Homework 11

Due Date: 111/05/18

Instruction. Do not submit part B. All fields are assumed to have characteristic $\neq 2$.

A Homework Problems

1. Let (V, H) be an n-dimensional quadratic space and β be a basis for V. Denote $A = \psi_\beta(H)$. Define
$$G = \{[\sigma]_\beta \,|\, \sigma \in \mathrm{O}(V)\}$$
to be the set of matrices representing the isometries on V. Prove that an $n \times n$ invertible matrix Q is in G if and only if $Q^t A Q = A$. Deduce that, when $F = \mathbb{R}$, H is an inner product on V, and β is an orthonormal basis of V, G equals the set of all orthogonal matrices.

> **Remark.** When $F = \mathbb{R}$, recall that the matrix of a non-degenerate bilinear form H is equivalent to $I_{p,q} = \mathrm{diag}(1, 1, ..., 1, -1, -1, ..., -1)$ (p entries of 1 and q entries of -1 on the diagonal). In this case the set $G = \{Q \in M_n(\mathbb{R}) | Q^t I_{p,q} Q = I_{p,q}\}$ is denoted by $\mathrm{O}(p, q)$, which is called the **orthogonal group of signature** (p, q).

2. Let (V, H) be an n-dimensional non-degenerate quadratic space and $\sigma \in \mathrm{O}(V)$ is a product of n reflections. Prove that the first reflection in the product can be arbitrarily chosen. That is, for any reflection $\tau \in \mathrm{O}(V)$, there exist $n-1$ reflections $\tau_1, \tau_2, ..., \tau_{n-1}$ such that $\sigma = \tau\tau_1\tau_2...\tau_{n-1}$.

3. Let (V, H) be an n-dimensional non-degenerate quadratic space. Cartan-Dieudonné theorem states that any isometry $\sigma \in \mathrm{O}(V)$ is a product of at most n reflections. Prove that this result is optimal: there is an isometry $\sigma \in \mathrm{O}(V)$ which can NOT be written as a product of less than n reflections. (*Hint.* Each reflection fixes a $(n-1)$-dimensional subspace. What does a product of two reflections fixes? Also, a precise counterexample is $-I$, the negative identity.)

B Supplementary Problems

4. Let (V, H) is an n-dimensional non-degenerate quadratic space over F and $\sigma \in \mathrm{O}(V)$. Show that σ is a product of at most $n - \dim N(\sigma - I)$ reflections.

5. (**Symplectic groups**) Let V be a $2n$-dimensional vector space over F and H be a non-degenerate **skew-symmetric** bilinear form on V. Denote by $\mathrm{Sp}(V)$ the set of all invertible linear transformations $\sigma : V \to V$ with $H(x, y) = H(\sigma x, \sigma y)$ for all $x, y \in V$. By theorem A.2 in Homework 9, H has a matrix representation
$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
Denote this matrix by J_n. By problem 1, the set $\mathrm{Sp}(V)$ is equivalent to the set of all $2n \times 2n$ matrices Q with $Q^t J_n Q = J_n$. This set of matrices is called the **symplectic group** of order n, denoted by $\mathrm{Sp}(n, F)$.

(a) For an invertible $M \in M_{2n}(F)$, we write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_n(F)$. Show that $M \in \mathrm{Sp}(n, F)$ if and only if $M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$.

(b) Prove that any matrix $M \in \mathrm{Sp}(n, F)$ is a product of J_n and some matrices in the sets

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} | A \in M_n(F), \det A \neq 0 \right\}$$

and

$$\left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} | B \in M_n(F), B = B^t \right\}.$$

Deduce that if $M \in \mathrm{Sp}(n, F)$, then $\det(M) = 1$.

(c) For a non-zero vector $v \in V$ and a scalar $c \in F$, we define a **symplectic transvection** operator $T_{v,c}$ on V to be

$$T_{v,c}(x) = x + cH(x, v)\, v.$$

It is clear that $T_{v,c} \in \mathrm{Sp}(V)$ and $\det T_{v,c} = 1$. Show that every element of $\mathrm{Sp}(V)$ is a product of at most $2n$ symplectic transvections.