IPB University — Bogor Indonesia | Departemen Statistika

Bentuk Kuadratik

Responsi 2 STA1333 Pengantar Model Linear

1. Diketahui matriks A sebagai berikut

$$A = \begin{bmatrix} 3 & 1 & 8 \\ 1 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix}$$

Tentukan $z = y'Ay$ dan $\frac{\partial z}{\partial y}$.

Jawab:

$$z = y'Ay = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 8 \\ 1 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2$$

$$z = 3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2$$

$$\frac{\partial z}{\partial y} = \begin{bmatrix} \dfrac{\partial z}{\partial y_1} \\ \dfrac{\partial z}{\partial y_2} \\ \dfrac{\partial z}{\partial y_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial(3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2)}{\partial y_1} \\ \dfrac{\partial(3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2)}{\ } \\ \dfrac{\partial(3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2)}{\partial y_3} \end{bmatrix}$$

$$\frac{\partial z}{\partial y} = \begin{bmatrix} 6y_1 + 2y_2 + 10y_3 \\ 2y_1 + 2y_3 \\ 10y_1 + 2y_2 - 8y_3 \end{bmatrix}$$

2. Jika $\mathbf{y} = (y_1 \quad y_2 \quad y_3)'$ merupakan vektor acak dengan nilai tengah $\boldsymbol{\mu} = (1 \quad 3 \quad 2)'$. Asumsikan bahwa $\sigma_{ij} = 0, i \neq j$, dan $\sigma_i^2 = 4$, $i = 1,2,3.$

a. Tentukan ragam $\mathbf{y}$

b. Misalkan $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$, tentukan $E(\mathbf{y}'\mathbf{A}\mathbf{y})$

Jawab:

Ragam y = $V = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$, tentukan E[y'Ay]

Teorema

Diketahui $\underline{y}$ adalah vektor acak berukuran kx1 dengan $E[\underline{y}] = \mu$ dan var($\underline{y}$)=V. misal A dalah matriks berukuran kxk, maka

$$E[y'Ay] = \mathrm{tr}(AV) + \mu' A \mu$$

sehingga

$$AV = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 4 \\ 4 & 8 & 0 \\ -4 & 24 & 4 \end{bmatrix}$$ dan tr(AV)=20

$$AV = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 4 \\ 4 & 8 & 0 \\ -4 & 24 & 4 \end{bmatrix}$$ dan tr(AV)=20

$$\mu' A \mu = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 54$$

sehingga

$$E[y'Ay] = tr(AV) + \mu' A \mu = 20 + 54 = 74$$

Cara lain :

$$y'Ay = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 - 2y_1y_2 + 2y_2^2 + 6y_2y_3 + y_3^2$$

$$E[y'Ay] = E[2y_1^2 - 2y_1y_2 + 2y_2^2 + 6y_2y_3 + y_3^2]$$

$$= 2E[y_1^2] - 2E[y_1y_2] + 2E[y_2^2] + 6E[y_2y_3] + E[y_3^2]$$

$$\text{var}\ (yi) = \sigma_i^2 = E\left[y_i^2\right] - \mu_i^2 \ \rightarrow\ E\left[y_i^2\right] = \sigma_i^2 + \mu_i^2$$

Untuk i=1 maka $E[y_1^2] = 4 + 1 = 5$

Untuk i=2 maka $E[y_2^2] = 4 + 9 = 13$

Untuk i=3 maka $E[y_3^2] = 4 + 4 = 8$

$$cov\left(y_i\, y_j\right) = \sigma_i\, \sigma_j = E\ \left[y_i\, y_j\right] - \mu_i \mu_j \ \rightarrow\ E\left[y_i\, y_j\right] = \sigma_i\, \sigma_j + \mu_i \mu_j$$

Untuk i=1 dan j=2 maka $E[y_1 y_2] = 0 + (1)(3) = 3$

Untuk i=2 dan j=3 maka $E[y_2 y_3] = 0 + (3)(2) = 6$

$$
\begin{aligned}
E[y'Ay] &= E[2y_1^2 - 2y_1y_2 + 2y_2^2 + 6y_2y_3 + y_3^2] \\
&= 2E[y_1^2] - 2E[y_1y_2] + 2E[y_2^2] + 6E[y_2y_3] + E[y_3^2] \\
&= 2(5) - 2(3) + 2(13) + 6(6) + 8 \\
&= 74
\end{aligned}
$$

Terima Kasih

IPB University
Bogor Indonesia

Departemen Statistika
Fakultas Matematika dan Ilmu Pengetahuan Alam
Jl Meranti Wing 22 Level 4
Kampus IPB Darmaga - Bogor 16680
0251-8624535 | http://stat.ipb.ac.id