Homework 03

Due Date: 111/03/09

Instruction. Do not submit part B.

A Homework Problems

1. (**Parseval's identity**) Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal basis for a finite-dimensional inner product space V. For $x, y \in V$, prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

Deduce that $\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$.

2. Let $V = \mathbb{R}[x]$ be the space of all polynomials with real coefficients equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t,$$

and let $W \subseteq V$ be the subspace consisting of all polynomials with degree ≤ 1. Find the orthogonal projection of $4 + 3x - 2x^2$ on W.

3. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Recall that $S^\perp$ is a subspace of V even if S itself is not. Prove the following results.

   (a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.

   (b) $S \subseteq \left(S^\perp\right)^\perp$; so span$(S) \subseteq \left(S^\perp\right)^\perp$.

   (c) $W = \left(W^\perp\right)^\perp$.

   (d) $V = W \oplus W^\perp$.

   Remark. (c) and (d) may fail in infinite dimensional cases. See exercise 23 in section 6.2 of textbook, for example.

4. Let H_1, H_2 be two inner product spaces over a same field. Let $A : H_1 \to H_2$ be a linear transformation. Show that if H_1 is finite dimensional, then A is bounded. That is, there exists a constant $M \in \mathbb{R}$ such that $\|A(x)\|_{H_2} \leq M\|x\|_{H_1}$ for all $x \in H_1$.

B Supplementary Problems

5. (**Quantum search**) Let $V = \mathbb{C}^n$ equipped with standard inner product, and $\{e_1, e_2, ..., e_n\}$ be its standard orthonormal basis. Fix an integer k, $1 \leq k \leq n$. For an unit vector $v \in V$, we denote the reflection operator across span$\{v\}^\perp$ by I_v. That is to say, I_v sends v to $-v$ and fixes all vectors orthogonal to v.

(a) Put $w := \frac{1}{\sqrt{n}}(e_1 + e_2 + ... + e_n)$. Find an orthonormal basis $\beta = \{v_1, v_2\}$ for the subspace P spanned by $\{e_k\}$ and w.

(b) Define $\mathcal{G} := -I_w I_{e_k}$. Compute the matrix representation of $\mathcal{G}|_{\mathcal{P}}$ using your basis β in (a), and deduce that it acts as a rotation on P.

Remark. $\mathcal{G}$ is called *Grover's operator*, which is used in the quantum search algorithm. The result from this problem is the main idea to the algorithm. k denotes the (unknown) target of the search and we "rotate" the vector w to make it "closer" to e_k by applying $\mathcal{G}$.

6. (**Two-Norm Theorem**) Let $V = \mathbb{R}^n$.

(a) Let $v = (v_1, v_2, ..., v_n)^t$ be a vector in V. Show that both $||v||_1 := \sum_{i=1}^n |v_i|$ and $||v||_\infty := \max\{|v_i| | i = 1, 2, ..., n\}$ are norms on V. That is, prove that these two functions satisfy the axioms of norm.

(b) Let $|| \cdot ||_a$ and $|| \cdot ||_b$ be two norms. Show that there is a constant $C \in \mathbb{R}$ such that $||x||_a \leq C||x||_b$ for all $x \in V$.
Remark. Hence in $\mathbb{R}^n$, no matter which norm we use, the convergence/divergence of sequences remain the same. Thus we say any two norms in $\mathbb{R}^n$ are *equivalent*. This, however, does not generalize to infinite-dimensional vector spaces.

7. The space $\ell^2(\mathbb{Z})$ is the set of all two-sided infinite sequences in $\mathbb{C}$ which are square-summable. That is to say,

$$\ell^2(\mathbb{Z}) := \left\{ (\ldots, a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k, \ldots) \,\middle|\, a_n \in \mathbb{C} \text{ for all } n, \sum_{n\in\mathbb{Z}} |a_n|^2 \text{ converges} \right\}.$$

The addition and scalar multiplication is defined componentwisely.

(a) Prove that if $A, B \in \ell^2(\mathbb{Z})$, we have $A + B \in \ell^2(\mathbb{Z})$. (So it is a vector space.)

(b) For $A, B \in \ell^2(\mathbb{Z})$, define

$$\langle A, B\rangle := \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

Show that it is an inner product on $\ell^2(\mathbb{Z})$.

(c) Prove that $\ell^2(\mathbb{Z})$ is complete.

Definition B.1. Let $(V, \langle -, -\rangle_V)$ and $(W, \langle -, -\rangle_W)$ be inner product spaces over the same field. A linear map $T : V \to W$ is called a *morphism* of inner product spaces if $\langle T(x), T(y)\rangle_W = \langle x, y\rangle_V$ for all $x, y \in V$.

8. (**Completion**) Let $(V, \langle -, -\rangle)$ be an inner product space.

(a) Let $c(V)$ be the space of sequences $f : \mathbb{N} \to V$ under the usual addition and scalar multiplication. Prove that

$$c_c(V) = \{\text{all Cauchy sequences in } V\}$$

is a subspace of $c(V)$, and

$$c_0(V) = \{\text{all sequences in } V \text{ converging to } 0\}$$

is a subspace of $c_c(V)$.

(b) Let $f, g \in c_c(V)$. Prove that $\lim_{n\to\infty} \langle f(n), g(n)\rangle$ converges. Deduce also that if moreover $f \in c_0(V)$, then $\lim_{n\to\infty} \langle f(n), g(n)\rangle = 0$. (*Hint*: Cauchy-Schwarz inequality.)

(c) Define

$$\langle -, -\rangle : c_c(V)/c_0(V) \to \mathbb{C}, \quad \langle f + c_0(V), g + c_0(V)\rangle = \lim_{n\to\infty} \langle f(n), g(n)\rangle.$$

Prove that $(c_c(V)/c_0(V), \langle -, -\rangle)$ is an inner product space. (*Hint*: Use (8b) to see that the limit converges and is independent of the choice of the representative of coset.)

Write $\widehat{V} = c_c(V)/c_0(V)$ and $\widehat{f} = f + c_0(V)$ for elements in $\widehat{V}$ for simplicity. $\widehat{V}$ is called the *completion* of V.

(d) Prove that $\left(\widehat{V}, \langle -, -\rangle\right)$ is a Hilbert space. (*Hint*: Let $\left\{\widehat{f_n}\right\}$ be a Cauchy sequence in $\widehat{V}$. For each $n \in \mathbb{N}$, since f_n is a Cauchy sequence in V, there is $N_n \in \mathbb{N}$ such that $|f_n(m) - f_n(k)| < 1/n$ for all $m, k \geq N_n$. Define a sequence $g(n) = f_n(N_n)$ in V. Prove that g is a Cauchy sequence in V and $\{\widehat{f_n}\}$ converges to $\widehat{g}$ in $\widehat{V}$.)

(e) Prove that the map

$$i : V \to \widehat{V}, \quad i(v) = (v, v, v, \ldots)$$

is a morphism of inner product space.

(f) Let W be a Hilbert space and $T : V \to W$ a morphism of inner product spaces. Prove that there is a unique morphism $\widehat{T} : \widehat{V} \to W$ of inner product spaces such that the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "i"'] & W \\ \widehat{V} \arrow[ur, "\widehat{T}"'] & \end{tikzcd}$$

commutes. The exercise (8f) is called the universal property of completion.

Definition B.2. Let $(V, \langle -, -\rangle)$ be an inner product space and S a subset of V. We say $\sum_{v\in S} v$ converges to an element x in V *unordered* if for any $\varepsilon > 0$, there is a finite subset I of S such that for any finite subset J of S containing I, we have $\|x - \sum_{v\in J} v\| < \varepsilon$. In this case, we write $\sum_{v\in S} v = x$.

9. Let $(V, \langle -, -\rangle)$ be a Hilbert space and S an orthonormal subset of V. Prove that the following statements are equivalent:

(a) S is a maximal orthonormal subset of V.

(b) For any $x \in V$, there is a unique expression $x = \sum_{v\in S} a_v v$ such that $a_v \in \mathbb{F}$ are zero for all but countably many $v \in S$.

(c) $\langle x, y\rangle = \sum_{v\in S} \langle x, v\rangle \overline{\langle y, v\rangle}$ for all $x, y \in V$.

(d) $\|x\|^2 = \sum_{v\in S} |\langle x, v\rangle|^2$ for all $x \in V$.

All of the sums are unordered sum.

10. Let H be a finite-dimensional inner product space, $\dim H = n$ and $\{\varphi_i\}_{i=1}^n$ be an orthonormal basis for H. Let $A : H \to H$ be a linear transformation and set $\psi_i := A\varphi_i$. It turns out we may write

$$Ax = \sum_{i=1}^{n} \langle x, \varphi_i \rangle \psi_i$$

for $x \in H$. Show that

(a) $I - A$ is invertible if and only if $\det(M) \neq 0$, where M is an $n \times n$ matrix with $M_{ij} = \delta_{ij} - \langle \psi_j, \varphi_i \rangle$.

(b) If $I - A$ is invertible then for all $y \in H$

$$(I - A)^{-1} y = y - \frac{1}{\det(M)} \sum_{k=1}^{n} M_k \psi_k$$

where M_k is the $n \times n$ matrix obtained by replacing the k-th column of M by $\begin{pmatrix} \langle y, \varphi_1 \rangle \\ \langle y, \varphi_2 \rangle \\ \vdots \\ \langle y, \varphi_n \rangle \end{pmatrix}$. Alternatively, the sum $\sum_{k=1}^{n} M_k \psi_k$ can be also represented by a determinant:

$$\sum_{k=1}^{n} M_k \psi_k = \det \begin{pmatrix} & & & & \langle y, \varphi_1 \rangle \\ & & M & & \langle y, \varphi_2 \rangle \\ & & & & \vdots \\ & & & & \langle y, \varphi_n \rangle \\ \psi_1 & \psi_2 & \cdots & \psi_n & 0 \end{pmatrix}$$

(c) We have

$$N(I - A) = \left\{ \sum_{i=1}^{n} a_i \psi_i \;\middle|\; (\delta_{ij} - \langle \psi_j, \varphi_i \rangle)(a_i) = 0 \right\}.$$

11. Let H be a Hilbert space and $A : H \to H$ is linear with $\|A\| < 1$. Show that

(a) $I - A$ is invertible and for all $y \in H$

$$(I - A)^{-1} y = \sum_{i=0}^{\infty} A^i y.$$

(b) We have

$$\lim_{n \to \infty} \left\| (I - A)^{-1} - \sum_{i=0}^{\infty} A^i \right\| = 0$$

and

$$\|(I - A)^{-1}\| \leqslant \frac{1}{1 - \|A\|}.$$

(c) Moreover, if $A, B : H \to H$ are linear with A invertible and $\|A - B\| < \frac{1}{\|A^{-1}\|}$ then B is invertible,

$$B^{-1}y = \sum_{i=0}^{\infty} [A^{-1}(A - B)]^i A^{-1} y$$

and

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A^{-1}\| \|A - B\|}.$$