Advanced Networking Assignment 4: Queuing Theory

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1 Maximum Rate Possible in a Network of Queues

Consider the network of queues shown in figure1 . There are three nodes with the following

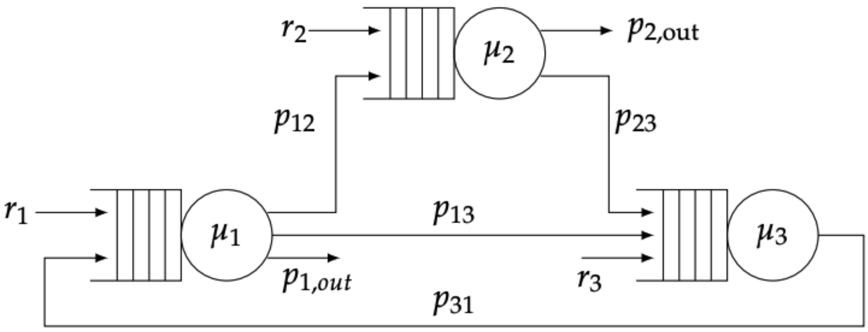

Figure 1: A network of queues with probabilistic routing.

characteristics:

- Each server has a mean service rate of $\mu = 10$ jobs/sec.
- External arrival rates: $r_2 = 1$ job/sec, $r_3 = 1$ job/sec, and r_1 is the unknown arrival rate to be determined.
- Routing probabilities:
  - From node 1: $p_{12} = 0.8$ (to node 2), $p_{13} = 0.2$ (to node 3), and $p_{1,\text{out}} = 0$.
  - From node 2: $p_{23} = 0.2$ (to node 3) and $p_{2,\text{out}} = 0.8$.
  - From node 3: $p_{31} = 1$ (all jobs are routed back to node 1).

Let λ_1, λ_2, and λ_3 denote the total arrival rates at nodes 1, 2, and 3, respectively. Based on the routing and external arrivals, we can establish the following balance equations:

$$\lambda_1 = r_1 + p_{31}\lambda_3 = r_1 + \lambda_3, \tag{1}$$

$$\lambda_2 = r_2 + p_{12}\lambda_1 = 1 + 0.8\,\lambda_1, \tag{2}$$

$$\lambda_3 = r_3 + p_{23}\lambda_2 + p_{13}\lambda_1 = 1 + 0.2\,\lambda_2 + 0.2\,\lambda_1. \tag{3}$$

Substituting equation (2) into (3) gives:

$$\lambda_3 = 1 + 0.2\,(1 + 0.8\,\lambda_1) + 0.2\,\lambda_1 = 1.2 + 0.36\,\lambda_1.$$

Now substitute this expression into (1):

$$\lambda_1 = r_1 + (1.2 + 0.36\,\lambda_1).$$

Solving for λ_1:

$$\begin{aligned} \lambda_1 - 0.36\,\lambda_1 &= r_1 + 1.2, \\ 0.64\,\lambda_1 &= r_1 + 1.2, \\ \lambda_1 &= \frac{r_1 + 1.2}{0.64}. \end{aligned}$$

Stability of the system requires that for each node the arrival rate must be less than the service rate ($\lambda_i < 10$ jobs/sec). The most restrictive condition is at node 1:

$$\frac{r_1 + 1.2}{0.64} < 10 \quad \Rightarrow \quad r_1 + 1.2 < 6.4 \quad \Rightarrow \quad r_1 < 5.2.$$

Conclusion: To ensure system stability, the external arrival rate r_1 must satisfy

$$r_1 < 5.2 \quad \text{(jobs/sec)}.$$

2 Little's Theorem and its Application to the Emergency Room

In this problem, the following are given:

- The mean waiting time in the Emergency Room (ER) is $W = 3$ hours.
- Patients arrive on average every 5 minutes, which implies an arrival rate of:

$$\lambda = \frac{60\,\text{min/hour}}{5\,\text{min/patient}} = 12\,\text{patients/hour}.$$

Little's Theorem states that:

$$L = \lambda W,$$

where L is the average number of customers (or patients) in the system.

Thus, the average number of patients waiting (or being treated) in the ER is:

$$L = 12\,\text{patients/hour} \times 3\,\text{hours} = 36\,\text{patients}.$$

Discussion on Waiting Room Capacity: The computed $L = 36$ represents an *average* value. Due to random fluctuations inherent in queuing systems, there is a nonzero probability of having more than 36 patients at any given time. Therefore, if one requires the waiting room to always accommodate arriving patients (i.e., ensuring zero loss), the capacity would need to be infinite. In practical designs, however, the waiting room is sized to limit the probability of overflow to an acceptably small level.

3 Proof of the M/M/1 Queue Length Formula

For a standard M/M/1 queue, the stationary distribution (including the customer in service) is given by:

$$P(N = n) = (1 - \rho)\,\rho^n, \quad n \geq 0,$$

with the utilization $\rho = \lambda/\mu$. The expected number of customers in the system is:

$$E[N] = \frac{\rho}{1 - \rho}.$$

Since the server can serve at most one customer at a time, the mean number of customers in service is ρ. Thus, the average number of customers waiting in the queue (excluding the one in service) is:

$$E[N_Q] = E[N] - \rho = \frac{\rho}{1 - \rho} - \rho.$$

Simplify the expression:

$$\begin{aligned} E[N_Q] &= \frac{\rho - \rho(1 - \rho)}{1 - \rho} \\ &= \frac{\rho - \rho + \rho^2}{1 - \rho} \\ &= \frac{\rho^2}{1 - \rho}. \end{aligned}$$

This completes the proof that:

$$E[N_Q] = \frac{\rho^2}{1 - \rho}.$$

4 M/M/1 Queue Analysis for a Packet Transmission Link

Consider a transmission link with the following parameters:

- Packets arrive according to a Poisson process with rate $\lambda = 450$ packets/sec.
- The service times are exponentially distributed.
- The average packet size is 250 bytes (i.e., $250 \times 8 = 2000$ bits).
- The link capacity is $1\,\text{Mbps} = 10^6$ bits/sec.

Service Rate Calculation

The service rate μ is determined by the transmission time for one packet:

$$\mu = \frac{\text{Link Capacity}}{\text{Packet Size}} = \frac{10^6\,\text{bits/sec}}{2000\,\text{bits}} = 500\,\text{packets/sec}.$$

Thus, the system's utilization is:

$$\rho = \frac{\lambda}{\mu} = \frac{450}{500} = 0.9.$$

Steady-State Probabilities

The steady-state probability for there being n packets in the system is:

$$P(N = n) = (1 - \rho)\,\rho^n.$$

Hence,

$$\begin{aligned} P(N = 1) &= 0.1 \times (0.9)^1 = 0.09, \\ P(N = 2) &= 0.1 \times (0.9)^2 = 0.081, \\ P(N = 10) &\approx 0.1 \times (0.9)^{10} \approx 0.0349. \end{aligned}$$

Average Number of Packets and Waiting Time

The average number of packets in the system is:

$$E[N] = \frac{\rho}{1 - \rho} = \frac{0.9}{0.1} = 9.$$

And the average number in the queue (excluding the packet in service) is:

$$E[N_Q] = \frac{\rho^2}{1 - \rho} = \frac{(0.9)^2}{0.1} = 8.1.$$

Using Little's Theorem, the average delay (or total time in the system) is given by:

$$W = \frac{E[N]}{\lambda} = \frac{9}{450} = 0.02\,\text{sec} \quad (20\,\text{ms}).$$

The mean waiting time in the queue is:

$$W_Q = \frac{E[N_Q]}{\lambda} \approx \frac{8.1}{450} \approx 0.018\,\text{sec} \quad (18\,\text{ms}),$$

with an additional average service time of:

$$\frac{1}{\mu} = \frac{1}{500} = 0.002\,\text{sec}.$$

Thus, the overall delay $W = W_Q + 1/\mu \approx 0.018 + 0.002 = 0.02$ sec.

5 Conclusion

In this report, we have:

1. Derived the condition for stability in a network of queues with probabilistic routing, showing that to keep the system stable, the external arrival rate r_1 must satisfy $r_1 < 5.2$ jobs/sec.

2. Applied Little's Theorem to an ER setting to calculate an average of 36 patients in the system and discussed the implications for waiting room capacity.

3. Provided a detailed proof of the result $E[N_Q] = \frac{\rho^2}{1-\rho}$ for an M/M/1 queue.

4. Analyzed an M/M/1 model for a packet transmission link by calculating steady-state probabilities, the average number of packets in the system and queue, as well as the mean delay.

These analyses illustrate the practical applications of queuing theory in both networking systems and everyday scenarios. They also reinforce the theoretical concepts presented in *Elements of Queuing Theory* [1].

Acknowledgements

This report is based on assignment materials and the theoretical background provided in the textbook *Elements of Queuing Theory* by A. Carzaniga [1]. Additional insights were derived from class lectures and related academic resources.

References

[1] A. Carzaniga, *Elements of Queuing Theory*, March 5, 2025.