STK333
Pengantar Model Linear

Model Linier Penuh

- Model Penuh
- Pendugaan Parameter Model

Model Penuh

Model Linier:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots\ldots + \beta_k x_k + \varepsilon_i, i = 1, 2, \ldots\ldots, n$$

Model ini disebut model linier karena parameter-parameternya linier

Dengan notasi matriks:

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

di mana: $\mathbf{y}_{n\times 1}$ = vektor respon
$X_{n\times(k+1)}$ = matriks peubah bebas (*independent*)
$\boldsymbol{\beta}_{(k+1)\times 1}$ = vektor parameter
$\boldsymbol{\epsilon}_{n\times 1}$ = vektor peubah acak

Model Penuh

Contoh Model non Linier:

$$y = e^{\beta_0 + \beta_1 x} + \epsilon$$

$$y = \frac{1}{(1 + e^{-\beta_0 - \beta_1 x + \epsilon})}$$

Model Penuh

Contoh:

$$y = \beta_0 + \beta_1 x + \epsilon$$

$$\begin{gathered} y_1 = \beta_0 + \beta_1 x_1 + \epsilon_1 \\ y_2 = \beta_0 + \beta_1 x_2 + \epsilon_2 \\ \vdots \\ y_n = \beta_0 + \beta_1 x_n + \epsilon_n \end{gathered}$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \varepsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \quad \boldsymbol{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

Pendugaan koefisien regresi $\beta_0 \mathrm{dan}\ \beta_1$ berdasarkan sistem persamaan linier dengan dua bilangan yang tidak diketahui.

Model Penuh

Model Linier disebut model penuh bila r(X) = k+1 → $|X'X| \neq 0$

$$X'X = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}$$

Model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$

Asumsi: $E[\boldsymbol{\epsilon}] = \mathbf{0}$, $V[\boldsymbol{\epsilon}] = \sigma^2 I$, $\sigma^2 > 0$

maka $E[\mathbf{y}] = E[X\boldsymbol{\beta} + \boldsymbol{\epsilon}] = X\boldsymbol{\beta} + E[\boldsymbol{\epsilon}] = X\boldsymbol{\beta}$

$\boldsymbol{\epsilon} = \mathbf{y} - X\boldsymbol{\beta} = \mathbf{y} - E[\mathbf{y}]$

Bila ada penduga parameter $\widehat{\boldsymbol{\beta}}$ = **b** dan $\hat{\boldsymbol{\epsilon}}$ = **e** maka **e**=$\mathbf{y} - X\mathbf{b}$

Penduga parameter $\widehat{\boldsymbol{\beta}}$ diperoleh dengan metode kuadrat terkecil (*Least Square*) dengan cara meminimumkan $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}$ → LS estimator

Model Penuh

Penentuan *least square estimator* bagi $\beta_0, \beta_1, \ldots, \beta_k$

$$
\begin{aligned}
y_1 &= b_0 + b_1 x_{11} + b_2 x_{12} + \cdots + b_k x_{1k} + e_1 \\
y_2 &= b_0 + b_1 x_{21} + b_2 x_{22} + \cdots + b_k x_{2k} + e_2 \\
&\vdots \\
y_n &= b_0 + b_1 x_{n1} + b_2 x_{n2} + \cdots + b_k x_{nk} + e_n
\end{aligned}
$$

$$
X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & & x_{nk} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}
$$

Model Penuh

Penentuan *least square estimator* bagi $\beta_0, \beta_1, \ldots, \beta_k$

Model $\mathbf{y} = \mathrm{X}\mathbf{b} + \mathbf{e}$

Metode kuadrat terkecil meminimumkan jumlah kuadrat sisaan (*residual*) untuk mendapatkan penduga koefisien regresi $\mathrm{b}_0, \mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}$ yaitu meminimumkan

$$\mathbf{e}'\mathbf{e} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{e}_\mathrm{i}^2 = (\mathbf{y} - \mathrm{X}\mathbf{b})'(\mathbf{y} - \mathrm{X}\mathbf{b}) = y'y - y'Xb - (\mathrm{Xb})'\mathrm{y} + (\mathrm{Xb})'\mathrm{Xb}$$

$$= y'y - y'Xb - b'X'\mathrm{y} + b'X'\mathrm{Xb}$$

Model Penuh

Penentuan *least square estimator* (LSE) bagi $\beta_0, \beta_1, \dots, \beta_k$

$$\frac{\partial \mathbf{e}'\mathbf{e}}{\partial \mathbf{b}} = -2X'\mathbf{y} + (X'X)\mathbf{b} + (X'X)'\mathbf{b}$$

$$= -2X'\mathbf{y} + 2(X'X)\mathbf{b}$$

$$-2X'\mathbf{y} + 2(X'X)\mathbf{b} = \mathbf{0}$$

$$(X'X)\mathbf{b} = X'\mathbf{y}$$

$$\mathbf{b} = (X'X)^{-1}X'\mathbf{y}$$

Model Penuh

Contoh 1:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

$$\mathbf{y} = \begin{bmatrix} 50 \\ 40 \\ 52 \\ 47 \\ 65 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 5 & 2 \\ 1 & 10 & 2 \\ 1 & 20 & 3 \end{bmatrix} \qquad X'X = \begin{bmatrix} 5 & 41 & 9 \\ 41 & 551 & 96 \\ 9 & 96 & 19 \end{bmatrix} \qquad X'\mathbf{y} = \begin{bmatrix} 254 \\ 2280 \\ 483 \end{bmatrix}$$

Model Penuh

Contoh 1:

$$(X'X)^{-1} = \begin{bmatrix} 2.307551 & 0.1565378 & -1.88398 \\ 0.1565378 & 0.02578269 & -0.20442 \\ -1.88398 & -0.20442 & 1.977901 \end{bmatrix}$$

$$(X'X)^{-1}X'\mathbf{y} = \begin{bmatrix} 2.307551 & 0.1565378 & -1.88398 \\ 0.1565378 & 0.02578269 & -0.20442 \\ -1.88398 & -0.20442 & 1.977901 \end{bmatrix} \begin{bmatrix} 254 \\ 2280 \\ 483 \end{bmatrix}$$

$$\doteq \begin{bmatrix} 33.06 \\ -0.189 \\ 10.718 \end{bmatrix}$$

Model Penuh

Contoh 1:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

$$y = b_0 + b_1 x_1 + b_2 x_2 + e$$

$$y = 33.06 - 0.189 x_1 + 10.718 x_2 + e$$

Model Penuh

Contoh 2:

$$y = b_0 + b_1 x + e$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

Model Penuh

Contoh 2:

$$X'X = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \qquad X'\mathbf{y} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2} \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{bmatrix}$$

Model Penuh

Contoh 2:

$$\mathbf{b} = (X'X)^{-1}X'\mathbf{y}$$

$$= \frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$= \frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i \\ n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \end{bmatrix}$$

Model Penuh

Contoh 2:

$$y = b_0 + b_1 x + e$$

$$\frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\begin{bmatrix} \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i \\ n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \end{bmatrix}$$

$$b_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b_0 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Model Penuh

Teorema: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $r(X) = k+1$, $E[\boldsymbol{\epsilon}] = \mathbf{0}$, $V[\boldsymbol{\epsilon}] = \sigma^2 I$

maka penduga $\boldsymbol{\beta}$ adalah $\mathbf{b} = (X'X)^{-1}X'\mathbf{y}$

$$E(\mathbf{b}) = E[(X'X)^{-1}X'\mathbf{y}] = (X'X)^{-1}X'E(\mathbf{y}) = (X'X)^{-1}X'X\boldsymbol{\beta} = \boldsymbol{\beta}$$

$$\begin{aligned} \mathrm{var}(\mathbf{b}) &= \mathrm{var}[(X'X)^{-1}X'\mathbf{y} = (X'X)^{-1}X'[\mathrm{var}(\mathbf{y})][(X'X)^{-1}X']' \\ &= (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1} \\ &= \sigma^2 (X'X)^{-1}X'X(X'X)^{-1} \\ &= \sigma^2 (X'X)^{-1} \end{aligned}$$

$\boxed{\rightarrow \mathrm{var}(A\mathbf{y}) = A[\mathrm{var}(y)]A'}$

$\boxed{\mathbf{b} \text{ tidak berbias dan } \mathrm{var}(\mathbf{b}) = \sigma^2(X'X)^{-1}}$

Model Penuh

Teorema Gauss-Markoff:

$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $r(X) = k+1$, $E[\boldsymbol{\epsilon}] = \mathbf{0}$, $V[\boldsymbol{\epsilon}] = \sigma^2 I$

maka penduga $\boldsymbol{\beta}$ adalah $\mathbf{b} = (X'X)^{-1}X'\mathbf{y}$

$\mathbf{b}$ adalah **penduga tak bias linier terbaik** (BLUE) bagi $\boldsymbol{\beta}$

Teorema:

$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $r(X) = k+1$, $E[\boldsymbol{\epsilon}] = \mathbf{0}$, $V[\boldsymbol{\epsilon}] = \sigma^2 I$

bila $\mathbf{t}_{(k+1)\times 1}$ adalah vektor bilangan nyata tidak nol

maka BLUE $\mathbf{t}'\boldsymbol{\beta}$ adalah $\mathbf{t}'\mathbf{b}$ dimana $\mathbf{b}$ adalah penduga $\boldsymbol{\beta}$

Model Penuh

Contoh 3:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

$$y = 33.06 - 0.189x_1 + 10.718x_2 + e$$

$$\mathbf{t}' = [1 \quad 15 \quad 2.5]$$

$$\widehat{E[y]} = \mathbf{t}'\mathbf{b} = [1 \quad 15 \quad 2.5]\mathbf{b}$$

$$= b_0 + 15b_1 + 2.5b_2$$

$$\widehat{E[y]} = 33.06 - 0.189(15) + 10.718(2.5) = 57.02$$

Model Penuh

Latihan:

i	x	y
1	8	8
2	12	15
3	14	16
4	16	20
5	16	25
6	20	40

Berdasarkan data tersebut:

1. Tentukan vektor **y** dan matriks X.
2. Tentukan X'X, X'**y**, dan $(X'X)^{-1}$.
3. Tenukan penduga koefisien regresi, b_0 dan b_1, dengan $(X'X)^{-1}$ X'**y**.
4. Ada suatu vektor **t**'= [1 15], tentukan nilai harapan $\widehat{E[y]}$ = **t**'**b**.