Konsep Matriks dalam Analisis Peubah Ganda

Pertemuan 1 – STA1342

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Komponen Penilaian Praktikum

STA1342

- **KEAKTIFAN** 10%
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Konsep Matriks Dalam TPG

- Teras
- Determinan
- Operasi (+, -, x)
- Inverse
- Transpose
- Akar Ciri dan Vektor Ciri
- Matriks Koragam
- Matriks Korelasi

Teras Matriks

$$tr(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}$$

Contoh:

$$tr\left(\begin{bmatrix} 5 & 3 & 5 \\ 4 & -1 & 2 \\ -3 & 8 & 7 \end{bmatrix}\right) = 5 - 1 + 7 = 11$$

Transpose/Putaran Matriks

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix} \to \boldsymbol{A}^t = \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Determinan Matriks

Determinan matriks merupakan selisih antara perkalian elemen-elemen pada diagonal utama dengan perkalian elemen-elemen pada diagonal sekunder. Determinan dari matriks A dapat ditulis det(A) atau |A|.

Determinan Matriks Persegi Ordo 2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

Invers Matriks

Matriks invers dari suatu matriks A adalah matriks B yang apabila dikalikan dengan matriks A memberikan satuan I

Notasi matriks invers : A^{-1}

Matriks invers 2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Invers Matriks - Contoh

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\det(A) = (1)(2) - (3)(4) = -10$$

$$A^{-1} = \frac{1}{-10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$$

Perkalian Matriks

Syarat perkalian matriks:

Banyaknya kolom matriks pertama = Banyaknya baris matriks kedua

$$A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$$

Perkalian Matriks - Contoh

Contoh:

$$\boldsymbol{A}_{1\times 3} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$\boldsymbol{B}_{3\times 1} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A} \times \boldsymbol{B} = [(3 \times 3) + (2 \times 1) + (1 \times 0)] = [11]$$

$$\boldsymbol{B} \times \boldsymbol{A} = \begin{bmatrix} (3 \times 3) & (3 \times 2) & (3 \times 1) \\ (1 \times 3) & (1 \times 2) & (1 \times 1) \\ (0 \times 3) & (0 \times 2) & (0 \times 1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Mencari Matriks Korelasi dan Matriks Kovarian

Misalkan matriks **X** memiliki matriks kovarian **Σ** berikut:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Maka untuk mencari nilai ρ_{ij} dalam matriks **ρ** sebagai berikut:

$$\rho_{ij} = \frac{cov(x_i, x_j)}{\sqrt{var(x_i)\, var(x_j)}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

$$\boldsymbol{\rho} = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} & \frac{\sigma_{13}}{\sqrt{\sigma_{11}\sigma_{33}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}\sigma_{22}}} & \frac{\sigma_{23}}{\sqrt{\sigma_{22}\sigma_{33}}} \\ \frac{\sigma_{31}}{\sqrt{\sigma_{33}\sigma_{11}}} & \frac{\sigma_{32}}{\sqrt{\sigma_{33}\sigma_{22}}} & \frac{\sigma_{33}}{\sqrt{\sigma_{33}\sigma_{33}}} \end{bmatrix}$$

Mencari Matriks Korelasi dan Matriks Kovarian - Contoh

Diketahui

$$\boldsymbol{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

Maka dengan rumus sebelumnya, diperoleh matriks **ρ** sebagai berikut::

$$\boldsymbol{\rho} = \begin{bmatrix} \frac{25}{\sqrt{25 \times 25}} & \frac{-2}{\sqrt{25 \times 4}} & \frac{4}{\sqrt{25 \times 9}} \\ \frac{-2}{\sqrt{4 \times 25}} & \frac{4}{\sqrt{4 \times 4}} & \frac{1}{\sqrt{4 \times 9}} \\ \frac{4}{\sqrt{9 \times 25}} & \frac{1}{\sqrt{9 \times 4}} & \frac{9}{\sqrt{9 \times 9}} \end{bmatrix} = \begin{bmatrix} \frac{25}{25} & \frac{-2}{10} & \frac{4}{15} \\ \frac{-2}{10} & \frac{4}{4} & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & \frac{9}{9} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.2 & 0.267 \\ -0.2 & 1 & 0.167 \\ 0.267 & 0.167 & 1 \end{bmatrix}$$

Mencari Simpangan Baku dari Matriks Kovarian

Misal **X** memiliki matriks kovarian yakni $\boldsymbol{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$

Maka tentukan $\boldsymbol{V}^{1/2}$

Jawab:

Untuk mencari matriks $\boldsymbol{V}^{1/2}$ digunakan persamaan sebagai berikut : $\boldsymbol{V}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix}$

$$\boldsymbol{V}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} \sqrt{25} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{9} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Mencari Akar Ciri dan Vektor Ciri

Sebuah matriks **A** dikatakan ***positif definit*** jika $\boldsymbol{x}^t\boldsymbol{A}\boldsymbol{x} > 0$ untuk sembarang vektor $\boldsymbol{x} \neq 0$.

Karena vektor **x** tidak diketahui, maka untuk menunjukkan **A** merupakan definit positif maka dicari **vektor ciri**nya terlebih dahulu menggunakan persamaan berikut:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

Mencari Akar Ciri dan Vektor Ciri - Contoh

Diketahui:

$$\boldsymbol{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

$$\left|\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right| = 0$$

$$[(9-\lambda)(6-\lambda) - (-2)(-2)] = 0$$

$$[54 - 9\lambda - 6\lambda + \lambda^2 - 4] = 0$$

$$\lambda^2 - 15\lambda + 50 = 0$$

$$(\lambda - 10)(\lambda - 5) = 0$$

Diperoleh:

$$\lambda_1 = 10$$

$$\lambda_2 = 5$$

Akar Ciri

Mencari Akar Ciri dan Vektor Ciri - Contoh

$$\lambda_1 = 10$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

$$\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right) \overset{B_{2+1(-2)}}{\sim} \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-x_1 - 2x_2 = 0$$

$$-x_1 = 2x_2$$

$$misal\ x_2 = s$$

$$x_1 = -2s$$

$$\boldsymbol{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Jadi, vektor ciri dari akar ciri $\lambda_1 = 10$ adalah $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

A definit positif dibuktikan dengan:

$$\boldsymbol{x}^t\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} -2 & 1 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -20 & 10 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 50 > 0$$

Karena $\boldsymbol{x}^t\boldsymbol{A}\boldsymbol{x} = 50 > 0$ maka terbukti bahwa **A** merupakan definit positif.

Mencari Akar Ciri dan Vektor Ciri - Contoh

$$\lambda_2 = 5$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

$$\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) \overset{B_{2+1}\left(\frac{1}{2}\right)}{\sim} \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$4x_1 - 2x_2 = 0$$

$$4x_1 = 2x_2$$

$$x_1 = \frac{1}{2}x_2$$

$$misal\ x_1 = t$$

$$x_2 = 2t$$

$$\boldsymbol{x} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Jadi, vektor ciri dari akar ciri $\lambda_2 = 5$ adalah $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A definit positif dibuktikan dengan:

$$\boldsymbol{x}^t\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 25 > 0$$

Karena $\boldsymbol{x}^t\boldsymbol{A}\boldsymbol{x} = 25 > 0$ maka terbukti bahwa **A** merupakan definit positif.

Latihan Soal 1

Diketahui matriks A, B, dan C berikut

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a. Tentukan akar ciri dan vektor ciri dari matriks **A**, **B**, dan **C**.
b. Apakah matriks **A**, **B**, dan **C** merupakan matriks definit positif?

Jawaban Soal 1

a. Tentukan akar ciri dan vektor ciri dari matriks **A**, **B**, dan **C**.

✓ Untuk matriks $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

akar ciri akan diperoleh dengan menyelesaikan $|\boldsymbol{A} - \lambda\mathbf{I}| = 0$

$$\left|\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}\right| = 0$$

$$(2-\lambda)(4-\lambda)(3-\lambda) = 0$$

Maka, akar ciri untuk matriks A:

$2-\lambda = 0$	$4-\lambda = 0$	$3-\lambda = 0$
$\lambda_1 = 2$	$\lambda_2 = 4$	$\lambda_3 = 3$

Jawaban Soal 1

Vektor ciri akan diperoleh dengan menyelesaikan persamaan $\boldsymbol{Ax} = \lambda \boldsymbol{x}$

- $\lambda_1 = 2$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 2x_1$
$4x_2 = 2x_2$ → diperoleh $x_1 = t, t \in R , x_2 = 0 \; dan \; x_3 = 0$ sehingga
$3x_3 = 2x_3$

$$\mathbf{x} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t \text{ , t} \in R$$

vektor ciri untuk $\lambda_1 = 2$ adalah :

$$\boldsymbol{e} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x'x}}}$$

$$\boldsymbol{e} = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Jawaban Soal 1

- $\lambda_2 = 4$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 4x_1$
$4x_2 = 4x_2$ → diperoleh $x_1 = 0$, $x_2 = t$, $t \in R$ dan $x_3 = 0$ sehingga
$3x_3 = 4x_3$

$$\mathbf{x} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t \text{ , t} \in R$$

vektor ciri untuk $\lambda_2 = 4$ adalah :

$$\boldsymbol{e} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}}$$

$$\boldsymbol{e} = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Jawaban Soal 1

- $\lambda_3 = 3$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 3x_1$
$4x_2 = 3x_2$ → diperoleh $x_1 = 0$, $x_2 = 0$ dan $x_3 = t, t \in R$ sehingga
$3x_3 = 3x_3$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \text{ , t} \in R$$

vektor ciri untuk $\lambda_3 = 3$ adalah :

$$e = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x}'\boldsymbol{x}}}$$

$$e = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Jawaban Soal 1

✓ Untuk matriks $\mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Akar ciri akan diperoleh dengan menyelesaikan $|\square - \lambda\mathbf{I}| = 0$

$$\left| \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \right| = 0$$

$(2-\lambda)(4-\lambda)(-\lambda) = 0$

Maka, nilai ciri untuk matriks **B**:

$2 - \lambda = 0 \quad 4 - \lambda = 0 \quad -\lambda = 0$

$\lambda_1 = 2 \quad \lambda_2 = 4 \quad \lambda_3 = 0$

Maka, akar ciri untuk matriks **B**:

$2 - \lambda = 0 \quad 4 - \lambda = 0 \quad -\lambda = 0$

$\lambda_1 = 2 \quad \lambda_2 = 4 \quad \lambda_3 = 0$

Jawaban Soal 1

Vektor ciri akan diperoleh dengan menyelesaikan persamaan $\mathbf{B}\boldsymbol{x} = \lambda\boldsymbol{x}$

- $\lambda_1 = 2$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 2x_1$

$4x_2 = 2x_2$ → diperoleh $x_1 = t, t \in R$, $x_2 = 0$ dan $x_3 = 0$ sehingga

$0 = 2x_3$

$$\mathbf{x} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t \text{ , t} \in R$$

vector ciri untuk $\lambda_2 = 2$ adalah :

$$\boldsymbol{e} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x}'\boldsymbol{x}}}$$

$$\boldsymbol{e} = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Jawaban Soal 1

- $\lambda_2 = 4$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 4x_1$

$4x_2 = 4x_2$ → diperoleh $x_1 = 0\,, x_2 = t\,, t \in R$ dan $x_3 = 0$ sehingga

$0 = 4x_3$

$$\mathbf{x} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t\,, \mathrm{t} \in R$$

vector ciri untuk $\lambda_2 = 4$ adalah :

$$\boldsymbol{e} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x'x}}}$$

$$\boldsymbol{e} = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Jawaban Soal 1

- $\lambda_3 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$2x_1 = 0$
$4x_2 = 0$ → diperoleh $x_1 = 0$, $x_2 = 0$ dan $x_3 = t, t \in R$ sehingga
$0 \;\; = 0$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \text{ , t} \in R$$

vector ciri untuk $\lambda_3 = 3$ adalah :

$$\boldsymbol{e} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}}$$

$$\boldsymbol{e} = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Jawaban Soal 1

✓ Untuk matriks $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Nilai ciri akan diperoleh dengan menyelesaikan $|C - \lambda\mathbf{I}| = 0$

$$\left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \right| = 0$$

$$(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

Maka, akar ciri untuk matriks **C**:

$1 - \lambda = 0$ $\quad 1 - \lambda = 0$ $\quad 1 - \lambda = 0$

$\lambda_1 = 1$ $\quad \lambda_2 = 1$ $\quad \lambda_3 = 1$

Jawaban Soal 1

Vektor ciri akan diperoleh dengan menyelesaikan persamaan $\boldsymbol{Cx} = \lambda\boldsymbol{x}$

- $\lambda_1 = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

→ diperoleh $x_1 = t, x_2 = t \; dan \; x_3 = t, t \in R$ sehingga

$$\mathbf{x} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t \text{ , t} \in R$$

Jawaban Soal 1

vector ciri untuk $\lambda_1 = \lambda_2 = \lambda_3 = 1$ adalah :

$$e = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{x}'\boldsymbol{x}}}$$

$$e = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Jawaban Soal 1

b. Apakah matriks A, B, dan C merupakan matriks definit positif?

- ✓ untuk matriks $\boldsymbol{A}$: $\lambda_1.\lambda_2.\lambda_3 = 2.4.3 > 0$, maka matriks **A** definit positif
- ✓ untuk matriks $\boldsymbol{B}$: $\lambda_1.\lambda_2.\lambda_3 = 2.4.0 \geq 0$, maka matriks **B** semi definit positif
- ✓ untuk matriks $\boldsymbol{C}$: $\lambda_1.\lambda_2.\lambda_3 = 1.1.1 > 0$, maka matriks **C** definit positif

Latihan Soal 2

Diketahui $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ adalah matriks simetris. Tunjukkan bahwa determinan

A adalah sama dengan akar ciri pertama kali akar ciri kedua, atau

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \lambda_1 \lambda_2$$

Jawaban Soal 2

Diketahui: $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Mencari akar ciri dari matriks **A**:

$$|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

Jawaban Soal 2

Bentuk ini analog dengan rumus abc untuk persamaan kuadrat $a\lambda^2 + b\lambda + c = 0$, *dimana*

$$\lambda_1\lambda_2 = \frac{c}{a}$$

Sehingga didapat:

$$\lambda_1\lambda_2 = \frac{a_{11}a_{22} - a_{12}a_{21}}{1} = a_{11}a_{22} - a_{12}a_{21}$$

- Mencari determinan dari matriks **A**:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$\therefore$ terbukti bahwa $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \lambda_1\lambda_2$.

Latihan Soal 3

Suatu matriks ragam peragam $\sum = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}$

a. Tentukan ρ_{13}
b. Tentukan korelasi X1 dan ½ X2+ ½ X3

Jawaban Soal 3

Suatu matriks ragam peragam $\sum = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}$

$$\rho_{13} = \frac{\sigma_{13}}{\sigma_1 \sigma_3} = \frac{4}{\sqrt{25}\sqrt{9}} = \frac{4}{(5 * 3)} = \frac{4}{15} = 0.27$$

Jawaban Soal 3

Suatu matriks ragam peragam $\sum = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}$

Misal,

$$Y = X_1$$

$$Z = \frac{1}{2}X_2 + \frac{1}{2}X_3$$

Maka untuk mencari simpangan baku dari Z,

$$\sigma_{zz} = Var\left(\frac{1}{2}X2 + \frac{1}{2}X3\right)$$

$$= \frac{1}{4}Var(X2) + \frac{1}{4}Var(X3) + 2\left(\frac{1}{2}\right)Cov(X2, X3)\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}(4) + \frac{1}{4}(9) + \frac{1}{2}(1) = \frac{15}{4} = 3.75$$

$$\sigma_z = \sqrt{\sigma_{zz}} = 1.94$$

Jawaban Soal 3

Dan kovarian dari Y dan Z diperoleh dengan

$$
\begin{aligned}
\sigma_{yz} &= Var\ (Y, Z) = COV\left(X1, \frac{1}{2}X2 + \frac{1}{2}X3\right) \\
&= Cov\left(X1, \frac{1}{2}X2\right) + Cov\left(X1, \frac{1}{2}X3\right) \\
&= \frac{1}{2}Cov(X1, X2) + \frac{1}{2}Cov(X1, X3) \\
&= \frac{1}{2}(-2) + \frac{1}{2}(4) \\
&= -1 + 2 \\
&= 1
\end{aligned}
$$

Sehingga didapatkan

$$
\rho_{yz} = \frac{\sigma_{yz}}{\sqrt{\sigma_{yy}\sigma_{zz}}} = \frac{\sigma_{yz}}{\sigma_y \sigma_z}
$$

$$
\rho_{yz} = \frac{1}{5(1.94)}
$$