Normal Random Number

STK473 – Praktikum 4

Normal Random Variable

PDF random variables `X ~ Normal (μ,σ²)`

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty < x < \infty$$

We need to generate n random number $x_1, x_2, \dots, x_n$ that normally distributed.

Uniform (0,1) $\longrightarrow$ Normal (0,1)

We have
3 ways

Method 1: Central Limit Theorem Approach

$$U_1, U_2, \ldots, U_n \sim U(0,1)$$

$$N = \sum_{i=1}^{n} U_i \sim_{n\to\infty} N(\mu, \sigma^2)$$

- So how generate $N(0,1)$ from $U_1, U_2, \ldots, U_n \sim U(0,1)$?
- How many n we need?

Uniform(a, b)

pdf	$f(x\|a, b) = \frac{1}{b-a}, \quad a \le x \le b$
mean and variance	$\mathrm{E}X = \frac{b+a}{2}, \quad \mathrm{Var}\,X = \frac{(b-a)^2}{12}$
mgf	$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

Normal(μ, σ^2)

pdf	$f(x\|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$
mean and variance	$\mathrm{E}X = \mu, \quad \mathrm{Var}\,X = \sigma^2$
mgf	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$

$$E(U_i) = \frac{a+b}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$Var(U_i) = \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12} = \frac{1}{12}$$

with

$E(N) = \mu$ and $Var(N) = \sigma^2$

So suppose $N = \sum_{i=1}^{n} U_i + k$

$$Var(N) = Var\left(\sum_{i=1}^{n} U_i + k\right) = \sum_{i=1}^{n} Var(U_i) = \frac{n}{12} = \sigma^2 \qquad n = 12\sigma^2$$

$$E(N) = E\left(\sum_{i=1}^{n} U_i + k\right) = \sum_{i=1}^{n} E(U_i) + k = \frac{n}{2} + k = \mu \qquad k = \mu - \frac{n}{2}$$

so

$N \sim N(0,1)$

$$N = \sum_{i=1}^{12} U_i - 6$$

$$Y = \sqrt{10}\, N + 5$$

$Y \sim N(5,10)$

$$Y = \sum_{i=1}^{120} U_i - 55$$

Application in R

```
i<-1000
mu<-0
sigma2<-1
n<-12*sigma2
k<-mu-n/2

U<-runif(n*i)
Um<-matrix(U,i)
N<-apply(Um,1,sum)+k

mean(N); var(N); hist(N)
```

```
i<-1000
mu<-5
sigma2<-10
n<-12*sigma2
k<-mu-n/2

U<-runif(n*i)
Um<-matrix(U,i)
N<-apply(Um,1,sum)+k

mean(N); var(N); hist(N)
```

Application in R

```
i<-1000
mu<-0
sigma2<-1
n<-12*sigma2
k<-mu-n/2

U<-runif(n*i)
Um<-matrix(U,i)
N<-apply(Um,1,sum)+k

Y<-sqrt(10)*N+5
mean(Y); var(Y); hist(Y)
```

$$N = a\left(\sum_{i=1}^{n} U_i - b\right)$$

$$Var(N) = a^2 \frac{n}{12} = \sigma^2 \qquad n = \frac{12\sigma^2}{a^2}$$

$$E(N) = a\left(\frac{n}{2} - b\right) = \mu \qquad b = \frac{n}{2} - \frac{\mu}{a}$$

```
i<-1000
mu<-40
var<-15
a<-3
n<-12*var/a^2
b<-n/2-mu/a

U<-runif(n*i)
Um<-matrix(U,i)
N<-a*(apply(Um,1,sum)-b)

mean(N); var(N); hist(N)
```

Method 2:
Box-Muller Method

$$U_1, U_2 \sim^{id} U(0,1)$$

So

$$N_1 = (-2lnU_1)^{1/2}\cos(2\pi U_2)$$
$$N_2 = (-2lnU_1)^{1/2}\sin(2\pi U_2)$$

Finally we have N_1 & $N_2 \sim^{id} N(0,1)$

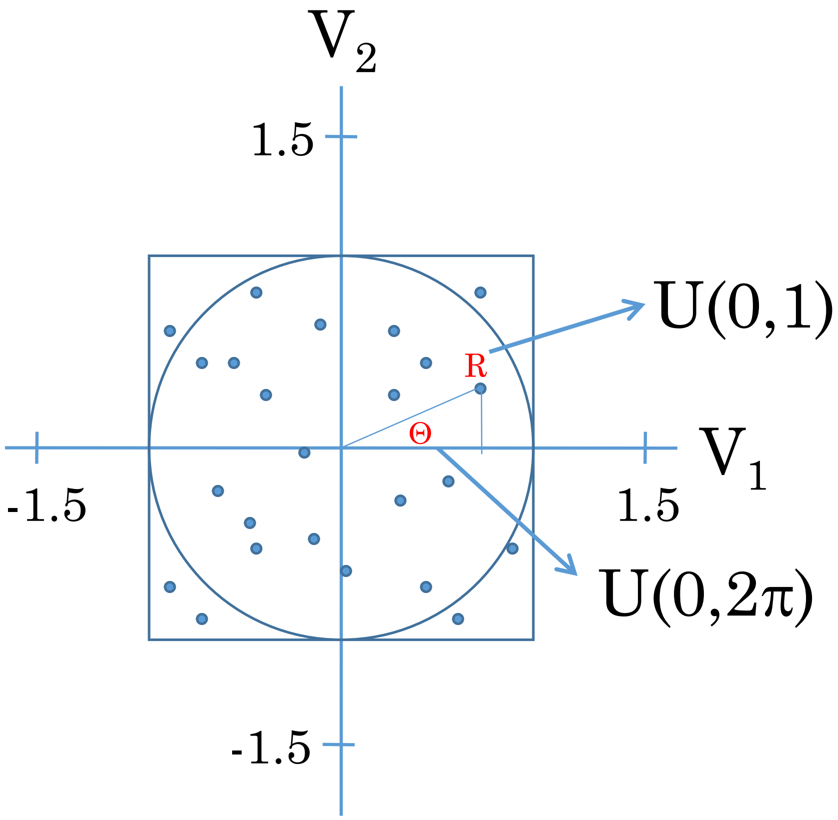
- It can be written:
  - $N_1 = R \cos \Theta$
  - $N_2 = R \sin \Theta$
- with
  - $R = (-2 \log_e U_1)^{1/2}$ and
  - $\Theta = 2\pi U_2 \sim U(0,2\pi)$

- $(N1, N2) \leftrightarrow (R, \Theta)$
- Cartesius Coordinate$\leftrightarrow$ polar coordinate

Application in R

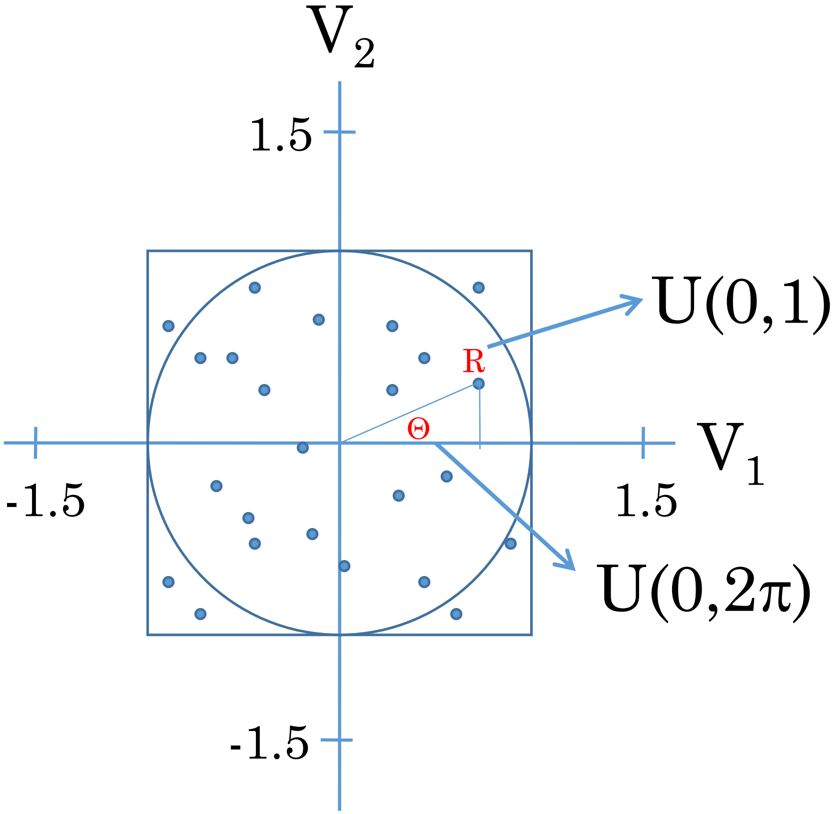
```
i<-1000
U1<-runif(i)
U2<-runif(i)
R<-sqrt(-2*log(U1))
Theta<-2*pi*U2
N1<-R*cos(Theta)
N2<-R*sin(Theta)
mean(N1); var(N1); hist(N1)
mean(N2); var(N2); hist(N2)
```

Method 3:
Polar Marsaglia



- if U ~ U(0,1)
  - 2U ~ U(0,2)
  - V = (2U – 1) ~ U(-1,1)

- V_1,V_2 ~ U(-1,1)
  - $R^2 = V_1^2 + V_2^2$
  - $\tan \Theta = V_2/V_1$



- from Box-Muller
  - $N_1 = (-2 \log_e U_1)^{1/2} \cos (2\pi U_2)$
  - $N_2 = (-2 \log_e U_1)^{1/2} \sin (2\pi U_2)$

R

$\sin \Theta = V_2/R$

$\cos \Theta = V_1/R$

$V_1(V_1^2+V_2^2)^{-1/2}$

$V_2(V_1^2+V_2^2)^{-1/2}$

- $N_1 = (-2 \log_e U_1)^{1/2} \cos (2\pi U_2)$
- $N_2 = (-2 \log_e U_1)^{1/2} \sin (2\pi U_2)$

- $N_1 = (-2 \log_e R^2)^{1/2} V_1(V_1^2+V_2^2)^{-1/2}$
- $N_2 = (-2 \log_e R^2)^{1/2} V_2(V_1^2+V_2^2)^{-1/2}$

- $N_1 = (-2 \log_e (V_1^2+V_2^2))^{1/2} V_1(V_1^2+V_2^2)^{-1/2}$
- $N_2 = (-2 \log_e (V_1^2+V_2^2))^{1/2} V_2(V_1^2+V_2^2)^{-1/2}$

- $N_1 = V_1\{(-2 \log_e W)/W\}^{1/2}$
- $N_2 = V_2\{(-2 \log_e W)/W\}^{1/2}$

- with $W = V_1^2+V_2^2$

$$V_1, V_2 \sim^{iid} U(-1,1)$$

- So

$$N_1 = V_1 \left[\frac{-2lnW}{W}\right]^{1/2}$$

$$N_2 = V_2 \left[\frac{-2lnW}{W}\right]^{1/2}$$

- With $W = V_1^2 + V_2^2$
- Finally we have N_1 & $N_2 \sim^{iid} N(0,1)$
- Simulation???

Application in R

```
i<-1000
U1<-runif(i)
U2<-runif(i)
V1<-2*U1-1
V2<-2*U2-1
W<-V1^2+V2^2

W1<-W[W<1]
V1<-V1[W<1]
V2<-V2[W<1]

i_1<-i-length(W1)
while (i_1>0) {
  U_1<-runif(i_1)
  U_2<-runif(i_1)
  V_1<-2*U_1-1
  V_2<-2*U_2-1
  W_1<-V_1^2+V_2^2
  W1<-c(W1,W_1[W_1<1])
  V1<-c(V1,V_1[W_1<1])
  V2<-c(V2,V_2[W_1<1])
  i_1<-i-length(W1)
}

N1<-V1*sqrt(-2*log(W1)/W1)
N2<-V2*sqrt(-2*log(W1)/W1)

mean(N1);var(N1);hist(N1)
mean(N2);var(N2);hist(N2)
```

thank you!

MRJ