## Homework 09

Due Date: 111/05/04

**Instruction.** Do not submit part B.

## **A Homework Problems**

1. For the following quadratic form K on  $\mathbb{R}^2$ , find a symmetric bilinear form H on  $\mathbb{R}^2$  such that K(x) = H(x,x) for all  $x \in \mathbb{R}^2$ . Also, find an orthonormal basis  $\beta$  for  $\mathbb{R}^2$  (equipped with standard inner product) such that  $\psi_{\beta}(H)$  is a diagonal matrix.

$$K \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -2t_1^2 + 4t_1t_2 + t_2^2.$$

- 2. Let V be a finite-dimensional vector space over a field F, dim V=n and  $H_1, H_2$  be two bilinear forms on V. We say  $H_1$  and  $H_2$  are **equivalent** on V if there exists bases  $\beta_1, \beta_2$  of V such that  $\psi_{\beta_1}(H_1) = \psi_{\beta_2}(H_2)$ . Under this definition, the set  $\mathcal{B}(V)$  of all bilinear forms on V is partitioned into several **equivalent classes**.
  - (a) When  $F = \mathbb{R}$ , determine the number of equivalent classes of symmetric bilinear forms on V. (This is a translation of exercise 26, which is explained in teacher's lecture. Still, you need to give out all details.)
  - (b) When  $F = \mathbb{C}$ , determine the number of equivalent classes of symmetric bilinear forms on V. (*Hint*. First show that every symmetric bilinear form on V has a matrix representation using only **non-negative** numbers as entries.)

**Definition A.1.** A bilinear form H on V is called **skew-symmetric** if H(x,y)=-H(y,x) for all  $x,y\in V$ .

(c) Let F be a field with char  $F \neq 2$ . Use the following theorem A.2 to determine the number of equivalent classes of skew-symmetric bilinear forms on V.

**Theorem A.2.** Let F be a field with char  $F \neq 2$ , V be a finite dimensional vector space over F, dim V = n, and H be a skew-symmetric bilinear form on V. Then there exists a basis  $\beta$  of V such that the matrix representation  $\psi_{\beta}(H)$  takes the form

$$\begin{pmatrix}
0_k & I_k & 0_{k \times (n-2k)} \\
-I_k & 0_k & 0_{k \times (n-2k)} \\
0_{(n-2k) \times k} & 0_{(n-2k) \times k} & 0_{n-2k}
\end{pmatrix}$$

3. Let (V, H) be a quadratic space over F with char  $F \neq 2$ . For a non-isotropic vector  $v \in V$ , define the **reflection**  $R_v : (V, H) \to (V, H)$  to be

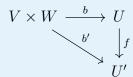
$$R_{v}(x) = x - \frac{2H(x, v)}{H(v, v)}v.$$

- (a) Prove that  $R_v$  is an isometry.
- (b) Let  $u \in V$  be another non-isotropic vector and  $T:(V,H) \to (V,H)$  be an isometry. Show that  $TR_uT^{-1}=R_{T(u)}$ .

## **B** Supplementary Problems

4. Let V, W be two finite dimensional vector spaces over a field F.

**Definition B.1.** A **tensor product** of V, W over F is a pair (U, b), where U is a vector space over F, and b is a bilinear function  $V \times W \to U$  satisfying the following universal property: For any pair (U', b'), where U' is a vector space over F, b' is a bilinear function  $V \times W \to U$ , there exist a unique map  $f: U \to U'$  such that  $b' = f \circ b$ . Namely, the diagram



commutes.

- (a) Show that the tensor product of V, W over F is unique up to isomorphism. We denote the vector space U by  $V \otimes W$ .
- (b) Show that  $\dim V \otimes W = \dim V \times \dim W$ .
- (c) Let W' be another finite dimensional vector space over F. Show that

$$V \otimes (W \oplus W') \simeq (V \otimes W) \oplus (V \otimes W').$$

- 5. Let M be the  $n \times n$  symmetric matrix with its diagonal entries equalling o and other entries equalling 1. Find the rank, index and signature of M.
- 6. Let V be a finite-dimensional vector space over F. Let  $T:V\to V^*$  be an isomorphism such that

$$T(x)(y) = 0$$
 implies  $T(y)(x) = 0$  for all  $x, y \in V$ .

Show that  $H\left(x,y\right)=T\left(x\right)\left(y\right)$  defines either a symmetric or alternating bilinear form on V.

- 7. Let  $H_1, H_2$  be two skew-symmetric bilinear forms on a finite-dimensional vector space V over  $\mathbb{R}$ . Prove that there is an invertible linear map T on V with  $H_1(T(x), T(y)) = H_2(x,y)$  for all  $x,y \in V$  if and only if  $H_1$  and  $H_2$  have the same rank. (*Hint*. Use theorem A.2.)
- 8. Prove theorem A.2.
- 9. Let H be a bilinear form on V. Set

$$V^{\perp_{L}}=\left\{ v\in V:H\left( v,w\right) =0\text{ for all }w\in V\right\}$$

and

$$V^{\perp_{R}}=\{v\in V:H\left( w,v\right) =0\text{ for all }w\in V\}.$$

Since H(v, w + w') = H(v, w) whenever  $w' \in V^{\perp_R}$ ,  $\mathscr{L}_H$  (see homework 8) induces a linear map  $V \to (V/V^{\perp_R})^*$ . Show that this linear map has kernel  $V^{\perp_L}$ , so we get a linear embedding  $V/V^{\perp_L} \hookrightarrow (V/V^{\perp_R})^*$ . Moreover, show that this embedding map is an isomorphism if V is finite-dimensional.