STA1333
Pengantar Model Linear

Bentuk Kuadratik

- Bentuk Kuadrat
- Turunan suatu Fungsi
- Nilai Harapan suatu vektor
- Ragam Peubah Acak

Bentuk Kuadrat

Definisi bentuk kuadrat:

$$A_{kxk}, \underline{y}_{kx1} \Rightarrow q = \underline{y}' A \underline{y} \qquad q = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} y_i y_j \qquad \mathbf{y}_{kx1} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

$\underline{y}' A \underline{y}$ dikatakan **definit positif,** bila $\underline{y}' A \underline{y} > 0 \; \forall \; \underline{y} \neq \underline{0}$

$\underline{y}' A \underline{y}$ dikatakan **semidefinit positif,** bila $\underline{y}' A \underline{y} \geq 0 \forall \underline{y} \neq \underline{0} \wedge \exists | \underline{y}' A \underline{y} = \underline{0}$

Bentuk Kuadrat

Contoh:

$$\mathbf{y}_{\mathrm{kx1}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix} \qquad \mathbf{y}'A\mathbf{y} = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{y}'A\mathbf{y} = {} & a_{11}y_1y_1 + a_{21}y_1y_2 + a_{31}y_1y_3 + a_{12}y_1y_2 + a_{22}y_2y_2 \\ & + a_{32}y_2y_3 + a_{13}y_1y_3 + a_{23}y_2y_3 + a_{33}y_3y_3 \end{aligned}$$

Bentuk Kuadrat

Contoh:

$$\mathbf{y}'A\mathbf{y} = [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [2y_1 + y_2 + 4y_3 \quad 3y_1 + 2y_2 + 6y_3 \quad y_1 + 3y_3]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + y_1 y_2 + 4y_1 y_3 + 3y_1 y_2 + 2y_2^2 + 6y_2 y_3 + y_1 y_3 + 3y_3^2$$

$$= 2y_1^2 + 2y_2^2 + 3y_3^2 + 4y_1 y_2 + 5y_1 y_3 + 6y_2 y_3$$

Bentuk Kuadrat

Teorema:

$A' = A$ **definit positif** $\Longleftrightarrow$ $\boldsymbol{\lambda_i > 0, \forall \iota}$

$A' = A$ semi**definit positif** $\Longleftrightarrow$ $\boldsymbol{\lambda_i \geq 0} \wedge \exists | \lambda_i = 0$

A matriks definit positif disekat menjadi $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

dimana A_{11} dan A_{22} adalah matriks persegi.

Bila $B=A^{-1}$ yang disekat seperti A maka $A_{11}^{-1} = B_{11} - B_{12}\, B_{22}^{-1}\, B_{21}$

Bentuk Kuadrat

Contoh:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad (\boldsymbol{A} - \lambda \mathrm{I}) = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$$

$|A - \lambda I| = (2 - \lambda)^2 - 1$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 3 > 0$ dan $\lambda = 1 > 0$

A adalah matriks definit positif

Bentuk Kuadrat

Contoh:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{y}' A\mathbf{y} = [y_1 \quad y_2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 + 2y_2^2 - 2y_1 y_2$$

$$2y_1^2 + 2y_2^2 - 2y_1 y_2 = \mathbf{y}' A\mathbf{y} > 0$$

A adalah matriks definit positif

Bentuk Kuadrat

Contoh:

$$\boldsymbol{A} = \left[\begin{array}{cc|c} 2 & -1 & 2 \\ -1 & 2 & 1 \\ \hline 2 & 1 & 4 \end{array}\right] \qquad \boldsymbol{B} = \boldsymbol{A}^{-1} = -\frac{1}{2}\left[\begin{array}{cc|c} 7 & 6 & -5 \\ 6 & 2 & -4 \\ \hline -5 & -4 & 3 \end{array}\right]$$

$$\boldsymbol{A}_{11}^{-1} = \mathrm{B}_{11} - \mathrm{B}_{12}\,\mathrm{B}_{22}^{-1}\,\mathrm{B}_{21} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Bentuk Kuadrat

Latihan:

1. Diketahui matriks A berikut $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

   Tunjukan bahwa A adalah matriks definit positif,

   a. dengan menunjukkan $2y_1^2 + 2y_2^2 - 2y_1 y_2 = \mathbf{y}' A \mathbf{y} > 0$
   b. berdasarkan nilai akar ciri matriks A

2. Diketahui matriks B berikut $\mathrm{B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

   Apakah B adalah matriks definit positif atau semidefinit positif ?

Turunan suatu fungsi

Definisi:

Bila $\mathbf{y}_{kx1}$ dan z = f($\mathbf{y}$) maka yang dikatakan turunan partial adalah $\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}} = \begin{bmatrix} \frac{\partial \boldsymbol{z}}{\partial y_1} \\ \frac{\partial \boldsymbol{z}}{\partial y_2} \\ \vdots \\ \frac{\partial \boldsymbol{z}}{\partial y_k} \end{bmatrix}$

1. $z = \mathbf{a}'\mathbf{y} \Rightarrow \frac{\partial \mathbf{z}}{\partial \mathbf{y}} = \mathbf{a}$, bila **a** adalah vektor konstanta

2. $z = \mathbf{y}'\mathbf{y} \Rightarrow \frac{\partial \mathbf{z}}{\partial \mathbf{y}} = 2\mathbf{y}$

3. $z = \mathbf{y}'\mathrm{A}\mathbf{y} \Rightarrow \frac{\partial \mathbf{z}}{\partial \mathbf{y}} = \mathrm{A}\mathbf{y} + \mathrm{A}'\mathbf{y}$; A adalah matriks bilangan nyata

Turunan suatu fungsi

Contoh:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 3 \\ 2 & 3 & 2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \\ \mathrm{y}_3 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y}'\mathrm{A}\mathbf{y} = y_1^2 - 2y_1y_2 + y_2^2 + 4y_1y_3 + 2y_3^2 + 6y_2y_3$$

$$\begin{aligned} \partial z/\partial y_1 &= 2y_1 - 2y_2 + 4y_3 \\ \partial z/\partial y_2 &= 2y_2 - 2y_1 + 6y_3 \\ \partial z/\partial y_3 &= 4y_3 + 4y_1 + 6y_2 \end{aligned} \quad \Rightarrow \quad \partial z/\partial \mathbf{y} = \begin{bmatrix} 2y_1 - 2y_2 + 4y_3 \\ 2y_2 - 2y_1 + 6y_3 \\ 4y_3 + 4y_1 + 6y_2 \end{bmatrix}$$

Turunan suatu fungsi

Contoh:

$$A\mathbf{y} = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & -2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + 2y_3 \\ y_1 - y_2 - 2y_3 \\ 3y_1 + 4y_2 + 5y_3 \end{bmatrix}$$

$$A'\mathbf{y} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & -2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_2 + 3y_3 \\ -y_2 + 4y_3 \\ 2y_1 - 2y_2 + 5y_3 \end{bmatrix}$$

Turunan suatu fungsi

Contoh:

$$
A\mathbf{y} + A'\mathbf{y} = \begin{bmatrix} 2y_1 + 2y_3 \\ y_1 - y_2 - 2y_3 \\ 3y_1 + 4y_2 + 5y_3 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 + 3y_3 \\ -y_2 + 4y_3 \\ 2y_1 - 2y_2 + 5y_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4y_1 + y_2 + 5y_3 \\ y_1 - 2y_2 + 2y_3 \\ 5y_1 + 2y_2 + 10y_3 \end{bmatrix} = \partial z/\partial \mathbf{y}
$$

Turunan suatu fungsi

Contoh:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & -2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \\ \mathrm{y}_3 \end{bmatrix} \qquad \mathbf{z} = \mathbf{y}'\mathrm{A}\mathbf{y}$$

$$z = 2y_1^2 - y_2^2 + 5y_3^2 + y_1y_2 + 5y_1y_3 + 5y_2y_3$$

$$\partial z/\partial \mathbf{y} = \begin{bmatrix} 4y_1 + y_2 + 5y_3 \\ -2y_2 + y_1 + 2y_3 \\ 10y_3 + 5y_1 + 2y_2 \end{bmatrix}$$

Turunan suatu fungsi

Latihan:

1. Diketahui vektor **a** dan **y** berikut $\mathbf{a} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$ $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

   a. Tuliskan bentuk kuadrat $z = \mathbf{a}'\mathbf{y}$
   b. Tentukan turunan dari z, yaitu $\partial z/\partial y_1, \partial z/\partial y_2, \partial z/\partial y_3$
   c. Apakah turunan dari z sama dengan **a**

2. Diketahui matriks A dan z berikut

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & -2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \mathbf{z} = \mathbf{y}'\mathrm{A}\mathbf{y}$$

   Tentukan $\dfrac{\partial \boldsymbol{z}}{\partial \boldsymbol{y}}$

Nilai Harapan *(Expected Value)* suatu Vektor

Definisi:

$$\mathbf{y}_{\mathrm{kx1}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \qquad \mathrm{E}[\mathbf{y}] = \begin{bmatrix} \mathrm{E}[\mathrm{y}_1] \\ \mathrm{E}[\mathrm{y}_2] \\ \vdots \\ \mathrm{E}[\mathrm{y}_\mathrm{k}] \end{bmatrix}$$

1. $\mathrm{E}[\mathbf{a}] = \mathbf{a}$
2. $\mathrm{E}[\mathbf{a}'\mathbf{y}] = \mathbf{a}'\mathrm{E}[\mathbf{y}] = \mathbf{a}'\boldsymbol{\mu}$
3. $\mathrm{E}[A\mathbf{y}] = A\mathrm{E}[\mathbf{y}] = \mathrm{A}\boldsymbol{\mu}$

Nilai Harapan *(Expected Value)* suatu Vektor

Contoh:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \mathrm{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\mathrm{E}[Y_1] = 10 \text{ dan } \mathrm{E}[Y_2] = 20$$

$$\boldsymbol{\mu} = \begin{bmatrix} E[Y_1] \\ E[Y_2] \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$A\mathbf{Y} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2Y_1 + 3Y_2 \\ Y_1 + 4Y_2 \end{bmatrix} \qquad E[A\mathbf{Y}] = \begin{bmatrix} E[2Y_1 + 3Y_2] \\ E[Y_1 + 4Y_2] \end{bmatrix} = \begin{bmatrix} 80 \\ 90 \end{bmatrix}$$

$$A\boldsymbol{\mu} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 80 \\ 90 \end{bmatrix} = E[A\mathbf{Y}]$$

Nilai Harapan *(Expected Value)* suatu Vektor

Latihan:

Diketahui

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Tentukan $\mathrm{E}[\mathbf{a}'\mathbf{y}]$

Ragam (*Variance*) Peubah Acak

Definisi:

$$V(\mathbf{y}) = \mathrm{E}[(\mathbf{y}-\boldsymbol{\mu})(\mathbf{y}-\boldsymbol{\mu})'] = \mathrm{E}[\mathbf{y}\mathbf{y}'] - \boldsymbol{\mu}\,\boldsymbol{\mu}'$$

Rules:

1. $z = \mathbf{a}'\mathbf{y}$ dan V(**y**)=V $\Longrightarrow$ V(z) = V(**a’y**) = a’Va
   bila **a** adalah vektor konstanta

2. $\mathrm{z} = \mathrm{A}\mathbf{y}$ dan V(**y**)=V $\Longrightarrow$ V(z) = V(A**y**) = AVA’
   bila A adalah matriks konstanta

Teorema:

$$\mathbf{y}_{\mathrm{kx1}},\ \mathrm{E}[\mathbf{y}] = \boldsymbol{\mu}, \mathrm{V}(\mathbf{y}) = \mathrm{V} \Longrightarrow \mathrm{E}[\mathbf{y}'\mathrm{A}\mathbf{y}] = \mathbf{tr}(\mathrm{AV}) + \boldsymbol{\mu}'\mathrm{A}\boldsymbol{\mu}$$

Ragam (*Variance*) Peubah Acak

Contoh:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\mathbf{y}$ adalah suatu vektor acak dengan ragam σ^2 untuk i=1,2,3. Ketiga peubah acak, y_1 , y_2 , dan y_3 mempunyai ragam yang sama. Bila diasumsikan bahwa ketiga peubah acak itu bebas, sehingga matriks ragam-peragam V adalah

$$V = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} = \sigma^2 I$$

$$V(\mathbf{y}) = \sigma^2 I \text{ dan } \mathbf{z} = \underbrace{(X'X)^{-1}X'}_{A}\mathbf{y} \Rightarrow V(\mathbf{z}) = \sigma^2 (X'X)^{-1}$$

$$V(\mathbf{z}) = V(Ay) = AVA' = (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1} = \sigma^2 (X'X)^{-1}$$

Ragam (*Variance*) Peubah Acak

Contoh:

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathrm{V} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \qquad \mathrm{A} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{y}'\mathrm{A}\mathbf{y} = 4y_1^2 + 2y_1y_2 + 2y_2^2$$

$$\mathbf{E}(\mathbf{y}'\mathrm{A}\mathbf{y}) = 4\mathrm{E}[y_1^2] + 2E[y_1y_2] + 2E[y_2^2] = \mathbf{tr}(\mathrm{AV}) + \boldsymbol{\mu}'\mathrm{A}\boldsymbol{\mu}$$

$$2 = \mathrm{var}(y_1) = \mathrm{E}[y_1^2] - E[y_1]^2 = \mathrm{E}[y_1^2] - 1 \longrightarrow 2 = \mathrm{E}[y_1^2] - 1 \longrightarrow \mathrm{E}[y_1^2] = 3$$

$$5 = \mathrm{var}(y_2) = \mathrm{E}[y_2^2] - E[y_2]^2 = \mathrm{E}[y_1^2] - 9 \longrightarrow 5 = \mathrm{E}[y_1^2] - 9 \longrightarrow \mathrm{E}[y_1^2] = 14$$

$$1 = \mathrm{cov}(y_1, y_2) = \mathrm{E}[y_1, y_2] - \mathrm{E}[\mathrm{y}_1]\mathrm{E}[\mathrm{y}_2] \longrightarrow 1 = \mathrm{E}[y_1, y_2] - (1)(3)$$

$$\longrightarrow \mathrm{E}[y_1, y_2] = 1 + (1)(3) = 4$$

Ragam (*Variance*) Peubah Acak

Contoh:

$$\mathbf{E}(\mathbf{y}'\mathrm{A}\mathbf{y}) = 4\mathrm{E}[y_1^2] + 2E[y_1 y_2] + 2[y_2^2] = 4(3) + 2(4) + 2(14) = 48$$

$$\mathrm{AV} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 4 & 11 \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{tr}(\mathrm{AV}) = 20 \qquad \boldsymbol{\mu}'\mathrm{A}\boldsymbol{\mu} = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 28$$

$$\mathrm{E}[\mathbf{y}'\mathrm{A}\mathbf{y}] = \mathbf{tr}(\mathrm{AV}) + \boldsymbol{\mu}'\mathrm{A}\boldsymbol{\mu} = 20 + 28 = 48$$

Ragam (*Variance*) Peubah Acak

Latihan:

Diketahui

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$

Asumsikan bahwa σ_{ij}=0, i≠j, dan $\sigma_i^2 = 4$, untuk i=1,2,3

a. Tentukan ragam $\mathbf{y}$
b. Tentukan $E(\mathbf{y}' A \mathbf{y})$