Discretization Methods
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Exercise Sheet 3

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Exercise 1: Fourier–Galerkin for Variable Coefficient

Consider the variable coefficient problem

$$\frac{\partial u}{\partial t} + \sin(x)\frac{\partial u}{\partial x} = 0, \tag{1}$$

subject to periodic boundary conditions.

Fourier–Galerkin Approximation

We seek an approximate solution in the form:

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx} \tag{2}$$

The Galerkin method requires:

$$\left\langle \frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x}, e^{imx} \right\rangle = 0, \quad \forall m = -N, \dots, N \tag{3}$$

where $\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$.

Expanding the terms:

$$\frac{\partial u_N}{\partial t} = \sum_{k=-N}^{N} \frac{d\hat{u}_k}{dt} e^{ikx}$$
$$\frac{\partial u_N}{\partial x} = \sum_{k=-N}^{N} ik\hat{u}_k e^{ikx}$$

The variable coefficient $\sin(x)$ can be written as:

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \tag{4}$$

So

$$\begin{aligned}\sin(x)\frac{\partial u_N}{\partial x} &= \sum_{k=-N}^{N} ik\hat{u}_k \cdot \frac{1}{2i}(e^{i(k+1)x} - e^{i(k-1)x}) \\ &= \sum_{k=-N}^{N} \frac{k\hat{u}_k}{2}(e^{i(k+1)x} - e^{i(k-1)x})\end{aligned}$$

Projecting onto e^{imx} and using orthogonality:

$$\begin{aligned}\left\langle \sin(x)\frac{\partial u_N}{\partial x}, e^{imx} \right\rangle &= \int_0^{2\pi} \sum_{k=-N}^{N} \frac{k\hat{u}_k}{2}(e^{i(k+1)x} - e^{i(k-1)x})\overline{e^{imx}}dx \\ &= \pi\left[(m-1)\hat{u}_{m-1} - (m+1)\hat{u}_{m+1}\right]\end{aligned}$$

Thus, the Galerkin system is:

$$\frac{d\hat{u}_m}{dt} + \frac{1}{2}\left[(m-1)\hat{u}_{m-1} - (m+1)\hat{u}_{m+1}\right] = 0, \quad m = -N, \ldots, N \tag{5}$$

Is $P_N u = u_N$?

No, in general $P_N u \neq u_N$ for variable coefficient problems. $P_N u$ is the L^2 projection of the true solution onto the Fourier space, while u_N is the Galerkin approximation, which only coincides with $P_N u$ for constant coefficient problems.

Exercise 2: Fourier–Galerkin with Dirichlet BCs

Consider

$$\frac{\partial u}{\partial t} + \sin(x)\frac{\partial u}{\partial x} = 0, \tag{6}$$

with homogeneous Dirichlet boundary conditions:

$$u(0,t) = u(\pi,t) = 0. \tag{7}$$

Sine Basis Expansion

We expand u in a sine series:

$$u_N(x,t) = \sum_{k=1}^{N} \hat{u}_k(t)\sin(kx) \tag{8}$$

The Galerkin condition:

$$\left\langle \frac{\partial u_N}{\partial t} + \sin(x)\frac{\partial u_N}{\partial x}, \sin(mx) \right\rangle = 0, \quad m = 1, \ldots, N \tag{9}$$

Compute the terms:

$$\begin{aligned}\frac{\partial u_N}{\partial t} &= \sum_{k=1}^{N} \frac{d\hat{u}_k}{dt}\sin(kx) \\ \frac{\partial u_N}{\partial x} &= \sum_{k=1}^{N} k\hat{u}_k\cos(kx)\end{aligned}$$

Expand $\sin(x)\cos(kx)$:

$$\sin(x)\cos(kx) = \frac{1}{2}[\sin((k+1)x) + \sin((k-1)x)] \tag{10}$$

So

$$\begin{aligned}\sin(x)\frac{\partial u_N}{\partial x} &= \sum_{k=1}^{N} k\hat{u}_k \cdot \frac{1}{2}[\sin((k+1)x) + \sin((k-1)x)] \\ &= \frac{1}{2}\sum_{k=1}^{N} k\hat{u}_k\sin((k+1)x) + \frac{1}{2}\sum_{k=1}^{N} k\hat{u}_k\sin((k-1)x)\end{aligned}$$

Project onto $\sin(mx)$ using orthogonality:

$$\left\langle \sin(x)\frac{\partial u_N}{\partial x}, \sin(mx) \right\rangle = \frac{\pi}{2}[(m+1)\hat{u}_{m+1} + (m-1)\hat{u}_{m-1}]$$

Thus, the Galerkin system is:

$$\frac{d\hat{u}_m}{dt} + \frac{1}{2}[(m+1)\hat{u}_{m+1} + (m-1)\hat{u}_{m-1}] = 0, \quad m = 1, \ldots, N \tag{11}$$

Exercise 3: Tau Approximation

Consider the same problem as Exercise 2, but approximate

$$u_N(x,t) = \sum_{n=0}^{N+N_b} \hat{u}_n(t)\cos(nx). \tag{12}$$

with $u(0,t) = u(\pi,t) = 0$.

We define the residual

$$R_N(x,t) = u_{N,t}(x,t) + \sin(x)u_{N,x}(x,t).$$

Expand u_N and its derivatives:

$$u_N(x,t) = \sum_{n=0}^{N+N_b} \hat{u}_n(t)\cos(nx),$$

$$u_{N,t}(x,t) = \sum_{n=0}^{N+N_b} \dot{\hat{u}}_n(t)\cos(nx),$$

$$u_{N,x}(x,t) = -\sum_{n=0}^{N+N_b} n\hat{u}_n(t)\sin(nx).$$

Using the trigonometric identity:

$$\sin(x)\sin(nx) = \frac{1}{2}\left[\cos((n-1)x) - \cos((n+1)x)\right]$$

Therefore,

$$\begin{aligned}
\sin(x)u_{N,x}(x,t) &= -\sum_{n=0}^{N+N_b} n\hat{u}_n(t)\cdot\frac{1}{2}\left[\cos((n-1)x) - \cos((n+1)x)\right] \\
&= -\frac{1}{2}\sum_{n=0}^{N+N_b} n\hat{u}_n(t)\cos((n-1)x) + \frac{1}{2}\sum_{n=0}^{N+N_b} n\hat{u}_n(t)\cos((n+1)x)
\end{aligned}$$

Expand all terms in the cosine basis:

$$R_N(x,t) = \sum_{m=0}^{N+N_b} r_m(t)\cos(mx)$$

where

$$r_m(t) = \dot{\hat{u}}_m + \frac{m-1}{2}\hat{u}_{m-1} - \frac{m+1}{2}\hat{u}_{m+1}, \quad m = 0, \ldots, N+N_b,$$

with $\hat{u}_{-1} = \hat{u}_{N+N_b+1} = 0$.

The Tau method proceeds as follows:

- The first N spectral equations vanish: $r_m = 0$, $m = 0, 1, \ldots, N-1$, i.e., the residual projected onto the cosine basis is zero.
- The remaining $N_b + 1$ equations are replaced by the boundary conditions:

$$u_N(0,t) = \sum_{n=0}^{N+N_b} \hat{u}_n(t) = 0, \quad u_N(\pi,t) = \sum_{n=0}^{N+N_b} (-1)^n \hat{u}_n(t) = 0.$$

This gives a total of $N + N_b + 2$ equations for the unknowns $\hat{u}_0, \ldots, \hat{u}_{N+N_b}$.

Exercise 4: Fourier–Collocation for Burgers' Equation

Consider Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \tag{13}$$

subject to periodic boundary conditions.

Let $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx}$ be the truncated Fourier series. Choose the collocation points

$$x_j = \frac{2\pi j}{2N+1}, \quad j = 0, 1, \ldots, 2N,$$

Require the PDE to hold at each x_j:

$$\frac{\partial u_N}{\partial t}(x_j,t) + \frac{1}{2}\frac{\partial}{\partial x}[u_N^2](x_j,t) = \varepsilon \frac{\partial^2 u_N}{\partial x^2}(x_j,t).$$

The spectral expressions for the derivatives are:

$$\frac{\partial u_N}{\partial x}(x_j) = \sum_{k=-N}^{N} ik\hat{u}_k e^{ikx_j}, \quad \frac{\partial^2 u_N}{\partial x^2}(x_j) = \sum_{k=-N}^{N} -k^2 \hat{u}_k e^{ikx_j}.$$

The nonlinear term $u_N^2(x_j)$ is computed in physical space, then transformed back to modal space using FFT.

Write the PDE at each x_j as an ODE system:

$$\frac{du_N(x_j,t)}{dt} + \frac{1}{2}\frac{\partial (u_N^2)}{\partial x}(x_j,t) = \varepsilon \frac{\partial^2 u_N}{\partial x^2}(x_j,t)$$

Equivalently, in modal space:

$$\dot{\hat{u}}_k + \frac{ik}{2} \sum_{m=-N}^{N} \hat{u}_m \hat{u}_{k-m} = -\varepsilon k^2 \hat{u}_k, \quad k = -N, \ldots, N.$$

This semi-discrete system is the Fourier–Collocation approximation. Time integration can then be performed by any suitable ODE solver in modal space.