LINEAR ALGEBRA I – HOMEWORK XI
請在 DEC 14 2021 5PM前線上繳交

(1) Let $A = \begin{pmatrix} 0 & -2 & 5 & 5 \\ 2 & 5 & -2 & -2 \\ 1 & 2 & 2 & 0 \\ -2 & -3 & 2 & 4 \end{pmatrix}$. Find the Jordan canonical form of A.

(2) Let $A, B \in M_{n\times n}(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (*i.e.* there exists an invertible matrix $P \in M_{n\times n}(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute. (Hint: Let λ be one of the eigenvalues of A. Show $\ker(A - \lambda I)$ is B-invariant.)

(3) [Section 7.2, ex.17, FIS] Let T be a linear transformation on a finite-dimensional vector space V such that the characteristic polynomial of T splits, and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T. Let $S : V \to V$ be the mapping defined by

$$S(x) = \lambda_1 v_1 + \cdots + \lambda_k v_k,$$

where v_i is the unique vector in K_{λ_i} such that $x = v_1 + \cdots + v_k$.
(a) Prove that S is a diagonalizable linear transformation on V.
(b) Let $U = T - S$. Prove that U is nilpotent and commutes with S.

(4) Let $A \in M_{n\times n}(\mathbb{C})$. Suppose that $\operatorname{rank}(A) + \operatorname{rank}(I_n + A) = n$. Show that $\operatorname{trace}(A) + \operatorname{rank}(A) = 0$. (Hint: You may apply Jordan canonical form of A.)