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TUGAS 1

STA1333 PENGANTAR MODEL LINEAR

1. Apakah Matriks X berikut *full rank*? Apakah Matriks X non singular?

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Matriks tersebut tidak full rank karena ukurannya 5x3. Sedangkan rank paling besar yang mungkin adalah 3. Sehingga Matriks X tersebut tidak mungkin full rank. Matriks X juga bukanlah matriks non singular. Karena matriks tersebut tidak memiliki determinan. Sehingga matriks X tidak mungkin memiliki invers. Maka matriks X adalah matriks singular.

2. Tentukan invers dari

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

1	0	0	0	1	0	0	0	
1/4	1	0	0	0	1	0	0	X2 4
1/3	1/3	1	0	0	0	1	0	X3 3
1/2	1/2	1/2	1	0	0	0	1	X4 2
1	0	0	0	1	0	0	0	
1	4	0	0	0	4	0	0	X2 - X1
1	1	3	0	0	0	3	0	X3 - X2
1	1	1	2	0	0	0	2	X4 - X3
1	0	0	0	1	0	0	0	
0	4	0	0	-1	4	0	0	X2 1/4
0	-3	3	0	0	-4	3	0	X3 + X2 3/4
0	0	-2	2	0	0	-3	2	X4 1/2
1	0	0	0	1	0	0	0	
0	1	0	0	- 1/4	1	0	0	
0	0	3	0	- 3/4	-1	3	0	X3 1/3
0	0	-1	1	0	0	-1 1/2	1	X4 + X3 1/3
1	0	0	0	1	0	0	0	
0	1	0	0	- 1/4	1	0	0	
0	0	1	0	- 1/4	- 1/3	1	0	
0	0	0	1	- 1/4	- 1/3	- 1/2	1	

3. Jika $\boldsymbol{y} = (y_1 \quad y_2 \quad y_3)'$ merupakan vektor acak dengan nilai tengah $\boldsymbol{\mu} = (3 \quad 2 \quad 1)'$. Asumsikan bahwa $\sigma_{ij} = 0$, $i \neq j$, dan $\sigma^2 = 4$, $i = 1,2,3$.
   a. Tentukan ragam $\boldsymbol{y}$
   b. Misalkan $A = \begin{bmatrix} 5 & 3 & -3 \\ 4 & -2 & 1 \\ -1 & 0 & 2 \end{bmatrix}$, tentukan E($\mathbf{y}'\mathbf{A}\mathbf{y}$)

a.) $\rightarrow \sigma_{ij} = 0$, $i \neq j$, $\sigma^2 = 4$, $i = 1,2,3$

ragam y $= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ //

b.) $E(y'Ay) = tr(AV) + \mu' A \mu$

$\rightarrow AV = \begin{bmatrix} 5 & 3 & -3 \\ 4 & -2 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 20 & 12 & -12 \\ 16 & -8 & 4 \\ -4 & 0 & 8 \end{bmatrix}$

$tr(AV) = 20$

$\rightarrow \mu' A \mu = (3 \; 2 \; 1) \begin{bmatrix} 5 & 3 & -3 \\ 4 & -2 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$= (15+8-1 \quad 9-4 \quad -9+2+2) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$= (22 \quad 5 \quad -5) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = (66 + 10 - 5) = 71$

$E(y'Ay) = tr(AV) + \mu' A \mu$

$= 20 + 71$

$= 91$ //

✗ Cara Panjang :

$E(y'Ay) = E\left[(y_1 \; y_2 \; y_3) \begin{pmatrix} 5 & 3 & -3 \\ 4 & -2 & 1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right]$

$= E\left[(5y_1 + 4y_2 - y_3 \quad 3y_1 - 2y_2 \quad -3y_1 + y_2 + 2y_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right]$

$= E\left[5y_1^2 - 2y_2^2 + 2y_3^2 + 7y_1y_2 - 4y_1y_2 + y_2y_3 \right]$

$\rightarrow Var(y_i) = \sigma_i^2 = E[y_i^2] - \mu_i^2$

$E[y_i^2] = \sigma_i^2 + \mu_i^2$

$\rightarrow E[y_1^2] = 4 + (3)^2 = 13$

$E[y_2^2] = 4 + (2)^2 = 8$

$E[y_3^2] = 4 + (1)^2 = 5$

$\rightarrow Cov(y_i y_j) = \sigma_i \sigma_j = E(y_i y_j) - \mu_i \mu_j$

$E(y_i y_j) = \sigma_i \sigma_j + \mu_i \mu_j$

$\rightarrow E[y_1 y_2] = 0 + 6$

$E[y_1 y_3] = 0 + 3$

$E[y_2 y_3] = 0 + 2$

$E(y'Ay) = 5(13) - 2(8) + 2(5) + 7(6) - 4(3) + 2$

$= 91$ //

4. Misal (X,Y) mempunyai joint probability density $f(x,y) = x + y, 0 \le x \le 1, 0 \le y \le 1$, definisikan $U = \begin{bmatrix} X \\ Y \end{bmatrix}$, tentukan nilai $E(U)$

$E(U) = E\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} E(X) \\ E(Y) \end{bmatrix}$

$\rightarrow E(X) = \int_0^1\int_0^1 x(x+y)\,dy\,dx$

$= \int_0^1\int_0^1 x^2 + xy \, dy\,dx$

$= \int_0^1 \left(x^2 y\Big|_0^1 + \left(\frac{xy^2}{2}\right)\Big|_0^1 \right) dx$

$= \int_0^1 x^2 + \frac{x}{2}\, dx$

$= \left(\frac{1}{3}x^3 + \frac{1}{4}x^2 \right)\Big|_0^1$

$= \frac{1}{3} + \frac{1}{4}$

$= \frac{4+3}{12} = \frac{7}{12}$

$\rightarrow E(Y) = \int_0^1\int_0^1 y(x+y)\,dy\,dx$

$= \int_0^1\int_0^1 xy + y^2 \, dy\,dx$

$= \int_0^1 \left(\frac{1}{2}xy^2 + \frac{1}{3}y^3 \Big|_0^1 \right) dx$

$= \int_0^1 \frac{1}{2}x + \frac{1}{3}\, dx$

$= \left(\frac{1}{4}x^2 + \frac{1}{3}x \right)\Big|_0^1$

$= \frac{1}{4} + \frac{1}{3}$

$= \frac{7}{12}$

$\rightarrow E(U) = E\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} E(X) \\ E(Y) \end{bmatrix} = \begin{bmatrix} 7/12 \\ 7/12 \end{bmatrix}$ //

5. Misalkan $\mathbf{x} = (x_1 \quad x_2)'$ menyebar $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ dengan $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

$Q_1 = (x_1 - x_2)^2$

$Q_2 = (x_1 + x_2)^2$

a. Tunjukkan bahwa $\frac{Q_1}{2(1-\rho)} \sim \chi^2$

b. Periksa apakah Q_1 dan Q_2 saling bebas?

a.) $Q_1 = (x_1 - x_2)^2$

$= x_1^2 - 2x_1x_2 + x_2^2$

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

misal : $P = \frac{Q_1}{2(1-\rho)}$

$= \frac{x_1^2 - 2x_1x_2 + x_2^2}{2(1-\rho)}$

$A^* = \frac{1}{2(1-\rho)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = A^{*\prime}$ (simetrik) ✓

$\rightarrow A^* V = A^* \Sigma$

$= \frac{1}{2(1-\rho)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$= \frac{1}{2(1-\rho)} \begin{bmatrix} 1-\rho & \rho-1 \\ \rho-1 & 1-\rho \end{bmatrix}$

$\rightarrow$ rank $(A^*V) \rightarrow \begin{vmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{vmatrix}$

$= \frac{1}{4} - (-\frac{1}{4})$

$= \frac{1}{2} \neq 0$

Rank $(A^*V) = 2$ ✓

$\to (A^*V)(A^*V) = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$= \frac{1}{2}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{4}\begin{pmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{pmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$ (Idempoten) ✓

Jadi $\frac{Q1}{2(1-\rho)} \sim \chi^2$ //

b.) $Q_1 = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = A'$ (simetrik) ✓

$Q_2 = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$

$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = B'$ (simetrik) ✓

$AVB = A\Sigma B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{pmatrix} 1-\rho & \rho-1 \\ \rho-1 & 1-\rho \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1-\rho+\rho-1 & 1-\rho+\rho-1 \\ \rho-1+1-\rho & \rho-1+\rho-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ ✓

Jadi Q_1 dan Q_2 saling bebas