Homework 12

Due Date: 111/05/25

Instruction. Do not submit part B.
Congratulations that this is the last homework in this course. You have indeed done a great job. Good luck to your final exams and feel free to ask questions. Also, please check out whether the grades on CEIBA are correct as soon as possible.

A Homework Problems

1. Let $f(x), g(x) \in \mathbb{C}[x]$ and $d(x)$ be a common divisor of $f(x)$ and $g(x)$ having maximum degree (gcd(f, g)). Let $A_{f,g}$ be the resultant matrix of f, g. That is, $A_{f,g}$ has the same expression as $R_{f,g}$ but does not take the determinant. Prove that the nullity of $A_{f,g}$ equals the degree of $d(x)$. (*Hint.* This is a slight generalization to a proposition discussed in the note [class26]. Going through the detail in the proof of that proposition yields this result.)

2. Let $f(x, y) \in \mathbb{C}[x, y]$ be a nonzero homogeneous polynomial.

   (a) Prove that if $f(x, y) = g(x, y)h(x, y)$ for some $g(x, y), h(x, y) \in \mathbb{C}[x, y]$, then both $g(x, y)$ and $h(x, y)$ are homogeneous.

   (b) Prove that if $f(a, b) = 0$ for some $(a, b) \neq (0, 0)$, then $ay - bx$ is a divisor of $f(x, y)$.

   (c) Prove that
   $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = d \cdot f(x, y)$$
   where d is the degree of f. (*Hint.* Differentiate $f(tx, ty) = t^d f(x, y)$ with respect to t.)

3. Let $f(x) \in \mathbb{C}[x]$. $\alpha \in \mathbb{C}$ is called a **multiple root** of the equation $f(x) = 0$ if $(x - \alpha)^2$ is a divisor of $f(x)$.

   (a) Show that α is a multiple root of $f(x) = 0$ if and only if $(x - \alpha)$ is a common divisor of $f(x)$ and $f'(x)$, the derivative of $f(x)$.

   (b) Let $f(x) = x^3 + bx + c$. Show that $f(x) = 0$ has no multiple root if and only if $4b^3 + 27c^2 \neq 0$.

4. Let $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m \in \mathbb{C}$ be $n + m$ complex numbers. Consider
   $$f(x) = \prod_{i=1}^{n}(x - \alpha_i);$$
   $$g(x) = \prod_{j=1}^{m}(x - \beta_j).$$
   Show that
   $$R_{f,g} = \prod_{1\leq i\leq n, 1\leq j\leq m}(\beta_j - \alpha_i).$$
   (*Hint.* Regard $R_{f,g}$ as a polynomial in $\mathbb{C}[\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m]$. What is the degree of $R_{f,g}$? What are its divisors?)

B Supplementary Problems

5. Find the number of solutions $(x, y) \in \mathbb{C}^2$ to the following system:

$$\begin{cases} xy^2 - y + x^2 + 1 = 0 \\ x^2y^2 + y - 1 = 0 \end{cases}.$$

6. A projective line in $\mathbb{CP}^2$ is defined to be the set of solutions of an equation of the form

$$ax + by + cz = 0$$

for some $a, b, c \in \mathbb{C}$, not all zero.

(a) Prove that any two distinct points in $\mathbb{CP}^2$ in contained in a unique line.

(b) Prove that any two distinct lines in $\mathbb{CP}^2$ intersect at a unique point.

Definition B.1. Let $f(x), g(x) \in \mathbb{C}[x]$ and $n = \max \deg f(x), \deg g(x)$. By appending zeroes to the coefficients, we may write

$$f(x) = f_n x^n + f_{n-1}x^{n-1} + \dots + f_0,$$

$$g(x) = g_n x^n + g_{n-1}x^{n-1} + \dots + g_0.$$

The $n \times n$ matrix $B_{f,g}$ satisfying

$$\frac{f(x)g(y) - g(x)f(y)}{x - y} = \begin{bmatrix} 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix} B_{f,g} \begin{bmatrix} 1 \\ y \\ y^2 \\ \vdots \\ y^{n-1} \end{bmatrix}$$

is called the **Bézout matrix** (or Bézoutian) of f and g. In the following, we also denote

$$H_f = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_2 & f_3 & f_4 & \cdots & 0 \\ f_3 & f_4 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad T_f = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{n-1} \\ 0 & f_0 & f_1 & \cdots & f_{n-2} \\ 0 & 0 & f_0 & \cdots & f_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_0 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(that is, Z is the "opposite identity".)
All matrices above are $n \times n$ matrices. Also recall the notation in problem 1: $A_{f,g}$ is the resultant matrix.

In the following few questions, we are to prove that the nullity of $B_{f,g}$ and $A_{f,g}$ are same, and that we can compute $B_{f,g}$ by an easier formula. Thus we obtain an improvement to problem 1.

7. Using the notations above, prove that

   (a) $T_f T_g = T_g T_f$

   (b) $T_f^t = Z T_f Z$

   (c) $H_f Z H_g = H_g Z H_f$

   (d) $B_{x^n,1} = Z$.

   (e) $A_{f,g} = \begin{bmatrix} T_f & ZH_f \\ T_g & ZH_g \end{bmatrix}$

8. Set $b(x,y) = \frac{f(x)g(y)-g(x)f(y)}{x-y}$, $v_n(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}$. Prove that

$$(x^n - y^n) b(x,y) = v_{2n}(x)^t \begin{bmatrix} 0 & -B_{f,g} \\ B_{f,g} & 0 \end{bmatrix} v_{2n}(y)$$

   and

$$(x^n - y^n) b(x,y) = v_{2n}(x)^t A_{f,g}^t \begin{bmatrix} 0 & Z \\ -Z & 0 \end{bmatrix} A_{f,g} v_{2n}(y).$$

   Deduce that $A_{f,g}^t \begin{bmatrix} 0 & Z \\ -Z & 0 \end{bmatrix} A_{f,g} = \begin{bmatrix} 0 & -B_{f,g} \\ B_{f,g} & 0 \end{bmatrix}$ and $B_{f,g} = H_f T_g - H_g T_f$.

9. Prove that the nullity of $B_{f,g}$ equals the nullity of $A_{f,g}$.