{bmatrix} \qquad \mathrm{X}\mathbf{b} = \begin{bmatrix} 2.27 \\ 2.92 \\ 3.57 \\ 4.22 \\ 4.87 \\ 5.52 \end{bmatrix} \qquad \mathbf{y} - \mathrm{X}\mathbf{b} = \begin{bmatrix} 1.9 \\ 2.7 \\ 4.2 \\ 4.8 \\ 4.8 \\ 5.1 \end{bmatrix} - \begin{bmatrix} 2.27 \\ 2.92 \\ 3.57 \\ 4.22 \\ 4.87 \\ 5.52 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.22 \\ 0.63 \\ 0.58 \\ -0.07 \\ -0.42 \end{bmatrix} = \boldsymbol{e}$$

Teladan: [Exp 3.3.1]

$$s^2 = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{(n-p)} = \frac{\boldsymbol{e}'e}{(n-p)} = \frac{\mathrm{JK_{Res}}}{(n-p)}$$

$$= \frac{\boldsymbol{e}'e}{(6-2)} = \sum_{i=1}^{6} \frac{e_i^2}{4}$$

$$\mathbf{e} = \begin{bmatrix} -0.37 \\ -0.22 \\ 0.63 \\ 0.58 \\ -0.07 \\ -0.42 \end{bmatrix}$$

$$= \frac{[(-0.37)^2 + (-0.22)^2 + \cdots + (-0.42)^2)}{4} = 0.2749$$

$$s = \sqrt{0.2749} = 0.5243$$

Selang kepercayaan 95% bagi $\mathbf{x}_*'\mathbf{b}$ untuk $\mathbf{x} = \begin{bmatrix} 1 \\ 5.5 \end{bmatrix}$ adalah:

$$\mathbf{x}_*'\mathbf{b} \pm t_{(n-p,\frac{\alpha}{2})} s\sqrt{\mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}$$

$$\begin{bmatrix} 1 & 5.5 \end{bmatrix} \begin{bmatrix} 0{,}97 \\ 0.65 \end{bmatrix} \pm (2.45)(0.5243)\sqrt{\begin{bmatrix} 1 & 5.5 \end{bmatrix} \frac{1}{105} \begin{bmatrix} 139 & -27 \\ -27 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 5.5 \end{bmatrix}}$$

$$4.545 \pm (2.45)(0.5243)(0.4731)$$

$$4.545 \pm 0.608 \rightarrow (3.937\,;5.153)$$

Selang kepercayaan 95% bagi dugaan <u>SATU</u> pengamatan untuk $\mathbf{x} = \begin{bmatrix} 1 \\ 5.5 \end{bmatrix}$ adalah:

$$\mathbf{x}_*'\mathbf{b} \pm t_{(n-p,\frac{\alpha}{2})} s\sqrt{1 + \mathbf{x}_*'(X'X)^{-1}\mathbf{x}_*}$$

$$[1 \quad 5.5]\begin{bmatrix} 0,97 \\ 0.65 \end{bmatrix} \pm (2.45)(0.5243)\sqrt{1 + [1 \quad 5.5]\frac{1}{105}\begin{bmatrix} 139 & -27 \\ -27 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 5.5 \end{bmatrix}}$$

$$4.545 \pm (2.45)(0.5243)(1.1063)$$

$$4.545 \pm 1.421 \rightarrow (3.124; 5.966)$$

Definisi: Bila $\chi^2_{\gamma 1}$ dan $\chi^2_{\gamma 2}$ saling bebas, maka $\dfrac{\chi^2_{\gamma 1}/\gamma 1}{\chi^2_{\gamma 2}/\gamma 2} \sim F_{(\gamma 1, \gamma 2)}$

Daerah kepercayaan bagi $\boldsymbol{\beta}$

$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$ $\quad$ $r(X) = p$

$\boldsymbol{\epsilon} \sim \mathbf{N}(\mathbf{0}, \sigma^2 I)$ $\quad$ $\mathbf{b} \sim \mathbf{N}(\boldsymbol{\beta}, (X'X)^{-1}\sigma^2)$

$$\frac{(b-\beta)}{\sigma\sqrt{(X'X)-1}} \sim N(0,\ 1)$$

$$\frac{1}{\sigma^2}(\mathbf{b}-\boldsymbol{\beta})'(X'X)(\mathbf{b}-\boldsymbol{\beta}) \sim \chi^2_{r,\lambda}$$

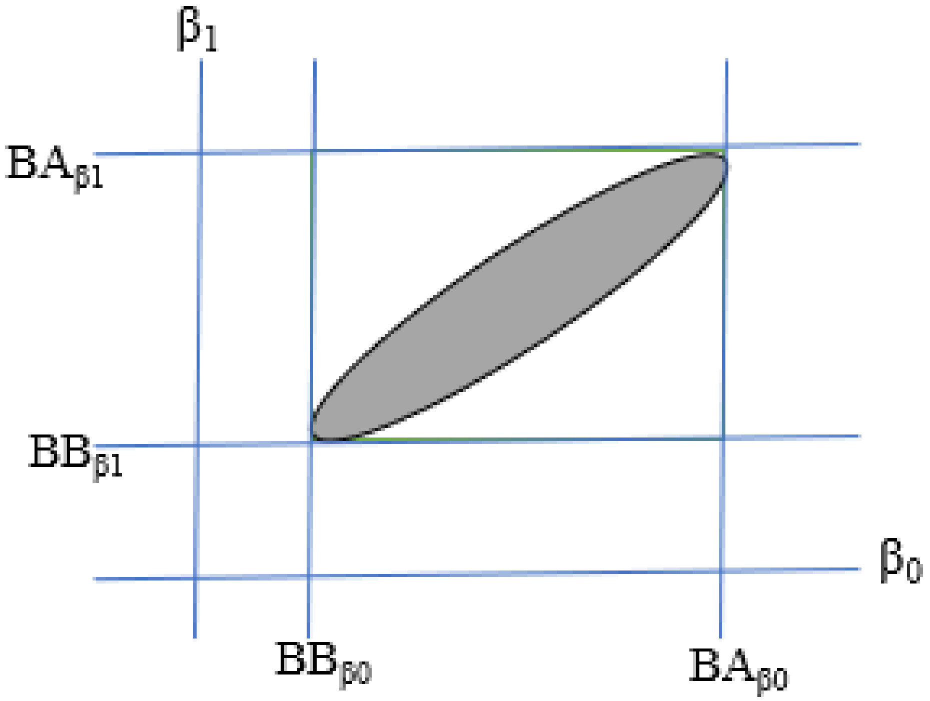
$r = r((X'X)^{-1}) = p$, $\lambda = 0$

$$\frac{(\mathbf{b}-\boldsymbol{\beta})'(X'X)(\mathbf{b}-\boldsymbol{\beta})/\sigma^2 p}{(n-p)s^2/\sigma^2(n-p)} \sim F_{p,(n-p)}$$

$$\frac{(\mathbf{b}-\boldsymbol{\beta})'(X'X)(\mathbf{b}-\boldsymbol{\beta})}{s^2 p} \sim F_{p,(n-p)}$$

Daerah kepercayaan $\boldsymbol{\beta}$

$$\mathbf{P}\left[(\mathbf{b}-\boldsymbol{\beta})'(\mathrm{X}'\mathrm{X})(\mathbf{b}-\boldsymbol{\beta}) \leq s^2 p F_{p,(n-p)}\right] = 1-\alpha$$



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Teladan: [Exp 3.3.1]

$$s^2 = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{(n-p)} = \frac{\boldsymbol{e}'e}{(n-p)} = \frac{\mathrm{JK}_{\mathrm{Res}}}{(n-p)}$$

$$= \frac{\boldsymbol{e}'e}{(6-4)} = \sum_{i=1}^{6} \frac{e_1^2}{4}$$

$$= \frac{[(-0.37)^2 + (-0.22)^2 + \cdots + (-0.42)^2)}{4} = 0.2749$$

$$s = \sqrt{0.2749} = 0.5243$$

Teladan:

$$\beta_0: \; 0.97 \pm (2.776)(0.5243)\left(\sqrt{\frac{139}{105}}\right) \rightarrow 0.97 \pm 1.67 \rightarrow (-0.70; 2.64)$$

$$\beta_1: 0.65 \pm (2.776)(0.5243)\left(\sqrt{\frac{6}{105}}\right) \rightarrow 0.65 \pm 0.35 \rightarrow (0.30; 1.00)$$

$$\mathrm{P}\left[(\mathbf{b} - \boldsymbol{\beta})'(\mathrm{X}'\mathrm{X})(\mathbf{b} - \boldsymbol{\beta}) \leq \mathrm{s}^2\mathrm{p}\mathrm{F}_{\mathrm{p},(\mathrm{n}-\mathrm{p})}\right] = 1 - \alpha$$

Daerah kepercayaan bersamaan untuk β_0 dan β_1:

$$\mathrm{P}\left[\left(\begin{bmatrix}0.97\\0.65\end{bmatrix} - \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right)'\begin{bmatrix}6 & 27\\27 & 139\end{bmatrix}\left(\begin{bmatrix}0.97\\0.65\end{bmatrix} - \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right) \leq (0.2749)(2)(6.94)\right] = 1 - 0.05$$

$$\mathrm{P}\left[\left(\begin{bmatrix}0.97\\0.65\end{bmatrix} - \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right)'\begin{bmatrix}6 & 27\\27 & 139\end{bmatrix}\left(\begin{bmatrix}0.97\\0.65\end{bmatrix} - \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right) \leq 3.82\right] = 0.95$$

Teladan:

Daerah kepercayaan $\boldsymbol{\beta}$

Selang kepercayaan 95% untuk β_0 dan β_1:

$\beta_0 : (-0.70; 2.64)$ $\beta_1 : (0.30; 1.00)$

$$\mathrm{P}\left[\left(\begin{bmatrix}0.97\\0.65\end{bmatrix}-\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right)'\begin{bmatrix}6 & 27\\27 & 139\end{bmatrix}\left(\begin{bmatrix}0.97\\0.65\end{bmatrix}-\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\right)\leq 3.82\right]=0.95$$

$$\mathrm{P}[6\beta_0^2 - 46.68\beta_0 + 139\beta_1^2 - 233.08\,\beta_1 + 54\beta_0\,\beta_1 + 98.42 \leq 3.82]=0.95$$

$$\mathrm{P}[6\beta_0^2 - 46.68\beta_0 + 139\beta_1^2 - 233.08\,\beta_1 + 54\beta_0\,\beta_1 + 102.24 \leq 0]=0.95$$

Teladan:

Daerah kepercayaan $\boldsymbol{\beta}$

$$6\beta_0^2 - 46.68\beta_0 + 139\beta_1^2 - 233.08\,\beta_1 + 54\beta_0\,\beta_1 + 102.24 \leq 0$$

$\beta_0 : (-0.70; 2.64)$

$\beta_1 : (0.30; 1.00)$