Homework 08

Due Date: 111/04/27

Instruction. Do not submit part B.

A Homework Problems

Definition A.1. Let V be a finite-dimensional vector space, and H be a bilinear form on V. The **rank** of H is defined to be the rank of any matrix representation of H. If $\text{rank}(H) = \dim(V)$, we call H **non-degenerate**.

1. Let V be a finite-dimensional vector space, and H be a bilinear form on V. Prove that for any two bases β_1, β_2 of V, the matrices $\psi_{\beta_1}(H)$ and $\psi_{\beta_2}(H)$ has the same rank. Hence the rank of H is well-defined.
2. For the following matrix $A \in M_3(\mathbb{R})$, find a diagonal matrix D with all its entries equalling to $1, -1$ or 0, and an invertible matrix $Q \in M_3(\mathbb{R})$ such that $Q^t A Q = D$.
$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$
3. Let V and W be vector spaces over the same field $\mathbb{F}$, and let $T\colon V \to W$ be a linear transformation. For any $H \in \mathcal{B}(W)$, define the mapping $\widehat{T}(H)$ on $V \times V$ by
$$\begin{aligned} \widehat{T}(H)\colon V \times V &\longrightarrow \mathbb{F} \\ (x, y) &\longmapsto H(T(x), T(y)) \end{aligned}$$
Prove the following results.
   (a) Show that $\widehat{T}(H)$ is also a bilinear form on V.
   (b) Show that $\widehat{T}\colon \mathcal{B}(W) \to \mathcal{B}(V)$ is a linear transformation.
   (c) If T is an isomorphism, then so is $\widehat{T}$.

   Such $\widehat{T}(H)$ is called the **pull-back** bilinear form of H on V.
4. Let V be a vector space over a field $\mathbb{F}$ (not necessarily finite-dimensional). For any $H \in \mathcal{B}(V)$, define a function $\mathscr{L}_H\colon V \to V^*$ by $(\mathscr{L}_H(v))(w) := H(v, w)$.
   (a) Show that $\mathscr{L}_H$ is a linear transformation.
   (b) Show that the map
$$\begin{aligned} \mathcal{B}(V) &\xrightarrow{\ \varphi\ } \mathcal{L}(V, V^*) \\ H &\longmapsto \mathscr{L}_H \end{aligned}$$
   is a linear transformation.
   (c) Show that φ is an isomorphism by constructing an inverse linear transformation. (You need to explain that your function is linear, and is inverse to φ.)

B Supplementary Problems

5. For $v \in \mathbb{R}^3$, let $L_v : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by $L_v(w) = v \times w$ (cross product in $\mathbb{R}^3$). Set $B(v, w) = \operatorname{tr}(L_v L_w)$. Show that B is a symmetric bilinear form on $\mathbb{R}^3$ and compute its matrix representation relative to the standard basis of $\mathbb{R}^3$. Is it non-degenerate?

6. Let V be a finite-dimensional vector space over F and $H : V \times V \to \mathbb{F}$ be a non-degenerate bilinear form on V. Suppose $T : V \to V$ is a linear operator on V.

   (a) Show that there exists a unique linear operator $T^* : V \to V$ such that
   $$H(T(v), w) = H(v, T^*(w))$$
   for all $v, w \in V$. Such linear operator T^* is called the **adjoint** of T relative to H.

   (b) Fix a basis β of V. Show that the matrix representations $[T]_\beta$, $[T^*]_\beta$, and $\psi_\beta(H)$ satisfy the relation
   $$[T^*]_\beta = [H]_\beta^{-1} [T]_\beta^t [H]_\beta .$$

7. Let V be a finite-dimensional vector space over $\mathbb{F}$ and $H\colon V \times V \to \mathbb{F}$ be a non-degenerate bilinear form on V. Suppose $T : V \to V$ is a linear operator on V. Show that the following properties are equivalent:

   (a) $H(v, w) = 0$ implies $H(Tv, Tw) = 0$ for all $v, w \in V$.

   (b) There is a constant $c \in \mathbb{F}$ such that $H(Tv, Tw) = cH(v, w)$ for all $v, w \in V$.

   (c) There is a constant $c \in \mathbb{F}$ such that $T^*T = c\,\mathrm{id}_V$.

8. Let $V = C([0, 1])$ be the space of real-valued continuous functions on $[0, 1]$. The function
   $$B(f, g) = \int_0^1 f(x)g(x)\,dx$$
   is a symmetric bilinear form on V. Also, for any continuous function $k : [0, 1]^2 \to \mathbb{R}$, the function
   $$B_k(f, g) = \iint_{[0,1]^2} f(x)g(y)k(x, y)\,dxdy$$
   is a bilinear form on V. Prove or disprove that there exists a function k that makes $B_k = B$. (The function $k = 1$ does not work.)

9. (**Duality**) Let V be a finite-dimensional vector space over $\mathbb{F}$ and $H\colon V \times V \to \mathbb{F}$ be a bilinear form on V. Similar as Problem 4, define the mappings
   $$\begin{aligned} \mathscr{L}_H\colon \quad V &\to V^* \\ v &\mapsto H(v, -) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathscr{R}_H\colon \quad V &\to V^* \\ v &\mapsto H(-, v) \end{aligned}$$
   If we identify the double dual space V^{**} by V in the canonical way (that is, using $\phi\colon V \to V^{**}$ defined by $\phi(v)f = f(v)$ for all $f \in V^*$ and $v \in V$), show that $\mathscr{L}_H^*\colon V^{**} \to V^*$ is the same mapping as $\mathscr{R}_H\colon V \to V^*$. Roughly speaking, we have $\mathscr{L}_H^* = \mathscr{R}_H$.

10. (2019 NTU Math Master Entrance Exam) Let V be a finite-dimensional vector space over $\mathbb{R}$ and $H: V \times V \to \mathbb{R}$ be a **symmetric** bilinear form on V.

   (a) Let W be the vector subspace of V and let

   $$W^\perp = \{u \in V \mid H(u,v) = 0, \text{for all } v \in W\}.$$

   Prove that if $\dim W = m$, and $W^\perp$ is a vector sunspace of V with $\dim W^\perp \geqslant n-m$. (*Hint.* Choose a basis $\{v_i\}_{i=1}^m$ of W and consider the map

   $$u \mapsto (B(u,v_1), \ldots, B(u,v_m))$$

   from V to $\mathbb{R}^n$.)

   (b) Prove that $V = W \oplus W^\perp$ if and only if the restriction of H to W is non-degenerate.

   (c) Prove that if H is non-degenerate on V, then there is a non-negative integer p with $p \leqslant n$ and a basis $\{v_i\}_{i=1}^n$ of V such that

   $$H(v_i, v_j) = \begin{cases} 1 & , 1 \leqslant i = j \leqslant p; \\ -1 & , p+1 \leqslant i = j \leqslant n; \\ 0 & , i \neq j. \end{cases}$$

   One may regard this as a "Gram-Schmidt orthonormalization" on V with respect to H.

11. (**Sylvester's Law of Inertia**) Let V be a finite-dimensional vector space over $\mathbb{R}$ and $H: V \times V \to \mathbb{R}$ be a **symmetric** bilinear form on V. Show that

   (a) the number of positive diagonal entries and

   (b) the number of negative diagonal entries

   in any diagonal matrix representation of H are each independent of the diagonal representation. (*Hint*: Use Courant-Fischer min-max theorem from Homework 5, or see Theorem 6.38 in textbook.) The number of positive diagonal entries is called the **index** of the bilinear form H, and the number of positive diagonal entries minus the number of negative diagonal entries is called the **signature** of the bilinear form H.

12. Let V be a n-dimensional vector space over $\mathbb{R}$ ($n \geqslant 2$) and f, $g \in V^*$ be two linearly independent functionals. Define

   $$\begin{array}{rccc} H: & V \times V & \to & \mathbb{R} \\ & (v,w) & \mapsto & f(v)g(w) + f(w)g(v) \end{array}$$

   (a) Show that H is a symmetric bilinear form.

   (b) Show that the index of H is 1 and the signature of H is 0.