Homework 09

Due Date: 111/05/04

Instruction. Do not submit part B.

A Homework Problems

1. For the following quadratic form K on $\mathbb{R}^2$, find a symmetric bilinear form H on $\mathbb{R}^2$ such that $K(x) = H(x,x)$ for all $x \in \mathbb{R}^2$. Also, find an orthonormal basis β for $\mathbb{R}^2$ (equipped with standard inner product) such that $\psi_\beta(H)$ is a diagonal matrix.
$$K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -2t_1^2 + 4t_1t_2 + t_2^2.$$

2. Let V be a finite-dimensional vector space over a field F, $\dim V = n$ and H_1, H_2 be two bilinear forms on V. We say H_1 and H_2 are **equivalent** on V if there exists bases β_1, β_2 of V such that $\psi_{\beta_1}(H_1) = \psi_{\beta_2}(H_2)$. Under this definition, the set $\mathcal{B}(V)$ of all bilinear forms on V is partitioned into several **equivalent classes**.
   (a) When $F = \mathbb{R}$, determine the number of equivalent classes of symmetric bilinear forms on V. (This is a translation of exercise 26, which is explained in teacher's lecture. Still, you need to give out all details.)
   (b) When $F = \mathbb{C}$, determine the number of equivalent classes of symmetric bilinear forms on V. (*Hint.* First show that every symmetric bilinear form on V has a matrix representation using only **non-negative** numbers as entries.)

   Definition A.1. A bilinear form H on V is called **skew-symmetric** if $H(x,y) = -H(y,x)$ for all $x, y \in V$.

   (c) Let F be a field with $\operatorname{char} F \neq 2$. Use the following theorem A.2 to determine the number of equivalent classes of skew-symmetric bilinear forms on V.

Theorem A.2. Let F be a field with $\operatorname{char} F \neq 2$, V be a finite dimensional vector space over F, $\dim V = n$, and H be a skew-symmetric bilinear form on V. Then there exists a basis β of V such that the matrix representation $\psi_\beta(H)$ takes the form
$$\begin{pmatrix} 0_k & I_k & 0_{k\times(n-2k)} \\ -I_k & 0_k & 0_{k\times(n-2k)} \\ 0_{(n-2k)\times k} & 0_{(n-2k)\times k} & 0_{n-2k} \end{pmatrix}$$

3. Let (V,H) be a quadratic space over F with $\operatorname{char} F \neq 2$. For a non-isotropic vector $v \in V$, define the **reflection** $R_v : (V,H) \to (V,H)$ to be
$$R_v(x) = x - \frac{2H(x,v)}{H(v,v)}v.$$
   (a) Prove that R_v is an isometry.
   (b) Let $u \in V$ be another non-isotropic vector and $T : (V,H) \to (V,H)$ be an isometry. Show that $TR_uT^{-1} = R_{T(u)}$.

B Supplementary Problems

4. Let V, W be two finite dimensional vector spaces over a field F.

Definition B.1. A **tensor product** of V, W over F is a pair (U, b), where U is a vector space over F, and b is a bilinear function $V \times W \to U$ satisfying the following universal property: For any pair (U', b'), where U' is a vector space over F, b' is a bilinear function $V \times W \to U$, there exist a unique map $f : U \to U'$ such that $b' = f \circ b$. Namely, the diagram

$$\begin{tikzcd} V \times W \arrow[r, "b"] \arrow[dr, "b'"'] & U \arrow[d, "f"] \\ & U' \end{tikzcd}$$

commutes.

(a) Show that the tensor product of V, W over F is unique up to isomorphism. We denote the vector space U by $V \otimes W$.

(b) Show that $\dim V \otimes W = \dim V \times \dim W$.

(c) Let W' be another finite dimensional vector space over F. Show that

$$V \otimes (W \oplus W') \simeq (V \otimes W) \oplus (V \otimes W').$$

5. Let M be the $n \times n$ symmetric matrix with its diagonal entries equalling 0 and other entries equalling 1. Find the rank, index and signature of M.

6. Let V be a finite-dimensional vector space over F. Let $T : V \to V^*$ be an isomorphism such that

$$T(x)(y) = 0 \text{ implies } T(y)(x) = 0 \text{ for all } x, y \in V.$$

Show that $H(x, y) = T(x)(y)$ defines either a symmetric or alternating bilinear form on V.

7. Let H_1, H_2 be two skew-symmetric bilinear forms on a finite-dimensional vector space V over $\mathbb{R}$. Prove that there is an invertible linear map T on V with $H_1(T(x), T(y)) = H_2(x, y)$ for all $x, y \in V$ if and only if H_1 and H_2 have the same rank. (*Hint.* Use theorem A.2.)

8. Prove theorem A.2.

9. Let H be a bilinear form on V. Set

$$V^{\perp_L} = \{v \in V : H(v, w) = 0 \text{ for all } w \in V\}$$

and

$$V^{\perp_R} = \{v \in V : H(w, v) = 0 \text{ for all } w \in V\}.$$

Since $H(v, w + w') = H(v, w)$ whenever $w' \in V^{\perp_R}$, $\mathscr{L}_H$ (see homework 8) induces a linear map $V \to (V/V^{\perp_R})^*$. Show that this linear map has kernel $V^{\perp_L}$, so we get a linear embedding $V/V^{\perp_L} \hookrightarrow (V/V^{\perp_R})^*$. Moreover, show that this embedding map is an isomorphism if V is finite-dimensional.