Homework 10

Due Date: 111/05/11

Instruction. Do not submit part B. All fields are assumed to have characteristic $\neq 2$.

A Homework Problems

1. [Lam, exercise 2] Let $V = M_n(F)$ be the n^2-dimensional vector space of all $n \times n$ matrices with entries in F.

   (a) Define $H(X, Y) = \text{tr}(XY)$. Show that H is a non-degenerate symmetric bilinear form on V and find a basis β for V such that

   $$\psi_\beta(H) = \begin{bmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & 1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix}$$

   where there are $\frac{n(n-1)}{2}$ copies of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and n copies of $[1]$. That is to say, the quadratic space (V, B) is isometric to $n\langle 1\rangle \oplus \frac{n(n-1)}{2}\mathbb{H}$. This notation is used in [Lam]. (*Hint.* Consider the matrices E_{ij}, which has an 1 in the (i, j)-entry and all other entries being 0.)

   (b) Do part (a) with $H(X, Y) = \text{tr}(XY^t)$ instead. This time, find a basis β so that

   $$\psi_\beta(H) = I_{n^2}.$$

2. [Lam, exercise 9] Let $a, b \in F$ such that $a^2 + b^2 = c \neq 0$. Consider the quadratic form $Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - cx_3^2 - cx_4^2$ defined on $V = F^4$. Prove that Q is hyperbolic. That is to say, the quadratic space (V, H_Q) is a direct sum of some hyperbolic planes. (*Hint.* In matrix form, it suffices to prove that we can change the basis of V by

   $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = P \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

   so that $Q(y_1, y_2, y_3, y_4) = y_1^2 - y_2^2 + y_3^2 - y_4^2$.)

3. [Lam, exercise 16(b)(c)] For a quadratic space (V, H), denote by $i(V, H)$ or simply by $i(H)$ the Witt index of (V, H). Let (V_1, H_1) and (V_2, H_2) be two non-degenerate finite-dimensional quadratic spaces over the same field F.

(a) Prove that $i(H_1 \oplus H_2) \leq i(H_1) + \dim V_2$.

(b) Suppose that (V_1, H_1) is isometric to a subspace $(W, (H_2)|_W)$ of (V_2, H_2). Prove that

$$\dim V_2 - i(H_2) \geq \dim V_1 - i(H_1).$$

Deduce that if $\dim V_1 > \dim V_2 - i(H_2)$, then (V_1, H_1) must be isotropic. (*Hint*. Use part (a).)

B Supplementary Problems

4. [Lam, exercise 14,15]

   (a) Let U be a subspace of dimension $m + r$ in a hyperbolic space $\mathbb{H}^m$. Prove that $i(U) \geq r$.

   (b) Let U be a quadratic space of dimension k. Prove that U is isometric to a subspace of $\mathbb{H}^m$ if and only if $i(U) \geq k - m$.

5. [Lam, section 6 and exercise 16(a)] Let (V_1, H_1) and (V_2, H_2) be two non-degenerate finite-dimensional quadratic spaces over the same field F. The **tensor product** of the quadratic spaces (V_1, H_1) and (V_2, H_2) is defined by $(V_1 \otimes V_2, H_1 \cdot H_2)$, where $(V_1 \otimes V_2)$ is the tensor product of vector spaces defined in Homework 9, and $H_1 \cdot H_2$ is the bilinear form associated to the quadratic form $Q_1 \cdot Q_2$, which sends $v_1 \otimes v_2$ to $Q_1(v_1) \cdot Q_2(v_2)$. Prove that if (V_1, H_1) and (V_2, H_2) are non-degenerate, then $i(H_1 \otimes H_2) \geq i(H_1) \cdot \dim V_2$.