

# Discretization Methods

## Spring Semester 2025

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### Exercise Sheet 2

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#### Exercise 1

Repeat exercise 3 from homework 1 using the even method. Compare your results with the results you previously obtained using the odd method. Which method is more accurate?

#### Exercise 2

Using the Fourier differentiation matrix for even method, compute the derivative of the following functions defined on interval  $[0, 2\pi]$ :

(a)  $f(x) = \cos(10x)$

(b)  $f(x) = \cos(x/2)$

(c)  $f(x) = x$

Compute pointwise error  $L_{\text{inf}}$  and global error  $L_2$  for increasing values of  $N$  and discuss the different behaviors and convergence rates you observe. Can you explain the differences?

#### Exercise 3

This exercise is based on the example in the book.. but it will be useful for the exam project. Consider scalar hyperbolic problem given as

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t} &= -2\pi \frac{\partial u(x, t)}{\partial x}, \\ u(0, t) &= u(2\pi, t), \\ u(x, 0) &= \exp[\sin(x)],\end{aligned}$$

where  $u(x, t) \in C^\infty[0, 2\pi]$  is assumed periodic and the initial condition is assumed periodically extended.

The problem has an analytic solution given as

$$u(x, t) = \exp[\sin(x - 2\pi t)].$$

This solution represents the initial condition which propagates at the velocity  $2\pi$  towards increasing  $x$ . Construct a program that solves this problem using an equidistant grid given as

$$x_j = \frac{2\pi}{N+1}j = j\Delta x, \quad j \in [0, N],$$

and use the exact solution at  $t = 0$  as the initial condition.

To advance the equation in time, use 4th order Runge-Kutta method obtained by first defining the function,  $F(u^n)$ , as

$$\frac{du^n}{dt} = F(u^n) = -2\pi \frac{\partial u^n}{\partial x},$$

where  $u^n$  represents the solution at  $t = n\Delta t$  and  $\Delta t$  is the time step. The 4th order Runge-Kutta method for advancing from  $u^n$  to  $u^{n+1}$  is then defined as

$$u_1 = u^n + \frac{\Delta t}{2}F(u^n)$$

$$u_2 = u^n + \frac{\Delta t}{2}F(u_1)$$

$$u_3 = u^n + \Delta t F(u_2)$$

$$u^{n+1} = \frac{1}{3} \left( -u^n + u_1 + 2u_2 + u_3 + \frac{\Delta t}{2} F(u_3) \right),$$

where  $u_1$ ,  $u_2$  and  $u_3$  are introduced for convenience as help-functions/arrays.

Use sufficiently small time step,  $\Delta t$ , to ensure stability and to reduce time-stepping errors.

Use three different approximations of the spatial derivative:

- The local second order centered finite difference approximation to the spatial derivative of  $u(x, t)$  at  $x_j$

$$\frac{du}{dx} \Big|_{x_j} = \frac{u_{j+1} - u_{j-1}}{2\Delta x}.$$

- The local forth order centered finite difference approximation

$$\frac{du}{dx} \Big|_{x_j} = \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x}.$$

- The global infinite order approximation using Fourier differentiation matrix

$$\frac{du}{dx} \Big|_{x_j} = \sum_{i=0}^N \tilde{D}_{ji} u_i,$$

where the entries of the matrix operator are

$$\tilde{D}_{ji} = \begin{cases} \frac{(-1)^{j+i}}{2} \left[ \sin \left( \frac{(j-i)\pi}{N+1} \right) \right]^{-1} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}.$$

- Measure the  $L^\infty$ -error between the computed solution using three approximations of spatial derivative and the exact solution at time  $t = \pi$  for  $N = 8, 16, 32, 64, 128, 256, 512, 1024, 2048$ . What are the convergence rates observed and do they correspond to what you would expect? If you measure error for  $N = 2048$  using second order scheme, what  $N$  do you need to obtain the same error using fourth order and infinite order schemes?
- Compare the performance of second order and infinite order schemes for long time integration. For both schemes, plot the computed and the exact solutions at time  $t = 0, 100, 200$ . For second order scheme use  $N = 200$  and for infinite order scheme use  $N = 10$ .