Homework 01

Due Date: 111/02/23

Instruction. Do not submit part B.

A Homework Problems

1. Let V, W be finite dimensional vector spaces over a field $\mathbb{F}$ and $T\colon V \to W$ be a linear transformation. Let $T^t\colon W^* \to V^*$ be the dual (transpose) of T. Show that

   (a) T is injective if and only if T^t is surjective.

   (b) T is surjective if and only if T^t is injective.

   Remark. These results hold also for infinite-dimensional vector spaces.

2. Let V be the vector space over $\mathbb{C}$ consisting of all infinite sequences $(a_1, a_2, a_3, \dots)$ **with only finitely many nonzero entries**.[1] Let e_i be the sequence with only one nonzero entry being 1 at i-th position. It follows that $\beta := \{e_1, e_2, \dots\}$ is a basis for V. Show that, however, $\beta^* = \{e_1^*, e_2^*, \dots\}$ is **NOT** a basis for V^*.

   (*Hint.* Let $f \in V^*$ defined by $f(a_1, a_2, \dots) = \sum_{i=1}^{\infty} a_i$. Explain that f is well-defined and show that $f \notin \operatorname{span} \beta^*$. Note that in a vector space, "span" only collects finite linear combinations.)

   Remark. If $\dim V = \infty$, one can show that $\dim V^* > \dim V$.

3. Let V_1, V_2 be two subspaces of a vector space V. Show that there is an isomorphism

   $$\varphi\colon \frac{V_1}{V_1 \cap V_2} \to \frac{V_1 + V_2}{V_2}.$$

   (*Hint.* Construct a linear transformation from V_1 to $(V_1 + V_2)/V_2$ and use theorem in class.)

[1] The addition and the scalar multiplication are the component-wise ones as usual.

B Supplementary Problems.

4. Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows:

$$f_1(x,y,z) = x - 2y, \quad f_2(x,y,z) = x + y + z \quad \text{and} \quad f_3(x,y,z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ for V such that the dual basis is given by $\mathcal{B}^* = \{f_1, f_2, f_3\}$.

5. Let V_1, V_2 be two (finite dimensional) vector spaces over a field $\mathbb{F}$. Suppose $\psi_i \colon V_i \to V_i^{**}$ are the canonical embeddings given by

$$\begin{aligned} \psi_i \colon \quad V_i &\to \quad V_i^{**} \\ v &\mapsto [\widehat{v} \colon f \mapsto f(v)] \end{aligned}$$

for $i = 1, 2$. Do the following.

(a) Let $T \in \mathcal{L}(V_1, V_2)$ be a linear transformation from V_1 to V_2. Illustrate the linear transformation $T^{tt} \colon V_1^{**} \to V_2^{**}$ (That is, show that how it acts on V_1^{**})

(b) Show for all linear transformations $T \in \mathcal{L}(V_1, V_2)$, we have the following identity

$$\psi_2 \circ T = T^{tt} \circ \psi_1.$$

That is, the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\psi_1"] & V_2 \arrow[d, "\psi_2"] \\ V_1^{**} \arrow[r, "T^{tt}"] & V_2^{**} \end{tikzcd}$$

6. Let V denotes a finite-dimensional vector space over $\mathbb{F}$. For every subset S of V, define the **annihilator** S^0 of S as

$$S^0 = \{f \in V^* \mid f(x) = 0 \text{ for all } x \in S\}.$$

(a) Prove that S^0 is a subspace of V provided S is a subset of V.

(b) If W is a subspace of V, show that

$$\dim W + \dim W^0 = \dim V.$$

(c) Deduce form the previous problem that $W \simeq W^{00}$. (*Hint.* Build the isomorphism via the canonical isomorphism $\psi \colon V \to V^{**}$)

(d) If W is a subspace of V, show that $(V/W)^* \simeq W^0$.

(e) Suppose $V = W_1 \oplus W_2$, show that $V^* = W_1^0 \oplus W_2^0$.

Definition B.1. An *exact sequence* is a diagram

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

such that U, V, W are vector spaces over $\mathbb{F}$, and f, g are linear transformations such that

- f is injective.
- g is surjective.
- $\operatorname{Im} f = \operatorname{Ker} g$.

7. Assume
$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$
is an exact sequence. Prove that the sequence
$$0 \longrightarrow W^* \xrightarrow{g^t} V^* \xrightarrow{f^t} U^* \longrightarrow 0$$
is also exact.

8. Let $f\colon V \to W$ be a linear map. Denote
$$\begin{array}{rccc} \overline{f}\colon & V/\operatorname{Ker} f & \to & W \\ & v + \operatorname{Ker} f & \mapsto & f(v) \end{array}$$
the linear map induced by the universal property of quotient space. Prove that the sequence
$$0 \longrightarrow V/\operatorname{Ker} f \xrightarrow{\overline{f}} W \xrightarrow{\pi} W/\operatorname{Im} f \longrightarrow 0$$
is exact, where π is the quotient map
$$\begin{array}{rccc} \pi\colon & W & \to & W/\operatorname{Im} f \\ & w & \mapsto & w + \operatorname{Im} f. \end{array}$$

9. Use Problem 7 and Problem 8 to show that the sequence
$$0 \longrightarrow (W/\operatorname{Im} f)^* \xrightarrow{\pi^t} W^* \xrightarrow{\overline{f}^t} (V/\operatorname{Ker} f)^* \longrightarrow 0$$
is exact. Deduce that $\dim \operatorname{Im}(f^t) = \dim \operatorname{Im}(f)$ if V, W are finite-dimensional. This gives a categorical proof that the row-rank and column-rank of a matrix coincide. (*Hint.* Denote $p\colon V \to V/\operatorname{Ker} f$ the quotient map. Then $f = \overline{f} \circ p$ and so $f^t = p^t \circ \overline{f}^t$. Note that p is surjective, so p^t is injective. Hence $\dim \operatorname{Im}(f^t) = \dim \operatorname{Im}(p^t \circ \overline{f}^t) = \dim \operatorname{Im}(\overline{f}^t)$. Now use the exact sequence to see that $\dim \operatorname{Im}(\overline{f}^t) = \dim \operatorname{Im}(f)$).)

Definition B.2. Let I be any set and $\mathbb{F}$ a field. Define

$$\prod_{i\in I}\mathbb{F} = \{\text{all functions } I \to \mathbb{F}\},$$

a vector space over $\mathbb{F}$ with valuewise addition and scalar multiplication, and

$$\bigoplus_{i\in I}\mathbb{F} = \left\{ f \in \prod_{i\in I}\mathbb{F} : f(i) = 0 \text{ for all but finitely many } i \in I \right\},$$

a subspace of $\prod_{i\in I}\mathbb{F}$.

10. For $i \in I$, define

$$\varepsilon_i : I \to \mathbb{F}, \quad \varepsilon_i(j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases} \quad j \in I.$$

Show that $\{\varepsilon_i\}_{i\in I}$ is a basis for $\bigoplus_{i\in I}\mathbb{F}$.

11. Define

$$\Phi : \left(\bigoplus_{i\in I}\mathbb{F}\right)^* \to \prod_{i\in I}\mathbb{F}, \quad \Phi(f)(i) = f(\varepsilon_i), \quad f \in \left(\bigoplus_{i\in I}\mathbb{F}\right)^*, \quad i \in I.$$

Prove that Φ is an isomorphism.

12. Assume I is infinite, so there is an injection

$$\mathbb{N} \to I, \quad n \mapsto i_n.$$

For $c \in \mathbb{F}$, define $f_c \in \prod_{i\in I}\mathbb{F}$ by

$$f_c : I \to \mathbb{F}, \quad f_c(i) = \begin{cases} c^n & \text{if } i = i_n \text{ for some } n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\left\{ f_c \in \prod_{i\in I}\mathbb{F} : c \in \mathbb{F} \text{ and } c \neq 0 \right\}$$

is a linearly independent subset of $\prod_{i\in I}\mathbb{F}$. (Hint: Use the Vandermonde matrix.)

13. Use the fact (B.3) below to show that $\dim(\prod_{i\in I}\mathbb{F}) > \dim(\bigoplus_{i\in I}\mathbb{F})$ if I is infinite. Deduce that $\dim V^* > \dim V$ if V is an infinite-dimensional vector space. (*Hint*. Note that we have $\dim(\prod_{i\in I}\mathbb{F}) = |\prod_{i\in I}\mathbb{F}|$ by (12) and (B.3), and $|\prod_{i\in I}\mathbb{F}| = |\mathbb{F}^I| \geq |\{0,1\}^I| > |I|$ by Cantor's theorem, where for any sets A and B, we denote B^A the set of all functions from A to B.)

Theorem B.3. Let V be an infinite dimensional vector space over $\mathbb{F}$ such that $\dim_{\mathbb{F}} V \geq |\mathbb{F}|$. Then $\dim_{\mathbb{F}} V = |V|$.