STA1333
Pengantar Model Linear

ALJABAR MATRIKS

STK333 Pengantar Model Linear

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Pendahuluan

Model Regresi:

$$\underset{\substack{\uparrow\\ \text{Response}}}{y} = \underbrace{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3}_{\substack{\uparrow\\ \text{Explained}\\ \text{variation}}} + \underset{\substack{\uparrow\\ \text{Random}\\ \text{variation}}}{\varepsilon}$$

Pendahuluan

Model Regresi:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \qquad \boldsymbol{X} = \begin{pmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 1 & x_{12} & x_{22} & x_{32} \\ 1 & x_{13} & x_{23} & x_{33} \\ 1 & x_{14} & x_{24} & x_{34} \\ 1 & x_{15} & x_{25} & x_{35} \\ 1 & x_{16} & x_{26} & x_{36} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = ?$$

Pendahuluan

	y	x1	x2	x3
1	1.8	130	4	1
2	2.0	110	2	0
3	2.5	100	0	0
4	1.7	120	1	0
5	3.0	90	1	1
6	2.3	100	0	0

$$1.8 = \beta_0 + \beta_1(130) + \beta_2(4) + \beta_3(1) + \varepsilon_1$$
$$2.0 = \beta_0 + \beta_1(110) + \beta_2(2) + \beta_3(0) + \varepsilon_2$$
$$2.5 = \beta_0 + \beta_1(100) + \beta_2(0) + \beta_3(0) + \varepsilon_3$$
$$1.7 = \beta_0 + \beta_1(120) + \beta_2(1) + \beta_3(0) + \varepsilon_4$$
$$3.0 = \beta_0 + \beta_1(90) + \beta_2(1) + \beta_3(1) + \varepsilon_5$$
$$2.3 = \beta_0 + \beta_1(100) + \beta_2(0) + \beta_3(0) + \varepsilon_6$$

Pendahuluan

Model Regresi:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

$$\boldsymbol{y} = \begin{pmatrix} 1.8 \\ 2.0 \\ 2.5 \\ 1.7 \\ 3.0 \\ 2.3 \end{pmatrix} \quad \boldsymbol{X} = \begin{pmatrix} 1 & 130 & 4 & 1 \\ 1 & 110 & 2 & 0 \\ 1 & 100 & 0 & 0 \\ 1 & 120 & 1 & 0 \\ 1 & 90 & 1 & 1 \\ 1 & 100 & 0 & 0 \end{pmatrix} \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{y}$$

Definisi

Notasi: $X_{nxk} = (x_{ij})_{nxk}, Y_{nxk} = (y_{ij})_{nxk}$

$$X_{nxk} = \begin{pmatrix} x_{11} & x_{12} & . & . & x_{1k} \\ x_{21} & x_{22} & . & . & x_{2k} \\ . & . & . & . & . \\ . & . & . & . & . \\ x_{n1} & x_{n2} & . & . & x_{nk} \end{pmatrix}_{nxk} = \left(\underline{x}_1, \underline{x}_2, \ldots\ldots, \underline{x}_k\right)$$

Definisi

Contoh:

$$X = \begin{bmatrix} 2 & 7 & 8 \\ 1 & 1 & 0 \\ 0 & 4 & -5 \end{bmatrix}; \qquad Y = [2 \quad 3 \quad 7 \quad 8]; \qquad Z = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$$

Operasi Dasar Matriks

$$X_{nxk}, Y_{nxk}, X + Y = C, c_{ij} = x_{ij} + y_{ij},$$

$$X - Y = C, c_{ij} = x_{ij} - y_{ij}$$

$$kX = C, c_{ij} = kx_{ij}$$

$$X_{nxk} Y_{kxm} = C_{nxm}, c_{ij} = \sum_{s=1}^{k} x_{is} y_{sj}$$

Operasi Dasar Matriks

Properties:

$$X + Y = Y + X$$
$$X + (Y + Z) = (X + Y) + Z$$
$$X(YZ) = (XY)Z$$
$$(Y + X)Z = YZ + XZ$$
$$X(kY) = k(XY)$$
$$(a + b)X = aX + bX$$

$$a(X + Y) = aX + aY$$
$$X + 0 = 0 + X = X, 0 \text{ matriks nol}$$
$$Y + X = 0, X \text{ negatif matriks } Y$$
$$\text{Umumnya } XY \neq YX$$
$$\text{Matriks tersekat} : X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$$

Operasi Dasar Matriks

Contoh:

$$X = \begin{bmatrix} 2 & 3 & 1 \\ 8 & 0 & 4 \end{bmatrix}, \qquad Y = \begin{bmatrix} -1 & 0 & 6 \\ 7 & 2 & -3 \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 2 & 3 & 1 \\ 8 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 6 \\ 7 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 15 & 2 & 1 \end{bmatrix}$$

$$X - Y = \begin{bmatrix} 2 & 3 & 1 \\ 8 & 0 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 6 \\ 7 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -5 \\ 1 & -2 & 7 \end{bmatrix}$$

$$cX = 3\begin{bmatrix} 2 & 3 & 1 \\ 8 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 3 \\ 24 & 0 & 12 \end{bmatrix}$$

Putaran (*Transpose*) Matriks

$$X_{nxk}, X'_{kxn} \Leftrightarrow X' = (x_{ji})_{kxn}, X = (x_{ij})_{nxk}$$

Properties:

$$(cX)' = cX'$$

$$(X + Y)' = X' + Y'$$

$$(X - Y)' = X' - Y'$$

$$(X')' = X$$

$$(XY)' = Y'X'$$

Putaran Matriks

Teorema:

$$X_{nxk} \Rightarrow (X'X)_{kxk} = (X'X)'_{kxk}, \text{ unsur diag JK lainnya JHK}$$

$$\underline{x}_{kx1} \Rightarrow (\underline{x}'\underline{x})_{1x1} = c, \text{ konstanta}$$

$$\underline{x}_{kx1} \Rightarrow (\underline{x}\underline{x}')_{kxk} = (\underline{x}\underline{x}')_{kxk}$$

Definisi Simetrik :

bila $X' = X$, matrix X disebut matrik simetrik

Putaran Matriks

Contoh:

$$X = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 7 & 2 \end{bmatrix} \longrightarrow X' = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

Matriks Simetrik

Matriks A disebut simetrik, jika A' = A atau ($a_{ij} = a_{ji}$)

Semua matriks simetrik berbentuk segi (*square*)

Contoh:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 6 \\ 2 & 10 & -7 \\ 6 & -7 & 9 \end{pmatrix} \qquad \mathrm{diag}(\mathbf{A}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Matriks Identitas

$$I_{n x n} = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \end{pmatrix}$$

$$AI = IA = A$$

Kebalikan Matriks

$$\text{Y kebalikan dari X, bila } XY = YX = I,\ Y = X^{-1}$$

$$\textit{Bila}\ \text{Y ada} \rightarrow \text{X non-singular}$$

$$\text{Y tidak ada} \rightarrow \text{X singular}$$

Properties:

$$\text{X non-singular} \Rightarrow X^{-1}\text{non-singular},\ (X^{-1})^{-1} = X$$

$$\text{X non-singular} \Rightarrow (X')^{-1} = (X^{-1})'$$

$$\text{X dan Y non-singular, kompatibel} \Rightarrow (XY)^{-1} = Y^{-1}X^{-1}$$

Kebalikan Matriks

Contoh:

$$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow X^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

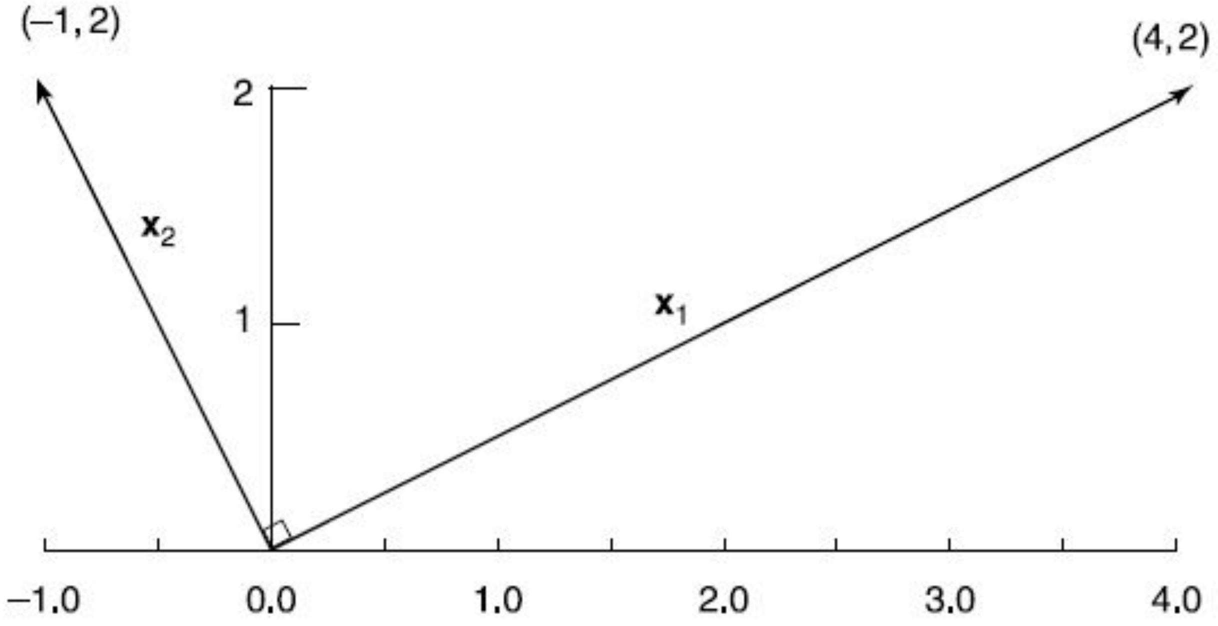
$$X = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \longrightarrow X^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & -1 \end{bmatrix}$$

Matriks Ortogonal

$\underline{x}_{nx1}$ dan $\underline{y}_{nx1}$ orthogonal bila $\underline{x}'\underline{y} = 0$

X_{kxk}, $X'X = I \Rightarrow X$ orthogonal

Contoh:



$\mathbf{x}_2 = (-1, 2)$

$\mathbf{x}_1 = (4, 2)$

Matriks Ortogonal

panjang vektor $\underline{x}_{kx1}$ atau $|\underline{x}| = \sqrt{\underline{x}'\underline{x}} = \sqrt{\sum_{i=1}^{k} x_i^2}$

$\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$ adalah satu set vektor orthogonal ukuran nx1.

bila $\underline{x}_i'\underline{x}_i = 1, \forall i \Rightarrow \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$ orthonormal

Matriks Ortogonal

Contoh:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \qquad \mathrm{X} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Akar Ciri (*Eigen value*) Matriks

$A_{kxk}, \underline{x} \neq \underline{0}$, akar ciri A adalah λ

bila $A\underline{x} = \lambda\underline{x}$ dan $\underline{x}$ adalah vektor ciri

atau $(A - \lambda I)\underline{x} = \underline{0} \Rightarrow |A - \lambda I| = 0$ karena $\underline{x} \neq \underline{0}$

Akar Ciri (*Eigen value*) Matriks

Ingat!!!!:

1. $|X|$ adalah determinant X
2. $|X| = |X'|$
3. Bila X dan Y conformable $|XY| = |X||Y|$

Akar Ciri (*Eigen value*) Matriks

Properties:

$$A'_{kxk} = A_{kxk} \Rightarrow \lambda_i \text{ real/nyata } \forall i$$

$$A_{kxk}, C_{kxk} \text{orthogonal } C'C = I \Rightarrow \text{akar ciri } A = \text{akar ciri } C'AC$$

$$A'_{kxk} = A_{kxk}, \lambda_i \neq \lambda_{i'} \Rightarrow \underline{x}_i \perp \underline{x}_{i'}$$

Teorema:

$$A' = A_{kxk} \; \exists | P, P'P = I \ni P'AP = D(\lambda_i), i = 1,2 \ldots k$$

Akar Ciri (*Eigen value*) Matriks

Contoh:

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad |A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (1-\lambda)(4-\lambda) - (-2) \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

$\lambda_1 = 2$ and $\lambda_2 = 3$

Pangkat (*Rank*) Matriks

Definisi:

$$\{\underline{x}_1, \underline{x}_2, \ldots\ldots, \underline{x}_k\}, \exists | a_i \neq 0, i = 1,2,\ldots,k \ni \sum_{i=1}^{k} a_i \underline{x}_i = \underline{0} \Rightarrow \{\underline{x}_1, \underline{x}_2, \ldots\ldots, \underline{x}_k\}$$

tidak bebas linear (*linearly dependent*)

Bila tidak ada a_i tsb,

$\{\underline{x}_1, \underline{x}_2, \ldots\ldots, \underline{x}_k\}$ bebas linear (*linearly independent*)

Pangkat (*Rank*) Matriks

Properties:

$$X_{nxk}, n \geq k, r(X) = k \Rightarrow r(X) = r(X') = r(X'X) = k$$

$$X_{kxk}, \Rightarrow X^{-1} \text{ ada jjk } r(X) = k$$

$$X_{nxk}, \left|P_{nxn}\right| \neq 0, \left|Q_{kxk}\right| \neq 0 \Rightarrow r(X) = r(PX) = r(XQ)$$

$$r(D) \text{ adalah banyaknya elemen diagonal non-zero}$$

$$r(XY) \leq r(X), r(XY) \leq r(Y)$$

Pangkat (*Rank*) Matriks

Contoh:

$$X = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$$

$$\mathbf{x}_3 = \tfrac{1}{2}\mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}_2$$

$$r(X) = 2.$$

Matriks Idempoten

Definisi:

$$A_{kxk} \text{ idempoten bila } AA = A^2 = A$$

Contoh:

$$X_{nxk}, n \geq k, r(X) = k \Rightarrow H = X(X'X)^{-1}X' \text{ idempoten}$$

Teras (*Trace*) Matriks

Definisi:

$$X_{kxk} \text{ maka teras } X \text{ adalah } tr(X) = \sum_{i=1}^{k} x_{ii}$$

Properties:

$$X_{kxk}, c \text{ konstanta} \Rightarrow tr(cX) = c\, tr(X)$$

$$X_{kxk}, Y_{kxk} \Rightarrow tr(X+Y) = tr(X) + tr(Y),$$
$$tr(X-Y) = tr(X) - tr(Y)$$

$$X_{nxp}, Y_{pxn} \Rightarrow tr(XY) = tr(YX)$$

Teras (*Trace*) Matriks

Contoh:

$$X = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\operatorname{tr}(X) = 2 + 1 + (-1) = 2$$

$$\operatorname{tr}(Y) = (-1) + 4 + 3 = 6$$