Homework 07

Due Date: 111/04/20

Instruction. Do not submit part B.

A Homework Problems

1. Let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$
   (a) Find the singular value decomposition of A.
   (b) Find the pseudoinverse of A.
2. Let A be a square matrix with polar decomposition $A = WP$.
   (a) Prove that A is normal if and only if $WP^2 = P^2W$.
   (b) Prove that A is normal if and only if $WP = PW$.
3. Let V and W be finite-dimensional inner product spaces, and let $T\colon V \to W$ be linear. Prove the following results.
   (a) $TT^\dagger T = T$.
   (b) $T^\dagger T T^\dagger = T^\dagger$.
   (c) Both $TT^\dagger$ and $T^\dagger T$ are self-adjoint.
4. Let $\mathbb{F}_q$ be the field of $q = p^k$ elements.
   (a) Let GL$(n, \mathbb{F}_q)$ be the subset of invertible $n \times n$ matrices over $\mathbb{F}_q$, show that
$$|\mathrm{GL}(n, \mathbb{F}_q)| = \prod_{i=1}^{n-1} \left(q^n - q^i\right) = (q^n - 1)(q^n - q) \cdots \left(q^n - q^{n-1}\right).$$
   (*Hint.* Consider filling the matrix column by column, there are $q^n - 1$ choices for the first column since we cannot fill in the zero vector; there are $q^n - q$ choices for the second column after fixing the first column since there are exactly q column vectors that is a multiple of the first column; the rest follows similarly.)
   (b) Let SL$(n, \mathbb{F}_q)$ be the subset of $n \times n$ matrices over $\mathbb{F}_q$ with determinant 1, show that
$$|\mathrm{SL}(n, \mathbb{F}_q)| = \frac{|\mathrm{GL}(n, \mathbb{F}_q)|}{q - 1}.$$

(*Hint.* Let S_a be the subset of $\mathrm{GL}(n, \mathbb{F}_q)$ consisting matrices with determinant $a \neq 0$. Show that $|S_a| = |S_1| = |\mathrm{SL}(n, \mathbb{F}_q)|$ for all $a \neq 0$ by considering the mapping

$$\begin{array}{cccc} \varphi_{q,a}: & S_1 & \to & S_a \\ & \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} & \mapsto & \begin{pmatrix} a \cdot x_{11} & a \cdot x_{12} & \cdots & a \cdot x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \end{array}$$

Show that $\varphi_{q,a}$ is well-defined and bijective for all $a \neq 0$. Recall that for any nonzero element a in a field, it must have a multiplicative inverse a^{-1}.)

Remark. $\mathrm{GL}(n, \mathbb{F})$ is called the **general linear group** of degree n over $\mathbb{F}$ and $\mathrm{SL}(n, \mathbb{F})$ is called the **special linear group** of degree n over $\mathbb{F}$.

B Supplementary Problems

5. Let A be an $n \times n$ matrix whose characteristic polynomial splits. Recall that $\mathrm{tr}(A)$ equals the sum of its eigenvalues (count with multiplicity), $\sum_{i=1}^{n} \lambda_i$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be its singular values. Prove that

$$\mathrm{tr}(A^*A) = \sum_{i=1}^{n} \sigma_i^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 \quad \text{and} \quad |\,\mathrm{tr}(A)| \leq \sum_{i=1}^{n} \sigma_i.$$

Definition B.1 (Generalized Inverse). Let $A \in M_{m\times n}(\mathbb{C})$ be a $m \times n$ matrix. A **generalized inverse** of A is any $n \times m$ matrix G such that

$$AGA = A. \tag{B.1}$$

(i) If a generalized inverse G satisfies $GAG = G$, we say that G is a **reflexive generalized inverse** of A.

(ii) If a generalized inverse G satisfies $(GA)^* = GA$, we say that G is a **minimum-norm generalized inverse** of A.

(iii) If a generalized inverse G satisfies $(AG)^* = AG$, we say that G is a **least squares generalized inverse** of A.

(iv) If a generalized inverse G satisfies (B.1), $GAG = G$, $(GA)^* = GA$ and $(AG)^* = AG$, we say that G is a **Moore-Penrose (generalized) inverse** of A.

6. (**Characterization of Generalized Inverse**) Suppose $A \in M_{m\times n}(\mathbb{C})$ has singular value decomposition

$$A = U\widetilde{\Sigma}V^* = U \begin{pmatrix} \Sigma & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{pmatrix} V^*$$

where $U \in M_{m\times m}(\mathbb{C})$, $V \in M_{n\times n}(\mathbb{C})$ are unitary matrices, $r = \mathrm{rank} A$ and $\Sigma \in M_{r\times r}(\mathbb{C})$ is a diagonal matrix consisting of the singular values of A.

(a) Show that G is a generalized inverse of A if and only if

$$G = V \begin{pmatrix} \Sigma^{-1} & X \\ Y & Z \end{pmatrix} U^* \tag{B.2}$$

for some $X \in M_{r\times(m-r)}(\mathbb{C})$, $Y \in M_{(n-r)\times r}(\mathbb{C})$, $Z \in M_{(n-r)\times(m-r)}(\mathbb{C})$

(b) Show that G is a reflexive generalized inverse of A if and only if G is of the form (B.2) and $Z = Y\Sigma X$.

(c) Show that G is a minimum-norm generalized inverse of A if and only if G is of the form (B.2) and $Y = O_{r\times(m-r)}$.

(d) Show that G is a least squares generalized inverse of A if and only if G is of the form (B.2) and $X = O_{(n-r)\times r}$.

(e) Show that the Moore-Penrose inverse (pseudoinverse) $G = A^\dagger$ of A is unique by showing that G is of the form (B.2) and $X = O_{(n-r)\times r}$, $Y = O_{r\times(m-r)}$ and $Z = O_{(n-r)\times(m-r)}$.

7. (**Reading comprehension.**) We compute the number of pairs of commuting matrices in $M_{2\times 2}(\mathbb{F}_q)$.

> **Lemma B.2.** Let T be a linear operator on a finite-dimensional vector space V with rational form decomposition
>
> $$V = \bigoplus_i K_{\phi_i} = \bigoplus_i \bigoplus_{j=1}^{k_i} C_{ij}$$
>
> where ϕ_i are distinct irreducible factors of the characteristic polynomial of T, and C_{ij} are T-cyclic subspaces with minimal polynomial $\phi_i(x)^{p_j}$ for some p_j such that $p_1 \geq \cdots \geq p_{k_i}$. Then, we have
>
> $$\dim\{S : V \to V \mid ST = TS\} = \sum_i \deg \phi_i \cdot \sum_{m,n=1}^{k_i} \min\{p_m, p_n\} \tag{$\star$}$$

Now, we try to write matrices in $M_{2\times 2}(\mathbb{F}_q)$ in their Jordan forms or rational forms.

> **Lemma B.3.** Any matrix in $M_{2\times 2}(\mathbb{F}_q)$ is similar to a matrix belonging to exactly one of the following types:
>
> - $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{F}_q$ and for each $\alpha \in \mathbb{F}_q$, there is only one matrix in $M_{2\times 2}(\mathbb{F}_q)$ similar to it.
> - $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{F}_q$ and for each $\alpha \in \mathbb{F}_q$, there are $q^2 - 1$ matrices in $M_{2\times 2}(\mathbb{F}_q)$ similar to it.
> - $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ for some distinct $\alpha, \beta \in \mathbb{F}_q$ and for each pair of distinct $\alpha, \beta \in \mathbb{F}_q$, there are $q(q+1)$ matrices in $M_{2\times 2}(\mathbb{F}_q)$ similar to it.
> - $\begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}$ such that $x^2 - \beta x - \alpha$ is an irreducible polynomial in $\mathbb{F}_q[x]$ and for

> each degree 2 irreducible polynomial in $\mathbb{F}_q[x]$, there are $q(q-1)$ matrices in $M_{2\times 2}(\mathbb{F}_q)$ similar to it.
>
> Also, there are exactly $\frac{q(q-1)}{2}$ monic irreducible polynomials of degree 2 in $\mathbb{F}_q[x]$. This classifies all q^4 matrices in $M_{2\times 2}(F)$ into four parts:
>
> $$q^4 = 1 \cdot q + \left(q^2-1\right) \cdot q + q(q+1) \cdot \frac{q(q-1)}{2} + q(q-1) \cdot \frac{q(q-1)}{2}.$$

Using the information above, you are asked to compute the number of ordered pairs of commuting matrices in $M_{2\times 2}(\mathbb{F}_q)$. For each of the four types in Lemma B.3, compute the number of matrices commuting with it by applying the formula ($\star$). Then derive the final answer and compare it to the one given in class.

8. Prove the two lemmas in Problem 7.