STK333
Pengantar Model Linear

Model Linier Penuh

- Penduga Ragam
- Selang Kepercayaan

Penduga Ragam

$$\widehat{\sigma^2} = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{\mathrm{n}}$$

$$\mathrm{E}\left(\widehat{\sigma^2}\right) = \mathrm{E}\left[\frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{\mathrm{n}}\right]$$

$$= \frac{1}{n}\mathrm{E}[(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})]$$

$$= \frac{1}{n}\mathrm{E}[(\mathbf{y} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathbf{y})'(\mathbf{y} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathbf{y})]$$

$$= \frac{1}{n}\mathrm{E}[\mathbf{y}'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\mathbf{y}]$$

Penduga Ragam

$$\mathrm{E}\left(\widehat{\sigma^2}\right) = \frac{1}{n}\mathrm{E}[\mathbf{y}'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\mathbf{y}]$$

$$= \frac{1}{n}\mathrm{E}[\mathbf{y}'\underbrace{(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')}_{\mathrm{A}}\mathbf{y}]$$

$\boxed{\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}' \text{ adalah idempoten}}$

$\boxed{\text{Teorema 2.2.1: } \mathrm{E}[\mathbf{y}'\mathrm{A}\mathbf{y}] = \mathrm{tr}(\mathrm{AV}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}}$

$$= \frac{1}{\mathrm{n}}[\mathrm{tr}(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\boldsymbol{\sigma}^{\mathbf{2}}\mathrm{I} + \mathrm{E}[\mathbf{y}'](\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\mathrm{E}[\boldsymbol{y}]$$

$\boxed{\mathrm{E}[\boldsymbol{y}] = X\boldsymbol{\beta}}$

$$= \frac{1}{\mathrm{n}}[\mathrm{tr}(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\boldsymbol{\sigma}^{\mathbf{2}}\mathrm{I} + (\mathrm{X}\boldsymbol{\beta})'(\mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')\mathrm{X}\boldsymbol{\beta}]$$

Penduga Ragam

$$E\left(\widehat{\sigma^2}\right) = \frac{1}{\mathrm{n}}(\sigma^2[\mathrm{tr}(\mathrm{I}) - \mathrm{tr}(\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')] + \boldsymbol{\beta}'\mathrm{X}'\mathrm{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathrm{X}'\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}'\mathrm{X}\boldsymbol{\beta}]$$

$$= \frac{1}{\mathrm{n}}(\sigma^2[\mathrm{tr}(\mathrm{I}) - \mathrm{tr}(\mathrm{X}(\mathrm{X}'\mathrm{X})^{-1}\mathrm{X}')] + \boldsymbol{\beta}'\mathrm{X}'\mathrm{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathrm{X}'\mathrm{X}\boldsymbol{\beta}]$$

n k+1

$$= \frac{1}{n}\sigma^2[\mathrm{n} - (\mathrm{k} + 1)]$$

Penduga Ragam

$$E\left(\widehat{\sigma^2}\right) = \frac{1}{n}\sigma^2[n - (k + 1)]$$

$$= \frac{n - (k + 1)}{n}\sigma^2 \quad \Rightarrow \quad E\left(\widehat{\sigma^2}\right) \neq \sigma^2$$ {tidak ‘**tak bias**’]

Agar penduga ragam ‘**tak bias**’, penduga ragam σ^2 dikalikan dengan

$$\frac{n}{n - (k + 1)}$$ sehingga

$$E\left(\widehat{\sigma^2}\right) = \left[\frac{n - (k + 1)}{n}\sigma^2\right]\left[\frac{n}{n - (k + 1)}\right] = \sigma^2$$

Penduga Ragam

$$\widehat{\sigma^2} = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{\mathrm{n}}$$

$\boxed{\mathrm{E}\left(\widehat{\sigma^2}\right) \neq \sigma^2 \quad \text{\{tidak `\textbf{tak bias}']}}$

$$\mathrm{s}^2 = \frac{\mathrm{n}}{\mathrm{n} - (\mathrm{k} + 1)}\left[\frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{\mathrm{n}}\right] = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{\mathrm{n} - (\mathrm{k} + 1)}$$

$$\mathrm{s}^2 = \frac{\mathrm{SS}_{\mathrm{Res}}}{\mathrm{n} - \mathrm{p}}$$

$\boxed{\mathrm{E}\left(\widehat{\sigma^2}\right) = \sigma^2 \quad \text{\{`\textbf{tak bias}']}}$

Penduga Ragam

Contoh:

$$y = \beta_0 + \beta_1 x + \epsilon$$

x	y
2.0	1.9
3.0	2.7
4.0	4.2
5.0	4.8
6.0	4.8
7.0	5.1

$$\mathbf{y} = \begin{bmatrix} 1.9 \\ 2.7 \\ 4.2 \\ 4.8 \\ 4.8 \\ 5.1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{bmatrix}, \quad \text{and} \quad X\mathbf{b} = \begin{bmatrix} 2.27 \\ 2.92 \\ 3.57 \\ 4.22 \\ 4.87 \\ 5.52 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.97 \\ 0.65 \end{bmatrix}$$

Penduga Ragam

Contoh:

$$s^2 = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{n - (k + 1)}$$

$$\mathbf{y} - X\mathbf{b} = \begin{bmatrix} 1.9 \\ 2.7 \\ 4.2 \\ 4.8 \\ 4.8 \\ 5.1 \end{bmatrix} - \begin{bmatrix} 2.27 \\ 2.92 \\ 3.57 \\ 4.22 \\ 4.87 \\ 5.52 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.22 \\ 0.63 \\ 0.58 \\ -0.07 \\ -0.42 \end{bmatrix} = \mathbf{e}$$

Penduga Ragam

Contoh:

$$s^2 = \frac{(\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b})}{n - (k + 1)} = \frac{\mathbf{e}'\mathbf{e}}{6 - 2} = \sum_{i=1}^{6} \frac{e_i^2}{4}$$

$$= \frac{[(-0.37)^2 + (-0.22)^2 + \ \ldots + (-0.42)^2}{4} = 0.2749$$

Penduga Ragam

Latihan:

$$y = \beta_1 x + \epsilon$$

Tentukan :

1. matriks X sesuai model tersebut.
2. X'X, $(X'X)^{-1}$, dan X'**y**
3. penduga koefisien regresi β_1.
4. Penduga ragam σ^2.

Penduga Ragam

Latihan:

$$y = \beta_1 x + \epsilon$$

Berdasarkan data berikut, tentukan penduga β_1, σ^2, ragam b_1.

Amount of Fuel in Gallons (y)	Time Motor Runs in Hours (x)
3	0.6
5	2.0
7	2.1
9	2.0
10	2.4

Selang Kepercayaan

$$\mathbf{b} \sim \mathrm{N}(\boldsymbol{\beta}, (\mathrm{X}'\mathrm{X})^{-1}\sigma^2)$$

Ragam **b** adalah:

$$(\mathrm{X}'\mathrm{X})^{-1}\sigma^2 = \begin{pmatrix} c_{00} & c_{01} & . & c_{0k} \\ c_{10} & c_{11} & . & c_{1k} \\ . & . & . & . \\ c_{k0} & c_{k1} & . & c_{kk} \end{pmatrix}\sigma^2$$

Penduga ragam **b** adalah $s_{\mathbf{b}}^2 = (\mathrm{X}'\mathrm{X})^{-1}s^2$

Selang Kepercayaan

$$b_i \sim N(\beta_i, c_{ii}\sigma^2) \Rightarrow \boxed{\frac{b_i - \beta_i}{\sigma\sqrt{c_{ii}}} \sim N(0,1)}$$

saling bebas

$$\boxed{\frac{JK_{res}}{\sigma^2} = \frac{(n-p)s^2}{\sigma^2} \sim \chi^2_{n-p}}$$

maka

$$\boxed{\frac{\dfrac{b_i - \beta_i}{\sigma\sqrt{c_{ii}}}}{\sqrt{\dfrac{(n-p)s^2}{\sigma^2}/(n-p)}} = \frac{b_i - \beta_i}{s\sqrt{c_{ii}}} \sim t_{(n-p)}}$$

t dengan derajat bebas (n-p)

Selang Kepercayaan

SelangKepercayaan(1-α) untuk β_i adalah

$$P[-t_{\left(\frac{\alpha}{2}\right)} \leq \frac{b_i - \beta_i}{s\sqrt{c_{ii}}} \leq t_{\left(\frac{\alpha}{2}\right)}] = 1 - \alpha$$

$$P[-t_{\left(\frac{\alpha}{2}\right)} s\sqrt{c_{ii}} \leq b_i - \beta_i \leq t_{\left(\frac{\alpha}{2}\right)} s\sqrt{c_{ii}}] = 1 - \alpha$$

$$P[b_i - t_{\left(\frac{\alpha}{2}\right)} s\sqrt{c_{ii}} \leq \beta_i \leq b_i + t_{\left(\frac{\alpha}{2}\right)} s\sqrt{c_{ii}}] = 1 - \alpha$$

$$\boxed{b_i \pm t_{\left(n-p,\frac{\alpha}{2}\right)} s\sqrt{c_{ii}}}$$

Selang Kepercayaan

Contoh:

Temperature °C (x)	g/liter (y)
0	2.1
10	4.5
20	6.1
30	11.2
40	13.8
50	17.0

$$y = \beta_0 + \beta_1 x + \varepsilon$$

$$\widehat{\beta_0} = b_0 = 1.438$$

$$\widehat{\beta_1} = b_1 = 0.307$$

$$\widehat{\sigma^2} = s^2 = 0.745$$

$$\hat{\sigma} = s = 0.863$$

$$\widehat{E[y]} = b_0 + b_1 x = 1.438 + 0.307x$$

Selang Kepercayaan

Contoh:

$$(X'X)^{-1} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{bmatrix}$$

$$\sum_{i=1}^{6} x_i = 150 \qquad \sum_{i=1}^{6} x_i^2 = 5500$$

$$c_{00} = \frac{\sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{5500}{6(5500) - (150)^2} \doteq 0.524$$

Selang Kepercayaan

Contoh:

Selang kepercayaan 95% bagi β_0

$$b_0 \pm t_{\alpha/2} s \sqrt{c_{00}} = 1.438 \pm 2.776(0.863)\sqrt{0.524}$$
$$= 1.438 \pm 1.734$$

Selang kepercayaan 95% bagi β_1

$$c_{11} = \frac{n}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{6}{6(5500) - (150)^2} = 0.000571$$

$$\mathrm{b}_1 \pm \mathrm{t}_{(4,0.025)} \mathrm{s}\sqrt{\mathrm{c}_{11}} = 0.307 \pm 2.776(0.863)\sqrt{0.000571}$$
$$= 0.307 \pm 0.057268$$

Selang Kepercayaan

Contoh: [Exp 3.2.1]

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

$$\mathbf{y} = \begin{bmatrix} 50 \\ 40 \\ 52 \\ 47 \\ 65 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 5 & 2 \\ 1 & 10 & 2 \\ 1 & 20 & 3 \end{bmatrix} \quad X'X = \begin{bmatrix} 5 & 41 & 9 \\ 41 & 551 & 96 \\ 9 & 96 & 19 \end{bmatrix} \quad X'\mathbf{y} = \begin{bmatrix} 254 \\ 2280 \\ 483 \end{bmatrix}$$

Selang Kepercayaan

Contoh:

$$(X'X)^{-1} = \begin{bmatrix} 2.307551 & 0.1565378 & -1.88398 \\ 0.1565378 & 0.02578269 & -0.20442 \\ -1.88398 & -0.20442 & 1.977901 \end{bmatrix}$$

$$(X'X)^{-1}X'\mathbf{y} = \begin{bmatrix} 2.307551 & 0.1565378 & -1.88398 \\ 0.1565378 & 0.02578269 & -0.20442 \\ -1.88398 & -0.20442 & 1.977901 \end{bmatrix} \begin{bmatrix} 254 \\ 2280 \\ 483 \end{bmatrix}$$

$$\doteq \begin{bmatrix} 33.06 \\ -0.189 \\ 10.718 \end{bmatrix}$$

$$\mathrm{y} = 33.06 - 0.189x_1 + 10.718x_2 + e$$

$$\hat{\mathrm{y}} = 33.06 - 0.189x_1 + 10.718x_2$$

Selang Kepercayaan

Latihan:

Tentukan :

1. Penduga ragam b_0, ragam b_1, dan ragam b_2.
2. Penduga peragam (b_0, b_1), peragam (b_0,b_2), dan peragam (b_1,b_2)
3. Selang kepercayaan 95% bagi β_1 dan β_2.

$$s^2 = \frac{(\mathbf{y} - X\mathbf{b})'(\mathbf{y} - X\mathbf{b})}{n - (k + 1)}$$

Penduga ragam **b** adalah $s_{\mathbf{b}}^2 = (X'X)^{-1}s^2$