Algoritma *Steepest Descent* (Cauchy)

$x^{(1)}$

1. Menentukan nilai awal $\boldsymbol{X}_1$, dan nomor urut iterasi $i = 1$
2. Menentukan arah $\boldsymbol{S}_i$, yaitu: $\boldsymbol{S}_i = -\nabla f(\boldsymbol{X}_i)$ $S^{(1)}$
3. Menentukan panjang langkah optimal λ_i^* pada arah $\boldsymbol{S}_i$ dan
$$\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + \lambda_i^* \boldsymbol{S}_i = \boldsymbol{X}_i - \lambda_i^* \nabla f_i$$
4. Memeriksa keoptimalan pada titik $\boldsymbol{X}_{i+1}$, jika sudah optimum maka proses berhenti di sini. Jika tidak, lanjutkan ke langkah 5.
5. Menetapkan nomor urut iterasi $i = i + 1$, dan kembali ke langkah 2.

Ilustrasi Perhitungan Manual

Contoh 6.8 di Rao (2019)

Minimumkan $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1 x_2 + x_2^2$, dengan nilai awal $\boldsymbol{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

Iterasi 1 $\quad i=1$

$$X^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\hookrightarrow \nabla f = \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}$$

$$S_1 = -\nabla f(X^{(1)})$$
$$= -\nabla f(0,0)$$
$$= -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$f(X^{(1)} + \lambda_1 S_1) = f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$
$$= f(x_1 = -\lambda_1, \; x_2 = \lambda_1)$$
$$= -\lambda_1 - \lambda_1 + 2\lambda_1^2 - 2\lambda_1^2 + \lambda_1^2$$
$$= \lambda_1^2 - 2\lambda_1$$

$$\frac{\partial f(X^{(1)} + \lambda_1 S_1)}{\partial \lambda_1} = \frac{\partial(\lambda_1^2 - 2\lambda_1)}{\partial \lambda_1} = 2\lambda_1 - 2 = 0$$
$$\Leftrightarrow \quad \lambda_1^* = 1$$

$$X^{(2)} = X^{(1)} + \lambda_1^* S_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\nabla f_2 = 0 \; ? \rightarrow \nabla f_2 = \nabla f(X^{(2)})$$
$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \neq 0$$

$\hookrightarrow$ belum optimal

iterasi ke-2

$$X^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S_2 = -\nabla f_2 = -\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f\left(X^{(2)} + \lambda_2 S_2\right) = f\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= f\left(x_1 = -1+\lambda_2 \ , \ x_2 = 1+\lambda_2\right)$$

$$= 5\lambda_2^2 - 2\lambda_2 - 1$$

$$\frac{\partial (5\lambda_2^2 - 2\lambda_2 - 1)}{\partial \lambda_2} = 10\lambda_2 - 2 = 0 \Leftrightarrow \lambda_2^* = \frac{1}{5}$$

$$X^{(3)} = X^{(2)} + \lambda_2^* S_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 1.2 \end{pmatrix}$$

$$\nabla f_3 = \nabla f\left(X^{(3)}\right) = \begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix} \neq 0$$

$\hookrightarrow$ $X^{(3)}$ bukan titik optimum.

$\hookrightarrow$ # iterasi ke-3.

$$X^{(3)} = \begin{pmatrix} -0.8 \\ 1.2 \end{pmatrix} \rightarrow S_3 = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix} \rightarrow \lambda_3^* = 1$$

$$X^{(4)} = X^{(3)} + \lambda_3^* S_3 = \begin{pmatrix} -1 \\ 1.4 \end{pmatrix}$$

$$\nabla f_4 = \nabla f\left(X^{(4)}\right) = \begin{pmatrix} -0.2 \\ -0.2 \end{pmatrix} \neq 0$$

Titik Optimum

$\hookrightarrow x^* = \begin{pmatrix} -1.0 \\ 1.5 \end{pmatrix}$

Kriteria konvergensi.

1. $$\left| \frac{f(x^{(i+1)}) - f(x^{(i)})}{f(x^{(i)})} \right| \leq \varepsilon_1$$

2. $$\left| \frac{\partial f}{\partial x^{(i)}} \right| \leq \varepsilon_2$$

3. $$\left| x^{(i+1)} - x^{(i)} \right| \leq \varepsilon_3$$

$$\left.\begin{aligned} x^{(4)} &= \begin{pmatrix} -1.0 \\ 1.4 \end{pmatrix} \\ x^{(3)} &= \begin{pmatrix} -0.8 \\ 1.2 \end{pmatrix} \end{aligned}\right\} \left| x^{(4)} - x^{(3)} \right| = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix}$$

Algoritma Gradien Konjugat (Fletcher-Reeves)

$\rightarrow X^{(1)}$

1. Menentukan nilai awal $\boldsymbol{X}_1$, dan nilai urutan iterasi $i = 1$
2. Menentukan arah awal $\boldsymbol{S}_1 = -\nabla f(\boldsymbol{X}_1) = -\nabla f_1$
3. Menentukan nilai $\boldsymbol{X}_2$ berdasarkan persamaan berikut:

$$\boldsymbol{X}_2 = \boldsymbol{X}_1 + \lambda_1^* \boldsymbol{S}_1$$

dengan λ_1^* adalah panjang langkah optimal pada arah $\boldsymbol{S}_1$. Selanjutnya tetapkan nilai $i = 2$ dan lanjutkan ke langkah berikutnya.

4. Menentukan $\nabla f_i = \nabla f(\boldsymbol{X}_i)$, dan

$$\boldsymbol{S}_i = -\nabla f_i + \frac{|\nabla f_i|^2}{|\nabla f_{i-1}|^2} \boldsymbol{S}_{i-1}$$

5. Menghitung panjang langkah optimal λ_i^* pada arah $\boldsymbol{S}_i$, dan menentukan titik baru

$$\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + \lambda_i^* \boldsymbol{S}_i$$

6. Memeriksa keoptimalan pada titik $\boldsymbol{X}_{i+1}$, jika sudah optimum maka proses berhenti di sini. Jika tidak, kembali ke langkah 4.

Ilustrasi Perhitungan Manual

Contoh 6.9 pada Rao (2019)

Minimumkan $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$, dengan nilai awal $\boldsymbol{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

$$f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2 \qquad \nabla f = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}$$

$$X^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$i = 1 \qquad S_1 = -\nabla f(X^{(1)}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \nabla f_1 = \nabla f(X^{(1)}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$X^{(1)} + \lambda_1 S_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \\ \lambda_1 \end{pmatrix}$$

$$f(X^{(1)} + \lambda_1 S_1) = f(x_1 = \lambda_1, x_2 = -\lambda_1) = \lambda_1^2 - 2\lambda_1$$

$$\frac{\partial(\lambda_1^2 - 2\lambda_1)}{\partial \lambda_1} = 2\lambda_1 - 2 = 0 \iff \lambda_1^* = 1$$

$$X^{(2)} = X^{(1)} + \lambda_1^* S_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \\ \lambda_1 \end{pmatrix}$$

iterasi ke - 2

$$S_2 = -\nabla f_2 + \frac{|\nabla f_2|^2}{|\nabla f_1|^2} S_1 \qquad \nabla f_2 = \nabla f(X^{(2)}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$= -\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{(-1)^2 + (-1)^2}{(1)^2 + (-1)^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$f\left(X^{(2)} + \lambda_2 S_2\right)$$

$$X^{(2)} + \lambda_2 S_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1+2\lambda_2 \end{pmatrix}$$

$$f\left(x_1 = -1,\ x_2 = 1 + 2\lambda_2\right) = 4\lambda_2^2 - 2\lambda_2 - 1$$

$$\frac{\partial\left(4\lambda_2^2 - 2\lambda_2 - 1\right)}{\partial \lambda_2} = 8\lambda_2 - 2 = 0 \Leftrightarrow \lambda_2^* = \frac{1}{4}$$

$$X^{(3)} = X^{(2)} + \lambda_2^* S_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$$

$$\nabla f_3 = \nabla f\left(X^{(3)}\right) = \begin{pmatrix} 1 + 4(-1) + 2(1.5) \\ -1 + 2(-1) + 2(1.5) \end{pmatrix}$$

$$\boxed{\nabla f = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 - 4 + 3 \\ -1 - 2 + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \text{sudah optimal.}$$

dengan 3 iterasi $\rightarrow X^{(3)} = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$