Homework 04

Due Date: 111/03/16

Instruction. Do not submit part B.

A Homework Problems

1. Let V be the space $\mathbb{C}^2$ with the standard inner product $\langle (a_1, a_2), (b_1, b_2) \rangle = a_1\overline{b_1} + a_2\overline{b_2}$. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation defined by $T(1,0) = (1,-2)$ and $T(1,1) = (i,-1)$. Let T^* be the adjoint of T. Find $T^*(x_1, x_2)$ for any $(x_1, x_2) \in \mathbb{C}^2$.
2. Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. Prove that T is linear, show that T^* exists, and find an explicit expression for T^*.
   Remark. Using Dirac's bra-ket notation, T could be written as $|z\rangle \langle y|$.
3. Let V be an inner product space and $T : V \to V$ be a linear transformation such that T^* exists. Prove that:
   (a) $T^*T = 0$ implies $T = 0$.
   (b) $R(T^*)^{\perp} = N(T)$.
   (c) If W is a T-invariant subspace of V, then $W^{\perp}$ is a T^*-invariant subspace of V.

B Supplementary Problems

4. Let $(V, \langle -, - \rangle)$ be a Hilbert space over **F** and V' the space of bounded linear functionals on V equipped with the valued-wise addition and scalar multiplication. By Riesz representation theorem, every element in V' is of the form $f_v : V \to \mathbb{F}$ where $v \in V$ and $f_v(x) = \langle x, v \rangle$ for all $x \in V$. Define

$$\langle -, - \rangle_{\mathrm{op}} : V' \times V' \to \mathbb{F}, \quad \langle f_v, f_w \rangle_{\mathrm{op}} = \langle v, w \rangle.$$

   Prove that $(V', \langle -, - \rangle_{\mathrm{op}})$ is a Hilbert space.

5. Let $(V, \langle -, - \rangle)$ be a non-zero inner product space over $\mathbb{F}$ and $L : V \to \mathbb{F}$ a linear functional. Prove that

$$\begin{aligned} &\sup\{|L(h)| : \|h\| \leqslant 1\} \\ =&\sup\{|L(h)| : \|h\| = 1\} \\ =&\sup\{|L(h)|/\|h\| : h \in V, h \neq 0\} \\ =&\inf\{c > 0 : |L(h)| \leqslant c\|h\|, h \text{ in } V\}. \end{aligned}$$

   Denote this number by $\|L\|_{\mathrm{op}}$, called the *operator norm* of L. Prove $|L(x)| \leq \|L\|_{\mathrm{op}}\|x\|$ for all $x \in V$.

   Remark. The operator norm defines a norm on the space of bounded linear functional V'. In fact, it is easy to see that $\langle V', \|\cdot\|_{\mathrm{op}} \rangle$ is the normed space induced by the Hilbert space $(V', \langle -, - \rangle_{\mathrm{op}})$ defined in (4).

6. Let $(V, \langle -, - \rangle)$ be a non-zero inner product space. Fix $y \in V$ and define $L : V \to \mathbb{F}$ by $L(x) = \langle x, y \rangle$. Prove that L is a bounded linear functional and $\|L\|_{\text{op}} = \|y\|$. Deduce for any $v \in V$, there is a bounded linear functional $L : V \to \mathbb{F}$ such that $\|L\|_{\text{op}} = 1$ and $L(v) = \|v\|$. (the norm of a functional L is defined in (5).)

   Remark. This result holds as well if V is just a normed space, but the proof uses the Hahn-Banach theorem. In fact, even the existence of a non-zero bounded linear map on a normed space requires Hahn-Banach theorem in general; see [Karagila, 2020].

7. Let $(V, \langle -, - \rangle)$ be a non-zero inner product space over $\mathbb{C}$ and $L : V \to \mathbb{C}$ be a bounded linear map. Prove that the real part $\text{Re}\, L : V \to \mathbb{R}$ and the imaginary part $\text{Im}\, L : V \to \mathbb{R}$ of L are bounded $\mathbb{R}$-linear functionals with $\|\text{Re}\, L\|_{\text{op}} = \|\text{Im}\, L\|_{\text{op}} = \|L\|_{\text{op}}$.

8. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Suppose that $T : V \to V$ is a linear operator satisfying properties:

   (a) T is an abstract projection, i.e. $T^2 = T$.

   (b) $\|T(x)\| \le \|x\|$ for all $x \in V$.

   Show that T is an orthogonal projection from V to some subspace W.

9. (**Hodge decomposition**) Let U, V, W be three finite-dimensional inner product spaces. Let

$$\begin{tikzcd} U \arrow[r, bend left, "S"] & V \arrow[l, bend left, "S^*"] \arrow[r, bend left, "T"] & W \arrow[l, bend left, "T^*"] \end{tikzcd}$$

   be a sequence of linear transformations such that $TS = 0$ and T^*, S^* are adjoint of T, S, respectively. Let $\Delta : V \to V$ equal $SS^* + T^*T$ and let $H = \ker(\Delta)$. Show that

$$H = \ker(T) \cap \ker(S^*)$$

   and V has a natural orthogonal decomposition as

$$V = H \oplus \text{Im}(S) \oplus \text{Im}(T^*)$$

   with orthogonal decompositions

$$\ker(T) = H \oplus \text{Im}(S)$$

   and

$$\ker(S^*) = H \oplus \text{Im}(T^*).$$

10. Let $(V, \langle -, - \rangle)$ be a non-complete inner product space over $\mathbb{F}$. Prove that there is a bounded linear functional $L : V \to \mathbb{F}$ such that there are no $v \in V$ satisfy $L(x) = \langle x, v \rangle$ for all $x \in V$.

Definition B.1. Let $(V, \langle -, - \rangle)$ be an inner product space and $\{v_n\}$ a sequence in V. We say

1. $\sum_{n=1}^{\infty} v_n$ *converges absolutely* if $\sum_{n=1}^{\infty} \|v_n\|$ converges in $\mathbb{R}$.
2. $\sum_{n=1}^{\infty} v_n$ *converges unconditionally* if $\sum_{n=1}^{\infty} v_{\sigma(n)}$ converges for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

11. Let $(V, \langle -, - \rangle)$ be an inner product space and $\{v_n\}$ a sequence in V such that $\sum_{n=1}^{\infty} v_n$ converges unconditionally. Prove that $\sum_{n=1}^{\infty} v_{\sigma(n)}$ converges to the same element in V for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$. (*Hint*: Prove that $\sum_{n=1}^{\infty} v_n$ converges *unordered* to some element $v \in V$, that is, for any $\varepsilon > 0$, there is a finite set $I \subseteq \mathbb{N}$ such that for every finite set $J \subseteq \mathbb{N}$ containing I, we have $\left\| v - \sum_{n \in J} v_n \right\| < \varepsilon$.)

12. Let $(V, \langle -, - \rangle)$ be a Hilbert space and $\{v_n\}$ a sequence in V such that $\sum_{n=1}^{\infty} v_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} \|v_n\|$ converges unconditionally.

13. For every $n \in \mathbb{N}$, let $e_n : \mathbb{N} \to \mathbb{F}$ be the element in $l^2(\mathbb{N})$ defined by $e_n(m) = \delta_{mn}$ for all $m \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} n^{-1} e_n$ converges unconditionally in $l^2(\mathbb{N})$, but $\sum_{n=1}^{\infty} \|n^{-1} e_n\|$ diverges. Deduce that if $(V, \langle -, - \rangle)$ is a Hilbert space, then the following statements are equivalent:

    (a) Absolute convergence is equivalent to unconditional convergence in V.

    (b) V is finite-dimensional.

References

[Karagila, 2020] Karagila, A. (2020). Zornian functional analysis or: How i learned to stop worrying and love the axiom of choice.