Homework 05

Due Date: 111/03/23

Instruction. Do not submit part B.

A Homework Problems

1. Let T be a normal operator on a finite-dimensional inner product space V.
   (a) Prove that $\ker(T) = \ker(T^*)$ and $\operatorname{Im}(T) = \operatorname{Im}(T^*)$.
   (b) Assume V is a complex vector space and suppose W is a T-invariant subspace of V. Prove that W is also T^*-invariant.

2. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.
   (a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.
   (b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = 0$. (*Hint.* Replace x by $x + y$ and then by $x + iy$.)
   (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.
   (d) Does the results (2b), (2c) above hold if we assume that V is a real inner product space? Prove or give a counterexample.

Definition A.1.

(i) A linear operator T on a finite-dimensional inner product space is called **positive definite** [**positive semidefinite**] if T is self-adjoint and $\langle T(x), x\rangle > 0$ [$\langle T(x), x\rangle \geq 0$] for all $x \neq 0$.

(ii) An $n \times n$ matrix A with entries from $\mathbb{R}$ or $\mathbb{C}$ is called **positive definite** [**positive semidefinite**] if L_A is positive definite [positive semidefinite].

3. Let T be a self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove the following results.
   (a) T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [nonnegative].
   (b) T is positive definite if and only if
   $$\sum_{i,j=1}^{n} A_{ij} a_j \overline{a_i} > 0$$
   for all nonzero n-tuples $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. That is, $x^* A x > 0$ for all nonzero $x \in \mathbb{C}^n$.

(c) **(Cholesky decomposition)** T is positive semidefinite if and only if $A = B^*B$ for some square matrix $B \in M_n(\mathbb{C})$.

4. Let

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \in M_3(\mathbb{R}).$$

Find $B \in M_3(\mathbb{R})$ such that $B^*B = A$ (so A is positive semidefinite).

B Supplementary Problems

5. Let $A \in M_n(\mathbb{C})$ and have eigenvalues $\lambda_1, \cdots, \lambda_n$. Show that the following assertions are equivalent:

   (a) A is normal.

   (b) $I - A$ is normal.

   (c) Every eigenvector of A is an eigenvector of A^*.

   (d) $\text{tr}(A^*A) = \sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.

   (e) $\text{tr}(A^*A)^2 = \text{tr}\left((A^*)^2 A^2\right)$.

   (f) $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.

   (g) $A + A^*$ and $A - A^*$ commute.

   (h) $A^*A - AA^*$ is positive semidefinite.

   (i) A commutes with A^*A.

   (j) A commutes with $AA^* - A^*A$.

6. Let $A \in M_n(\mathbb{C})$. Show that the following assertions are equivalent:

   (a) A is Hermitian.

   (b) x^*Ax is real for all $x \in \mathbb{C}^n$.

   (c) $A^2 = A^*A$.

   (d) $A^2 = AA^*$.

   (e) $\text{tr}(A^2) = \text{tr}(A^*A)$.

7. **(Courant-Fischer min-max theorem)** Let $A \in M_n(\mathbb{C})$ be Hermitian and $\lambda_1 \leq \cdots \leq \lambda_n$ be its eigenvalues repeated with multiplicity. For $1 \leq k \leq n$, show that

$$\begin{aligned} \lambda_k &= \min_{\dim S_k = k} \max_{0 \neq x \in S_k} \frac{x^*Ax}{x^*x} \\ &= \max_{\dim S_{n-k+1} = n-k+1} \min_{0 \neq x \in S_{n-k+1}} \frac{x^*Ax}{x^*x}. \end{aligned}$$

where S_k and S_{n-k+1} runs over all subspaces of $\mathbb{C}^n$ with dimension k and $n - k + 1$ respectively. (*Hint*: Let $\{v_1, \ldots, v_n\}$ be an orthonormal set of eigenvectors of A. For the first equality, choose a non-zero vector in $S_k \cap \text{span}\{v_k, \ldots, v_n\}$ to prove $\geq$, and choose $S_k = \text{span}\{v_1, \ldots, v_k\}$ to prove $\leq$. Similar for the second equality.)

8. The following questions consider the **partial trace** of a matrix (usually Hermitian). Let V be an m-dimensional subspace of $\mathbb{C}^n$, the **partial trace** of a Hermitian matrix $A \in M_n(\mathbb{C})$ with respect to V is defined to be

$$\operatorname{tr}(A|_V) = \sum_{i=1}^{m} v_i^* A v_i \tag{B.1}$$

where $\{v_i\}_{i=1}^m$ is an orthonormal basis of V.

   (a) Show that the definition of partial trace (B.1) is independent of the choice of orthonormal basis $\{v_i\}_{i=1}^m$. That is, if both $\{v_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^m$ are orthonomal basis of V, we must have

$$\sum_{i=1}^{m} v_i^* A v_i = \sum_{i=1}^{m} w_i^* A w_i.$$

   Hence, the partial trace of A with respect to V only depends on V.

   (b) **(Extremal Partial Trace)** Let $A \in M_n(\mathbb{C})$ be Hermitian and $\lambda_1 \leq \cdots \leq \lambda_n$ be its eigenvalues repeated with multiplicity. For $1 \leq k \leq n$, show that

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_k &= \inf_{\dim S_k = k} \operatorname{tr}(A|_{S_k}) \\ \lambda_{n-k+1} + \lambda_{n-k+2} + \cdots + \lambda_n &= \sup_{\dim S_k = k} \operatorname{tr}(A|_{S_k}) \end{aligned}$$

   where S_k runs over all subspaces of $\mathbb{C}^n$ with dimension k. (*Hint*: Let $\{v_1, \ldots, v_n\}$ be an orthonormal set of eigenvectors of A. For the second identity, show $\leq$ by taking $S_k = \operatorname{span}\{v_1, \ldots, v_k\}$ and apply induction on k with the fact $\lambda_n \geq v^* A v$ for all $v \in \mathbb{C}^n$ to show $\geq$. The first identity follows similarly)

   Remark. In fact, this shows that $A \mapsto \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is a convex function and $A \mapsto \lambda_{n-k+1} + \lambda_{n-k+2} + \cdots + \lambda_n$ a concave function.

   (c) **(Schur-Horn Inequality)** Using (8b), show that

$$\lambda_{n-k+1} + \lambda_{n-k+2} + \cdots + \lambda_n \leq A_{i_1 i_1} + A_{i_2 i_2} + \cdots + A_{i_k i_k} \leq \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

   for any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ (k distinct integers between 1 and n), where A_{11}, $A_{22}, \ldots, A_{nn}$ are the diagonal entries of A.

Definition B.1. Let $\mathbb{F}$ be a field and $A \in M_n(\mathbb{F})$. For $1 \leq k \leq n$, define the *k-th principal submatrix of A*, denoted by $A_k \in M_k(\mathbb{F})$, by $(A_k)_{ij} = A_{ij}$ for $1 \leq i, j \leq k$.

9. **(Cauchy interlacing theorem)** Let $A \in M_n(\mathbb{C})$ be Hermitian and $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ be its eigenvalues repeated with multiplicity. Let B be the $(n-1)$-th principal submatrix of A (that is, B is obtained by deleting the n-th row and n-th column of A) and $\lambda_1(B) \leq \cdots \leq \lambda_{n-1}(B)$ be its eigenvalues repeated with multiplicity. Prove that

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \cdots \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \lambda_n(A).$$

(*Hint*: Use exercise (7).)

10. Let $A \in M_n(\mathbb{C})$ be Hermitian. Prove that the following statements are equivalent:

    (a) A is positive-definite.

    (b) **(Sylvester's criterion)** Every determinant of principal submatrix of A is positive. That is, the matrices $M_k = (m_{ij})_{1 \le i,j \le k}$ defined by $m_{ij} = a_{ij}$ have nonnegative determinant for all $1 \le k \le n$.

    (c) $A = B^*B$ for some $B \in M_n(\mathbb{C})$ invertible.

    (d) $X^*AX \ge 0$ for all $n \times m$ matrix X.

    (e) tr $(AX) \ge 0$ for all positive semidefinite matrix X.

    (f) $X^*AX \ge 0$ for all $n \times m$ matrix X.

    (*Hint*: For (10a) $\Rightarrow$ (10b), use exercises (3a) and (9). For (10b) $\Rightarrow$ (10c), consider the LDU decomposition of A.)