Homework 02

Due Date: 111/03/02

Instruction. Do not submit part B.

A Homework Problems

0. (**Do this yourself but do NOT submit this**) In each of the following vector spaces, verify that the given function $\langle \cdot, \cdot \rangle$ is an inner product. That is, check the four conditions given in section 6.1.

   (a) $V = \mathbb{C}^n$, $\langle (a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \rangle = \sum_{k=1}^{n} a_k \overline{b_k}$.

   (b) $V = M_n(\mathbb{C})$, $\langle A, B \rangle = \mathrm{Tr}(A^* B)$.

   (c) $V =$ the set of all continuous functions $\mathbb{R} \to \mathbb{R}$, $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$.

1. (2020 NTU Math Master Entrance Exam) Let $A \in M_{n \times n}(F)$. Show that

$$\mathrm{rank}(A^2) - \mathrm{rank}(A^3) \leq \mathrm{rank}(A) - \mathrm{rank}(A^2).$$

   (*Hint.* Here is one possible way: Construct an injective linear transformation

$$N(A^3)/N(A^2) \longrightarrow N(A^2)/N(A)$$

   either directly or by using isomorphism theorem.)

2. Let V be an inner product space over $\mathbb{C}$. For any $x, y, z \in V$, prove the followings.

   (a) (Parallelogram law) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

   (b) (Polarization identity) $\langle x, y \rangle = \dfrac{1}{4} \displaystyle\sum_{k=1}^{4} i^k \left\| x + i^k y \right\|^2$, where $i^2 = -1$.

   (c) (Ptolemy's inequality) $\|x-y\|\|z\| + \|y-z\|\|x\| \geq \|x-z\|\|y\|$.

3. Let $V = \mathbb{R}[x]_{\leq 2}$ be the vector space of all polynomials having degree at most 2 with coefficient over $\mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

   Find an orthonormal basis for V by applying Gram-Schmidt process on the set $\{1, x, x^2\}$.

B Supplementary Problems

4. (**Correspondence theorem**) Let V be a vector space over a field F, $W \subseteq V$ be a subspace, and $Q : V \to V/W$ be the quotient map $Q(v) := v + W$.

(a) Suppose $\overline{U}$ is a subspace of V/W. Show that $U := Q^{-1}(\overline{U})$ is a subspace of V containing W and $\overline{U} = U/W$.

(b) Suppose X is a subspace of V containing W. Show that $\overline{X} := X/W$ is a subspace of V/W and $Q^{-1}(\overline{X}) = X$.

5. (Modified from 2021 NTNU Math Master Entrance Exam) Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be the linear transformation given by $T(x) = Ax$ for $x \in \mathbb{R}^4$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{pmatrix}.$$

Let W denote the null space of T and $V = \mathbb{R}^4/W$ be the quotient space of $\mathbb{R}^4$ by W. Also let $\beta = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$.

(a) Determine whether $\overline{\beta} := \{e_1 + W, e_2 + W, e_3 + W, e_4 + W\}$ is linearly independent, and obtain an ordered basis γ of V from $\overline{\beta}$.

(b) Define $\overline{T} : V \to \mathbb{R}^5$ by $\overline{T}(v + W) := T(v)$. Find the matrix representation $[\overline{T}]_\gamma^{\mathcal{E}}$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^5$. Then, verify the isomorphism theorem, that is, $\overline{T}$ is an isomorphism $V = \mathbb{R}^4/W \to R(T)$.

6. Let $A, B \in M_{n\times n}(\mathbb{C})$. Show that

$$|\mathrm{Tr}\,(AB^*)| \le (\mathrm{Tr}\,(AA^*)\,\mathrm{Tr}\,(BB^*))^{1/2} \le \frac{1}{2}\,(\mathrm{Tr}\,(AA^*) + \mathrm{Tr}\,(BB^*)).$$

7. A vector space V over F ($F = \mathbb{R}$ or $\mathbb{C}$) is a **normed space** if there is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying the conditions:

(1) $\|x\| \ge 0$ for all $x \in V$. Also, $\|x\| = 0$ if and only if $x = 0$.

(2) $\|ax\| = |a| \cdot \|x\|$ for all $x \in V$ and $a \in F$.

(3) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

Such function $\|\cdot\|$ is called a **norm** on V.

(a) Let V be a normed space with norm $\|\cdot\|$. Show that there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that $\|x\|^2 = \langle x, x \rangle$ for all $x \in V$ if and only if the norm satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

for all $x, y \in V$.

(b) Consider V to be the set of real-valued continuous functions defined on $[0, 1]$. Show that the norm

$$\|f\| = \max_{x \in [0,1]} |f(x)|$$

on V is never induced by any inner product.

8. Let $(V, \langle -, - \rangle)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$.

(a) **(Bessel's Inequality)** Let $\{s_n\}_{n=1}^{\infty}$ be an orthonormal subset of V and $x \in V$. Prove that $\sum_{n=1}^{\infty} |\langle x, s_n \rangle|^2$ converges and $\sum_{n=1}^{\infty} |\langle x, s_n \rangle|^2 \leq \|x\|^2$.

(b) Let A be an orthonormal subset of V and $x \in V$. Prove that $\langle x, a \rangle = 0$ for all but countably many $a \in A$. (*Hint*: For each $n \geq 1$, let $A_n = \{a \in A : |\langle x, a \rangle| \geq 1/n\}$. By (8a), A_n is finite. But $\bigcup_{n=1}^{\infty} A_n = \{a \in A : \langle x, a \rangle \neq 0\}$.)

9. Let $(V, \langle -, - \rangle)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$.

(a) Prove that every orthonormal subset of V is contained in a maximal orthonormal subset of V. (*Hint*: Zorn's lemma.)

(b) Let S be an orthonormal subset of V. Prove that S is a maximal orthonormal subset of V if and only if for every $x \in V$, $\langle x, s \rangle = 0$ for all $s \in S$ implies $x = 0$.

(c) Find an example that a maximal orthonormal subset of V not being a basis for V.

(d) Show that every maximal orthonormal subset of V has the same cardinality. (*Hint*: Let A and B be maximal orthonormal subset of V. For $a \in A$, define $B(a) = \{b \in B : \langle a, b \rangle \neq 0\}$. Use (9b) to see that $B \subseteq \bigcup_{a \in A} B(a)$, and (8b) to see that $B(a)$ is countable for each $a \in A$.)

10. (**Legendre polynomials**) Let $V = \mathbb{R}[x]$ be the space of polynomials with coefficients in $\mathbb{R}$. Let $b > a$ be real numbers. Define the inner product on V by

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx.$$

For each positive integer n, define

$$q_{2n}(x) = (x-a)^n (x-b)^n.$$
$$p_n(x) = \frac{d^n}{dx^n}\left(q_{2n}(x)\right).$$

(a) Show that

$$\frac{d^{i-1} q_{2n}}{dx^{i-1}}(a) = \frac{d^{i-1} q_{2n}}{dx^{i-1}}(b) = 0$$

for all $i = 1, 2, \ldots, n$.

(b) Show that p_n has degree n.

(c) Show that $p_1, p_2, \ldots, p_n$ are orthogonal to each other.

Remark. After normalizing the polynomials $\{p_n\}_{n=1}^{\infty}$, we get the **Legendre polynomials** on $[a, b]$.